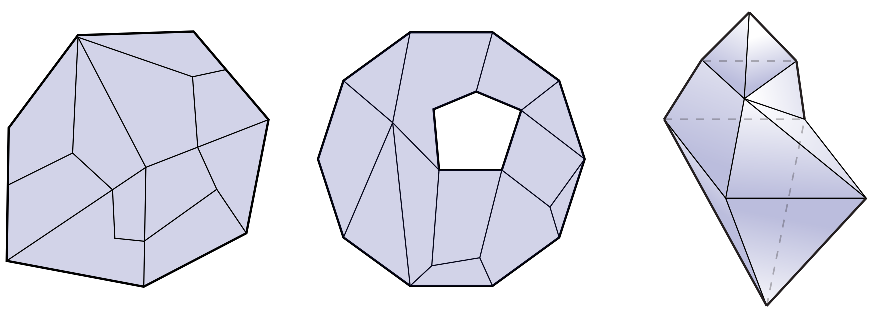
ASSIGNMENT 1

Topological Invariants of Discrete Surfaces

1.1. Euler Characteristic

A *topological disk* is, roughly speaking, any shape you can get by deforming the unit disk in the plane without tearing it, puncturing it, or gluing its edges together. Some examples of shapes that are disks include a flag, a leaf, and a glove. Some examples of shapes that are *not* disks include a circle (*i.e.*, a disk *without* its interior), a ball, a sphere, a donut, a washer, and a teapot. A *polygonal disk* is any topological disk constructed out of simple polygons. Similarly, a topological *sphere* is any shape resembling the standard sphere, and a *polyhedron* is a sphere made of polygons. More generally, a *piecewise linear* surface is any surface made by gluing together polygons along their edges; a *simplicial surface* is a special case of a piecewise linear surface where all the faces are triangles. The *boundary* of a piecewise linear surface is the set of edges that are contained in only a single face (all other edges are shared by exactly two faces). For example, a disk has a boundary whereas a polyhedron does not. You may assume that surfaces have no boundary unless otherwise stated.



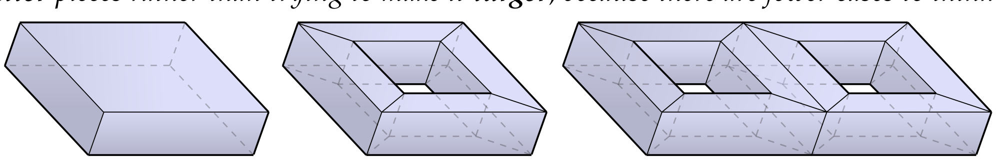
polygonal disk   *(neither)*   *polyhedron*

EXERCISE 1. **Polyhedral Formula.** Show that for any polygonal disk with V vertices, E edges, and F faces, the following relationship holds:

$$V - E + F = 1$$

and conclude that for any polyhedron $V - E + F = 2$.

Hint: use induction. Note that induction is generally easier if you start with a given object and decompose it into ***smaller*** *pieces rather than trying to make it* ***larger****, because there are fewer cases to think about.*



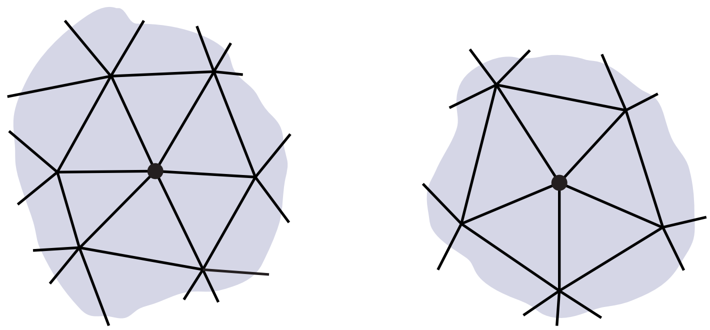
sphere ($g = 0$)   *torus* ($g = 1$)   *double torus* ($g = 2$)

Clearly not all surfaces look like disks or spheres. Some surfaces have additional *handles* that distinguish them topologically; the number of handles g is known as the *genus* of the surface (see illustration above for examples). In fact, among all surfaces that have no boundary and are connected (meaning a single piece), compact (meaning closed and contained in a ball of finite size), and orientable (having two distinct sides), the genus is the *only* thing that distinguishes two surfaces. A more general formula applies to such surfaces, namely

$$V - E + F = 2 - 2g,$$

which is known as the *Euler-Poincaré; formula.*

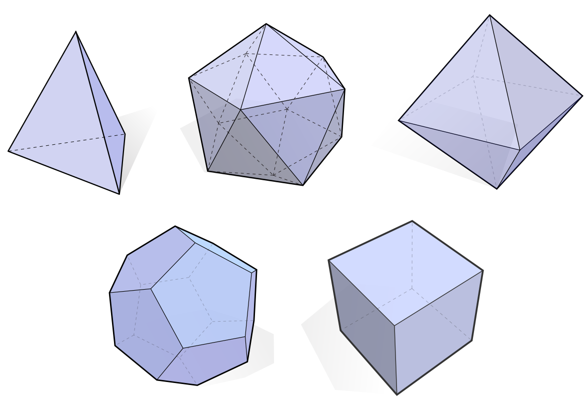
1.2. Regular Meshes and Average Valence



regular *irregular*

The *valence* of a vertex in a piecewise linear surface is the number of faces that contain that vertex. A vertex of a *simplicial* surface is said to be *regular* when its valence equals six. Many numerical algorithms (such as subdivision) exhibit ideal behavior only in the regular case, and generally behave better when the number of irregular valence vertices is small.

