1 Linear Algebra

1.1 Basic Vector Operations

Exercise 1. *Letting* $\mathbf{u} := (9, 6)$, $\mathbf{v} := (7, 4)$, $a := 8$ *and* $b := 2$, *calculate the following quantities:*

(a) $\mathbf{u} + \mathbf{v}$

(b) $b\mathbf{u}$

(c) $a\mathbf{u} - b\mathbf{v}$

Exercise 2. *Letting* $\mathbf{u} := (5, 4, 2)$ *and* $\mathbf{v} := (3, 4, 3)$, *calculate the following quantities:*

(a) $\mathbf{u} - \mathbf{v}$

(b) $\mathbf{u} + 2\mathbf{v}$

Exercise 3. *So far we have been working with vectors in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, *but it is important to remember that other objects, like functions, also behave like vectors in the sense that we can add them, subtract them, multiply them by scalars,* etc. *Calculate the following quantities for the two polynomials* $p(x) := 5x^2 + 4x + 2$ *and* $q(x) := 3x^2 + 4x + 3$, *and evaluate the result at the point* $x = 2$:

(a) $p(x) - q(x)$

(b) $p(x) + 2q(x)$

1.2 Inner Products and Norms

Exercise 4. *Let* $\mathbf{u} := (7, 6, 3)$ *and* $\mathbf{v} := (4, 9, 5)$, *and compute the value of* $\langle \mathbf{u}, \mathbf{v} \rangle$ *using the standard (Euclidean) inner product* $\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^{n} u_i v_i$. *Roughly speaking, an inner product measures how well two vectors "line up." If* $\mathbf{u}$ *is a unit vector, the inner product can also be interpreted as the extent of* $\mathbf{v}$ *along the direction* $\mathbf{u}$ *(and vice versa).*

Exercise 5. *Any inner product* $\langle \cdot, \cdot \rangle$ *determines a* norm, *given by* $|\mathbf{u}| := \sqrt{\langle \mathbf{u}, \mathbf{u}}$. *Using the standard Euclidean inner product, compute the norm* $|\mathbf{u}|$ *of the vector* $\mathbf{u} := (5, 4, 3)$.

Exercise 6. *Suppose we define an alternative operation on 2-vectors, given by*

$$\langle \mathbf{u}, \mathbf{v} \rangle := 8u_1v_1 + u_1v_2 + u_2v_1 + 7u_2v_2.$$

Compute the following quantities, thinking about how they help verify that $\langle \cdot, \cdot \rangle$ *is a valid inner product:*

(a) $\langle \mathbf{x}, \mathbf{x} \rangle$, *for* $\mathbf{x} := (1, 0)$.

(b) $\langle \mathbf{y}, \mathbf{y} \rangle$, *for* $\mathbf{y} := (0, 1)$.

(c) $\langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle$, *for* $\mathbf{u} := (4, 5)$ *and* $\mathbf{v} := (5, 2)$.

(d) $\langle 2\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle - (2\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle)$, *for* $\mathbf{u} = (6, 9)$, $\mathbf{v} = (4, 3)$, *and* $\mathbf{w} = (2, 6)$.

Exercise 7. *Just as we can take inner products of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can also take inner products of functions. In particular, if $f(x)$ and $g(x)$ are two real-valued functions over the unit interval $[0,1]$, we can define the so-called L_2 inner product*

$$\langle\langle f, g \rangle\rangle := \int_0^1 f(x)g(x)\,dx.$$

Intuitively, this inner product measures how much the two functions "line up" (just like the dot product in $\mathbb{R}^n$). Check that $\langle\langle \cdot, \cdot \rangle\rangle$ behaves like an inner product by evaluating the following expressions:

(a) $\langle\langle f, f \rangle\rangle$, *for* $f(x) := ax^2 + b$, *and evaluate the result for* $a = 2$, $b = 2$.

(b) $\langle\langle f, f \rangle\rangle$, *for* $f(x) := 6e^{8x}$.

(c) $\langle\langle f, g \rangle\rangle - \langle\langle g, f \rangle\rangle$, *for* $f(x) := 9x + 5$ *and* $g(x) := 3x^2$.

