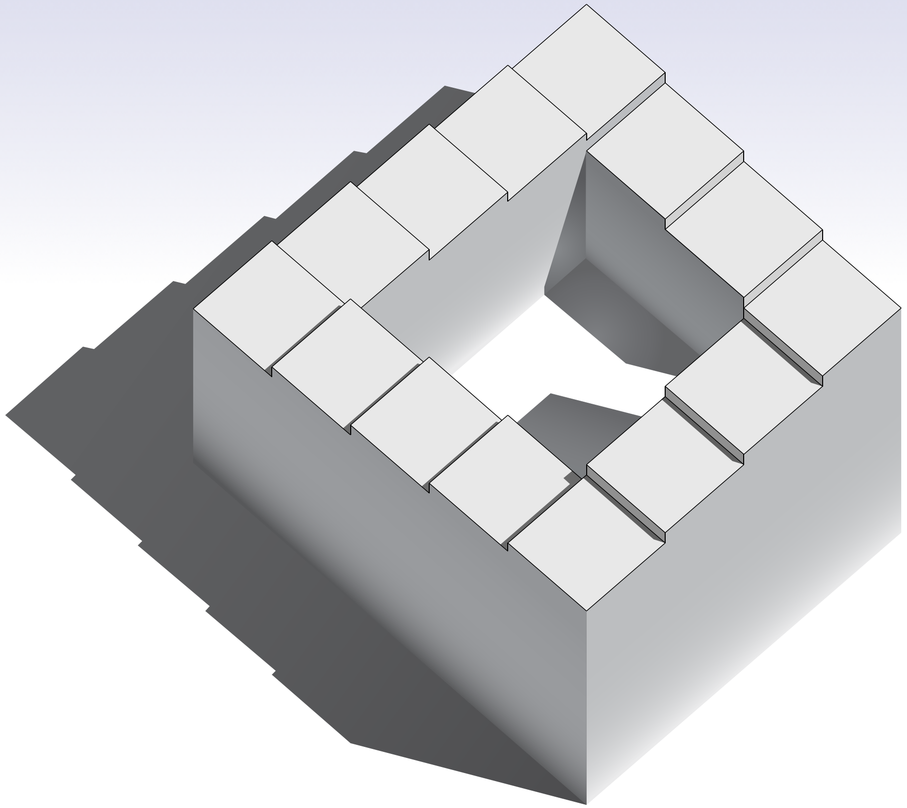
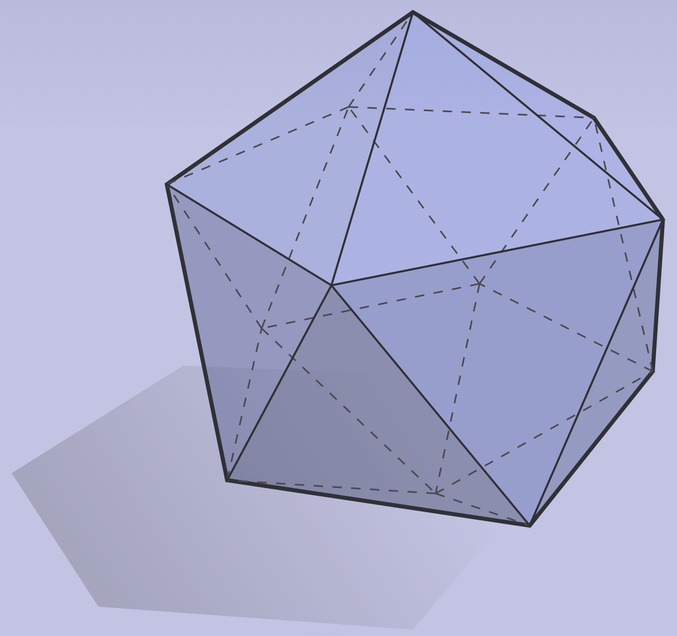


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

Lecture 1:

Overview



Discrete Differential Geometry: An Applied Introduction

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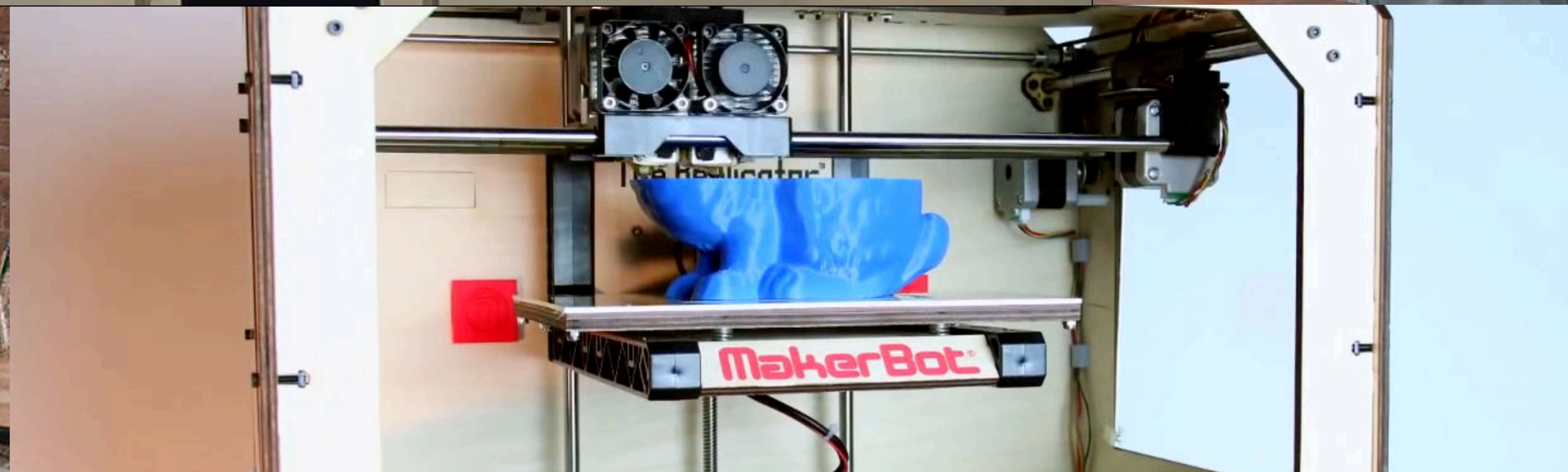
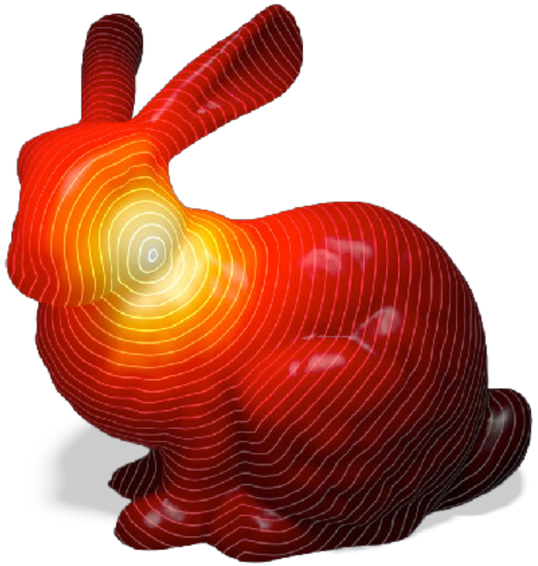
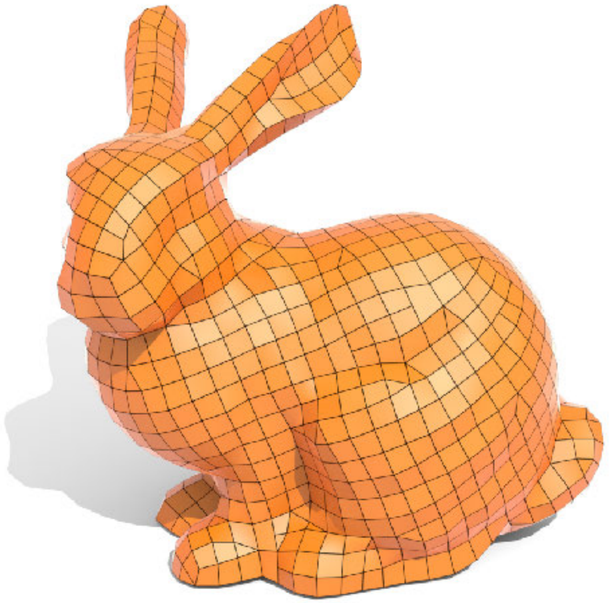
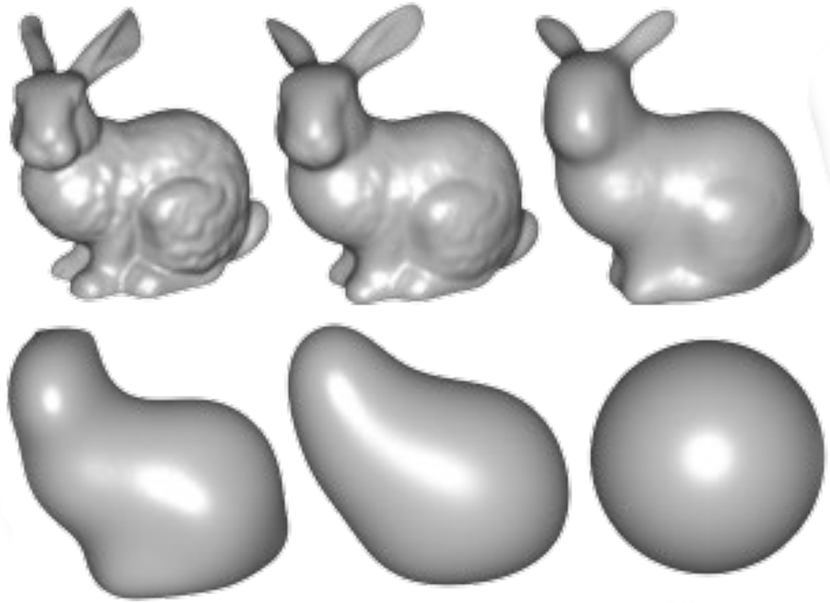
Geometry is Coming…

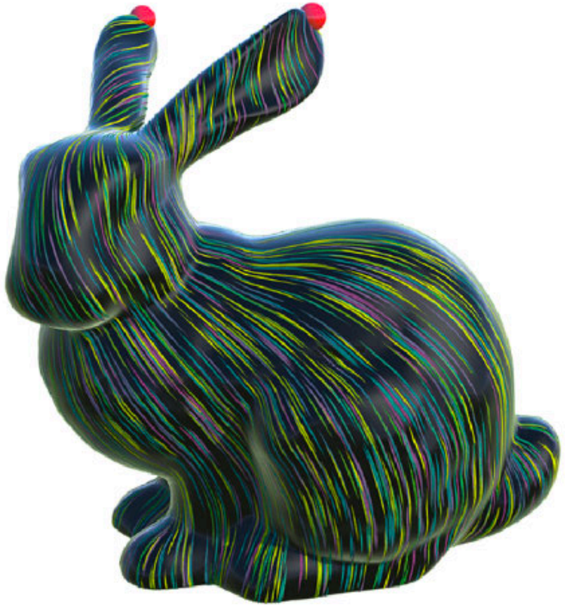


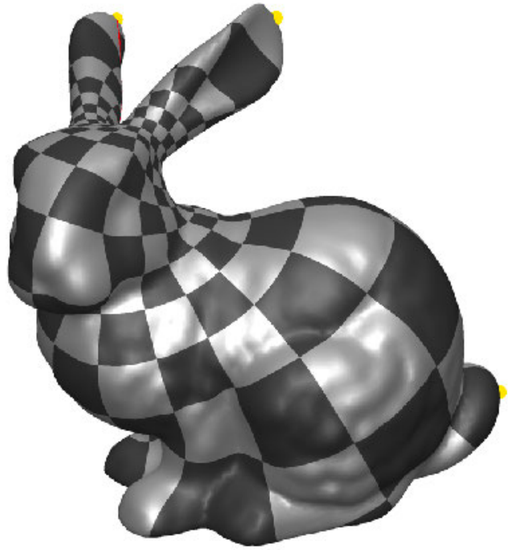
Applications of DDG: Geometry Processing

