DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
Keenan Crane • CMU 15-458/858B • Fall 2017
Lecture 10:

Discrete Curvature

Discrete Differential Geometry:

An Applied Introduction

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Discrete Curvature
Curvature of Surfaces

• In smooth setting, had many different curvatures (normal, principal, Gauss, mean, geodesic, …)

• In discrete setting, appear to be many different choices for discretization

• Actually, there is a unified viewpoint that helps explain many common choices…
A Unified Picture of Discrete Curvature

• By making some connections between smooth and discrete surfaces, we get a unified picture of many different discrete curvatures scattered throughout the literature

• To tell the full story we’ll need a few pieces:
  • geometric derivatives
  • Steiner polynomials
  • sequence of curvature variations
  • assorted theorems (Gauss-Bonnet, Schlafli, $\Delta f = 2HN$)
  • Let’s start with some of these assorted theorems…
Discrete Curvature—Visualized

area

mean

Gauss

maximum

minimum
Quantities & Conventions

Throughout we will consider the following basic quantities:

- $f_i$ — position of vertex $i$
- $e_{ij}$ — vector from $i$ to $j$
- $\ell_{ij}$ — length of edge $ij$
- $A_{ijk}$ — area of triangle $ijk$
- $N_{ijk}$ — unit normal of triangle $ijk$
- $\theta_{i}^{jk}$ — interior angle at vertex $i$ of triangle $ijk$
- $\varphi_{ij}$ — dihedral angle at oriented edge $ij$

$$\varphi_{ij} := \arctan2(\hat{e} \cdot N_{ijk} \times N_{jil}, N_{ijk} \cdot N_{jil}), \quad \hat{e}_{ij} := e_{ij} / \ell_{ij}$$

Q: Which of these quantities are discrete differential forms? (And what kind?)
Discrete Geometric Derivatives
Discrete Geometric Derivatives

• Practical technique for calculating derivatives of discrete geometric quantities

• Basic question: how does one geometric quantity change with respect to another?

• E.g., what’s the gradient of triangle area with respect to the position of one of its vertices?

• Don’t just grind out partial derivatives!

• Do follow a simple geometric recipe:
  
  • First, in which **direction** does the quantity change quickest?
  • Second, what’s the **magnitude** of this change?
  • Together, direction & magnitude give us the gradient vector
Dangers of Partial Derivatives

- Why not just take derivatives “the usual way?”
- usually takes way more work!
- can lead to expressions that are
  - inefficient
  - numerically unstable
  - hard to interpret
- Example: gradient of angle between two segments \((b,a), (c,a)\) w.r.t. coordinates of point \(a\)
Geometric Derivation of Angle Derivative

1. *(Direction)* What direction can we move the point $a$ to most quickly increase the angle $\theta$?

   **A:** Orthogonal to the segment $ab$.

2. *(Magnitude)* How much does the angle change if we move in this direction?

   **A:** Moving around a whole circle changes the angle by $2\pi$ over a distance $2\pi r$, where $r = |b-a|$. Hence, the instantaneous change is $1/|b-a|$.

• Multiplying the unit direction by the magnitude yields a final gradient expression.

\[ \nabla_a \theta = \frac{a-b}{|a-b|} \]
Gradient of Triangle Area

Q: What’s the gradient of triangle area with respect to one of its vertices $p$?

A: Can express via its unit normal $N$ and vector $e$ along edge opposite $p$:

$$\nabla_p A = \frac{1}{2} N \times e$$
Geometric Derivation

- In general, can lead to some pretty slick expressions (give it a try!)

\[ \nabla_{p_3} \theta = \frac{|e|}{2A_1} N_1 \]

\[ \nabla_p A = \frac{1}{2} N \times e \]

\[ d_{f_i} N(X) = \frac{\langle N, X \rangle}{2A} e_i \times N \]

\[ du(v) = \frac{v - \langle v, b - a \rangle (b - a)}{|b - a|^3} \]
Aside: Automatic Differentiation

- Geometric approach to differentiation greatly simplifies “small pieces” (gradient of a particular, angle, length, area, volume, …)

