From Curves to Surfaces

• Previously: saw how to talk about 1D curves (both smooth and discrete)

• Today: will study 2D curved surfaces (both smooth and discrete)

• Some concepts remain the same (e.g., differential); others need to be generalized (e.g., curvature)

• Still use exterior calculus as our lingua franca
Surfaces—Local vs. Global View

• So far, we’ve only studied exterior calculus in $\mathbb{R}^n$

• Will therefore be easiest to think of surfaces expressed in terms of patches of the plane (local picture)

• Later, when we study topology & smooth manifolds, we’ll be able to more easily think about “whole surfaces” all at once (global picture)

• Global picture is much better model for discrete surfaces (meshes)…
Parameterized Surfaces
A parameterized surface is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$:

$$f : U \to \mathbb{R}^n$$

The set of points $f(U)$ is called the image of the parameterization.
As an example, we can express a saddle as a parameterized surface:

\[ U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \]

\[ f : U \to \mathbb{R}^3; \quad (u, v) \mapsto (u, v, u^2 - v^2) \]
Reparameterization

• Many different parameterized surfaces can have the same image:

\[ U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \]

\[ f : U \to \mathbb{R}^3; (u, v) \mapsto (u + v, u - v, 4uv) \]

This “reparameterization symmetry” can be a major challenge in applications—e.g., trying to decide if two parameterized surfaces (or meshes) describe the same shape.

**Analogy:** graph isomorphism
Embedded Surface

• Roughly speaking, an **embedded** surface does not self-intersect.

• More precisely, a parameterized surface is an embedding if it is a continuous injective map, and has a continuous inverse on its image.
Differential of a Surface

Intuitively, the differential of a parameterized surface tells us how tangent vectors on the domain get mapped to vectors in space:

\[ f(X) \]

We say that \( df \) “pushes forward” vectors \( X \) into \( \mathbb{R}^n \), yielding vectors \( df(X) \).
In coordinates, the differential is simply the exterior derivative:

\[ f : U \to \mathbb{R}^3; (u, v) \mapsto (u, v, u^2 - v^2) \]

\[ df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = (1, 0, 2u) du + (0, 1, -2v) dv \]

**Pushforward of a vector field:**

\[ X := \frac{3}{4} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \]

\[ df(X) = \frac{3}{4} (1, -1, 2(u + v)) \]

E.g., at \( u = v = 0 \): \( \left( \frac{3}{4}, -\frac{3}{4}, 0 \right) \)
**Definition.** Consider a map \( f : \mathbb{R}^n \to \mathbb{R}^m \), and let \( x_1, \ldots, x_n \) be coordinates on \( \mathbb{R}^n \). Then the *Jacobian* of \( f \) is the matrix

\[
J_f := \begin{bmatrix}
\frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n}
\end{bmatrix},
\]

where \( f^1, \ldots, f^m \) are the components of \( f \) w.r.t. some coordinate system on \( \mathbb{R}^m \). This matrix represents the differential in the sense that \( df(X) = J_f X \).

(In solid mechanics, also known as the *deformation gradient.*)

**Note:** does not generalize to infinite dimensions! (E.g., maps between functions.)
Immersed Surface

- A parameterized surface $f$ is an immersion if its differential is nondegenerate, i.e., if $df(X) = 0$ if and only if $X = 0$.

**Intuition:** no region of the surface gets “pinched”
Consider the standard parameterization of the sphere:

\[ f(u, v) := (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \]

\[ df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = \left( \begin{array}{c} -\sin(u) \sin(v), \\
\cos(u) \sin(v), \\
\cos(v) \sin(u), \\
-\sin(v) \end{array} \right) du + \left( \begin{array}{c} 0, \\
\cos(u), \\
\sin(u), \\
-\sin(v) \end{array} \right) dv \]

Q: Is \( f \) an immersion?

A: No: when \( v = 0 \) we get

\[ \left( \begin{array}{c} 0, \\
\cos(u), \\
\sin(u), \\
-\sin(v) \end{array} \right) du + \left( \begin{array}{c} 0, \\
\cos(u), \\
\sin(u), \\
-\sin(v) \end{array} \right) dv \]

Nonzero tangents mapped to zero!
Immersion vs. Embedding

• In practice, ensuring that a surface is globally embedded can be challenging

• Immersions are typically “nice enough” to define local quantities like tangents, normals, metric, etc.