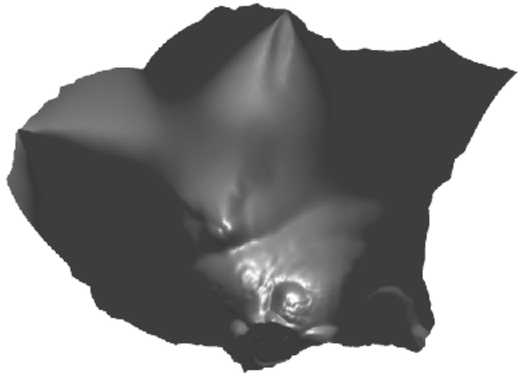




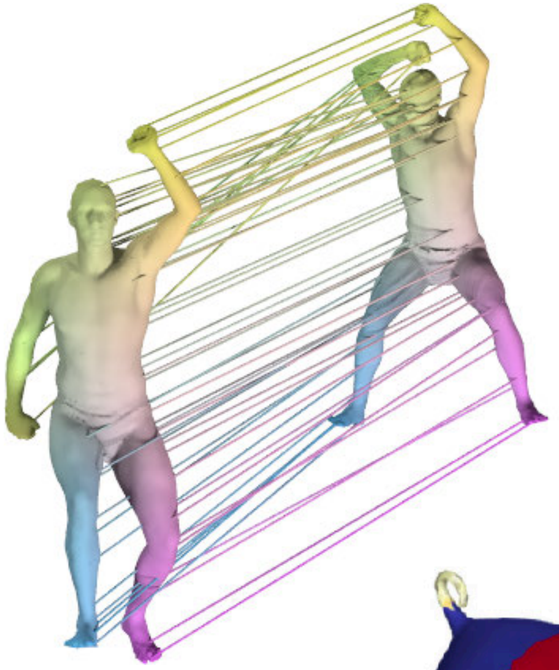


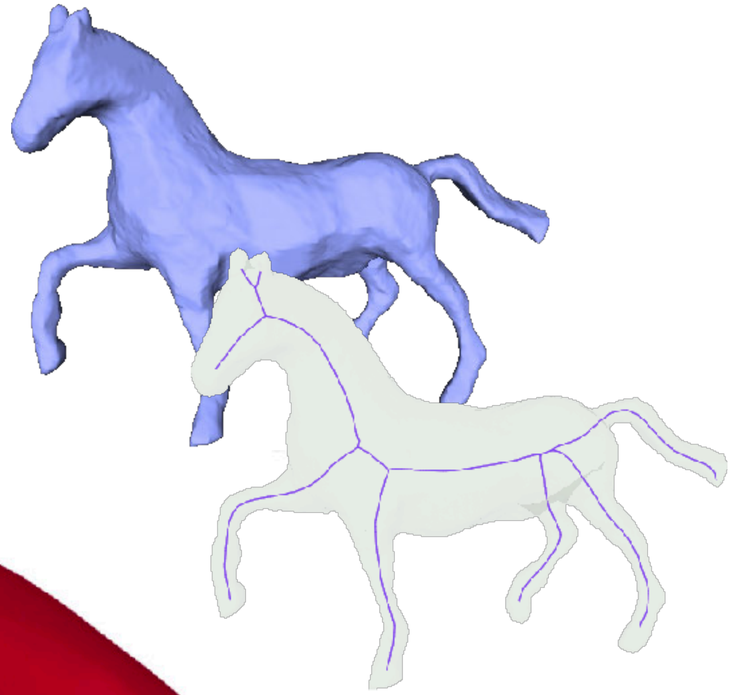


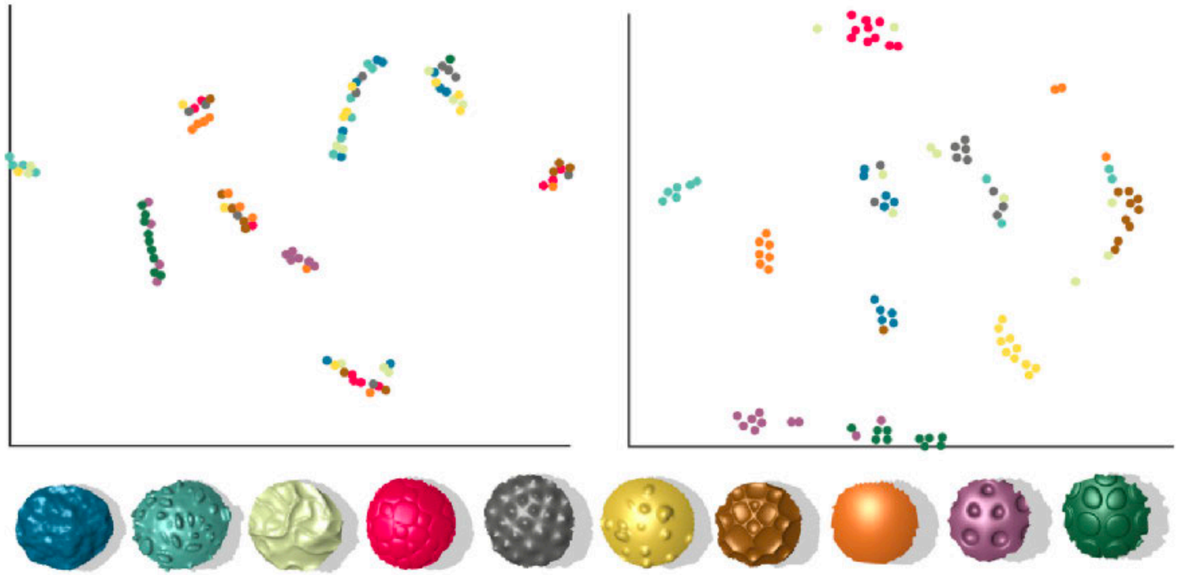


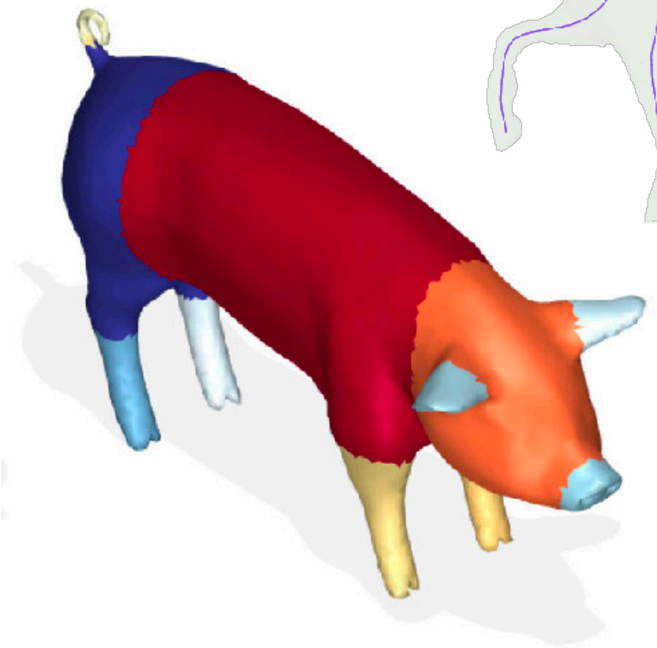


Applications of DDG: Shape Analysis

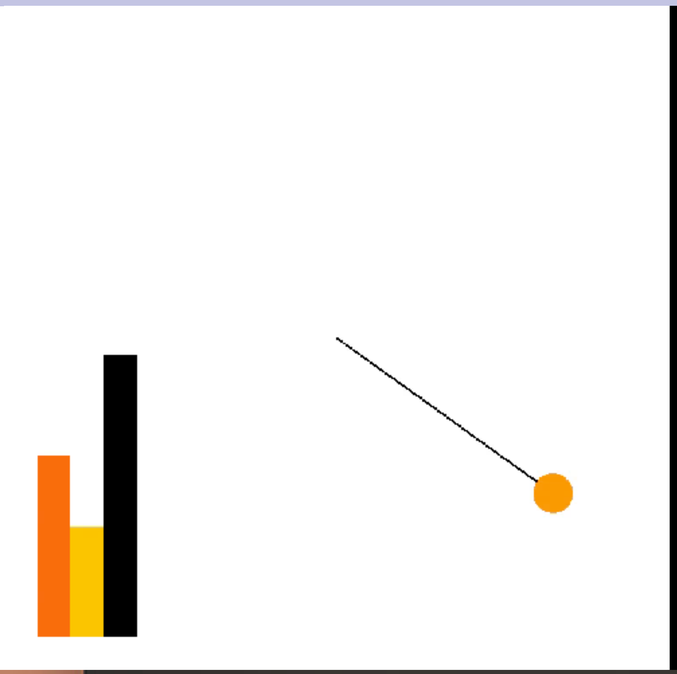


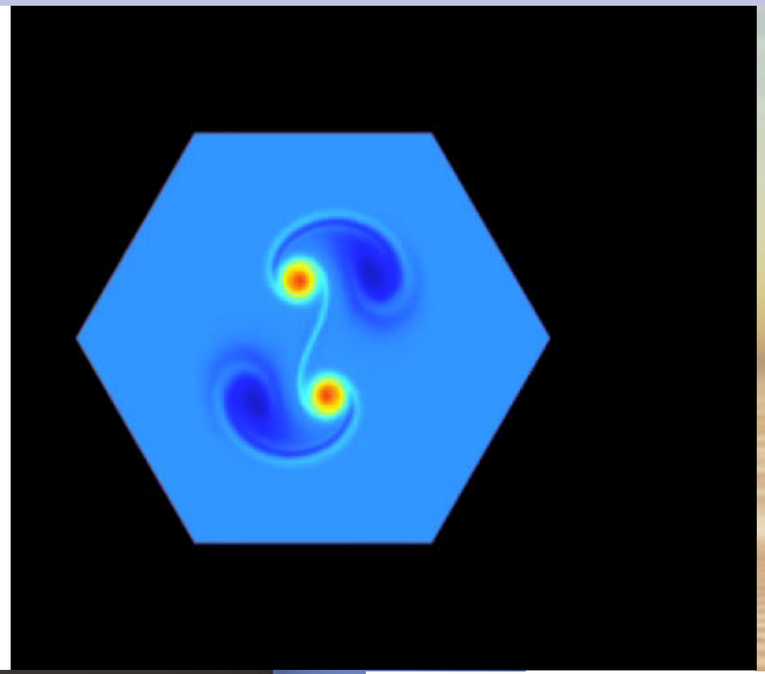


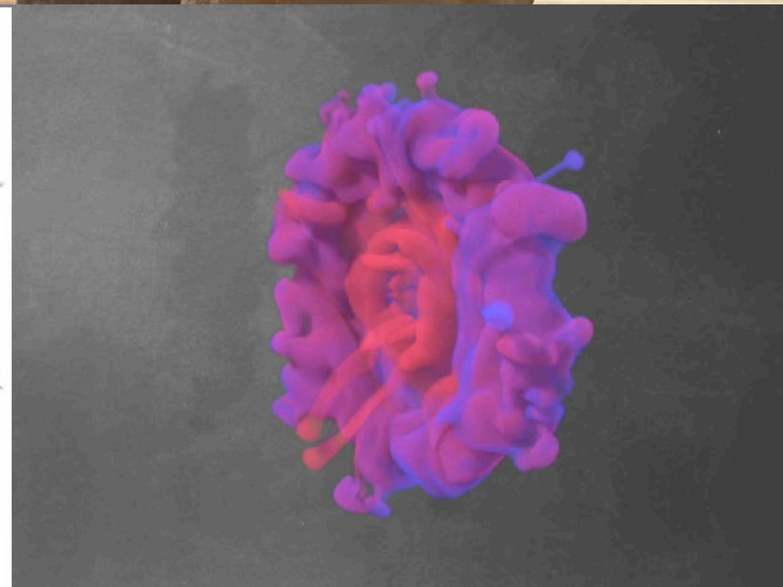




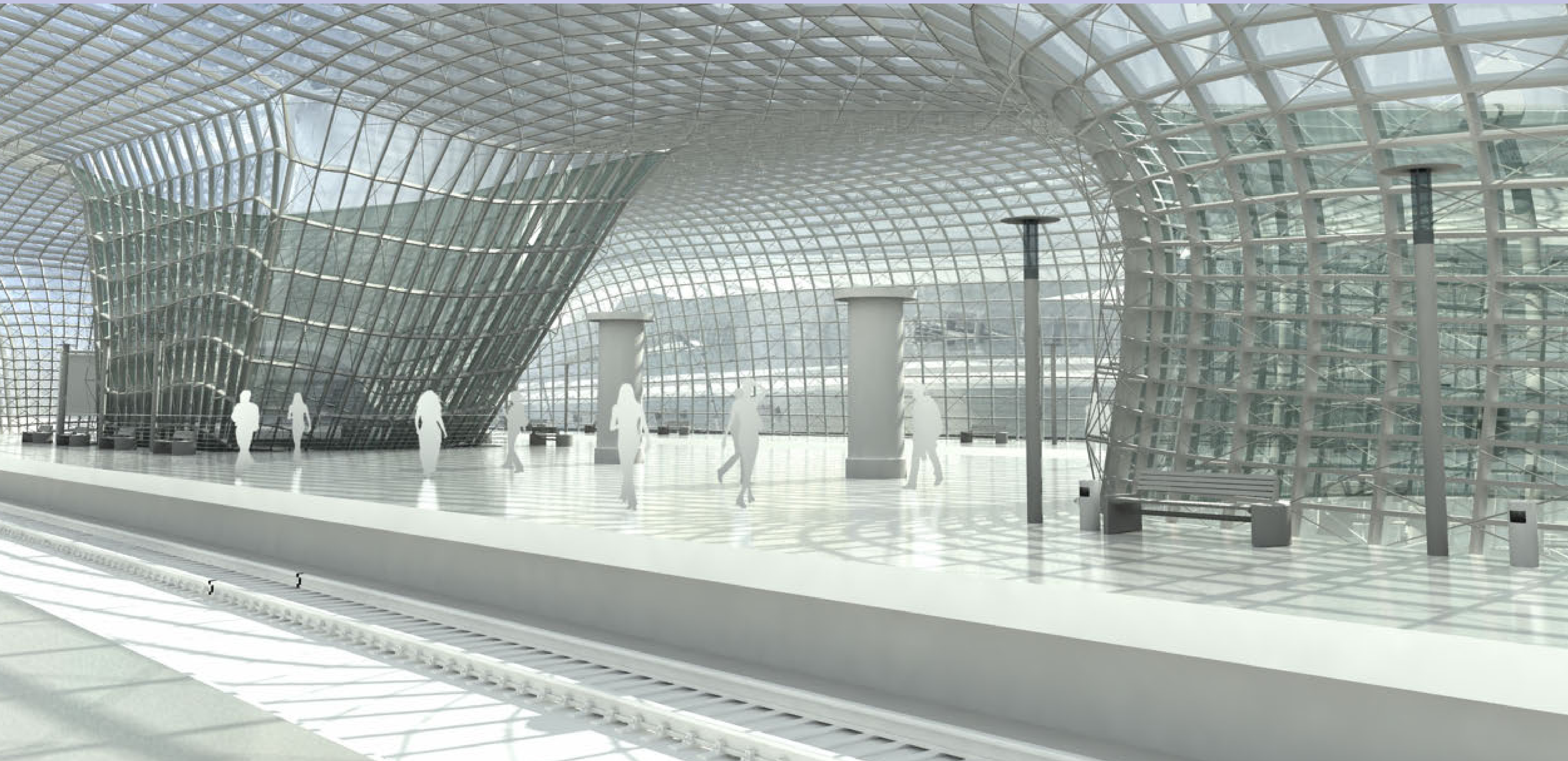
Applications of DDG: Numerical Simulation



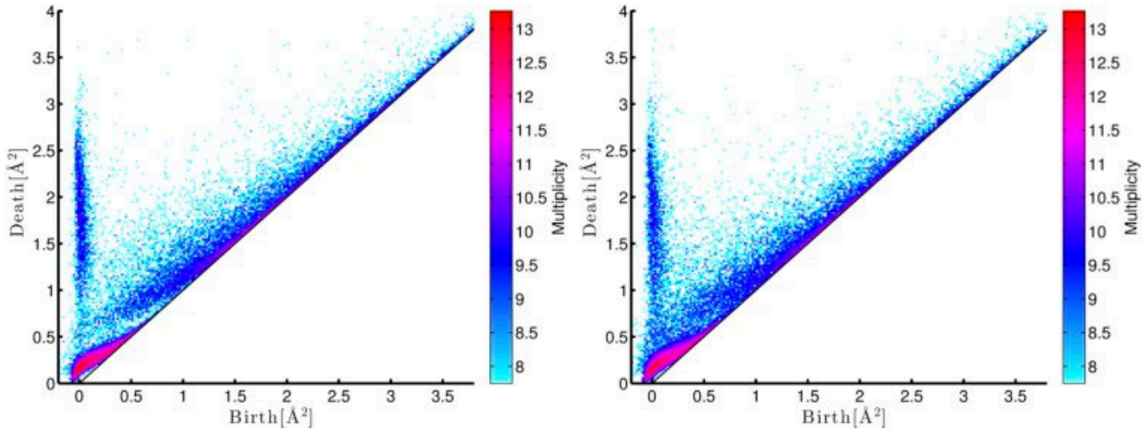


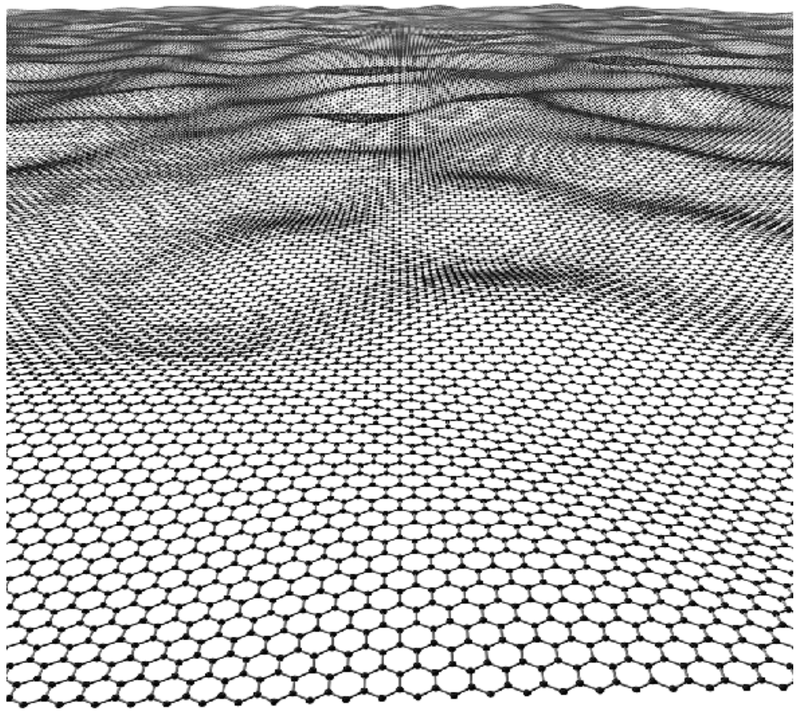


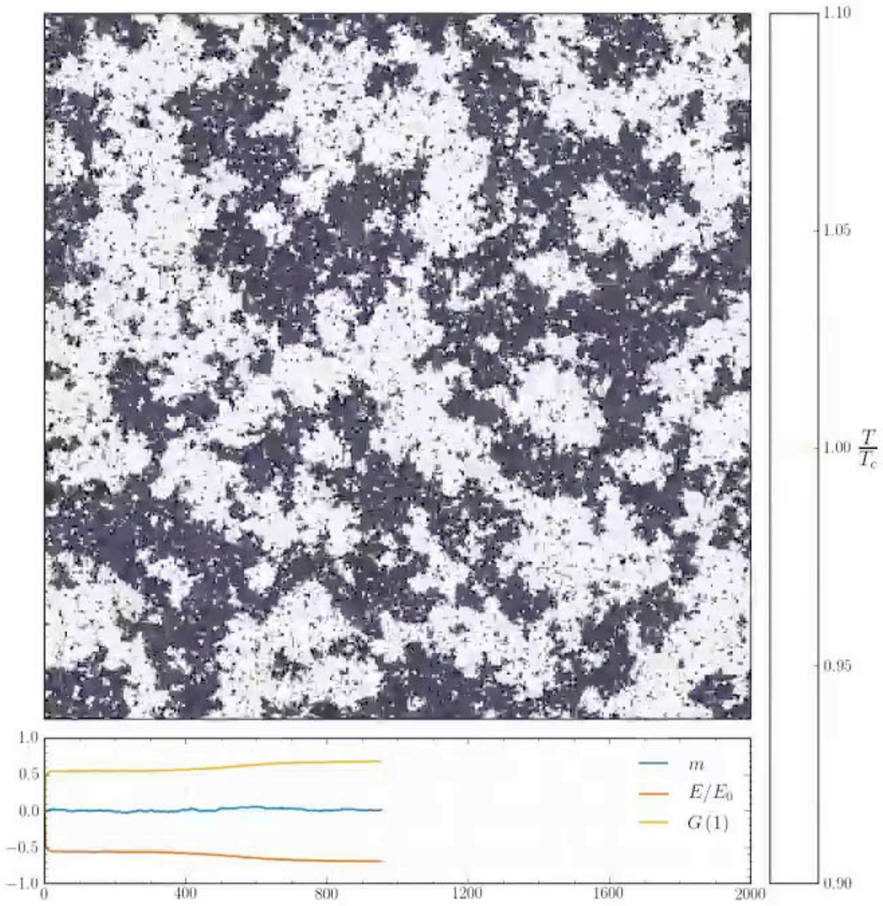
Applications of DDG: Architecture & Design



Applications of DDG: Discrete Models of Nature

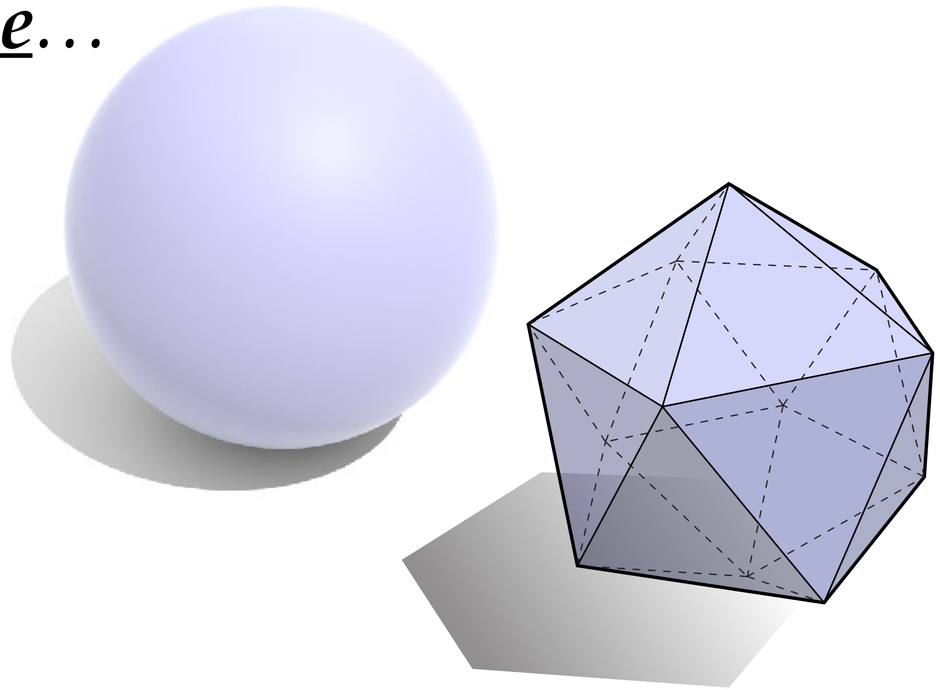






What Will We Learn in This Class?

- First and foremost: ***how to think about shape***…
  - …mathematically (differential geometry)
  - …computationally (geometry processing)
  - **Central Theme:** *link these two perspectives*



- Why? Shape is everywhere!
  - *computational biology, industrial design, computer vision, machine learning, architecture, computational mechanics, fashion, medical imaging…*
  - Flat images are old news :-)

Course Web Page

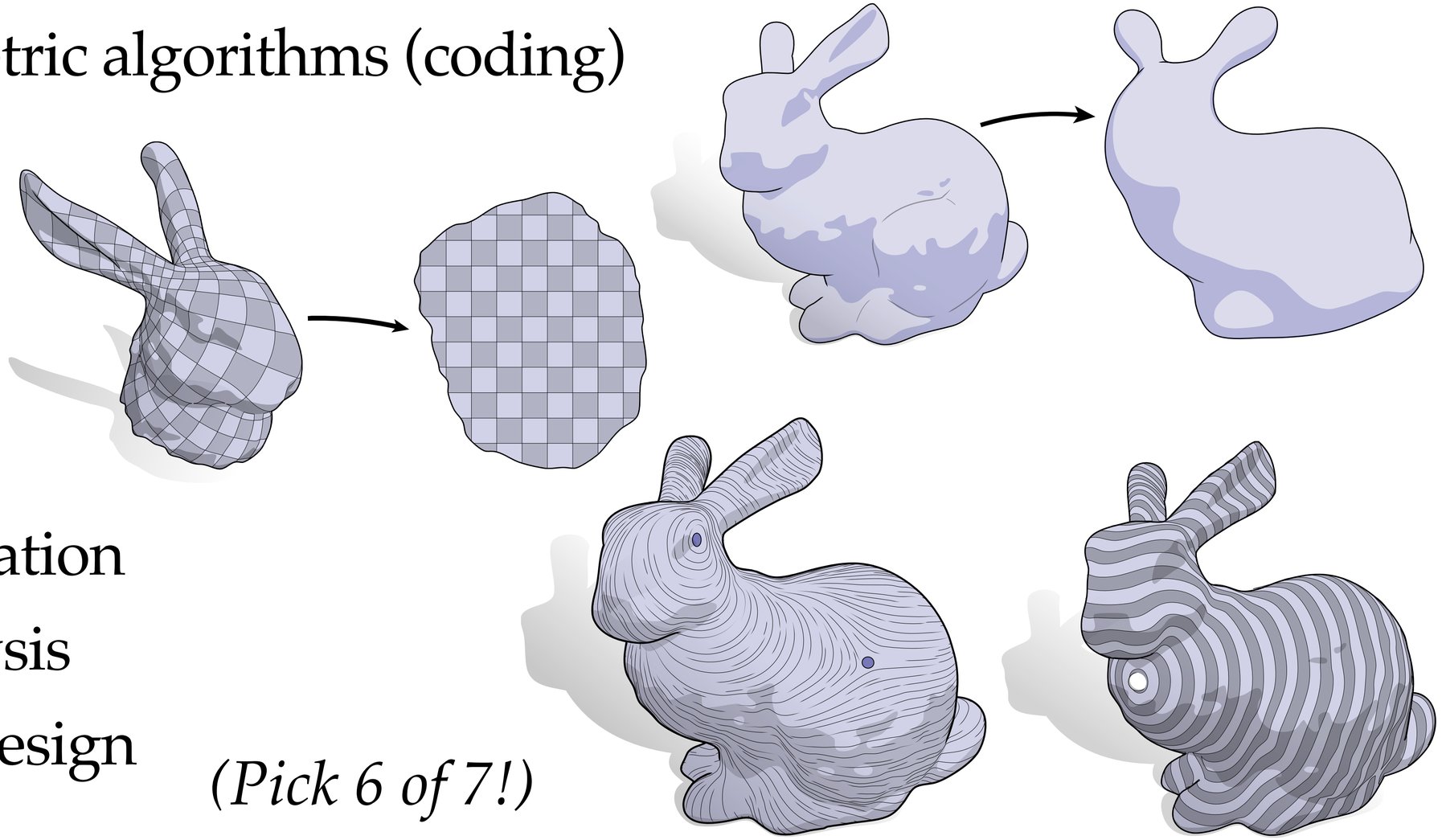
- All course information is spelled out in detail on course webpage:

`http://geometry.cs.cmu.edu/ddg`

- All communication goes through this site (assignments, discussion, etc.)
- Register account from link at end of menu:

Assignments

- **Derive** geometric algorithms from first principles (pen-and-paper)
- **Implement** geometric algorithms (coding)
  - Exterior calculus
  - Curvature
  - Smoothing
  - Parameterization
  - Distance computation
  - Vector field analysis
  - Direction Field Design



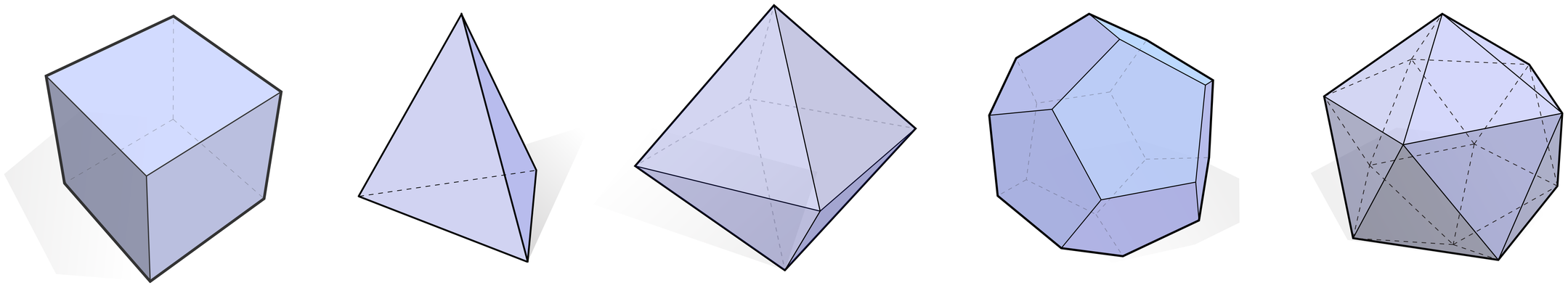
(Pick 6 of 7!)