- For larger expressions that combine many small pieces, approach of automatic differentiation is extremely useful*

- Basically does nothing more than automate repeated application of chain rule

- Simplest implementation: use pair that store both a value and its derivative; operations on these tuples apply operation & chain rule

*More recently known as backpropagation

Example.

```plaintext
// define how multiplication and sine // operate on (value,derivative) pairs // (usually done by an existing library) (a, a') * (b, b') := (a*b, a*b' + b*a') sin((a, a')) := (sin(a), a'*cos(a))

// to evaluate a function and its // derivative at a point, we first // construct a pair corresponding to the // identity function f(x) = x at the // desired evaluation point x = (5, 1) // derivative of x w.r.t x is 1

// now all we have to do is type a // function as usual, and it will yield // the correct value/derivative pair fx = sin(x*x) // (-0.132352, 9.91203)
```
Discrete Gaussian Curvature
Euler Characteristic

The Euler characteristic of a simplicial 2-complex $K=(V,E,F)$ is the constant

$$\chi := V - E + F$$

\[ \chi = 1 \quad \chi = 0 \quad \chi = 2 \]
**Fact.** (L’Huilier) For simplicial surfaces w/out boundary, the Euler characteristic is a topological invariant. E.g., for a torus of genus \( g \), \( \chi = 2 - 2g \) (independent of the particular tessellation).
Angle Defect

- The **angle defect** at a vertex $i$ is the deviation of the sum of interior angles from the Euclidean angle sum of $2\pi$:

$$\Omega_i := 2\pi - \sum_{ijk} \theta_{ijk}$$

**Intuition:** how “flat” is the vertex?
Angle Defect and Spherical Area

- Consider the discrete Gauss map...
- ...unit normals on surface become points on the sphere
- ...dihedral angles on surface become interior angles on sphere
- ...interior angles on surface become dihedral angles on the sphere
- ...angle defect on surface becomes area on the sphere
Total Angle Defect of a Convex Polyhedron

• Consider a closed convex polyhedron in $\mathbb{R}^3$

• **Q:** Given that angle defect is equivalent to spherical area, what might we guess about total angle defect?

• **A:** Equal to $4\pi!$ (Area of unit sphere)

• More generally, can argue that total angle defect is equal to $4\pi$ for any polyhedron with spherical topology, and $2\pi(2-2g)$ for any polyhedron of genus $g$

• Should remind you of Gauss-Bonnet theorem
Review: Gauss-Bonnet Theorem

• Classic example of *local-global* theorems in differential geometry

• Gauss-Bonnet theorem says total Gaussian curvature is always $2\pi$ times *Euler characteristic* $\chi$

• For tori, Euler characteristic expressed in terms of the *genus* (number of “handles”)

$$\chi := 2 - 2g$$

**Gauss-Bonnet**

$$\int_M K \, dA = 2\pi \chi$$
Gaussian Curvature as Ratio of Ball Areas

• Originally defined Gaussian curvature as product of principal curvatures
• Can also view it as “failure” of balls to behave like Euclidean balls

Roughly speaking,

\[ K \propto 1 - \frac{|B_g|}{|B_{\mathbb{R}^2}|} \]

More precisely:

\[ |B_g(p, \varepsilon)| = |B_{\mathbb{R}^2}(p, \varepsilon)| \left(1 - \frac{K}{12} \varepsilon^2 + O(\varepsilon^3)\right) \]
Discrete Gaussian Curvature as Ratio of Areas

• For small values of $\varepsilon$, we have

$$\frac{\varepsilon^2}{12} K \approx 1 - \frac{|B_g(\varepsilon)|}{|B_{\mathbb{R}^2}(\varepsilon)|}$$

Substitute

area of Euclidean ball $|B_{\mathbb{R}^2}(\varepsilon)| = \pi \varepsilon^2$

area of geodesic “wedge” $W_i(\varepsilon) = \frac{\theta_i}{2\pi} |B_{\mathbb{R}^2}| = \frac{1}{2} \varepsilon^2 \theta_i$

area of geodesic ball $|B_g(\varepsilon)| = \sum_i W_i(\varepsilon) = \frac{\varepsilon^2}{2} \sum_i \theta_i$