• Immersions are also a natural model for the way we typically think about meshes: most quantities (angles, lengths, etc.) are perfectly well-defined, even if there happen to be self-intersections
Sphere Eversion

Turning a Sphere Inside-Out (1994)

https://youtu.be/-6g3ZcmjJ7k
Riemannian Metric
Riemann Metric

• Many quantities on manifolds (curves, surfaces, etc.) ultimately boil down to measurements of lengths and angles of tangent vectors

• This information is encoded by the so-called Riemannian metric*

• Abstractly: smoothly-varying positive-definite bilinear form

• For immersed surface, can (and will!) describe more concretely / geometrically

*Note: not the same as a point-to-point distance metric $d(x,y)$
Metric Induced by an Immersion

• Given an immersed surface $f$, how should we measure inner product of vectors $X, Y$ on its domain $U$?

• We should **not** use the usual inner product on the plane! (Why not?)

• Planar inner product tells us *nothing* about actual length & angle on the surface (and changes depending on choice of parameterization!)

• Instead, use **induced metric**

\[ g(X, Y) := \langle df(X), df(Y) \rangle \]

**Key idea**: must account for “stretching”
Induced Metric—Matrix Representation

• Metric is a bilinear map from a pair of vectors to a scalar, which we can represent as a 2x2 matrix $I$ called the first fundamental form:

$$ g(X, Y) = X^T I Y $$

$$ \Rightarrow I_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \left\langle df \left( \frac{\partial}{\partial x^i} \right), df \left( \frac{\partial}{\partial x^j} \right) \right\rangle $$

• Alternatively, can express first fundamental form via Jacobian:

$$ g(X, Y) = \left\langle df(X), df(Y) \right\rangle = (J_f X)^T (J_f Y) = X^T (J_f^T J_f) Y $$

$$ \Rightarrow I = J_f^T J_f $$
Induced Metric — Example

Can use the differential to obtain the induced metric:

\[ f : U \rightarrow \mathbb{R}^3; (u, v) \mapsto (u, v, u^2 - v^2) \]

\[ df = (1, 0, 2u)du + (0, 1, -2v)dv \]

\[ J_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix} \]

\[ I = J_f^T J_f \]

\[ = \begin{bmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{bmatrix} \]
Conformal Coordinates

• As we’ve just seen, there can be a complicated relationship between length & angle on the domain (2D) and the image (3D)

• For curves, we simplified life by using an arc-length or isometric parameterization: lengths on domain are identical to lengths along curve

• For surfaces, usually not possible to preserve all lengths (e.g., globe). Remarkably, however, can always preserve angles (conformal)

• Equivalently, a parameterized surface is conformal if at each point the induced metric is simply a positive rescaling of the 2D Euclidean metric

\[ g(X, Y)_p = \phi_p \langle X, Y \rangle \]
Example (Enneper Surface)

Consider the surface

\[ f(u, v) := \begin{bmatrix} uv^2 + u - \frac{1}{3}u^3 \\ \frac{1}{3}v (v^2 - 3u^2 - 3) \\ (u - v)(u + v) \end{bmatrix} \]

Its Jacobian matrix is

\[ J_f = \begin{bmatrix} -u^2 + v^2 + 1 & 2uv \\ -2uv & -u^2 + v^2 - 1 \\ 2u & -2v \end{bmatrix} \]

Its metric then works out to be just a scalar function times the usual metric of the Euclidean plane:

\[ I = J_f^T J_f = (u^2 + v^2 + 1)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

This function is called the conformal scale factor.
Gauss Map
Gauss Map

- A vector is **normal** to a surface if it is orthogonal to all tangent vectors.

- **Q:** Is there a **unique** normal at a given point?

- **A:** No! Can have different magnitudes/directions.

- The **Gauss map** is a **continuous** map taking each point on the surface to a **unit** normal vector.

