Written Assignment 2:
Investigating Curvature

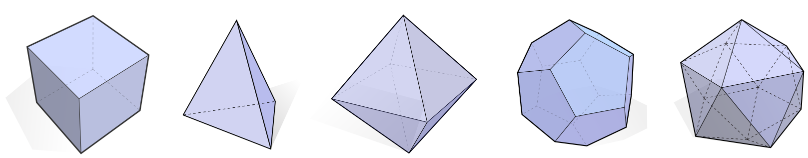
CMU 15-458/858 (Fall 2017)

Submission Instructions. Please submit your solutions to the exercises (whether handwritten, LaTeX, etc.) as a **single PDF file** by email to `Geometry.Collective@gmail.com`. *This email must also contain the `.zip` file for your coding solution.* Scanned images/photographs can be converted to a PDF using applications like *Preview* (on Mac) or a variety of free websites (*e.g.*, `http://imagetopdf.com`). Your submission email must include the string **DDG17A2** in the subject line. Your graded submission will (hopefully!) be returned to you at least one day before the due date of the next written assignment.

Grading. Please clearly show your work. Partial credit **will** be awarded for ideas toward the solution, so please submit your thoughts on an exercise even if you cannot find a full solution. **Note that you are required to complete only TWO exercises from each of sections 1, 2, 3, and 4!** (8 problems total.) You are of course welcome to do more. :-)

If you don't know where to get started with some of these exercises, just ask! A great way to do this is to leave comments on the course webpage under this assignment; this way everyone can benefit from your questions. We are glad to provide further hints, suggestions, and guidance either here on the website, via email, or in person. Office hours are listed on the course website, but let us know if youd like to arrange an individual meeting.

Late Days. Note that you have 5 no-penalty late days for the entire course, where a "day" runs from 6:00:00 PM Eastern to 5:59:59 PM Eastern the next day. No late submissions are allowed once all late days are exhausted. If you wish to claim one or more of your five late days on an assignment, please indicate which late day(s) you are using in your email submission. You must also draw **Platonic solids** corresponding to the late day(s) you are using (cube=1, tetrahedron=2, octahedron=3, dodecahedron=4, icosahedron=5). Use them wisely, as you cannot use the same polyhedron twice! If you are typesetting your homework on the computer, we have provided images that can be included for this purpose (in LaTeX these can be included with the `\includegraphics` command in the `graphicx` package).



1 2 3 4 5

Collaboration and External Resources. You are **strongly encouraged** to discuss all course material with your peers, including the written and coding assignments. You are especially encouraged to seek out new friends from other disciplines (CS, Math, Engineering, etc.) whose experience might complement your own. However, *your final work must be your own, i.e.*, direct collaboration on assignments is prohibited.

You are allowed to refer to any external resources *except* for homework solutions from previous editions of this course (at CMU and other institutions). If you use an external resource, cite such help on your submission. **If you are caught cheating, you will get a zero for the entire course.**

Warning! With probability 1, there are typos in this assignment. If *anything* in this handout does not make sense (or is blatantly wrong), let us know! We will be handing out extra credit for good catches. :-)

Format. This written assignment is cement your understanding of curvature on both continuous and discrete surfaces.

> This assignment is closely connected to Chapters 2, 4, and 5 of the course notes. Additionally, the course slides (as they are released) will be extremely helpful for completing the assignment.

1 Smooth Curves and Surfaces

1. Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(s) := (s, s^2, s^3).$$

   (a) Compute $\frac{d}{ds}\gamma$ and $T(s)$.

   (b) Compute $\kappa(s)$ and $N(s)$.

   (c) Compute $B(s)$ and $\tau(s)$.

2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(u,v) := \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1).$$

   (a) Verify that for all $(u,v) \in \mathbb{R}^2$, $f(u,v)$ lies on the unit sphere centered at the origin.

   (b) Explicitly compute the differential df.

   (c) Compute the metric g induced by f.

   (d) Compute the Gauss map N induced by f.

   (e) Compute the corresponding shape operator dN.

3. Consider the immersion of the torus $f : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$f(\theta, \varphi) := ((R + r\cos\varphi)\cos\theta, (R + r\cos\varphi)\sin\theta, r\sin\varphi)$$

   where $R > r > 0$.

   (a) Explicitly compute the differential df.

   (b) Compute the metric g induced by f.

   (c) Compute the Gauss map N induced by f.

   (d) Compute the corresponding shape operator dN.

4. Recall that the mean and Gaussian curvature can be expressed in terms of the principal curvatures κ_1, κ_2:

$$H := \frac{\kappa_1 + \kappa_2}{2}, \qquad K := \kappa_1\kappa_2.$$

   Derive expressions for κ_1 and κ_2 in terms of H and K. (For this exercise you do not need to worry about whether $\kappa_1 > \kappa_2$ or $\kappa_2 > \kappa_1$.)