Then

$$\frac{\varepsilon^2}{K} = 1 - \frac{1}{2\pi} \sum_i \theta_i \iff 2\pi - \sum_i \theta_i = \frac{1}{12} \pi \varepsilon^2 K$$

Angle defect is integrated curvature
Theorem. For a smooth surface of genus $g$, the total Gauss curvature is

$$\int_M K \, dA = 2\pi \chi$$

Theorem. For a simplicial surface of genus $g$, the total angle defect is

$$\sum_{i \in V} \Omega_i = 2\pi \chi$$
Approximating Gaussian Curvature

- Many other ways to approximate Gaussian curvature
- \textit{E.g.}, locally fit quadratic functions, compute smooth Gaussian curvature
- Which way is “best”?
  - values from quadratic fit won’t satisfy Gauss-Bonnet
  - angle defects won’t converge\(^1\) unless vertex valence is 4 or 6
- In general, no best way; each choice has its own pros & cons

\(^1\)Borrelli, Cazals, Morvan, “On the angular defect of triangulations and the pointwise approximation of curvatures”
Schläfli Formula
Schläfli Formula

- Consider a closed polyhedron in $\mathbb{R}^3$ with edge lengths $l_{ij}$ and dihedral angles $\varphi_{ij}$. Then for any motion of the vertices,

$$\sum_{ij \in E} l_{ij} \frac{d}{dt} \varphi_{ij} = 0$$
Curvature Normals
Curvature Normals

• Earlier we saw vector area, which was the integral of the 2-form $NdA$

• This 2-form is one of three quantities we can naturally associate with a surface:

\[
\frac{1}{2} \, df \wedge df = NdA \quad \text{(area normal)}
\]

\[
\frac{1}{2} \, dN \wedge dN = KNdA \quad \text{(mean curvature normal)}
\]

\[
\frac{1}{2} \, df \wedge dN = HNdA \quad \text{(Gauss curvature normal)}
\]

• Effectively "mixed areas" of change in position & normal (more later)
Curvature Normals—Derivation

- Let $X_1, X_2$ be principal curvature directions (recall that $dN(X_i) = \kappa_i df(X_i)$). Then

\[ df \wedge df(X_1, X_2) = df(X_1) \times df(X_2) - df(X_2) \times df(X_1) = 2df(X_1) \times df(X_2) = 2NdA(X_1, X_2) \]

\[ df \wedge dN(X_1, X_2) = df(X_1) \times dN(X_2) - df(X_2) \times dN(X_1) = \kappa_1 df(X_1) \times df(X_2) - \kappa_2 df(X_2) \times df(X_1) = (\kappa_1 + \kappa_2) df(X_1) \times df(X_2) = 2HNdA(X_1, X_2) \]

\[ dN \wedge dN(X_1, X_2) = dN(X_1) \times dN(X_2) - dN(X_2) \times dN(X_1) = \kappa_1 \kappa_2 df(X_1) \times df(X_2) - \kappa_2 \kappa_1 df(X_2) \times df(X_1) = 2Kdf(X_1) \times df(X_2) = 2KNdA(X_1, X_2) \]
Discrete Vector Area

- Recall smooth vector area: \( \int_{\Omega} N \, dA = \frac{1}{2} \int_{\Omega} df \wedge df = \frac{1}{2} \int_{\partial \Omega} f \times df \)

- Idea: Integrate \( NdA \) over dual cell to get normal at vertex \( p \)

\[
\frac{1}{3} \int_{\Omega} N \, dA = \frac{1}{6} \int_{\partial \Omega} f \times df = \\
\frac{1}{6} \sum_{ij \in \partial \Omega} \int_{e_{ij}} f \times df = \\
\frac{1}{6} \sum_{ij \in \partial \Omega} \frac{f_i + f_j}{2} \times (f_j - f_i) = \frac{1}{6} \sum_{ij \in \partial \Omega} f_i \times f_j
\]