EXERCISE 2. Even the ancient Greeks were interested in regular meshes. In particular, they knew that there are only five genus-zero polyhedra where every face and every vertex is identical—namely, the five *Platonic solids*: tetrahedron, icosahedron, octahedron, dodecahedron, cube. Show that this list is indeed exhaustive. *Hint: you do not need to use any facts about lengths or angles; just connectivity.*



EXERCISE 3. **Regular Valence.** Show that the only (connected, orientable) simplicial surface for which every vertex has regular valence is a torus ($g = 1$). You may assume that the surface has finitely many faces. *Hint: apply the Euler-Poincaré; formula.*

EXERCISE 4. Show that the minimum possible number of irregular valence vertices in a (connected, orientable) simplicial surface K of genus g is given by

$$m(K) = \begin{cases} 4, & g = 0 \\ 0, & g = 1 \\ 1, & g \geq 2, \end{cases}$$

assuming that all vertices have valence at least three and that there are finitely many faces.

EXERCISE 5. **Mean Valence.** Show that the mean valence approaches six as the number of vertices in a (connected, orientable) simplicial surface goes to infinity, and that the ratio of vertices to edges to faces hence approaches

$$V : E : F = 1 : 3 : 2.$$

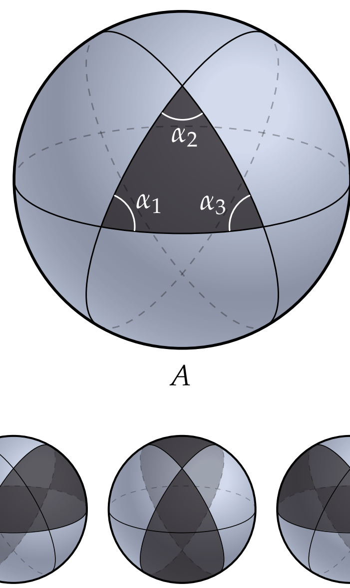
This fact can be useful when making decisions about algorithm design (e.g., it costs about three times as much to store a quantity on edges as on vertices), and simplifies discussions about asymptotic growth (since the number of different element types are essentially related by a constant).

1.3. Discrete Gauss-Bonnet

EXERCISE 6. **Area of a Spherical Triangle.** Show that the area of a spherical triangle on the unit sphere with interior angles α_1, α_2, α_3 is

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Hint: consider the areas A_1, A_2, A_3 of the three shaded regions (called "diangles") pictured below.



A

A_1 A_2 A_3

EXERCISE 7. **Area of a Spherical Polygon.** Show that the area of a spherical polygon with consecutive interior angles $\beta_1, \ldots, \beta_n$ is

$$A = (2 - n)\pi + \sum_{i=1}^{n} \beta_i.$$

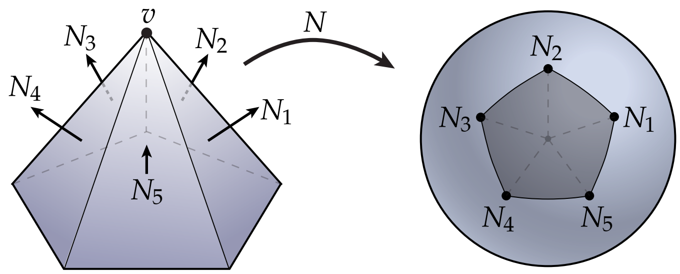
Hint: use the expression for the area of a spherical triangle you just derived!

EXERCISE 8. **Angle Defect.** Recall that for a discrete planar curve we can define the curvature at a vertex as the distance on the unit circle between the two adjacent normals. For a discrete *surface* we can define *discrete Gaussian curvature* at a vertex v as the *area* on the unit sphere bounded by a spherical polygon whose vertices are the unit normals of the faces around v. Show that this area is equal to the *angle defect*

$$d(v) = 2\pi - \sum_{f \in F_v} \angle_f(v)$$

where F_v is the set of faces containing v and $\angle_f(v)$ is the interior angle of the face f at vertex v.

Hint: consider planes that contain two consecutive normals and their intersection with the unit sphere.



EXERCISE 9. **Discrete Gauss-Bonnet Theorem.** Consider a (connected, orientable) simplicial surface K with finitely many vertices V, edges E and faces F. Show that a discrete analog of the Gauss-Bonnet theorem holds for simplicial surfaces, namely

$$\sum_{v \in V} d(v) = 2\pi\chi$$

where $\chi = |V| - |E| + |F|$ is the *Euler characteristic* of the surface.

This fact perfectly mirrors the smooth *Gauss-Bonnet theorem*, namely that the total Gaussian curvature K is always a constant multiple of the Euler characteristic:

$$\int_M K \, dA = 2\pi\chi.$$

We will study this theorem and its implications later on in class.