- Can visualize Gauss map as a map from the surface to the unit sphere.
Orientability

Not every surface admits a Gauss map (globally):

orientable

nonorientable
Gauss Map—Example

Can obtain unit normal by taking the cross product of two tangents*:

\[ f := (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \]

\[ df = \begin{pmatrix} -\sin(u) \sin(v), & \cos(u) \sin(v), & 0 \\ \cos(u) \cos(v), & \cos(v) \sin(u), & -\sin(v) \end{pmatrix} \, du + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \, dv \]

\[
\begin{align*}
df \left( \frac{\partial}{\partial u} \right) \times df \left( \frac{\partial}{\partial v} \right) &= \begin{bmatrix} -\cos(u) \sin^2(v) \\ -\sin(u) \sin^2(v) \\ -\cos(v) \sin(v) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

To get unit normal, divide by length. In this case, can just notice we have a constant multiple of the sphere itself:

\[ \Rightarrow N = -f \]

*Must not be parallel!
**Surjectivity of Gauss Map**

- Given a unit vector $u$, can we always find some point on a surface that has this normal? ($N = u$)
- Yes! **Proof** (Hilbert):

**Q:** Is the Gauss map *injective*?
Vector Area

- Given a little patch of surface $\Omega$, what’s the “average normal”?
- Can simply integrate normal over the patch, divide by area:

$$\frac{1}{\text{area}(\Omega)} \int_{\Omega} N \, dA$$

- Integrant $N \, dA$ is called the vector area. (Vector-valued 2-form)
- Can be easily expressed via exterior calculus*:

$$df \wedge df (X, Y) = df (X) \times df (Y) - df (Y) \times df (X) = 2df (X) \times df (Y) = 2NdA (X, Y)$$

$$\Rightarrow A = \frac{1}{2} df \wedge df$$
Vector Area, continued

• By expressing vector area this way, we make an interesting observation:

\[ 2 \int_{\Omega} N \, dA = \int_{\Omega} df \wedge df = \int_{\Omega} d(f \, df) = \int_{\partial \Omega} f \, df = \int_{\partial \Omega} f(s) \times df(T(s)) \, ds \]

• Hence, vector area is the same for any two patches with the same boundary
• Can define “normal” given only boundary (e.g., nonplanar polygon)
• **Corollary:** integral of normal vanishes for any closed surface
Curvature
The **Weingarten map** $dN$ is the differential of the Gauss map $N$.

At each point, it tells us the change in the normal vector along any given direction $X$.

Since change in *unit* normal cannot have any component in the normal direction, $dN(X)$ is always tangent to the surface.

Can also think of it as a vector tangent to the unit sphere $S^2$.

Q: Why is $dN(Y)$ “flipped”? 

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**Weingarten Map**

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- Can also think of it as a vector tangent to the unit sphere $S^2$.

Q: Why is $dN(Y)$ “flipped”?
Weingarten Map — Example

• Recall that for the sphere, $N = -f$. Hence, Weingarten map $dN$ is just $-df$:

$$f := (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v))$$

$$df = \begin{pmatrix} -\sin(u) \sin(v), & \cos(u) \sin(v), & 0 \\ \cos(u) \cos(v), & \cos(v) \sin(u), & -\sin(v) \end{pmatrix} du + \begin{pmatrix} \\ \end{pmatrix} dv$$

$$dN = \begin{pmatrix} \sin(u) \sin(v), & -\cos(u) \sin(v), & 0 \\ -\cos(u) \cos(v), & -\cos(v) \sin(u), & \sin(v) \end{pmatrix} du \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}$$

**Key idea:** computing the Weingarten map is no different from computing the differential of a surface.
Normal Curvature

• For curves, curvature was the rate of change of the tangent; for immersed surfaces, we’ll instead consider how quickly the normal is changing.*

• In particular, normal curvature is rate at which normal is bending along a given tangent direction:

\[ \kappa_N(X) := \frac{\langle df(X), dN(X) \rangle}{|df(X)|^2} \]

• Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve

*For plane curves, what would happen if we instead considered change in \(N\)?
Normal Curvature—Example