Homework Submission

- All homework must be submitted digitally
  - All source files in a **single zip file** called `solution.zip`
  - All written exercises in a **single PDF file** called `exercises.pdf`
    - Either typeset (e.g., LaTeX) or scans/photos of written work.
    - Convert images to PDF using Preview (Mac) or imagetopdf.com
  - Email to geometry.collective@gmail.com with requested string in the subject line (e.g., `DDG17A1`)
  - Will receive written feedback via email as marked-up PDF

Late Policy

- Assignments due at 5:59:59pm on due date (Eastern time zone)
- Can use **five late days** throughout semester
  - Must indicate which late day you're using (1–5) in email
  - If writing answers on paper, must also draw one of five Platonic solids:

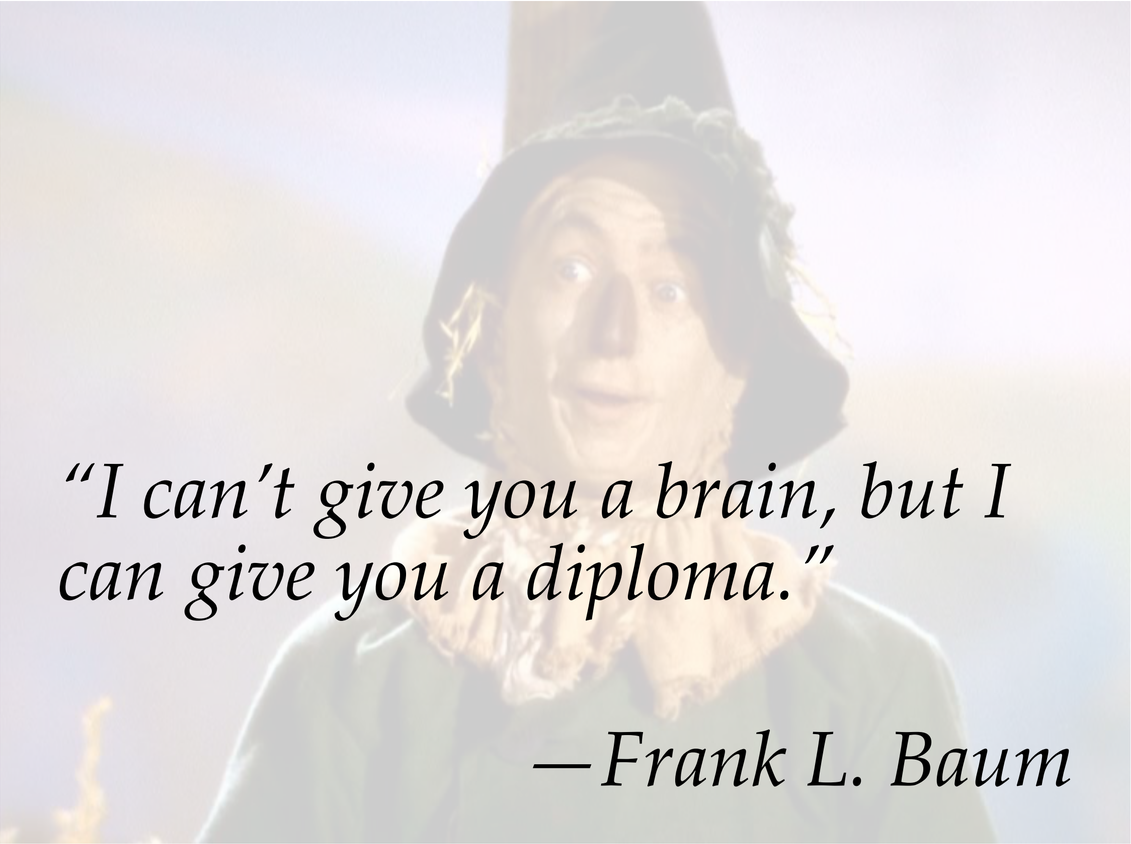


1 2 3 4 5

- *All subsequent late work will receive a zero!*

Grade Breakdown

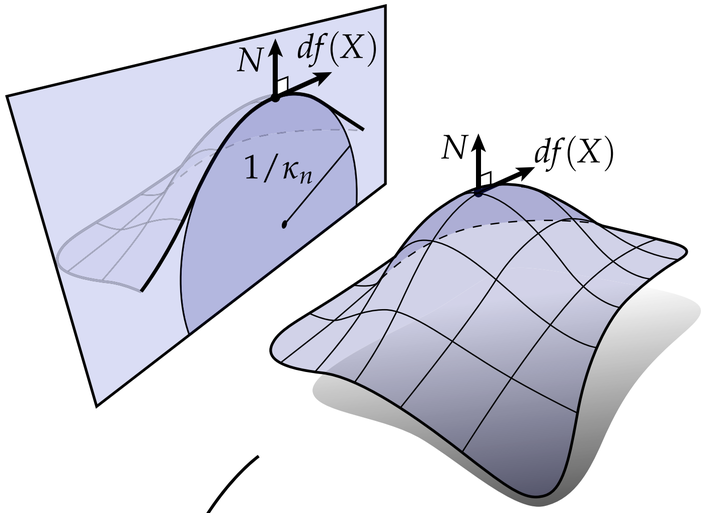
- **Assignments** – 78% (pick **6** out of 7*)
  - (13%) A1: *Exterior Calculus*
  - (13%) A2: *Normals & Curvature*
  - (13%) A3: *Surface Fairing*
  - (13%) A4: *Vector Field Processing*
  - (13%) A5: *Surface Parameterization*
  - (13%) A6: *Geodesic Distance*
  - (13%) A7: *Direction Field Design*
- **Final Exam** – 12%
- **Participation** – 10%
  - (5%) – in-class/web participation
  - (5%) – reading summaries/questions

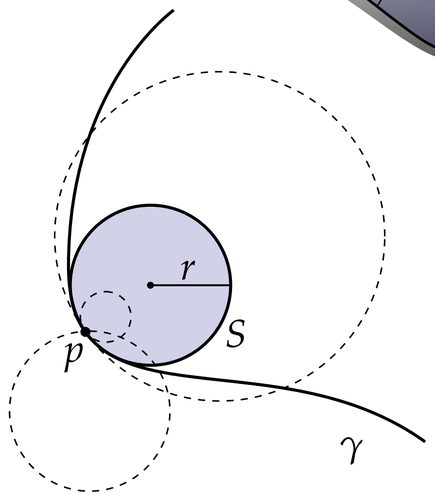


**Complete 7th assignment for up to 12% extra credit.*

What is Differential Geometry?

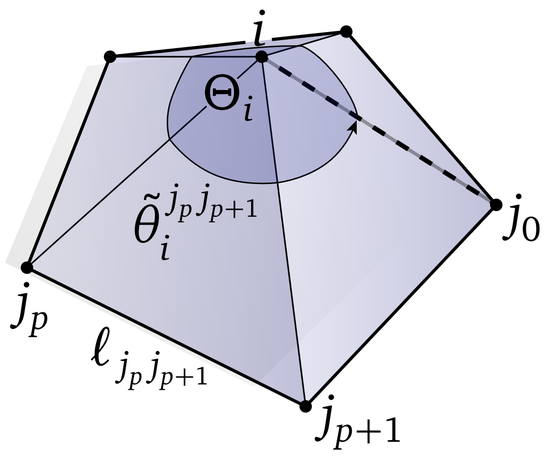
- **Language** for talking about *local properties of shape*
  - How fast are we traveling along a curve?
  - How much does the surface bend at a point?
  - etc.
- …and their connection to *global properties of shape*
  - So-called "local-global" relationships.
- Modern language of geometry, physics, statistics, …
- Profound impact on scientific & industrial development in 20th century



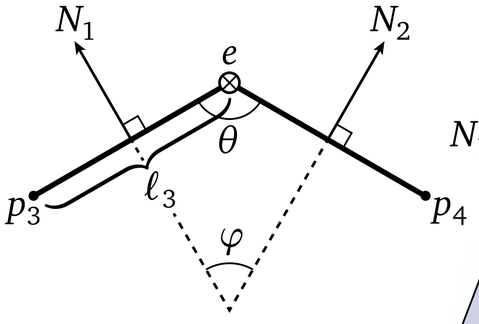


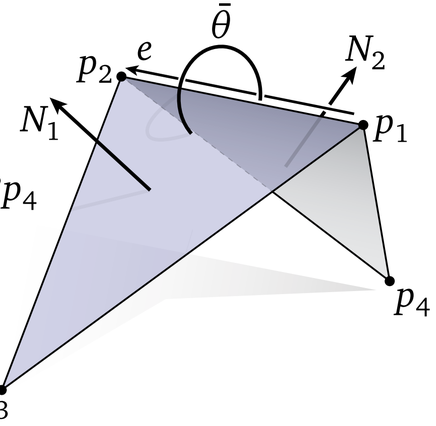
*What is **Discrete** Differential Geometry?*

- Also a language describing local properties of shape
  - *Infinity no longer allowed!*
  - No longer talk about derivatives, infinitesimals…
  - Everything expressed in terms of lengths, angles…



- Surprisingly little is lost!
  - Faithfully captures many fundamental ideas



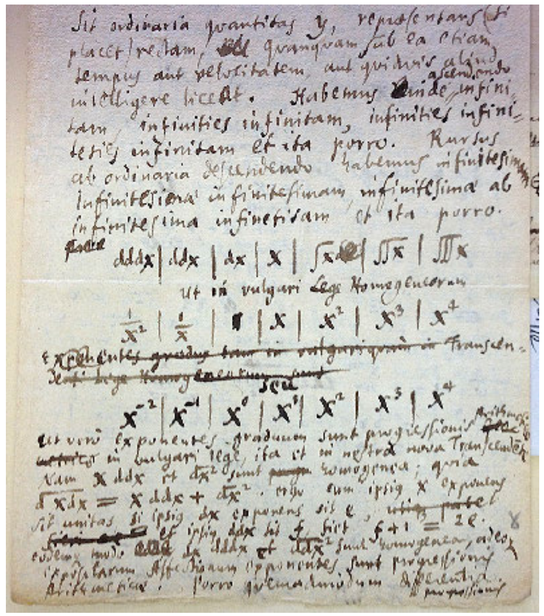


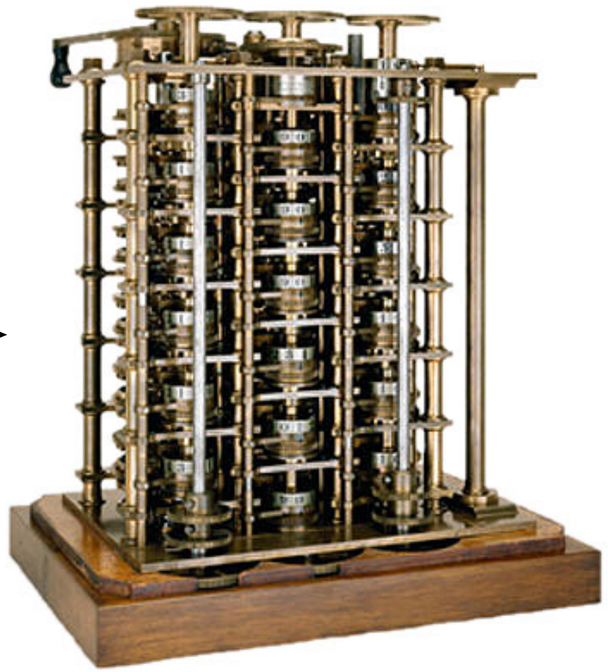
- Modern language of geometric computing
- Increasing impact on science & technology in 21st century.

Discrete Differential Geometry—Grand Vision

Grand Vision

Translate differential geometry into **language** suitable for *computation*.





How can we get there?

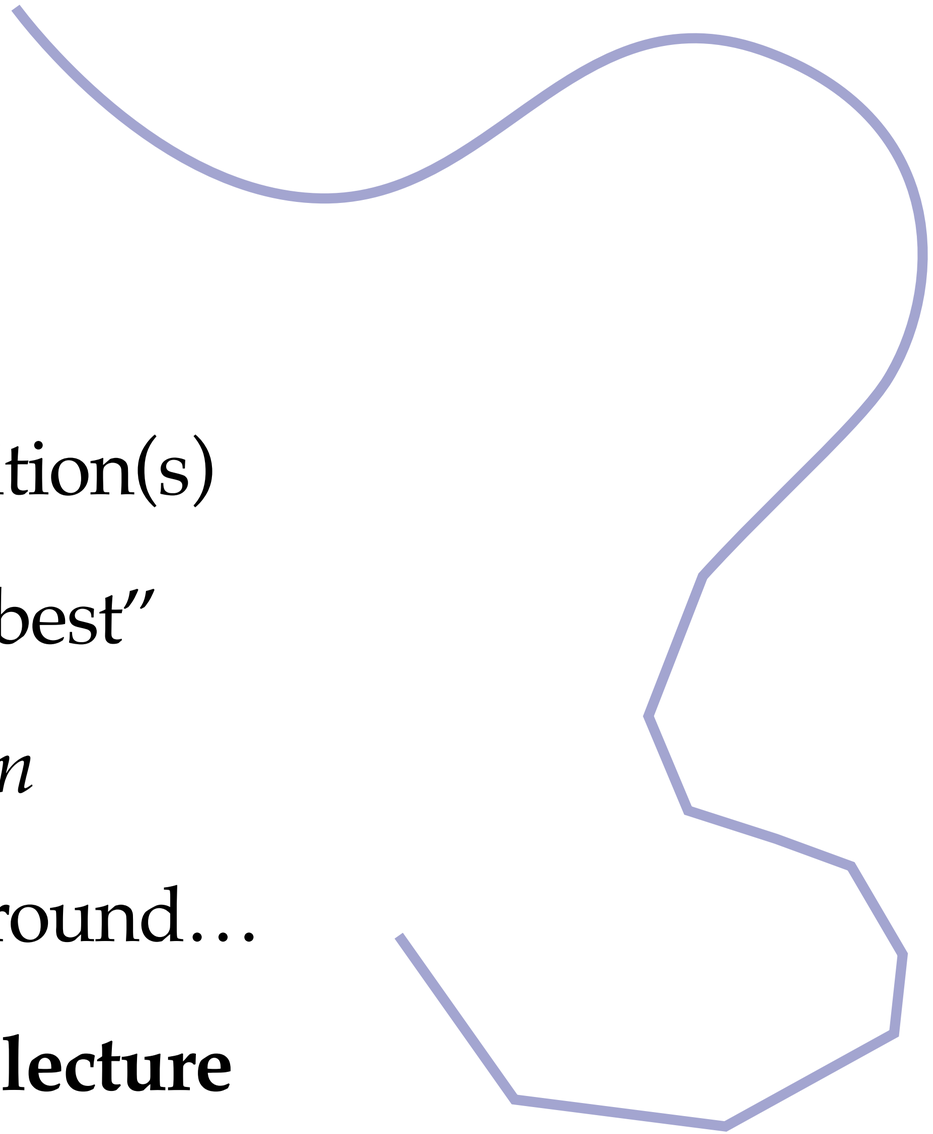
A common "game" is played in DDG to obtain discrete definitions:

1. Write down several **equivalent** definitions in the smooth setting.
2. Apply each smooth definition to an object in the discrete setting.
3. Determine which properties are captured by each resulting **inequivalent** discrete definition.

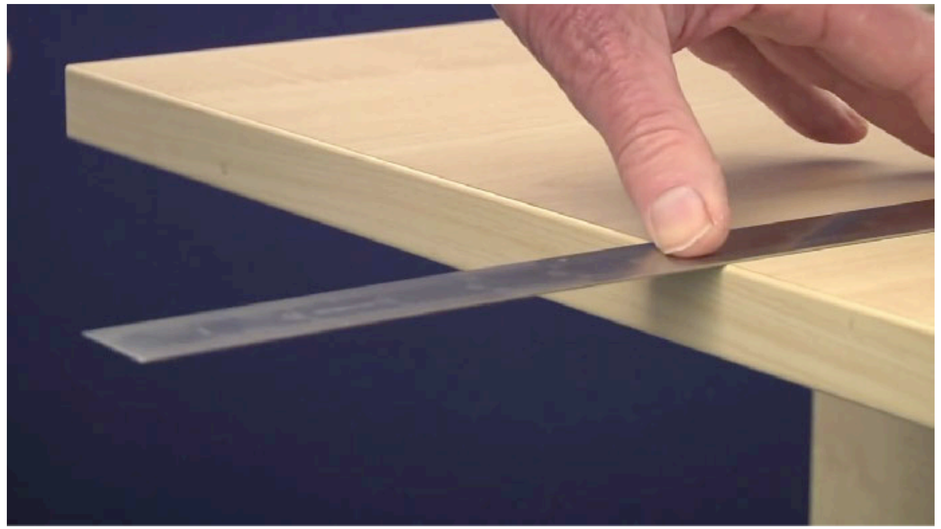
One often encounters a so-called *"no free lunch"* scenario: no single discrete definition captures *all* properties of its smooth counterpart.

