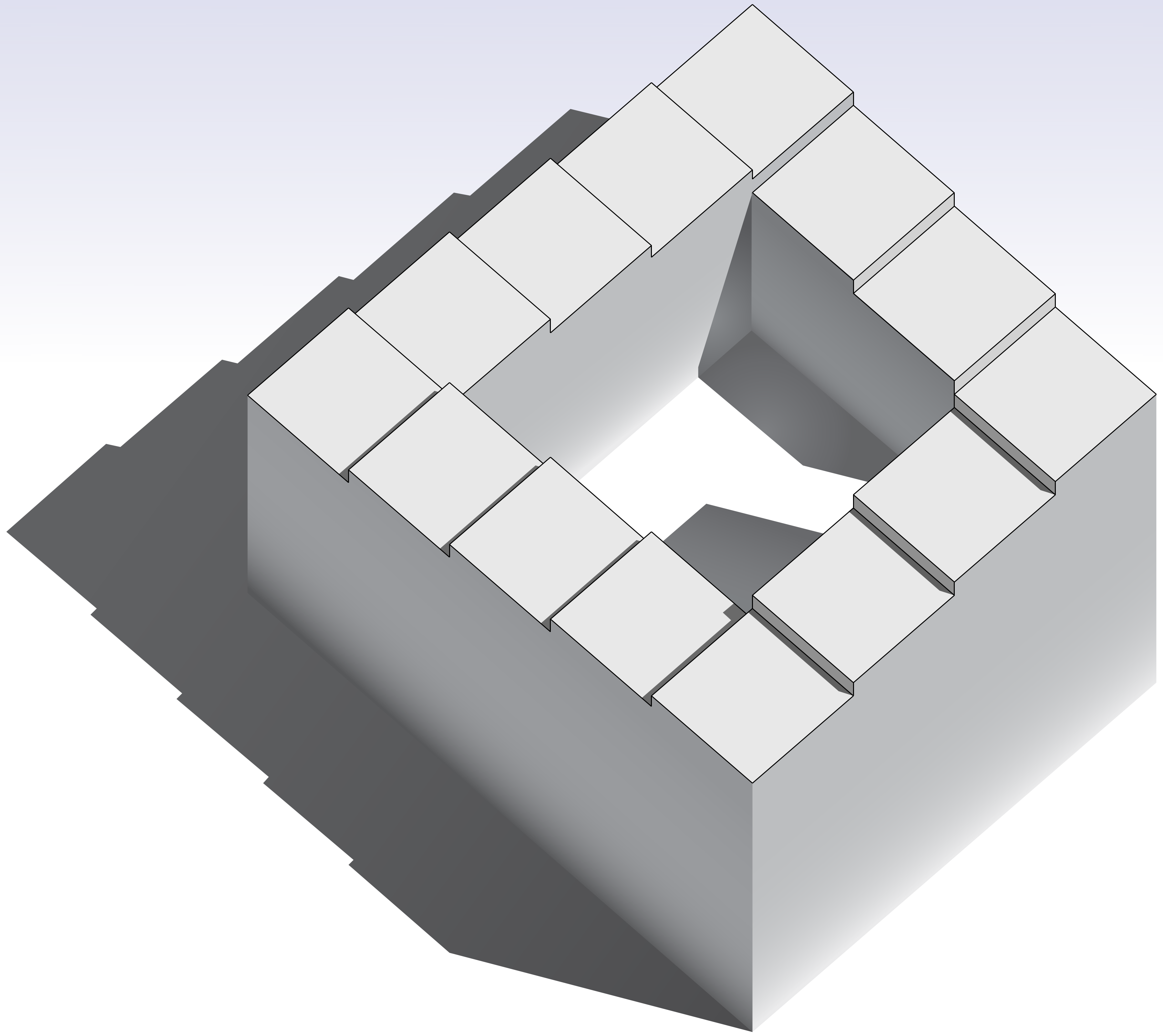
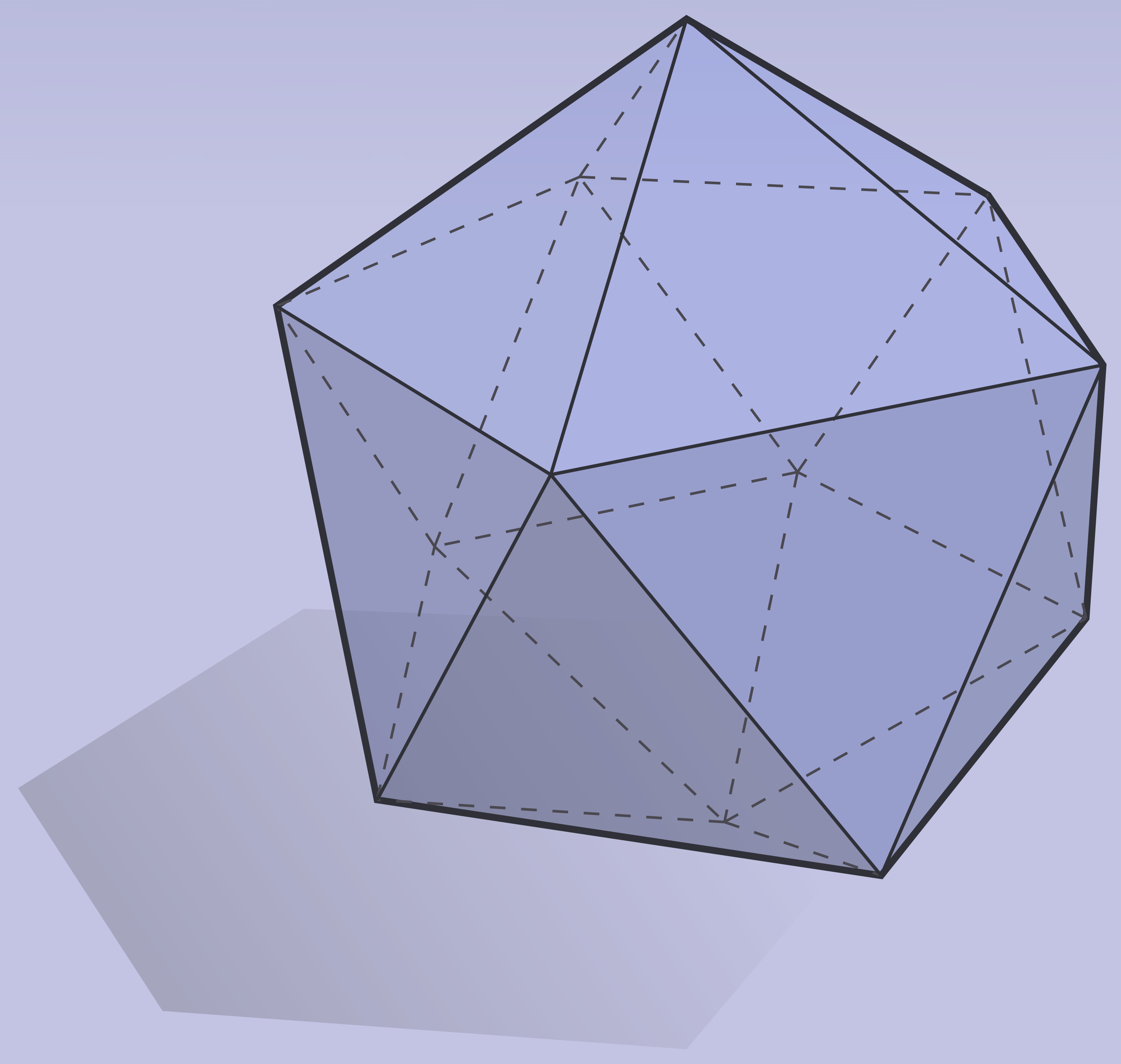


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-869(J) • Spring 2016

PART I:

TOPOLOGICAL STRUCTURE

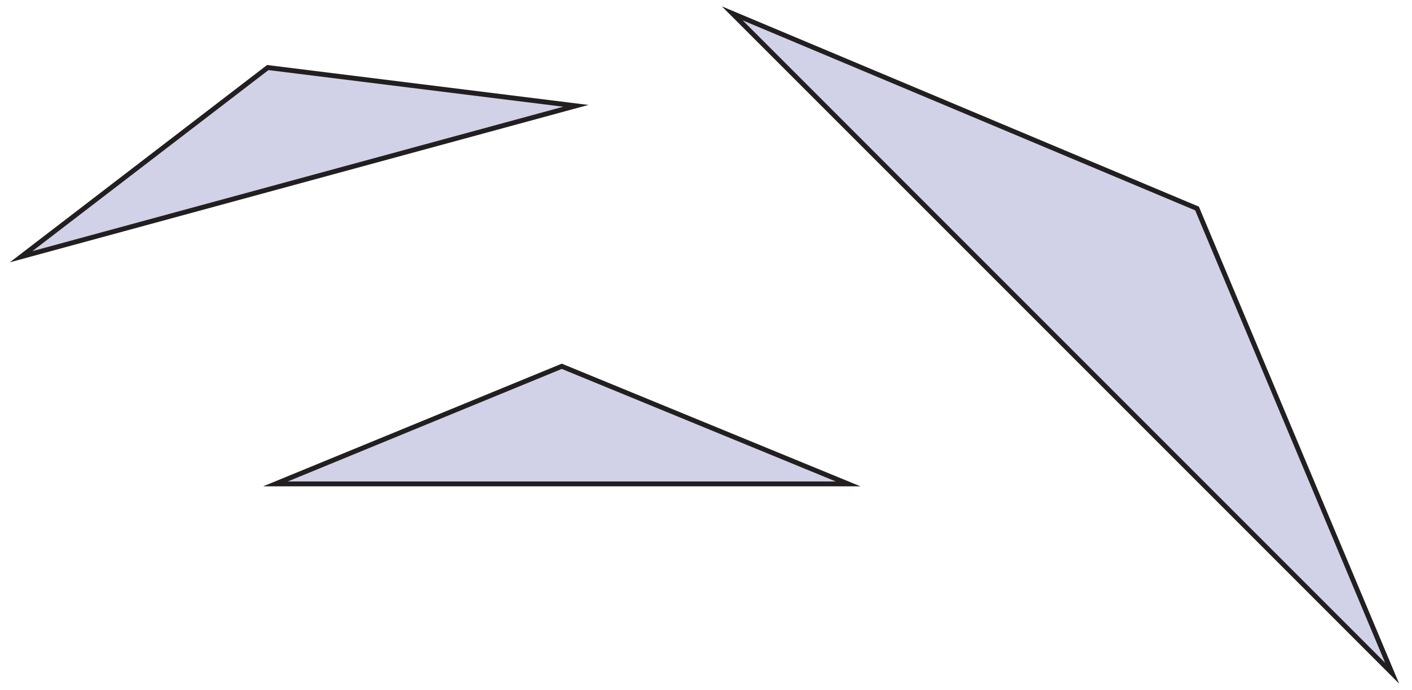


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-869(J) • Spring 2016

What is a "shape?"

Which of these objects are the same?



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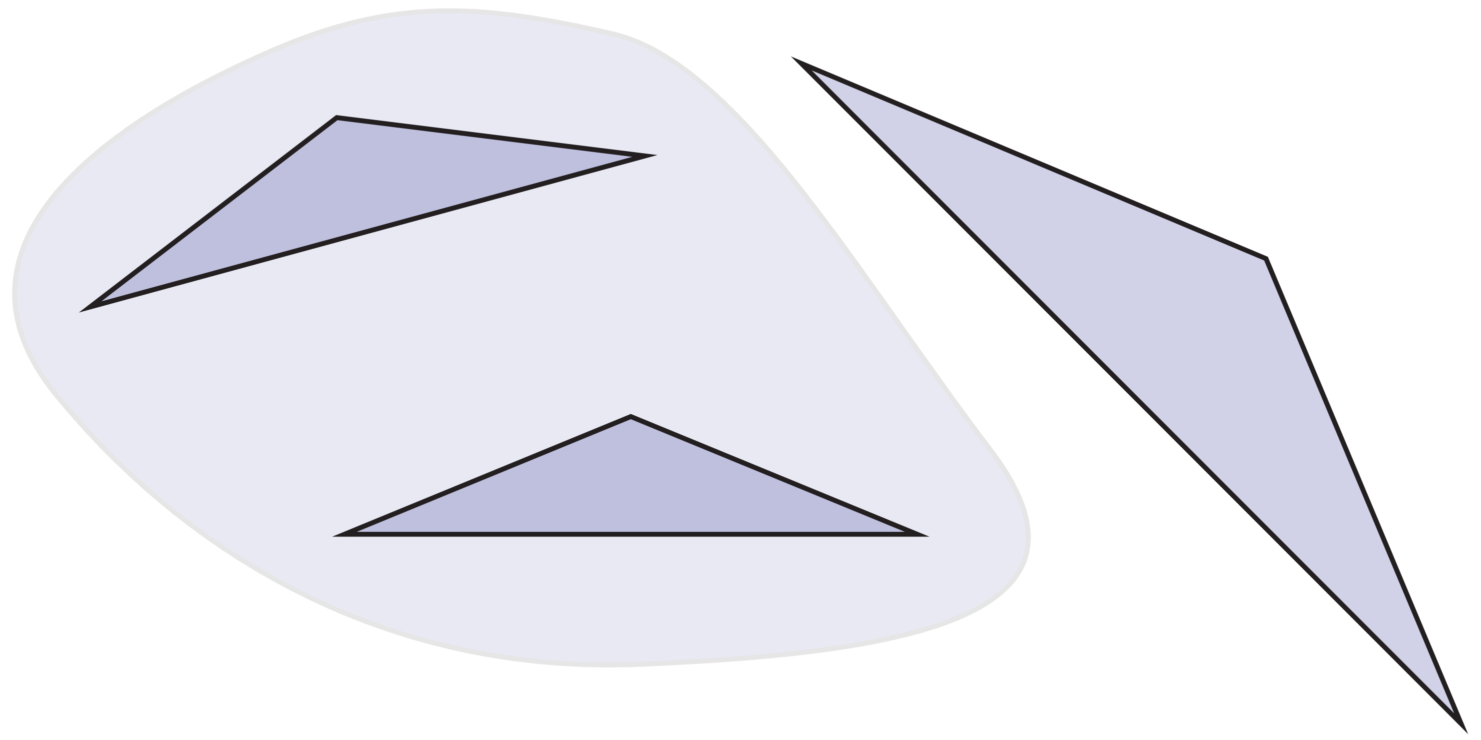

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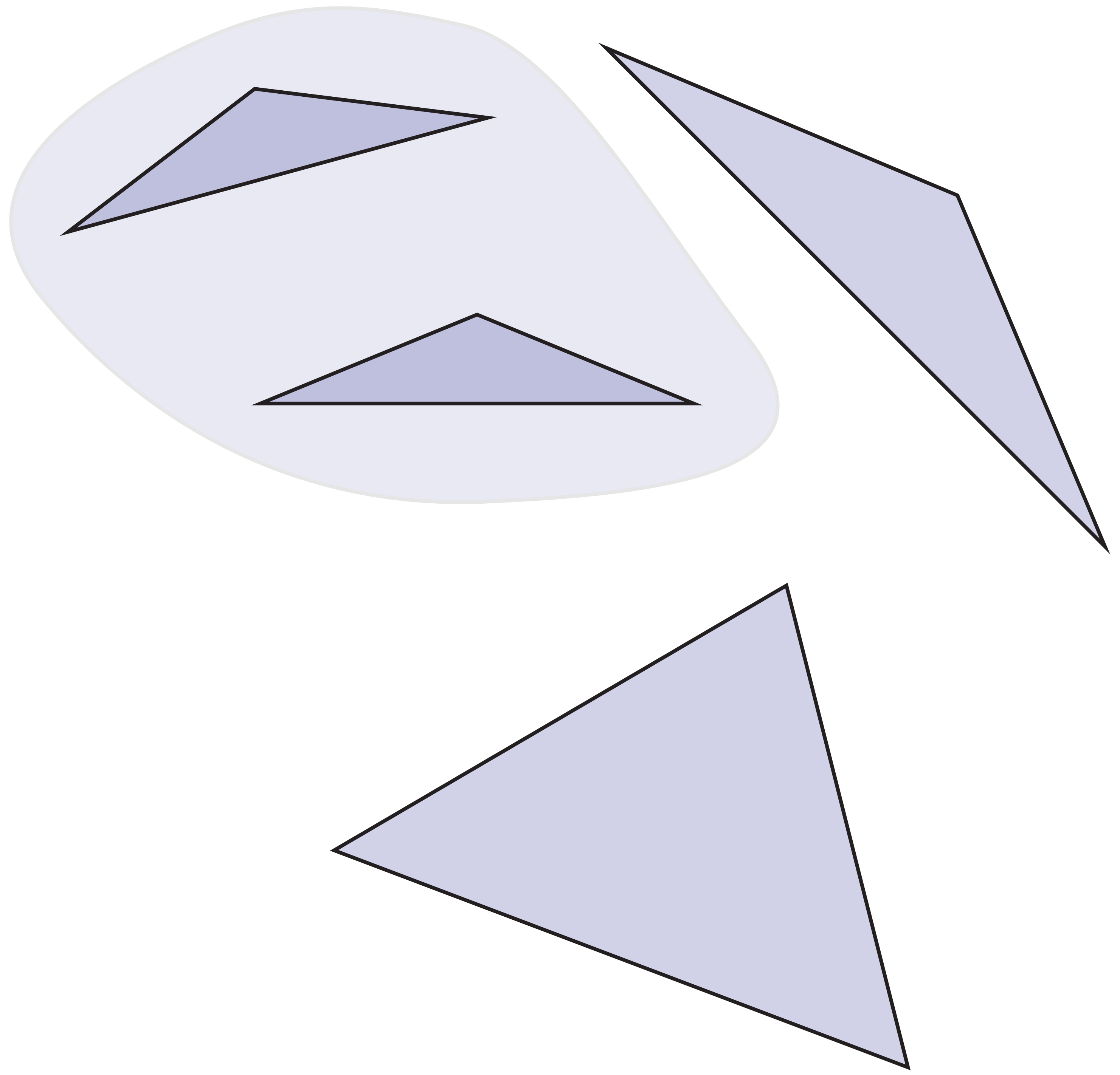

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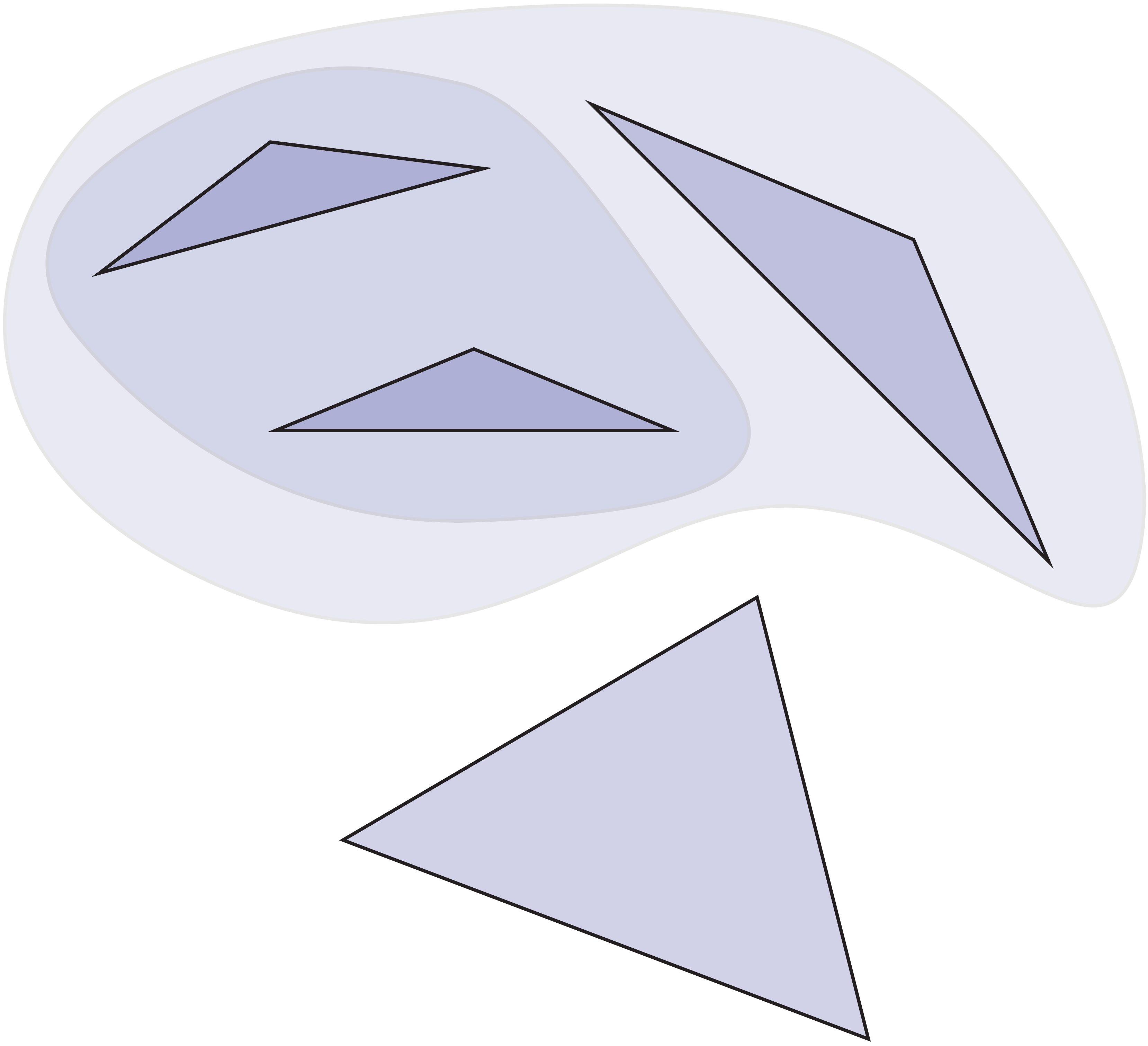

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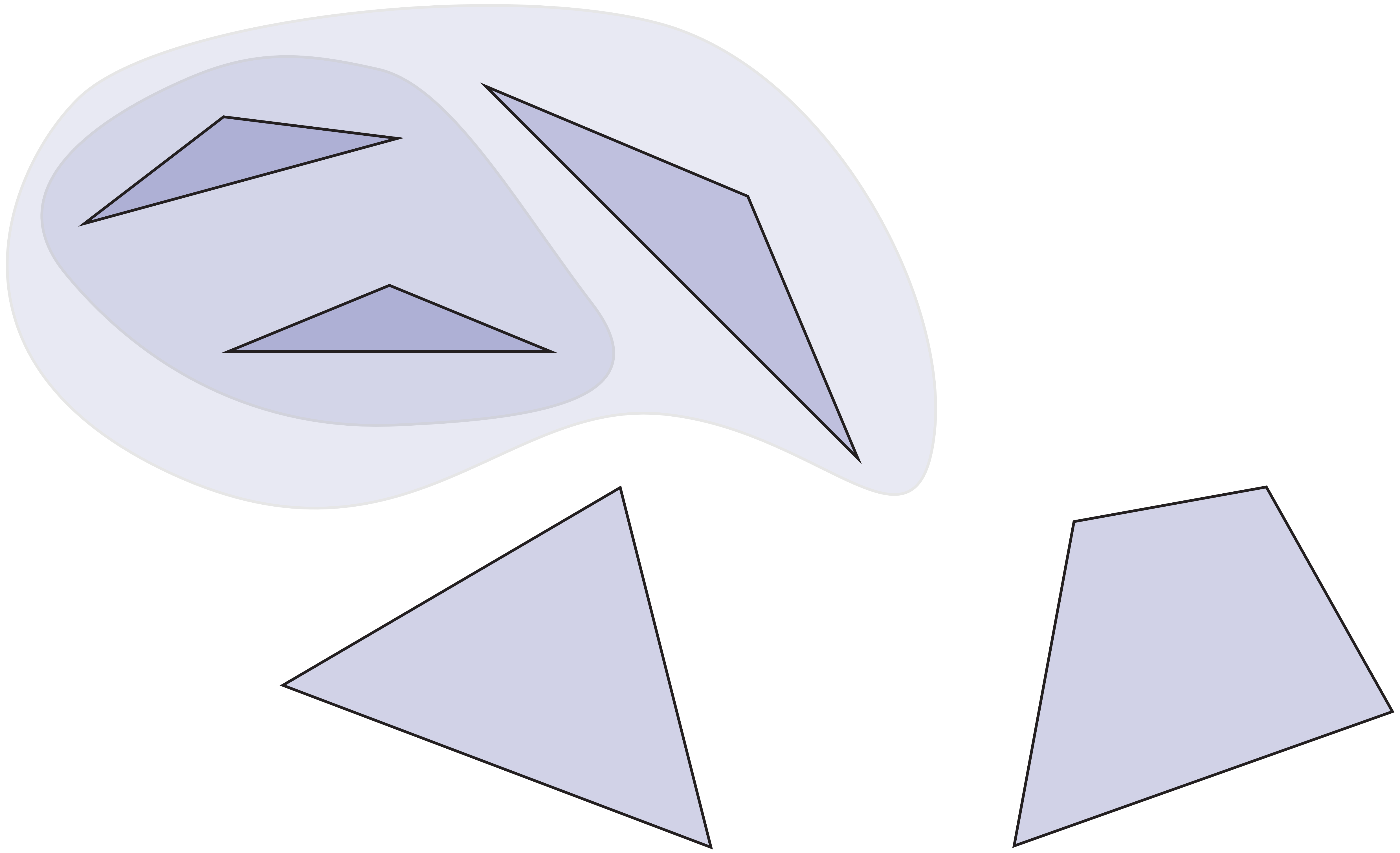

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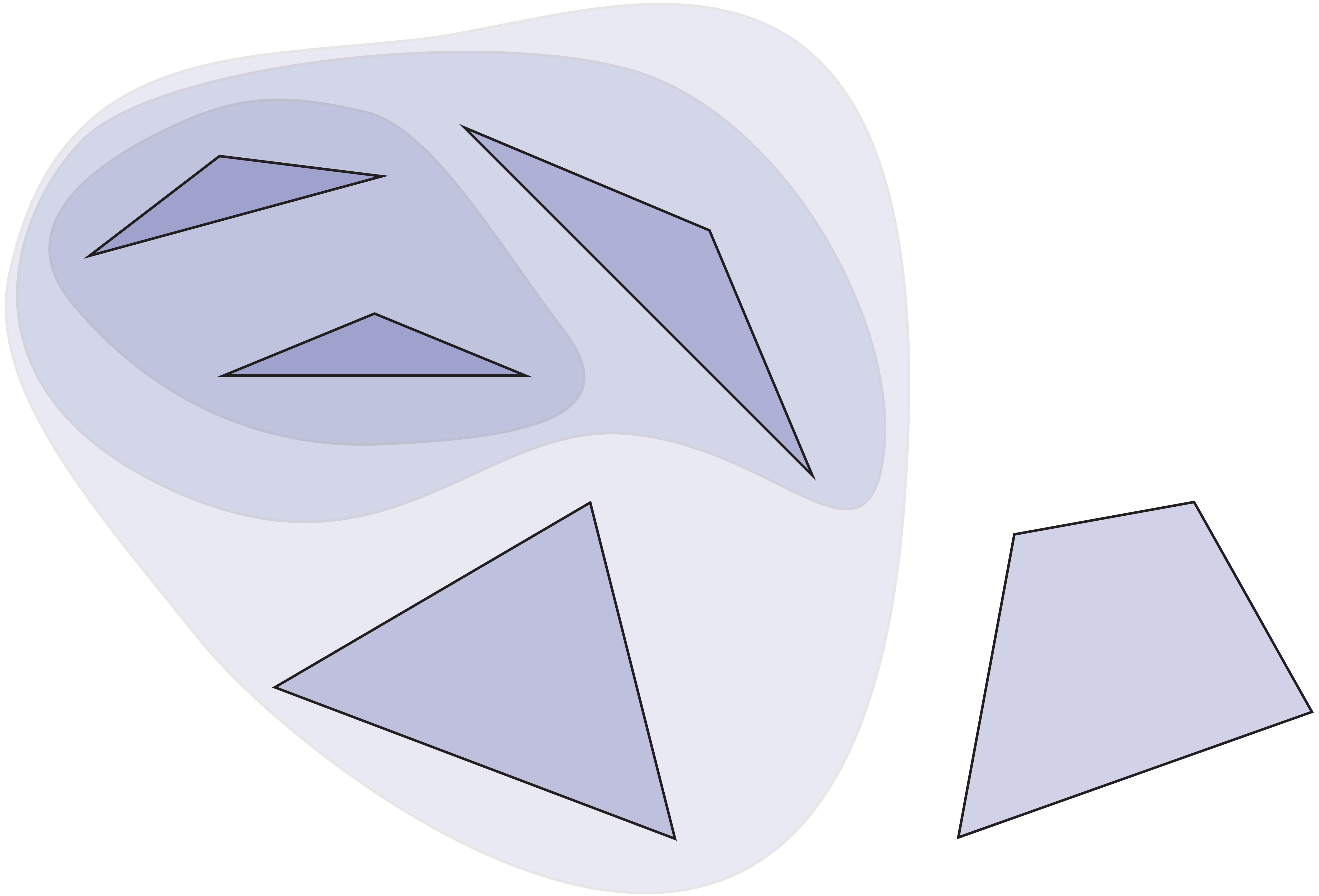

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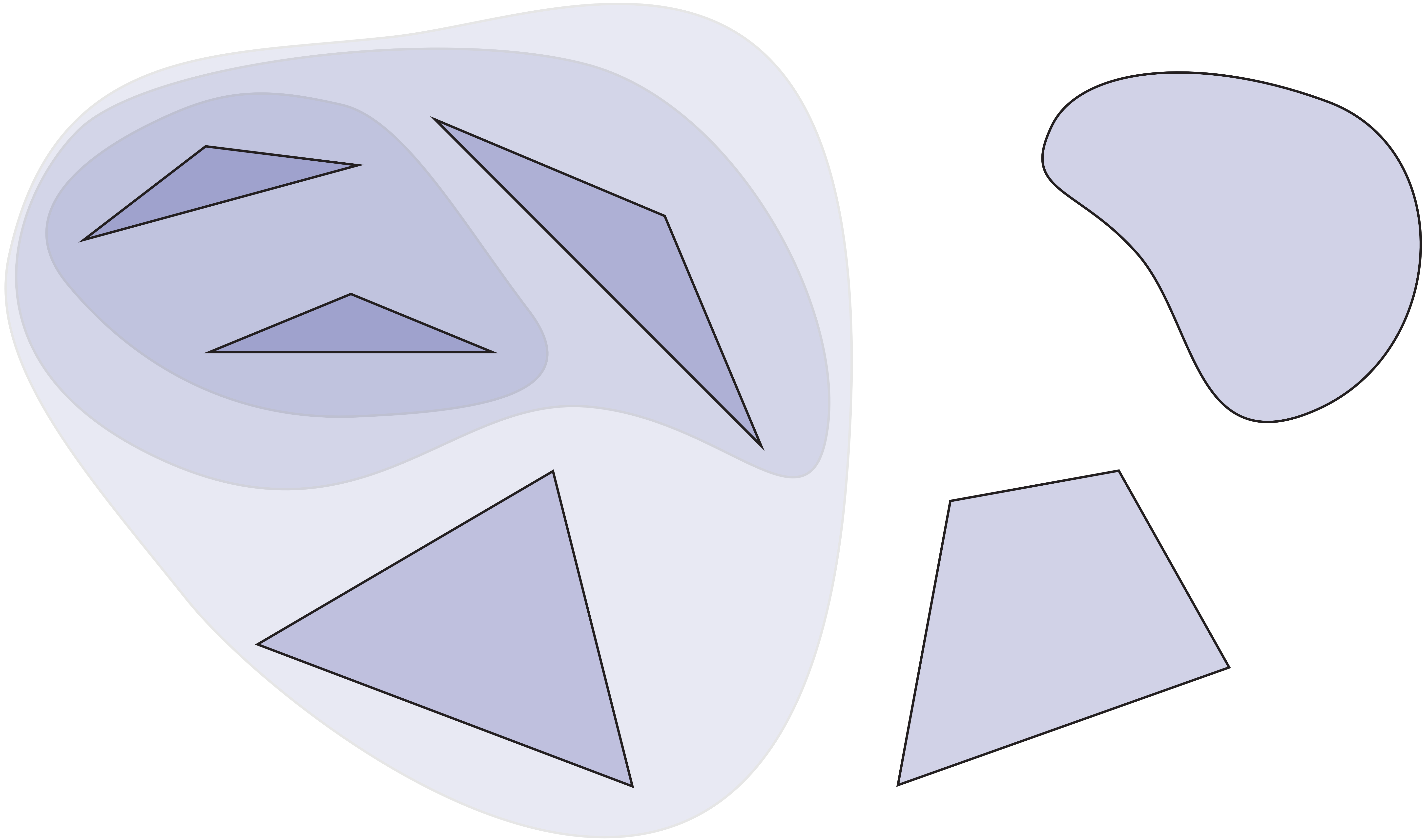

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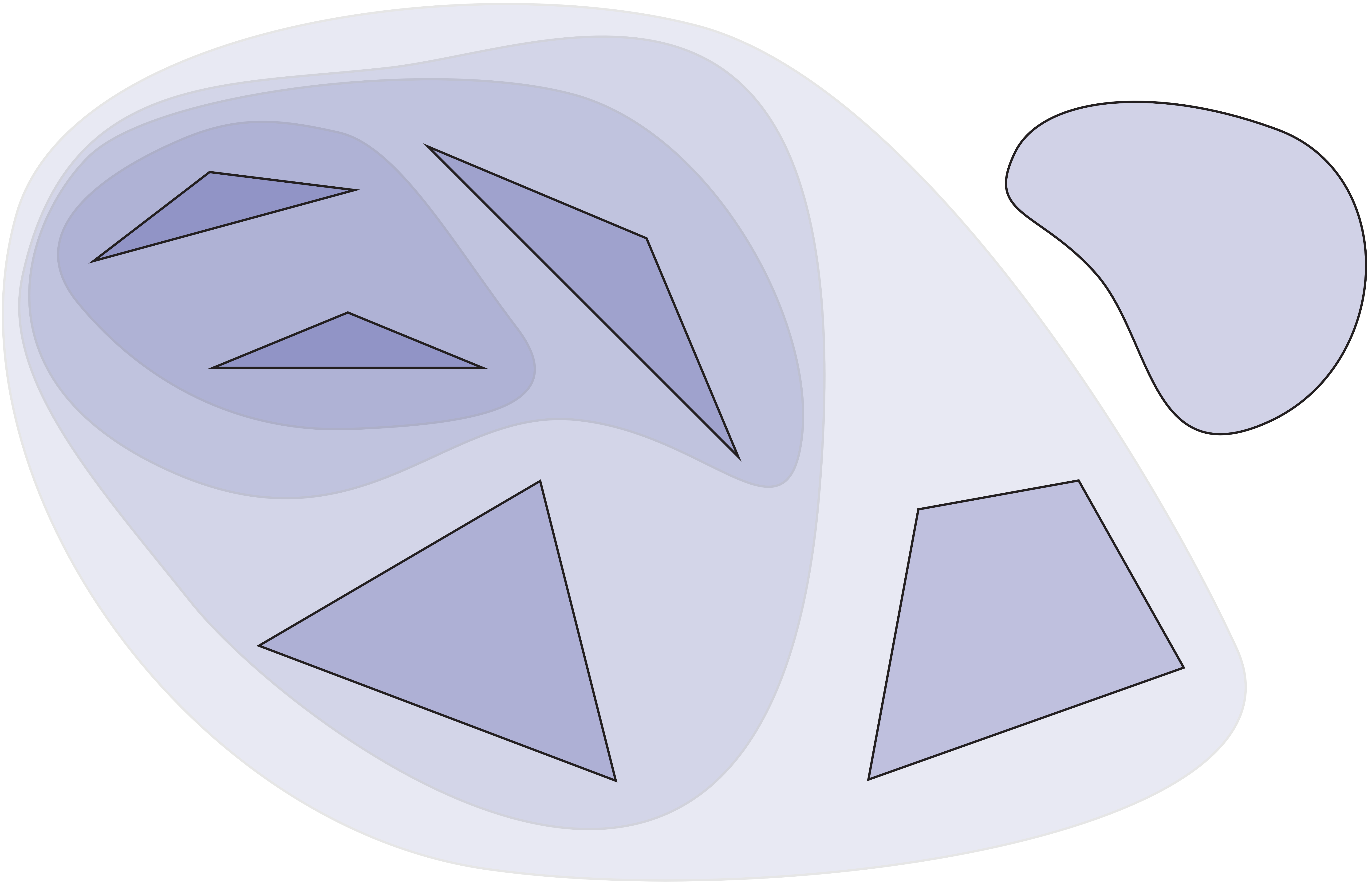

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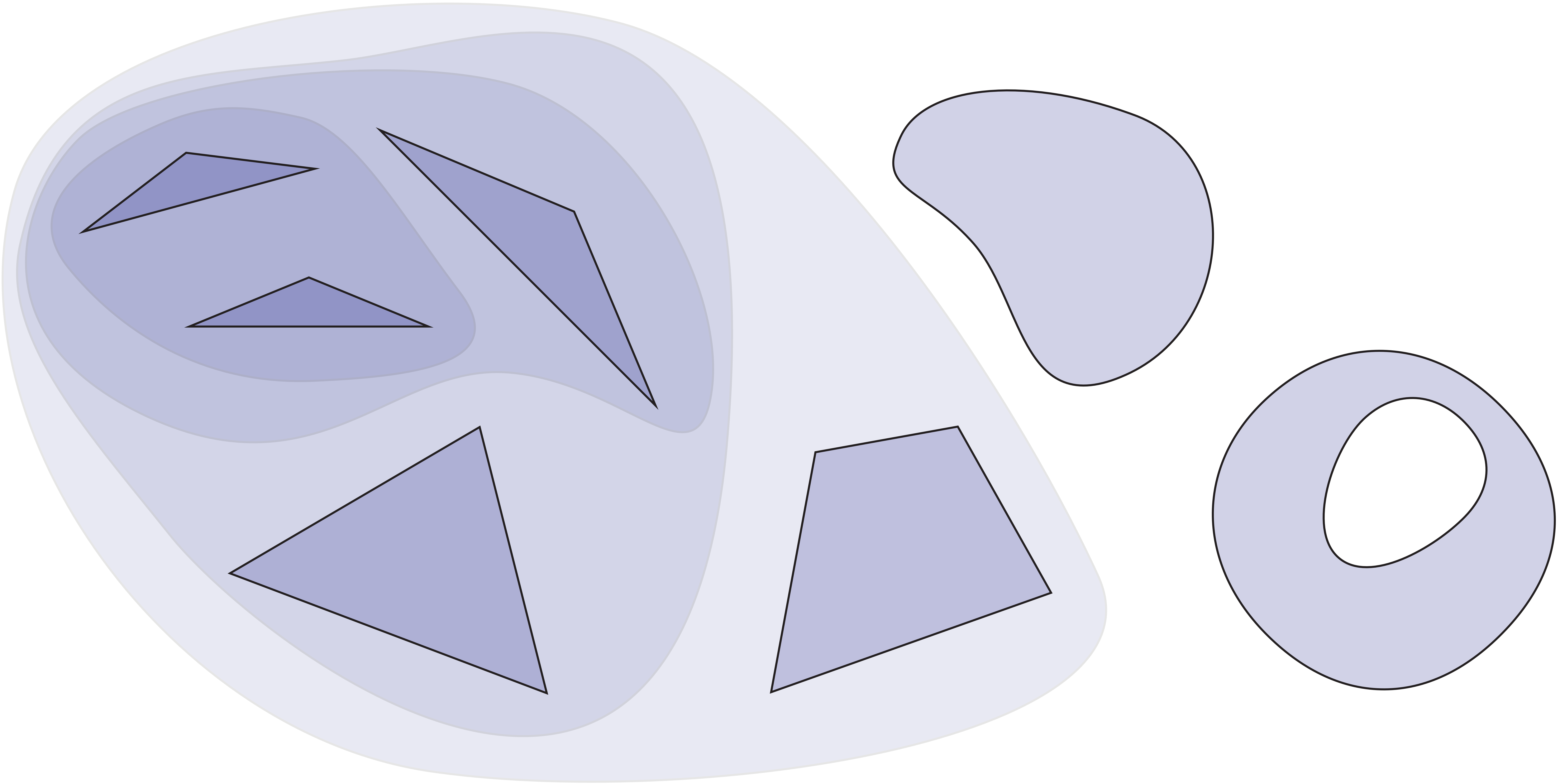

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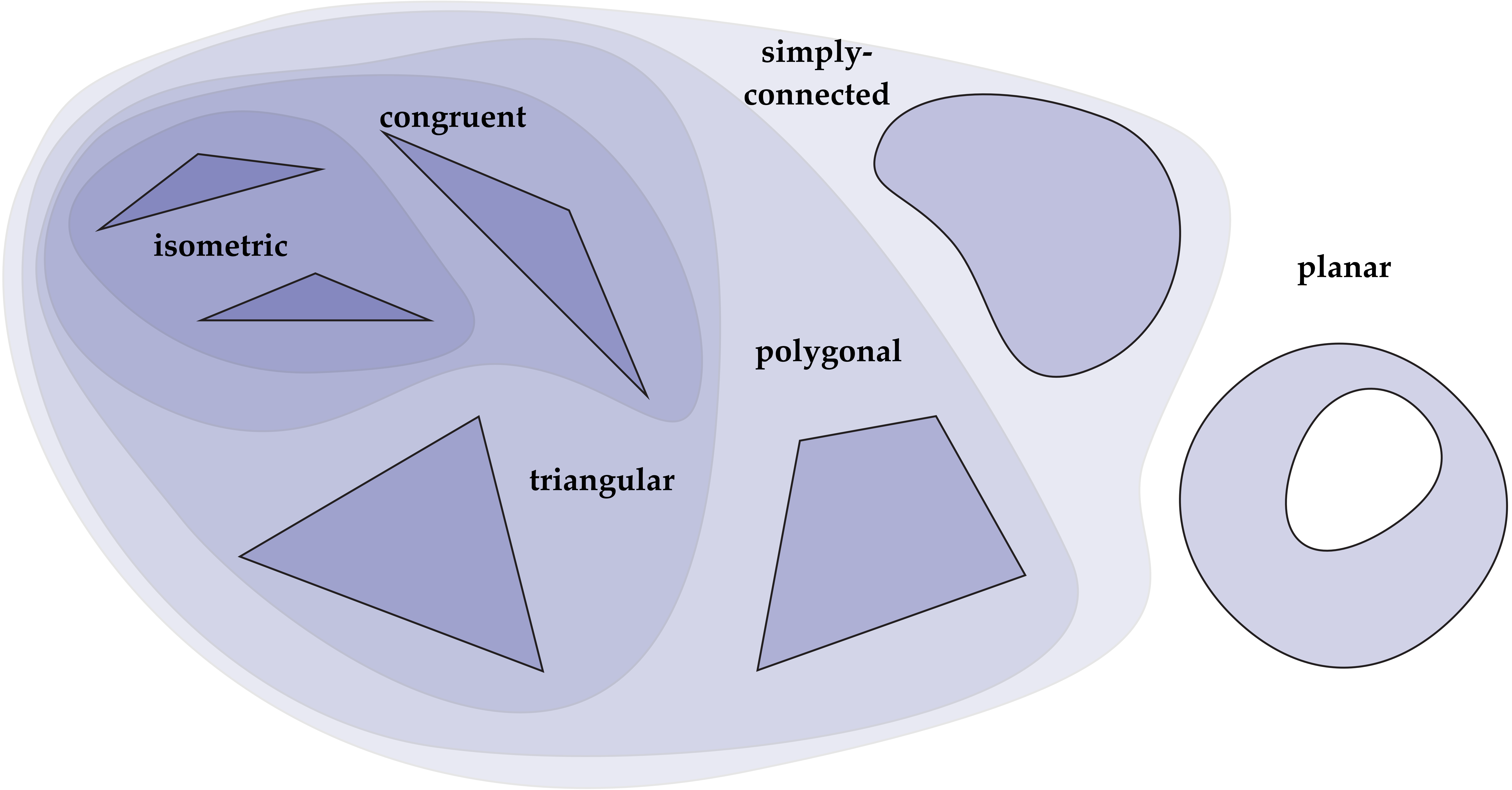
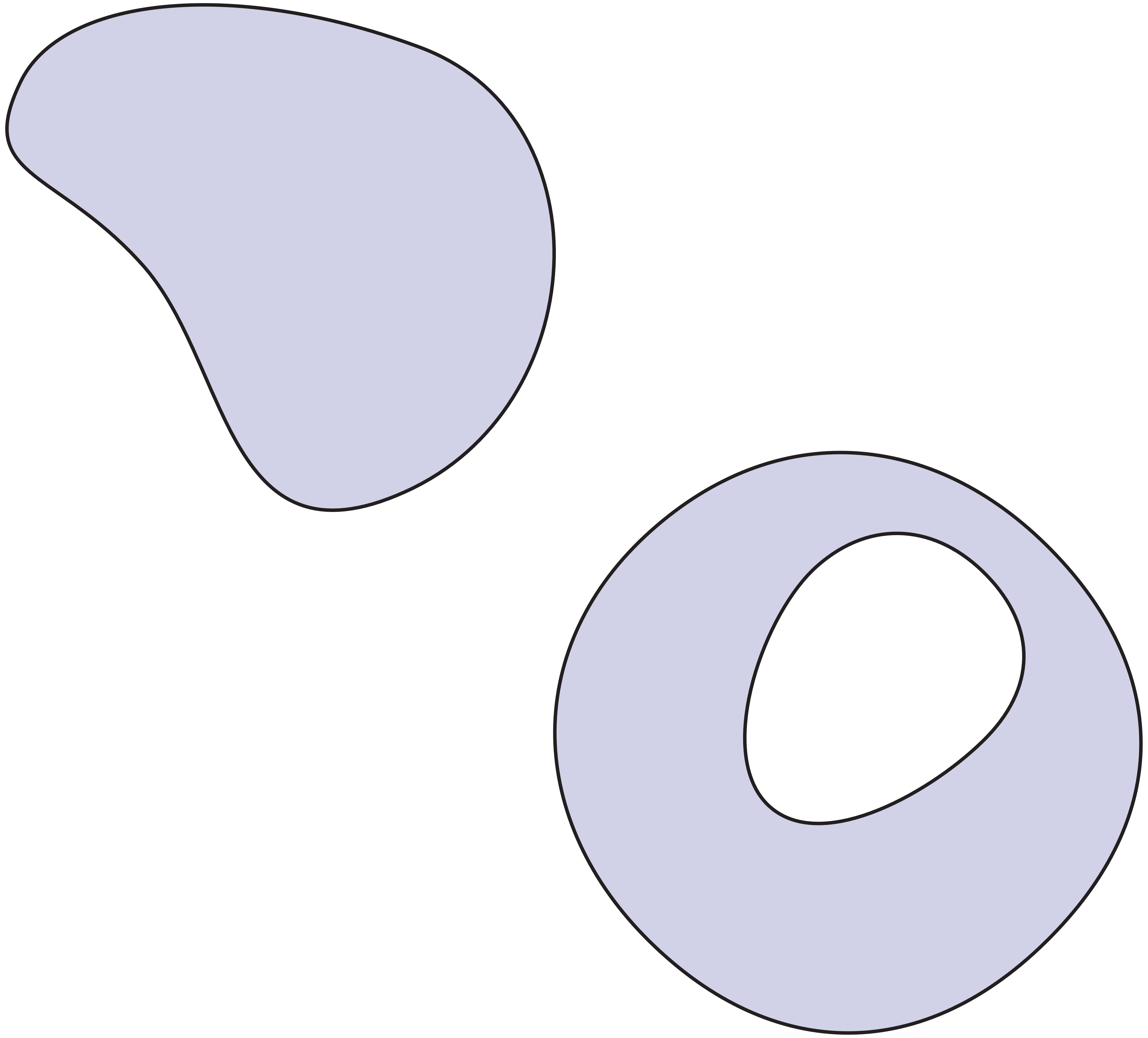


Hierarchy of Structures in Geometry

$$\begin{gathered}\textsc{Set}\\ \cup\\ \textsc{Topological Space}\\ \cup\\ \textsc{Topological Manifold}\\ \cup\\ \textsc{Smooth Manifold}\\ \cup\\ \textsc{Complex Manifold}\\ \cup\\ \textsc{Riemannian Manifold}\end{gathered}$$