Consider a parameterized cylinder:

\[ f(u, v) := (\cos(u), \sin(u), v) \]

\[ df = (-\sin(u), \cos(u), 0)\,du + (0, 0, 1)\,dv \]

\[ N = (-\sin(u), \cos(u), 0) \times (0, 0, 1) = (\cos(u), \sin(u), 0) \]

\[ dN = (-\sin(u), \cos(u), 0)\,du \]

\[ \kappa_N \left( \frac{\partial}{\partial u} \right) = \frac{\langle df \left( \frac{\partial}{\partial u} \right), dN \left( \frac{\partial}{\partial u} \right) \rangle}{|df \left( \frac{\partial}{\partial u} \right)|^2} = \frac{(-\sin(u), \cos(u), 0) \cdot (-\sin(u), \cos(u), 0)}{|(-\sin(u), \cos(u), 0)|^2} = 1 \]

\[ \kappa_N \left( \frac{\partial}{\partial v} \right) = \cdots = 0 \]

Q: Does this result make sense geometrically?
Principal Curvature

- Among all directions $X$, there are two principal directions $X_1, X_2$ where normal curvature has minimum/maximum value (respectively).

- Corresponding normal curvatures are the principal curvatures.

- Two critical facts*:
  1. $g(X_1, X_2) = 0$
  2. $dN(X_i) = \kappa_i df(X_i)$

Where do these relationships come from?
Shape Operator

• The change in the normal $N$ is always tangent to the surface

• Must therefore be some linear map $S$ from tangent vectors to tangent vectors, called the shape operator, such that

\[ df(SX) = dN(X) \]

• Principal directions are the eigenvectors of $S$

• Principal curvatures are eigenvalues of $S$

• Note: $S$ is not a symmetric matrix! Hence, eigenvectors are not orthogonal in $R^2$; only orthogonal with respect to induced metric $g$. 
Consider a nonstandard parameterization of the cylinder (sheared along $z$):

$$f(u,v) := (\cos(u), \sin(u), u + v)$$

$$df = ( -\sin(u), \cos(u), 1 ) du + (0, 0, 1) dv$$

$$N = (\cos(u), \sin(u), 0)$$

$$dN = ( -\sin(u), \cos(u), 0 ) du$$

\[
\begin{bmatrix}
-\sin(u) & 0 \\
\cos(u) & 0 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22} \\
\end{bmatrix}
= 
\begin{bmatrix}
-\sin(u) & 0 \\
\cos(u) & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
\Rightarrow S = 
\begin{bmatrix}
1 & 0 \\
-1 & 0 \\
\end{bmatrix}
\quad X_1 = 
\begin{bmatrix}
0 \\
1 \\
\end{bmatrix}
\quad X_2 = 
\begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}
\]

$$df(X_1) = (0, 0, 1) \quad \Rightarrow \kappa_1 = 0$$

$$df(X_2) = (\sin(u), -\cos(u), 0) \quad \Rightarrow \kappa_2 = 1$$

**Key observation:** principal directions orthogonal only in $R^3$. 
Umbilic Points

• Points where principal curvatures are equal are called **umbilic points**

• Principal *directions* are not uniquely determined here

• What happens to the shape operator $S$?
  
  • May still have full rank!

  • Just have repeated eigenvalues, 2-dim. eigenspace

\[
S = \begin{bmatrix} 1/r & 0 \\ 0 & 1/r \end{bmatrix} \quad \kappa_1 = \kappa_2 = \frac{1}{r} \quad \forall X, \; SX = \frac{1}{r}X
\]

Could still of course choose (arbitrarily) an orthonormal pair $X_1, X_2$...
Principal Curvature Nets

- Walking along principal direction field yields principal curvature lines
- Collection of all such lines is called the principal curvature network
Topological Invariance of Umbilic Count

Can classify regions around umbilics into three types based on behavior of principal network: *lemon*, *star*, and *monstar*

![Diagram](image)

**Fact.** If $k_1$, $k_2$, $k_3$ are number of umbilics of each type, then

$$\kappa_1 - \kappa_2 + \kappa_3 = 2\chi$$
Separatrices and Spirals

- If we walk along a principal curvature line, where do we end up?
- Sometimes, a curvature line terminates at an umbilic point in both directions; these so-called separatrices (can) split network into regular patches.
- Other times, we make a closed loop. More often, however, behavior is *not* so nice!
Application—Quad Remeshing

- Recent approach to meshing: construct net *roughly* aligned with principal curvature—but with separatrices & loops, not spirals.

from Knöppel, Crane, Pinkall, Schröder, “Stripe Patterns on Surfaces”
Gaussian and mean curvature also fully describe local bending:

**Gaussian** \( K := \kappa_1 \kappa_2 \)

**mean**\* \( H := \frac{1}{2} (\kappa_1 + \kappa_2) \)

\( K > 0 \)

“developable” \( K = 0 \)

\( H \neq 0 \)

“minimal” \( H = 0 \)

*Warning:* another common convention is to omit the factor of 1/2.
Gauss-Bonnet Theorem

- Recall that the total curvature of a closed plane curve was always equal to $2\pi$ times the turning number $k$.