2 Discrete Curvature, Part I: Surface Normals

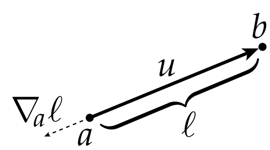
Warning! In several of the exercises in this section you are asked to derive an expression for the gradient of some geometric quantity. For these questions we ask that you use *geometric* arguments rather than simply taking the partial derivatives. (Solutions that compute the gradient via partial derivatives will not receive any credit.) The basic recipe for obtaining the gradient geometrically is:

- First, find the *unit vector* such that moving in that direction increases the quantity the fastest. This direction gives the direction of the gradient.
- Next, find the rate of change when moving in this direction. This quantity gives the magnitude of the gradient.

To make this idea clear, here is a concrete example:

Example. Let ℓ be the length of the vector $u := b - a$, where a and b are points in $\mathbb{R}^2$. Let $\hat{u} := u/\ell$ be the unit vector in the same direction as u. Show that the gradient of ℓ with respect to the location of the point a is given by

$$\nabla_a \ell = -\hat{u}.$$

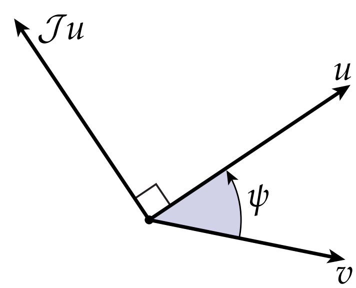


Solution. The fastest way to increase ℓ is to move a along the direction of $-u$ (since a small motion in the orthogonal direction Ju looks like a rotation, which does not change the length). Since moving a by one unit increases the length ℓ by one unit, the magnitude of the gradient is 1. Thus, $\nabla_a \ell = -\hat{u}$.

5. Consider two vectors u, v in the plane (not necessarily unit length), and let ψ be the angle from u to v. Argue that the gradient of ψ with respect to u is given by

$$\nabla_u \psi = \mathcal{J} u / |u|^2,$$

where $\mathcal{J}$ denotes a quarter turn in the counter-clockwise direction, and $|u|$ is the length of u.



6. Do Exercise 12 in Chapter 5 of the course notes (you will need to start reading at the beginning of Section 5.1).
7. Do Exercise 13 in Chapter 5 of the course notes (you will need to start reading on the preceding page).
8. Show that $df \wedge dN = 2HNdA$ and $dN \wedge dN = 2KNdA$ (Hint: evaluate the 2-forms on the left-hand side on the two principal curvature directions X_1, X_2.)
9. Do Exercise 14 in Chapter 5 of the course notes.
10. Do Exercise 17 in Chapter 5 of the course notes.

3 Discrete Curvature, Part II: Scalar Curvatures

3.1 Notation

In all of these exercises we will be considering a simplicial surface mesh $M = (V, E, F)$. It is embedded in $\mathbb{R}^3$ via a discrete 0-form $f : V \to \mathbb{R}^3$, i.e., just an assignment of coordinates to each vertex. (More precisely, M is a simplicial surface without boundary, and f is a discrete immersion—see the course slides on surfaces for definitions.) To make exercises easier, you may assume that the mesh bounds a convex region. (The boxed formulas are however ultimately valid for nonconvex meshes.) We use the following notation throughout:

- The set of vertices (0-simplices) of M is V. Each vertex is denoted by its index $i \in V$. The position of each vertex is denoted as $f_i \in \mathbb{R}^3$.
- The set of edges (1-simplices) of M is E. The edge connecting vertices i and j is denoted as $ij \in E$, and the vector along this edge will be denoted by $e_{ij} := f_j - f_i$. Note that $e_{ij} = -e_{ji}$. The length of each edge in the embedding is denoted as ℓ_{ij} (which is the same as ℓ_{ji}).
- The set of faces (2-simplices) of M is F. The face connecting vertices i, j, and k is denoted as $ijk \in F$. The area of each face in the embedding is denoted as A_{ijk}. The vector normal to the face is denoted as N_{ijk}.
- For each vertex i and face ijk, we use φ_i^{jk} to denote the so-called *interior angles* between edges e_{ij} and e_{ik}.
- For each edge ij there are exactly two faces ijk and $ji\ell$ containing ij. We use $\theta_{ij} \in [-\pi, \pi]$ to denote the so-called *dihedral angle* between their normals N_{ijk} and $N_{ij\ell}$.

3.2 Discrete Scalar Curvatures

In this section, we will derive the discrete scalar Gaussian curvature and discrete scalar mean curvature. The formulas we ultimately get are

$$\boxed{\begin{aligned} K_i &:= 2\pi - \sum_{ijk \in F} \varphi_i^{jk}, \\ H_i &:= \tfrac{1}{2} \sum_{ij} \theta_{ij} \ell_{ij}, \end{aligned}}$$

where the first sum is over triangles containing vertex i, and second is over edges containing vertex i.