Visual Topology

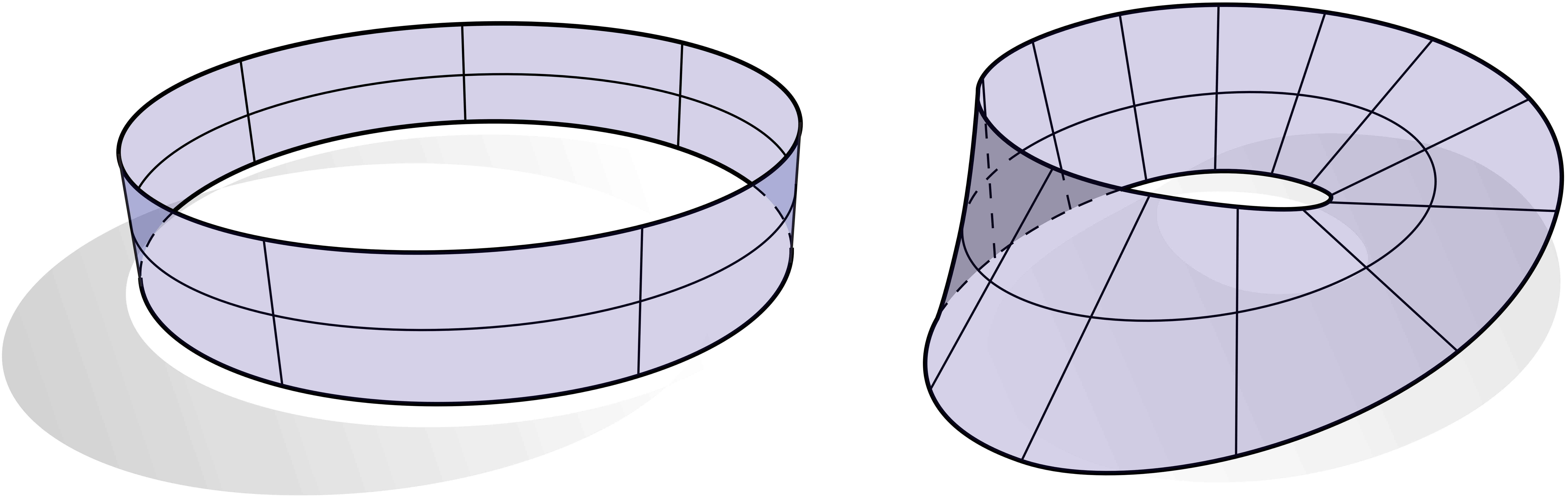


Visual Topology

Visual Topology

Visual Topology

Visual Topology



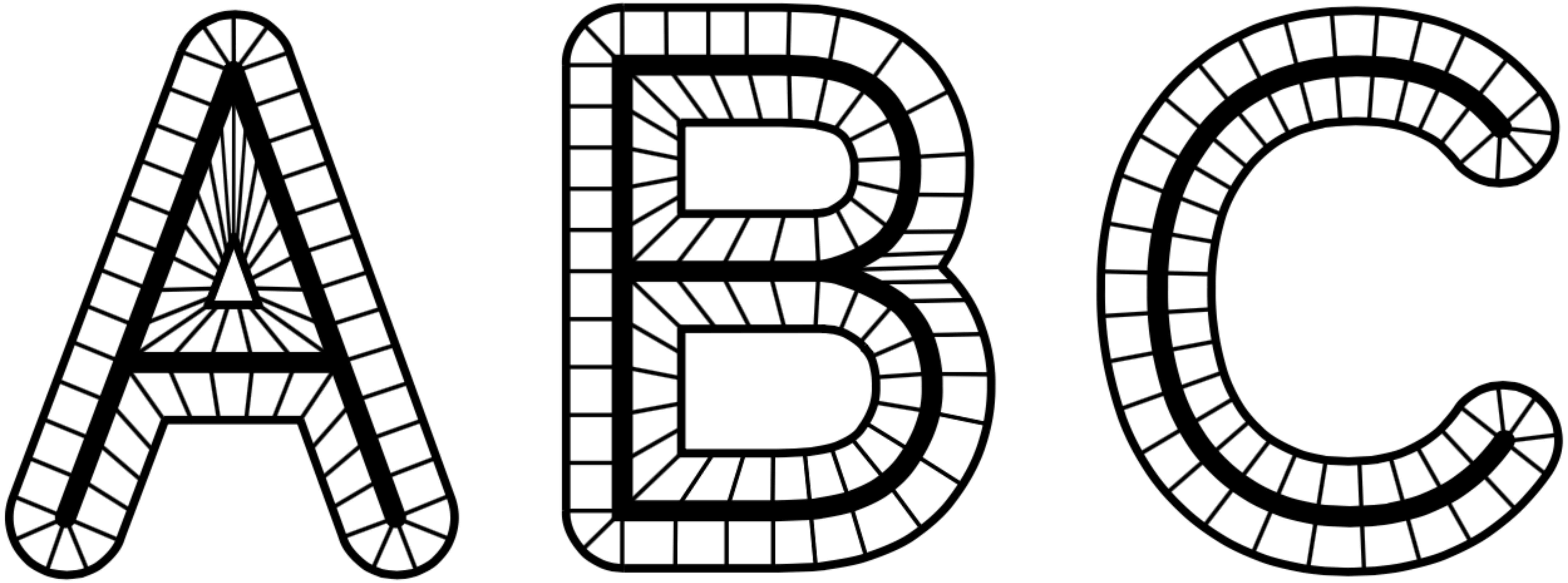
Visual Topology

Visual Topology

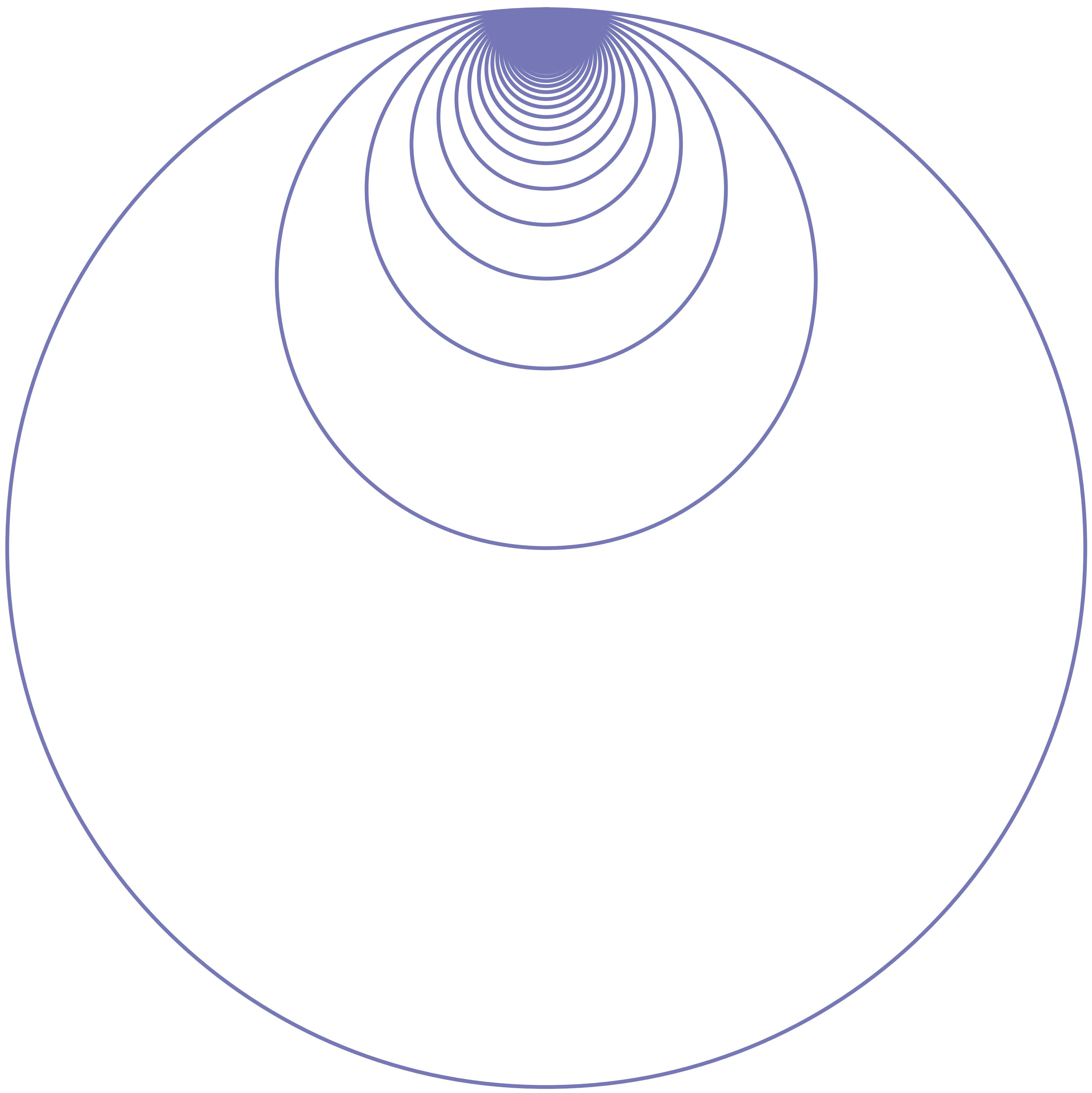
Crane et al, *"Robust Fairing via Conformal Curvature Flow"* / *Henry Segerman*

https://www.youtube.com/watch?v=9NlqYr6-TpA

Visual Topology

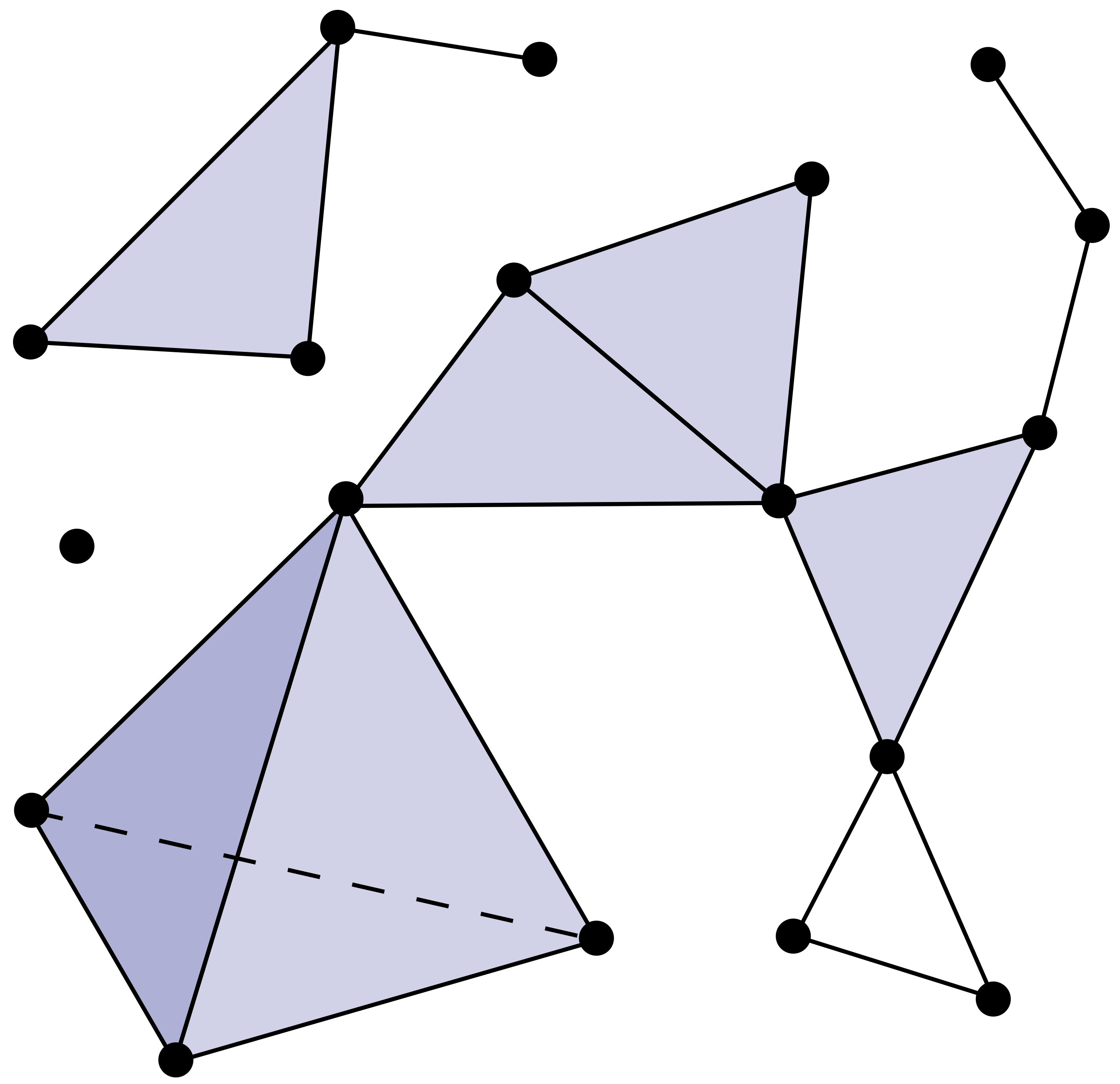


Visual Topology

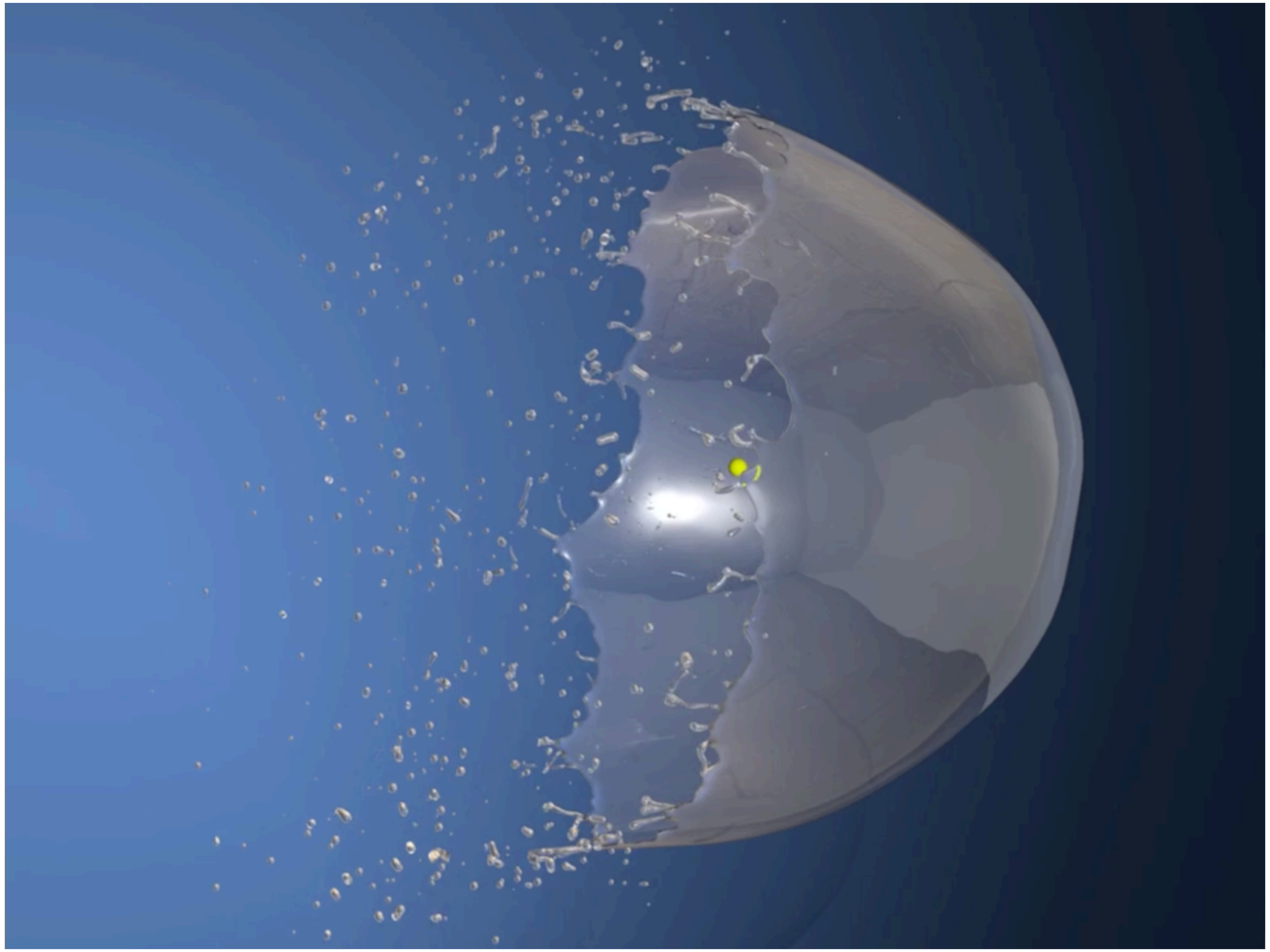


Discussion: The Simplicial Complex

- An *abstract simplicial complex* is a collection of (finite, nonempty) sets closed under the operation of taking subsets.
- A *(geometric) k-simplex* is the convex hull of k+1 (affinely-independent) points in n-dimensional Euclidean space, n ≥ k.
- Questions:
  - Empty set? Face vs. *proper* face?
  - Other questions?
  - Applications (& why not constant dimension?)



Applications: Simulation

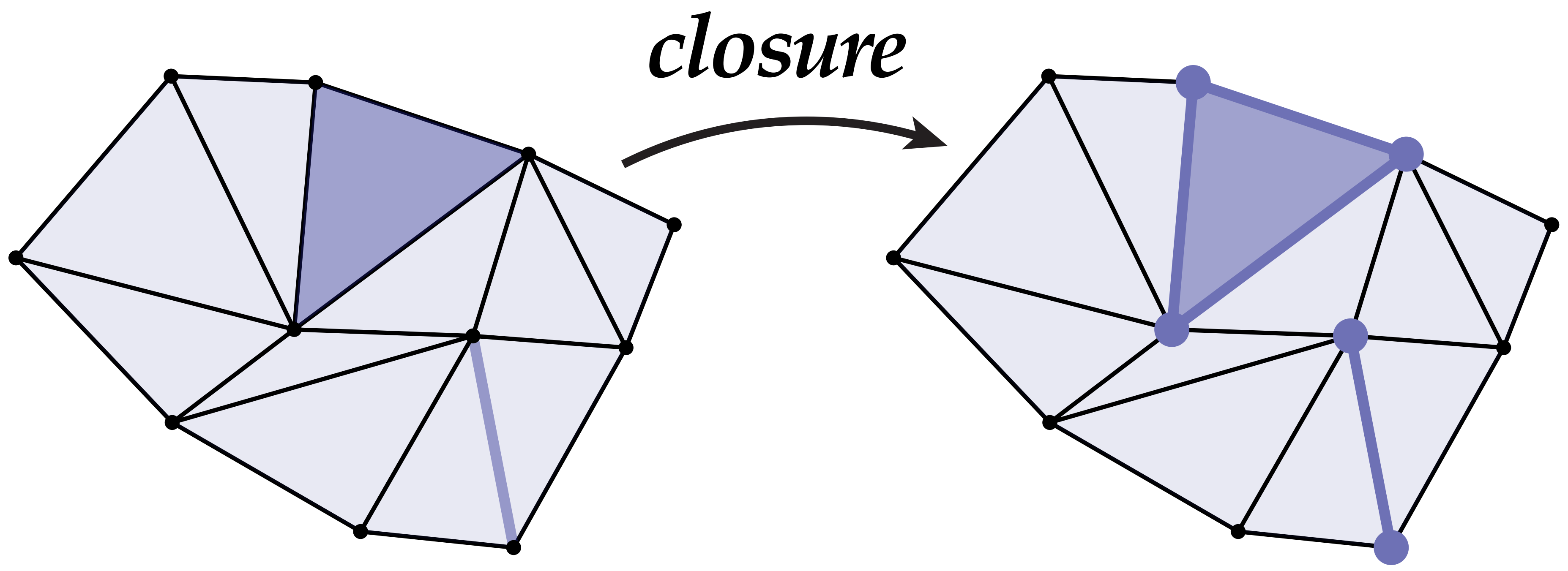


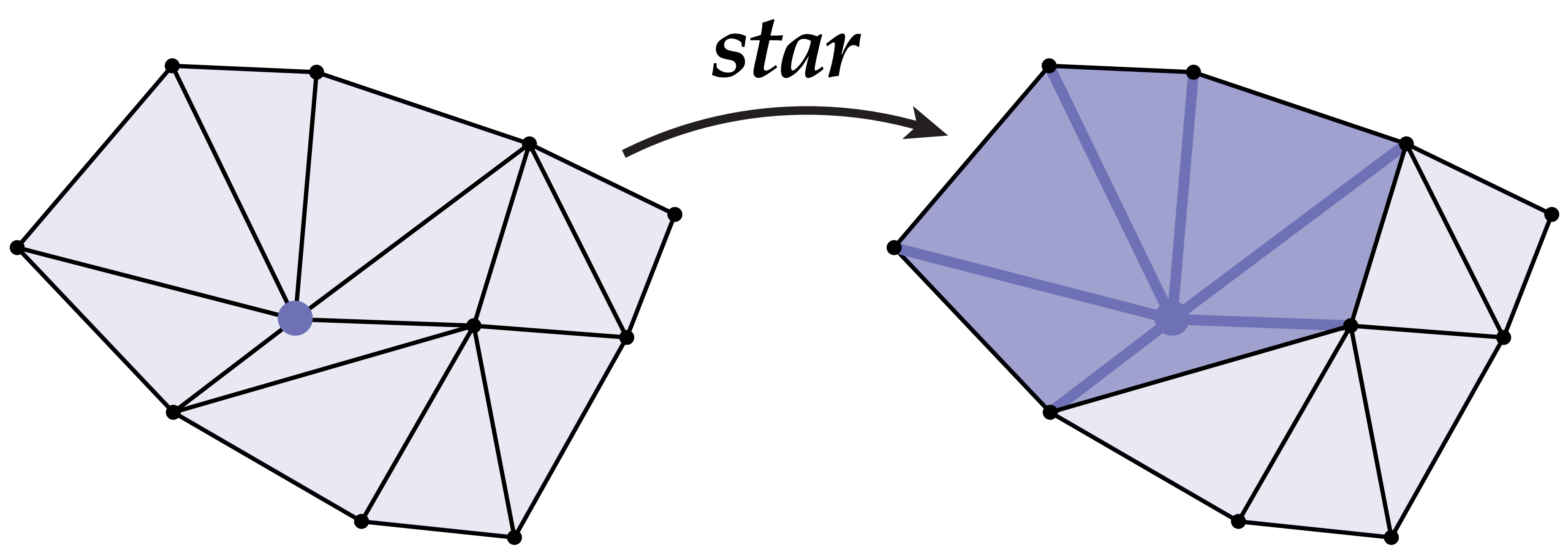
Zhu et al, *"Codimensional Surface Tension Flow on Simplicial Complexes"*

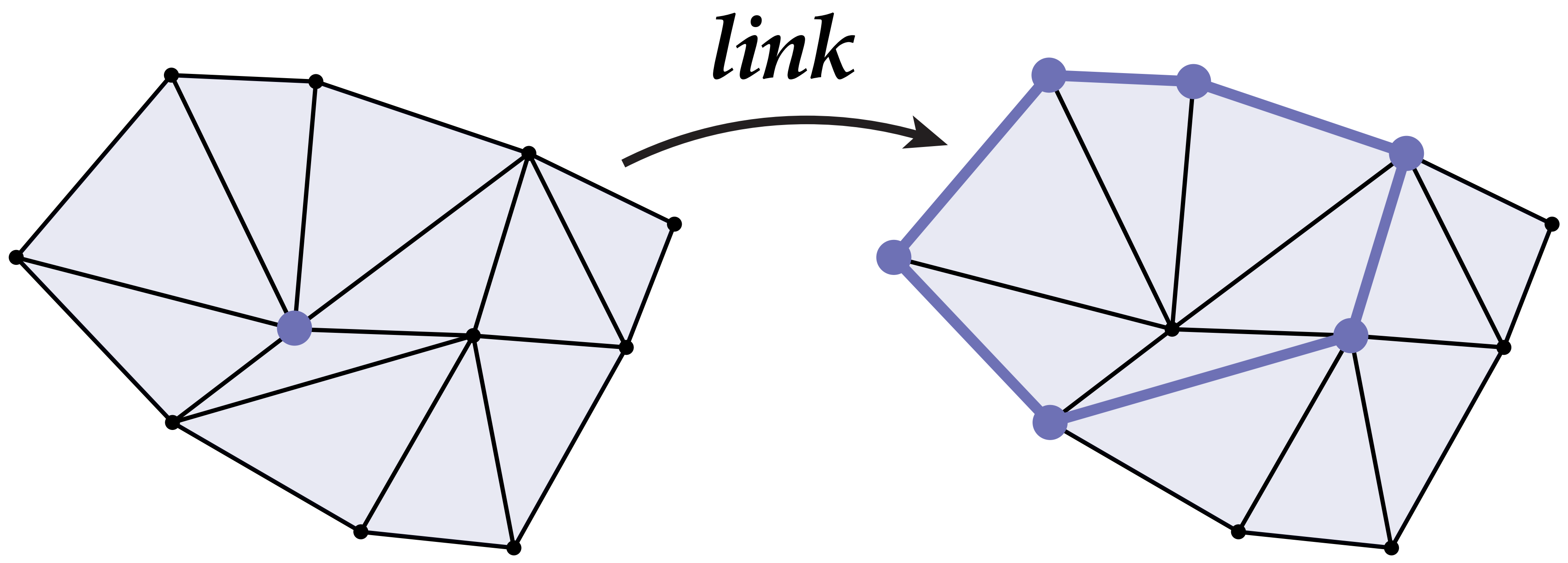
https://vimeo.com/94716977

Star, Link, and Closure

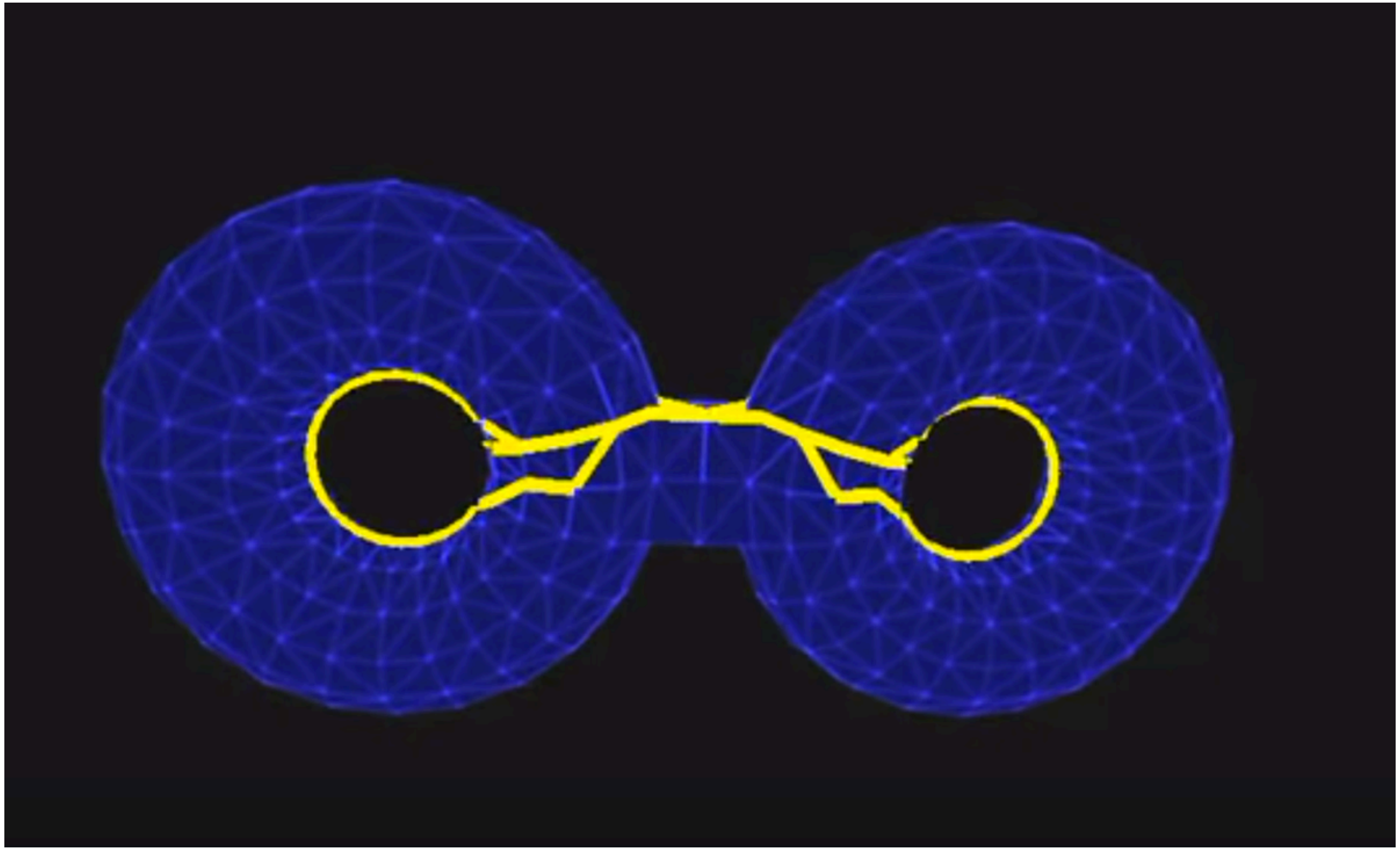
- **Closure:** smallest simplicial complex containing a given set of simplices
- **Star:** union of simplices containing a given subset of simplices
- **Link:** closure of the star minus the star of the closure







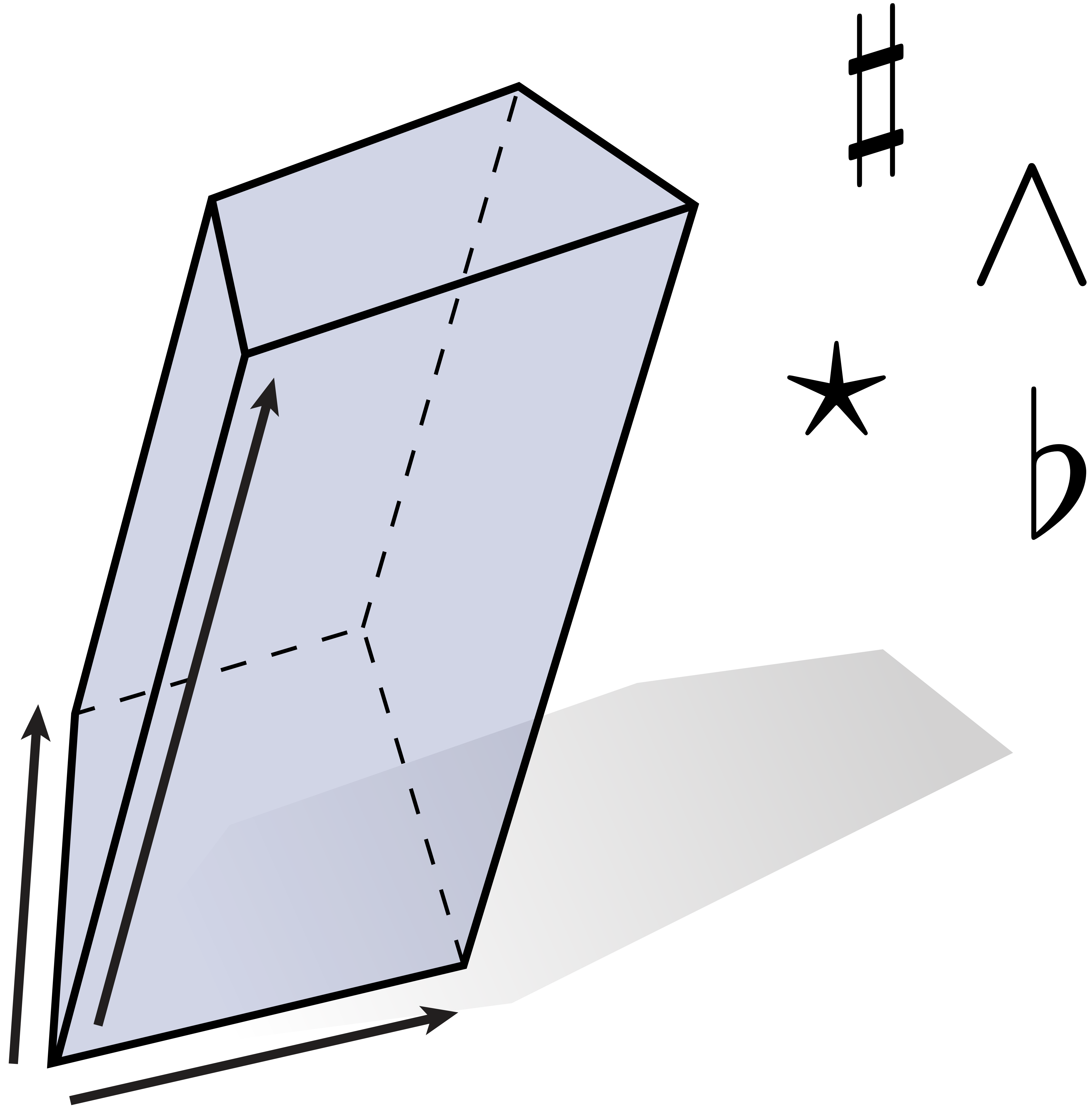
Motivation: Simplicial Homology

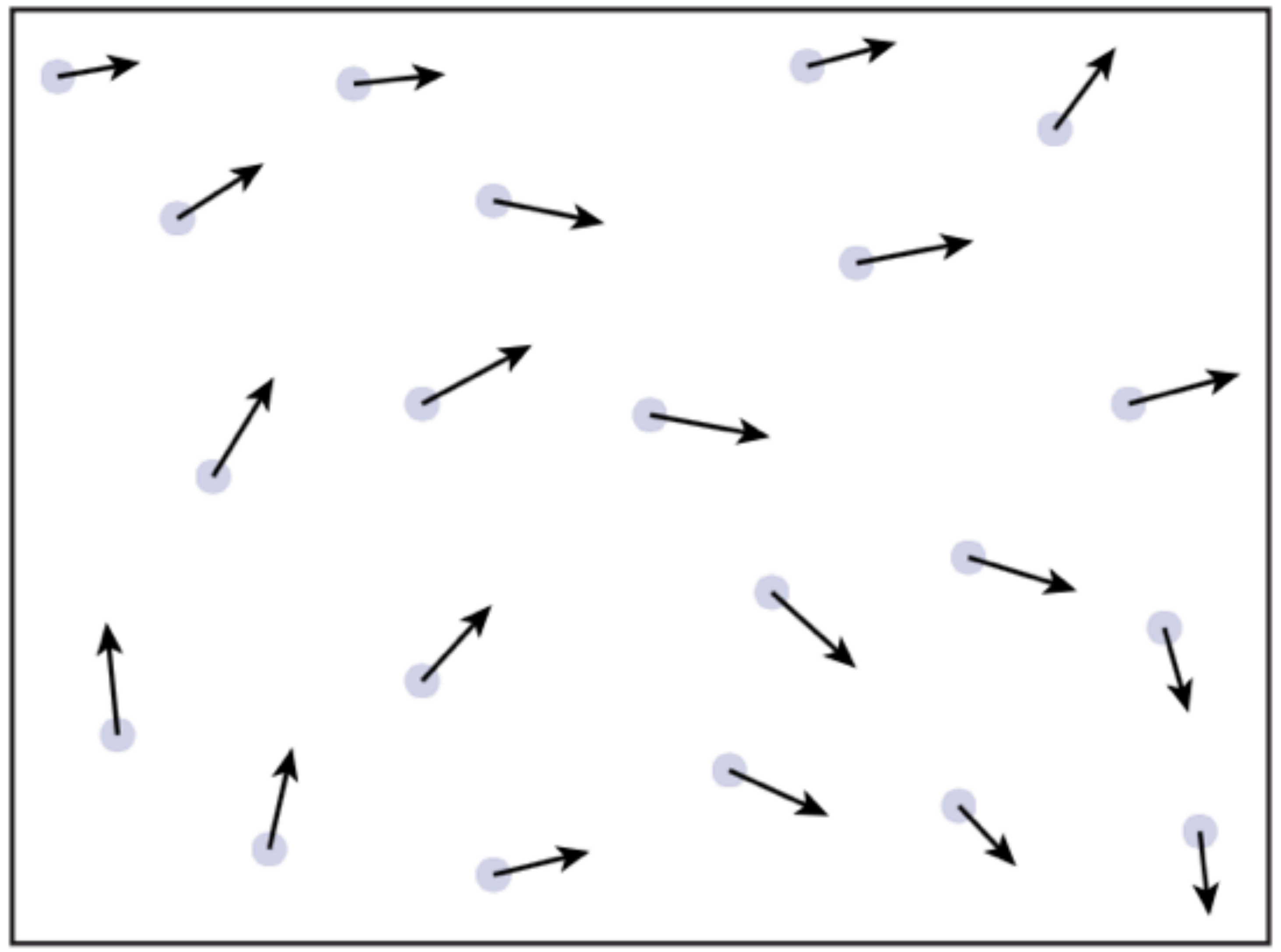


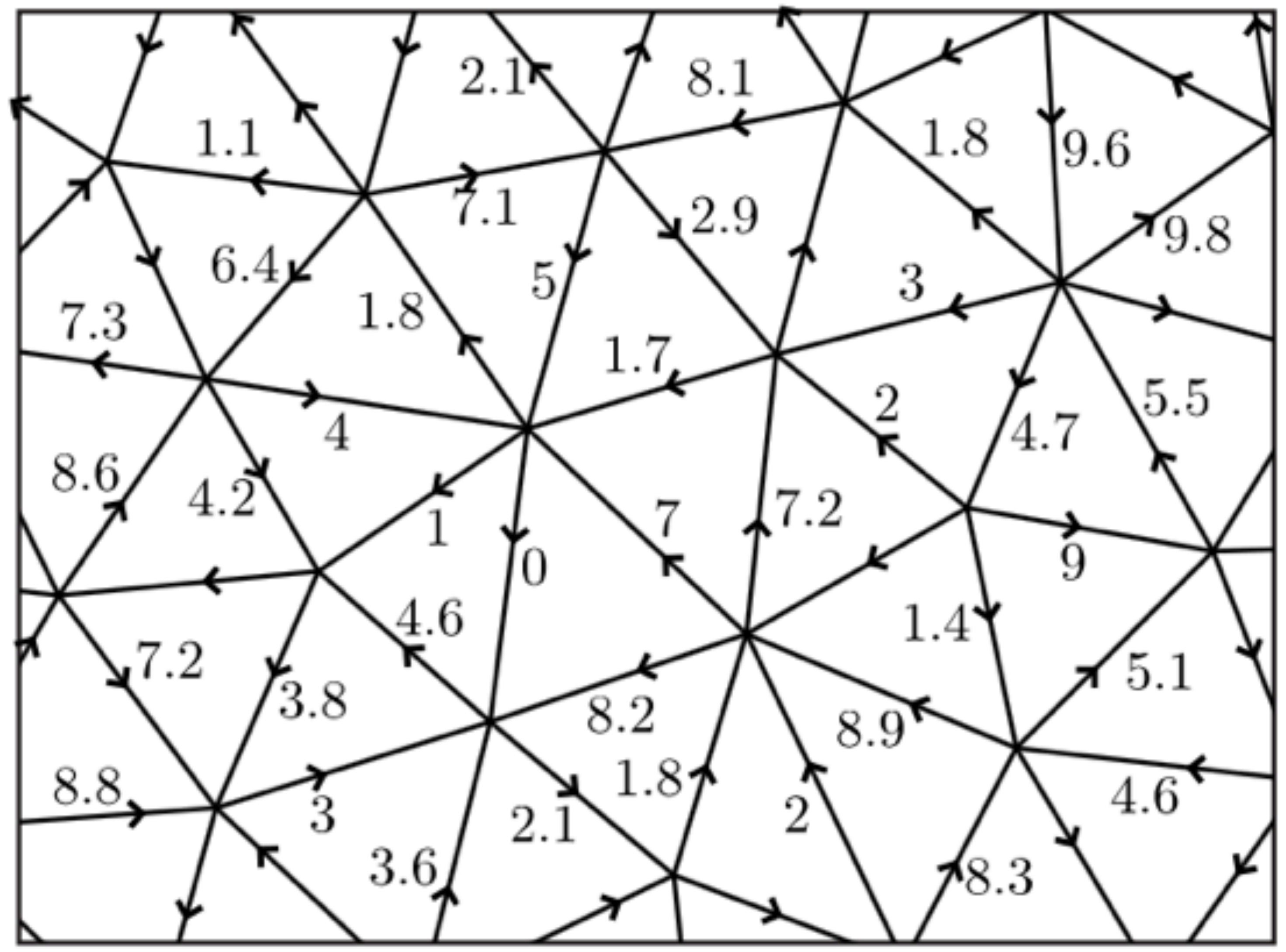
Erickson & Whittlesey, *"Greedy Optimal Homotopy and Homology Generators"*

https://www.youtube.com/watch?v=5ccXcEMFLbk

Motivation: Discrete Differential Forms







Topological Space

Definition. For any set X, a *topology* τ is a collection of subsets of X satisfying the following properties:

- The empty set $\emptyset$ and X itself are in τ.
- Any union of subsets from τ is also in τ.
- Any intersection of *finitely many* subsets from τ is in τ.

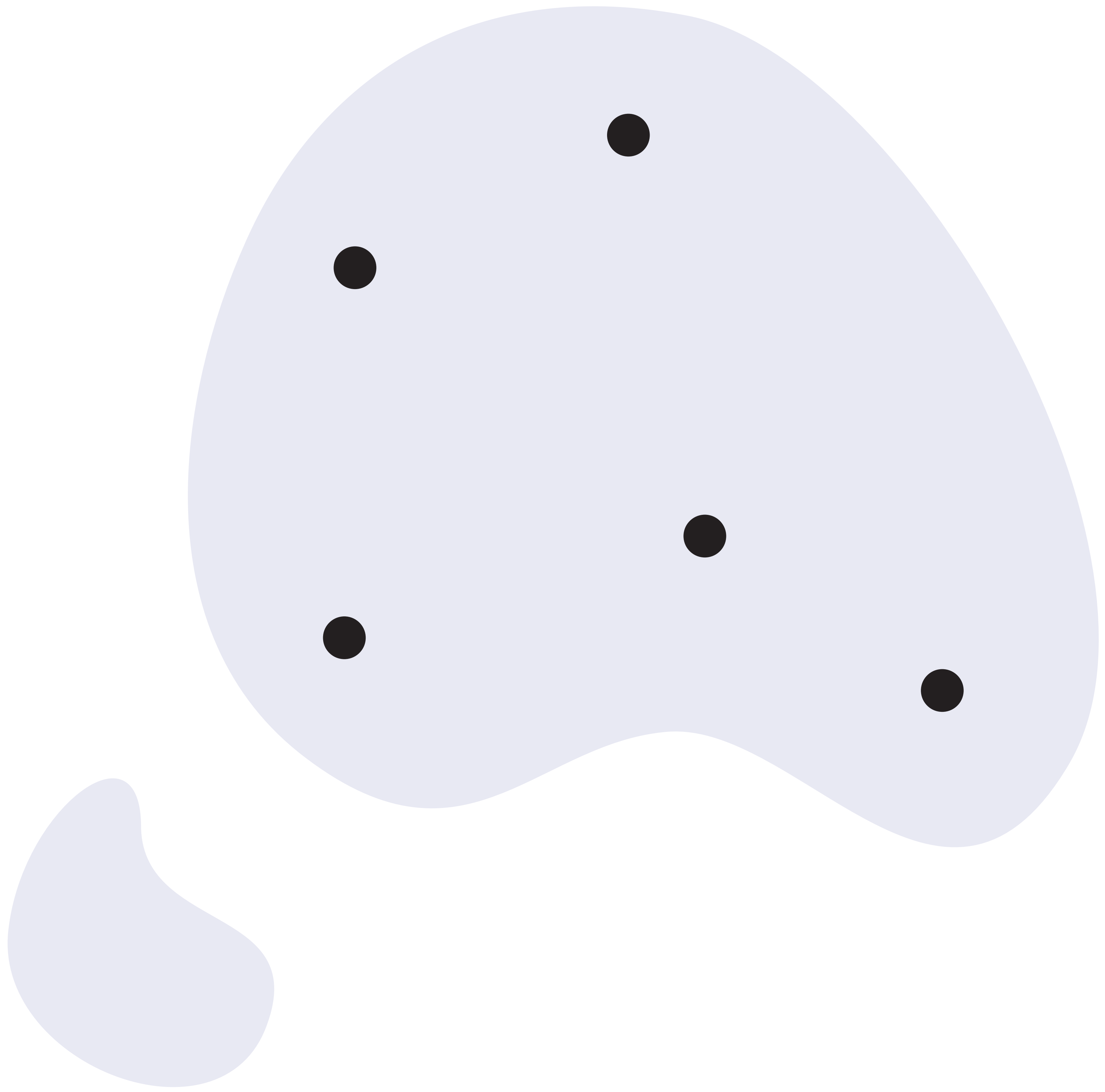
A *topological space* is a tuple (X, τ) where τ is a topology on X.

We refer to any element of τ as a *τ-open set* (or just *open set* when context is clear.) A set is *(τ-)closed* if it is the complement of a (τ-)open set.

Example—Trivial Topology

Smallest or *coarsest* possible topology:

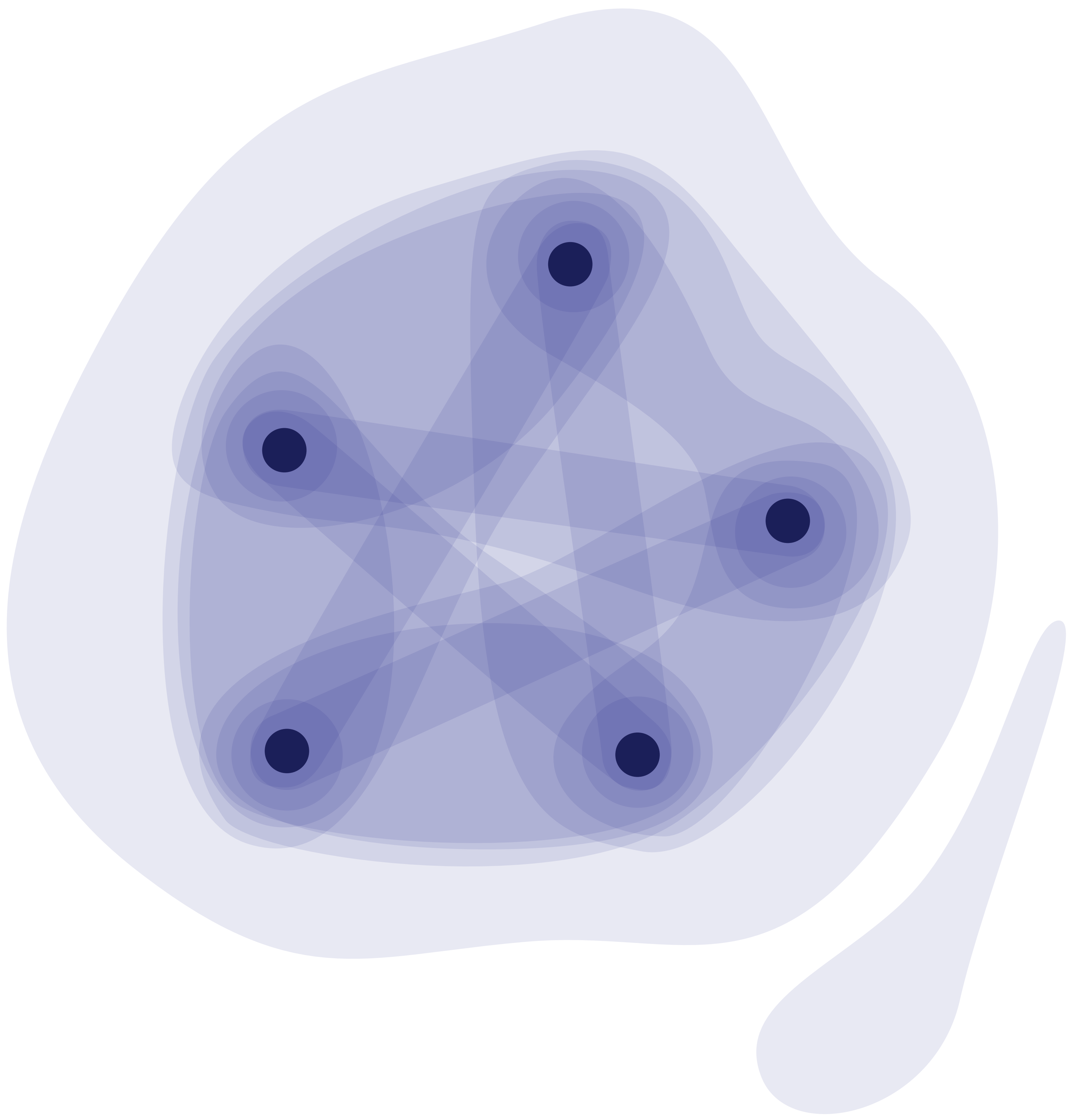
$$\tau = \{X, \emptyset\}$$



Example—Discrete Topology

Largest or *finest* possible topology:

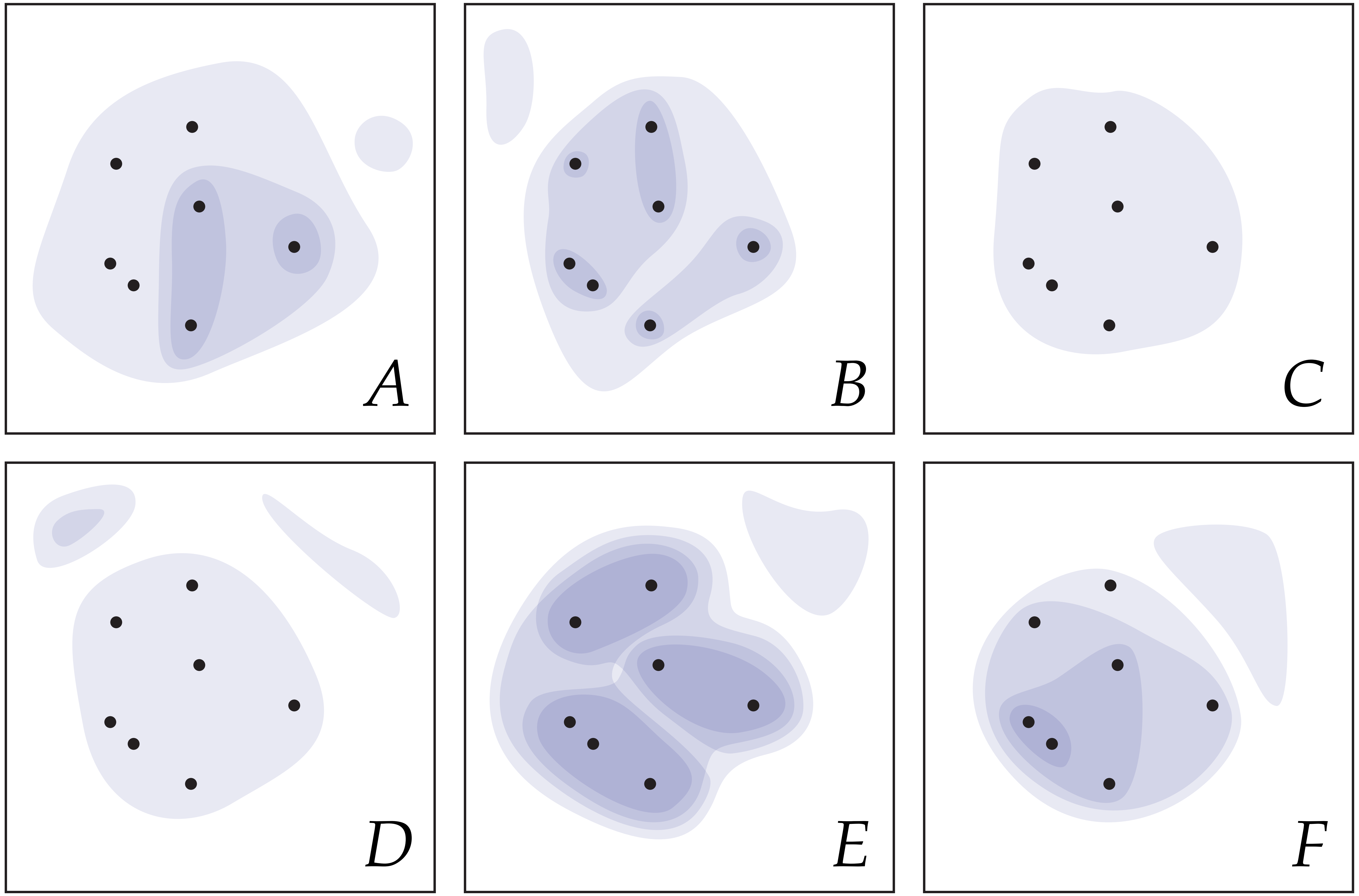
$$\tau = \mathcal{P}(X)$$



($\mathcal{P}$ is the *power set*, i.e., all possible subsets)

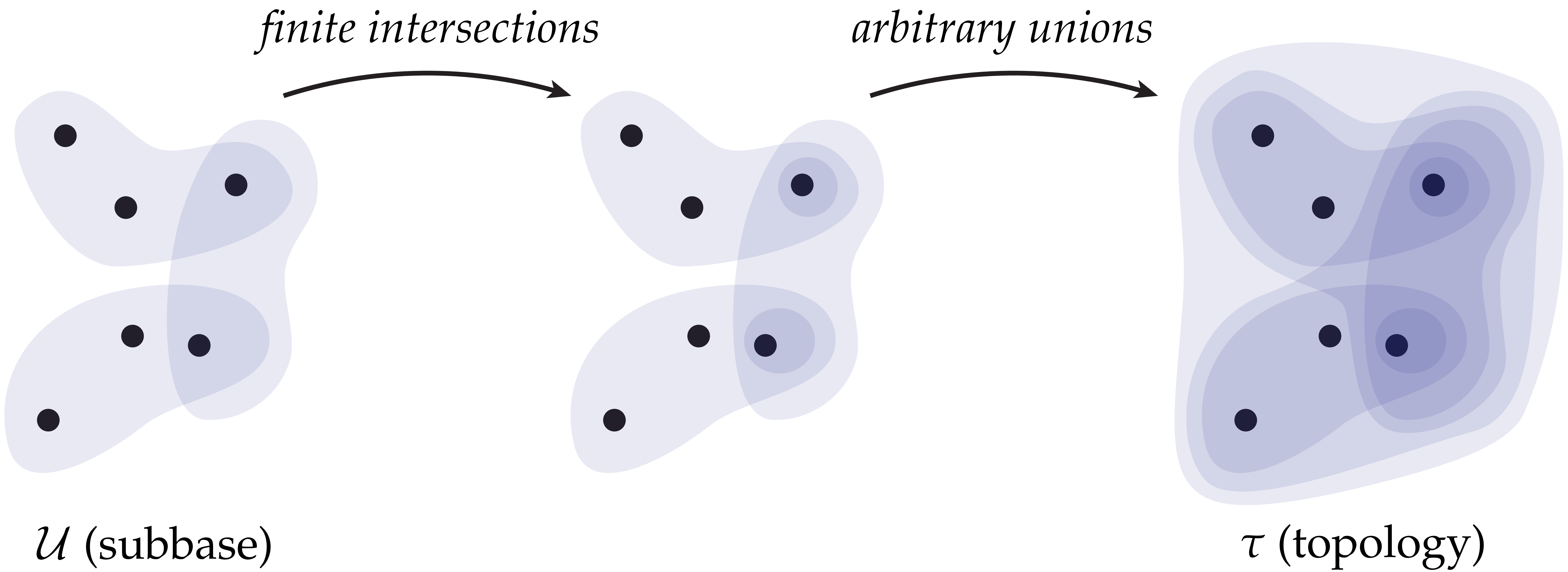
Topological Space — Visualized

Which of the following collections defines a *topology*?