- Q: Can we make an analogous statement about surfaces?

- A: Yes! Gauss-Bonnet theorem says total Gaussian curvature is always $2\pi$ times the Euler characteristic $\chi$.

- Euler characteristic can be expressed in terms of the genus (number of “handles”)

\[
\begin{align*}
\text{Curves} & \quad \int_0^L \kappa \, ds = 2\pi k \\
\text{Surfaces} & \quad \int_M K \, dA = 2\pi \chi
\end{align*}
\]
Total Mean Curvature?

**Theorem** (Minkowski): for a regular closed embedded surface,

\[ \int_M H \, dA \geq \sqrt{4\pi A} \]

**Q:** When do we get equality?

**A:** For a sphere.
Curvature of a Curve in a Surface

- Earlier, broke the “bending” of a space curve into curvature ($\kappa$) and torsion ($\tau$)

- For a curve in a surface, can instead break into normal and geodesic curvature:

  $\kappa_n := \langle N_M, \frac{d}{ds} T \rangle$
  $\kappa_g := \langle B_M, \frac{d}{ds} T \rangle$

- $T$ is still tangent of the curve; but unlike the Frenet frame, $N_M$ is the normal of the surface and $B_M := T \times B_M$

Q: Why no third curvature $\langle T_M, \frac{d}{ds} T \rangle$?
**Second Fundamental Form**

- Second fundamental form is closely related to principal curvature
- Can also be viewed as change in *first* fundamental form under motion in normal direction
- Why “fundamental?” First & second fundamental forms play role in important theorem…

\[ \mathbf{II}(X, Y) := \langle dN(X), df(Y) \rangle \]

\[ \kappa_N(X) := \frac{df(X), dN(X)}{|df(X)|^2} = \frac{\mathbf{II}(X, X)}{\mathbf{I}(X, X)} \]
**Fundamental Theorem of Surfaces**

- **Fact.** Two surfaces in $\mathbb{R}^3$ are congruent if and only if they have the same first and second fundamental forms.

- …However, not every pair of bilinear forms $I, II$ on a domain $U$ describes a valid surface—must satisfy the **Gauss Codazzi** equations.

- Analogous to fundamental theorem of plane curves: determined up to rigid motion by curvature.

- …However, for closed curves not every curvature function is valid (e.g., must integrate to $2k\pi$).
Other Descriptions of Surfaces?

- Classic question in differential geometry:
  
  "What data is sufficient to completely determine a surface in space?"

- Many possibilities…
  - First & second fundamental form (Gauss-Codazzi)
  - Mean curvature and metric (up to "Bonnet pairs")
  - Convex surfaces: metric alone is enough (Alexandrov/Pogorolev)
  - Gauss curvature essentially determines metric (Kazdan-Warner)
- …in general, still a surprisingly murky question!
Exterior Calculus on Immersed Surfaces
Exterior Calculus on Curved Domains

- Initial study of differential forms was in flat Euclidean $\mathbb{R}^n$
- How do we do exterior calculus on curved spaces?
- Recall that operators nicely “split up” topology & geometry:
  - (topology) wedge product ($\wedge$), exterior derivative ($d$)
  - (geometry) Hodge star ($\star$)
- For instance, discrete $d$ uses only mesh connectivity (topology); discrete $\star$ involves only ratios of volumes (geometry)
- Therefore, to get exterior calculus to work with curved spaces, we just need to figure out what the Hodge star looks like!
- Traditionally taught from abstract intrinsic point of view; we’ll start with the concrete extrinsic picture (which fewer people know… but is more directly relevant for real applications!)
Exterior Calculus on Immersed Surfaces

• For surface immersed in 3D, just need two pieces of data:
  
  • **Area form**—“how big is a given region?”
    