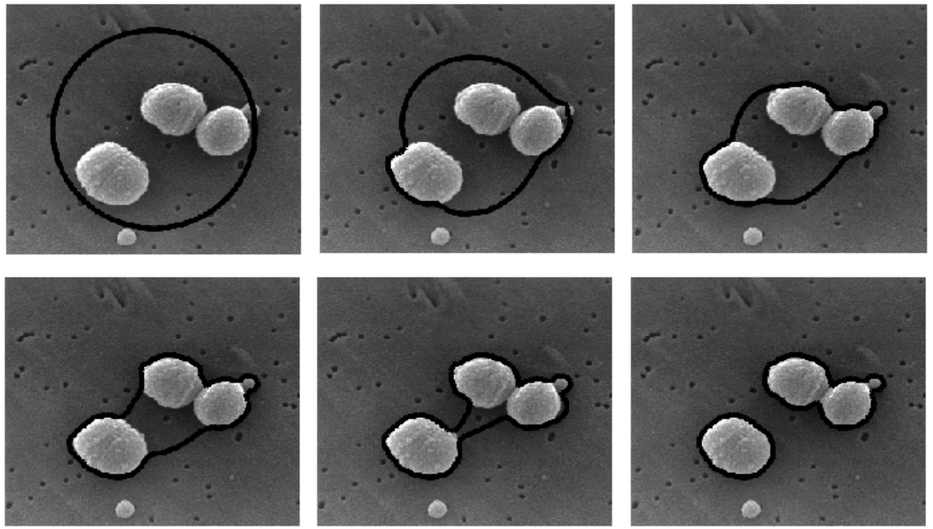
Example: Discrete Curvature of Plane Curves

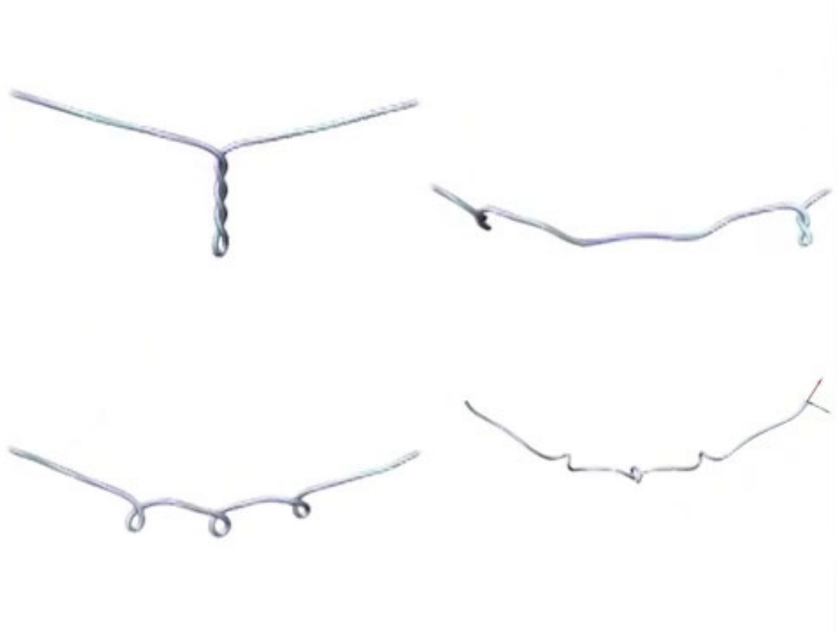
- **Toy example:** *curvature of plane curves*
  - Roughly speaking: "how much it bends"
  - First review smooth definition
  - Then play The Game to get discrete definition(s)
  - Will discover that no single definition is "best"
  - *Pick the definition best suited to the application*
- **Today** we're gonna quickly cover a lot of ground…
- Will start more slowly from the basics **next lecture**

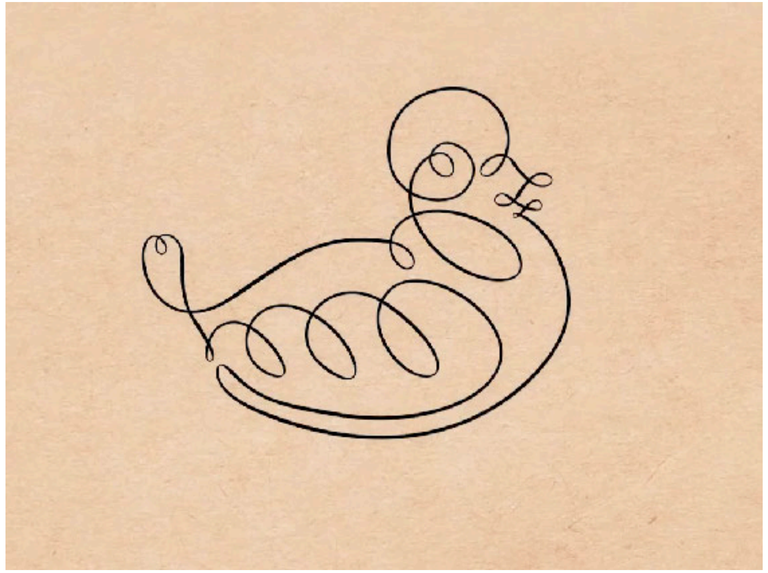


Curvature of a Curve—Motivation



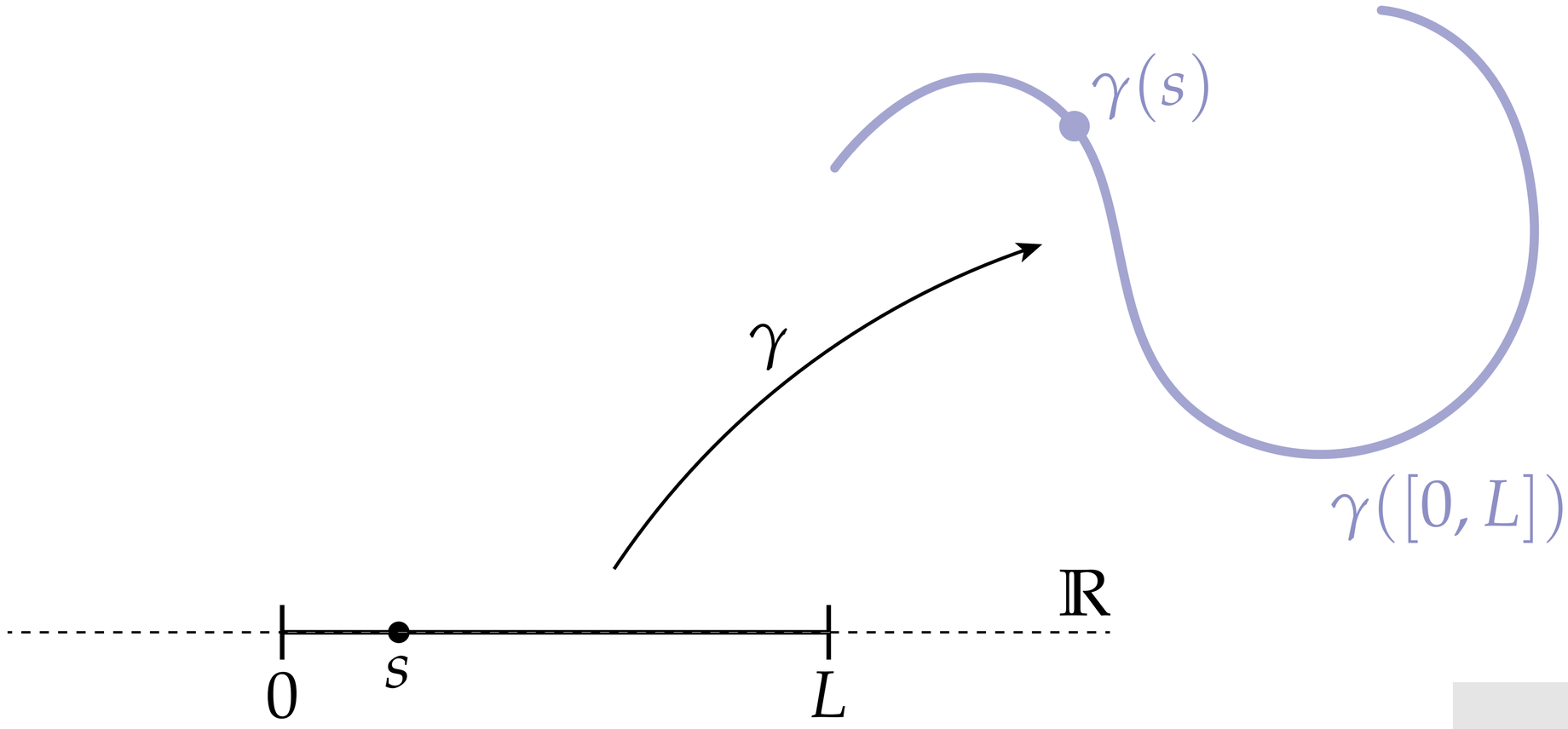






Curves in the Plane

- In the smooth setting, a **parameterized curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in the plane $\mathbb{R}^2$:



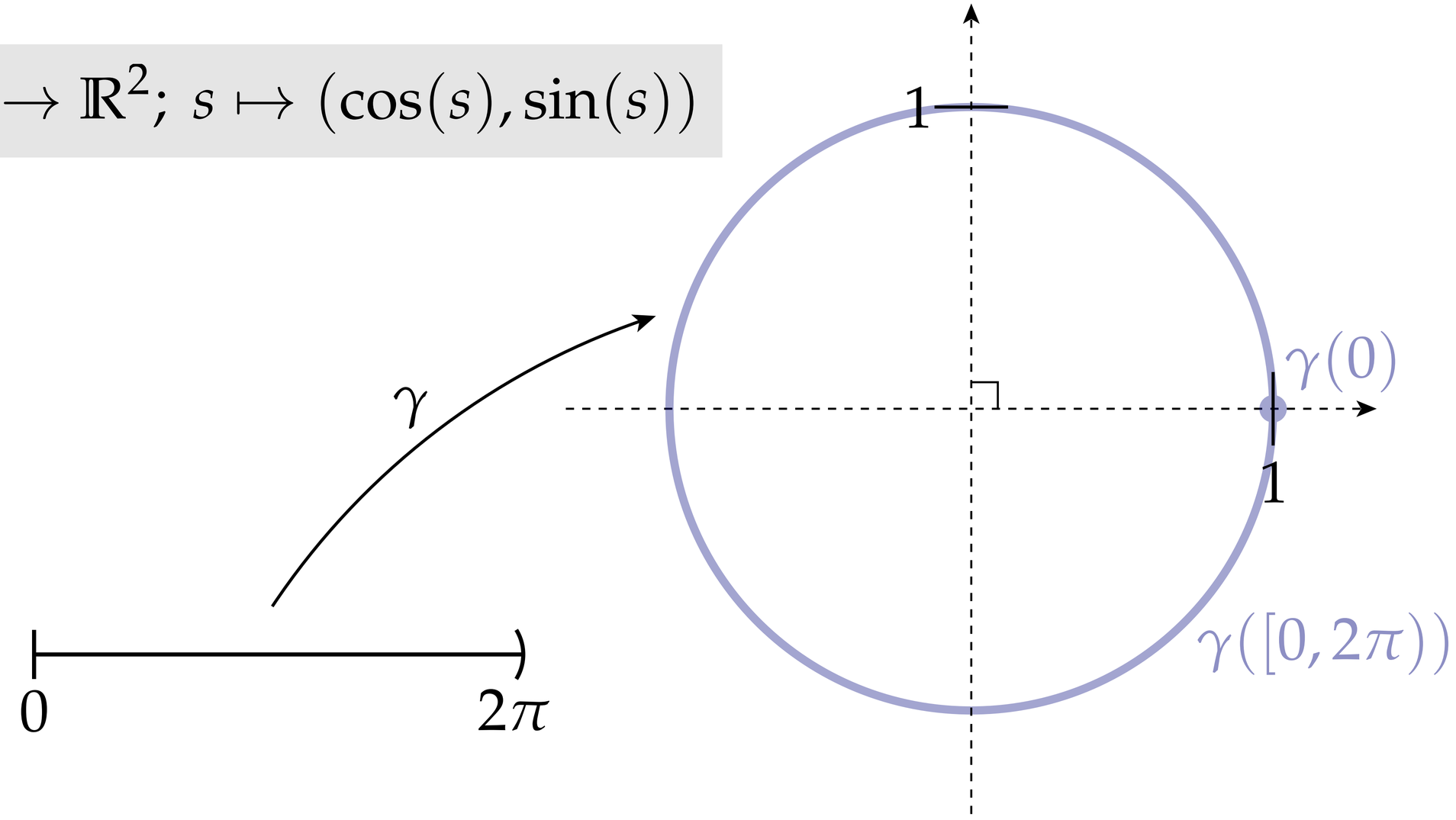
$$\gamma : [0, L] \to \mathbb{R}^2$$

*Continuous, differentiable, smooth…

Curves in the Plane—Example

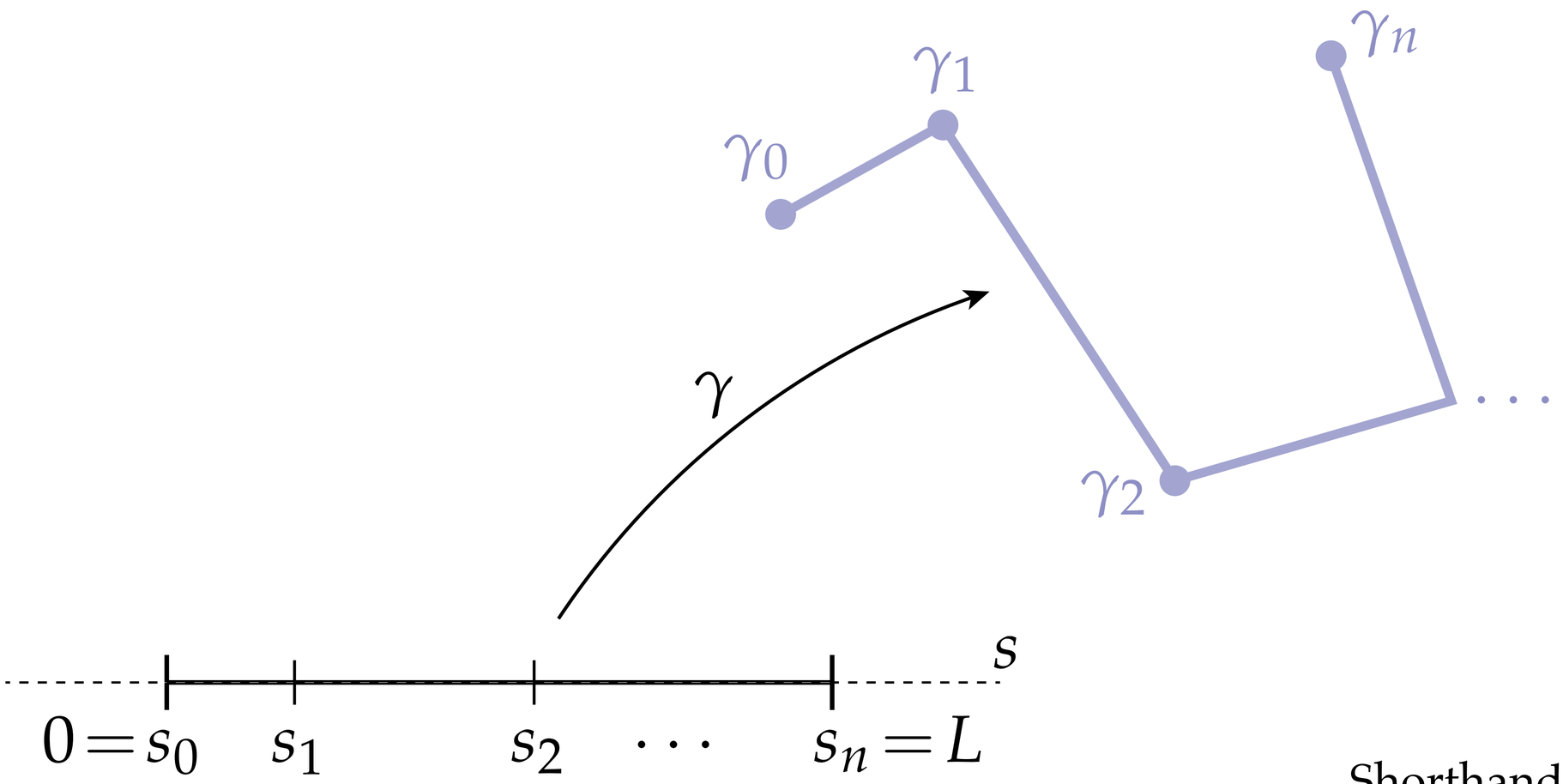
- As an example, we can express a circle as a parameterized curve γ:

$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$



Discrete Curves in the Plane

- Special case: a **discrete curve** is a *piecewise linear* parameterized curve, *i.e.*, it is a sequence of **vertices** connected by straight line segments:

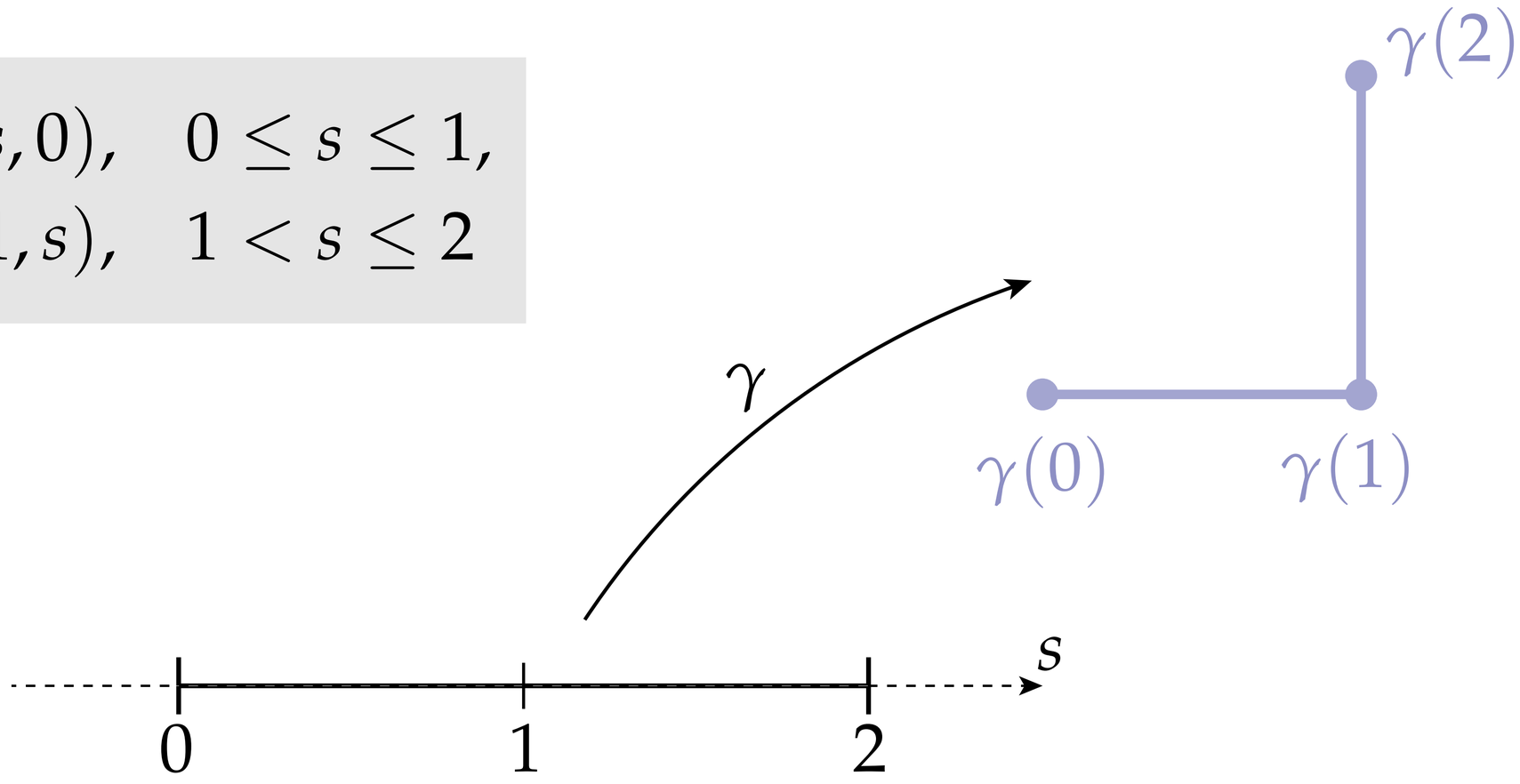


Shorthand: $\gamma_i := \gamma(s_i)$

Discrete Curves in the Plane—Example

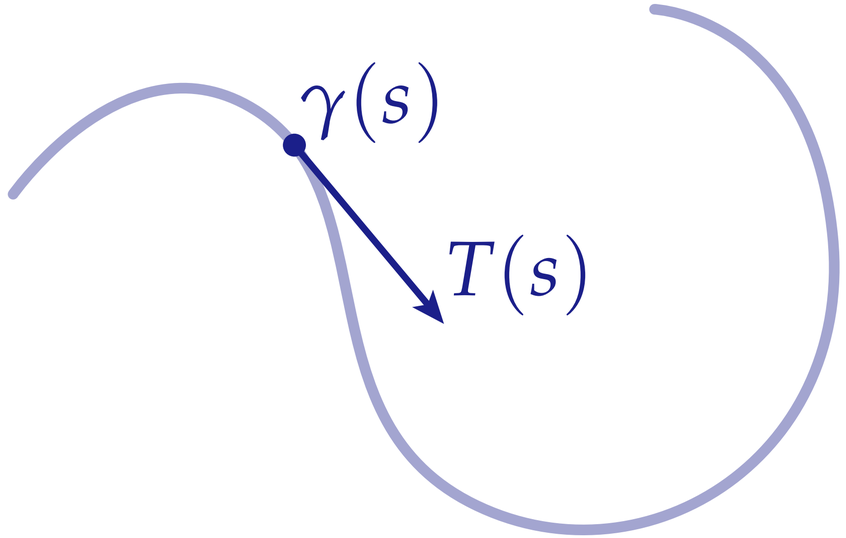
- A simple example is a curve comprised of two segments:

$$\gamma(s) := \begin{cases} (s,0), & 0 \leq s \leq 1, \\ (1,s), & 1 < s \leq 2 \end{cases}$$



Tangent of a Curve

- Informally, a vector is *tangent* to a curve if it "just barely grazes" the curve.