Exercise 8. *The L_2 norm of a function $f : [0,1] \to \mathbb{R}$ can be defined as $||f|| := \sqrt{\langle\langle f, f \rangle\rangle}$. Compute the L_2 norm of the function $f(x) = 3e^{5x}$.*

1.3 Linear Maps

A map f between vector spaces is *linear* if

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

and

$$f(a\mathbf{x}) = af(\mathbf{x}),$$

i.e., if it preserves sums and scalar products.

Exercise 9. *Determine whether the functions $f(x) := 9x + 4$, $g(x) := 8x$, and $h(x) := -x^2$ are linear by evaluating quantities below at $x = 7$ and $y = 4$. [**Hint:** You can make your life a whole lot easier by first evaluating these differences for a generic value x, only plugging in the values of x and y if the difference ends up being nonzero.]*

(a) $f(x + y) - (f(x) + f(y))$

(b) $f(2x) - 2f(x)$

(c) $g(x + y) - (g(x) + g(y))$

(d) $g(7x) - 7g(x)$

(e) $h(x + y) - (h(x) + h(y))$

(f) $h(3x) - 3h(x)$

Just as a linear function preserves weighted sums, an *affine* function f preserves *convex combinations*. In other words, if we have some set of weights $w_1, \ldots, w_n$ such that $\sum_{i=1}^n w_i = 1$, then a function f is affine if

$$f(w_1\mathbf{x}_1 + \cdots + w_n\mathbf{x}_n) = \sum_i w_i f(\mathbf{x}_i)$$

for any collection of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Exercise 10. *Check whether the functions $f(\mathbf{u}) = u_1 + u_2 + 2$ and $g(\mathbf{u}) := \langle \mathbf{u}, \mathbf{u} \rangle$ are affine by evaluating the differences below for the vectors $\mathbf{u} := (7,4)$, $\mathbf{v} := (3,2)$, and weights $w_1 := 0.900$, $w_2 := 0.100$. (Here $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on 2-vectors.)*

(a) $f(w_1\mathbf{u} + w_2\mathbf{v}) - (w_1 f(\mathbf{u}) + w_2 f(\mathbf{v}))$

(b) $g(w_1\mathbf{u} + w_2\mathbf{v}) - (w_1 g(\mathbf{u}) + w_2 g(\mathbf{v}))$

As discussed earlier, functions can also be thought of as vectors since they obey all the same rules (*e.g.*, you can add two functions, scale them by a constant, *etc.*). Just like we can have maps from scalars to scalars, or vectors to vectors, we also can consider maps F that take a function $f : \mathbb{R} \to \mathbb{R}$ as input, and yield a new function as output. For instance, the map $F(f) := f^2$ would map the function $\cos(x)$ to the function $\cos^2(x)$, e^x to e^{2x}, *etc.* A map F between functions is *linear* if $F(f+g) = F(f) + F(g)$ and $F(af) = aF(f)$ for all functions f, g and scalars a.

Exercise 11. *Determine whether the maps* $F(f)(x) := 9 + \frac{d}{dx} f(x)$, $G(f)(x) := \int_0^1 f(x)\,dx$, *and* $H(f)(x) := f(0)$ *are linear* [a] *by computing the quantities below for functions* $f(x) := \sin(x)$ *and* $g(x) := e^x$, *evaluating the final result at the point* $x = 2$. *[**Hint:** For some of these calculations you can save yourself a lot of time and trouble by first computing the difference in terms of generic functions* $f(x)$ *and* $g(x)$ *rather than immediately plugging in the functions* $\sin(x)$ *and* e^x, *or the point* $x = 2$.*]*

(a) $F(f+g) - (F(f) + F(g))$

(b) $F(4f) - 4F(f)$

(c) $G(f+g) - (G(f) + G(g))$

(d) $G(9f) - 9G(f)$

(e) $H(f+g) - (H(f) + H(g))$

(f) $H(3f) - 3H(f)$

[a] For those of you inclined to be pedantic: the function G can be considered linear only if we restrict its domain to functions f that have a well-defined integral over the interval $[0,1]$.