Q: What kind of quantity is the final expression? Does that matter?
Discrete Mean Curvature Normal

Similarly, integrating $HN$ over a circumcentric dual cell $C$ yields

$$\int_C HN \, dA = \int_C df \wedge dN = \int_C dN \wedge df = \int_C d(N \wedge df) =$$

$$\int_{\partial C} N \wedge df = \sum_j \int_{e_{ij}^*} N \wedge df = \sum_j N_a \times (m - a) + N_b \times (b - m)$$

- Since $N \times$ is an in-plane 90-degree rotation, each term in the sum is parallel to the edge vector $e_{ij}$
- The length of the vector is the length of the dual edge
- Ratio of dual/primal length is given by cotan formula, yielding

$$ (HN)_i := \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j) $$
Another well-known fact: mean curvature normal can be expressed via the Laplace-Beltrami operator \( \Delta \).

**Fact.** For any smooth immersed surface \( f \), \( \Delta f = 2HN \).

Can discretize \( \Delta \) via the cotangent formula, leading again to

\[
(\Delta f)_i = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)
\]

*Will say much more in upcoming lectures!"
Discrete Gauss Curvature Normal

- A similar calculation leads to an expression for the (discrete) Gauss curvature normal

\[ 2 \oint_C KN \, dA = \int_C dN \wedge dN = \int_C d(N \wedge dN) = \]

\[ \int_{\partial C} N \wedge dN = \int_{\partial C} N \times dN(\gamma') \, ds = \]

\[ \int_{\partial C} N \times T \, ds = \sum_j \int_{\partial C} \frac{e_{ij}}{|e_{ij}|} \, ds = \sum_j \frac{e_{ij}}{\ell_{ij}} \varphi_{ij} \]

\[ (KN)_i := \frac{1}{2} \sum_{ij \in E} \frac{\varphi_{ij}}{\ell_{ij}} (f_j - f_i) \]
## Discrete Curvature Normals—Summary

<table>
<thead>
<tr>
<th></th>
<th>area ((NdA))</th>
<th>mean ((HNdA))</th>
<th>Gauss ((KNdA))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>smooth</strong></td>
<td>(\frac{1}{2} df \wedge df)</td>
<td>(\frac{1}{2} df \wedge dN)</td>
<td>(\frac{1}{2} dN \wedge dN)</td>
</tr>
<tr>
<td><strong>discrete</strong></td>
<td>(\frac{1}{6} \sum_{ijk \in St(i)} f_j \times f_k)</td>
<td>(\frac{1}{2} \sum_{ij \in St(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j))</td>
<td>(\frac{1}{2} \sum_{ij \in St(i)} \frac{\varphi_{ij}}{\ell_{ij}}(f_j - f_i))</td>
</tr>
</tbody>
</table>
Steiner’s Formula
Steiner Approach to Curvature

• What’s the curvature of a discrete surface (polyhedron)?

• Simply taking derivatives of the normal yields a useless answer: zero except at vertices/edges, where derivative is ill-defined (“infinite”)

• Steiner approach: “smooth out” the surface; define discrete curvature in terms of this mollified surface
Minkowski Sum

- Given two sets $A, B$ in $\mathbb{R}^n$, their Minkowski sum is the set of points
  \[ A + B := \{a + b | a \in A, b \in B\} \]

Example.

Q: Does translation of $A$, $B$ matter?
Mollification of Polyhedral Surfaces

- Steiner approach mollifies polyhedral surface by taking Minkowski sum with ball of radius $\varepsilon > 0$
- Measure curvature, take limit as $\varepsilon$ goes to zero to get discrete definition
- (Have to think carefully about nonconvex polyhedra…)
**Mixed Area**

- Suppose we express the area of a Minkowski sum $sA + tB$ as
  \[
  \text{area}(sA + tB) = s^2 \text{area}(A) + 2st \text{area}(A,B) + t^2 \text{area}(B)
  \]