Subbase

Definition. Let τ be a topology on a set X. A collection $\mathcal{U}$ of subsets of X is a *subbase* for τ if τ is the smallest topology containing $\mathcal{U}$. We say that $\mathcal{U}$ *generates* τ.

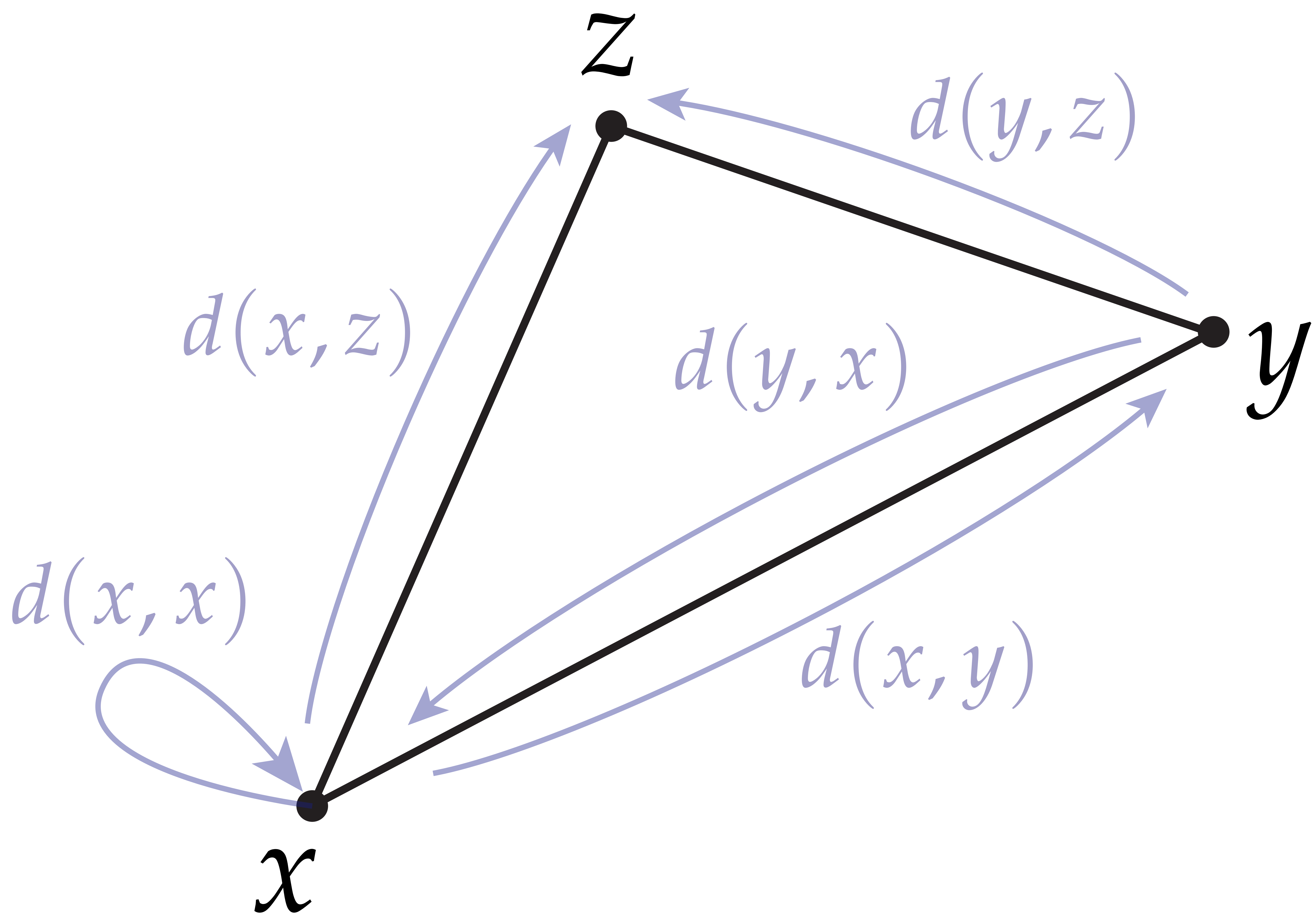


Metric Space

Definition. A *metric* on a set X is any function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

- $d(x, y) = d(y, x)$ (Symmetry)
- $d(x, y) \geq 0$ (Nonnegativity)
- $d(x, y) = 0 \iff x = y$ (Nondegeneracy)
- $d(x, y) + d(y, z) \geq d(x, z)$ (Triangle Inequality)

Metric Space — Visualized



Metric Space — Examples

Example. The *discrete metric* on any set X, given by $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$, otherwise.

Example. The *Euclidean metric* on $\mathbb{R}^n$, given by $d(x, y) := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Example. The *Manhattan distance* on $\mathbb{R}^n$, given by $d(x, y) := \sum_{i=1}^{n} |x_i - y_i|$.

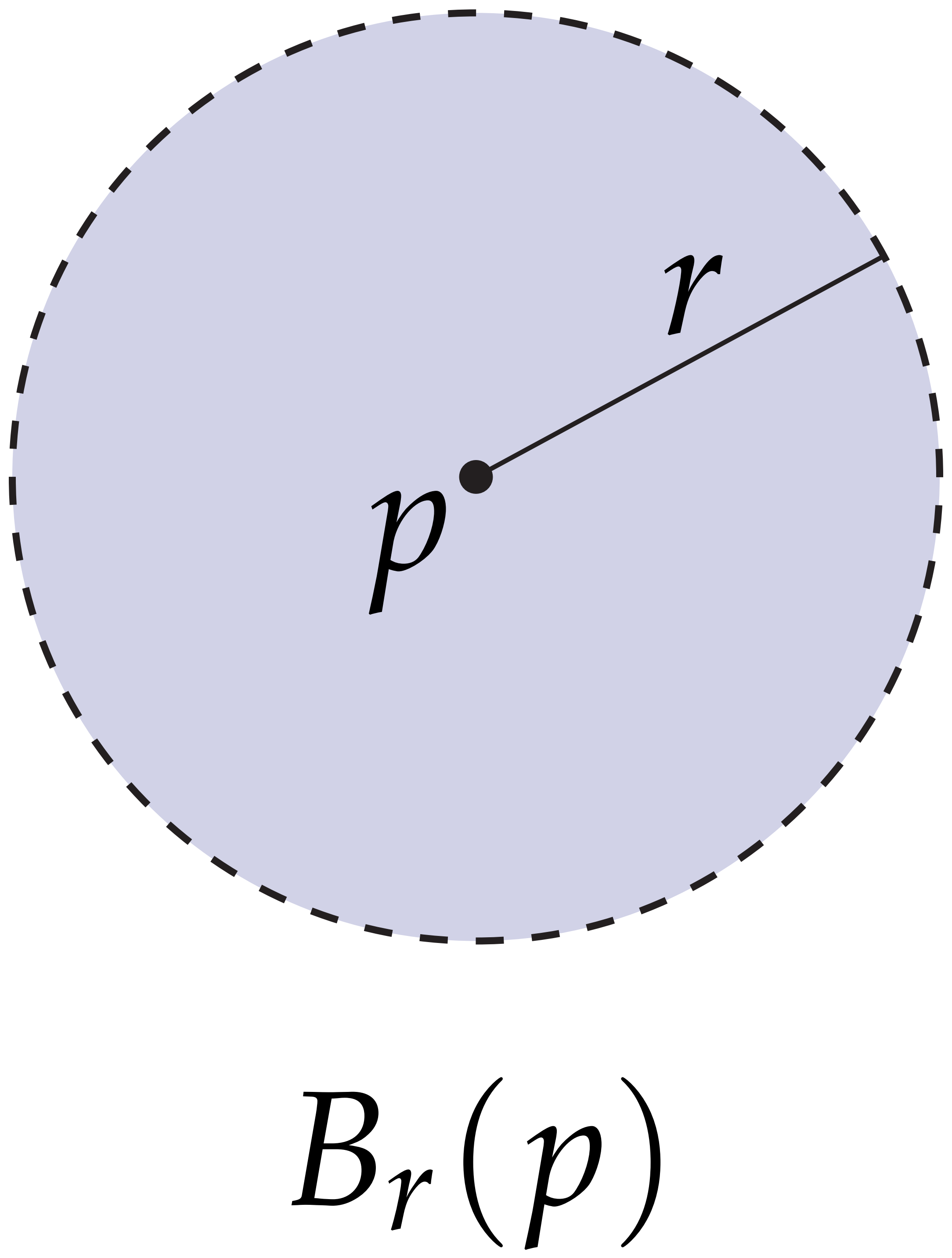
Example. The L^2 *distance* on functions $f : [0, 1] \to \mathbb{R}$, given by

$$d(f, g) := \left(\int_0^1 (f(x) - g(x))^2 \, dx\right)^{1/2}.$$

Open Ball

Definition. Given a metric d on a set X, the *open ball* of radius r around a point $p \in X$ is the subset

$$B_r(p) := \{x \in X \,|\, d(x,p) < r\}$$

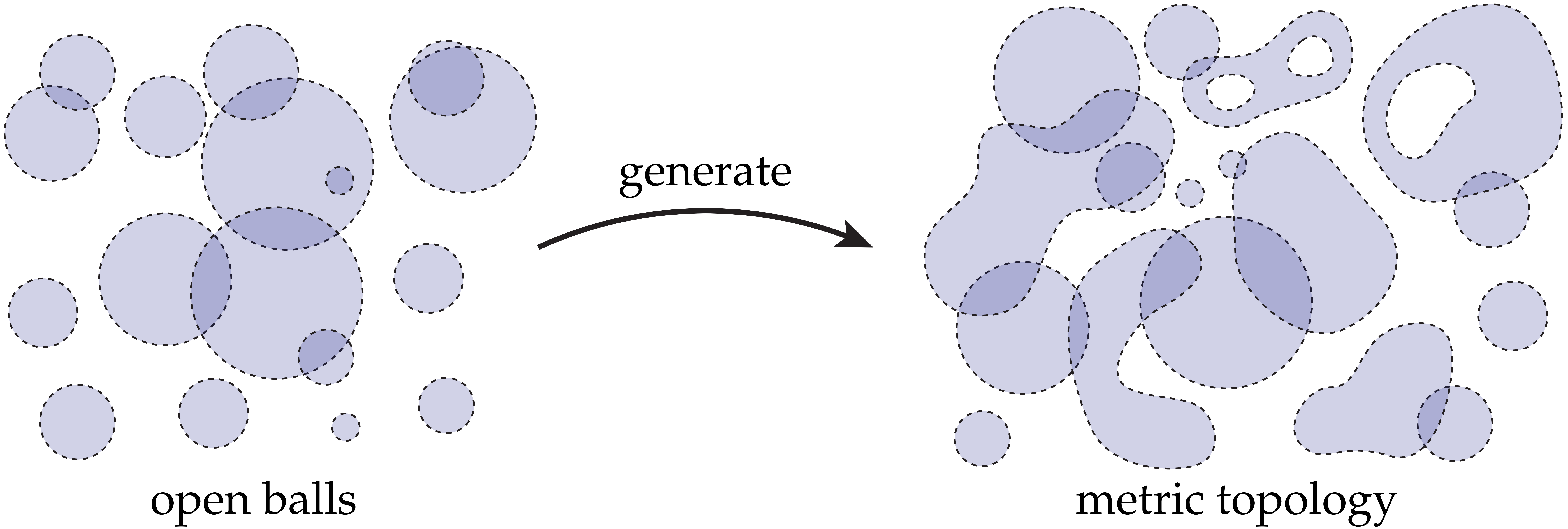


Metric Topology

Definition. The *metric topology* τ on any metric space (X, d) is the topology generated by the subbase

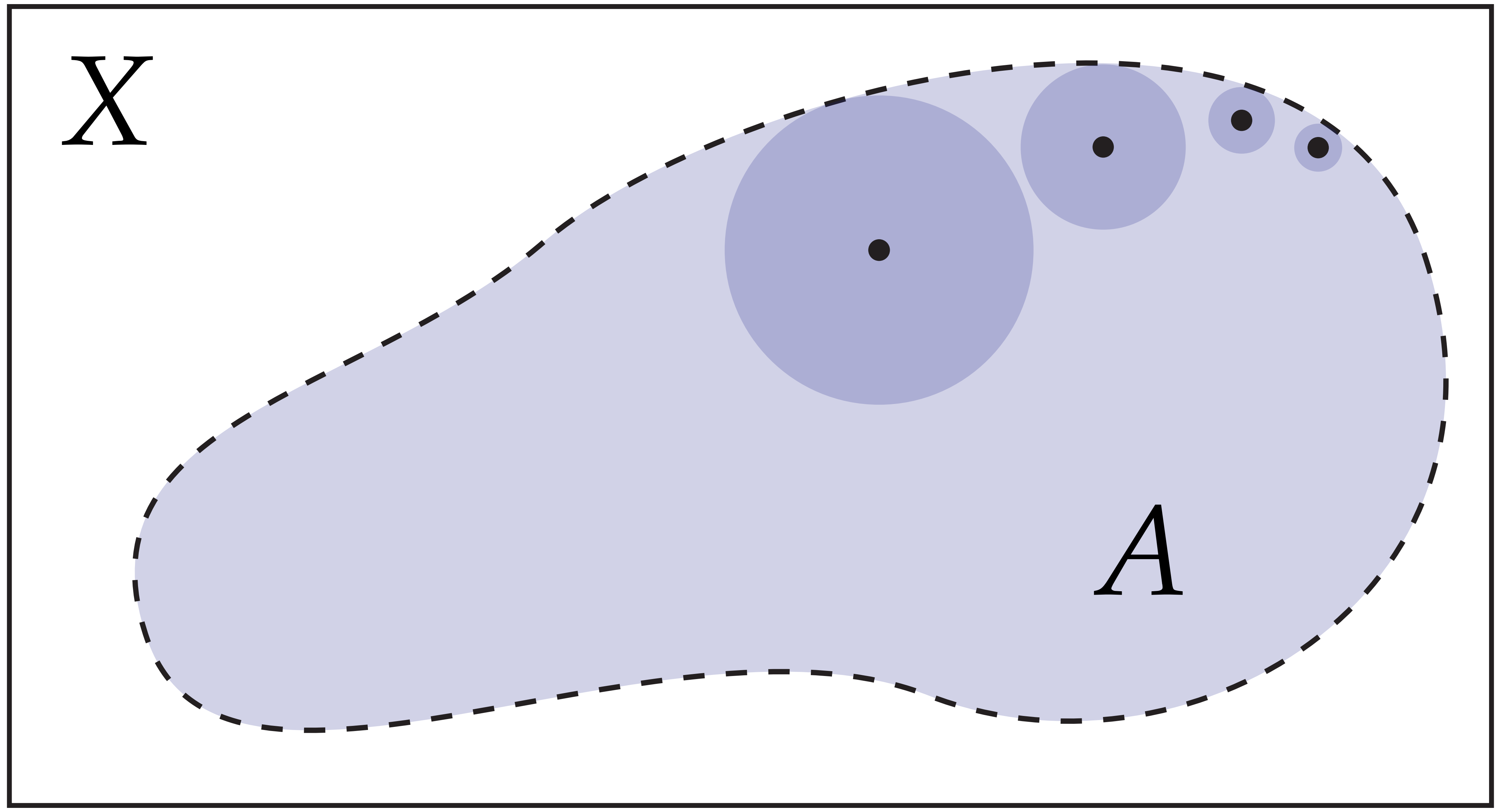
$$\mathcal{U} := \{B_r(p)\,|\,p \in X,\ r \geq 0\},$$

i.e., the collection of all open balls.

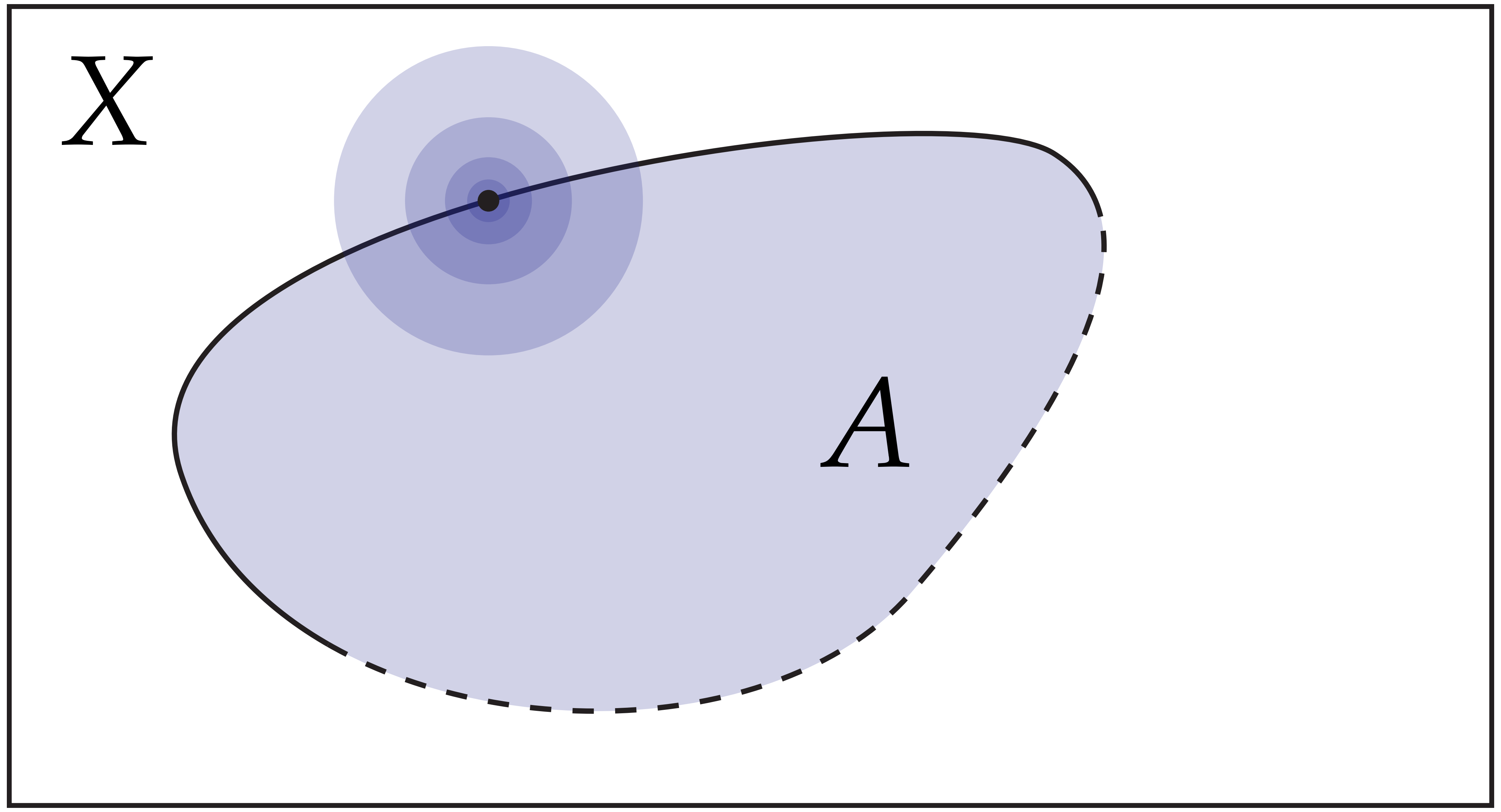


Open Sets—Metric Characterization

Fact. Let (X, d) be a metric space. A subset $A \subset X$ is open with respect to the metric topology on X if and only if for all $p \in A$ there exists an $\epsilon > 0$ such that $B_\epsilon(p)$ is contained in A.



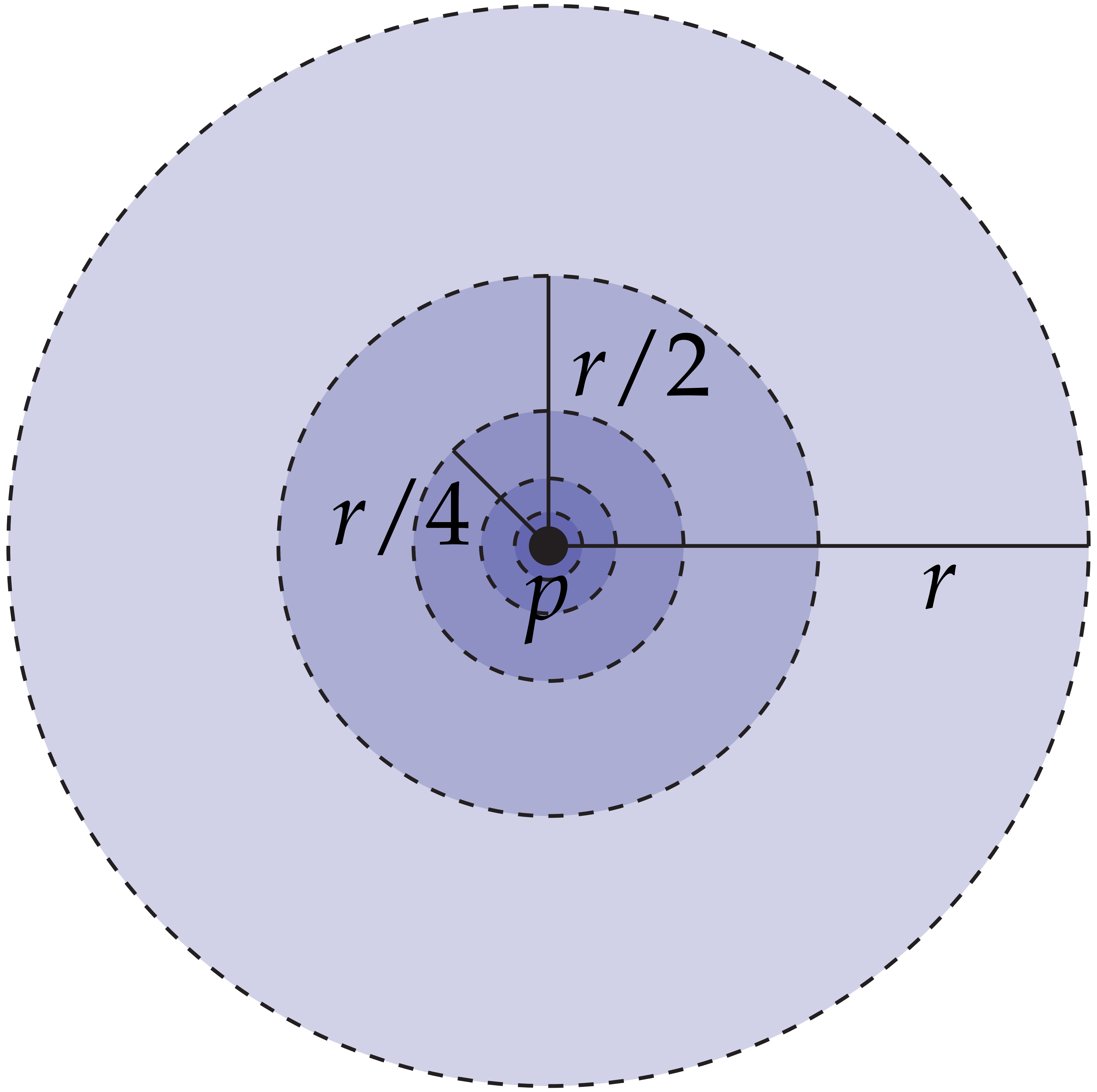
open



not open

Why Only ***Finite*** *Intersections?*

Suppose we allowed *arbitrary* intersections...



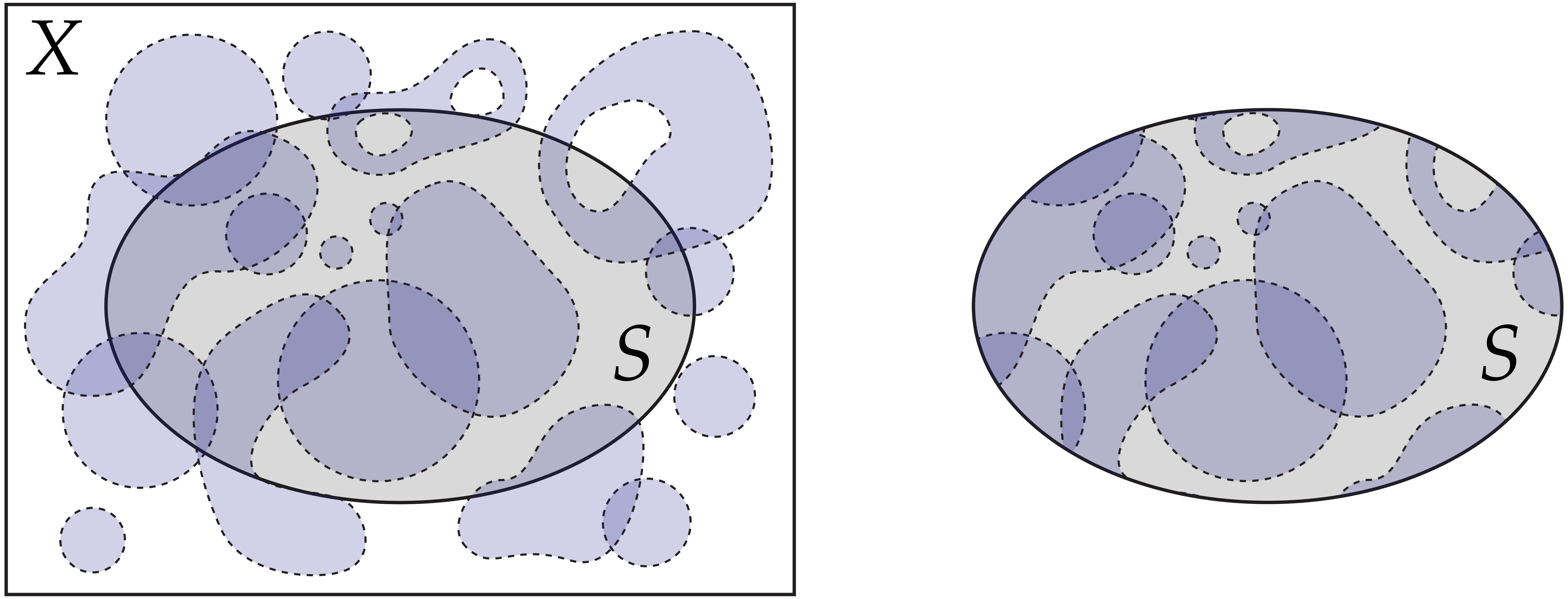
...then metric topology would always be the *discrete* topology!

Subspace Topology

Definition. If (X, τ) is a topological space and S is a subset of X, then the *subspace topology* on S is given by

$$\tau_S := \{S \cap U \,|\, U \in \tau\},$$

i.e., just the restriction of open sets on X to S.



Equivalence Relation

Definition. For any set X, consider a collection R of ordered pairs $(x, y) \in X \times X$, and let $x \sim y$ be shorthand for $(x, y) \in R$. Then R is an *equivalence relation* on X if for all $x, y, z \in X$

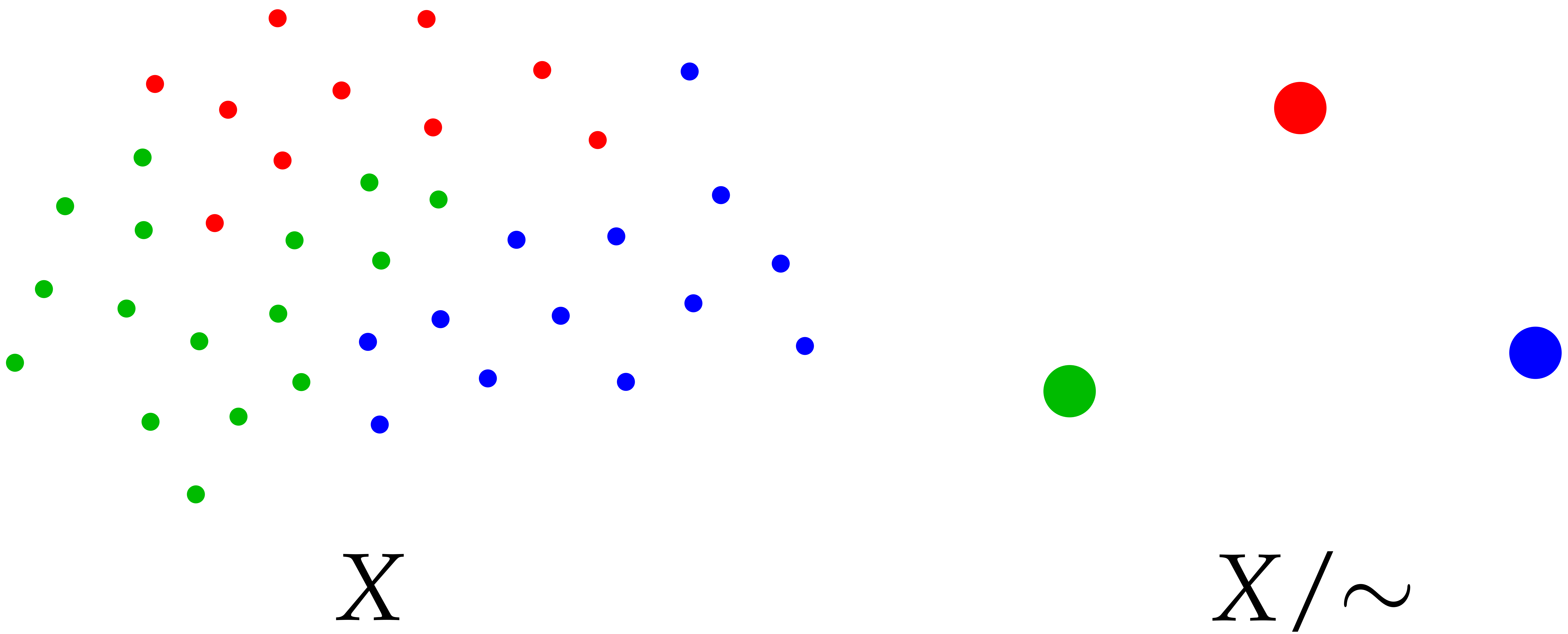
- $x \sim x$ (Reflexivity)
- $x \sim y \iff y \sim x$ (Symmetry)
- $x \sim y$ and $y \sim z \Rightarrow x \sim z$ (Transitivity)

The *equivalence class* associated with any element $x \in X$ is the subset

$$[x] := \{y \in X \,|x \sim y\}.$$

The *quotient space* $X/\sim$ of all equivalence classes *partitions* X, *i.e.*, every element $x \in X$ belongs to exactly one equivalence class.

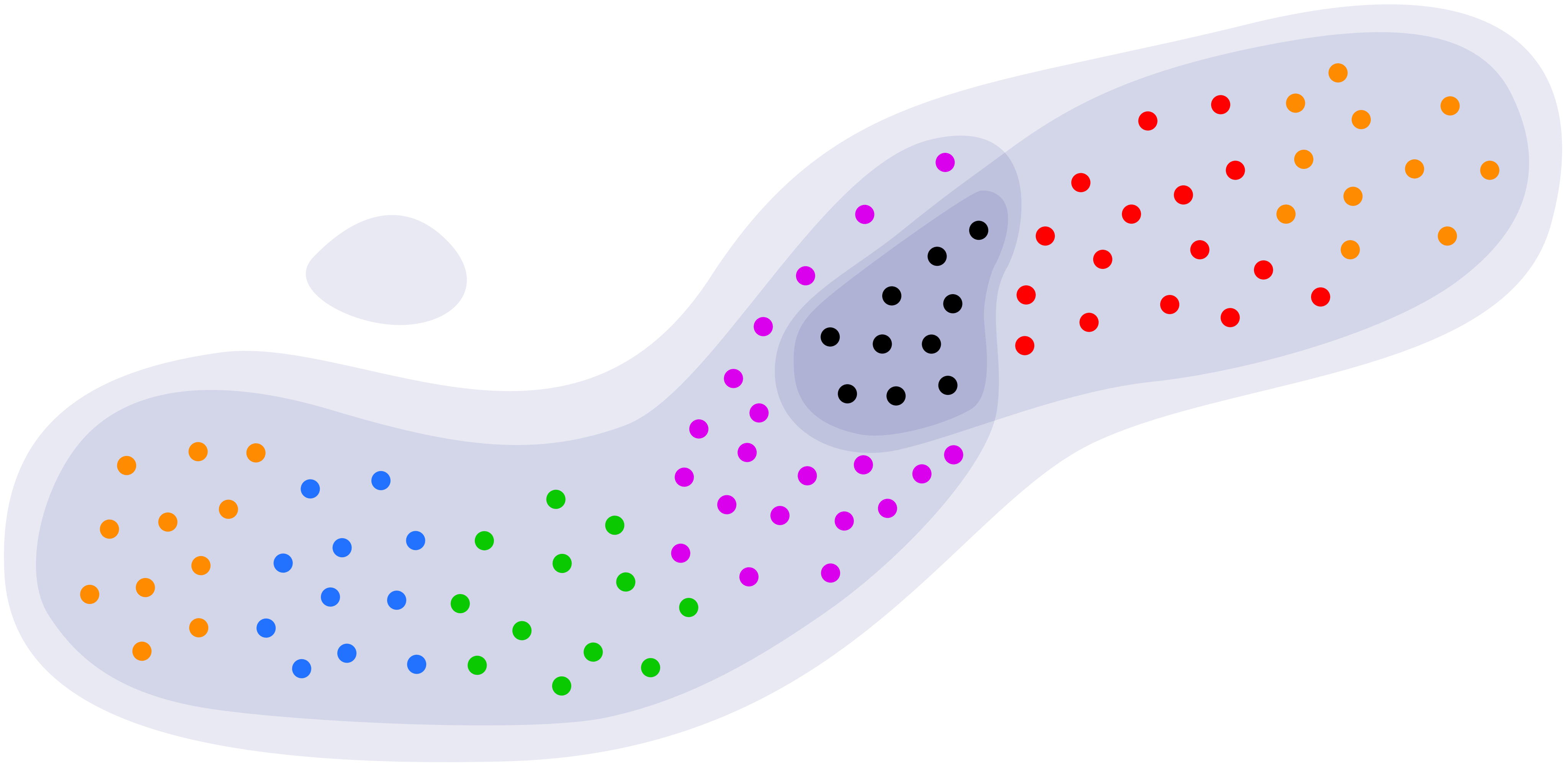
Equivalence Relation, Visualized



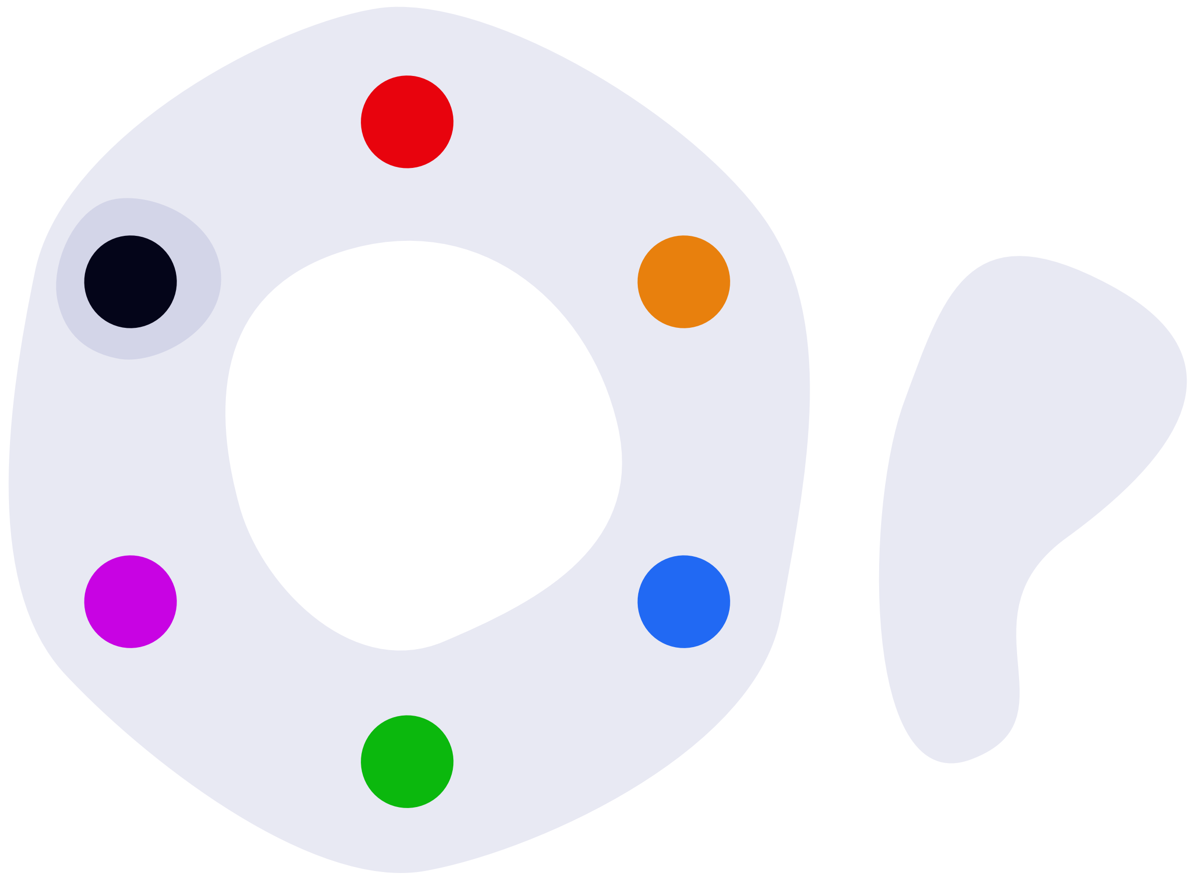
Quotient Topology

Definition. If (X, τ) is a topological space and $\sim$ is an equivalence relation on X, then the *quotient topology* on $X/\!\sim$ is the collection of subsets

$$\tilde{\tau} := \left\{ A \subset (X/\!\sim) \;\middle|\; \cup_{[a] \in A}[a] \in \tau \right\}$$



(X, τ)



$(X/\!\sim, \tilde{\tau})$

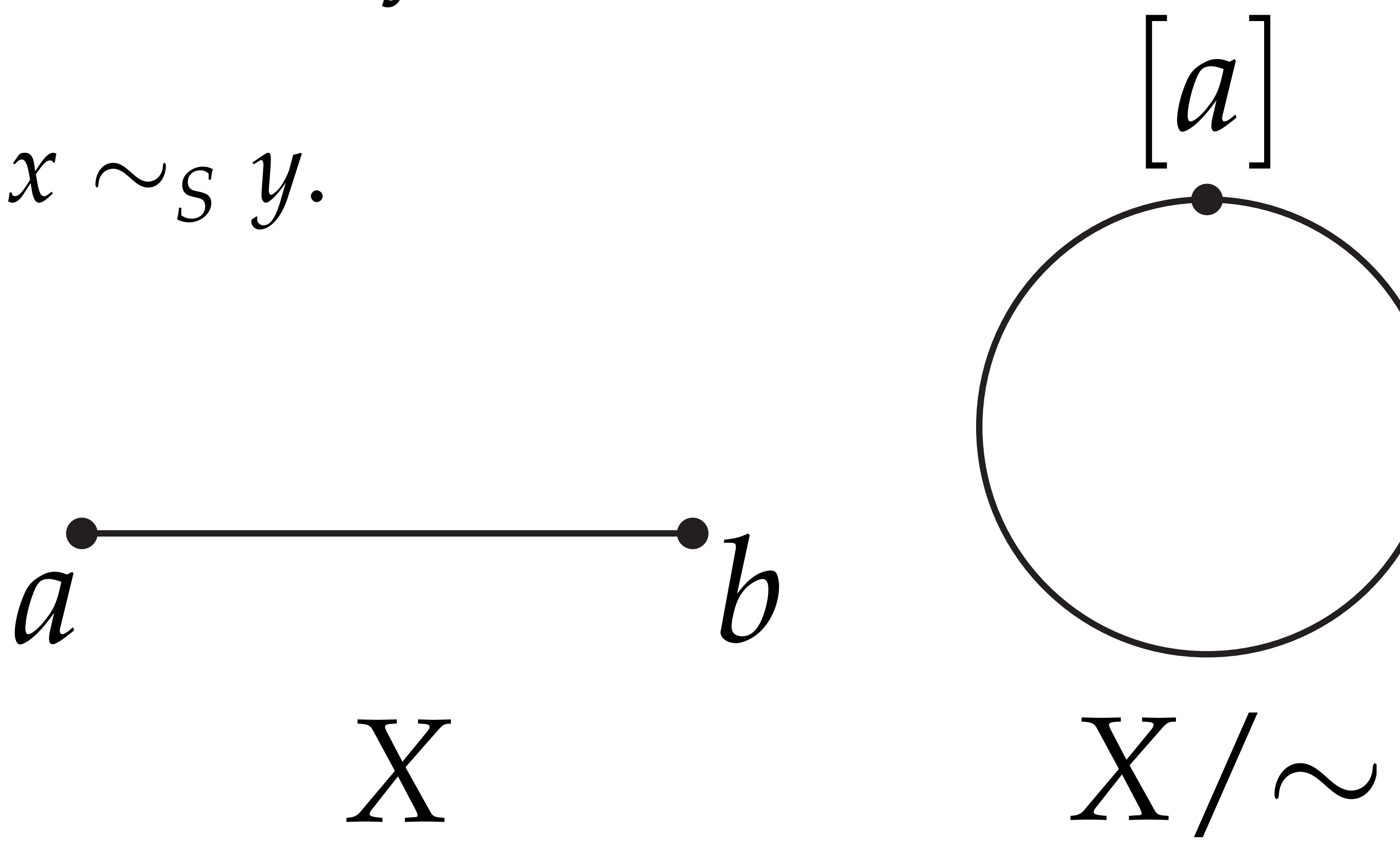
Quotient Topology—Gluing

Definition. If (X, τ) is a topological space and a, b are points in X, then *identifying* or *gluing* these two points means we construct the quotient topology $X/\sim$ with respect to the equivalence relation

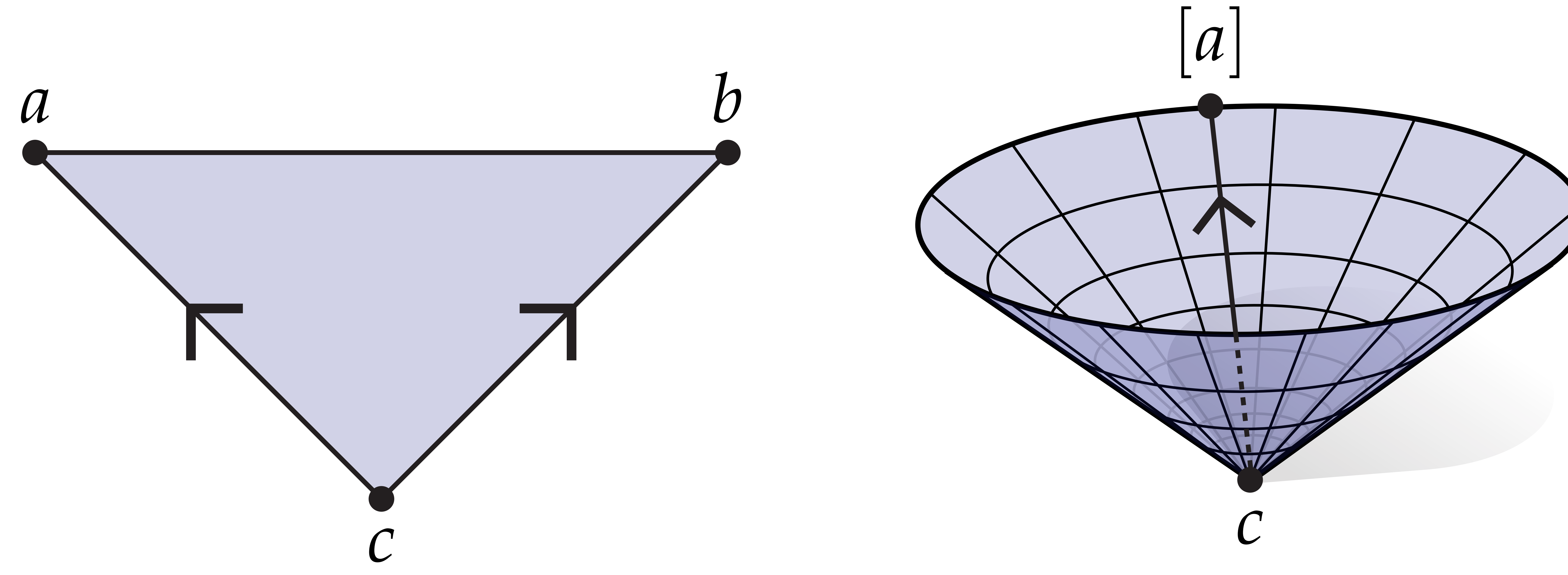
$$x \sim y \iff x = y \quad \text{or} \quad \{x, y\} = \{a, b\}.$$

Likewise, if $\sim_S$ is an equivalence relation on a subset S of X, then *gluing along S* means that we construct the quotient topology determined by

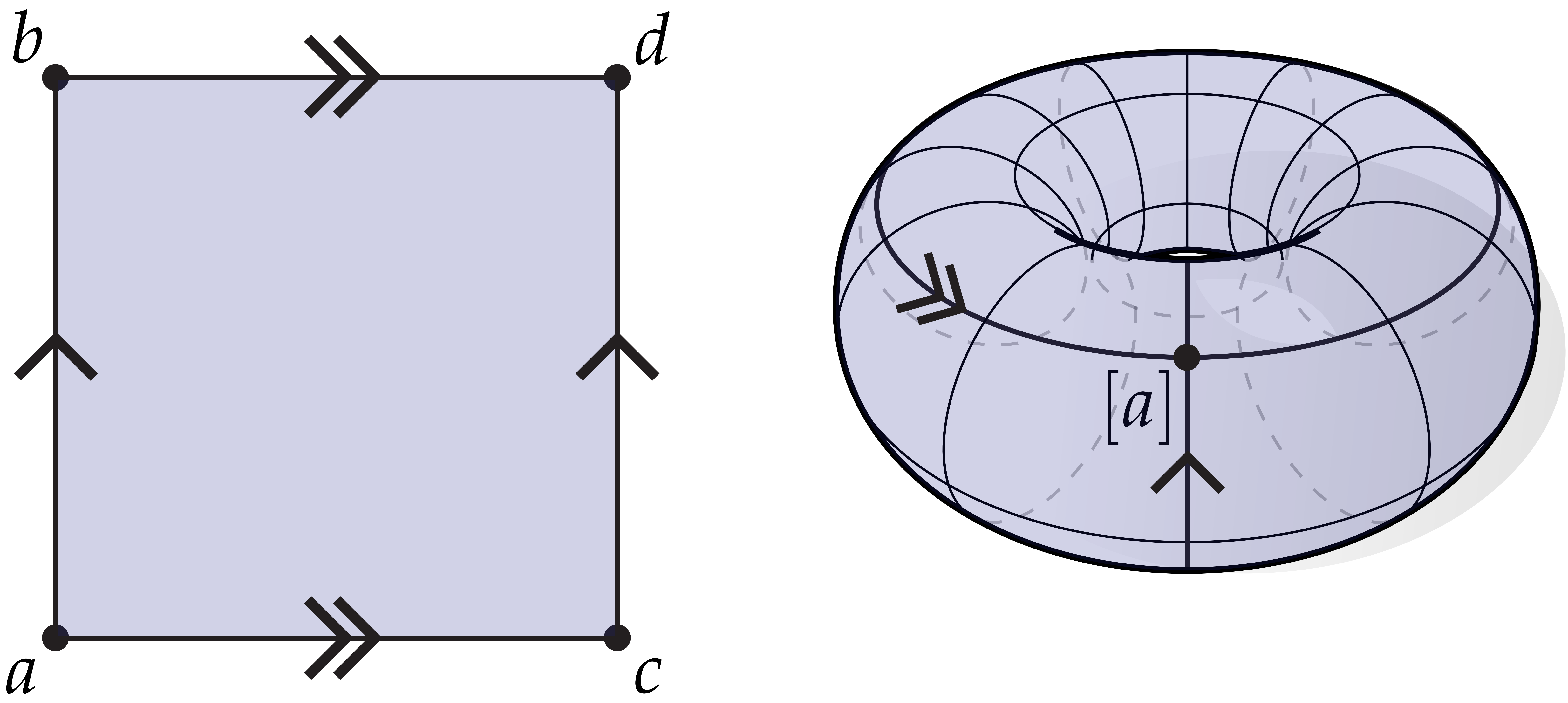
$$x \sim y \iff x = y \quad \text{or} \quad x \sim_S y.$$