1.4 Basis and Span

The *span* of a collection of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the set of all vectors $\mathbf{u}$ that can be expressed as

$$\mathbf{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n = \sum_{i=1}^{m} u_i \mathbf{e}_i$$

for some set of coefficients $u_1, \ldots, u_m \in \mathbb{R}$. If the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span all the vectors in $\mathbb{R}^n$, we say that the collection $\{\mathbf{e}_i\}$ is a *basis* for $\mathbb{R}^n$. If, in addition, $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$ for all i, and $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ for $i \neq j$, we say that $\{\mathbf{e}_i\}$ is an *orthonormal basis*. (Notice, by the way, that this definition depends on our choice of inner product $\langle \cdot, \cdot \rangle$.)

Exercise 12. *Consider the basis vectors* $\mathbf{e}_1 := (1/\sqrt{2}, 1/\sqrt{2})$ *and* $\mathbf{e}_2 := (-1/\sqrt{2}, 1/\sqrt{2})$, *and the vector* $\mathbf{u} := (4, 2)$ *(using the standard Euclidean inner product* $\langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + u_2 v_2$ *throughout).*

(a) *Compute the projection* $a := \langle \mathbf{e}_1, \mathbf{u} \rangle$ *of* $\mathbf{u}$ *onto* $\mathbf{e}_1$.

(b) *Compute the projection* b *of* $\mathbf{u}$ *onto* $\mathbf{e}_2$.

(c) *Compute the difference* $\mathbf{u} - (a\mathbf{e}_1 + b\mathbf{e}_2)$.

Exercise 13. *Consider the vectors* $\mathbf{e}_1 := (1, 0)$ *and* $\mathbf{e}_2 := (1, 1)$, *and compute the vector* $\langle \mathbf{u}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{u}, \mathbf{e}_2 \rangle \mathbf{e}_2$ *for* $\mathbf{u} := (2, 5)$. *(Think about why—unlike in the previous exercise—this vector ends up being different from* $\mathbf{u}$.*)*

Not all collections of vectors that span $\mathbb{R}^n$ provide an orthonormal basis. But given a collection of n vectors $\{\mathbf{e}_i\}$ that span $\mathbb{R}^n$, we can construct an orthonormal basis $\{\tilde{\mathbf{e}}_i\}$ by applying the *Gram-Schmidt procedure*.

Exercise 14. *Consider the vectors* $\mathbf{e}_1 := (8,4)$ *and* $\mathbf{e} := (9,4)$.

(a) *Compute* $\tilde{\mathbf{e}}_1 := \mathbf{e}_1/|\mathbf{e}_1|$.

(b) *Compute* $\hat{\mathbf{e}}_2 := \mathbf{e}_2 - \langle \mathbf{e}_2, \tilde{\mathbf{e}}_1\rangle \tilde{\mathbf{e}}_1$.

(c) *Compute* $\tilde{\mathbf{e}}_2 := \hat{\mathbf{e}}_2/|\hat{\mathbf{e}}_2|$.

(d) *Compute* $|\tilde{\mathbf{e}}_1|^2$.

(e) *Compute* $|\tilde{\mathbf{e}}_2|^2$.

(f) *Compute* $\langle \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\rangle$.

NOTE: If you were bases $\mathbf{e}_1$ and $\mathbf{e}_2$ that are **parallel**, there is no solution—please just delete any ? marks in the solution template for items (c) and (e) to receive credit.

Exercise 15. *The* determinant *of a collection of vectors* $\mathbf{e}_1, \ldots, \mathbf{e}_k \in \mathbb{R}^n$ *is the (signed) volume of the parallelepiped with these edge vectors. For instance, for two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, *the signed volume of the corresponding parallelogram is* $\det(u,v) := u_1v_2 - u_2v_1$.

(a) *Compute the determinant of the vectors* $\mathbf{e}_1 := (9,7)$ *and* $\mathbf{e}_2 := (18,14)$.

(b) *Compute the value* $a := \langle \mathbf{e}_1, \mathbf{w}\rangle$ *for* $\mathbf{w} = (2,3)$.

(c) *Compute the value* $b := \langle \mathbf{e}_2, \mathbf{w}\rangle$ *for the same vector* $\mathbf{w}$.