- More formally, the **unit tangent** (or just **tangent**) of a parameterized curve is the map obtained by normalizing its first derivative:

$$T(s) := \tfrac{d}{ds}\gamma(s) \Big/ \left|\tfrac{d}{ds}\gamma(s)\right|$$

- If the derivative already has unit length, then we say the curve is **arc-length parameterized** and can write the tangent as just

$$T(s) := \tfrac{d}{ds}\gamma(s)$$

Tangent of a Curve—Example

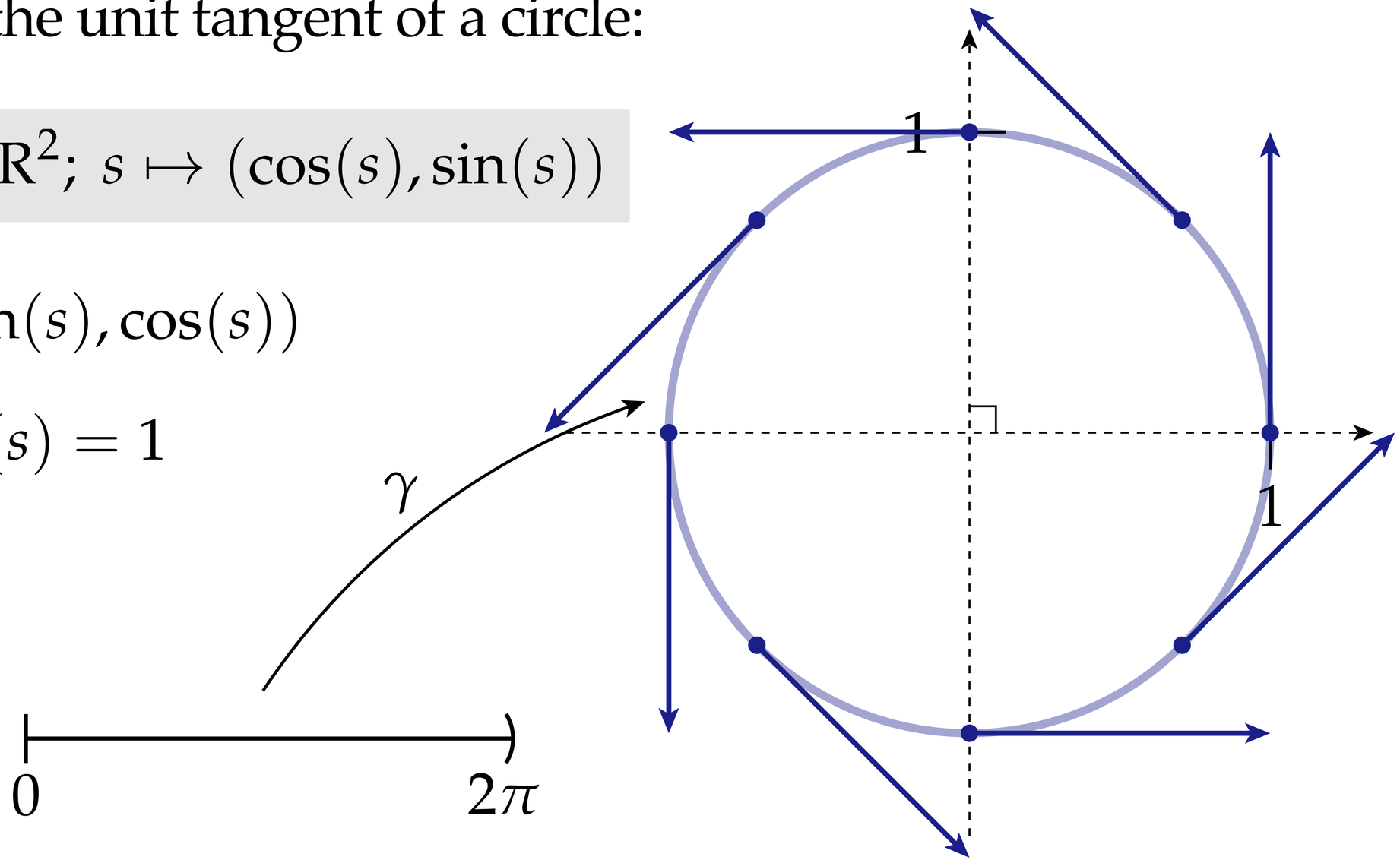
- Let's compute the unit tangent of a circle:

$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$\frac{d}{ds}\gamma(s) = (-\sin(s), \cos(s))$$

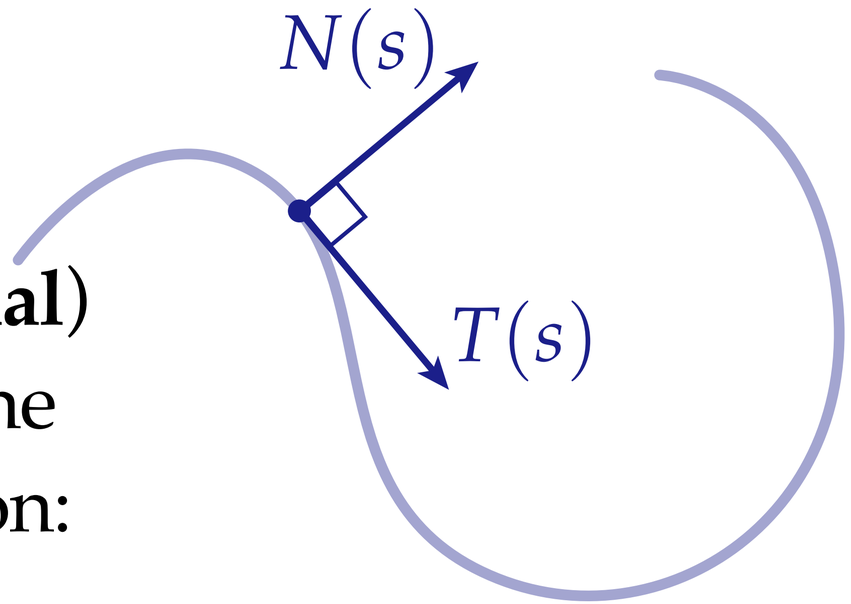
$$\cos^2(s) + \sin^2(s) = 1$$

$$\Rightarrow T = \frac{d}{ds}\gamma(s)$$



Normal of a Curve

- Informally, a vector is *normal* to a curve if it "sticks straight out" of the curve.



- More formally, the **unit normal** (or just **normal**) can be expressed as a quarter-rotation $\mathcal{J}$ of the unit tangent in the counter-clockwise direction:

$$N(s) := \mathcal{J}T(s)$$

- In coordinates (x,y), a quarter-turn can be achieved by* simply exchanging x and y, and then negating y:

$$(x, y) \overset{\mathcal{J}}{\mapsto} (-y, x)$$

*Why does this work?

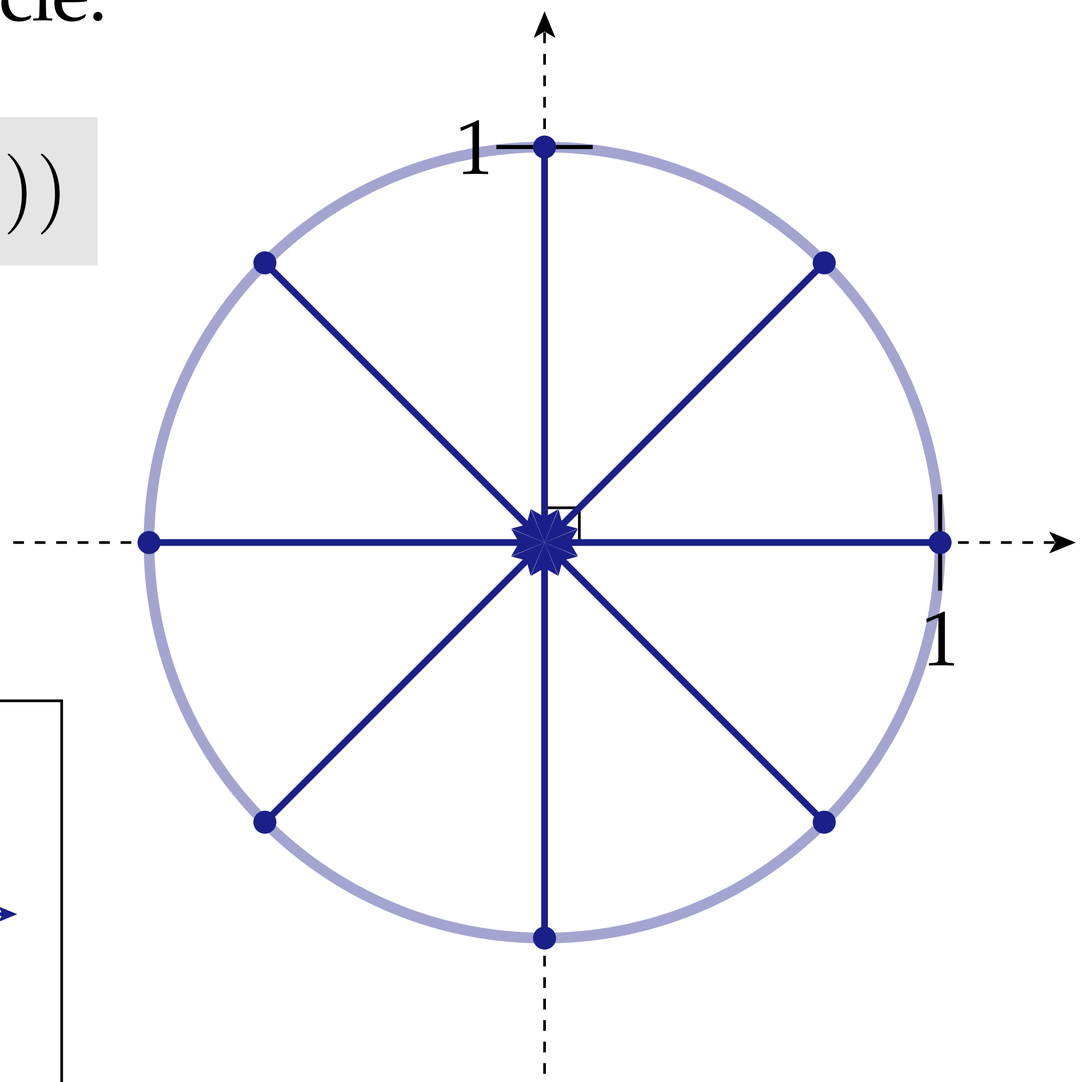
Normal of a Curve—Example

- Let's compute the unit normal of a circle:

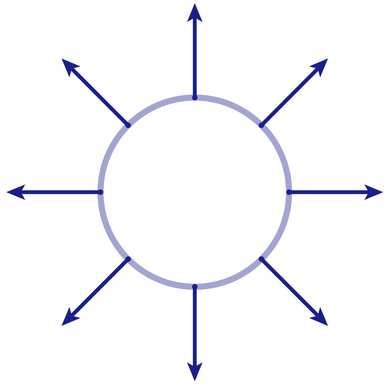
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$T(s) = (-\sin(s), \cos(s))$$

$$N(s) = \mathcal{J}T(s) = (-\cos(s), -\sin(s))$$



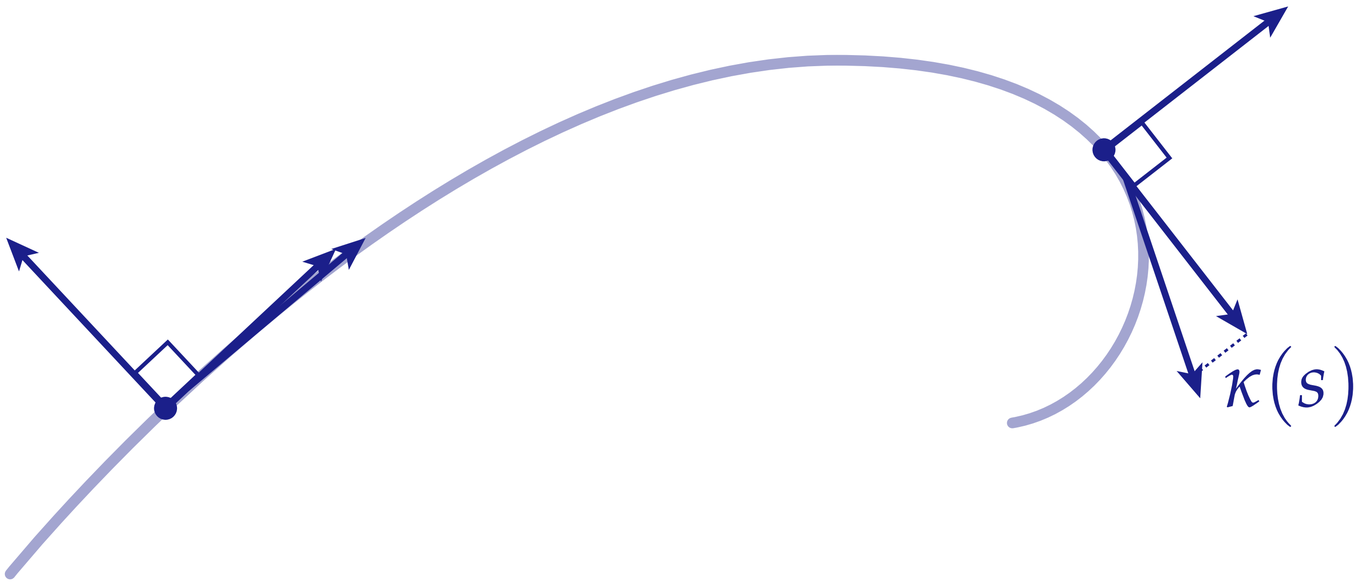
Note: could also adopt the convention $N = -\mathcal{J}T$. (Just remain consistent!)



Curvature of a Plane Curve

- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent*

$$\kappa(s) := \langle N(s), \tfrac{d}{ds}T(s)\rangle$$
$$= \langle N(s), \tfrac{d^2}{ds^2}\gamma(s)\rangle$$



KEY IDEA

Curvature is a *second derivative*.

*Here the angle brackets denote the usual dot product, i.e., $\langle (a,b),(x,y)\rangle := ax + by$.

Curvature: From Smooth to Discrete

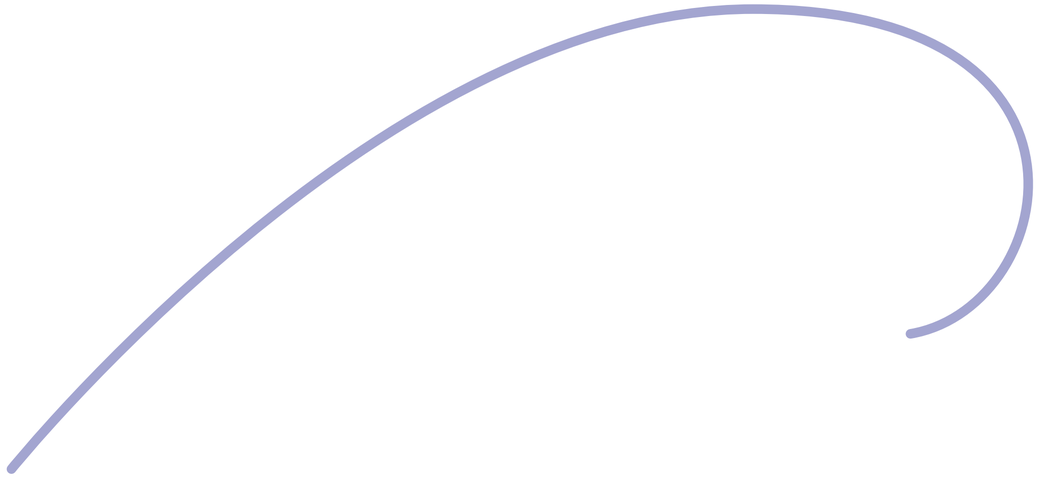
KEY IDEA

Curvature is a *second derivative*.

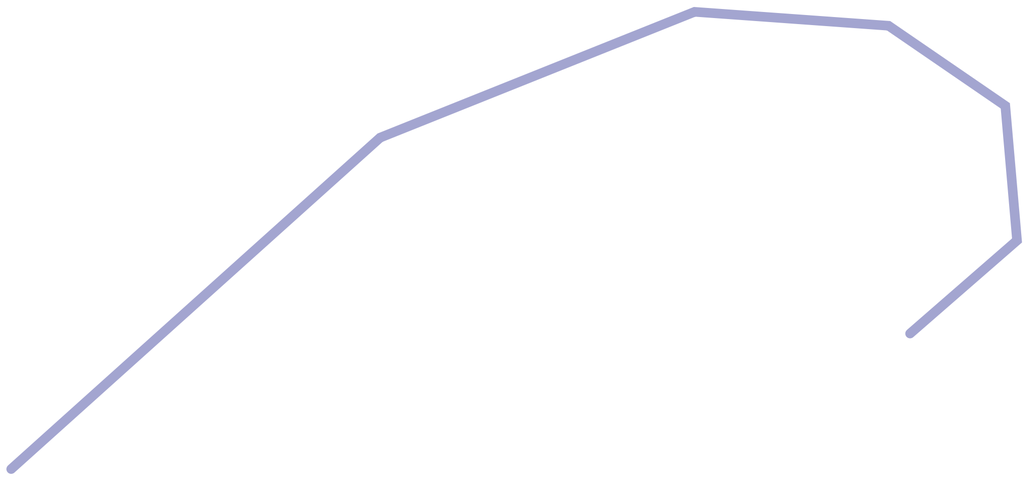
$$\kappa = \langle \mathcal{J} \frac{d}{ds}\gamma, \frac{d^2}{ds^2}\gamma \rangle$$

Can we directly apply this definition to a discrete curve?

SMOOTH



DISCRETE



No! Will get either zero or "∞". Need to think about it another way…

What is Discrete Curvature?

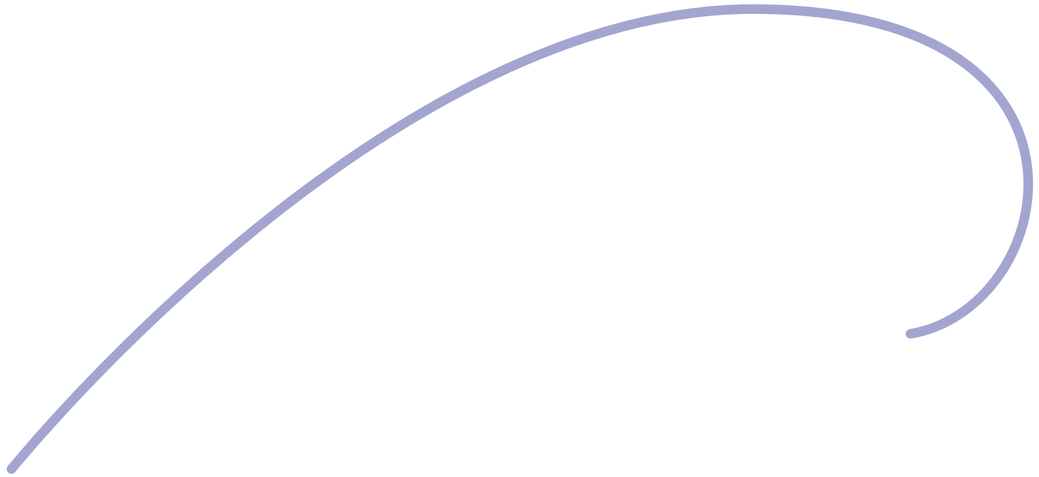
~~**KEY IDEA**~~

~~Curvature is a *second derivative*.~~

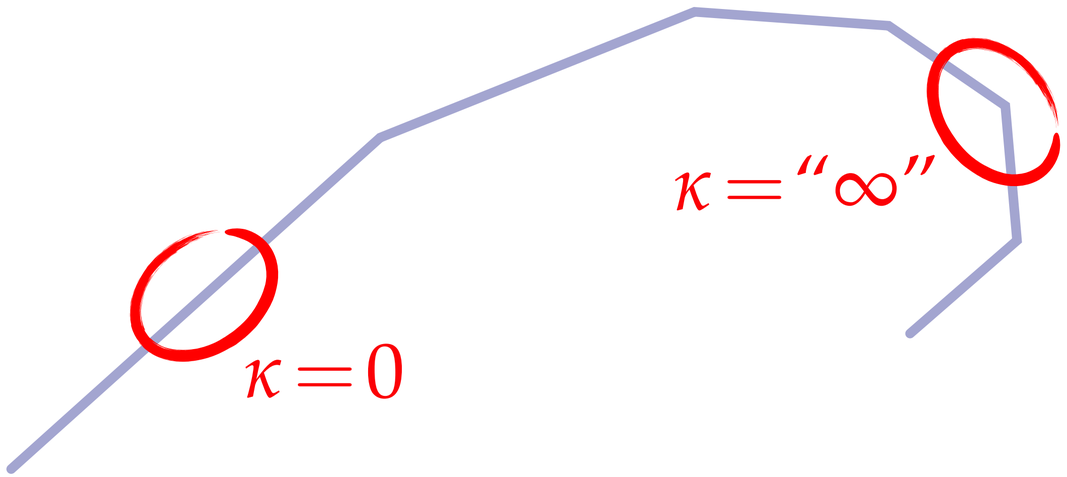
$$\cancel{\kappa = \left\langle \cancel{T}\, \frac{d}{ds}\gamma, \frac{d^2}{ds^2}\gamma \right\rangle}$$

Can we directly apply this definition to a discrete curve?