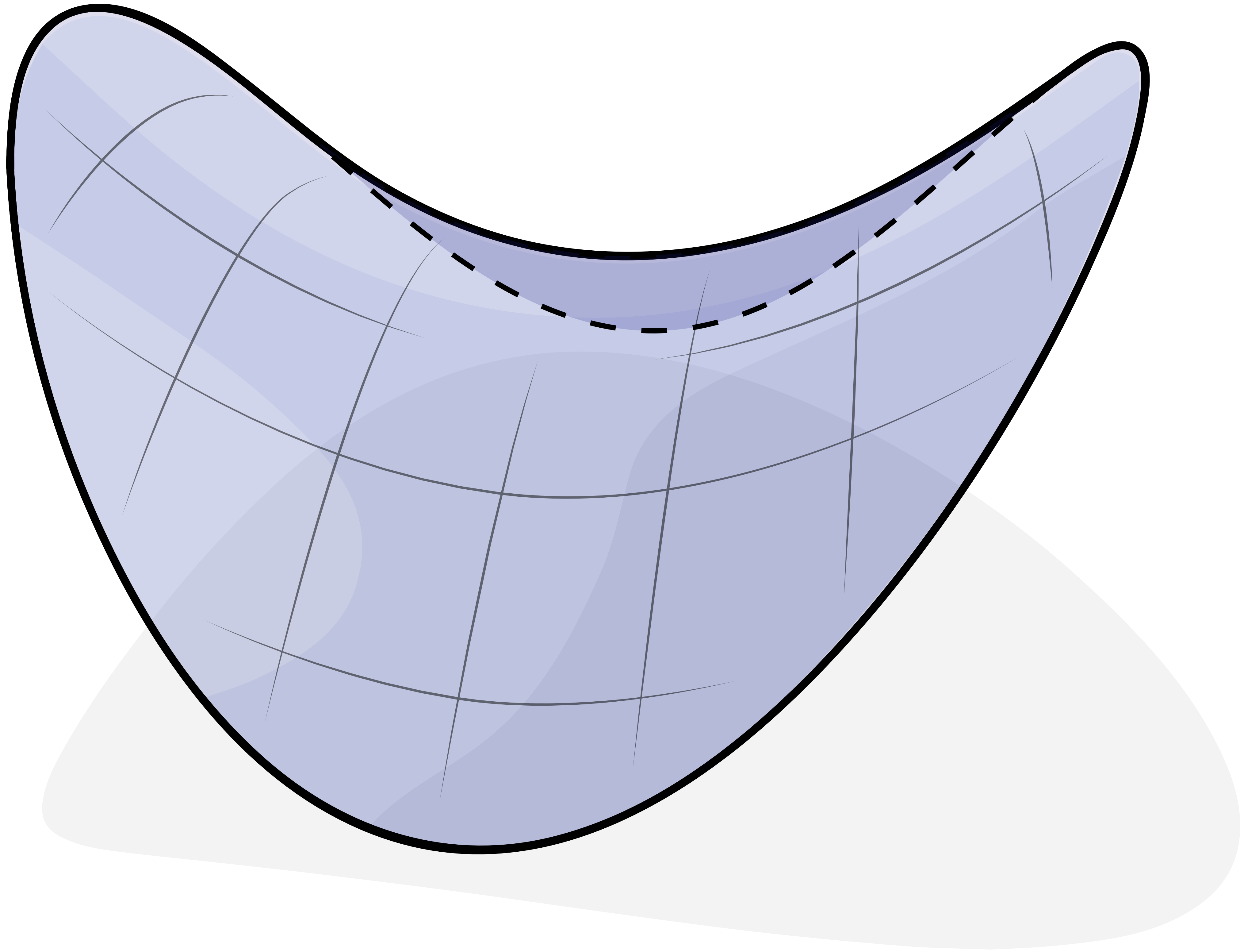
Gluing—Example

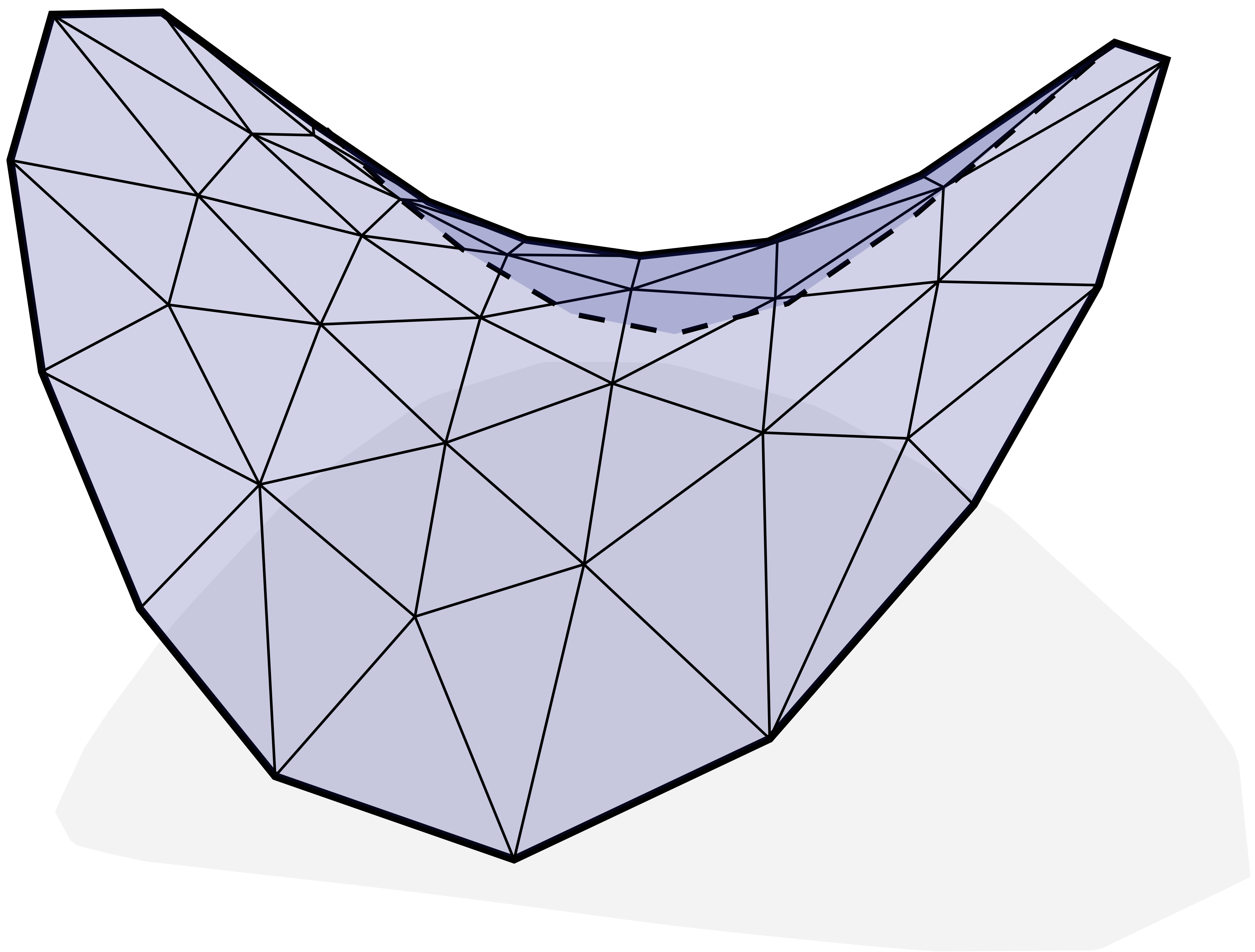


Gluing—Example



Topological Spaces—Continuous vs. "Discrete"





topological space $\longleftrightarrow$ simplicial complex

Abstract Simplicial Complex

Definition. An *abstract simplicial complex* K is a collection of finite subsets of a set X such that for each subset $\sigma \in K$, each element of $\mathcal{P}(X)$ is also in K. An element σ of K is called a *simplex*. A *k-simplex* is a simplex containing $k+1$ points; 0-, 1-, 2-, and 3-simplices are referred to as *vertices*, *edges*, *faces*, and *tetrahedra*, respectively. If $\sigma_1 \subseteq \sigma_2$, then σ_1 is a *facet* of σ_2. If $\sigma_1 \subset \sigma_2$, then σ_1 is a *proper facet* of σ_2.

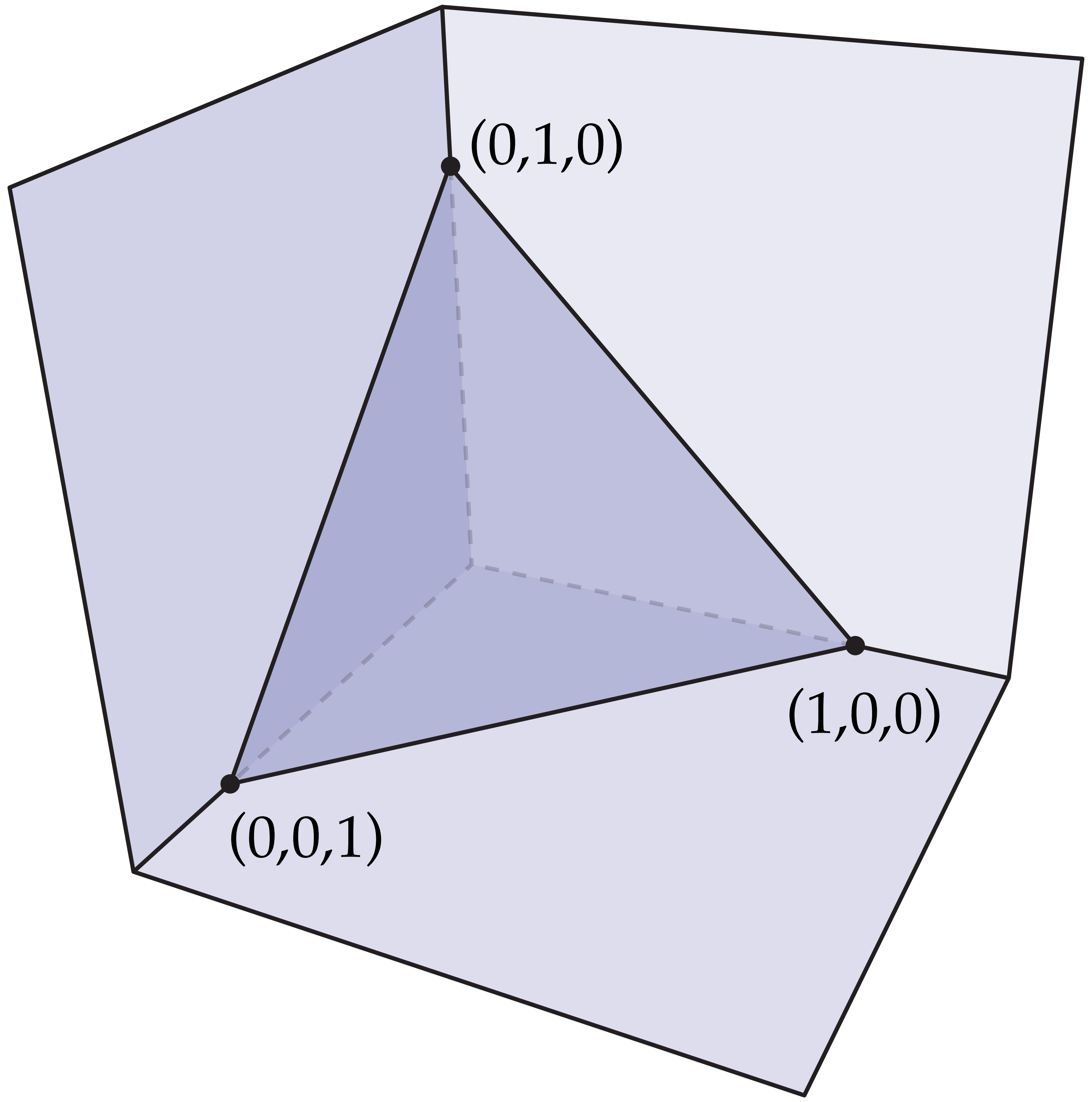
Example.

$$K = \{\{i,j,k\},\{j,k,l\},\{i,j\},\{i,k\},\{j,k\},\{j,l\},\{k,l\},\{i\},\{j\},\{k\},\{l\},\emptyset\}$$

Example. For any graph $G = (V, E)$, the union of E with all the singletons $\{v\}$, $v \in V$, plus the empty set, is a 1-simplicial complex.

Standard Simplex

Definition. Let $x_1, \ldots, x_{n+1}$ be coordinates for $\mathbb{R}^{n+1}$. The *standard n-simplex* σ_0 is the subset of $\mathbb{R}^n$ given by the intersection of the nonnegative orthant with the plane $\sum_{i=1}^{n+1} x_i = 1$. Its topology is the subspace topology induced by the usual metric topology on $\mathbb{R}^n$. For each point $p \in \sigma_0$, the coordinates $x_1, \ldots, x_{n+1}$ are its *barycentric coordinates*.

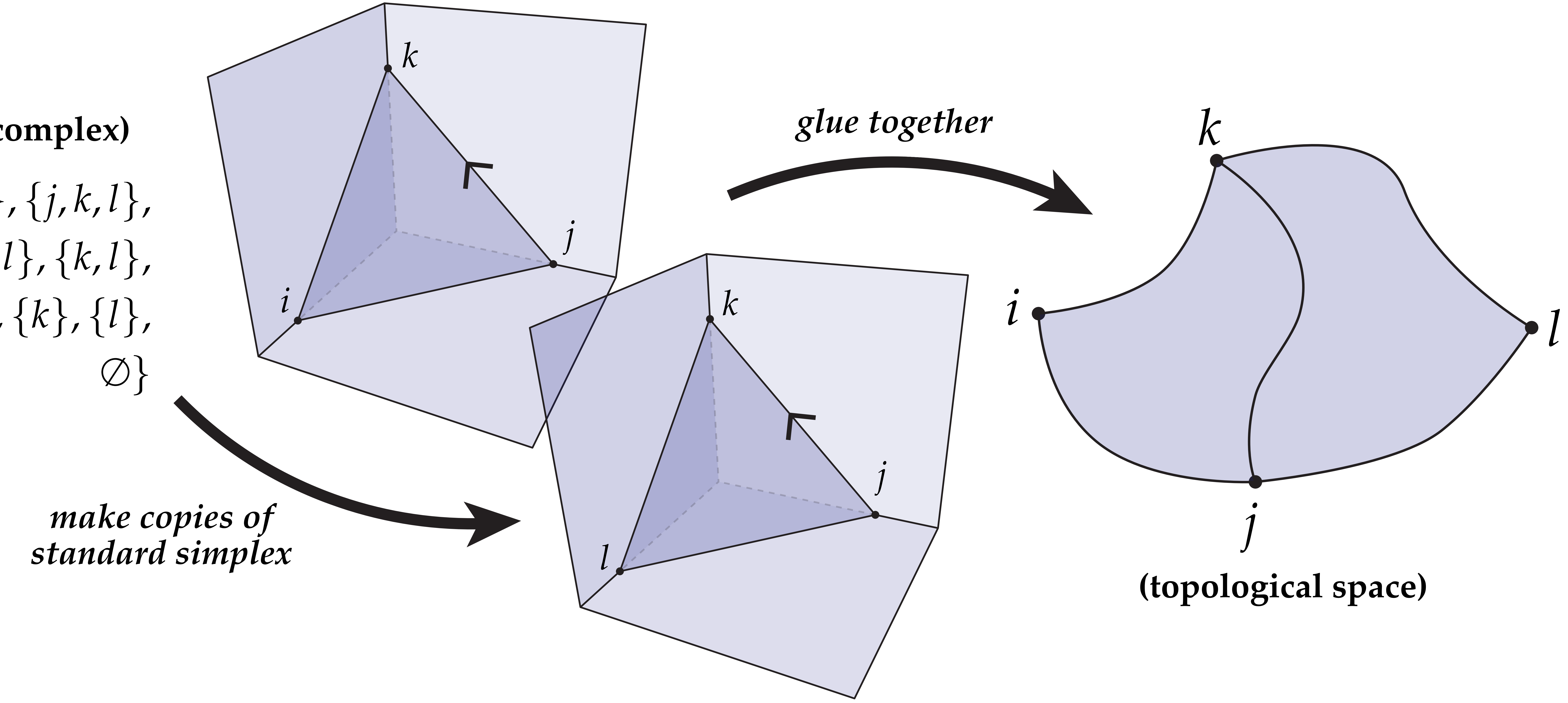


“Standard” Topology on a Simplicial Complex

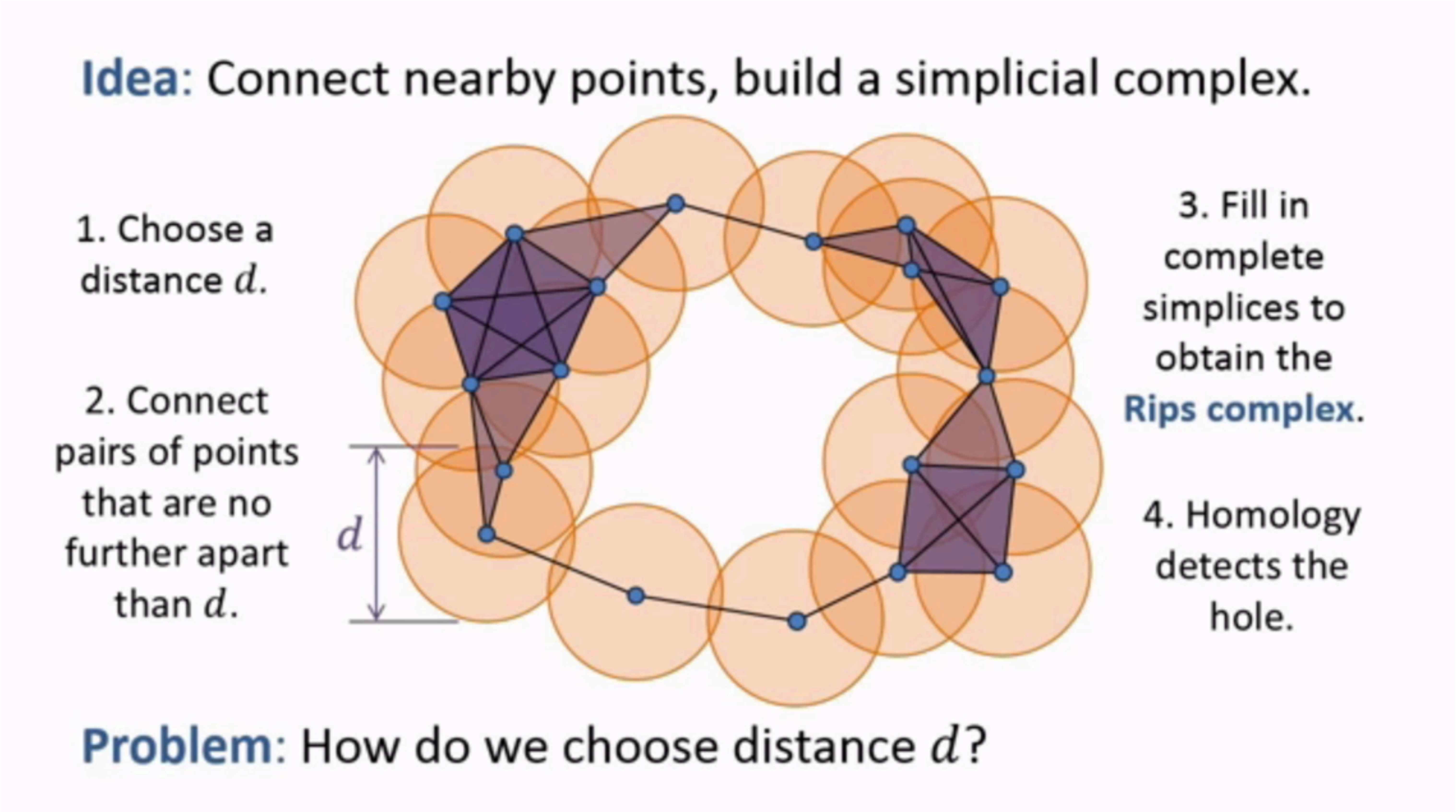
Sketch. Let K be an abstract simplicial complex. For each n-simplex $\sigma \in K$, make a copy of the standard n-simplex. Now for each pair of simplices that share a face, glue them together using barycentric coordinates to make identifications between shared points.

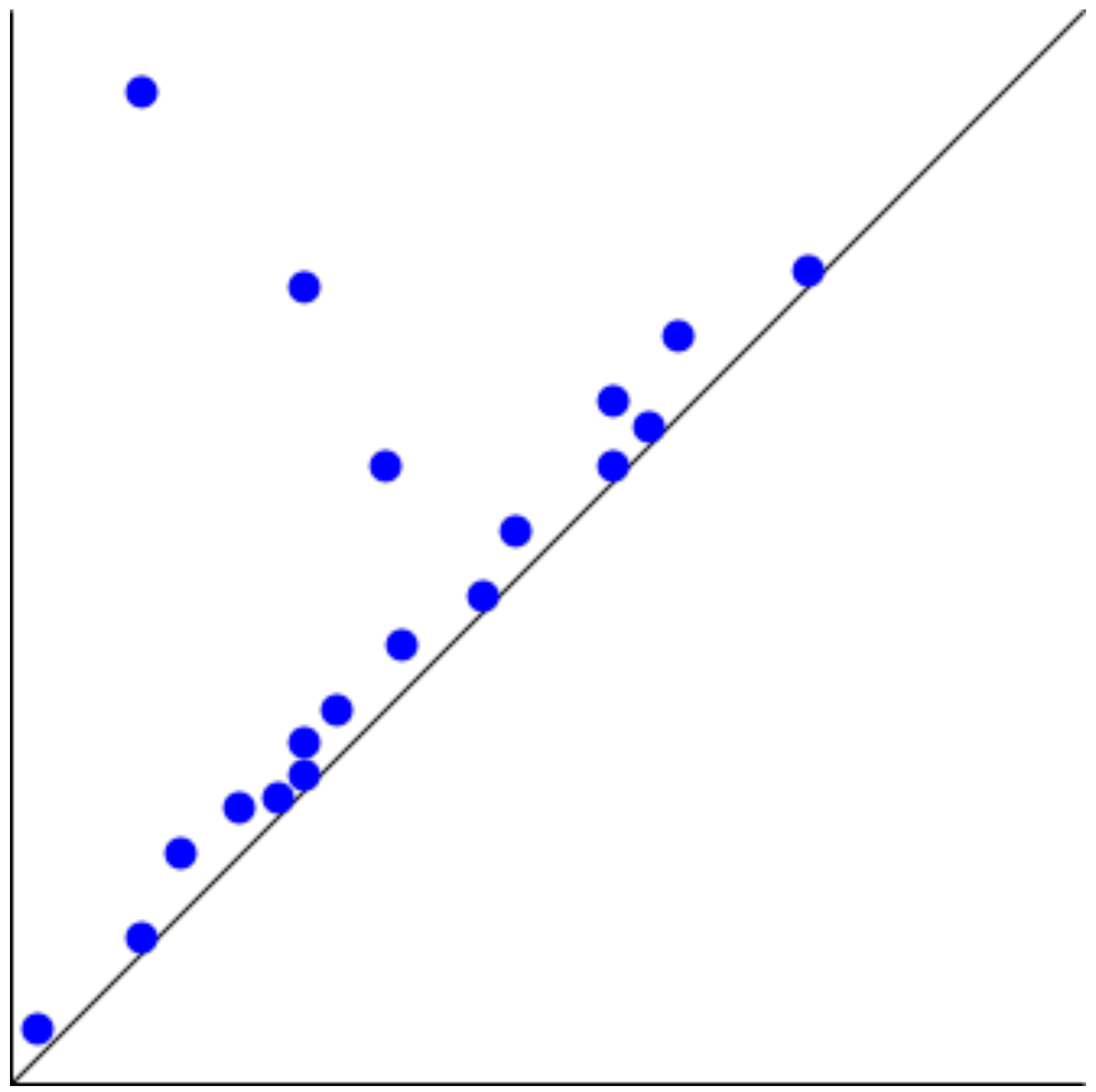
(abstract simplicial complex)

$$K = \{\{i,j,k\},\{j,k,l\},\{i,j\},\{i,k\},\{j,k\},\{j,l\},\{k,l\},\{i\},\{j\},\{k\},\{l\},\emptyset\}$$



Application: Persistence

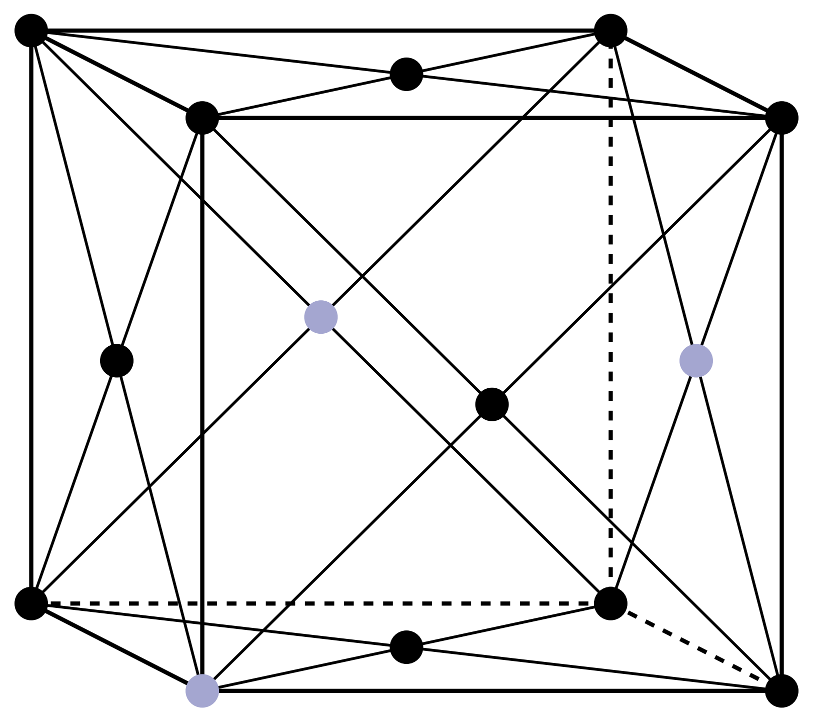


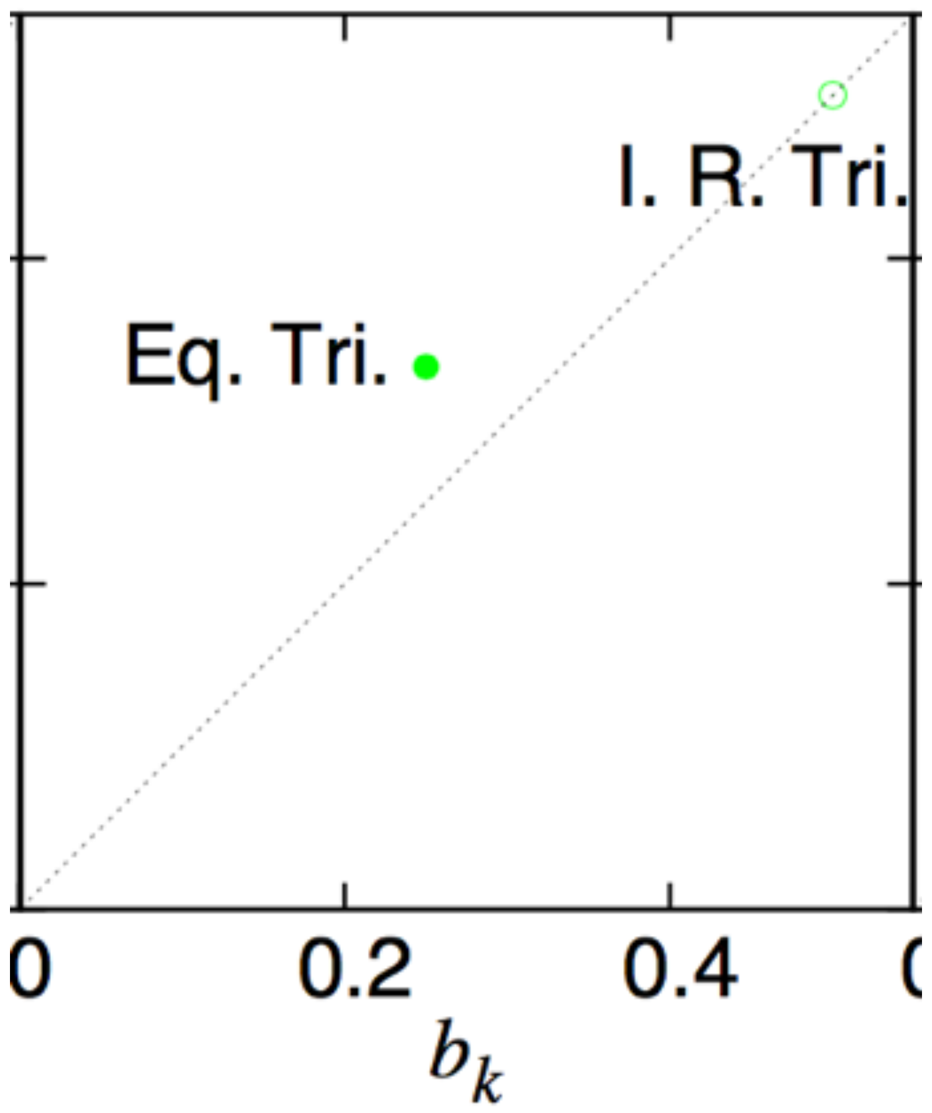


(persistence diagram)

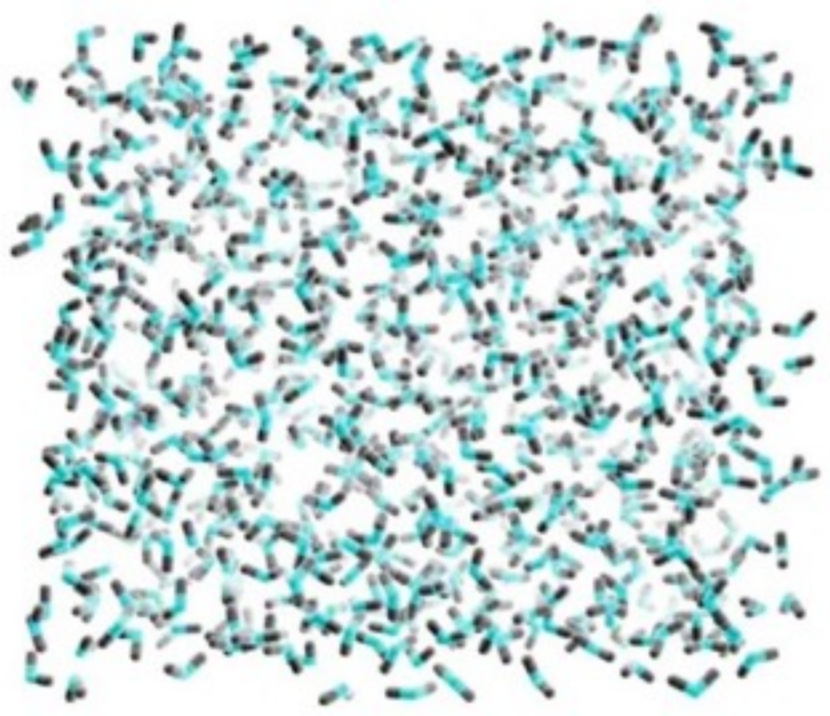
Application: Material Characterization

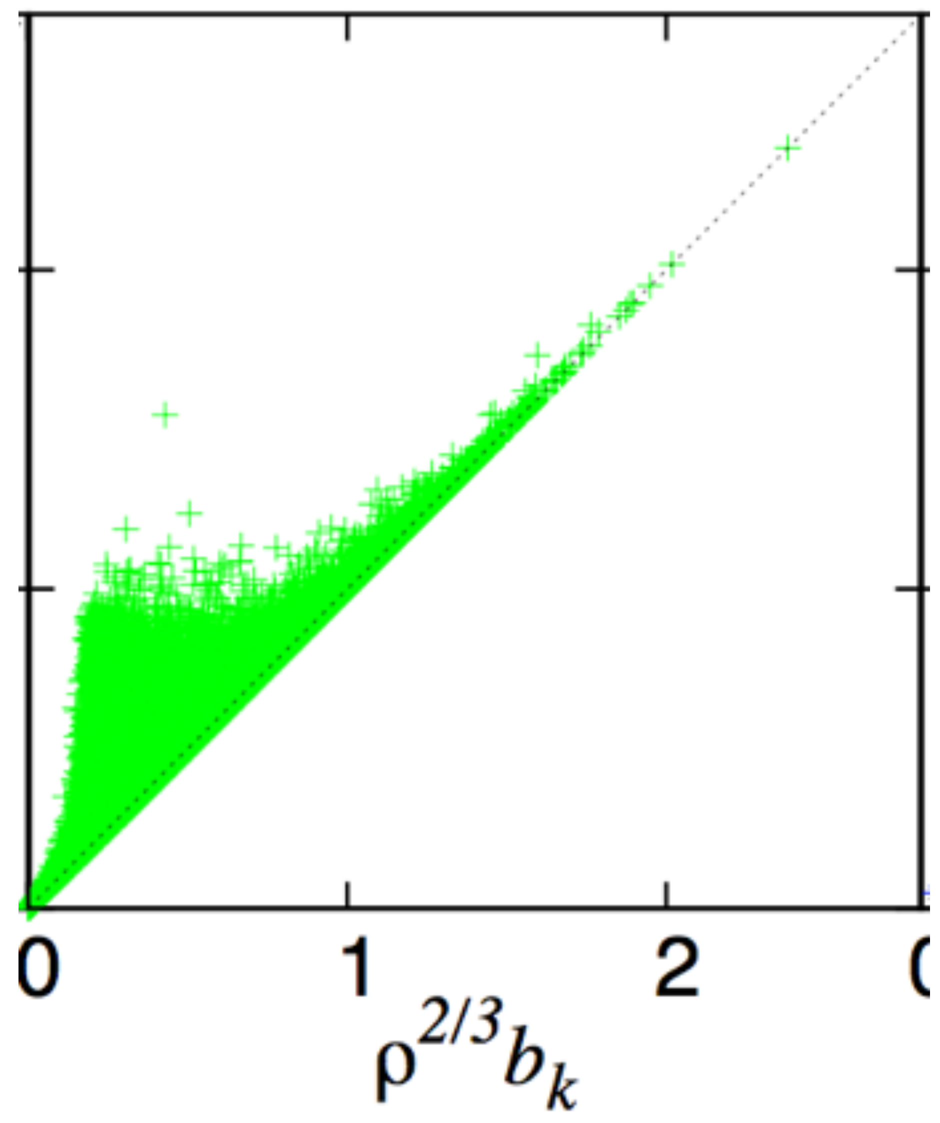
Regular



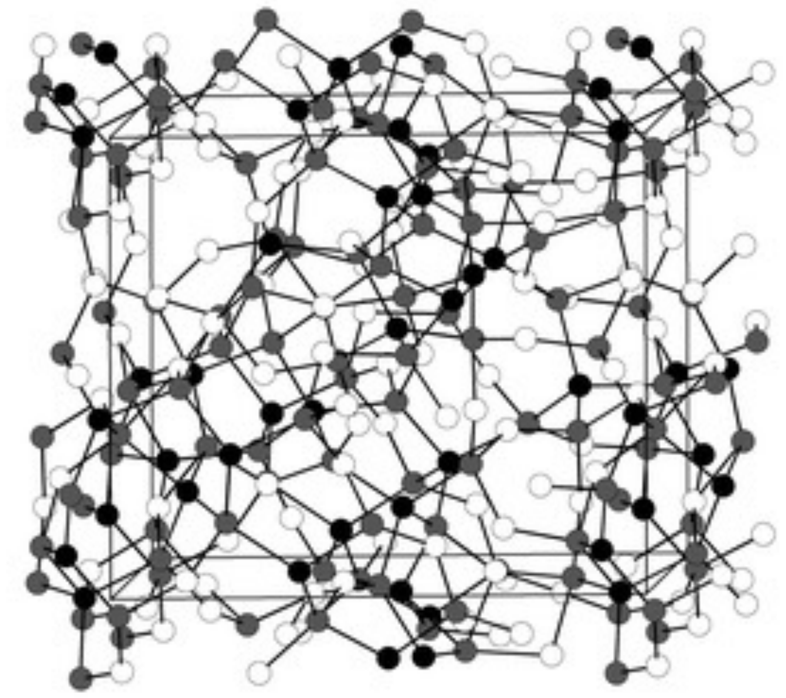


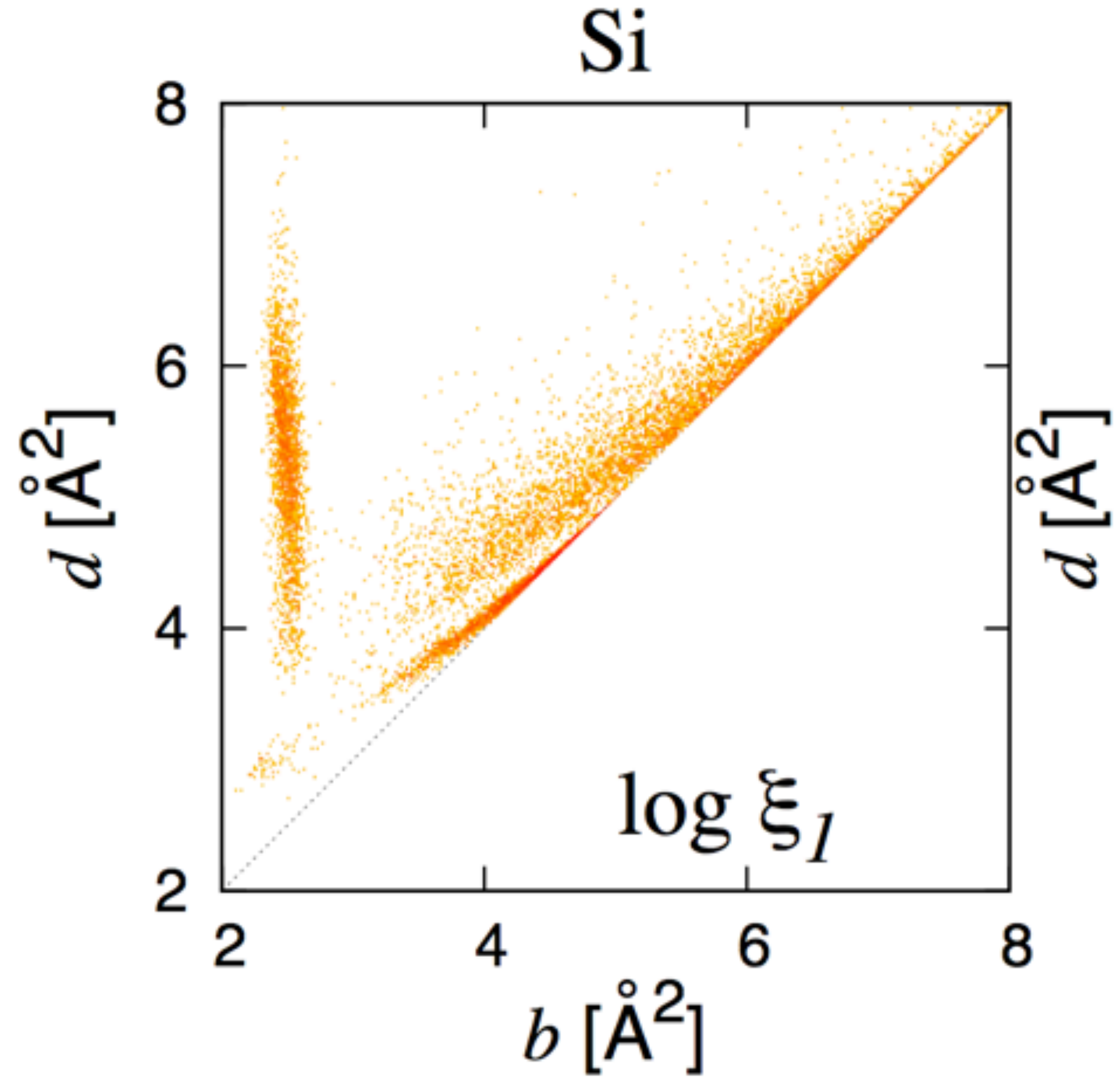
Random





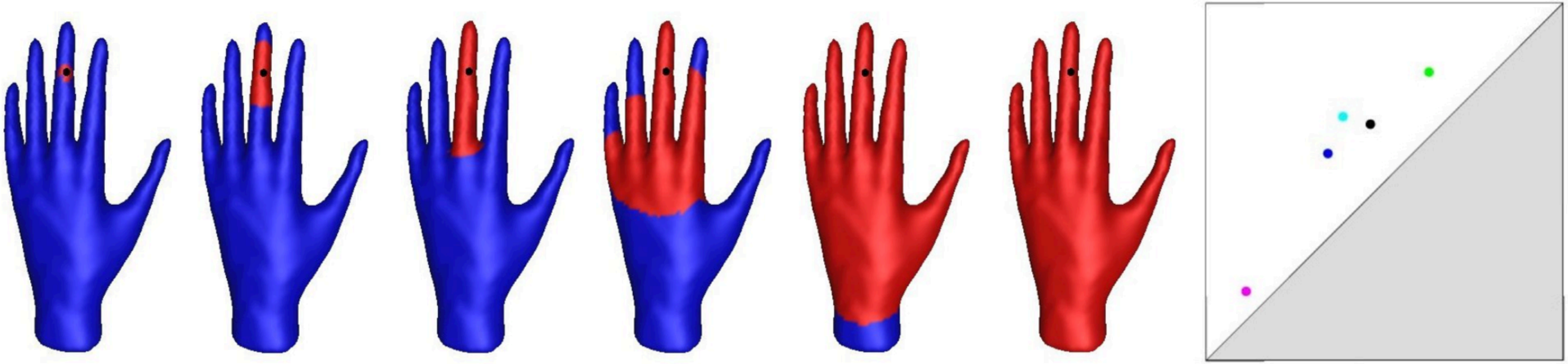
Glass





Nakamura et al, *"Persistent Homology and Many-Body Atomic Structure for Medium-Range Order in the Glass"*

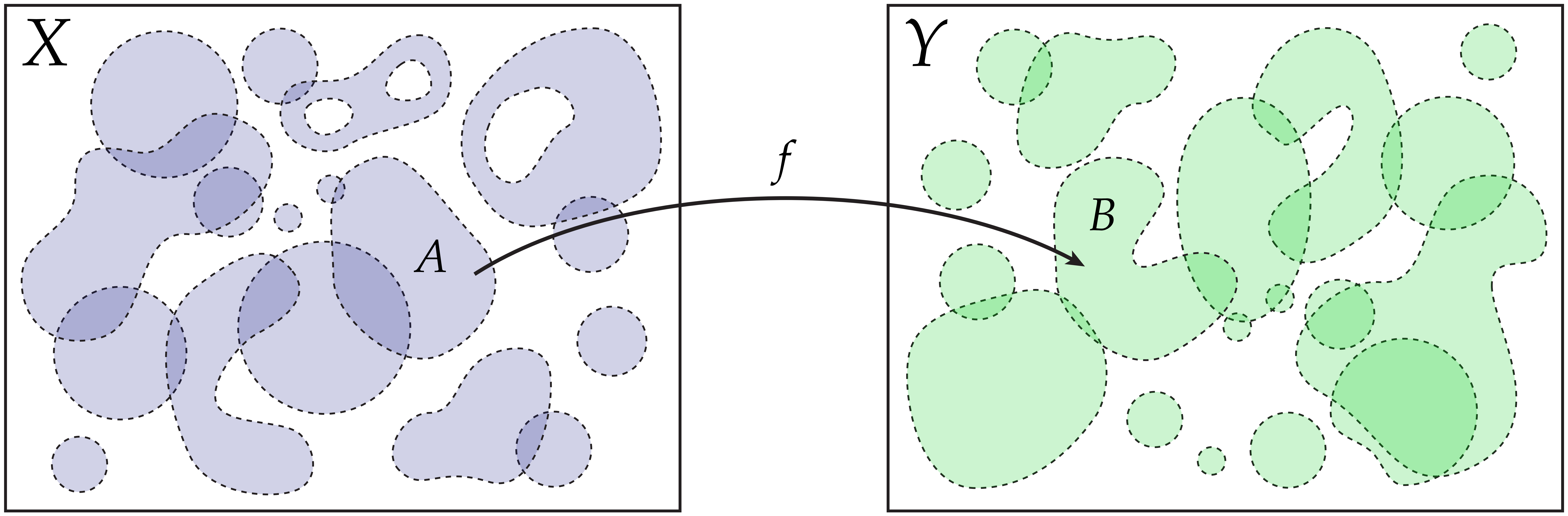
Application: Shape Matching via Persistence



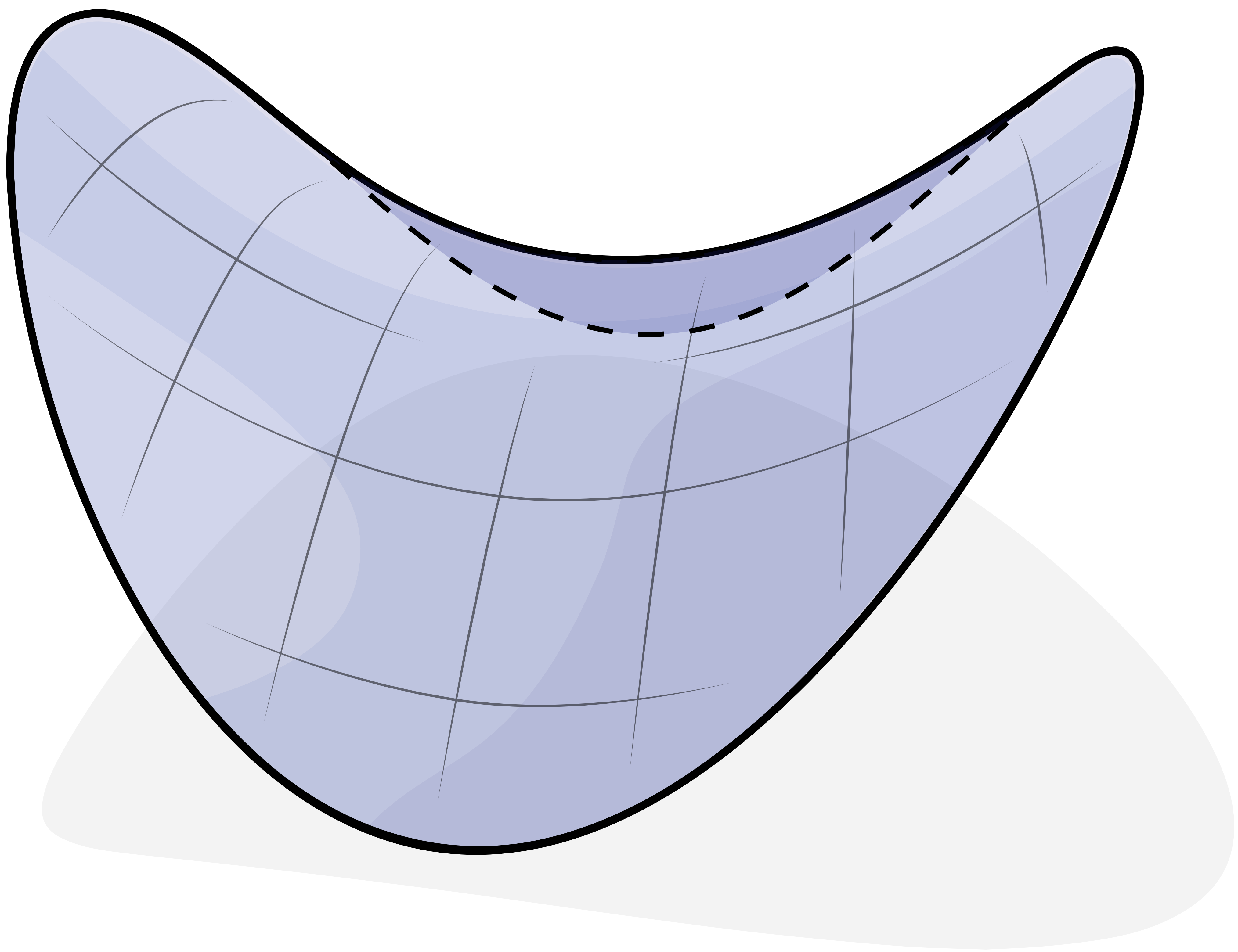
M. Carrière, S. Oudot, M. Ovsjanikov, *“Stable Topological Signatures for Points on 3D Shapes”*

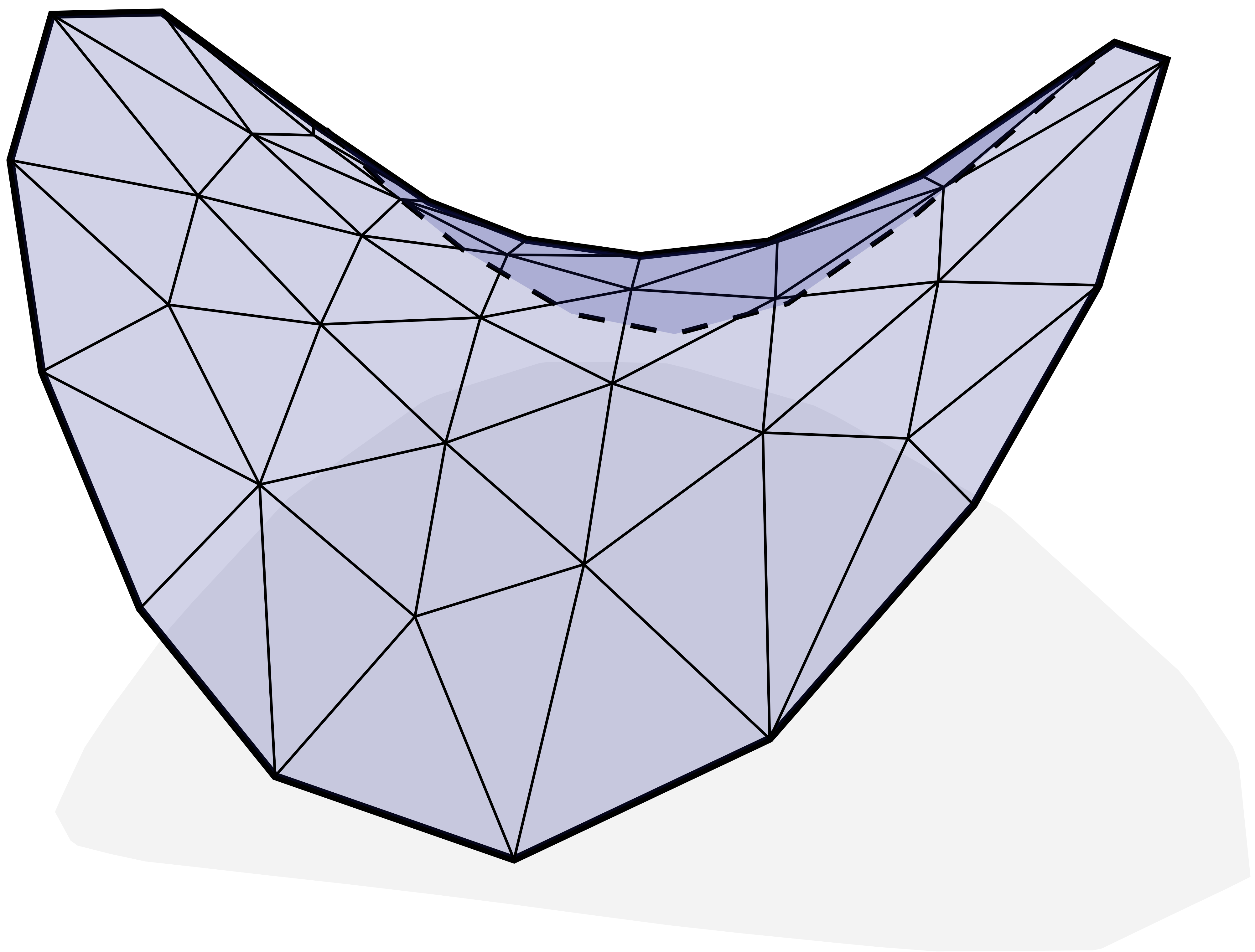
Continuous Map

Definition. Let (X, τ) and (Y, υ) be topological spaces. A map $f : (X, \tau) \to (Y, \upsilon)$ is *continuous* if for every υ-open set $B \in \upsilon)$, the preimage $A := f^{-1}(B)$ is τ-open.