(d) *Compute the vector* $a\mathbf{e}_1 + b\mathbf{e}_2$. *(Think about a couple different reasons why this procedure does or does not reconstruct the original vector* $\mathbf{w}$.*)*

1.5 Systems of Linear Equations

Given some set of variables $x_1, \ldots, x_k \in \mathbb{R}$, a *(real) linear equation* is any equation of the form

$$f(x_1, \ldots, x_k) = b$$

where f is a linear function and $b \in \mathbb{R}$ is a constant. Alternatively, it is any equation of the form

$$\hat{f}(x_1, \ldots, x_k) = 0$$

where $\hat{f}$ is affine. A *system* of linear equations is simply a collection of linear equations

$$\begin{aligned} f_1(x_1, \ldots, x_k) &= b_1 \\ &\vdots \\ f_p(x_1, \ldots, x_k) &= b_p, \end{aligned}$$

each of which shares the same set of variables. Solving a linear system means finding values for the variables $x_1, \ldots, x_k$ that satisfy all of the equations simultaneously. We will also sometimes refer to these variables as *degrees of freedom*.

Exercise 16. *Solve the system of linear equations*

$$\begin{aligned} 8x + 2y &= 3 \\ -2x + 8y &= 7 \end{aligned}$$

for the unknown pair $(x, y) \in \mathbb{R}^2$.

Exercise 17. *A linear system with fewer equations than unknowns is* underdetermined*, meaning that there are many possible solutions. In this situation, one can obtain a solution by imposing additional criteria* (i.e.*, by adding more equations!). Solve the system below, imposing the condition that the final solution* $(x, y, z) \in \mathbb{R}^3$ *must sit on the unit sphere,* i.e.*,* $x^2 + y^2 + z^2 = 1$*. If there are multiple solutions, pick the one for which* z *is larger. [**Hint:** First solve the linear part for* x *and* y*, treating* z *as a constant. Then at the very end plug in the final nonlinear "sphere" condition to obtain a value for* z*—this kind of strategy can always be applied when you have one nonlinear equation and* $n-1$ *linear equations.]*

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= \tfrac{1}{2} \end{aligned}$$

1.6 Bilinear and Quadratic Forms

In addition to linear maps, which might represent (for example) transformations of space, and linear systems of equations, which are commonly used to encode (say) dynamical constraints or physical processes, computer graphics makes heavy use of bilinear and quadratic forms, which are often used to describe (for instance) an energy that has to be minimized.

A *bilinear form* is any map B from a pair of vectors to a scalar that is linear in each argument. In other words, we have both

$$B(a\mathbf{x} + b\mathbf{y}, \mathbf{z}) = aB(\mathbf{x}, \mathbf{z}) + bB(\mathbf{y}, \mathbf{z})$$

and

$$B(\mathbf{x}, a\mathbf{y} + b\mathbf{z}) = aB(\mathbf{x}, \mathbf{y}) + bB(\mathbf{x}, \mathbf{z})$$

for all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and scalars a, b. In $\mathbb{R}^n$, any bilinear form $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be expressed as a homogeneous bivariate polynomial of degree 2. For instance, in $\mathbb{R}^2$, every bilinear form can be written as $B(\mathbf{x}, \mathbf{y}) = ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$ for some set of constant coefficients $a, b, c, d \in \mathbb{R}$.

A *quadratic form* is any map $Q : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$Q(a\mathbf{x}) = a^2 Q(\mathbf{x})$$

for all vectors $\mathbf{x} \in \mathbb{R}^n$ and scalars $a \in \mathbb{R}$. In $\mathbb{R}^n$, every quadratic form can be expressed as a homogeneous quadratic polynomial. For instance, in $\mathbb{R}^2$ every quadratic form can be written as $Q(\mathbf{x}) = ax_1^2 + bx_1x_2 + cx_2^2$ for some set of constant coefficients $a, b, c, d \in \mathbb{R}$.

The following exercises should help make these ideas more concrete.