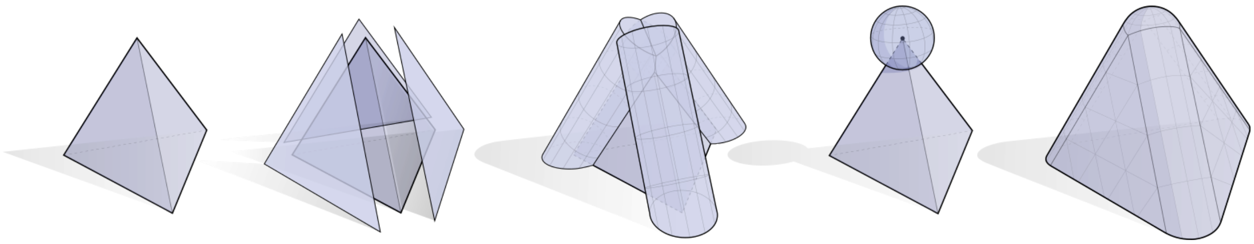
3.3 Discrete Gaussian Curvature

In class, we will study one approach to discretizing Gaussian curvature, based on the relationship between a Euclidean ball and a geodesic ball. For this homework you will take a different approach, based on the discrete Gauss map:

11. Exercise 8 in Section 4.3 of the course notes.
12. Exercise 9 in Section 4.3 of the course notes.
13. Exercise 10 in Section 4.3 of the course notes.
14. Exercise 11 in Section 4.3 of the course notes.

3.4 Steiner Formula for Simpicial Surfaces

To obtain a more holistic view of scalar curvatures on triangle meshes, we can consider the so-called *Steiner polynomial*. Pick some real number $r \geq 0$. Here we consider a *mollified surface* given by the set of points which are a positive distance r away from $f(M)$ in the normal direction. Below is an explanatory figure.



Let $\mathcal{V}(r)$ be the volume enclosed by the mollified surface. Note that $\mathcal{V}(0) = \mathcal{V}$ is just the volume enclosed by the original mesh. The goal of this section is to show that

$$\mathcal{V}(r) = \mathcal{V} + r \sum_{ijk \in F} A_{ijk} + \frac{r^2}{2} \sum_{ij \in E} H_{ij} + \frac{r^3}{6} \sum_{i \in V} K_i.$$

That is, our definitions of mean and Gaussian curvature perfectly capture how this volume is expanding.

15. Show that the flat part on top of ijk contributes a volume of rA_{ijk}.

16. Show that the curved part above each edge e_{ij} contributes a volume of $\frac{1}{2}r^2\theta_{ij}\ell_{ij}$.

17. Show that the sphere-like part above each vertex v_i contributes a volume of $\frac{1}{6}r^3K_i$. (Hint: use Exercise 9 from Section 4.3 of the course notes.)

18. Combine these parts to derive the formula for $\mathcal{V}(r)$.

4 Discrete Curvature, Part III: Curvature Normals

In the previous section, we derived the discretizations of mean and Gaussian curvature. In this section, we derive discretizations of their normal counterparts: HN and KN.

In the following exercises, the following theorems will be useful.

Discrete Gauss-Bonnet Theorem.

$$\sum_{i \in V} \left(2\pi - \sum_{ijk \in F} \varphi_i^{jk} \right) = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler Characteristic of the surface.

Schläfli formula. For every $i \in V$

$$\sum_{jk \in E} (\nabla_{f_i} \theta_{jk}) \ell_{jk} = 0.$$

In Chapter 5 of the course notes, it is shown that taking the gradient of total volume $\mathcal{V}$ with respect to the position f_i of vertex i yields a discretization for *area normal* NdA—specifically, we get the area-weighted sum of the face normals:

$$\nabla_{f_i} \mathcal{V} = \frac{1}{3} \sum_{ijk \in F} A_{ijk} N_{ijk}.$$

In the following exercises, you will show that the gradient of a sequence of quantities (total volume, total area, total mean curvature, total Gauss curvature) yields a corresponding sequence of discrete curvature normals (area normal, mean curvature normal, Gauss curvature normal, zero).

19. Show that when we take the gradient of the total surface area with respect to the position of one of the vertices, we instead get

$$\nabla_{f_i} \sum_{ijk \in F} A_{ijk} = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i),$$

where α_{ij} and β_{ij} are the two interior angles opposite edge ij. The right-hand-side is a discretization of the *mean curvature normal* HN at vertex v_i.

20. Next, show that the gradient of the total discrete scalar mean curvature gives

$$\nabla_{f_i} \frac{1}{2} \sum_{jk \in E} \theta_{jk} \ell_{jk} = \frac{1}{2} \sum_{ij \in E} \frac{\theta_{ij}}{\ell_{ij}} (f_j - f_i).$$

The right-hand-side is a discretization of the *Gauss curvature normal* KN at vertex v_i.

21. Finally, show that the gradient of the total discrete scalar Gauss curvature is zero:

$$\nabla_{f_i} \sum_{\ell \in V} \left[2\pi - \sum_{jk\ell \in F} \varphi_\ell^{jk} \right] = 0.$$

(Hint: there is a way to do this with almost no calculation.)