Topological Spaces—Continuous vs. “Discrete”

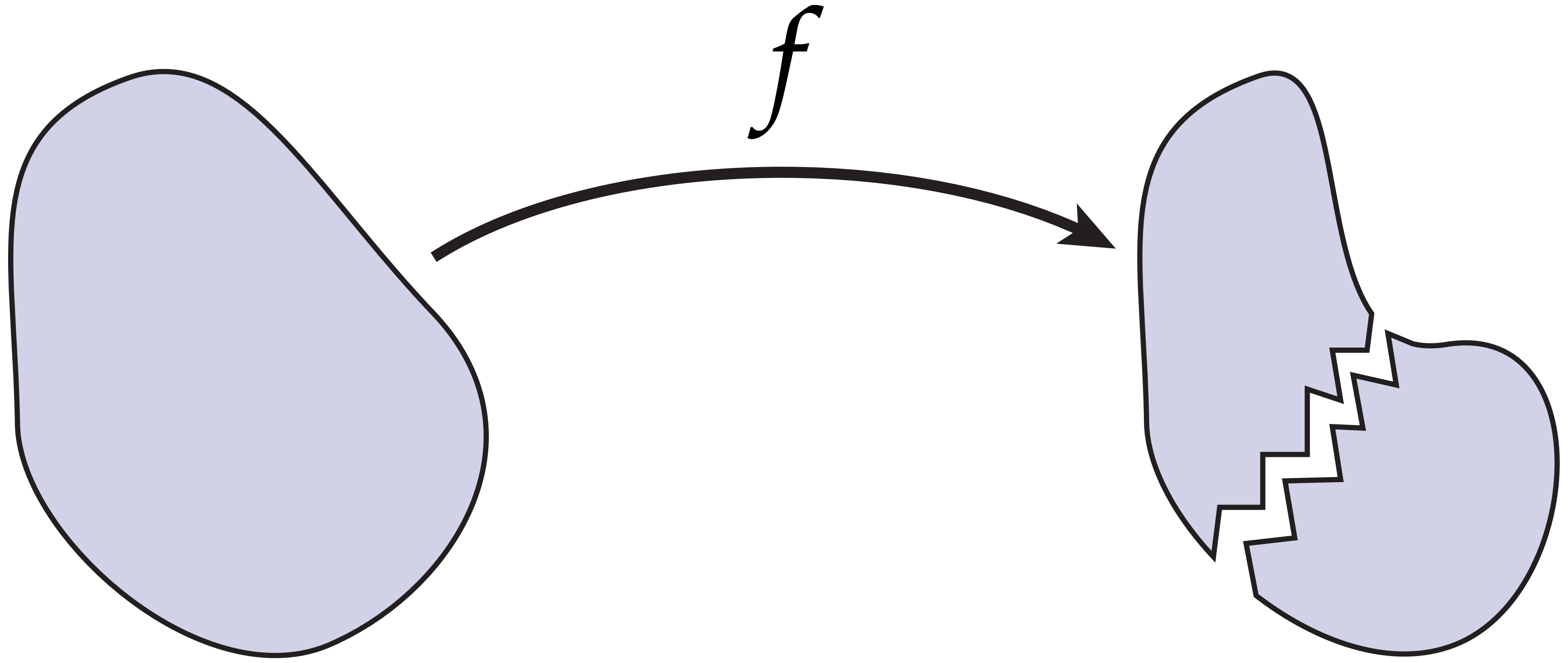


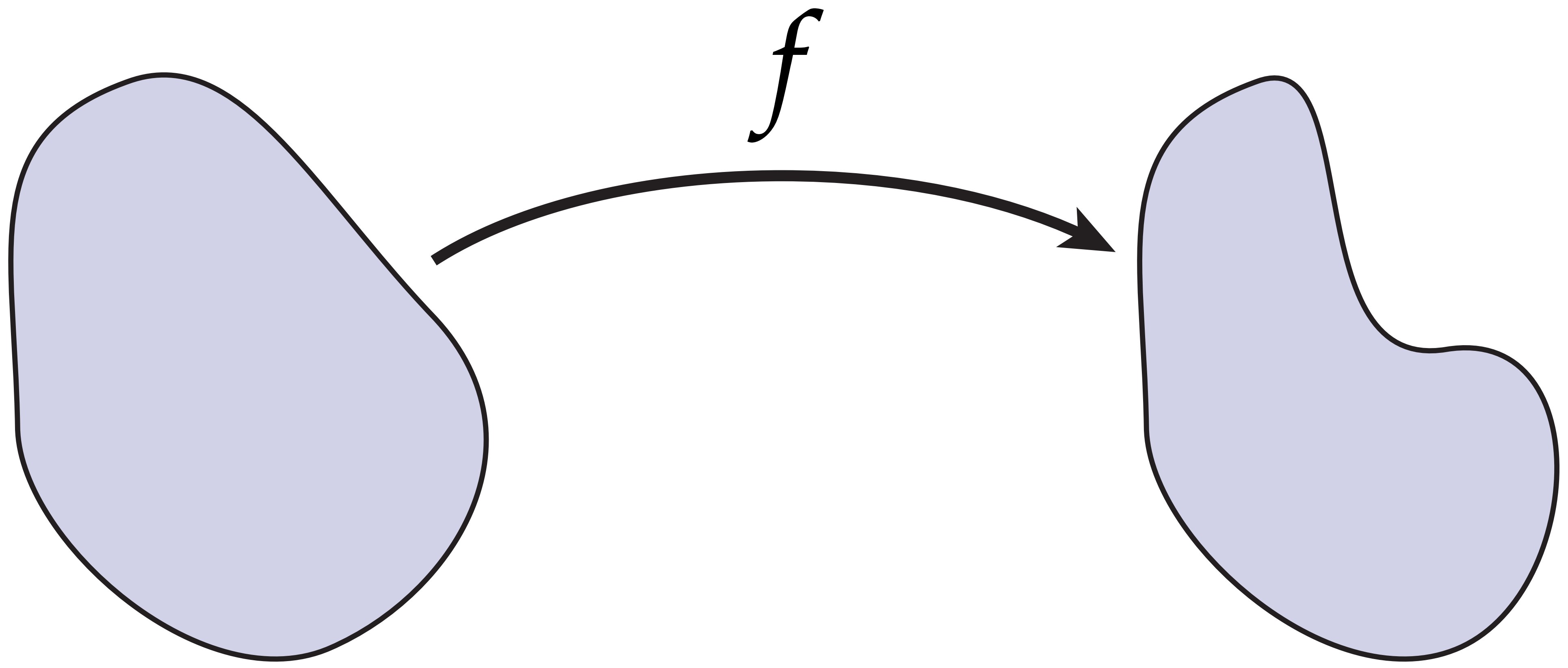


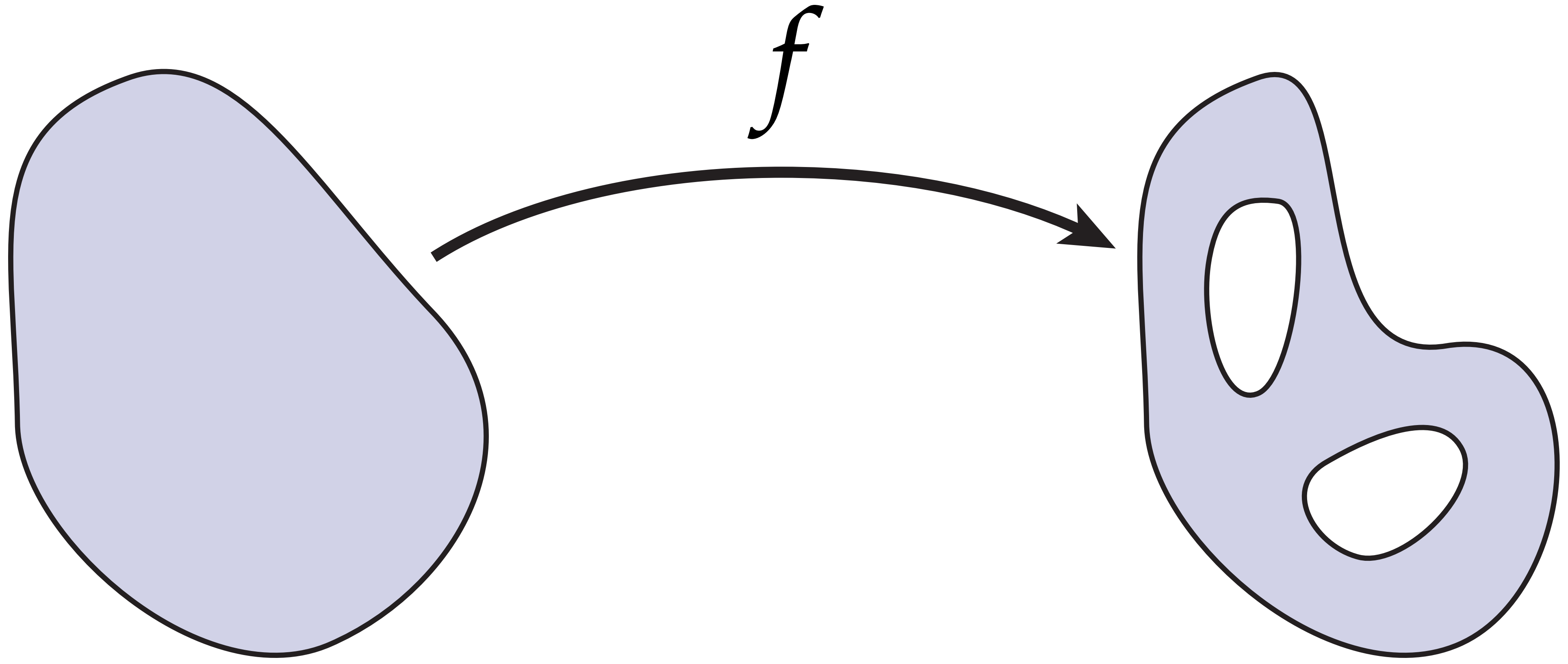
topological space $\longleftrightarrow$ simplicial complex

Continuous Map — Visualized

Which of these maps are continuous?

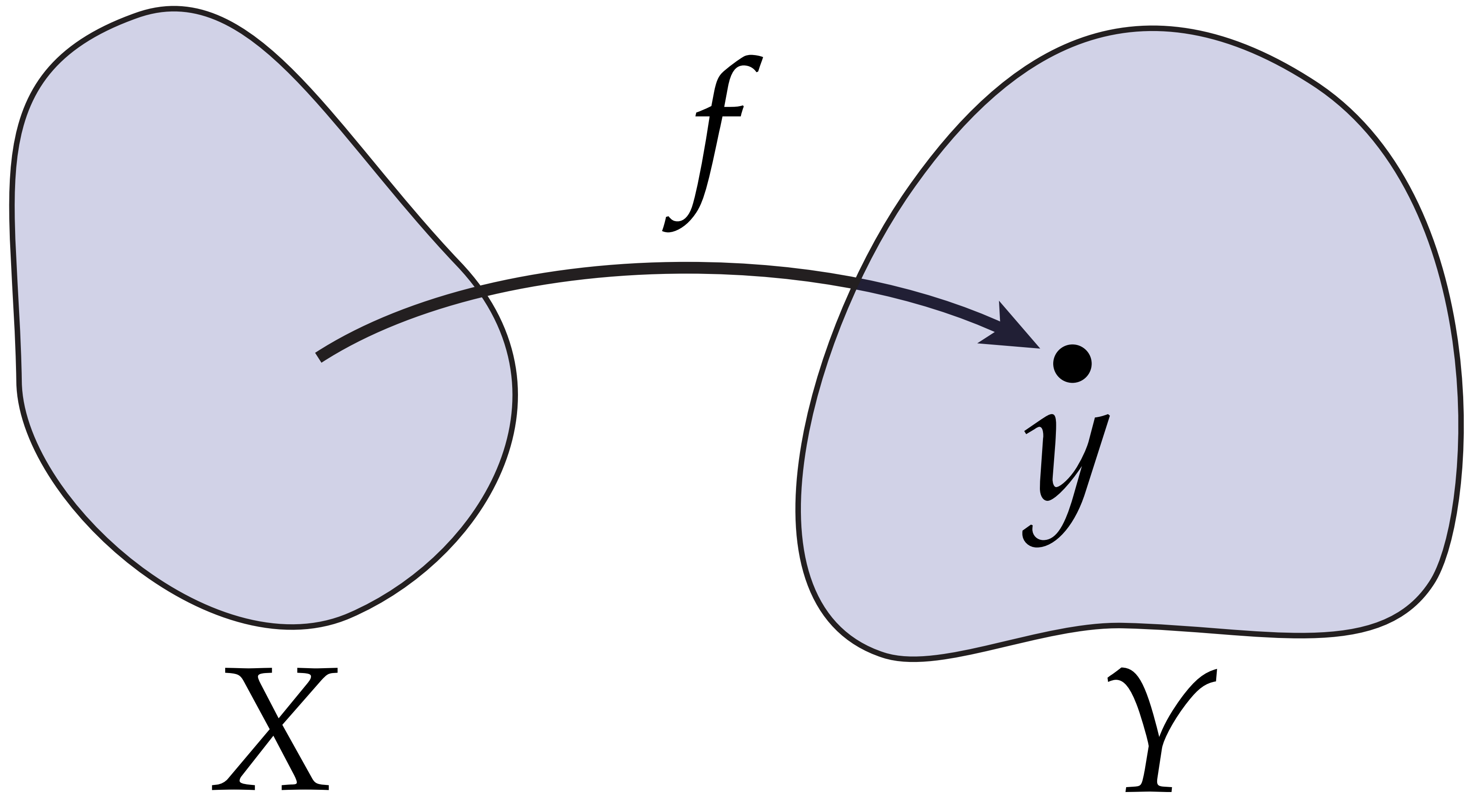






Continuous Map—Example?

Example.



(X, τ)—any topological space

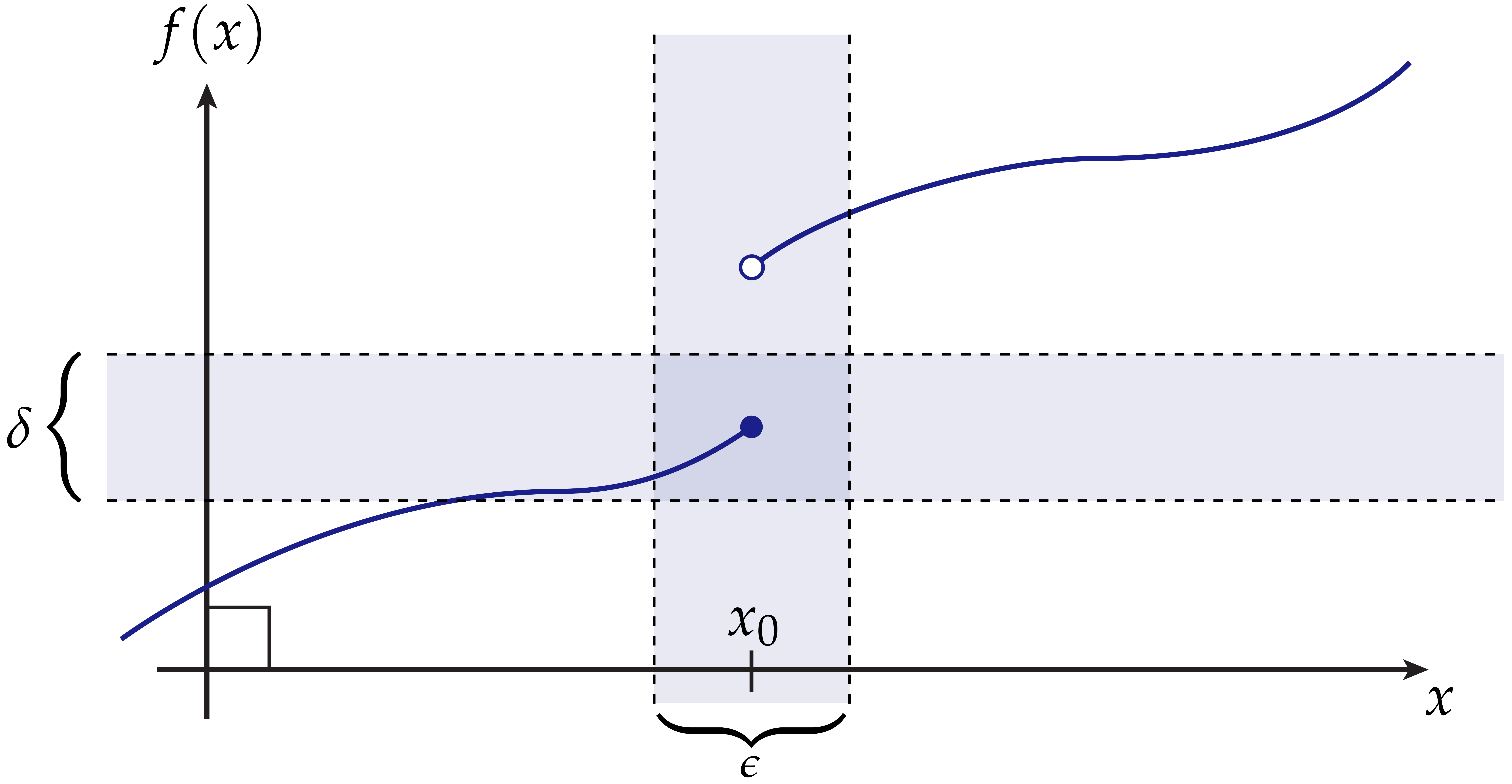
(Y, υ)—any topological space

For some fixed element $y \in Y$, $f(x) := y$ for all $x \in X$.

Q: Is f continuous?

A: Yes: every υ-open set either contains y (preimage is X), or it doesn't (preimage is the empty set).

Metric Continuity



"Discrete" Continuous Map

Definition. Let K_1, K_2 be abstract simplicial complexes, and let V_1, V_2 be the corresponding vertex sets. Then a map $f : V_1 \to V_2$ is an *abstract simplicial map* if for each simplex $\sigma \in K_1$, $f(\sigma) \in K_2$ (where f is applied to each vertex in the simplex). If one puts the standard topology on K_1 and K_2 (via the standard simplex), then f can be extended to a *simplicial map* that maps any point $p \in \sigma$ to a point $f(p) \in f(\sigma)$ via barycentric coordinates.

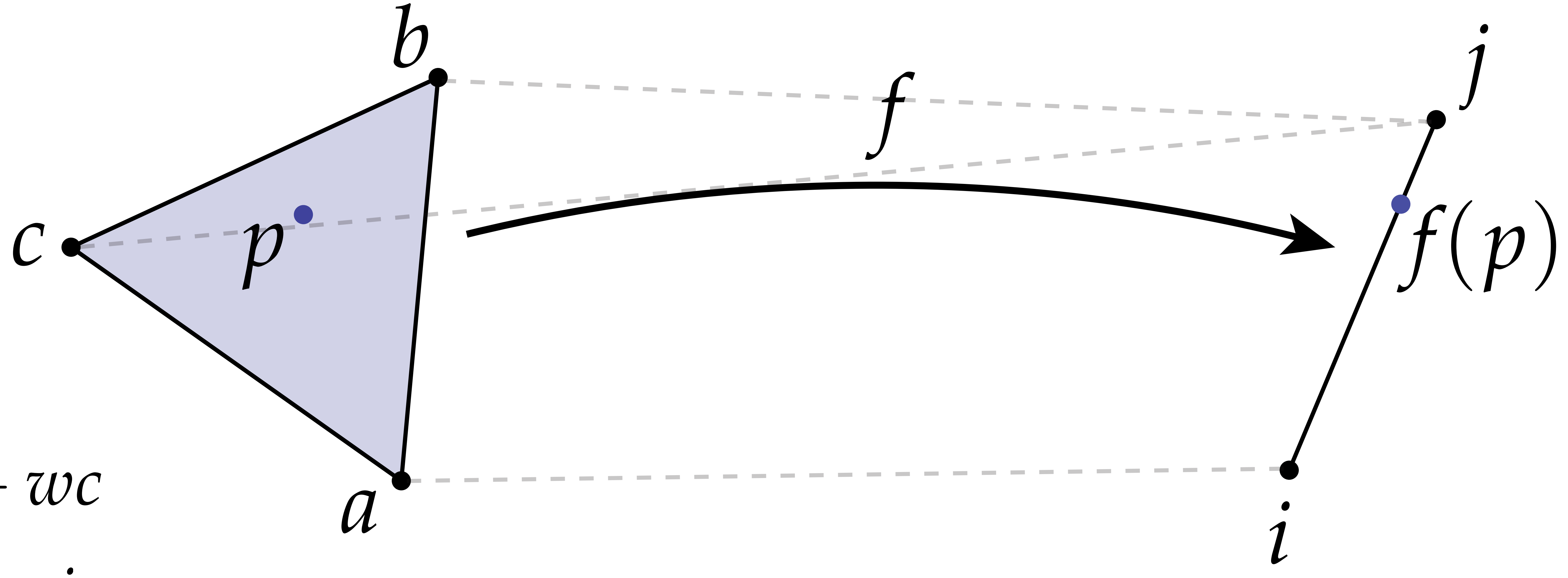
<u>abstract map:</u>

$$\begin{aligned} f(a) &= i \\ f(b) &= j \\ f(c) &= j \end{aligned}$$

<u>topological map:</u>

$$p = ua + vb + wc$$

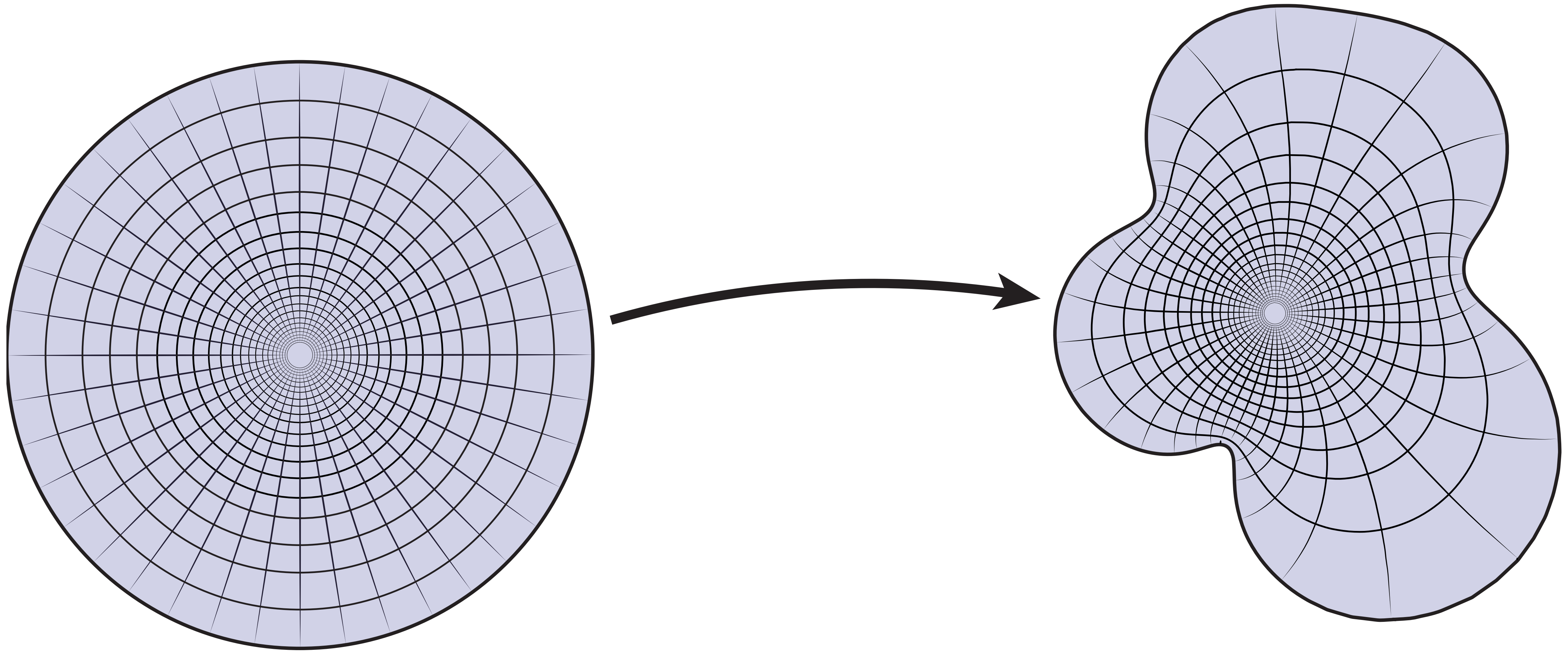
$$f(p) = ui + vj + wj$$



Homeomorphism

Definition. If (X, τ) and (Y, υ) are topological spaces, then a map $f : X \to Y$ is a *homeomorphism* if it is a continuous bijection with continuous inverse. Two spaces are *homeomorphic* if there exists a homeomorphism between them. If $f : X \to Y$ is not surjective but $f : X \to \mathrm{im}(f)$ is a homeomorphism with respect to the subspace topology, then we say f is a *homeomorphism onto its image*.

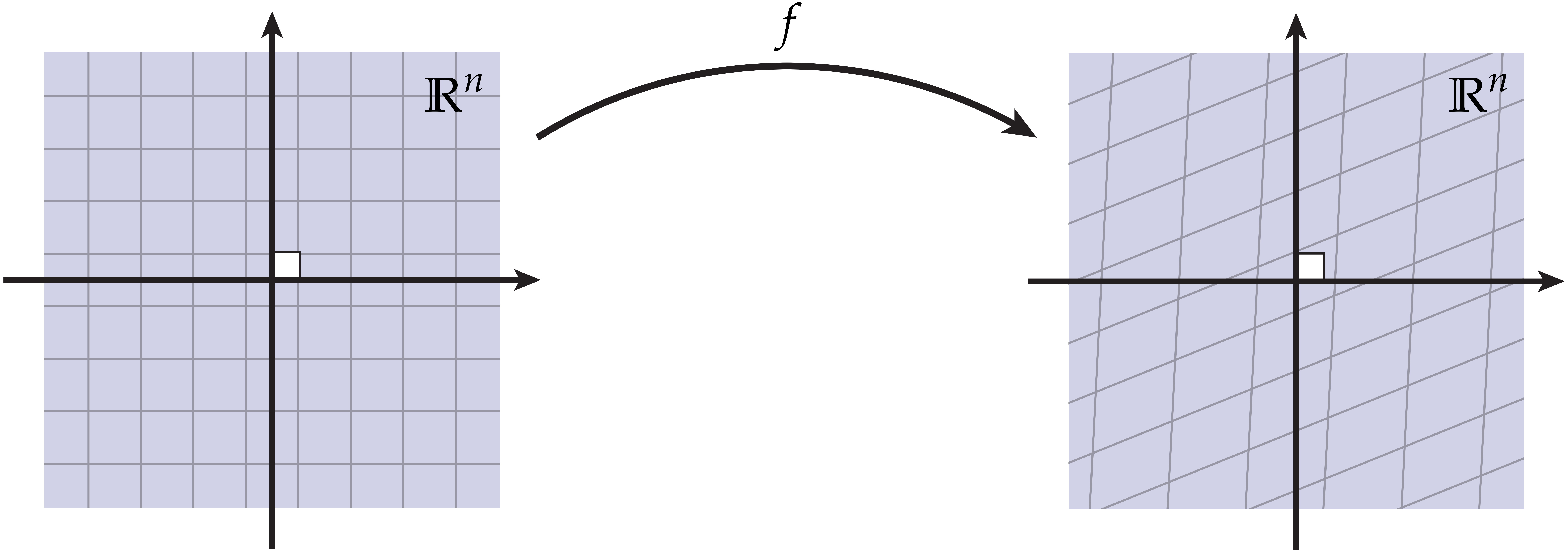
Key idea: Homeomorphisms preserve *everything* about topology.



(homeomorphism onto image)

Homeomorphism — Examples

Example. Consider $\mathbb{R}^n$ with its usual (Euclidean) metric topology. Then any bijective linear map or *linear isomorphism* $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.

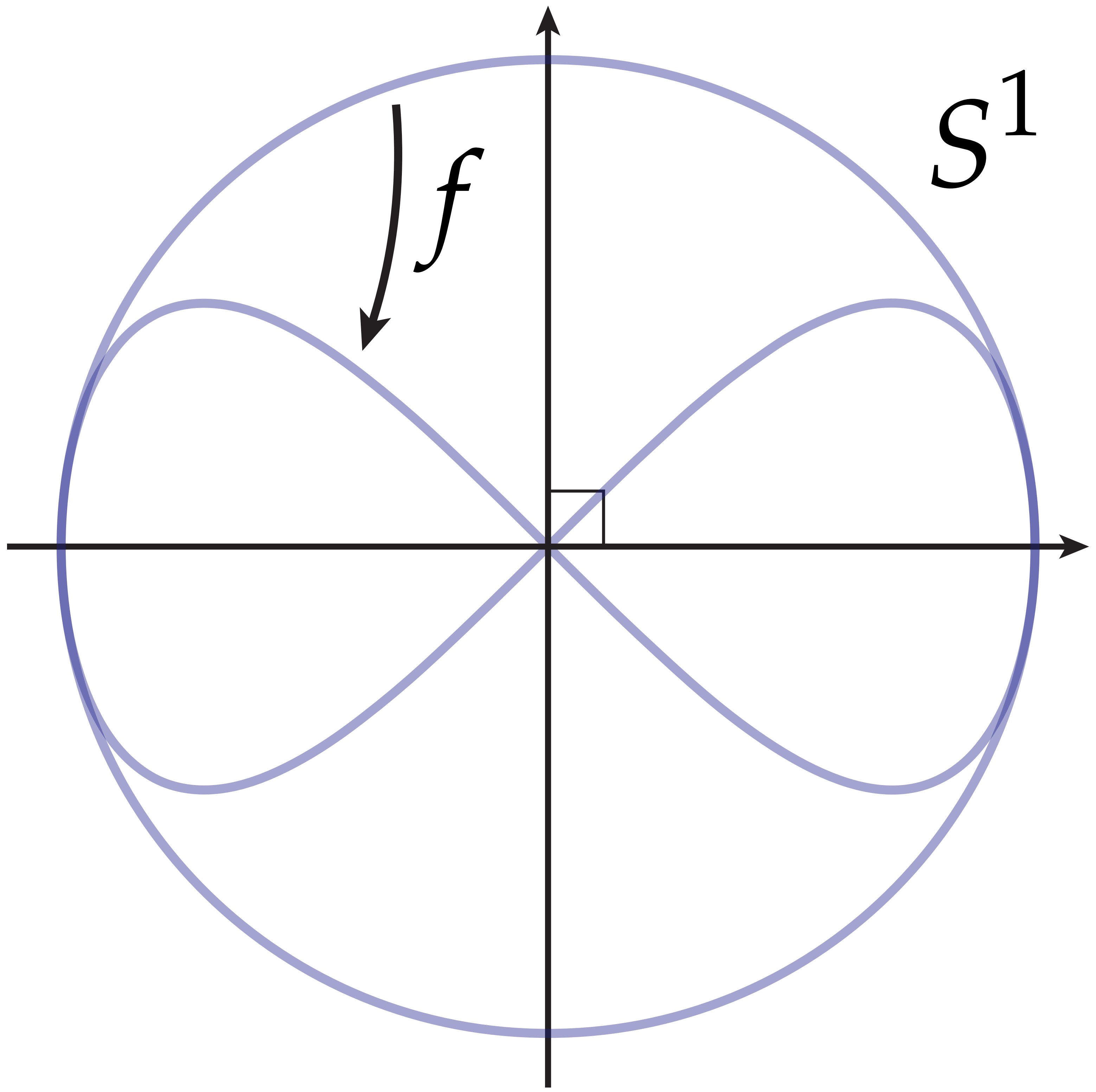


Homeomorphism — Examples

Example. Consider an equivalence relation on $\mathbb{R}^2$ given by $x \sim y$ whenever $x - y \in \mathbb{Z}^2$. The corresponding space $T^2 := \mathbb{R}^2/\sim$ is the *topological torus* (often we use the shorthand $\mathbb{R}^2/\mathbb{Z}^2$). Any homemorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x+n) = f(x) + n$ for all $n \in \mathbb{Z}^2$ induces a homeomorphism on T^2, given by $\tilde{f}([x]) := [f(x)]$.

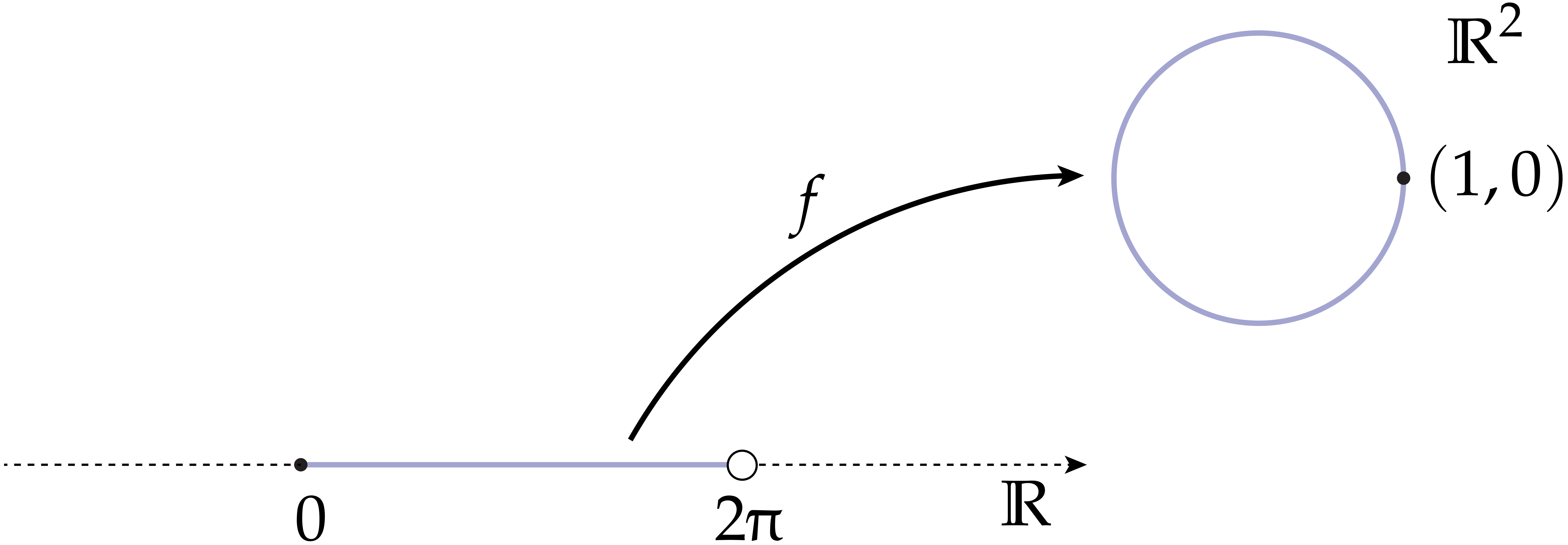
Continuous Map vs. Homeomorphism

Example. Consider the unit circle $S^1 := \{p \in \mathbb{R}^2 \,||p| = 1\}$ with the subspace topology induced by the usual Euclidean topology on $\mathbb{R}^2$, and the map $f : S^1 \to \mathbb{R}^2$ given by $f(x, y) := (x, \cos(x)y)$ (where again $\mathbb{R}^2$ has the Euclidean topology). Is f continuous? Is it a homeomorphism onto its image?



Continuous Bijection vs. Homeomorphism

Example. Let $I := [0, 2\pi) \subset \mathbb{R}$ have the subspace topology induced by the Euclidean topology on $\mathbb{R}$, and let $f : I \to \mathbb{R}^2$ be given by $f(s) := (\cos(s), \sin(s))$. For the usual metric topology on $\mathbb{R}^2$, is f bijective? Is it continuous? Is it homeomorphic (onto its image)?

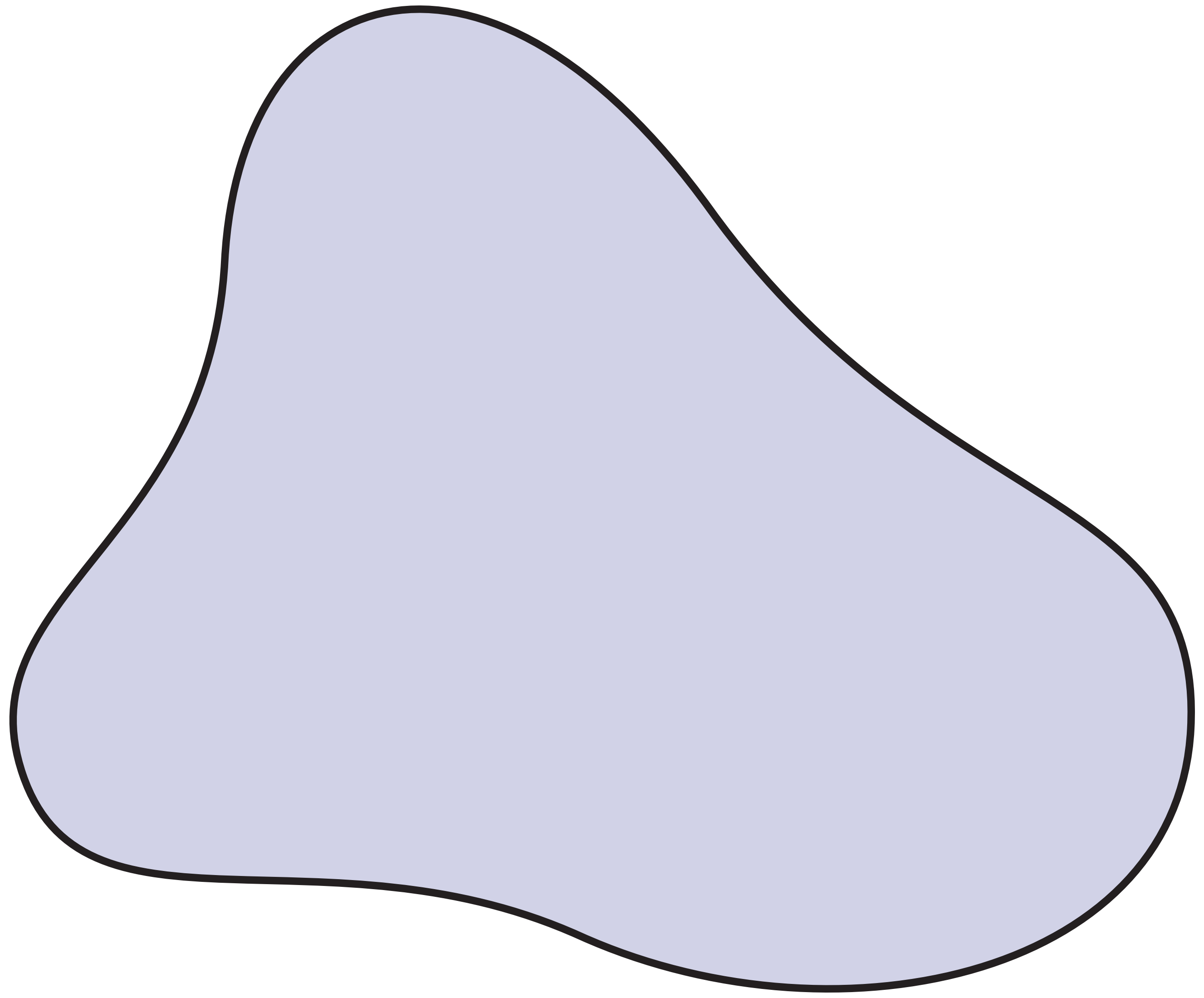


“Discrete” Homeomorphism?

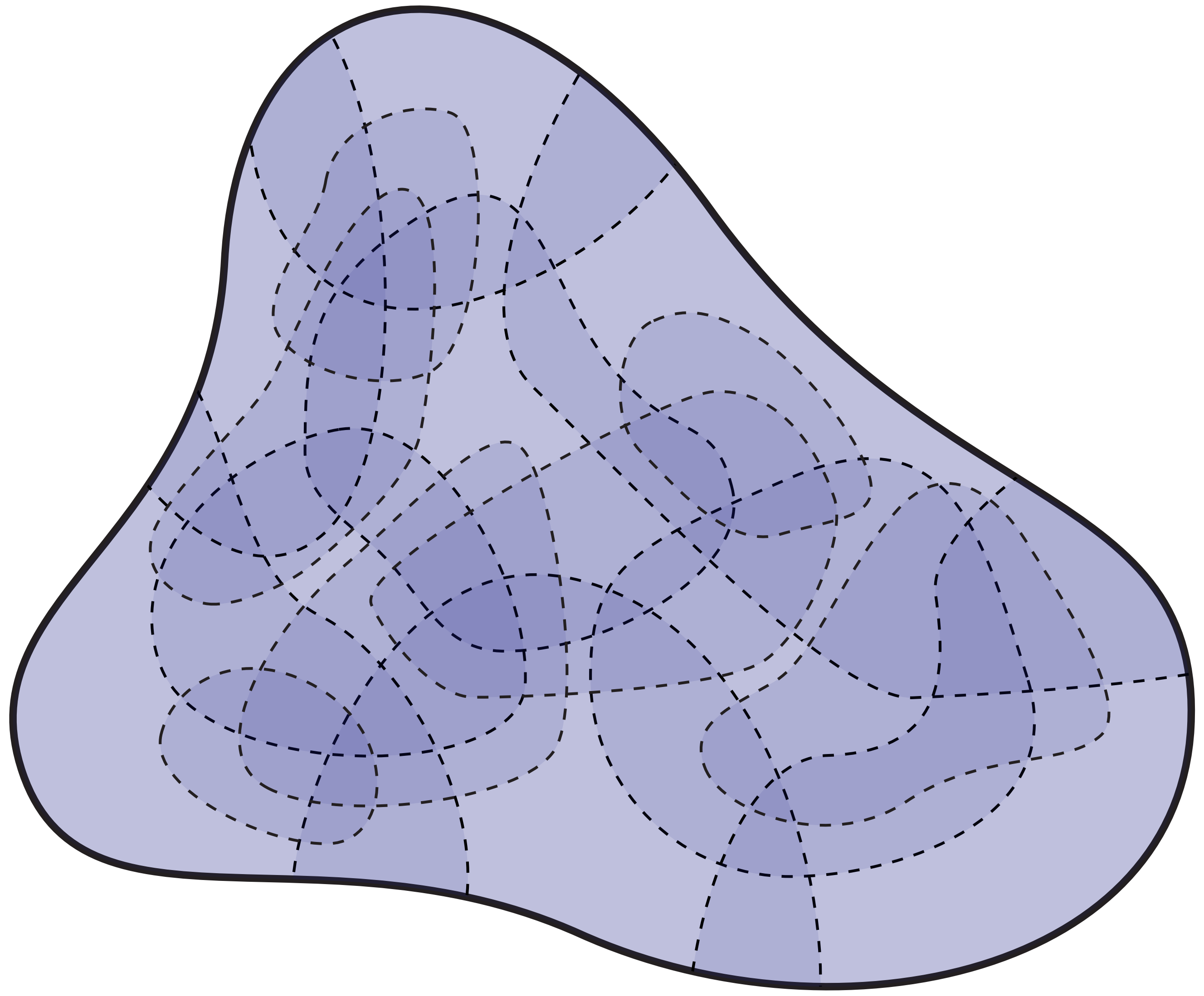
What do you think?

Open Cover

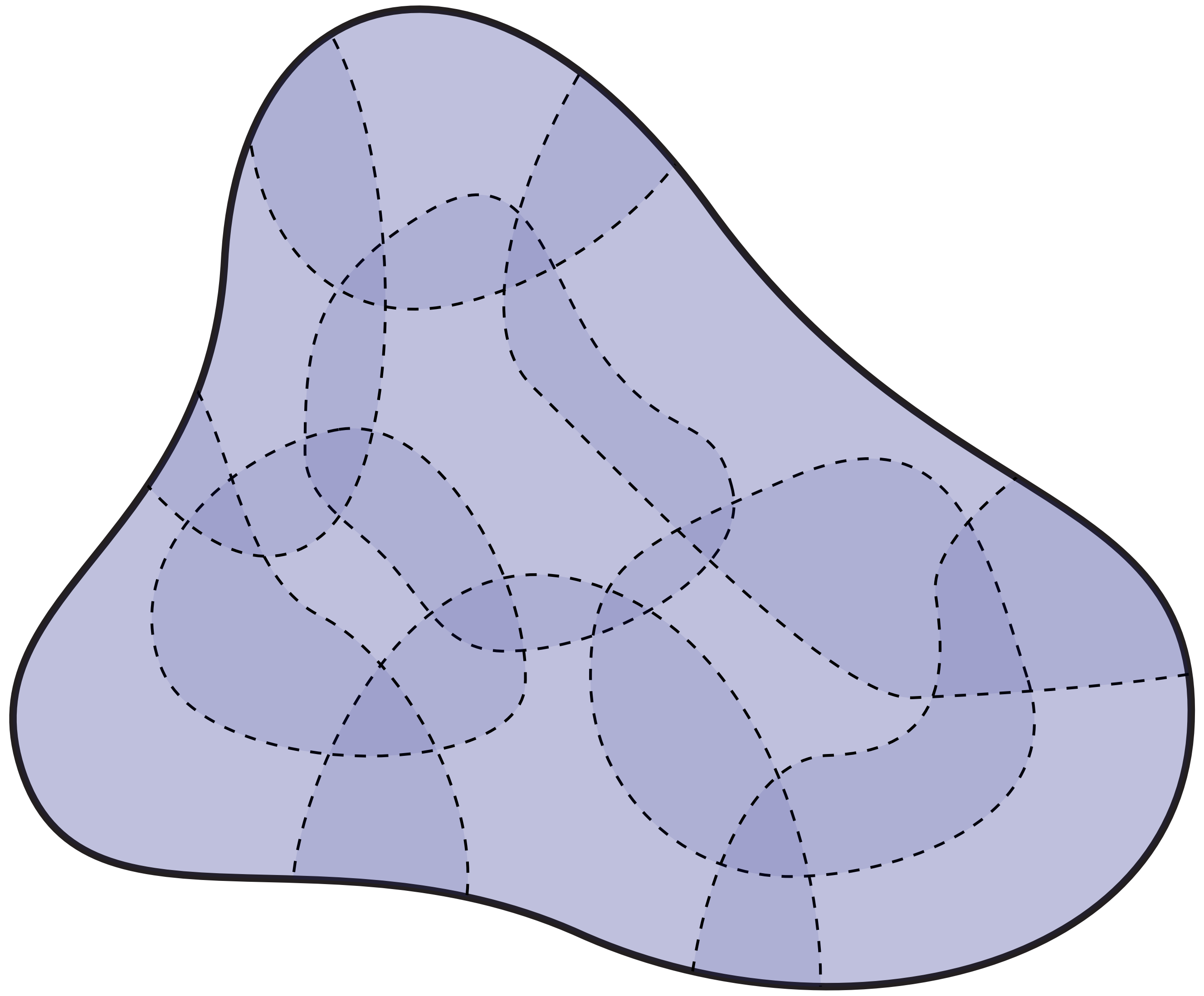
Definition. Let (X, τ) be a topological space. An *open cover* of X is a collection of τ-open sets $\mathcal{U} := \{U_i\}$ such that $\bigcup_i U_i = X$. If $\mathcal{V}$ is a subset of $\mathcal{U}$ that still covers X, then $\mathcal{V}$ is a *subcover* of $\mathcal{U}$.



topological space



open cover

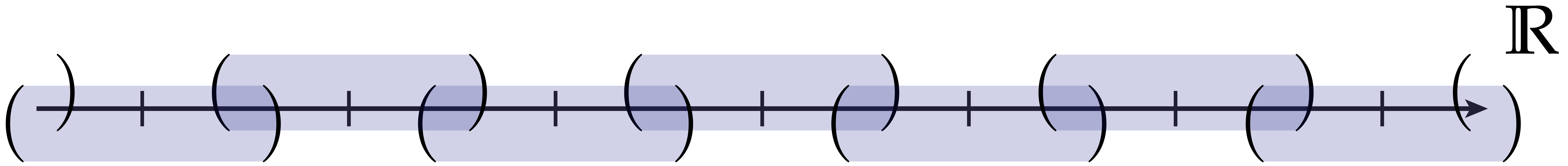


subcover

Compactness

Definition. A topological space is *compact* if **every** open cover has a finite subcover.