    • lets us define Hodge star on 0/2-forms
    
    • can express via cross product in $\mathbb{R}^3$
  
  • **Complex structure**—“how do we rotate by 90°?”
    
    • lets us define Hodge star on 1-forms
    
    • can express via cross product w/ surface normal
  
• All of this data also determined by induced metric
Induced Area 2-Form

• What signed area should we associate with a pair of vectors $X, Y$ on the domain?
• Not just their cross product! Need to account for “stretching” caused by immersion $f$
• What’s the signed area of the stretched vector? Let’s start here:

$$df \wedge df(X, Y) = df(X) \times df(Y) - df(Y) \times df(X) = 2df(X) \times df(Y)$$

Since $df(X)$ and $df(Y)$ are tangent, we get

$$df \wedge df(X, Y) = 2N dA(X, Y)$$

where $dA$ is the area 2-form on $f(M)$. Hence,

$$dA = \frac{1}{2} \langle N, df \wedge df \rangle$$
Induced Hodge Star on 0-Forms

• Given the area 2-form $dA$, can easily define Hodge star on 0-forms:

  $\phi \mapsto * \phi \, dA$

• **Meaning?** Applying this new 2-form to a unit area on the surface yields the original function value at that point.
Induced Hodge Star on 2-Forms

- To get the 2-form Hodge star, we just go the other way.

- Suppose $\omega$ is a 2-form on $f(M)$. Then its Hodge dual is the unique 0-form $\phi$ such that

\[ \omega = \phi \, dA \]
**Complex Structure**

- The *complex structure* tells us how to rotate by 90°.
- In $\mathbb{R}^2$, we just replace $(x, y)$ with $(-y, x)$:

  \[
  J_{\mathbb{R}^2} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad J_{\mathbb{R}^2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}
  \]

- For a surface immersed in $\mathbb{R}^3$, we can express a 90-degree rotation via a cross product with the unit normal $N$:

  \[
  df(J_fX) := N \times df(X)
  \]

- This relationship uniquely determines $J_f$.
- An immersion is conformal if and only if $J_f = J_{\mathbb{R}^2}$.

*Sometimes called *linear complex structure*; same thing for surfaces.*
Complex Structure in Coordinates

• Suppose we want to explicitly compute the linear complex structure*
• Similar strategy to shape operator: solve a matrix equation for $\mathcal{J}$

\[ \mathcal{N} := \begin{bmatrix} 0 & -N_z & N_y \\ N_z & 0 & -N_x \\ -N_y & N_x & 0 \end{bmatrix} \]  

cross product w/ normal  
\((N \times u = \mathcal{N}u)\)

\[ A := \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial f_y}{\partial u} & \frac{\partial f_y}{\partial v} \\ \frac{\partial f_z}{\partial u} & \frac{\partial f_z}{\partial v} \end{bmatrix} \]  

Jacobian

\[ J := \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \]  

complex structure

\[ df(\mathcal{J}X) = N \times df(X) \implies J = A^T \mathcal{N} A \]

*Note: not something you do much in practice, but may help make definition feel more concrete…
Induced Hodge Star on 1-Forms

• Recall that for a 1-form $\alpha$ in the plane, applying $\star \alpha$ to a vector $X$ is the same as applying $\alpha$ to a 90-degree rotation of $X$:

$$\star_{\mathbb{R}^2} \alpha(X) = \alpha(J_{\mathbb{R}^2}X)$$

• For 1-forms on an immersed surface $f$, we instead want to apply a 90-degree rotation with respect to the surface itself:

$$\star_f \alpha(X) = \alpha(J_fX)$$

• At this point we have everything we need to do calculus on curved surfaces: 0-, 1-, and 2-form Hodge star. (Will see more general/abstract/intrinsic definitions for $n$-manifolds later on.)
Sharp and Flat on a Surface

• Can use induced metric to translate between vector fields and 1-forms:

\[ X^b(Y) := g(X, Y) \quad \quad g(\alpha^b, Y) := \alpha(Y) \]

• No longer just a trivial “transpose” (as in Euclidean \( \mathbb{R}^n \))
• E.g., flat correctly encodes inner product on surface

\[ X \cdot Y \neq df(X) \cdot df(Y) \quad X^b(Y) = df(X) \cdot df(Y) \]
Thanks!

Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017