Exercise 18. *We've already seen one key example of bilinear and quadratic forms—the Euclidean inner product* $\langle \cdot, \cdot \rangle$ *on* $\mathbb{R}^2$ *and the corresponding norm (squared)* $|\mathbf{x}|^2 := \langle \mathbf{x}, \mathbf{x} \rangle$*. Verify that these objects really are bilinear and quadratic (respectively) by performing the following calculations for vectors* $\mathbf{x} := (3, 9)$*,* $\mathbf{y} := (8, 7)$*,* $\mathbf{z} := (7, 3)$ *and scalars* $a := 6$*,* $b := 5$*.*

(a) $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle$

(b) $a\langle \mathbf{x}, \mathbf{z} \rangle + b\langle \mathbf{y}, \mathbf{z} \rangle$

(c) $|a\mathbf{x}|^2$

(d) $a^2|\mathbf{x}|^2$

Given a bilinear form $B(\mathbf{x}, \mathbf{y})$, one can always construct a corresponding quadratic form $Q(\mathbf{x}) := B(\mathbf{x}, \mathbf{x})$. It is also possible to go the other direction: given any quadratic form $Q(\mathbf{x})$, one can construct a *symmetric* bilinear form via

$$B(\mathbf{x}, \mathbf{y}) := \tfrac{1}{2} \left(Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}) \right).$$

This bilinear form has the special property that applying it to two copies of the same argument recovers the original quadratic form: $B(\mathbf{x}, \mathbf{x}) = \frac{1}{2}(Q(2\mathbf{x}) - 2Q(\mathbf{x})) = \frac{1}{2}(4Q(\mathbf{x}) - 2Q(\mathbf{x})) = Q(\mathbf{x})$.

Exercise 19. *For the quadratic form* $Q(\mathbf{x}) := 9x_1^2 + 5x_1x_2 + 3x_2^2$*, compute the four coefficients* (a, b, c, d) *of the corresponding bilinear form* $B(\mathbf{x}, \mathbf{y}) = ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$.

Exercise 20. *An extremely common energy in computer graphics (used in image processing, geometry, physically-based animation, ...) measures the failure of the derivative of a function* f *to match some fixed function* u*. A simple version of this energy is given by the expression*

$$E(f) := \left|\left|\frac{df}{dx} - u\right|\right|^2,$$

where $||\cdot||$ *denotes the* L_2 *norm on functions over the unit interval,* i.e., $||f||^2 := \int_0^1 f(x)^2\, dx$.

If you expand the norm, you will get three terms: one that is quadratic in f *(*i.e.*, involving a square of* f*), one that is linear in* f *(*i.e.*, where* f *appears only once), and one that does not depend on* f *at all. Derive an expression for the bilinear form* B *corresponding to the quadratic part, which maps a pair of functions* $f, g : [0, 1] \to \mathbb{R}$ *to a real scalar value* $B(f, g)$*. Once you've determined* B*, evaluate it on the functions* $f(x) := 2x$ *and* $g(x) = e^{5x}$.

*[**Hint:** For most of this exercise, it will greatly simplify your calculations to write the* L_2 *norm and inner product as* $||\cdot||$ *and* $\langle\langle\cdot,\cdot\rangle\rangle$*, respectively, rather than expanding everything out in terms of integrals over the real line. Since we've already checked that these operations obey the same rules as for vectors in* $\mathbb{R}^n$*, you can manipulate functions just like you would manipulate ordinary vectors—this is part of the power of interpreting functions as vectors. However, if at any point you find these manipulations confusing, it may help to go back to the definition of the inner product in terms of an integral.]*

1.7 Matrices and Vectors

If you've studied linear algebra before, matrices may have been one of the very first objects you learned about. However, learning linear algebra from the perspective of matrices can often lead to confusion, because matrices are used to represent many different objects (linear maps, linear systems of equations, quadratic forms, ...), and the "rules" about what you can do with a matrix will depend on which of these objects it represents. Especially in computer graphics, matrices almost always have a very concrete geometric meaning (*e.g.*, a rotation, a re-scaling, an energy, *etc.*), and should never be thought about as just "a big block of numbers." It can be quite dangerous to manipulate matrices without understanding what they mean. For instance, inverting a matrix corresponding to a bijective linear map makes sense (you're just reversing the direction of the map), but what does it mean to invert the matrix of a quadratic form? Likewise, the same linear map represented in different coordinate systems will yield different matrices—even though the map itself is identical. In short, when someone hands you a matrix, the first thing you should ask is, *"what does this matrix **represent**? (And in what coordinate system?)"* You should also keep in mind that **matrices are not linear maps and linear maps are not matrices**. For instance, we already saw linear maps on functions (like integration and differentiation) that cannot be represented by any kind of matrix. Likewise, a given matrix may not represent a linear map at all—for instance, it could be the adjacency matrix for a graph, or just some rows from a medical database.