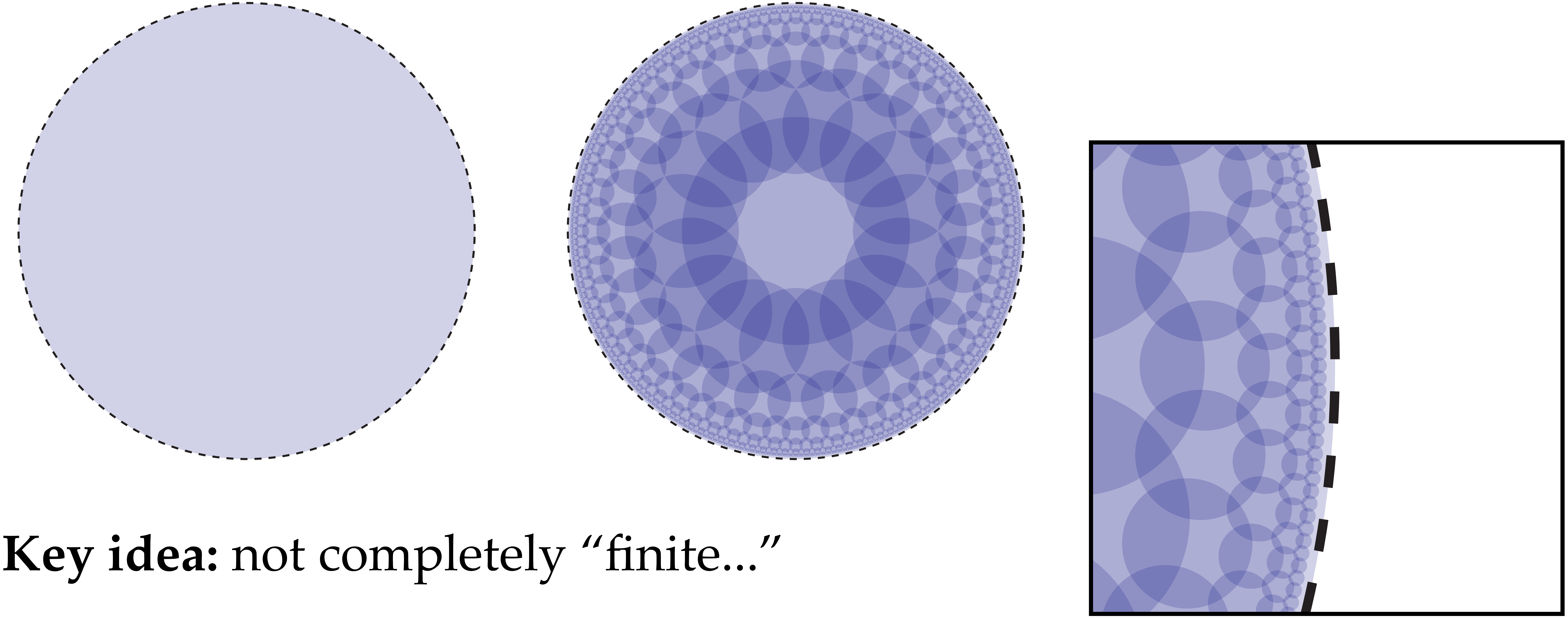
Example. Consider the real line $\mathbb{R}$ with its usual metric topology, which can be covered by open intervals $(n-\epsilon, n+\epsilon)$ for all $n \in \mathbb{Z}$ and some fixed $1/2 < \epsilon < 1$. Since there are infinitely many intervals, and $\mathbb{R}$ is no longer covered if we remove *any* interval, $\mathbb{R}$ is noncompact.



Key idea: captures notion of "finite size" for a topological space.

Compactness of Open Ball

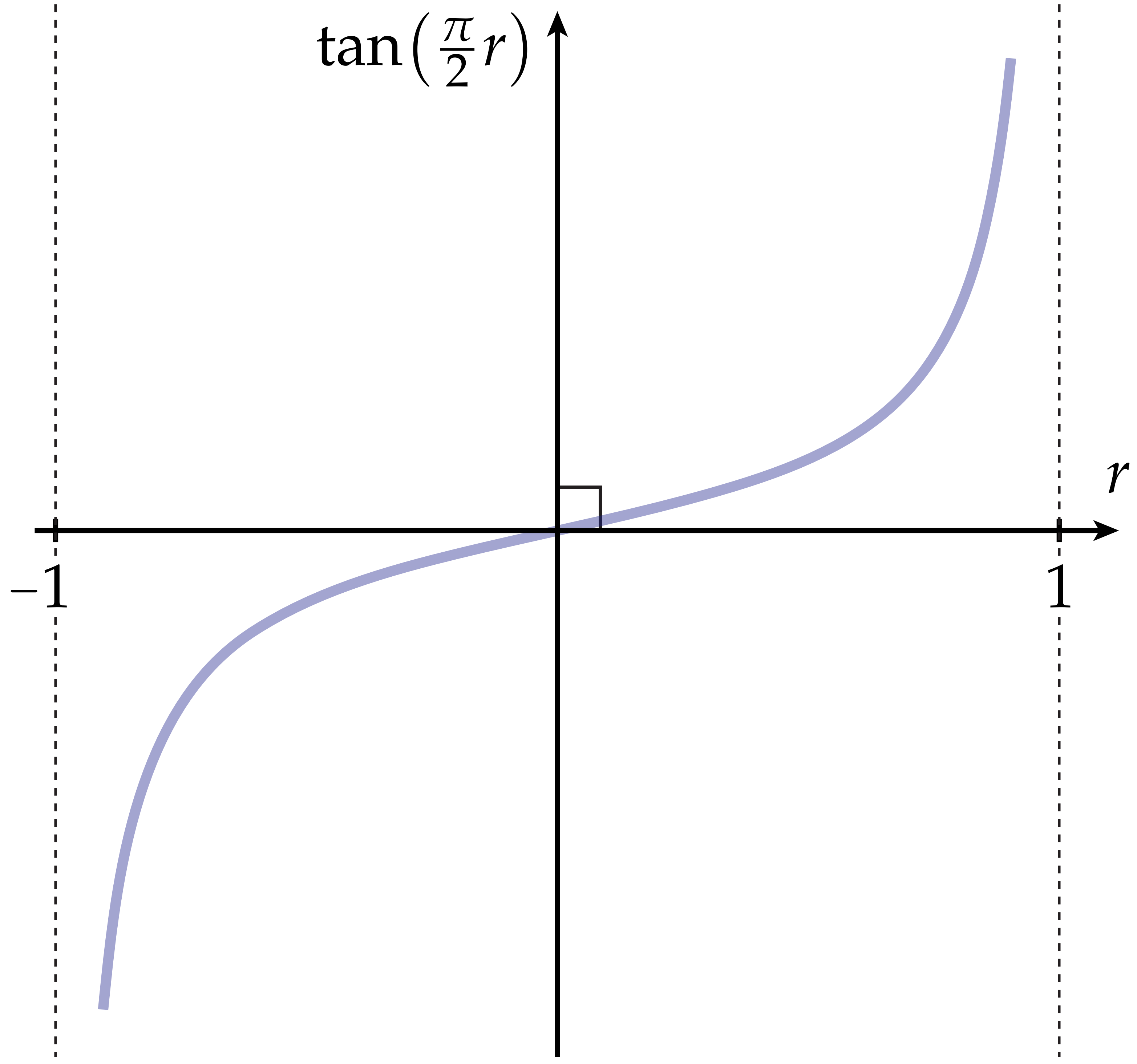
Example. Consider an open ball $B \subset \mathbb{R}^n$ with subspace topology τ induced by the Euclidean metric. Is (B, τ) a compact topological space?

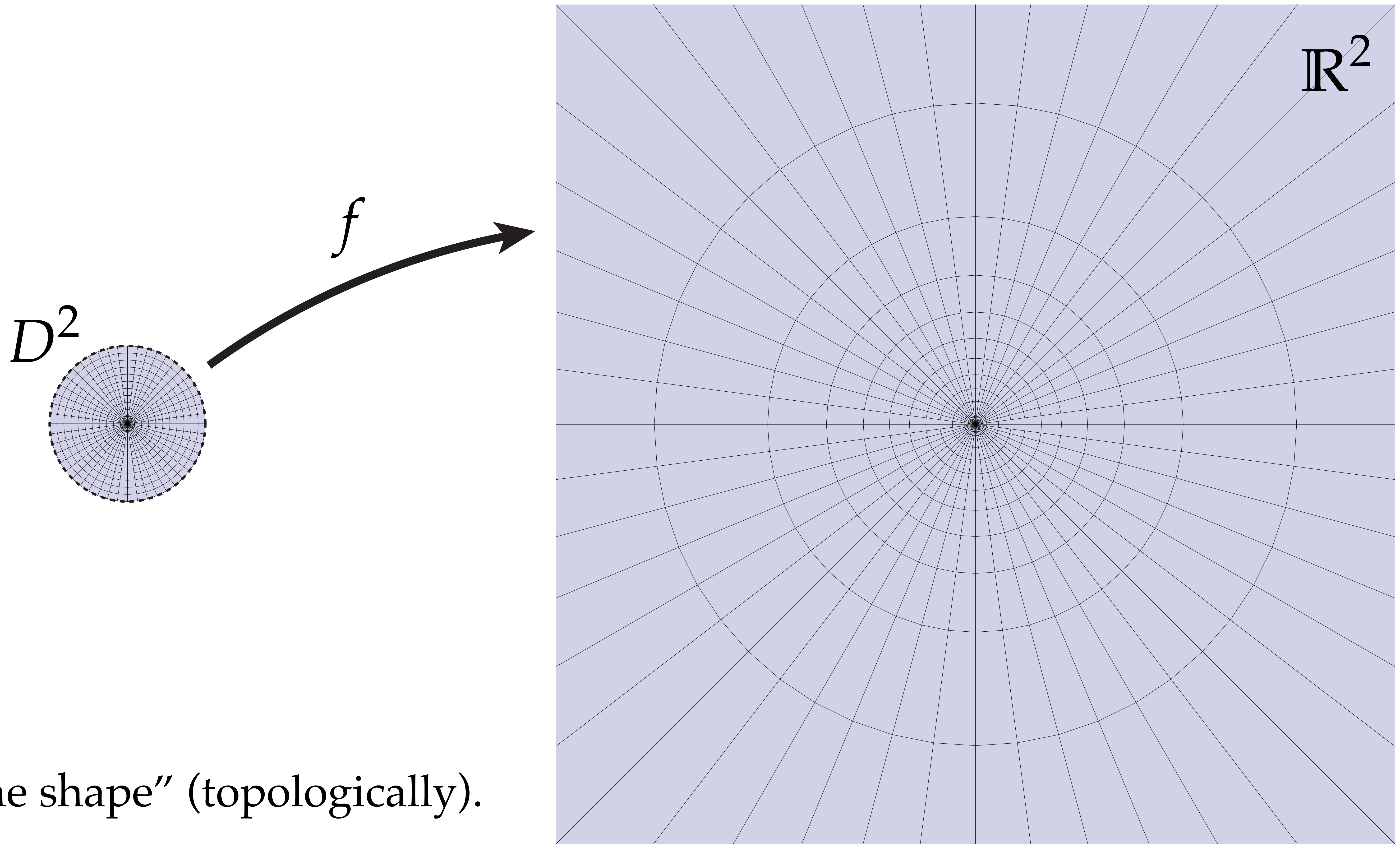


Key idea: not completely "finite..."

Open Disk vs. Plane

Example. Let $D^2 := B_1(0) \subset \mathbb{R}^2$ denote the open unit disk centered around the origin of the Euclidean plane. Consider the map $f : D^2 \to \mathbb{R}$ given in polar coordinates by $f(r, \theta) = (\tan(\frac{\pi}{2}r), \theta)$. Is this map a homeomorphism?

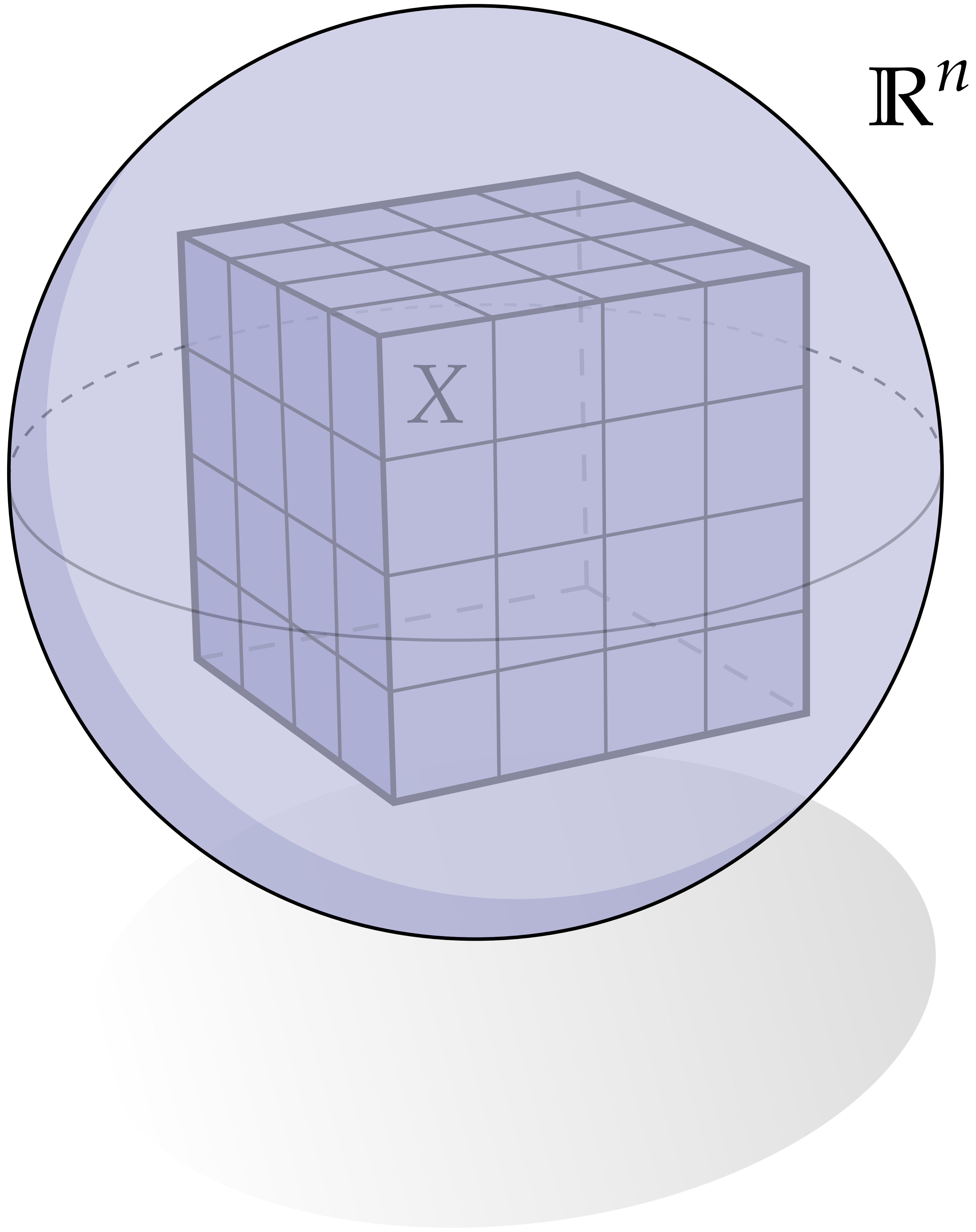




Yes. The open disk and the plane have the "same shape" (topologically).

Metric Compactness

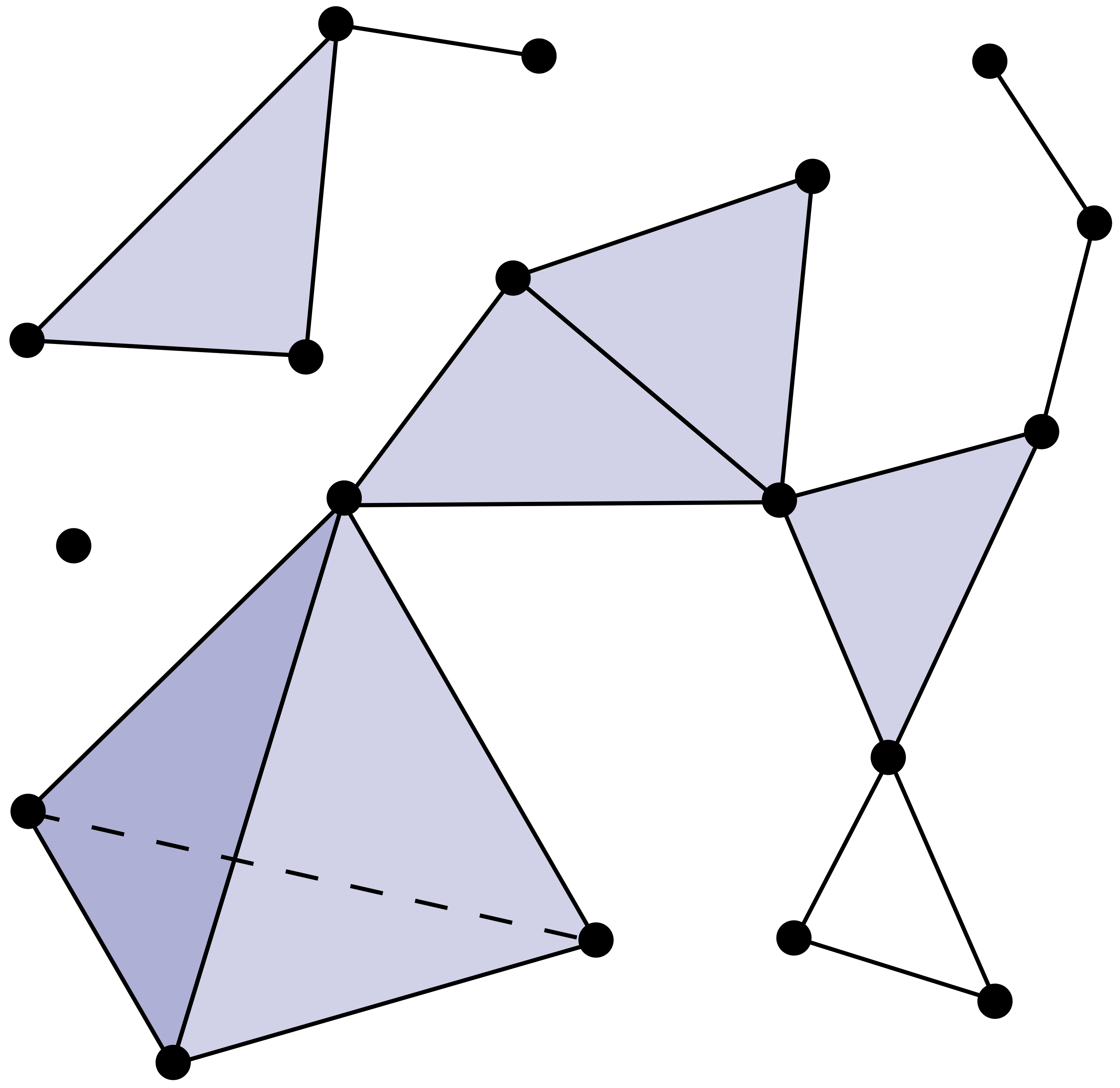
Theorem. (Heine-Borel) Consider $\mathbb{R}^n$ with its usual metric topology. A subset X of $\mathbb{R}^n$ is compact (with respect to the subspace topology) if and only if it is (i) closed and (ii) *bounded*, i.e., contained in a ball of finite radius.



Key idea: captures notion of "finite size."

Discrete Compactness

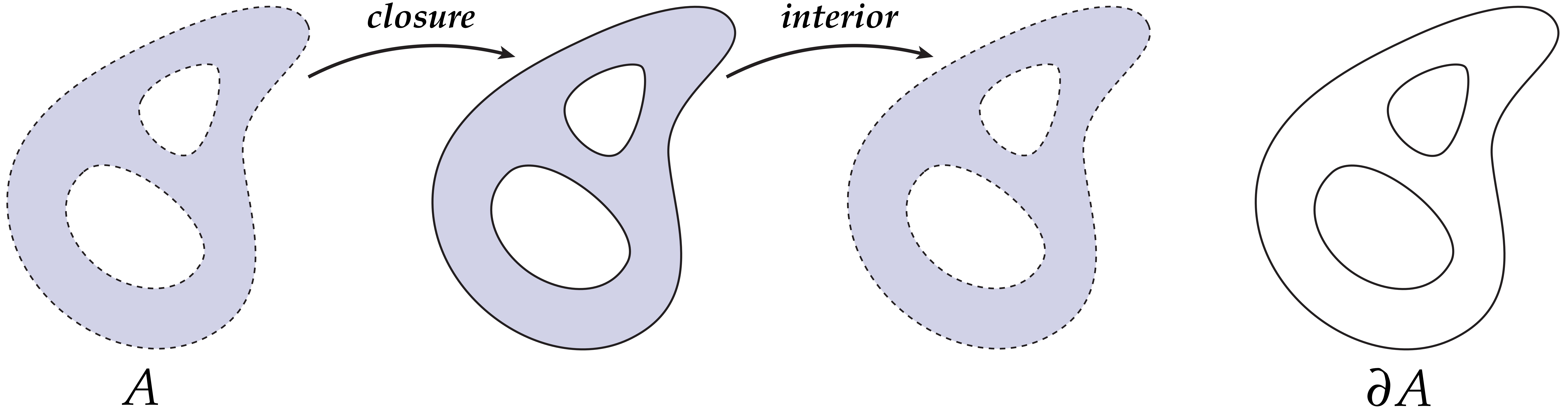
Theorem (Hatcher, Proposition A.1). A simplicial complex (with its standard topology) is compact if and only if it contains finitely many simplices.



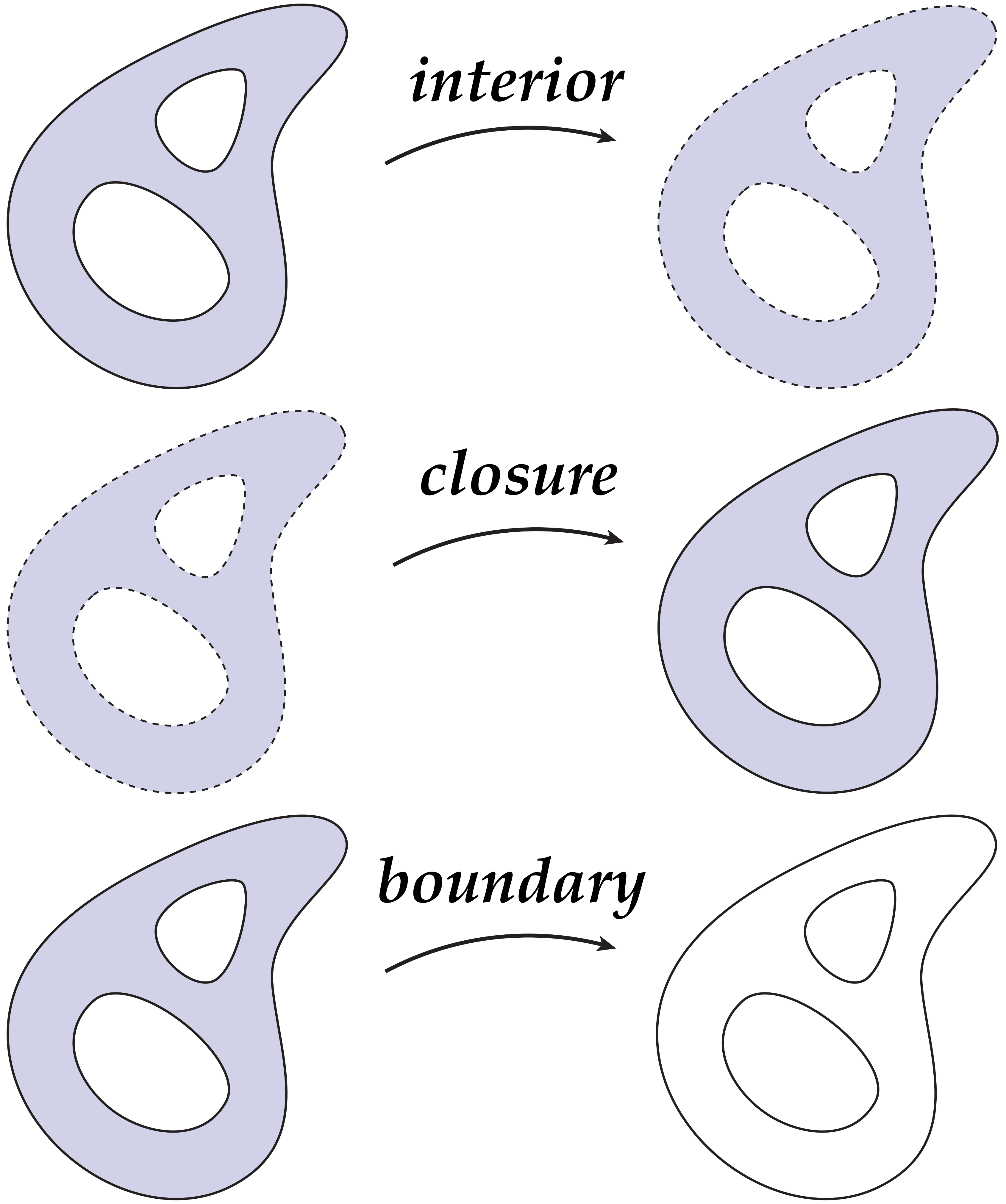
Philosophy: *isn't this the idea we wanted to capture in the first place?*

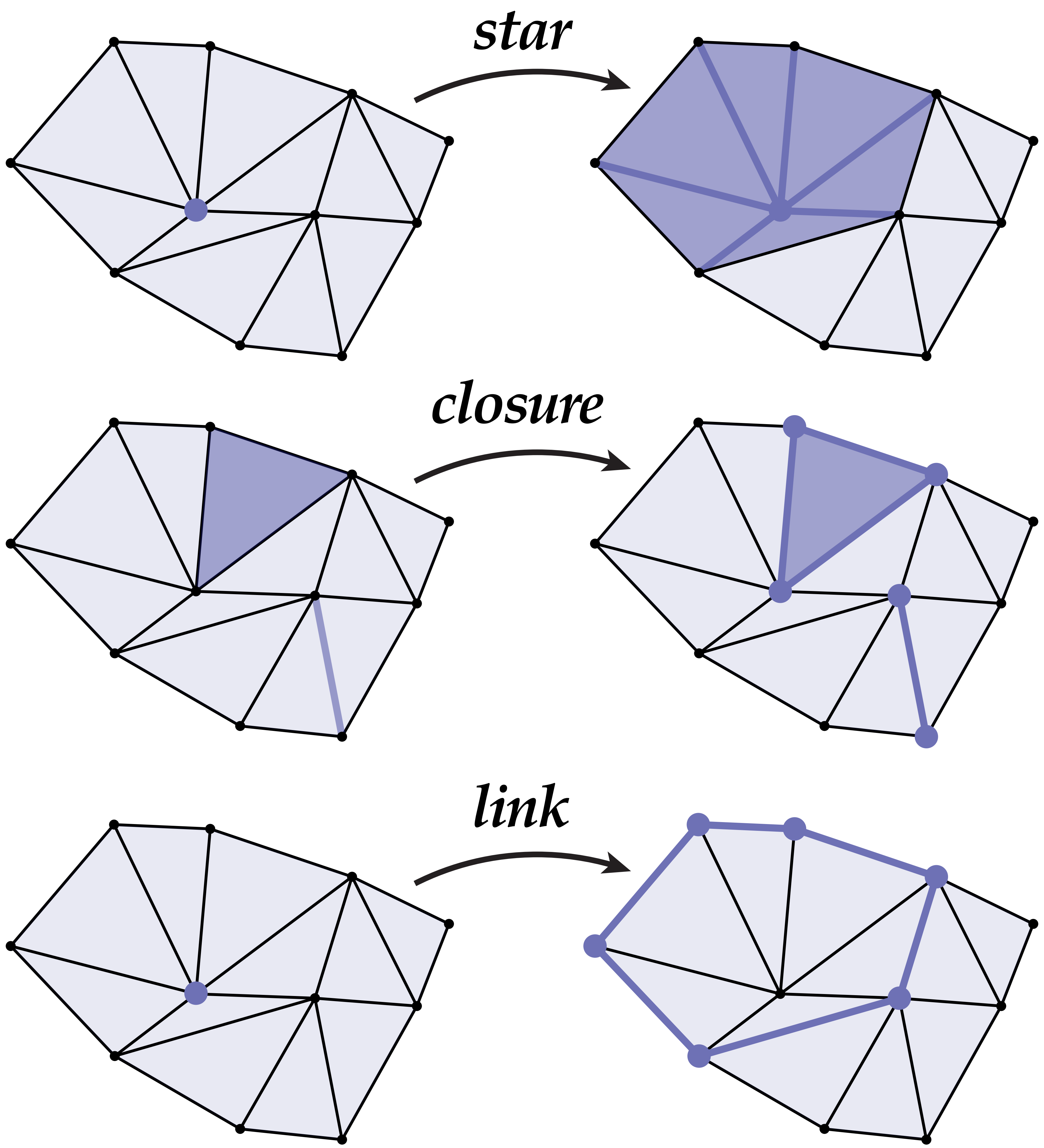
Closure, Interior, and Boundary

Definition. Let A be a subset of a topological space X. The *closure* $\mathrm{cl}(A)$ of A is the smallest closed set containing A; the *interior* $\mathrm{int}(A)$ of A is the largest open set contained by A, and the *boundary* ∂A of A is the intersection of $\mathrm{cl}(A)$ and $\mathrm{cl}(X \setminus A)$.



Star, Link, and Closure—Revisited

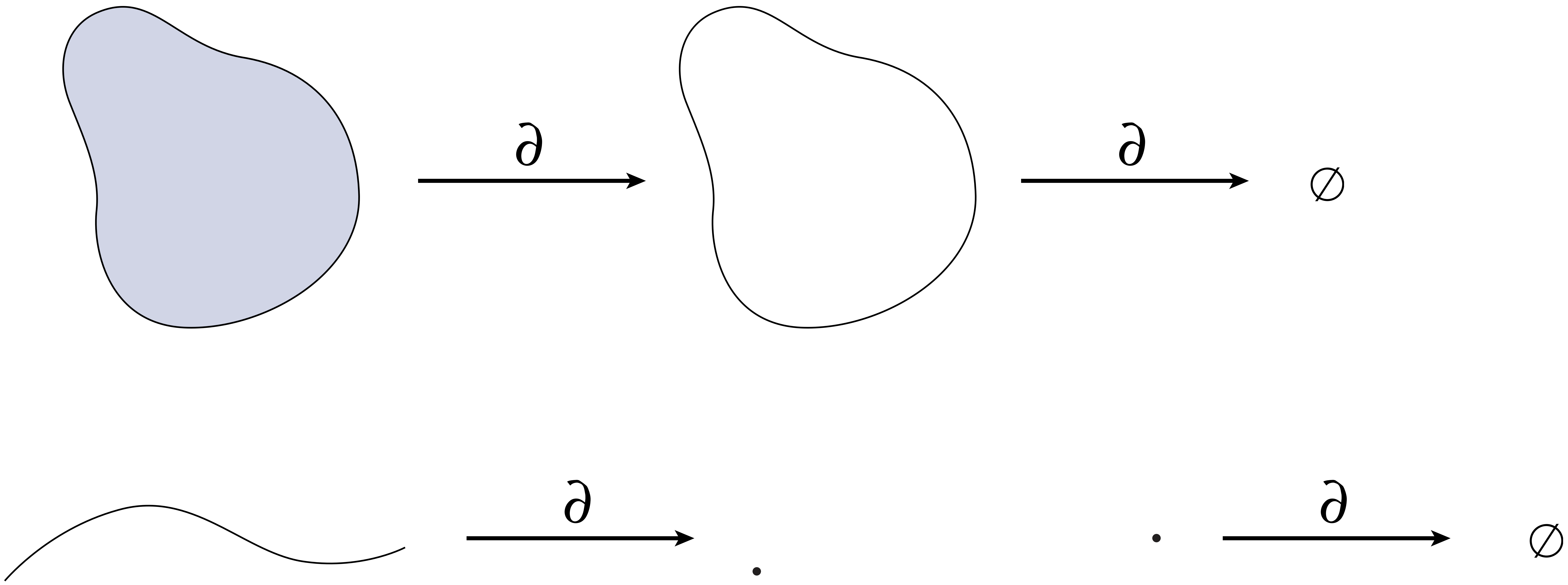




Q: What's the relationship?

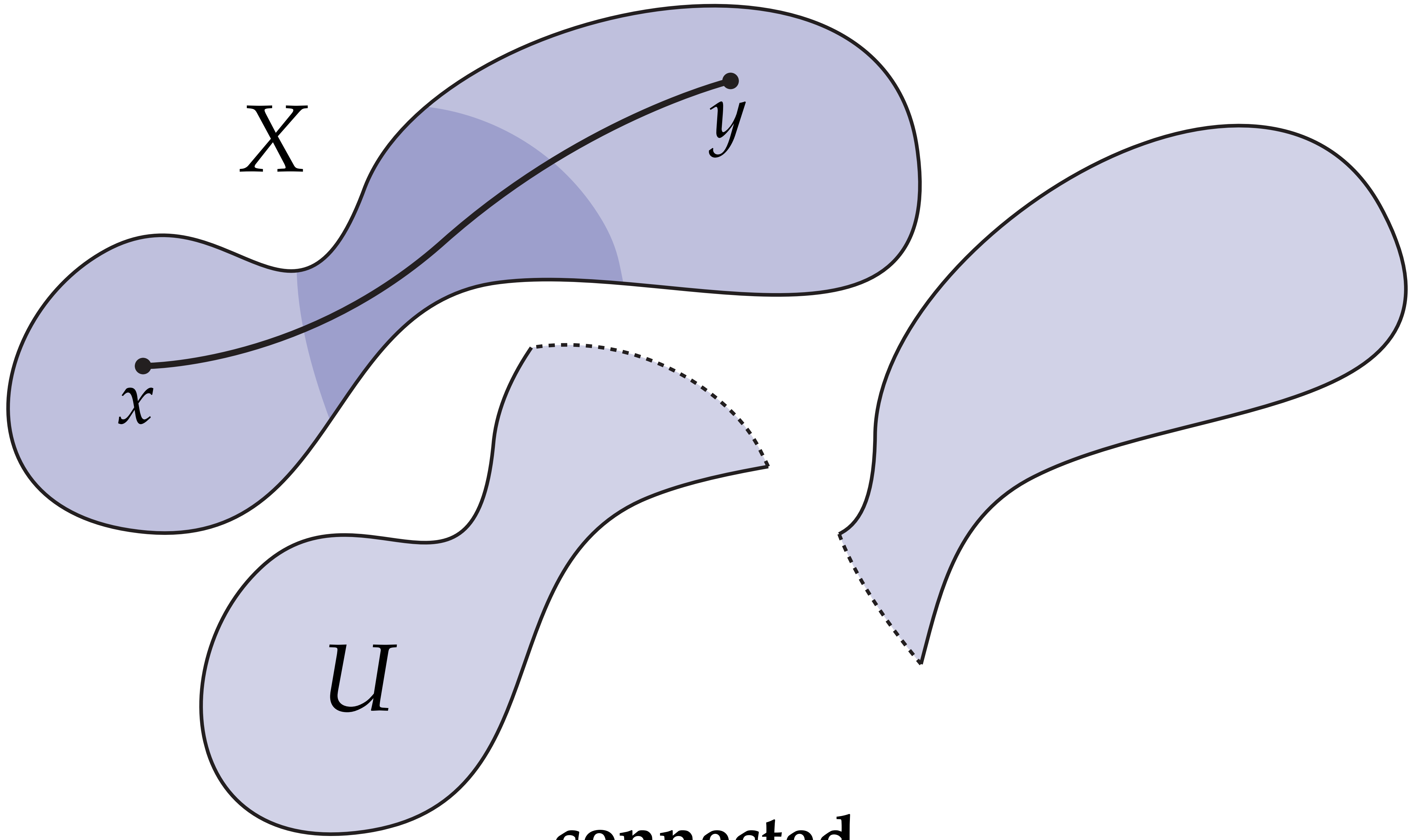
Boundary of a Boundary

Fact. Consider a topological space X. For any subset $A \subset X$, $\partial(\partial A) = \emptyset$, *i.e.*, a boundary has no boundary.

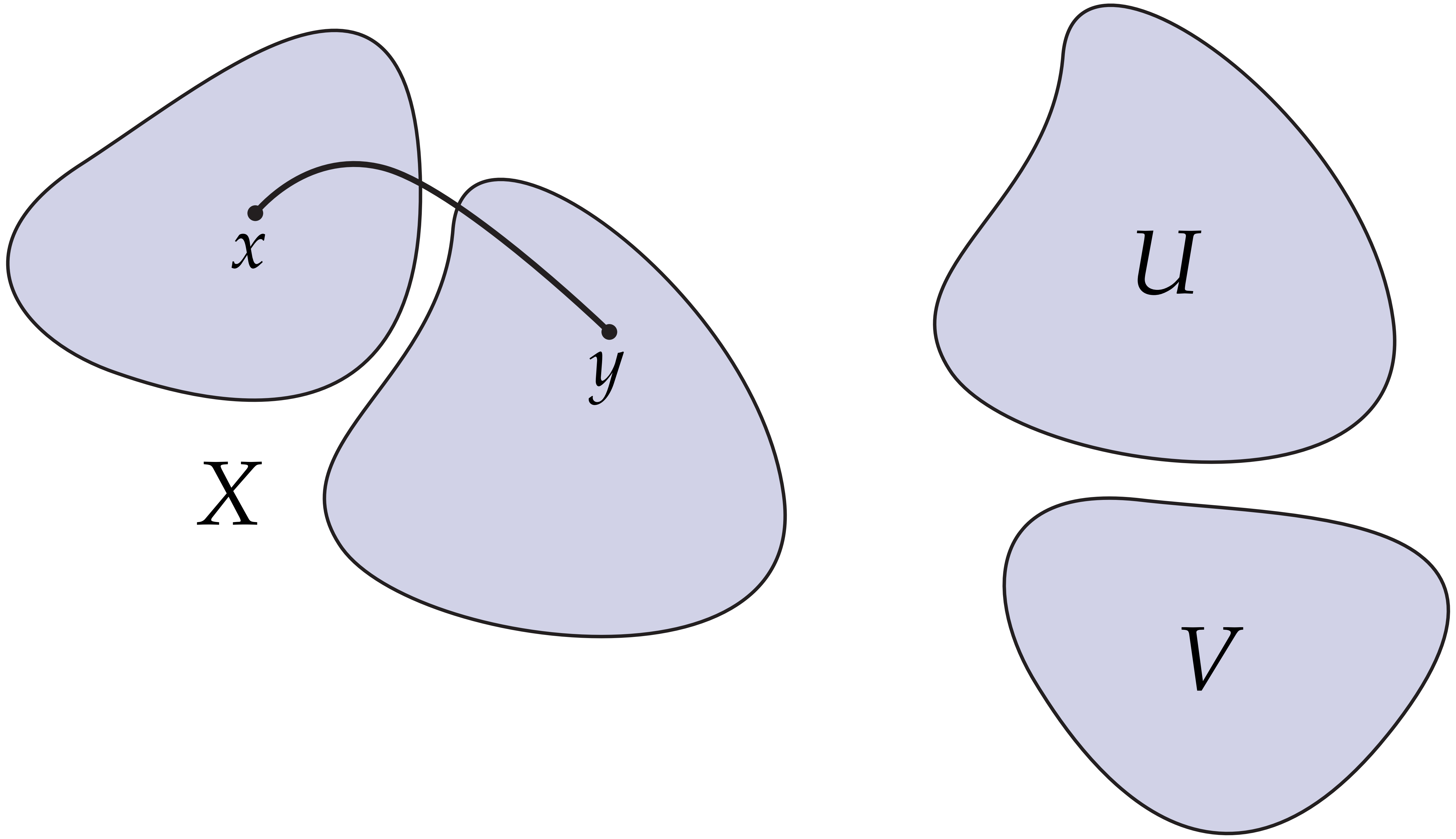


Connected Space

Definition. A topological space (X, τ) is *connected* if it cannot be expressed as the union of two nonempty disjoint open sets, *i.e.*, if there are no sets $U, V \in \tau$ such that $U \cup V = X$. It is *path connected* if for every pair of points $x, y \in X$ there is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ (where the interval has the usual metric topology).



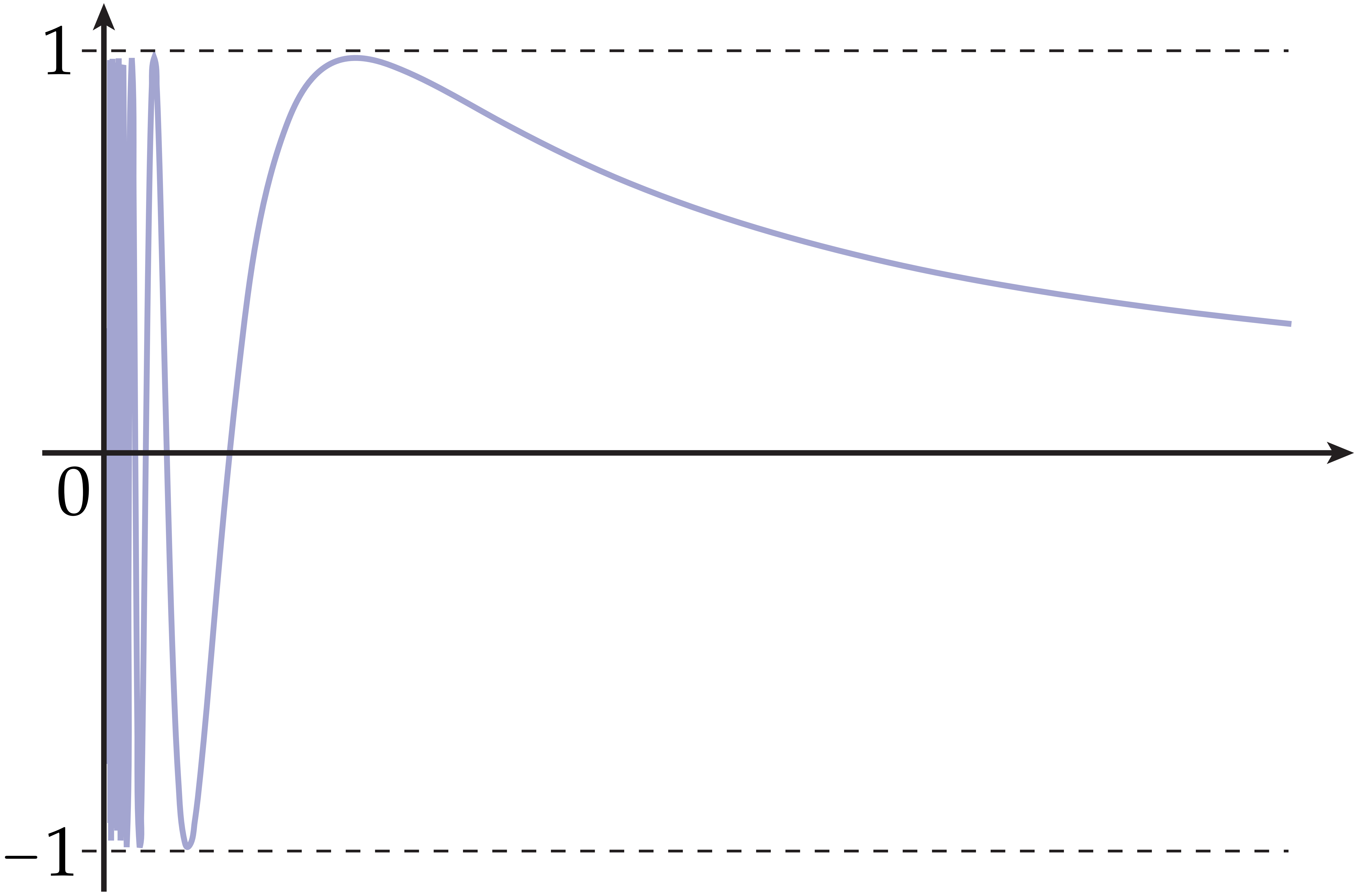
connected



not connected

Connected vs. Path Connected

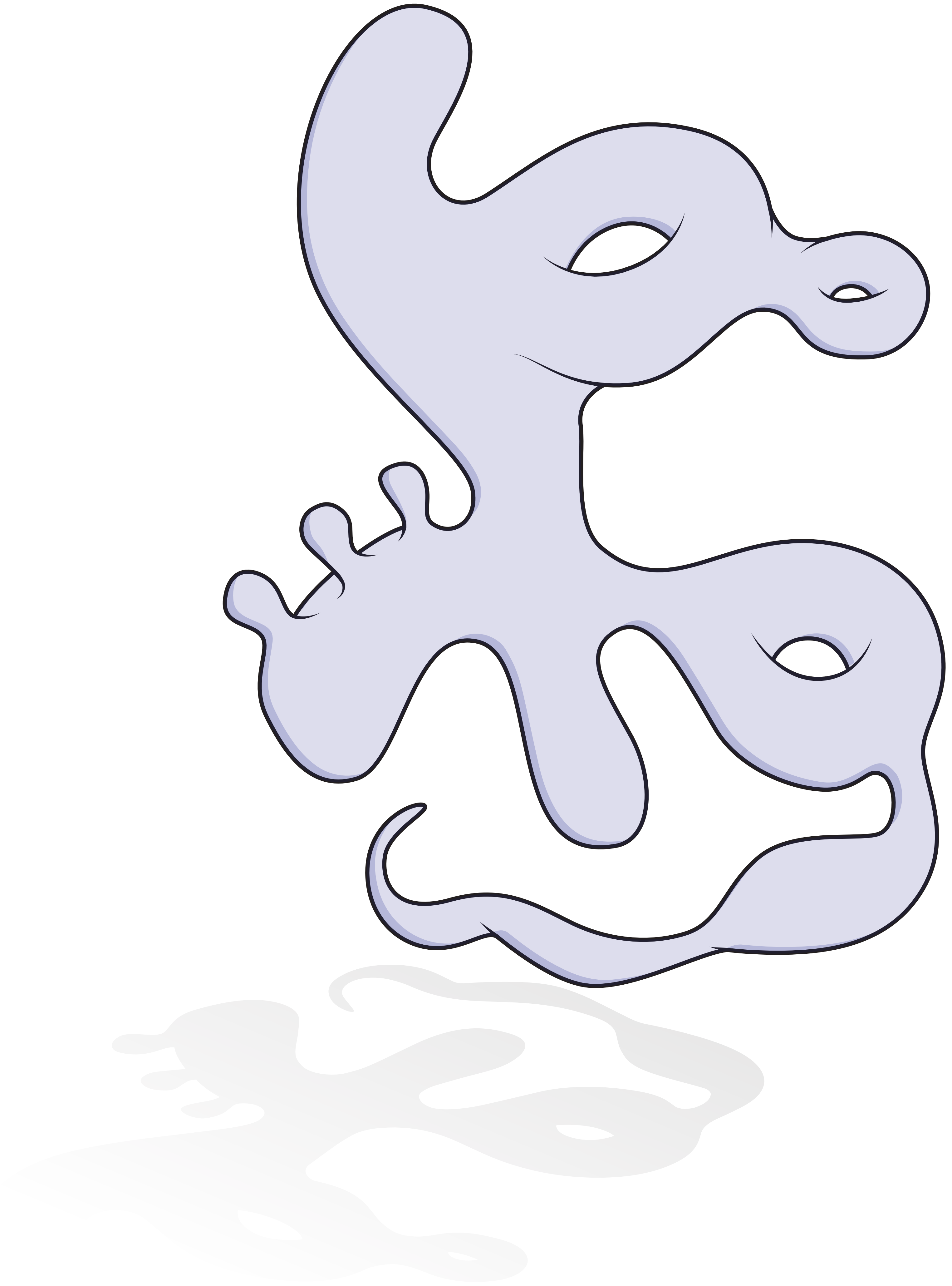
$$X := \{(0,0)\} \cup \{(x, \sin(1/x)) \mid x > 0\}$$