- This expression *defines* the **mixed area** $\text{area}(A,B)$
Mixed Area—Examples

\[
\text{area}(A, B) = \frac{1}{2}(\text{area}(A + B) - \text{area}(A) - \text{area}(B))
\]
Theorem. (Steiner) Let $A$ be any convex body in $\mathbb{R}^n$, and let $B_\varepsilon$ be a ball of radius $\varepsilon$. Then the volume of the Minkowski sum $A + B_\varepsilon$ can be expressed as a polynomial in $\varepsilon$:

$$\text{volume}(A + B_\varepsilon) = \text{volume}(A) + \sum_{k=1}^{n} \Phi_k(A)\varepsilon^k$$

Constant coefficients are called quermassintegrals, and determine how quickly the volume grows.

This volume growth in turn has to do with (discrete) curvature, as we are about to see…
Gaussian Curvature of Mollified Surface

• **Q:** Consider a closed, convex polyhedron in $R^3$; what’s the Gaussian curvature $K$ of the mollified surface for a ball of radius $\varepsilon$?

• **Triangles:** $K = 0$

• **Edges:** $K = 0$

• **Vertices?**
  
  • each contributes a piece of sphere of radius $\varepsilon$ ($K=1/\varepsilon^2$)
  
  • recall (unit) spherical area given by *angle defect* $\Omega_i$
  
  • *total* curvature associated with vertex $i$ is then

$$A_i K_i = \left( \frac{\Omega_i}{4\pi} \right) \frac{4\pi \varepsilon^2}{\varepsilon^2} = \Omega_i$$

(Spherical polygon is all normals associated with vertex.)
Mean Curvature of Mollified Surface

• **Q:** What’s the mean curvature $H$ of the mollified surface?
  
• **Faces:** $H = 0$

• **Edges?**
  
  • each contributes a piece of a cylinder ($H=1/2\varepsilon$)
  
  • area of cylindrical piece is $\ell_{ij} \varphi_{ij} \varepsilon$

  • total mean curvature for edge is hence $H_{ij} = \frac{1}{2} \ell_{ij} \varphi_{ij}$

• **Vertices?**

  • each contributes a piece of the sphere ($H=1/\varepsilon$)

  • area is $(\Omega_i/4\pi) 4\pi \varepsilon^2 = \Omega_i \varepsilon^2$

  • total mean curvature for vertex is then $H_i = \Omega_i \varepsilon$
**Area of a Mollified Surface**

- **Q:** What's the area of the mollified surface?
- **Faces:** just the original area $A_{ijk}$
- **Edges:** $\ell_{ij} \varphi_{ij} \varepsilon$
- **Vertices:** $\Omega_i \varepsilon^2$

Total area of the whole surface is then

$$\text{area}_\varepsilon(f) = \sum_{ijk \in F} A_{ijk} + \varepsilon \sum_{ij \in E} \ell_{ij} \varphi_{ij} + \varepsilon^2 \sum_{i \in V} \Omega_i$$

- By (discrete) Gauss-Bonnet, last term is also $2\pi \chi$
Volume of Mollified Surface

• Q: What’s the total volume of the mollified surface?

• Starting to see a pattern—if $V_0$ is original volume, then

\[
\text{volume}_{\varepsilon}(f) = V_0 + \varepsilon \sum_{ijk \in F} A_{ijk} + \varepsilon^2 \sum_{ij \in E} \ell_{ij} \varphi_{ij} + \varepsilon^3 \sum_{i \in V} \Omega_i
\]

• Q: How did we get here from our area expression?

• A: Increasing radius by $\varepsilon$ increases volume proportional to area
Steiner Polynomial for Polyhedra in $\mathbb{R}^3$

- Volume of mollified polyhedron is a polynomial in radius $\varepsilon$
- Derivatives w.r.t. $\varepsilon$ give total area, mean curvature, Gauss curvature

$$\text{volume}_\varepsilon(f) = V_0 + \varepsilon \sum_{ijk \in F} A_{ijk} + \varepsilon^2 \sum_{ij \in E} \ell_{ij} \varphi_{ij} + \varepsilon^3 \sum_{i \in V} \Omega_i$$

$$\frac{d}{d\varepsilon} \text{volume}_\varepsilon = \text{area}_\varepsilon$$
$$\frac{d}{d\varepsilon} \text{area}_\varepsilon = \text{mean}_\varepsilon$$
$$\frac{d}{d\varepsilon} \text{mean}_\varepsilon = \text{Gauss}_\varepsilon$$
$$\frac{d}{d\varepsilon} \text{Gauss}_\varepsilon = 0$$