In spite of this admonition, matrices are an extremely useful tool for doing practical calculations and numerical linear algebra, since they lead to easy symbolic manipulation and fast algorithms. Perhaps the most basic algorithm is matrix-vector multiplication, which applies a linear map to a vector. More precisely, suppose we have any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$. Then we can encode this map as a block of numbers or *matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ with entries

$$\mathbf{A}_{ij} := (f(\mathbf{e}_j))_i,$$

where $\mathbf{e}_i$ is the standard basis, *i.e.*, a "1" in coordinate i and "0" for all other coordinates. (The outer subscript i just means "take the ith component of the result.") Basically, $\mathbf{A}$ tells us what the linear map f does to each basis vector. But since any vector $\mathbf{u}$ can be expressed as a linear combination $\sum_i u_i\mathbf{e}_i$ of the basis vectors, we can of course also use $\mathbf{A}$ to compute $f(\mathbf{u})$. By definition, then, we say that

$$\mathbf{A}\mathbf{x} := f(\mathbf{x})$$

for all vectors $\mathbf{x}$ in the domain of f, where on the left-hand side we encode $\mathbf{x}$ as a $n \times 1$ matrix, *i.e.*, a *column vector*. Given this description, you should be able to reverse-engineer the algorithm for matrix-vector multiplication—even if you don't remember, or have never seen it before! (Of course, you are more than welcome to review matrix-vector multiplication in any standard reference.)

Exercise 21. *Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$f(\mathbf{u}) := (7u_1 + 9u_2, 6u_1 + 4u_2).$$

(a) Write the four entries of the matrix $A \in \mathbb{R}^{2\times 2}$ encoding f in the standard basis, putting the result in row-major, column-minor order, i.e., *the top row left-to-right, followed by the bottom row left-to-right.*

(b) For $\mathbf{x} := (3, 9)$, compute the matrix-vector product $\mathbf{Ax}$.

Exercise 22. *Suppose linear maps $f : \mathbb{R}^p \to \mathbb{R}^q$ and $g : \mathbb{R}^q \to \mathbb{R}^r$ are represented by matrices $\mathbf{A} \in \mathbb{R}^{q\times p}$ and $B \in \mathbb{R}^{r\times q}$, respectively. Then the product of these matrices encodes composition of the linear maps. In other words, for any vector $\mathbf{x} \in \mathbb{R}^p$*

$$\mathbf{BAx} := g(f(\mathbf{x})).$$

(a) In row-major order, give the matrix $\mathbf{A} \in \mathbb{R}^{3\times 2}$ representing the linear map

$$f : \mathbb{R}^2 \to \mathbb{R}^3; \quad (x_1, x_2) \mapsto (7x_1, 5x_2, x_1 + x_2).$$

(b) In row-major order, give the matrix $\mathbf{B} \in \mathbb{R}^{3\times 3}$ representing the linear map

$$g : \mathbb{R}^3 \to \mathbb{R}^3; \quad (x_1, x_2, x_3) \mapsto 9(x_2, x_3, x_1).$$

(c) Compute the matrix-matrix product $\mathbf{BA} \in \mathbb{R}^{3\times 2}$, which represents the map $g \circ f$.

Exercise 23. *A matrix $\mathbf{A}^{-1} \in \mathbb{R}^{n\times n}$ is called the* inverse *of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ if $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, i.e., the matrix representing the identity map $f(\mathbf{x}) := \mathbf{x}$.*

(a) For $n = 2$ (i.e., two dimensions), construct the matrix I corresponding to the identity map, giving the result in row-major form.

(b) Construct the matrix $A \in \mathbb{R}^{2\times 2}$ representing the function $f(x, y) := (6x + 5y, -10x + 18y)$.