SMOOTH



DISCRETE



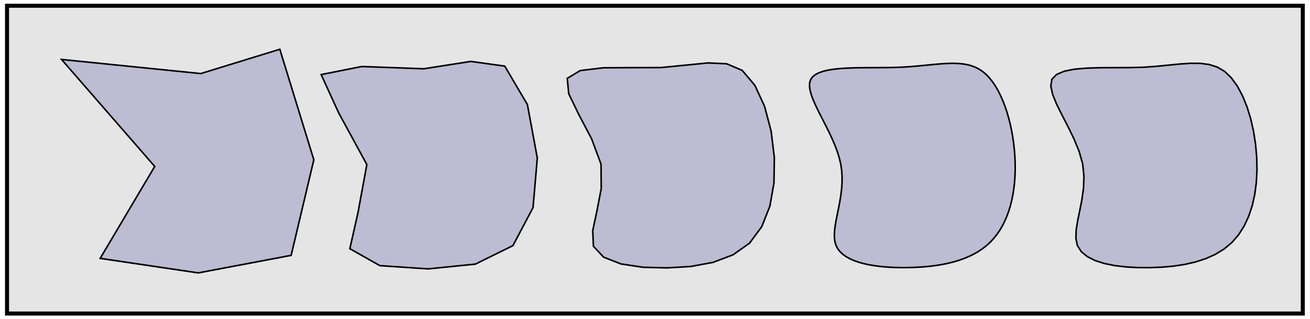
No! Will get either zero or "∞". Need to think about it another way…

When is a Discretization "Good?"

- How will we know if we came up with a good definition?
- Many different criteria for "good":
  - *satisfies (some of the) same properties/theorems as smooth curvature*
  - *converges to smooth value as we refine our curve*
  - *efficient to compute / solve equations*
  - …

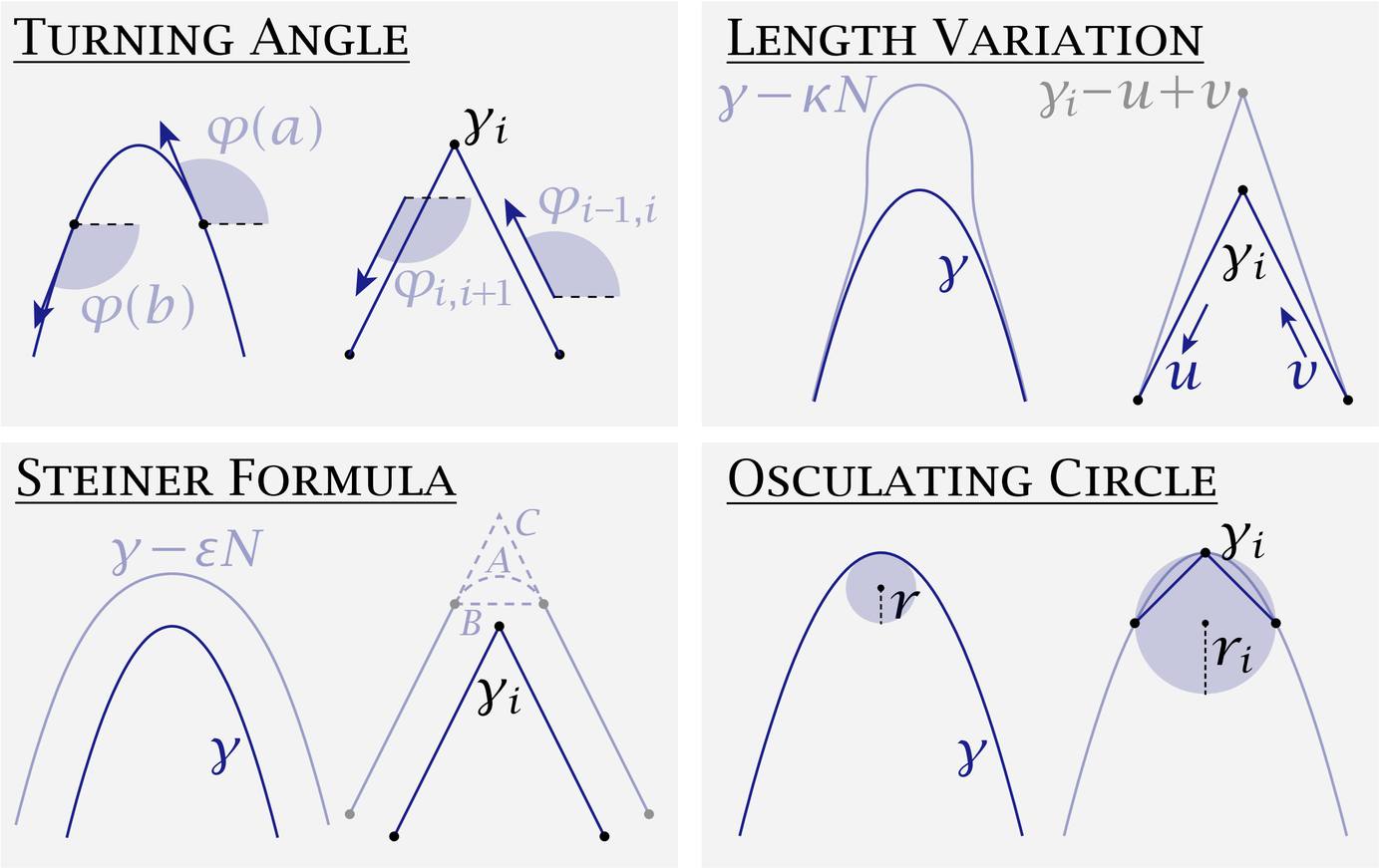
$$\oint \kappa \, ds \in 2\pi\mathbb{Z}$$

```
Complex Ta = gamma[i] - gamma[i-1];
Complex Tb = gamma[i+1] - gamma[i];
double kappa = (Tb*Ta.inv()).arg();
```



Curvature, Revisited

- In the **smooth** setting, there are several other **equivalent** definitions of curvature.
- **<u>IDEA:</u>** *perhaps some of these definitions can be applied directly to our discrete curve!*

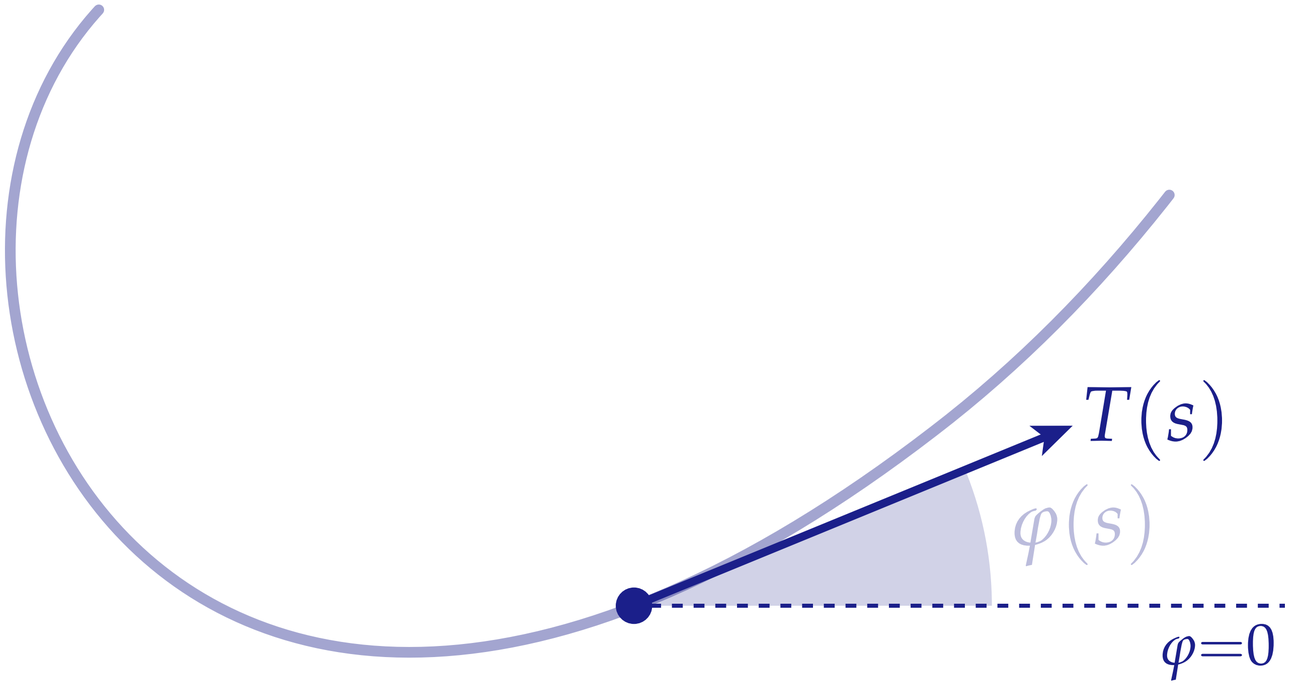


Turning Angle

- Our initial definition of curvature was the *rate of change of the tangent in the normal direction*.

$$\kappa(s) = \langle N(s), \tfrac{d}{ds}\gamma(s)\rangle$$

- Equivalently, we can measure the *rate of change of the angle the tangent makes with the horizontal*:



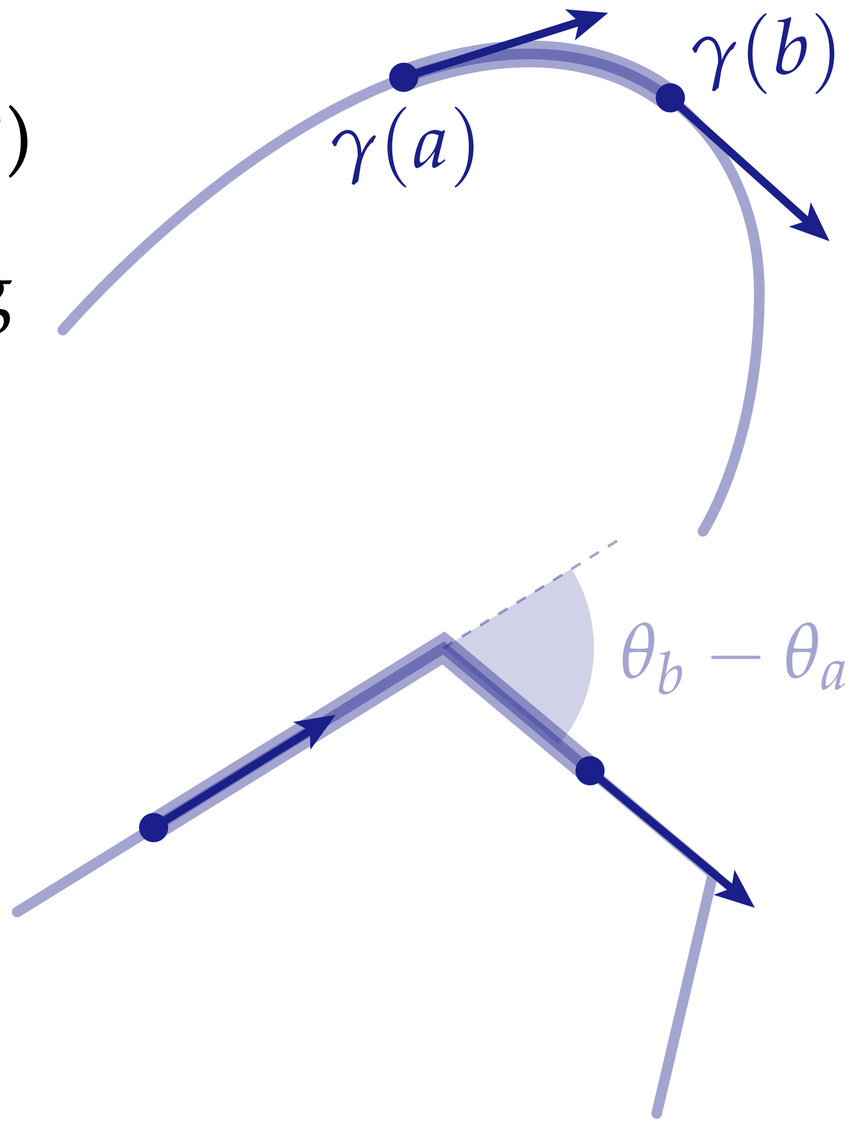
$$\boxed{\kappa(s) = \tfrac{d}{ds}\varphi(s)}$$

Integrated Curvature

- Still can't evaluate curvature at vertices of a discrete curve (*at what rate does the angle change?*)
- But let's consider the *integral* of curvature along a short segment:

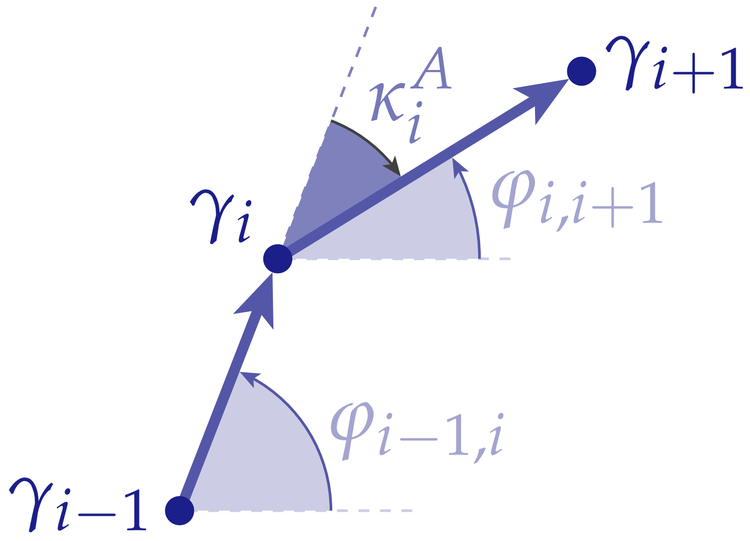
$$\int_a^b \kappa(s)\, ds = \int_a^b \left(\tfrac{d}{ds}\varphi(s)\right) ds = \varphi(b) - \varphi(a)$$

- Instead of *derivative* of angle, we now just have a *difference* of angles.
- **This definition works for our discrete curve!**



Discrete Curvature (Turning Angle)

- This formula gives us our first definition of discrete curvature, as just the *exterior angle* at the vertex of each curve:



$$\theta_i := \varphi_{i,i+1} - \varphi_{i-1,i} \quad \textbf{(exterior angle)}$$

$$\kappa_i^A := \theta_i \in (-\pi, \pi) \quad \textbf{(discrete curvature)}$$

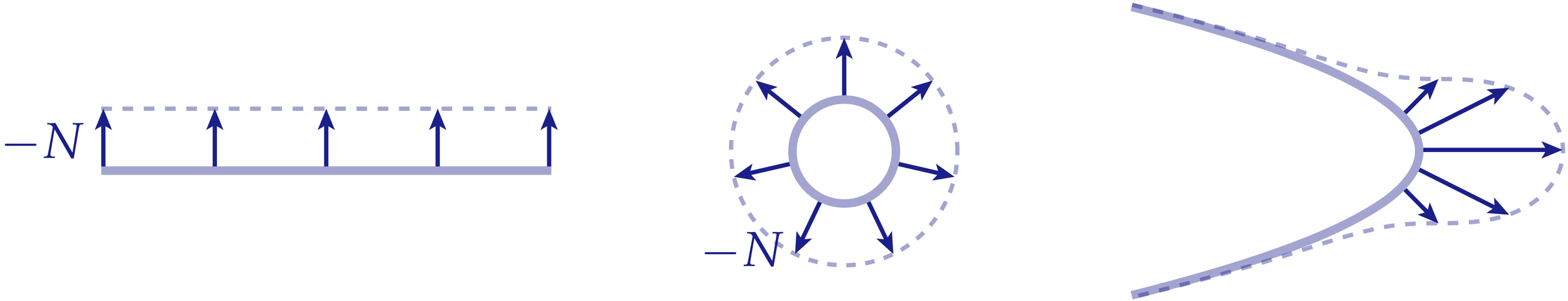
- Here we encounter another theme from discrete differential geometry: the quantities we want to work with are often more naturally expressed as <u>*integrated*</u> rather than <u>*pointwise*</u> quantities.

Length Variation

- Are there *other* ways to get a definition for discrete curvature?
- Well, here's a useful fact about curvature from the smooth setting:

> *The fastest way to decrease the length of a curve is to move it in the normal direction, with speed proportional to curvature.*

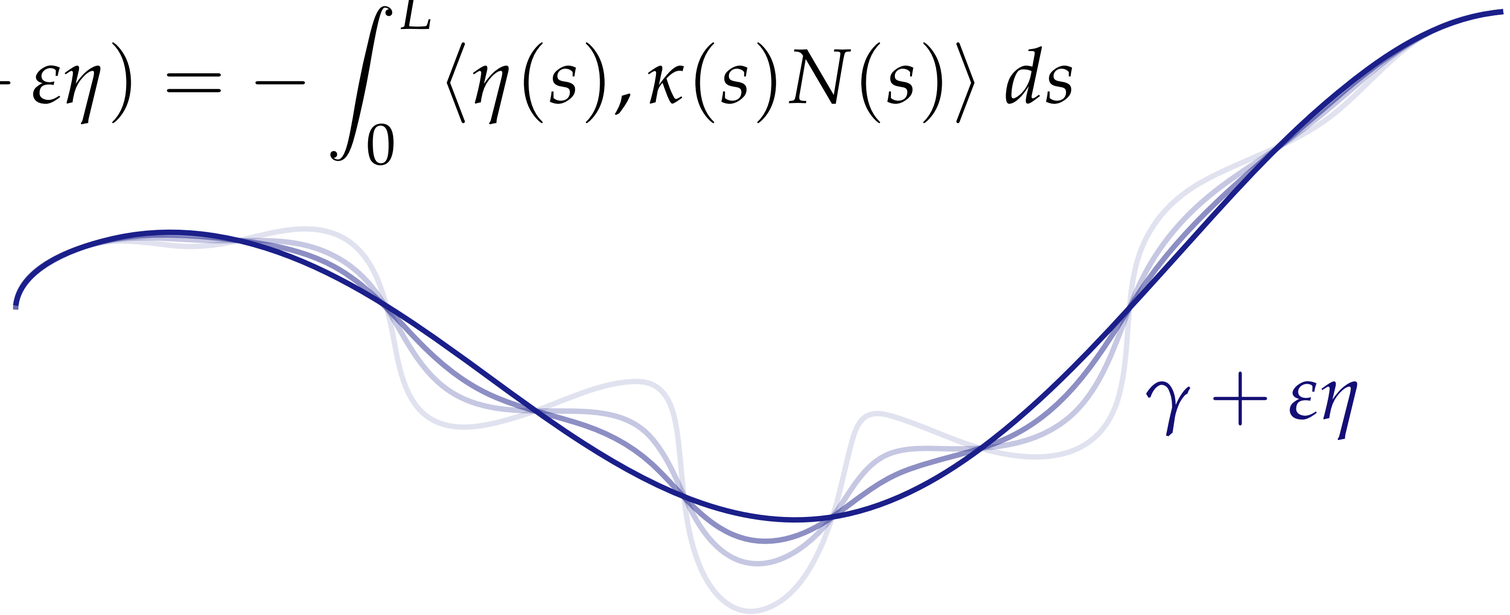
- **Intuition**: in flat regions, moving the curve doesn't change its length; in curved regions, the change in length (*per unit length*) is large:



Length Variation

- More formally, consider an *arbitrary* variation of the curve. *I.e.*, suppose we have another curve* $\eta : [0, L] \to \mathbb{R}^2$. One can show that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{length}(\gamma + \varepsilon\eta) = -\int_0^L \langle \eta(s), \kappa(s)N(s) \rangle \, ds$$

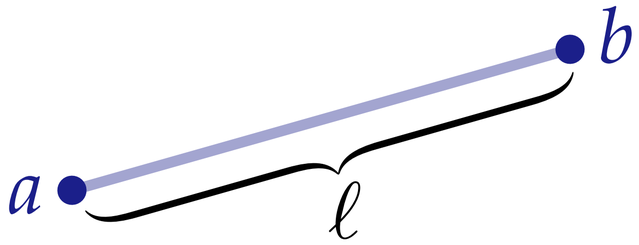


- Therefore, the motion that most quickly decreases length is $\eta = \kappa N$.