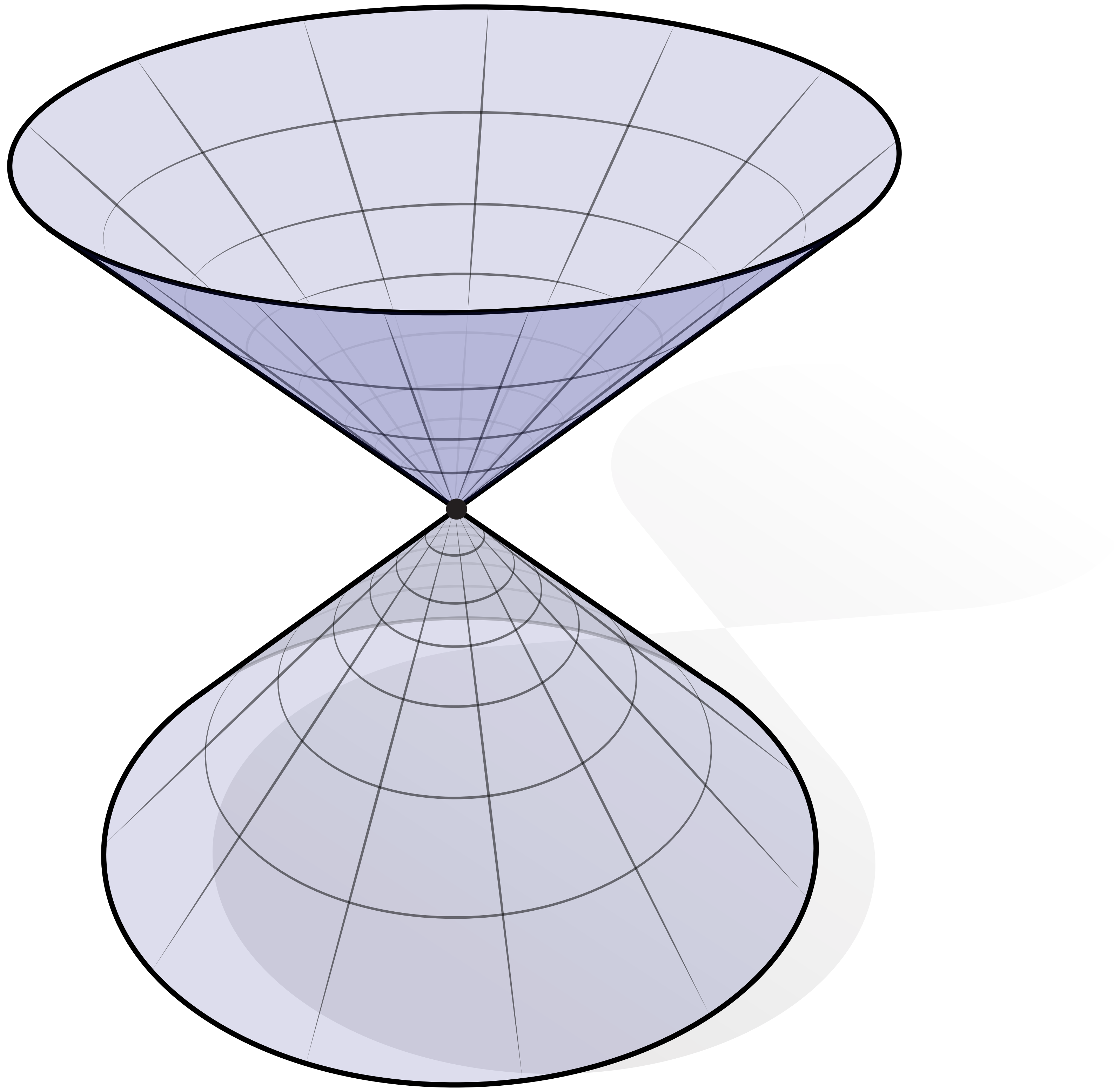
"topologist's sine curve"

Manifold — Visualized

manifold



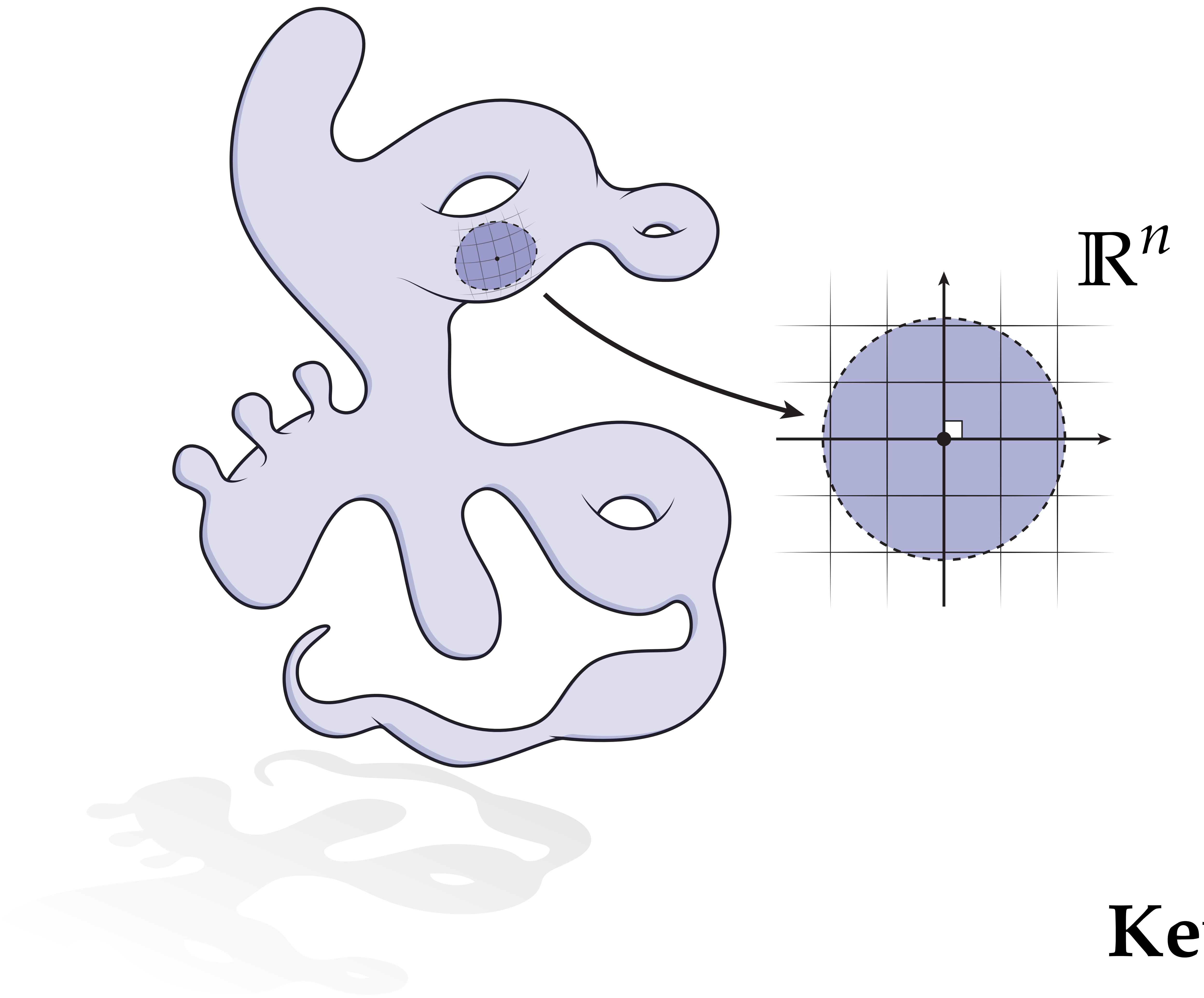
nonmanifold

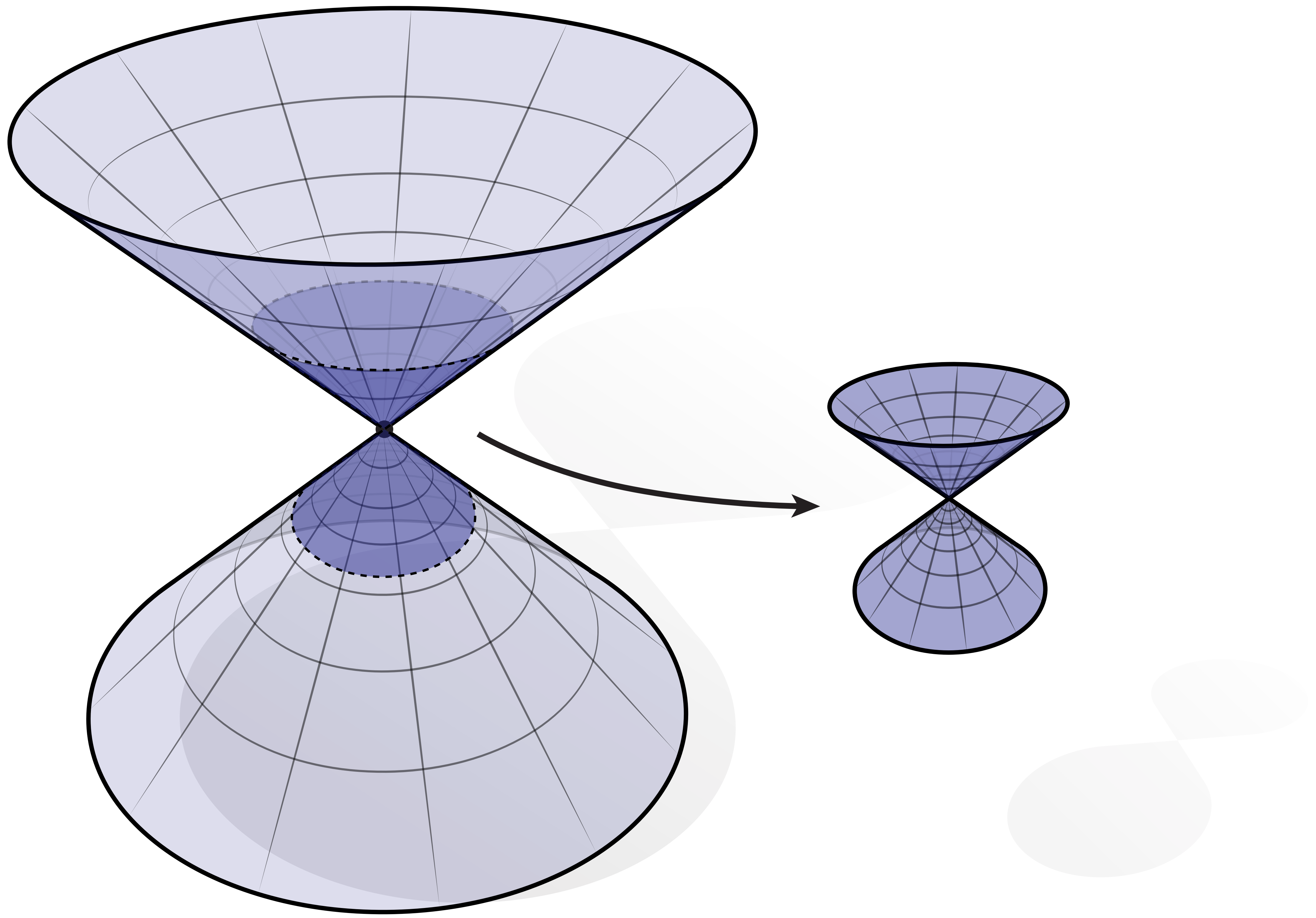


Manifold—Visualized

manifold

nonmanifold

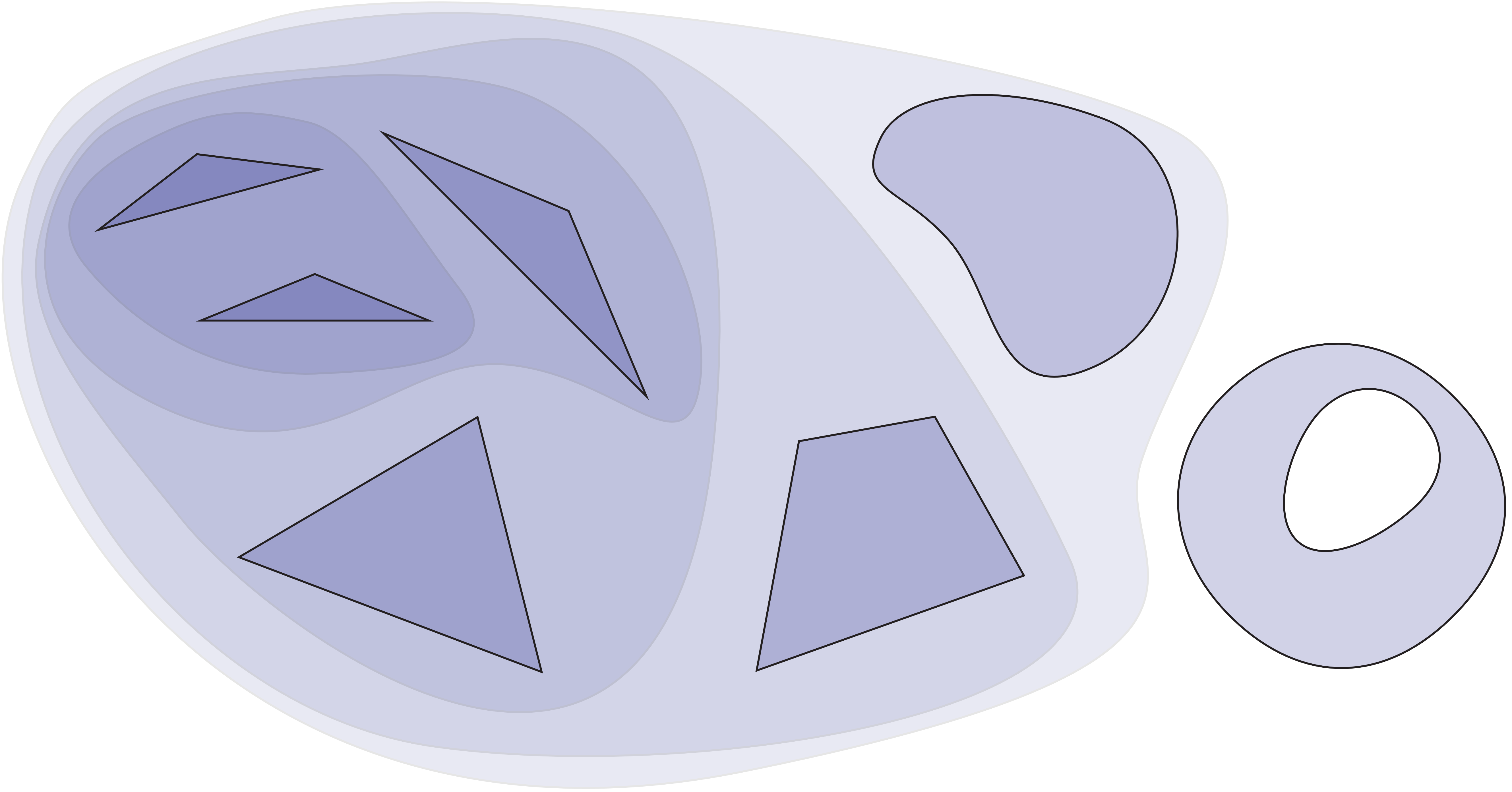




Key idea: “looks Euclidean” up close.

Manifold Structure

Remember...



...many ways to "look the same!"

SET

$\cup$

TOPOLOGICAL SPACE

$\cup$

TOPOLOGICAL MANIFOLD

$\cup$

SMOOTH MANIFOLD

$\cup$

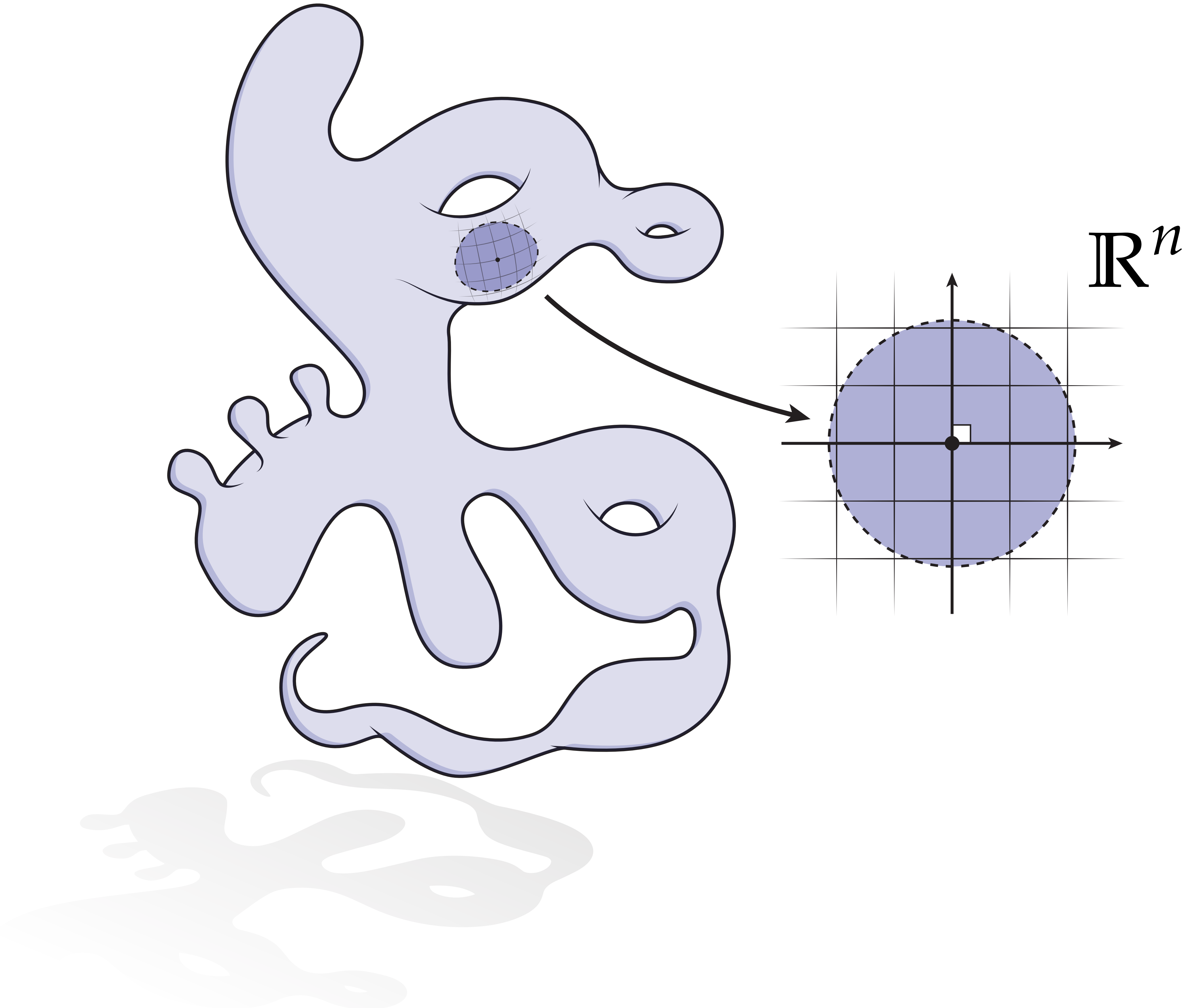
COMPLEX MANIFOLD

$\cup$

RIEMANNIAN MANIFOLD

Topological Manifold—Simple Definition

Definition. Let (M, τ) be a topological space. Then M is a *topological n-manifold* if every point $p \in M$ is contained in a τ-open subset homeomorphic to the unit open disk in $\mathbb{R}^n$, *i.e.*, if M is *locally homeomorphic to* $\mathbb{R}^n$.

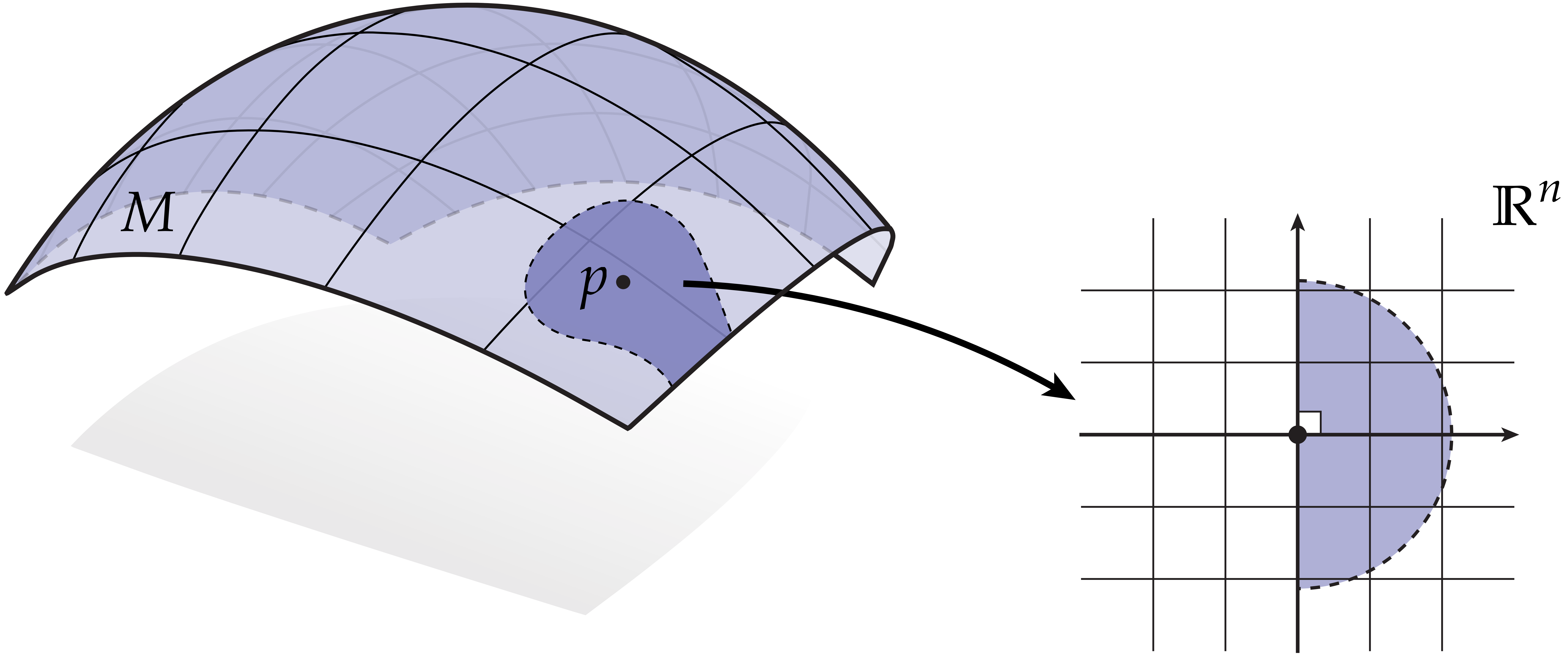


$n = 1$: **"curve"**

$n = 2$: **"surface"**

Topological Manifold with Boundary

Definition. Let (M, τ) be a topological space. Then M is a *topological n-manifold with boundary* if every point $p \in M$ is contained in a τ-open subset homeomorphic to either the unit open disk in $\mathbb{R}^n$, or the intersection of the open disk with a closed half space.



Topological Manifold—Atlas Description

Definition. Consider any set M. A *chart* φ on M is any bijection from a subset $U \subset M$ to an open subset of $\mathbb{R}^n$ (with its usual metric topology). For any two charts φ_i, φ_j, the *overlap map* is the composition

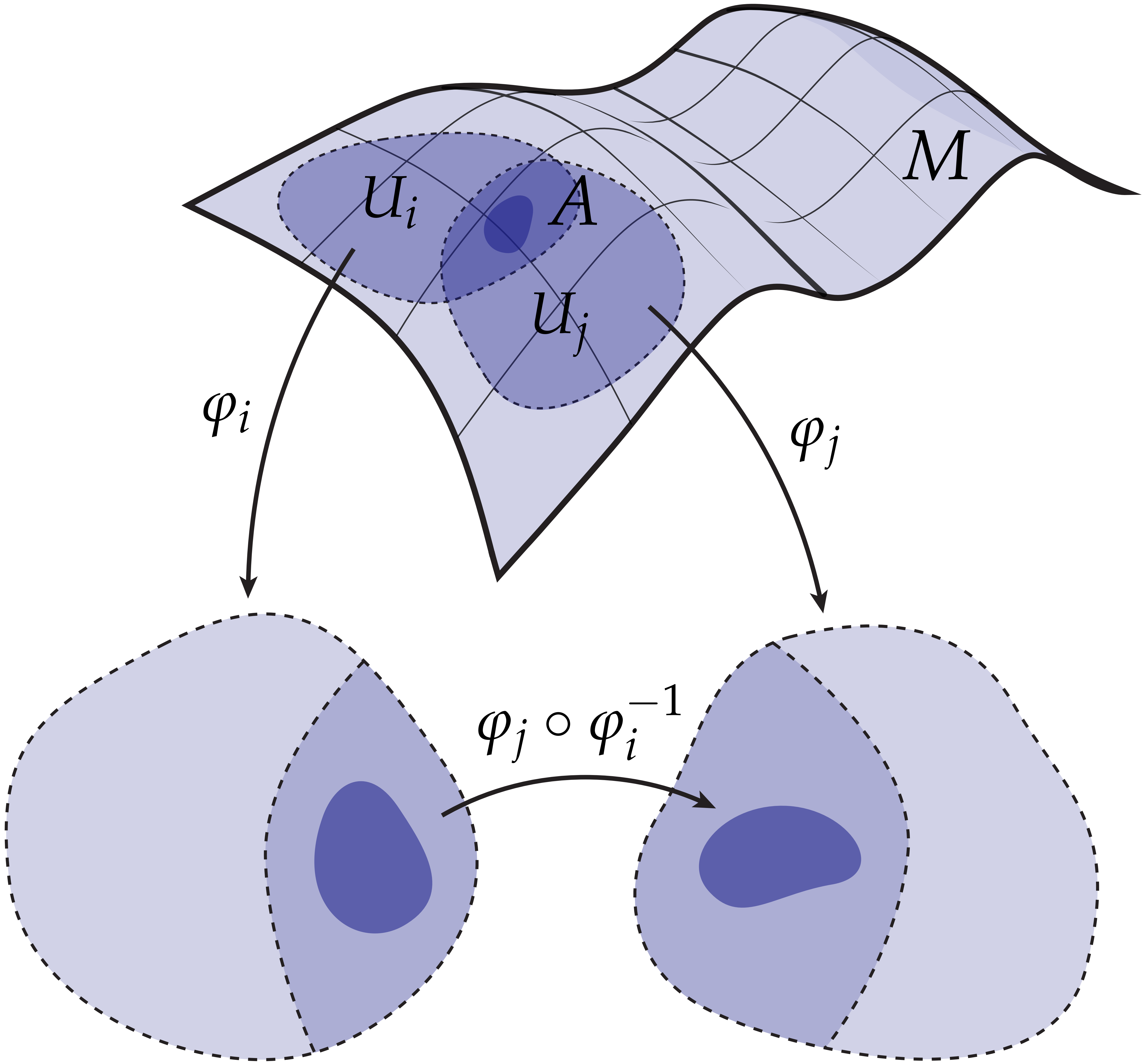
$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}\Big|_{\varphi_i(U_i \cap U_j)} .$$

An *atlas* is a collection of charts $\varphi_i : U_i \to \mathbb{R}^n$ such that

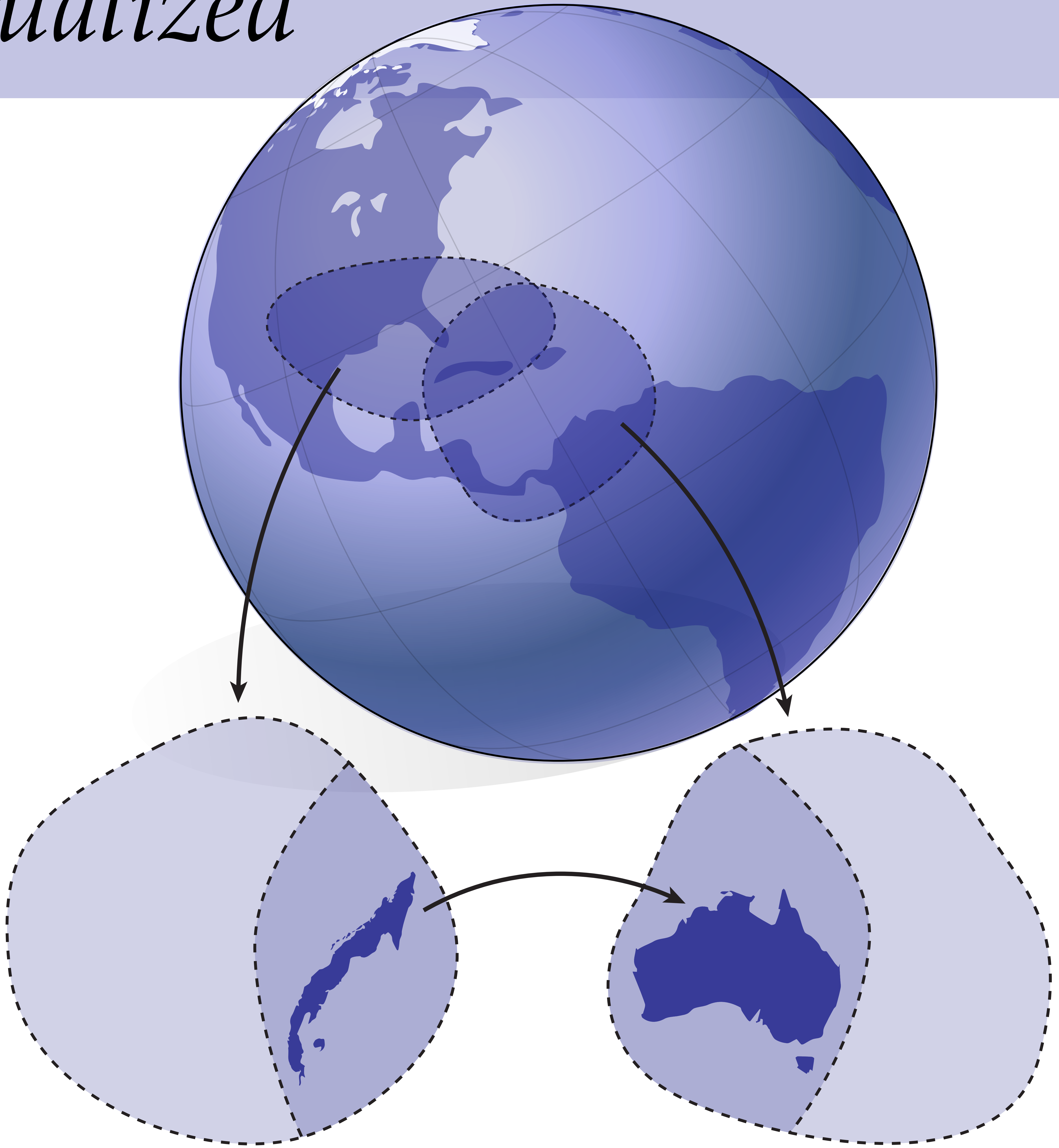
1. $\bigcup_i U_i = M$, and
2. φ_{ij} is a homeomorphism for all i, j.

If M admits an atlas over $\mathbb{R}^n$, then it is a *topological n-manifold*, and A is an open subset of M if and only if it can be expressed as the preimage of an open subset of $\mathbb{R}^n$ with respect to one of the charts φ_i.

Atlas — Visualized



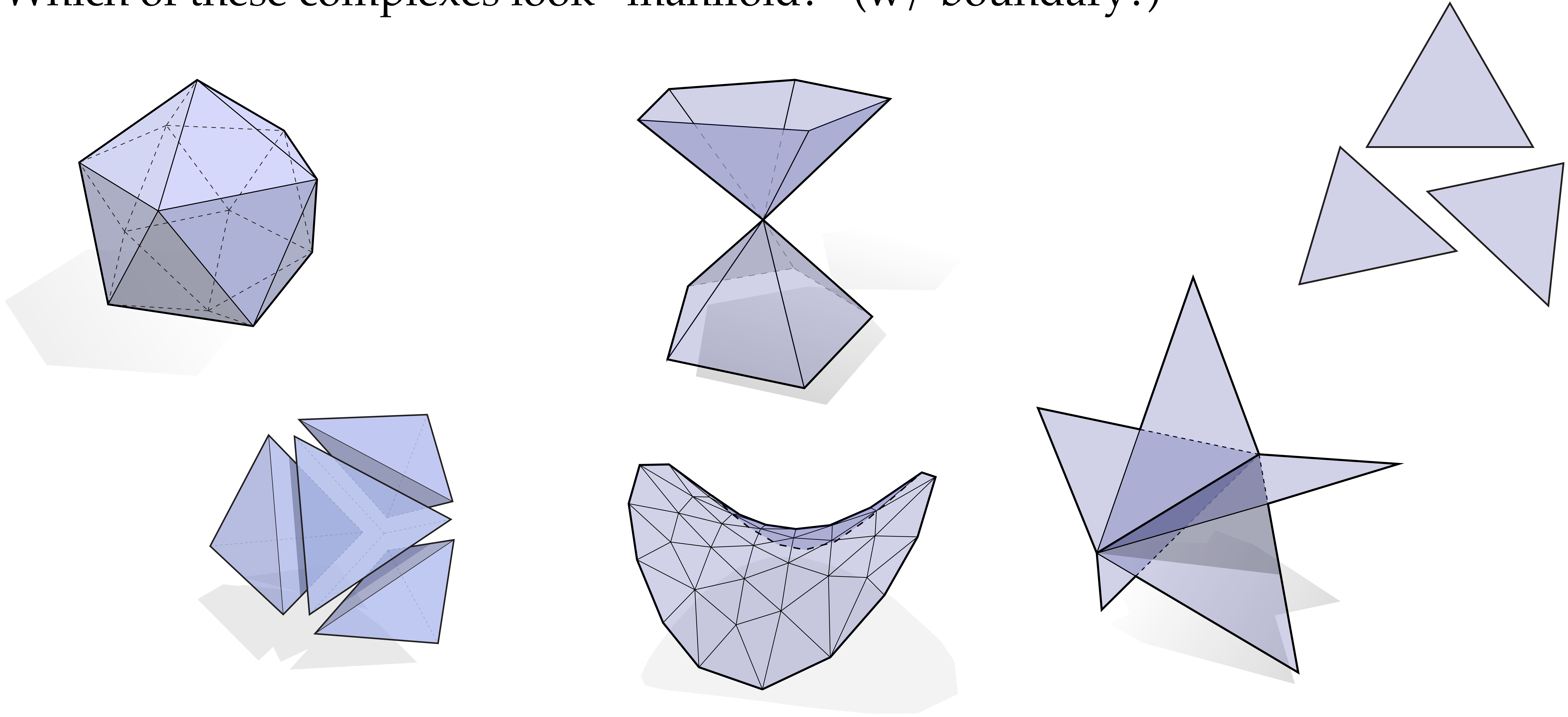
Atlas — Visualized



Charts must agree in
order to be meaningful!

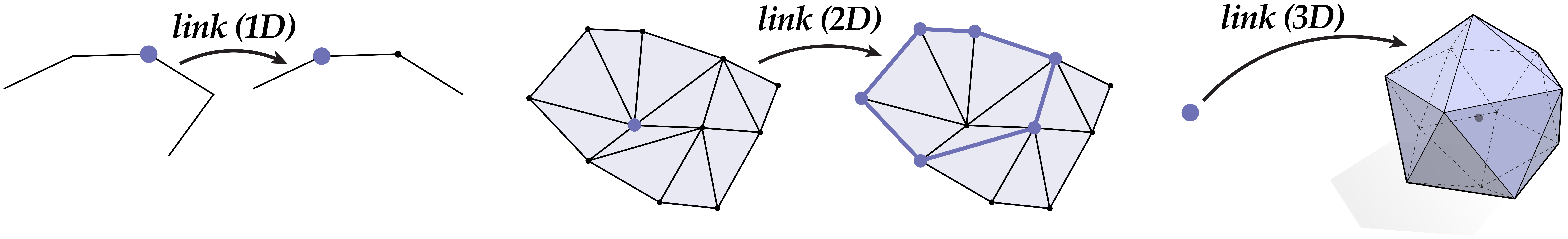
Discrete Manifold—Visualized

Which of these complexes look “manifold?” (w/ boundary?)



Discrete Topological Manifold

Fact. Let K be an abstract simplicial complex, and let τ be the topology on X that comes from gluing together standard simplices. Then (K, τ) is a topological n-manifold if and only if each vertex link is homeomorphic to the $(n-1)$-sphere.



Theorem (S. Schleimer)**.** 3-sphere detection is in NP.

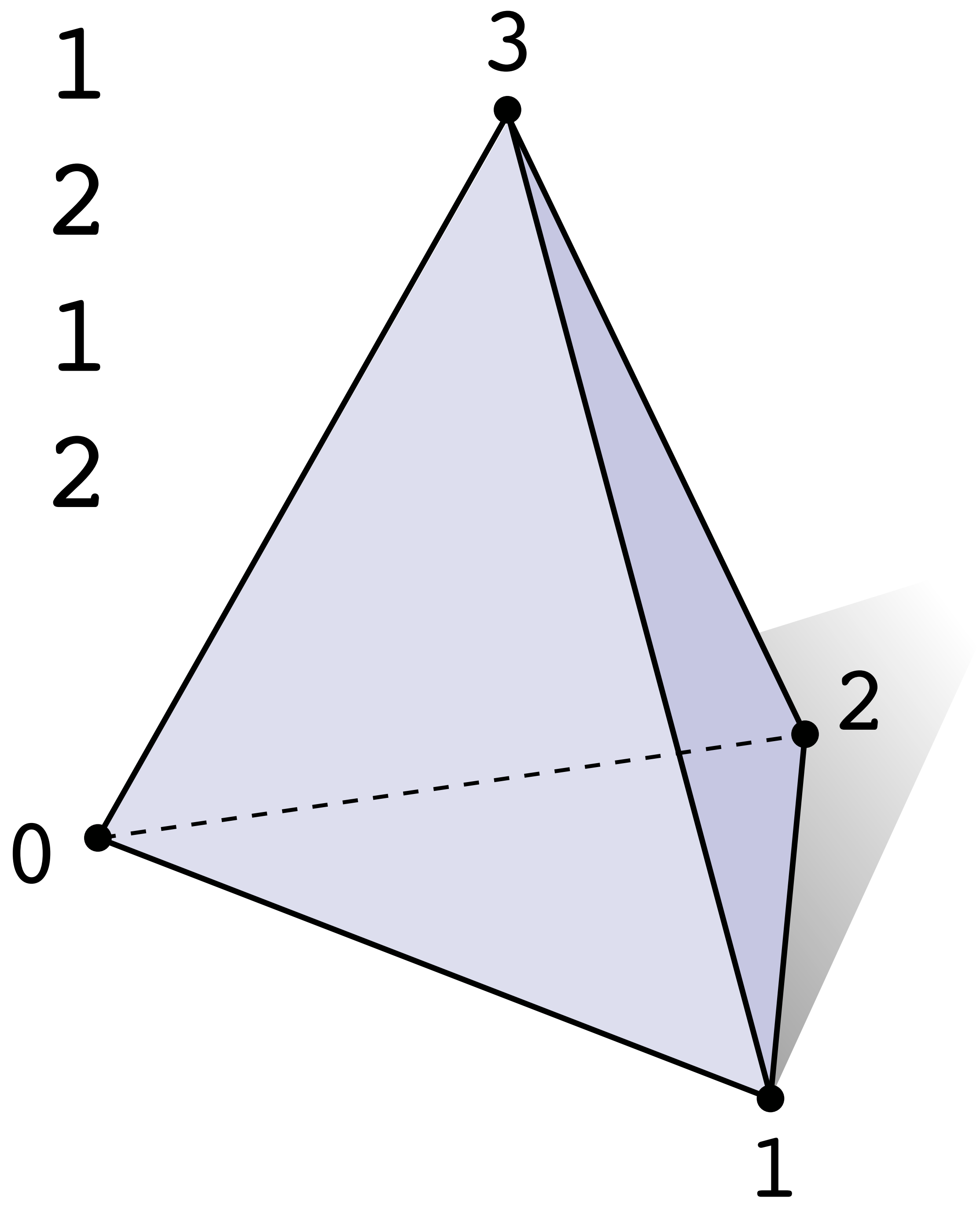
Checking whether a 4-simplicial complex is manifold is difficult. For dimension 5 and greater, we don't even have *algorithms*—independent of the question of complexity/decidability.

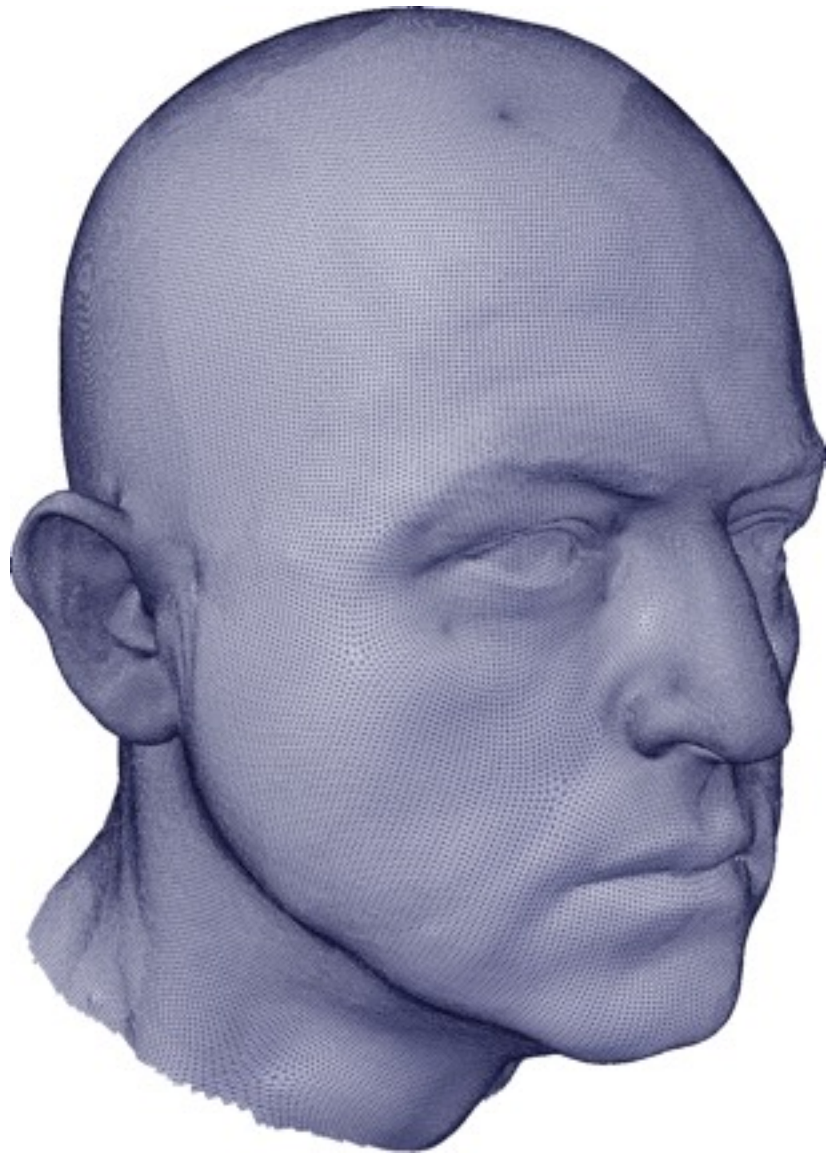
Topological Data Structures—Simplicial Soup

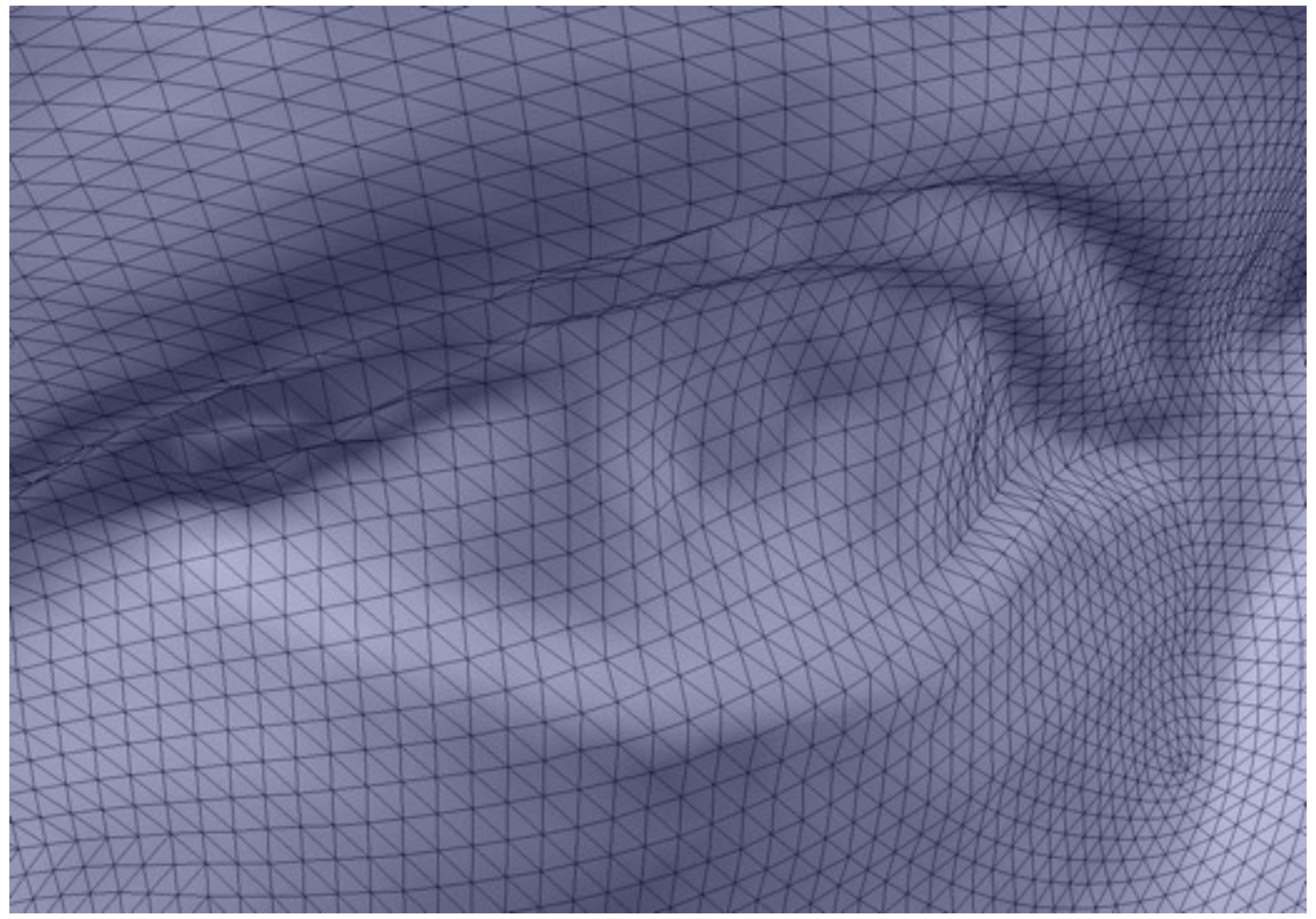
- Store only top-dimensional simplices
- Implicitly includes all facets
- Pros: simple, small storage cost
- Cons: hard to access neighbors

Example.

0	2	1
0	3	2
3	0	1
3	1	2







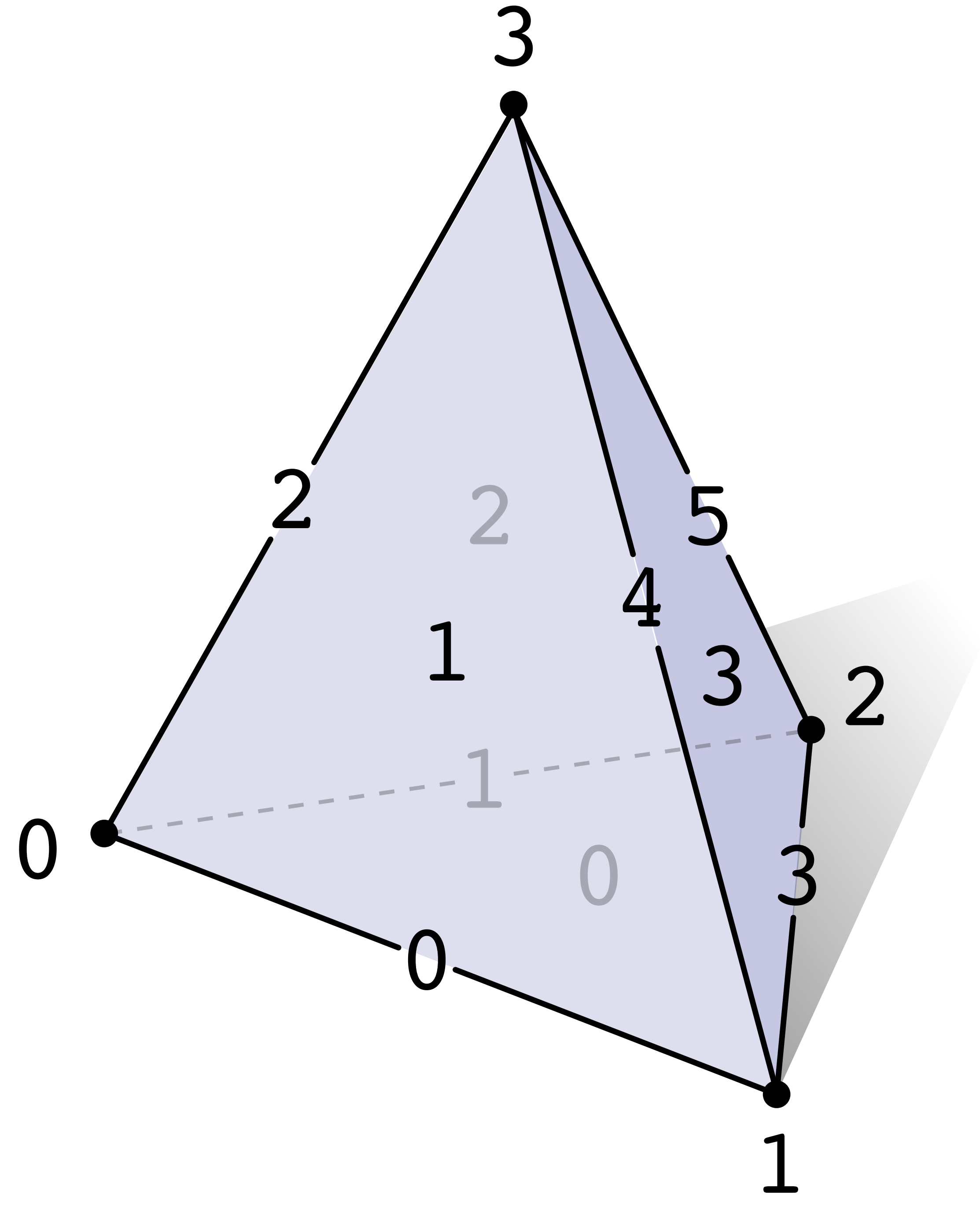
Q: How might you list all edges touching a given vertex? *What's the cost?*

Topological Data Structures—Incidence Matrix

Definition. Let K be a simplicial complex, let n_k denote the number of k-simplices in K, and suppose that for each k we give the k-simplices a canonical ordering so that they can be specified via indices $1, \ldots, n_k$. The kth *incidence matrix* is then a $n_{k+1} \times n_k$ matrix E^k with entries $E^k_{ij} = 1$ if the jth k-simplex is contained in the ith $(k+1)$-simplex, and $E^k_{ij} = 0$ otherwise.

Example.