Q: Why are there only four terms?
Curvature Variations
For a smooth surface $f: M \rightarrow R^3$ (without boundary), let

- $\text{volume}(f) := \frac{1}{3} \int_M N \cdot f \, dA$
- $\text{mean}(f) := \int_M H \, dA$
- $\text{area}(f) := \int_M dA$
- $\text{Gauss}(f) := \int_M K \, dA = 2\pi \chi$

How can we move the surface so that each of these quantities changes as quickly as possible? Remarkably enough…

$$\delta \text{volume}(f) = 2N$$
$$\delta \text{area}(f) = 2HN$$
$$\delta \text{mean}(f) = 2KN$$
$$\delta \text{Gauss}(f) = 0$$

| $\delta f$ | volume $\rightarrow$ area $\rightarrow$ mean $\rightarrow$ Gauss $\rightarrow$ 0 |
Discrete Normal via Volume Variation

• Recall that we still don’t have a clear definition for discrete normals at vertices, where the surface is not differentiable.

• However, in the smooth setting we know that the normal is equal to (half) the volume gradient.

• **Idea:** Since volume is perfectly well-defined for a discrete surface, why not use volume gradient to define vertex normals?

• Now just need to calculate the gradient of volume with respect to motion of one of the vertices, which we can do using our “geometric approach”…
Volume Enclosed by a Smooth Surface

- What’s the volume enclosed by a smooth surface \( f \)?

- One way: pick any point \( p \), integrate volume of “infinitesimal pyramids” over the surface.

- For a pyramid with base area \( b \) and height \( h \), the volume is \( V = bh/3 \) (no matter what shape the base is).

- For our infinitesimal pyramid, the height is the distance from the surface \( f \) to the point \( p \) along the normal direction: \( h = (f - p) \cdot N \).

- The area of the base is just the infinitesimal surface area \( dA \). Now we just integrate…

\[
\frac{1}{3} \int_M (f - p) \cdot N \, dA = \frac{1}{3} \int_M f \cdot N \, dA - p \cdot \int_M N \, dA = \frac{1}{3} \int_M f \cdot N \, dA
\]

Notice: doesn’t depend on choice of point \( p \)!
Volume Enclosed by a Discrete Surface

- What’s the volume enclosed by a discrete surface?
- Simply apply our smooth formula to a discrete $f$!
- **Exercise.** Show that the volume enclosed by a simplicial surface can be expressed as

$$\text{volume}(f) = \frac{1}{6} \sum_{ijk \in F} f_i \cdot (f_j \times f_k)$$
Discrete Volume Gradient

- Taking the gradient of enclosed volume with respect to the position $f_i$ of some vertex $i$ should now give us a notion of vertex normal:

$$\nabla_{f_i} \text{volume}(f) = \frac{1}{6} \nabla_{f_i} \sum_{ijk \in F} f_i \cdot (f_j \times f_k) = \frac{1}{6} \sum_{ijk \in F} f_j \times f_k$$

- But wait—this expression is the same as the discrete area vector!

- In other words: taking the gradient of discrete volume gave us exactly the same thing as integrating the normal over the dual cell.