*Technical note: must go to zero at endpoints (*i.e.*, pass through the origin).

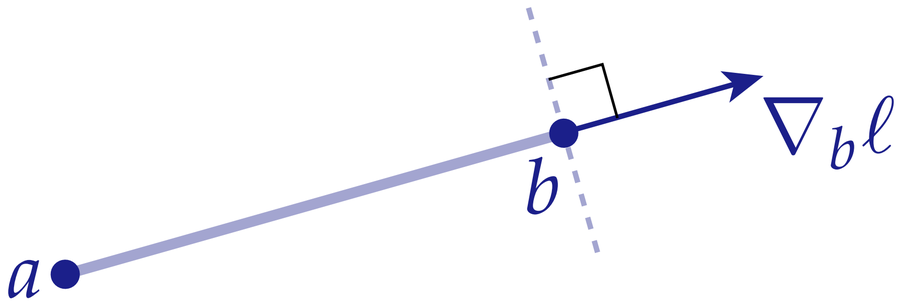
Gradient of Length for a Line Segment

- This all becomes much easier in the discrete setting: just take the gradient of length with respect to vertex positions.
- First, a warm-up exercise. Suppose we have a *single* line segment:



$$\ell := |b - a|$$

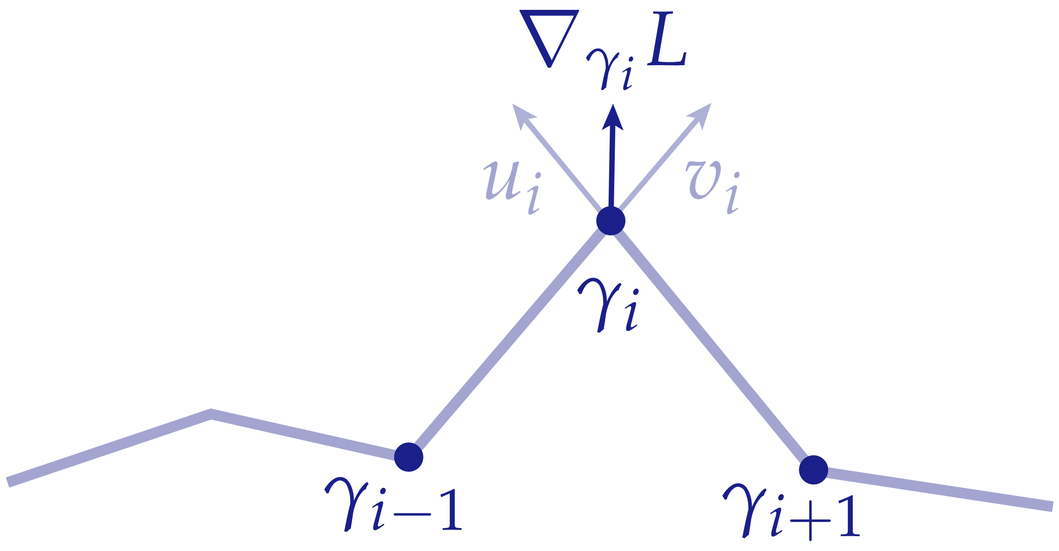
- How can we move the point b to most quickly increase its length?

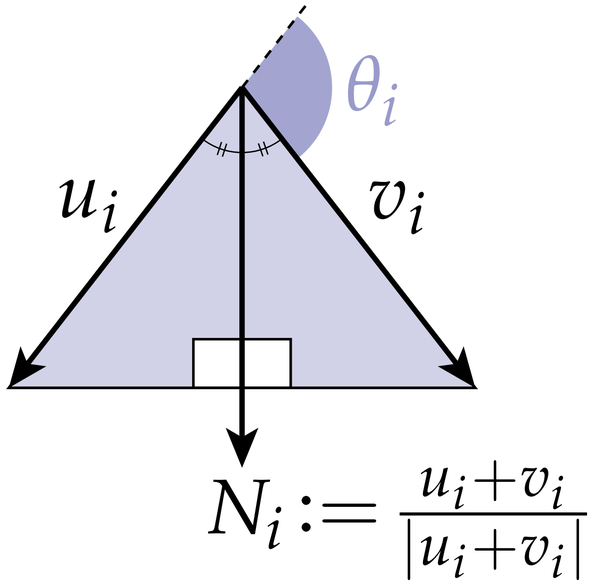


$$\nabla_b \ell = (b - a)/\ell$$

Gradient of Length for a Discrete Curve

- To find the motion that most quickly increases the *total* length L, we now just have to sum the contributions of each segment:



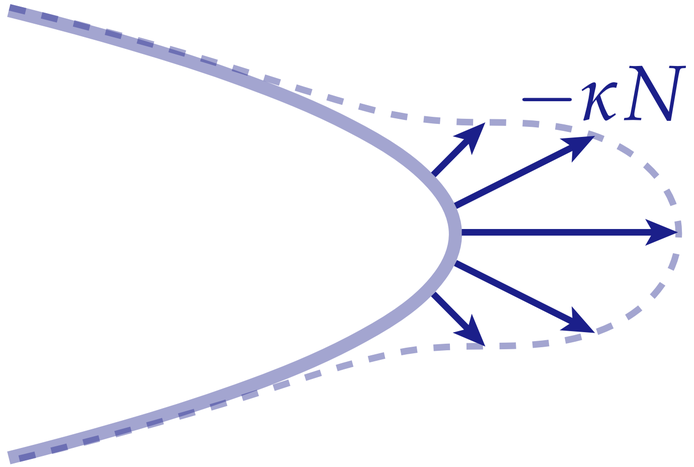


- Using some simple trigonometry, we can also express the length gradient in terms of the exterior angle θ_i and the angle bisector N_i:

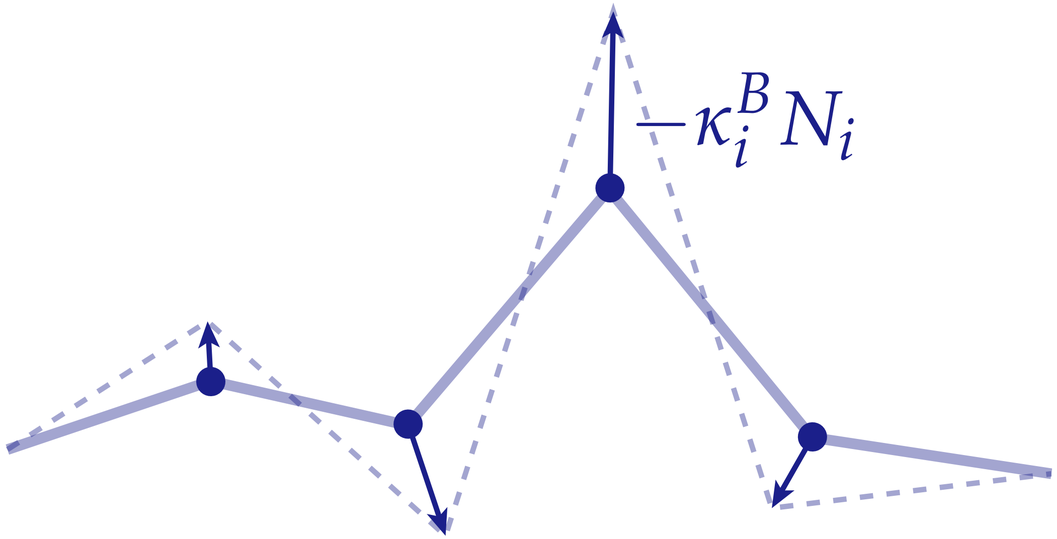
$$\nabla_{\gamma_i} L = 2 \sin(\theta_i / 2) N_i$$

Discrete Curvature (Length Variation)

- How does this help us define discrete curvature?
- Recall that in the smooth setting, the gradient of length is equal to the curvature times the normal.
- Hence, our expression for the *discrete* length variation provides a definition for the *discrete* curvature times the *discrete* normal.



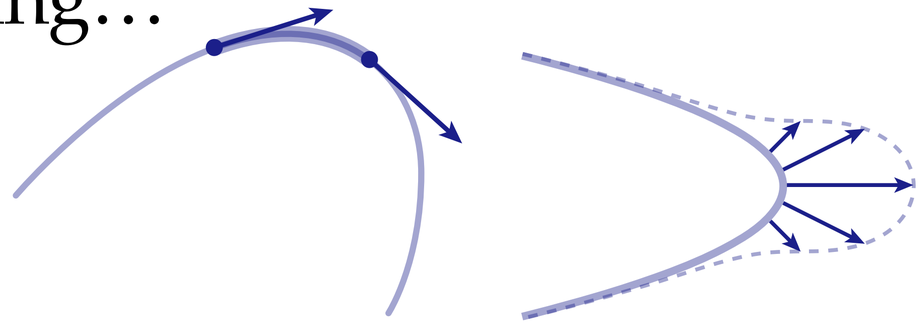
$$\kappa_i^B N_i := 2\sin(\theta_i/2) N_i$$



A Tale of Two Curvatures

- Let's recap what we've done so far. We considered starting points that are **equivalent** in the smooth setting…

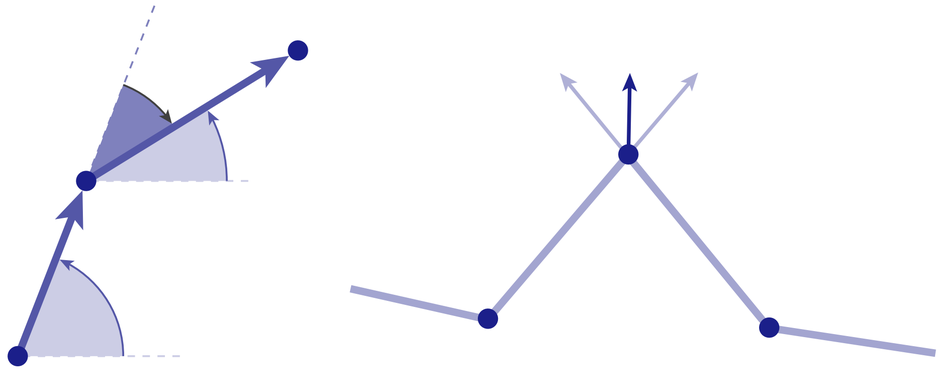
  1. turning angle
  2. length variation



- …which led to two **inequivalent** definitions of curvature in the discrete setting:

  1. $\kappa_i^A := \theta_i$

     $\kappa_i^B := 2\sin(\theta_i/2)$



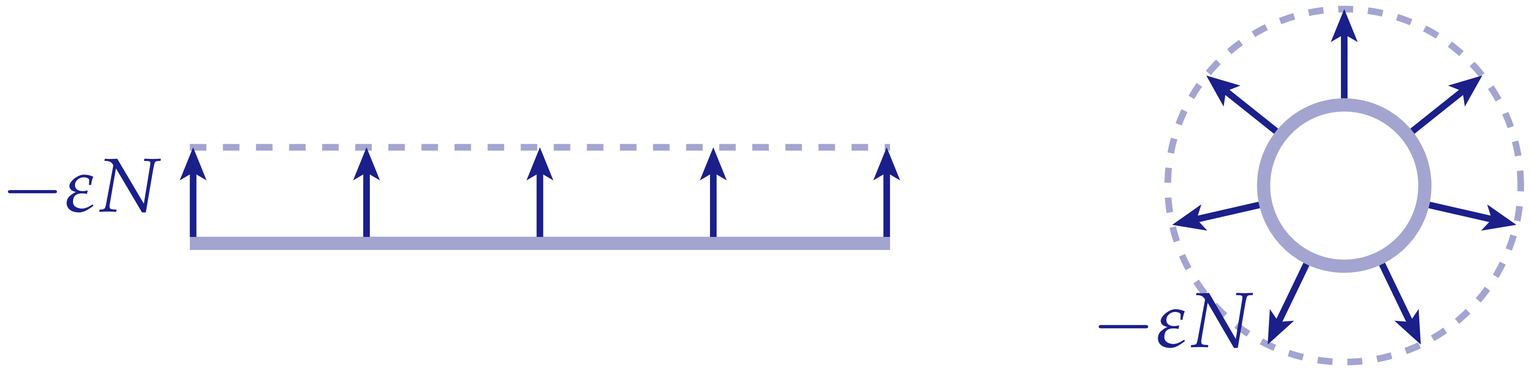
- For *small* angles, they agree. But in general, which one is "better"? And are there more possibilities? Let's keep going…

Steiner Formula

- Steiner's formula is closely related to our last approach: it says that if we move at a *constant* speed in the normal direction, then the change in length is proportional to curvature:

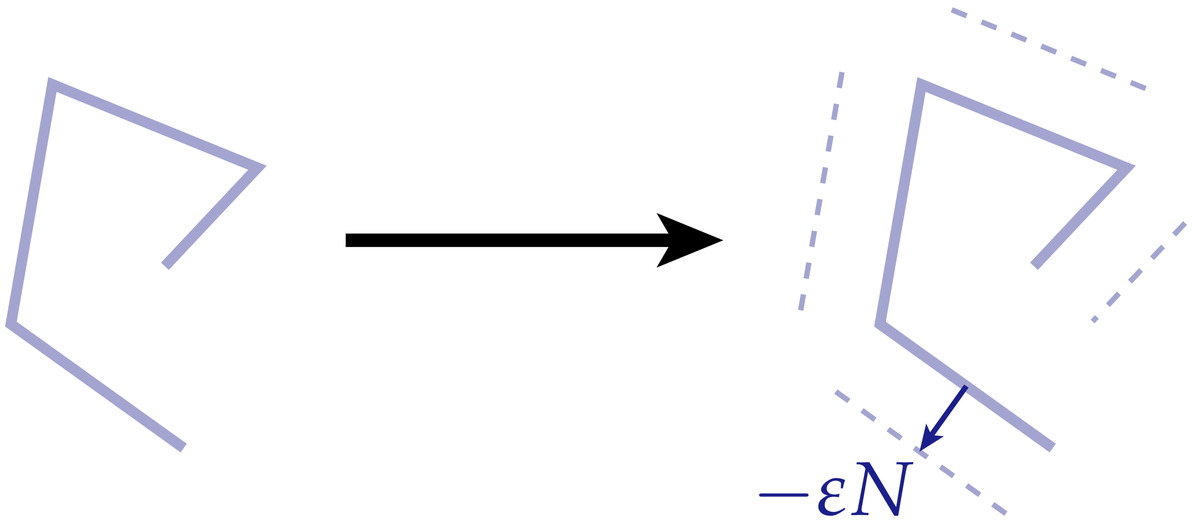
$$\text{length}(\gamma + \varepsilon N) = \text{length}(\gamma) - \varepsilon \int_0^L \kappa(s)\, ds$$

- The intuition is the same as before: for a constant-distance normal offset, length will change in curved regions but not flat regions:



Discrete Normal Offsets

- How do we apply normal offsets in the discrete case?
- The first problem is that *normals* are not defined at vertices!
- We can at very least offset individual edges along their normals:



- The question now is: how can we connect the normal-offset segments to get the final normal-offset curve?

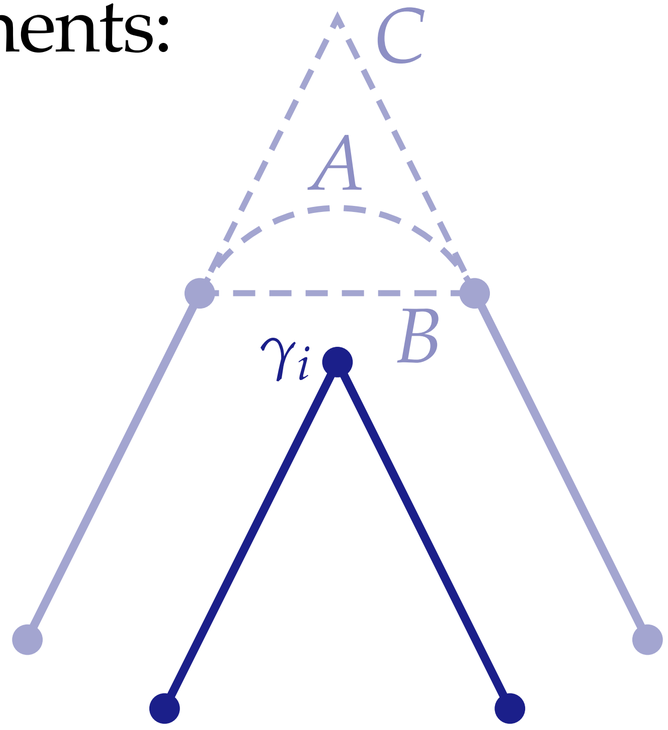
Discrete Normal Offsets

- There are then several natural ways to connect segments:

  (A) along a circular arc of radius ε

  (B) along a straight line

  (C) extend edges until they intersect

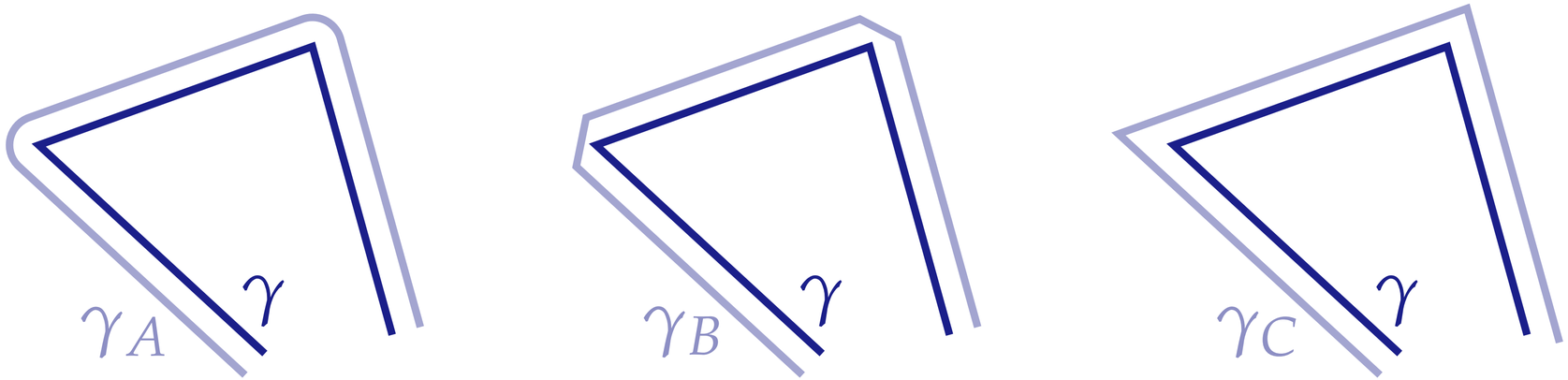


- If we now compute the total length of the connected curves, we get (after some work…):

$$
\begin{aligned}
\text{length}_A &= \text{length}(\gamma) - \varepsilon \textstyle\sum_i \theta_i \\
\text{length}_B &= \text{length}(\gamma) - \varepsilon \textstyle\sum_i 2\sin(\theta_i/2) \\
\text{length}_C &= \text{length}(\gamma) - \varepsilon \textstyle\sum_i 2\tan(\theta_i/2)
\end{aligned}
$$

Discrete Curvature (Steiner Formula)

- Since Steiner's formula says that the change in length is proportional to curvature, we get yet another definition for curvature by comparing the original and normal-offset lengths.