*(c) For the same function f as in part (b), construct the matrix $A \in \mathbb{R}^{2\times 2}$ representing the inverse function f^{-1}. [**Hint:** If we consider a general linear function $f(x, y) = (ax + by, cx + dy)$, then you want to solve the linear system $ax + by = u$, $cx + dy = v$ for x and y in terms of u, v, and the four constant coefficients a, b, c, d. This yields another linear map (the inverse), and from there you can build the corresponding matrix. (This exercise will also give you a general procedure for inverting 2x2 matrices.)]*

*(d) Compute the matrix-matrix product $\mathbf{A}^{-1}\mathbf{A}$, giving the result in row-major form. [**Hint:** Yes, you know what the answer* should *be, but if you actually carry out this calculation, you will be able to check your work for the previous two parts!]*

Exercise 24. *For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, its* transpose *is the matrix $\mathbf{A}^\mathsf{T} \in \mathbb{R}^{n\times m}$ with entries given by*

$$\mathbf{A}^\mathsf{T}_{ij} := \mathbf{A}_{ji}.$$

(a) *Compute the transpose of the matrix*

$$\mathbf{A} := \frac{1}{\sqrt{25}} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix},$$

giving the result in row-major form.

(b) *Compute the matrix-matrix product* $\mathbf{A}^\mathsf{T}\mathbf{A}$*, giving the result in row-major form. (Think about why you got this particular result!)*

Exercise 25. *Consider a linear map* f *represented as a matrix* $\mathbf{A} \in \mathbb{R}^{2\times 2}$*, and a quadratic form* Q *represented as a matrix* $\mathbf{B} \in \mathbb{R}^{2\times 2}$*. Suppose we originally expressed these matrices in terms of an orthonormal basis* $(\mathbf{u}, \mathbf{v})$*, but now we need to apply them to vectors expressed in a different orthonormal basis,* $(\mathbf{x}, \mathbf{y})$*. This situation is quite common in computer graphics, since (for instance) objects in a scene might experience some kind of transformation before we apply a given operation.*

(a) *Compute the matrix* $\mathbf{E} \in \mathbb{R}^{2\times 2}$ *expressing the change of basis from* $\mathbf{u} = (6/\sqrt{45}, 3/\sqrt{45})$, $\mathbf{v} = (-3/\sqrt{45}, 6/\sqrt{45})$ *to* $\mathbf{x} = (2/\sqrt{85}, 9/\sqrt{85})$, $\mathbf{y} = (-9/\sqrt{85}, 2/\sqrt{85})$*. In other words,* $\mathbf{E}$ *should satisfy the relationships* $\mathbf{Eu} = \mathbf{x}$ *and* $\mathbf{Ev} = \mathbf{x}$ (i.e., *the change of basis should be achieved via* ***left****-multiplication). The result should be given in row-major form. [**Hint:** Take an arbitrary vector* $\mathbf{a} = a_1\mathbf{u} + a_2\mathbf{v}$ *and project it onto the two new directions* $\mathbf{x}, \mathbf{y}$*. How can you express this operation as a matrix? You may want to double-check your answer by applying your final matrix to your vector u. It may also be helpful to work out the solution symbolically first, plugging in the numerical values only at the very end.]*

(b) *Suppose we now want to apply the linear map* $f(\mathbf{a}) := (8a_1 + 3a_2, 4a_1 + 3a_2)$ *to a vector* $\mathbf{a}$ *that is expressed in the basis* $(\mathbf{x}, \mathbf{y})$*, but express the result in the basis* $(\mathbf{u}, \mathbf{v}$*. Write (in row-major form) the matrix that represents this operation. [**Hint:** Be very careful about which direction you change bases! It may be helpful to consider the previous exercise (on the transpose).]*

(c) *Similarly, suppose we now want to apply the quadratic form* $Q(\mathbf{a}) := 8a_1^2 + 6a_1a_2 + 2a_2^2$ *to a vector* $\mathbf{a}$ *expressed in the basis* $(\mathbf{x}, \mathbf{y})$*. What matrix should we use (in row-major form)? [**Hint:** The matrix representation of a quadratic form generally looks like* $\mathbf{w}^\mathsf{T}\mathbf{B}\mathbf{w}$ *for some symmetric matrix* $\mathbf{B}$*, where* $\mathbf{w}$ *is the argument. Think very carefully about when and how the argument has to be transformed from one basis into another.]*