- Agrees with the first expression in our sequence of variations:

$$\delta \text{volume}(f) = N$$
Total Area of a Discrete Surface

- Total area of a discrete surface is simply the sum of the triangle areas:

\[
\text{area}(f) := \sum_{ijk \in F} A_{ijk}
\]

Q: Suppose \( f \) is not a discrete immersion. Is area well-defined? Differentiable?
Discrete Area Gradient

• Recall that the gradient of triangle area with respect to position $p$ of a vertex is just half the normal cross the opposite edge:

$$\nabla_p A = \frac{1}{2} N \times e$$

• By summing contribution of all triangles touching a given vertex, can show that gradient of total surface area with respect to vertex coordinate $f_i$ is

$$\nabla_{f_i} \text{area}(f) = \sum_{ij} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) (f_i - f_j)$$

• Agrees with second expression in our sequence:

$$\delta \text{area}(f) = HN = \frac{1}{2} \Delta f$$
Total Mean Curvature of a Discrete Surface

• From our Steiner polynomial, we know the total mean curvature of a discrete surface is

\[
\text{mean}(f) = \frac{1}{2} \sum_{ij \in E} \ell_{ij}\varphi_{ij}
\]

(In fact, total volume and area used for the previous two calculations also agree with Steiner polynomial...)

**Discrete Mean Curvature Gradient**

- What’s the gradient of total mean curvature with respect to a particular vertex position $f_i$?

\[
\nabla_{f_i} \text{mean}(f) = \frac{1}{2} \sum_{ij \in E} \nabla_{f_i} (\ell_{ij} \varphi_{ij}) = \\
0 \quad \text{(Schläfli)}
\]

\[
\frac{1}{2} \sum_{ij \in E} (\nabla_{f_i} \ell_{ij}) \varphi_{ij} + \ell_{ij} (\nabla_{f_i} \varphi_{ij}) = \\
\frac{1}{2} \sum_{ij \in E} \varphi_{ij} (f_i - f_j)
\]

- Agrees with third expression in our sequence:

\[
\delta \text{mean}(f) = KN
\]
Total Gauss Curvature

- Total Gauss curvature of a discrete surface is sum of angle defects:

\[
\text{Gauss}(f) = \sum_{i \in V} \left( 2\pi - \sum_{ijk} \theta^i_{jk} \right)
\]

- From (discrete) Gauss-Bonnet theorem, we know this sum is always equal to just \(2\pi \chi = 2\pi (V-E+F)\)

- Gradient with respect to motion of any vertex is therefore zero—sequence ends here!
Discrete Curvature—Panoramic View

\[
\begin{align*}
\text{Smooth} & \quad \text{Discrete} & \quad \text{Algebraic} \\
\text{vol}(M) & \rightarrow & \sum_{ijk \in F} V_{ijk} & \rightarrow & P(r) \\
\delta f & \downarrow & \gamma_j & \rightarrow & r \rightarrow 0 \\
N \, dA & \rightarrow & \sum_{j \in N(i)} A_j N_j & \leftarrow & \frac{d}{dr} P(r) \\
\delta t & \downarrow & \gamma_j & \rightarrow & \frac{d}{dr} \\
\int_M dA & \rightarrow & \sum_{ijk \in F} A_{ijk} & \rightarrow & P'(r) \\
\delta f & \downarrow & \gamma_j & \rightarrow & r \rightarrow 0 \\
HN \, dA & \rightarrow & \frac{1}{2} \sum_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i) & \rightarrow & P''(r) \\
\delta t & \downarrow & \gamma_j & \rightarrow & \frac{d}{dr} \\
\int_M H \, dA & \rightarrow & \sum_{ij \in E} \theta_{ij} \ell_{ij} & \rightarrow & P'''(r) \\
\delta f & \downarrow & \gamma_j & \rightarrow & r \rightarrow 0 \\
KN \, dA & \rightarrow & \frac{1}{2} \sum_{j \in N(i)} \theta_{ij} (f_j - f_i) & \rightarrow & \frac{d}{dr} \\
\delta t & \downarrow & \gamma_j & \rightarrow & \frac{d}{dr} \\
\int_M K \, dA & \rightarrow & \sum_{n \in V} \left(2\pi - \Sigma_{ijk \in F} \varphi_{ij}^k\right) & \rightarrow & P''''(r) \\
\delta f & \downarrow & \gamma_j & \rightarrow & r \rightarrow 0 \\
\text{(Gauss-Bonnet)} & \rightarrow & \frac{d}{dr} \\
0 & \rightarrow & \frac{d}{dr}
\end{align*}
\]
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