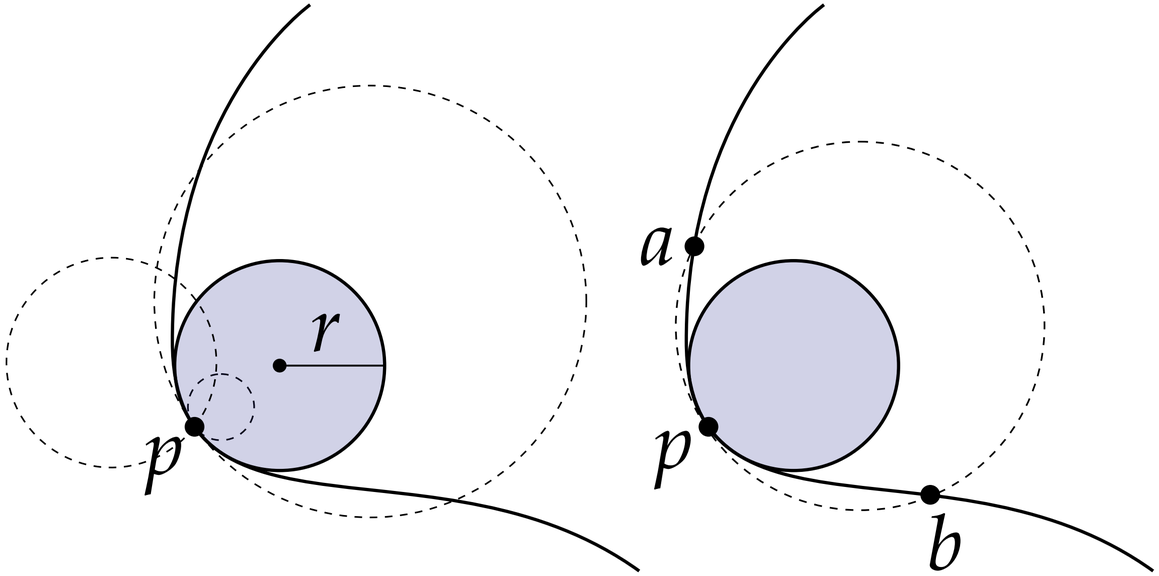


In fact, we get *three* definitions: two we've seen before and one we haven't:

$$\begin{aligned} \kappa_i^A &:= \theta_i \\ \kappa_i^B &:= 2\sin(\theta_i/2) \\ \kappa_i^C &:= 2\tan(\theta_i/2) \end{aligned}$$

Osculating Circle

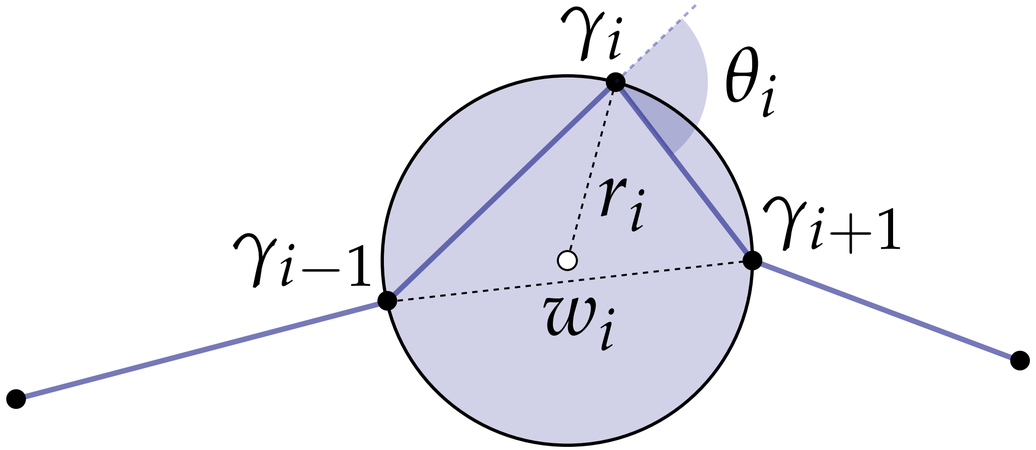
- One final idea is to consider the **osculating circle**, which is (roughly speaking) the circle that best approximates a curve at a given point p:



- More precisely, if we consider a circle passing through the point p itself and two equidistant points a and b to the "left" and "right" (resp.), the osculating circle is the limit as a and b approach p.
- The curvature is then the reciprocal of the radius: $\kappa(p) = \dfrac{1}{r(p)}$

Discrete Curvature (Osculating Circle)

- A natural idea, then, is to consider the *circumcircle* of three consecutive points on a discrete curve (*i.e.*, the circle touching all three points):



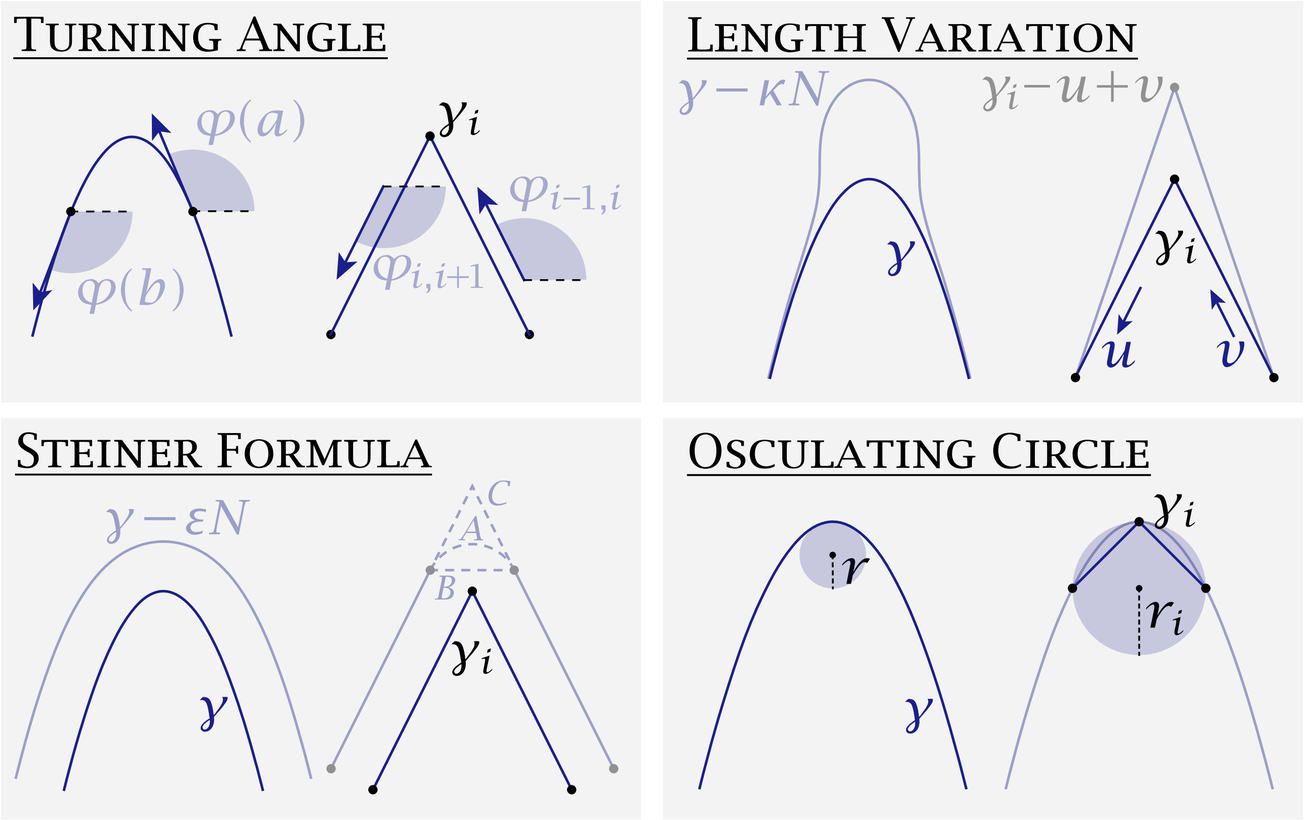
$$w_i := |\gamma_{i+1} - \gamma_{i-1}|$$

- We then get a *fourth* definition of discrete curvature:

$$\kappa_i^D := \frac{1}{r_i} = 2\sin(\theta_i)/w_i$$

A Tale of Four Curvatures

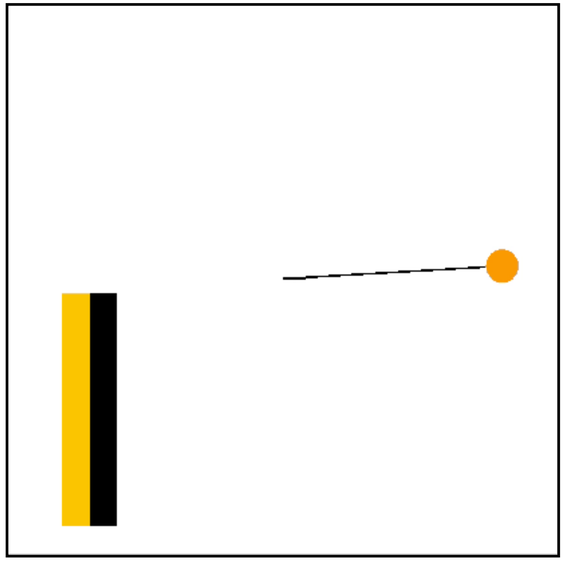
- Starting with four **equivalent** definitions of smooth curvature, we ended up with four **inequivalent** definitions for discrete curvature:



So… which one should we use?

Pick the Right Tool for the Job

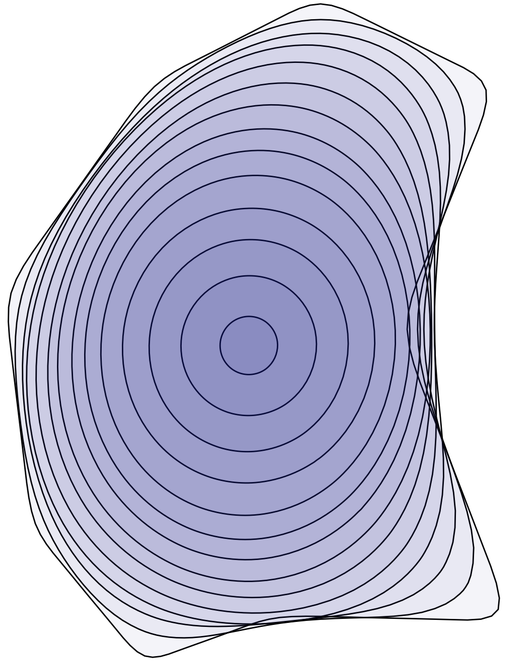
- **Answer:** pick the right tool for the job!
- For a given application, which properties are most important to us? Which quantities do we want to preserve? How much computation are we willing to do? Etc.
- *E.g.*, in a physical simulation we might care about preserving energy but not momentum—or momentum, but not energy.
- Ok… but what about curvature?



Toy Example: Curvature Flow

- A simple example is *curvature flow*, where a closed curve moves in the normal direction with speed proportional to curvature:

$$\frac{d}{dt}\gamma(s,t) = \kappa(s,t)N(s,t)$$

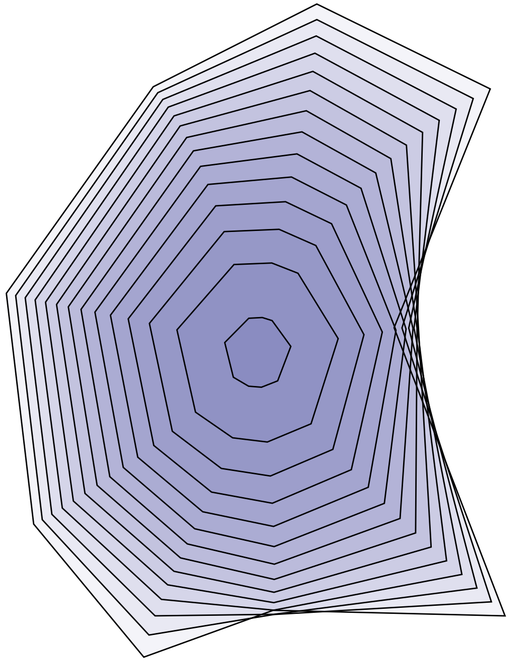


- Shows up in many places (finding silhouettes in images, annealing in metals, closed geodesics on manifolds…)

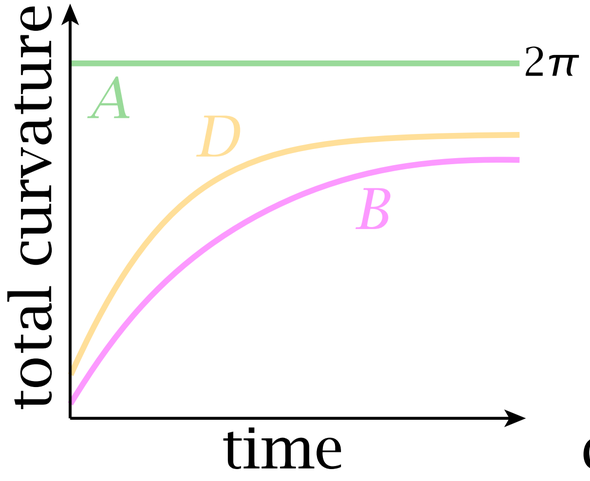
- Some key properties:
  - (**Total**) Total curvature remains constant throughout the flow.
  - (**Drift**) The center of mass does not drift from the origin.
  - (**Round**) Up to rescaling, the flow is stationary for circular curves.

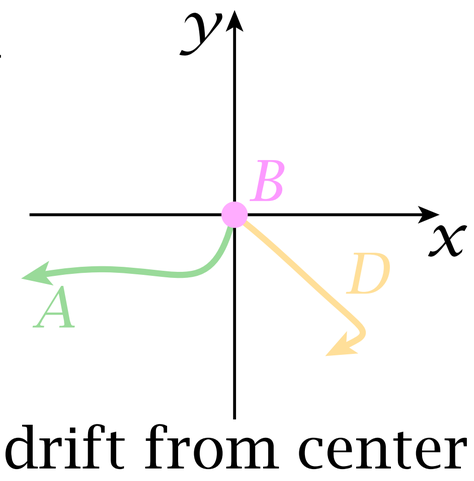
Discrete Curvature Flow—No Free Lunch

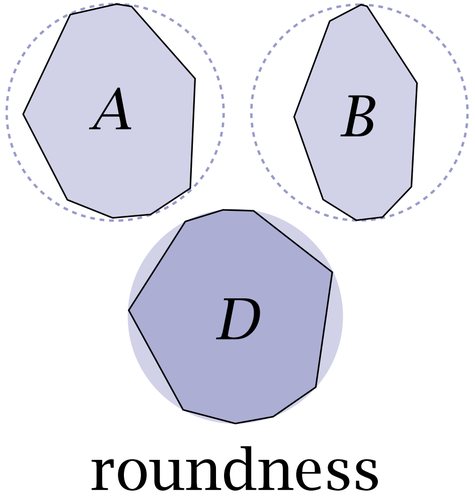
- We can approximate curvature flow by repeatedly moving each vertex a little bit in the direction of the discrete curvature normal: $$\gamma_i^{t+1} = \gamma_i^t + \tau \kappa_i N_i$$



- But **no** choice of discrete curvature simultaneously captures all three properties of the smooth flow*:







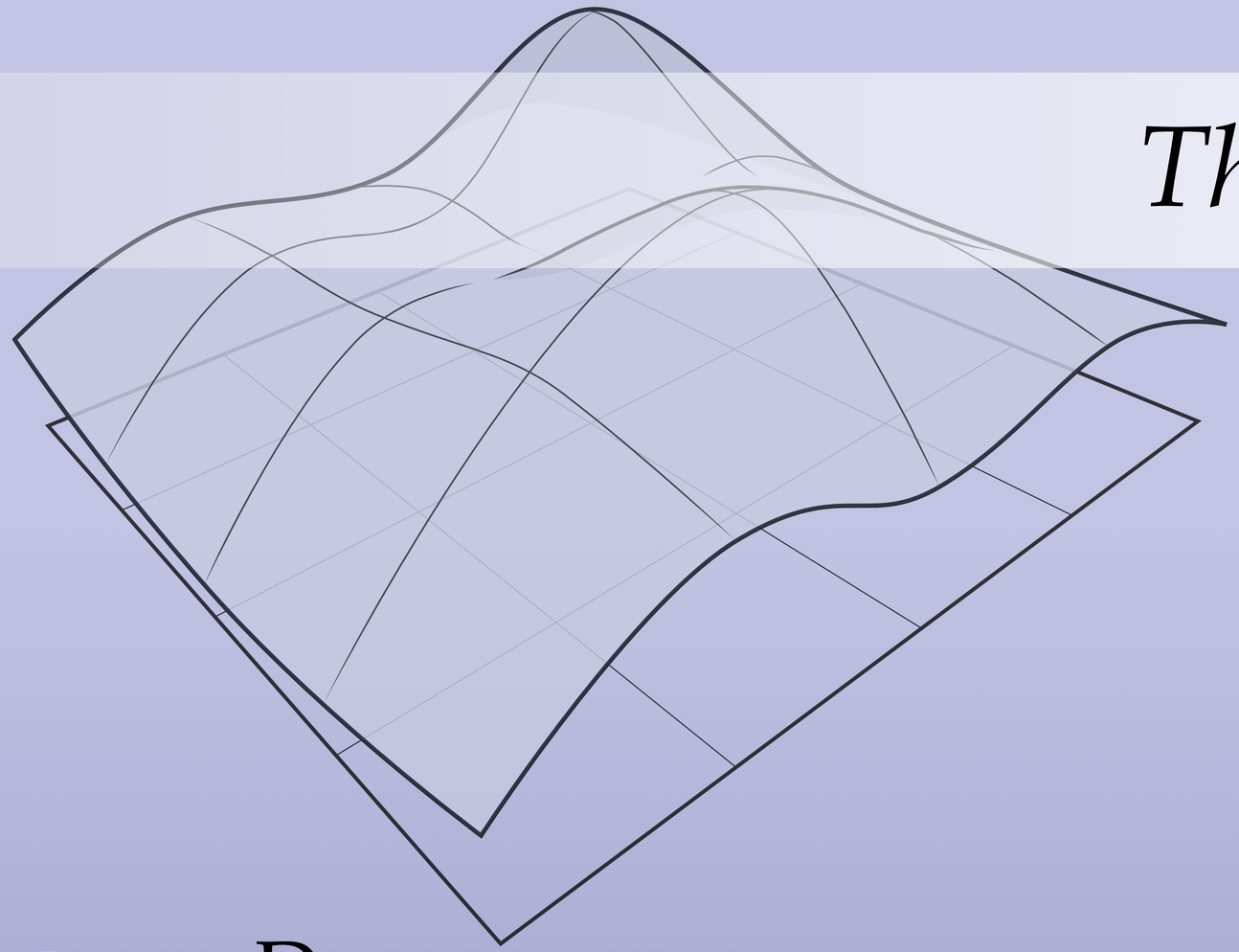
	Total	Drift	Round
κ^A	✓	×	×
κ^B	×	✓	×
κ^D	×	×	✓

*In fact, it's impossible!

No Free Lunch—Other Examples

- Beyond this "toy" problem, the *no free lunch* scenario is quite common in discrete differential geometry.
- Many examples (*e.g.*, Whitney-Graustein / Kirchoff analogy for curves; conservation of energy, momentum, and symplectic form for conservative time integrators; discrete Laplace operators…)
- At a more practical level: **The Game** played in DDG often leads to new & unexpected ways of formulating geometric algorithms. (E.g., faster, simpler, clearer guarantees, …)
- Will see *much* more of this as the course continues!

Thanks!



Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017