$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad E^1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

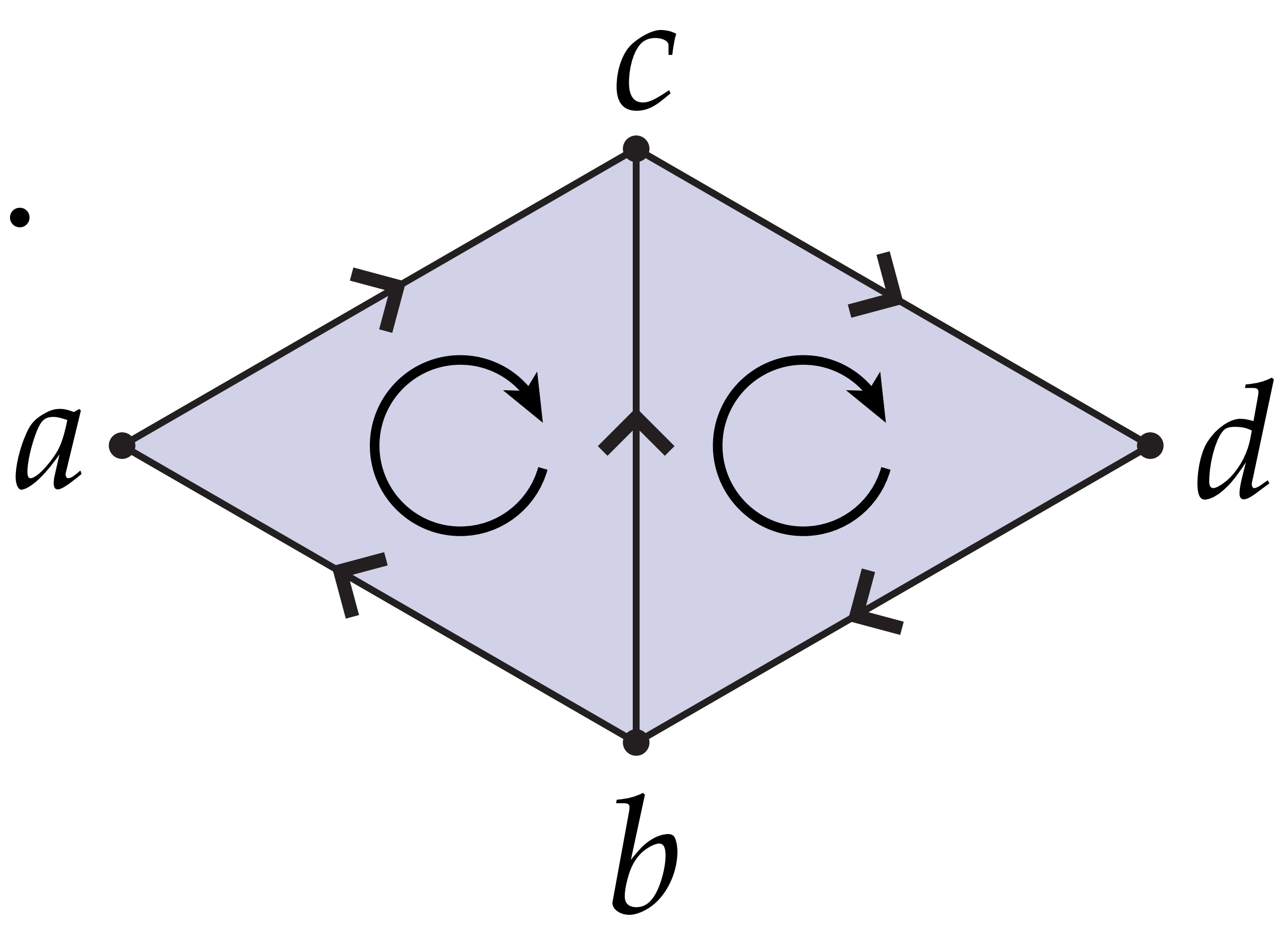


Q: Now what's the cost of finding edges incident on a given vertex?

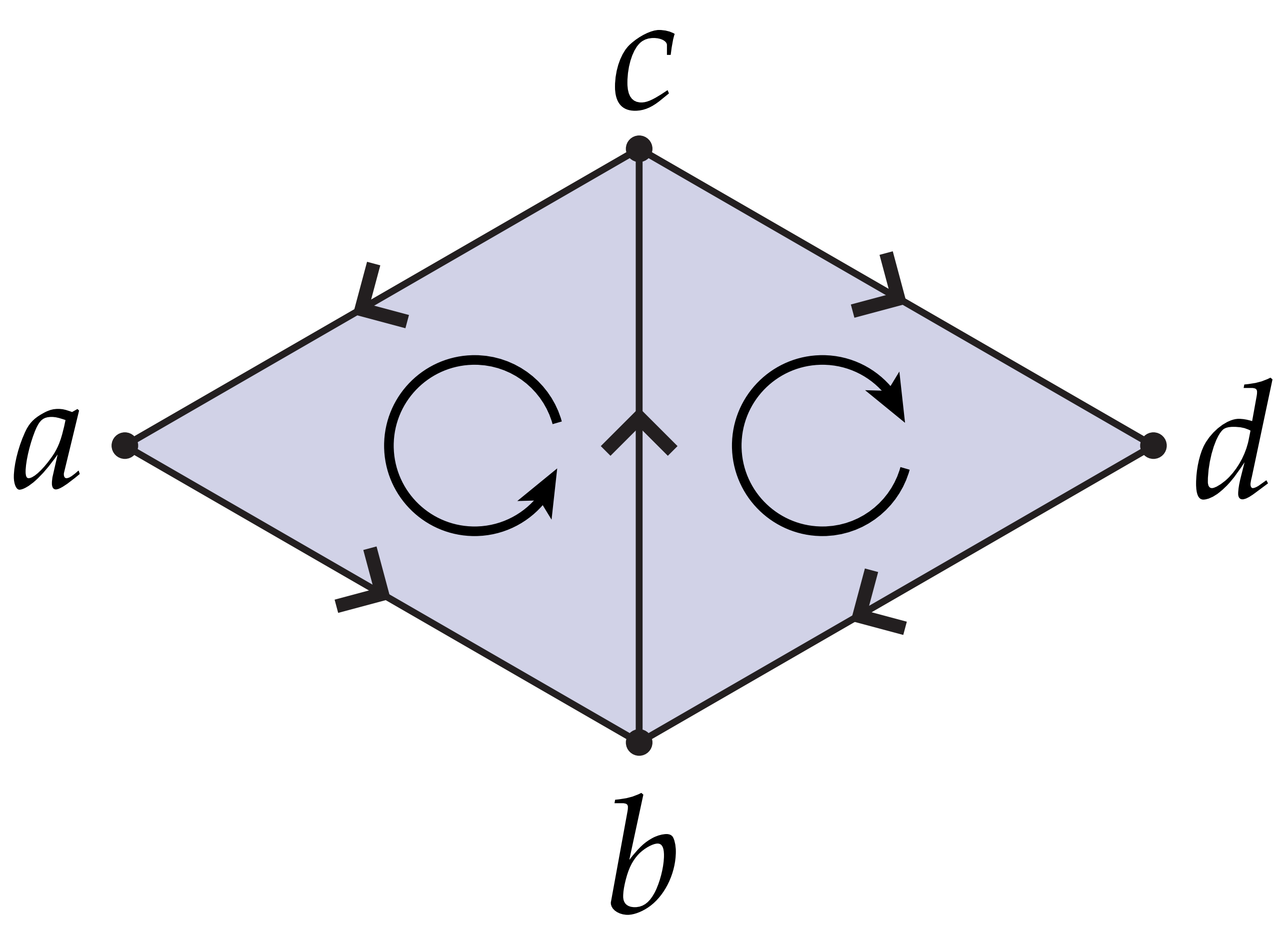
Oriented Simplicial Complex

Definition. An *orientation* of a simplex is an ordering of its vertices up to even permutation; one can specify an oriented simplex via one of its representative ordered tuples. An *oriented simplicial complex* is a simplicial complex where each simplex is given an ordering.

Example.



$$\{\emptyset, (a), (b), (c), (d),$$
$$(a,c), (b,a), (b,c), (c,d), (d,b),$$
$$(a,c,b), (b,c,d)\}$$

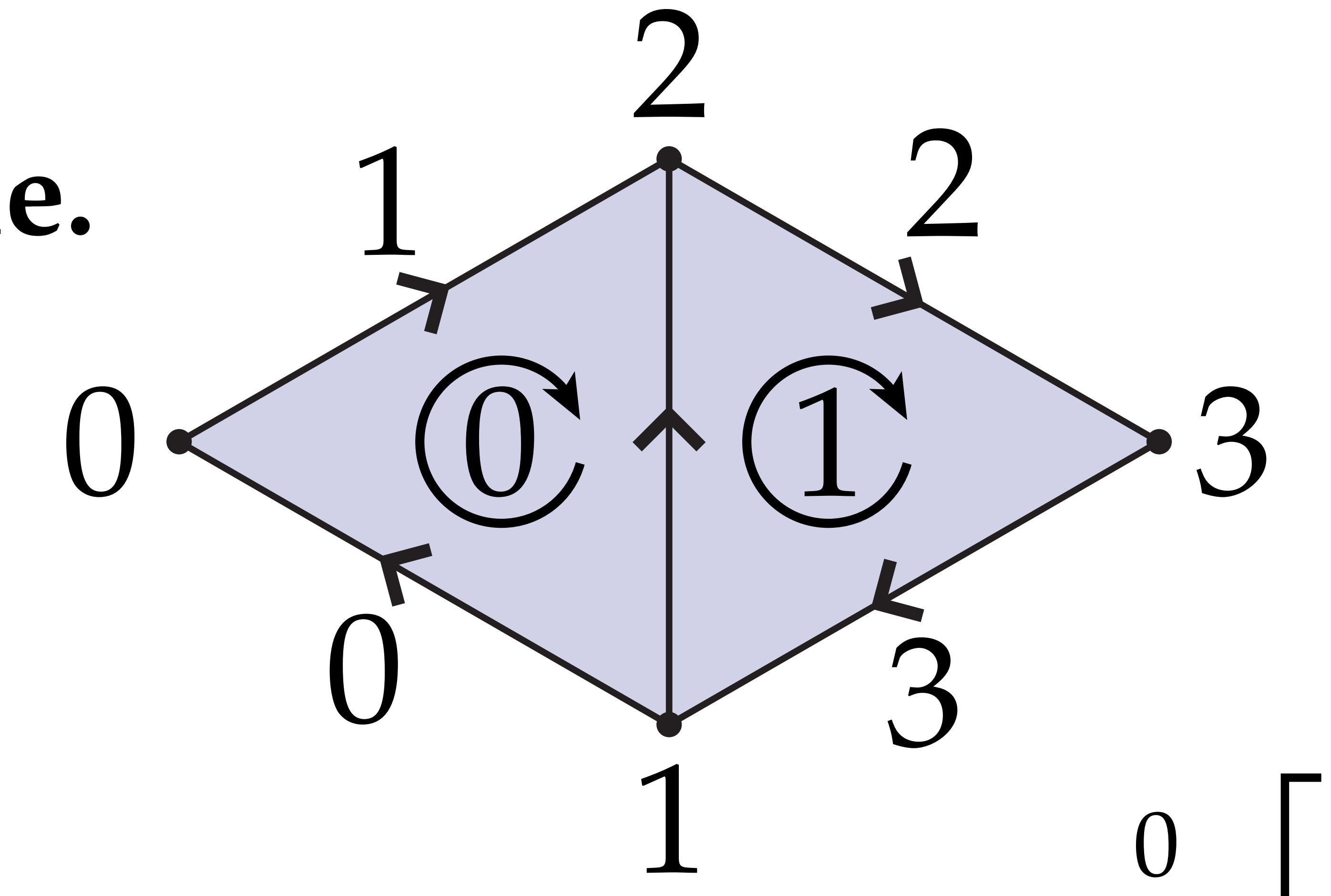


$$\{\emptyset, (a), (b), (c), (d),$$
$$(c,a), (a,b), (b,c), (c,d), (d,b),$$
$$(a,b,c), (b,c,d)\}$$

Data Structures—Signed Incidence Matrix

Definition. A *signed incidence matrix* is an incidence matrix where the sign of each nonzero entry is determined by the *relative orientation* of the two corresponding simplices: positive if they have the same ordering (up to even permutation) on the shared vertex set; negative otherwise. (**Exception:** vertices/edges.)

Example.



$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array} \qquad E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

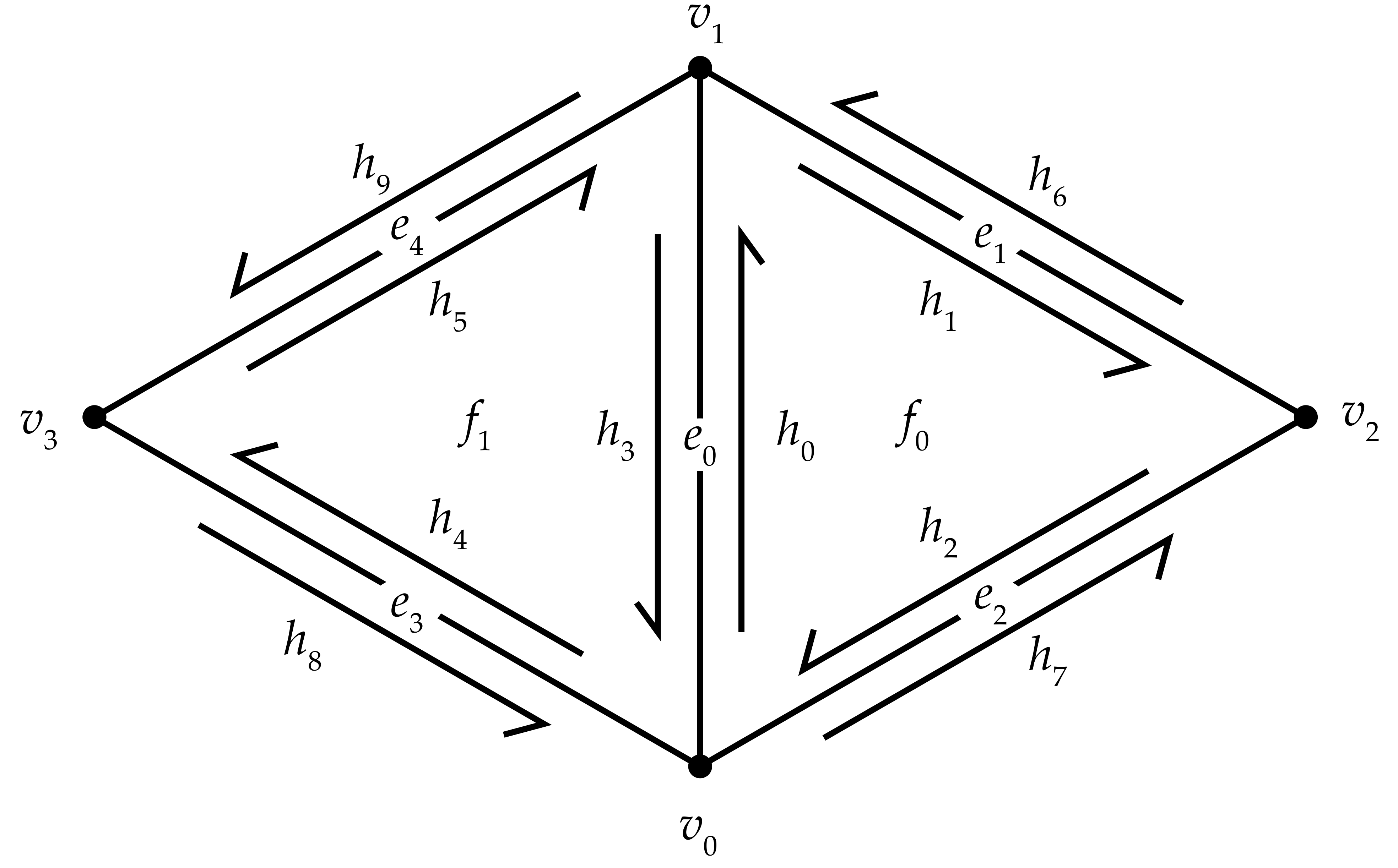
Data Structures—Half Edge

Definition. Let H be any set with an even number of elements, let $\rho : H \to H$ be any permutation of H, and let $\eta : H \to H$ be an involution without any fixed points, *i.e.*, $\eta \circ \eta = \mathrm{id}$ and $\eta(h) \neq h$ for any $h \in H$. Then (H, ρ, η) is a *half edge mesh*, the elements of H are called *half edges*, the orbits of η are *edges*, the orbits of ρ are *faces*, and the orbits of $\eta \circ \rho$ are *vertices*.

Fact. Every half edge mesh describes a compact oriented topological surface (without boundary).

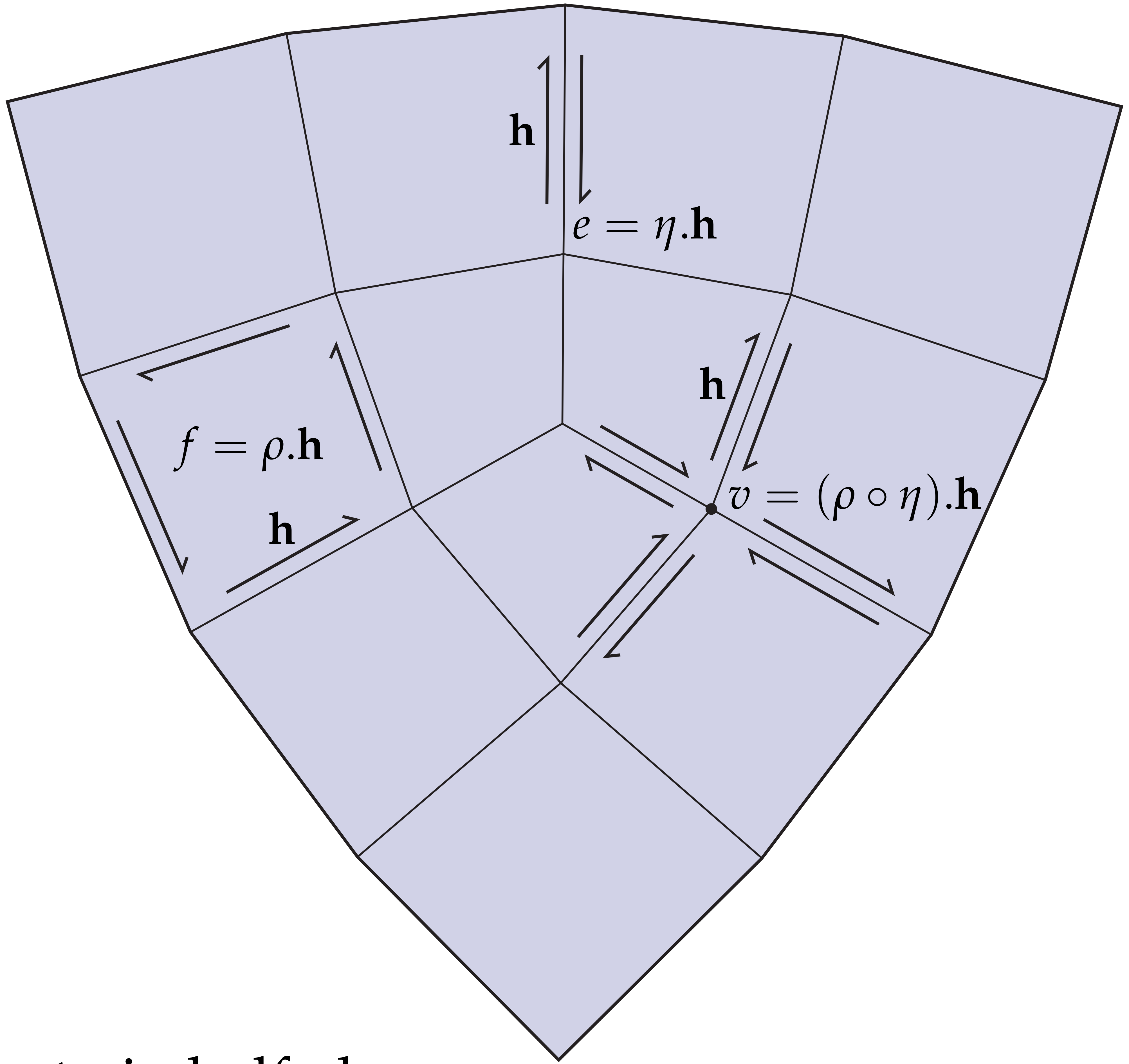
$$(h_1, \cdots, h_{10}) \underset{\textbf{“next”}}{\overset{\rho}{\mapsto}} (h_1, h_2, h_0, h_4, h_5, h_3, h_9, h_6, h_7, h_8)$$

$$(h_1, \cdots, h_{10}) \underset{\textbf{“twin”}}{\overset{\eta}{\mapsto}} (h_3, h_6, h_7, h_0, h_8, h_9, h_1, h_2, h_4, h_5)$$



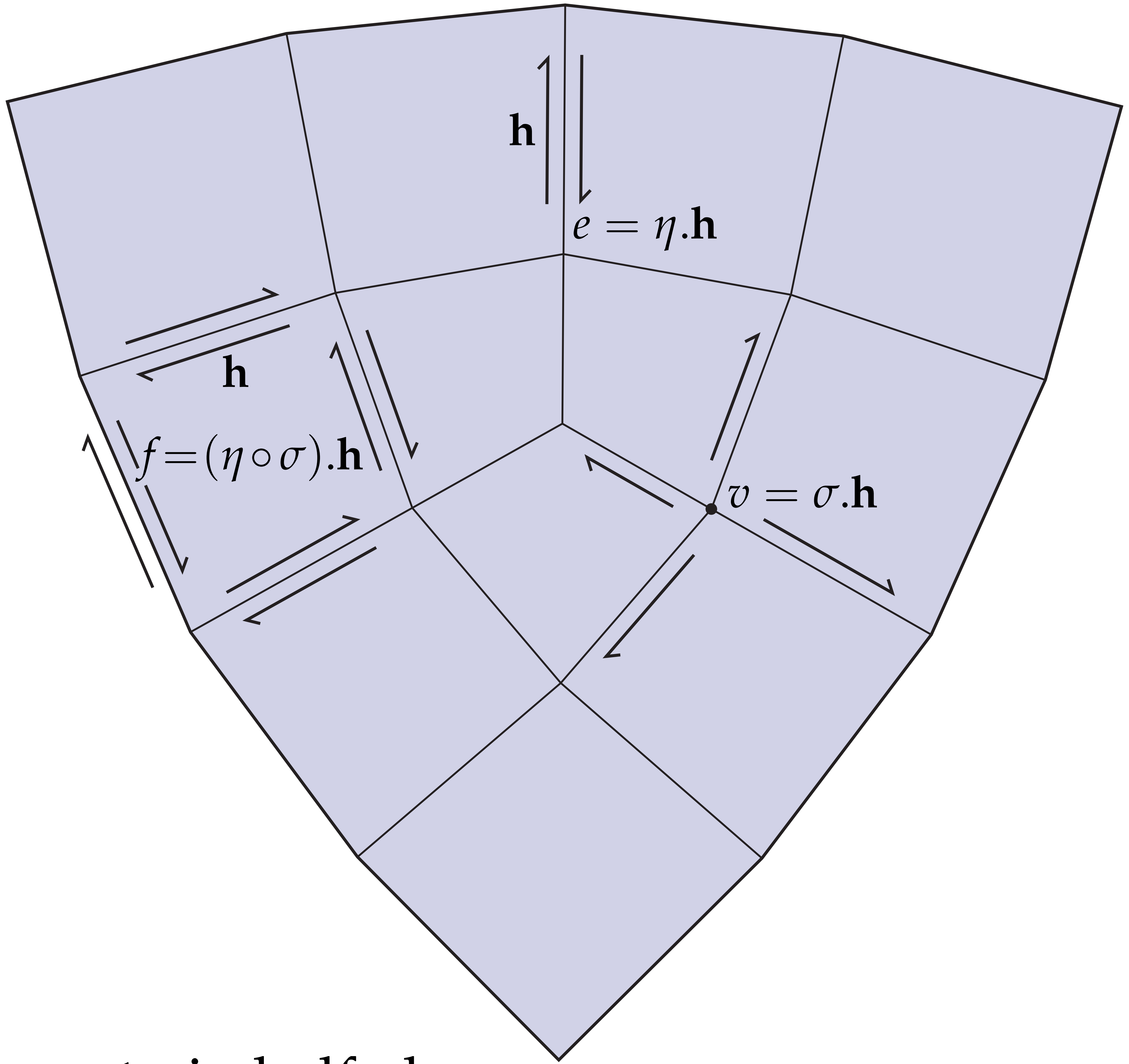
Half Edge vs. Rotation System

Half Edge



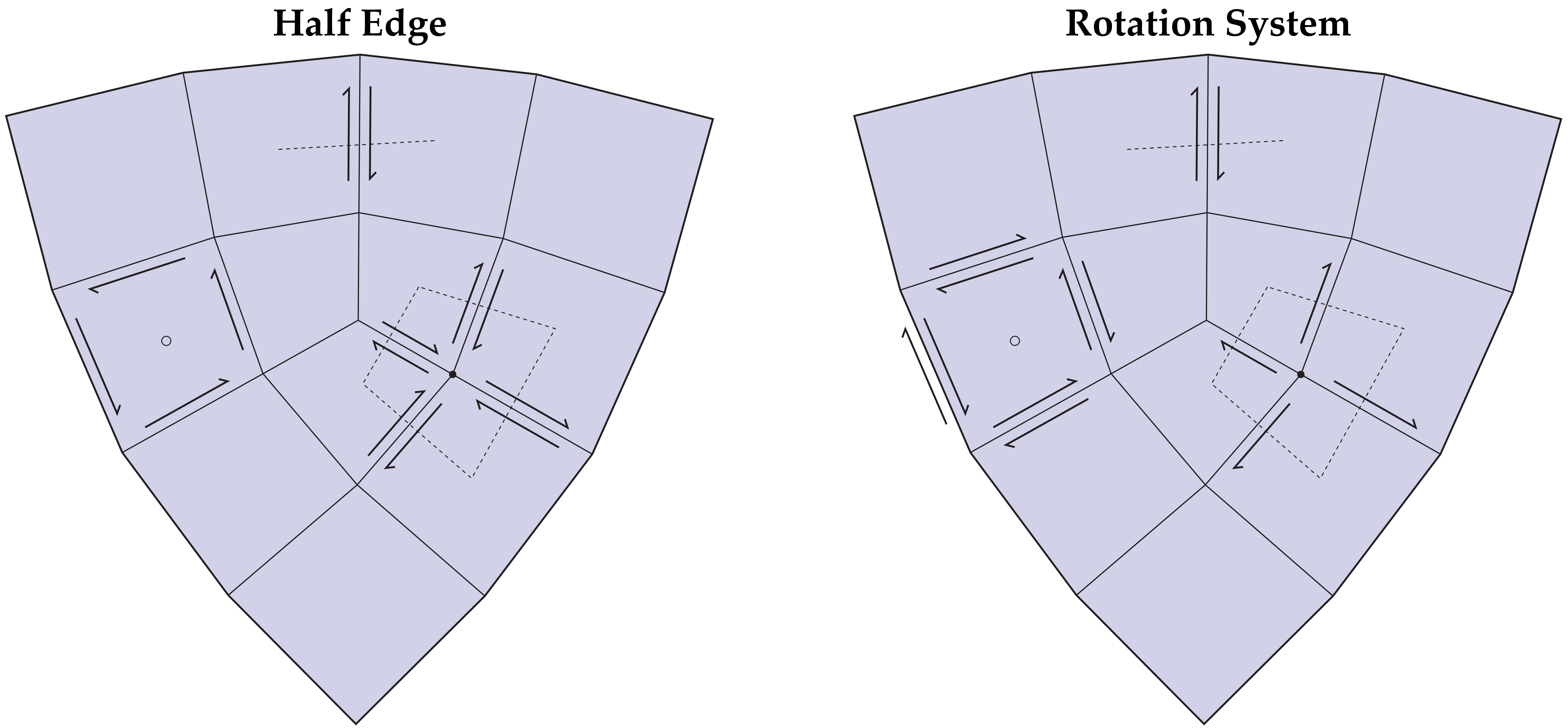
η: twin halfedge
ρ: next halfedge around face

Rotation System



η: twin halfedge
σ: next halfedge around vertex

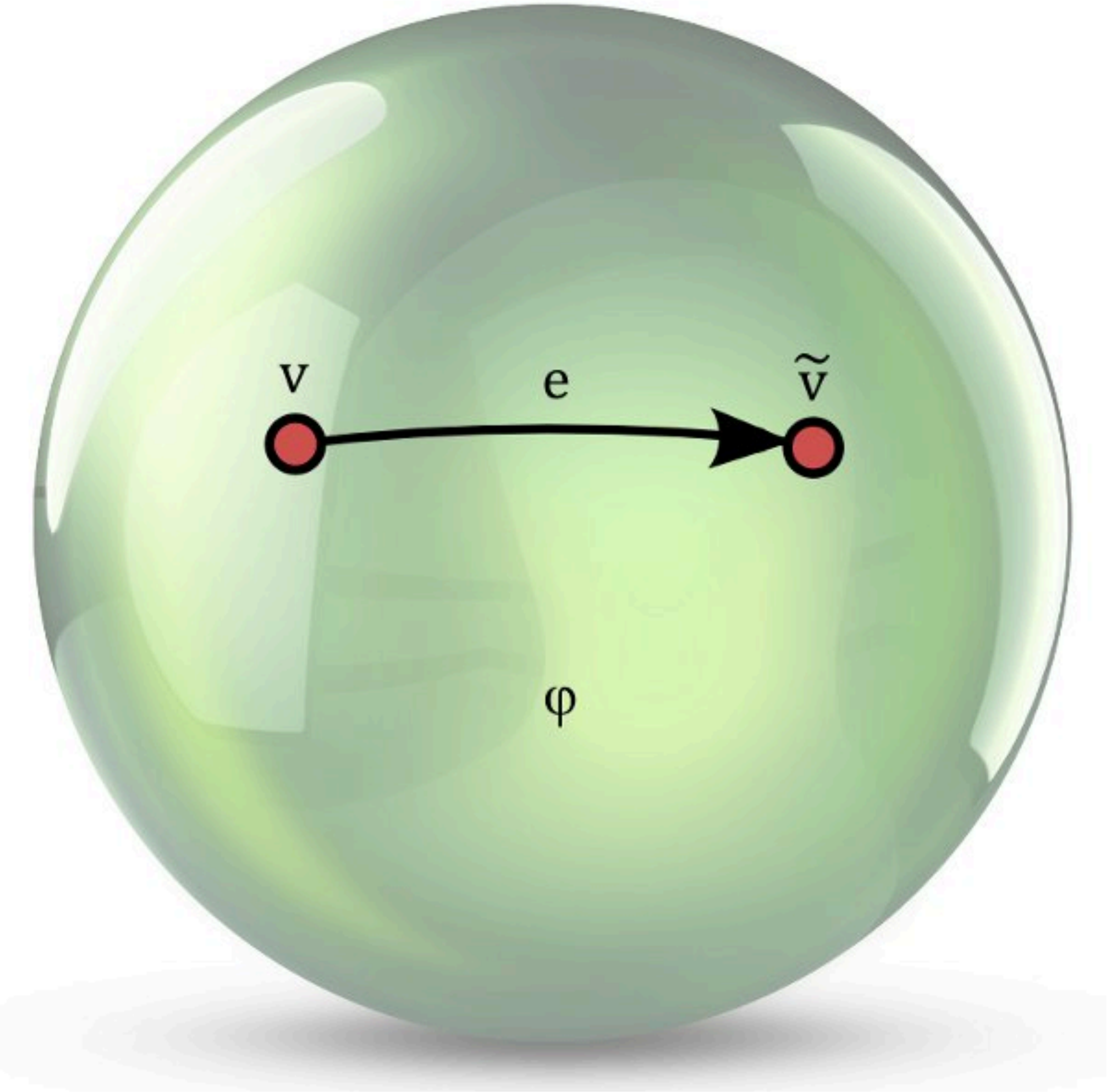
Half Edge vs. Rotation System

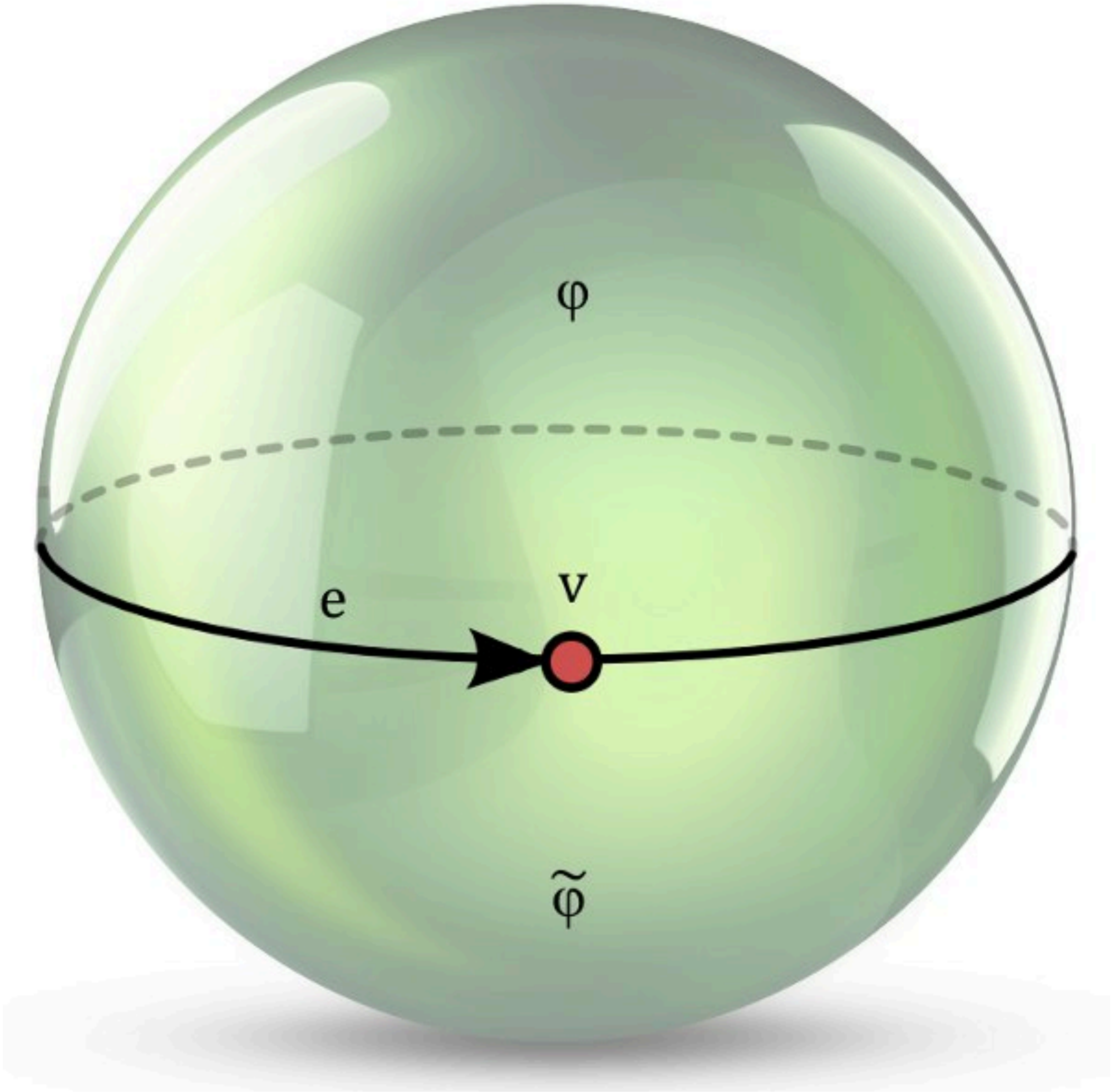


Rotation system is just half edge for the "dual" mesh (and vice-versa).

Half Edge—Example

Smallest examples (two half edges):

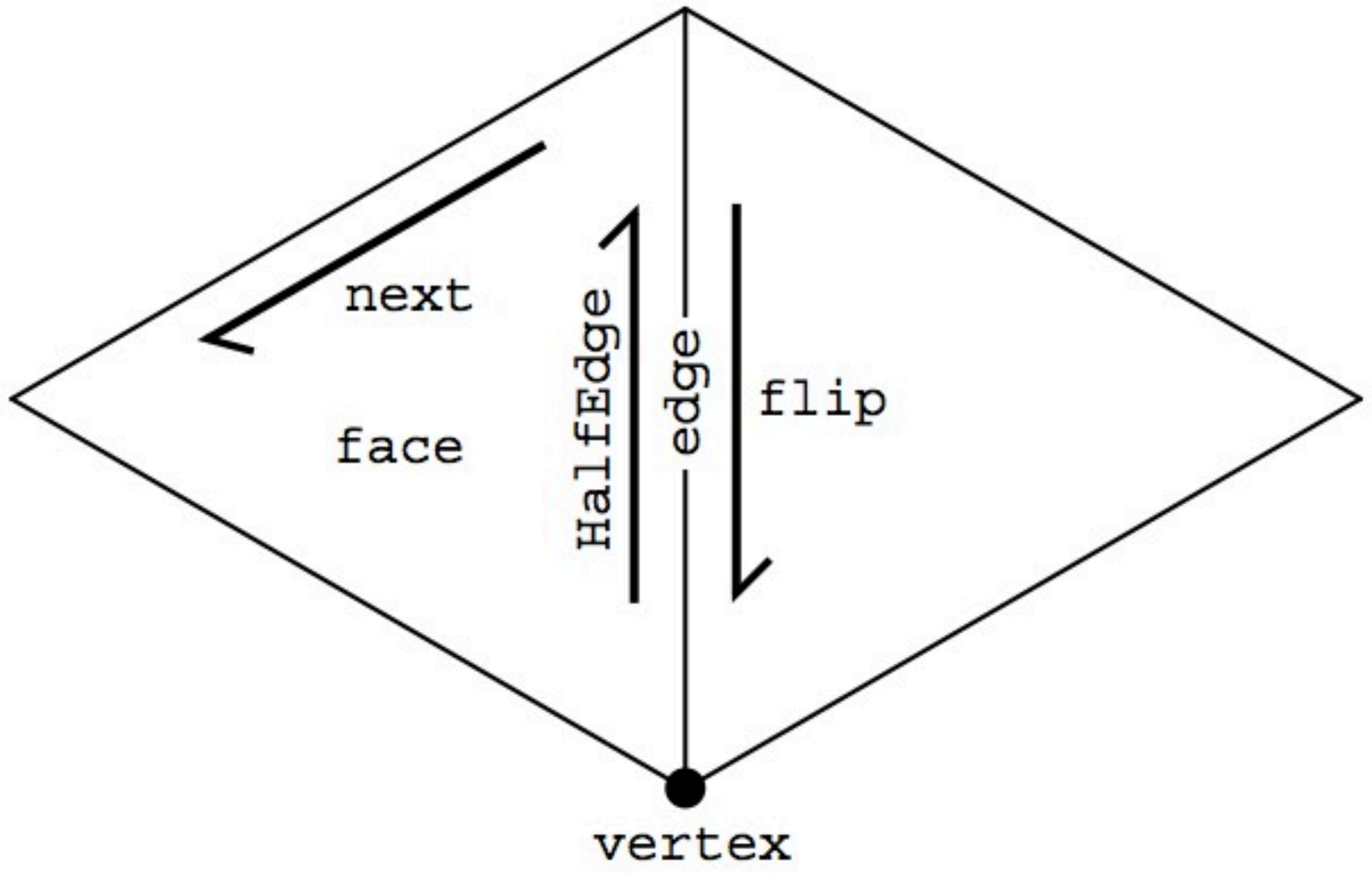


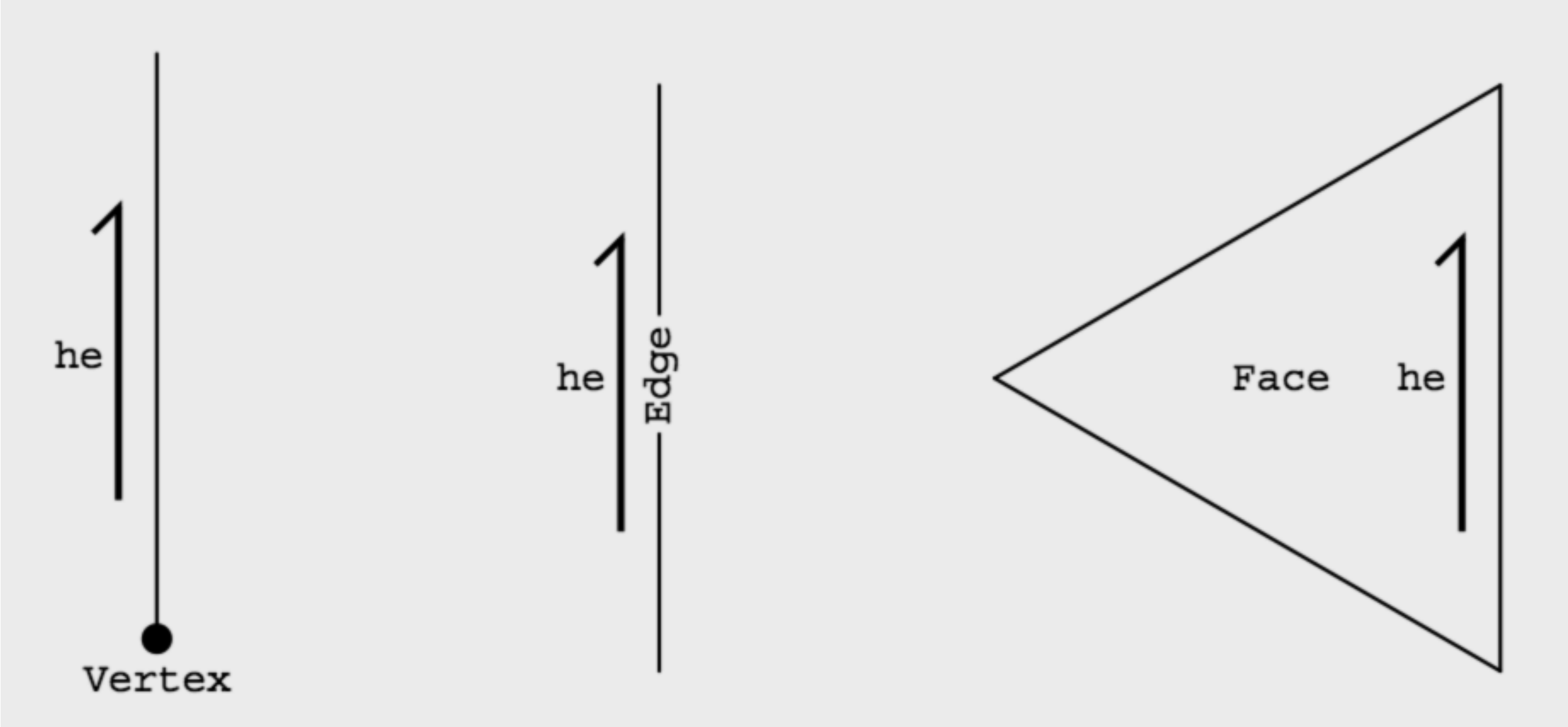


(image courtesy U. Pinkall)

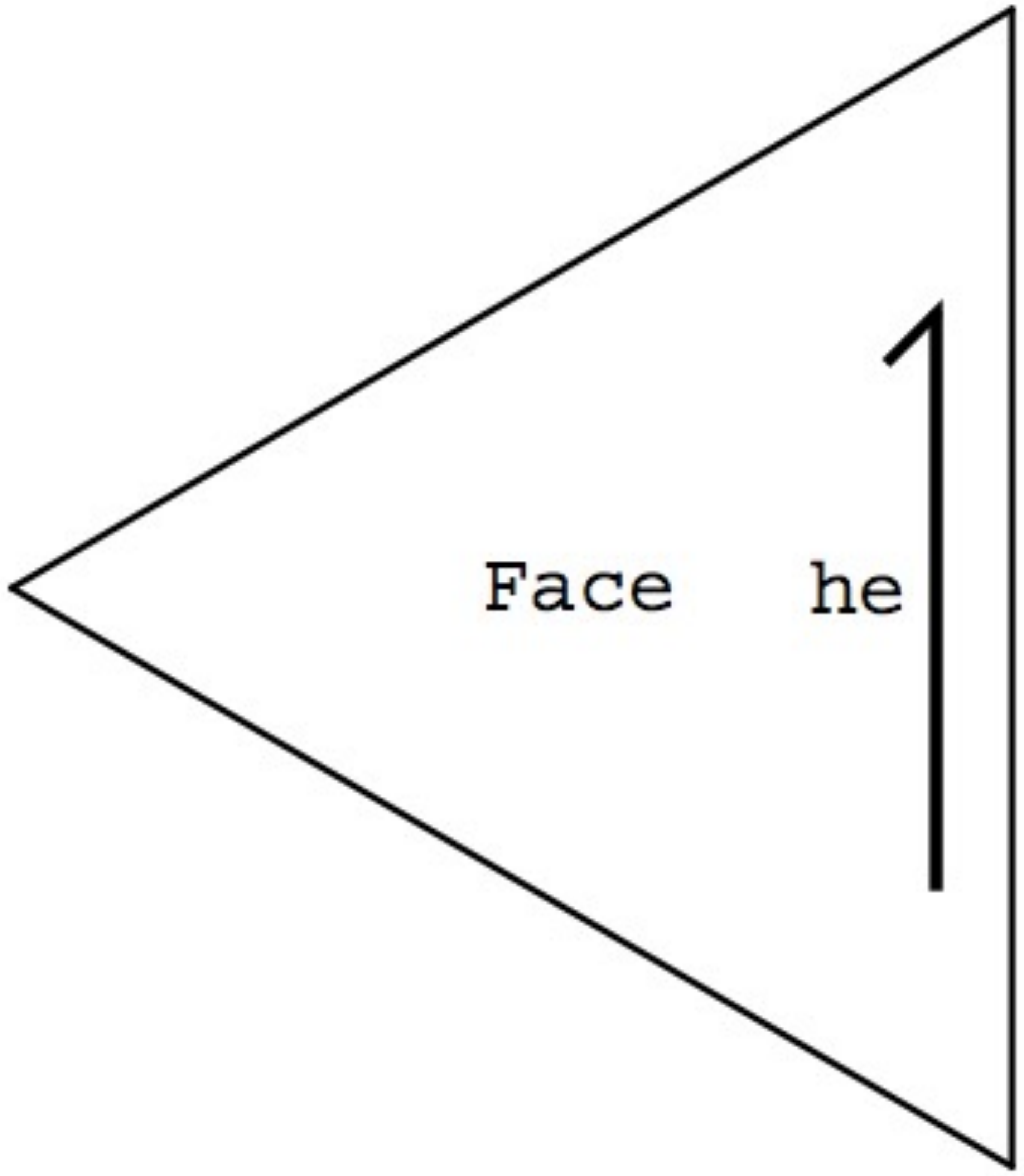
Half Edge—Practical Data Structure

```
class HalfEdge
{
   public:
      HalfEdgeIter next;
      HalfEdgeIter flip;
      VertexIter vertex;
      EdgeIter edge;
      FaceIter face;
};
```





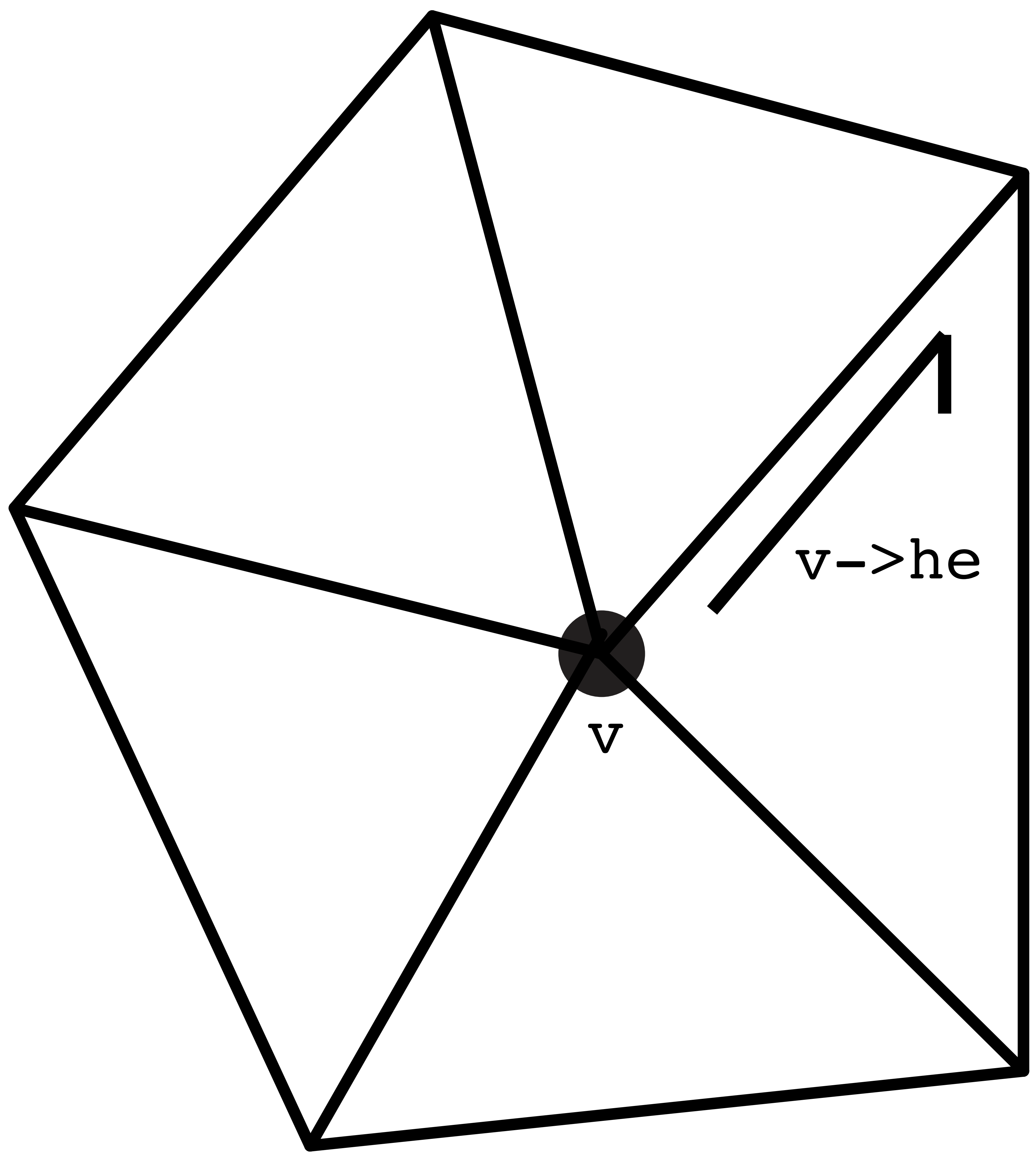
Iterating over Vertices of a Face




```
HalfEdgeIter he = f->he;
do
{
   // do something with he->vertex

   he = he->next;
}
while( he != f->he );
```

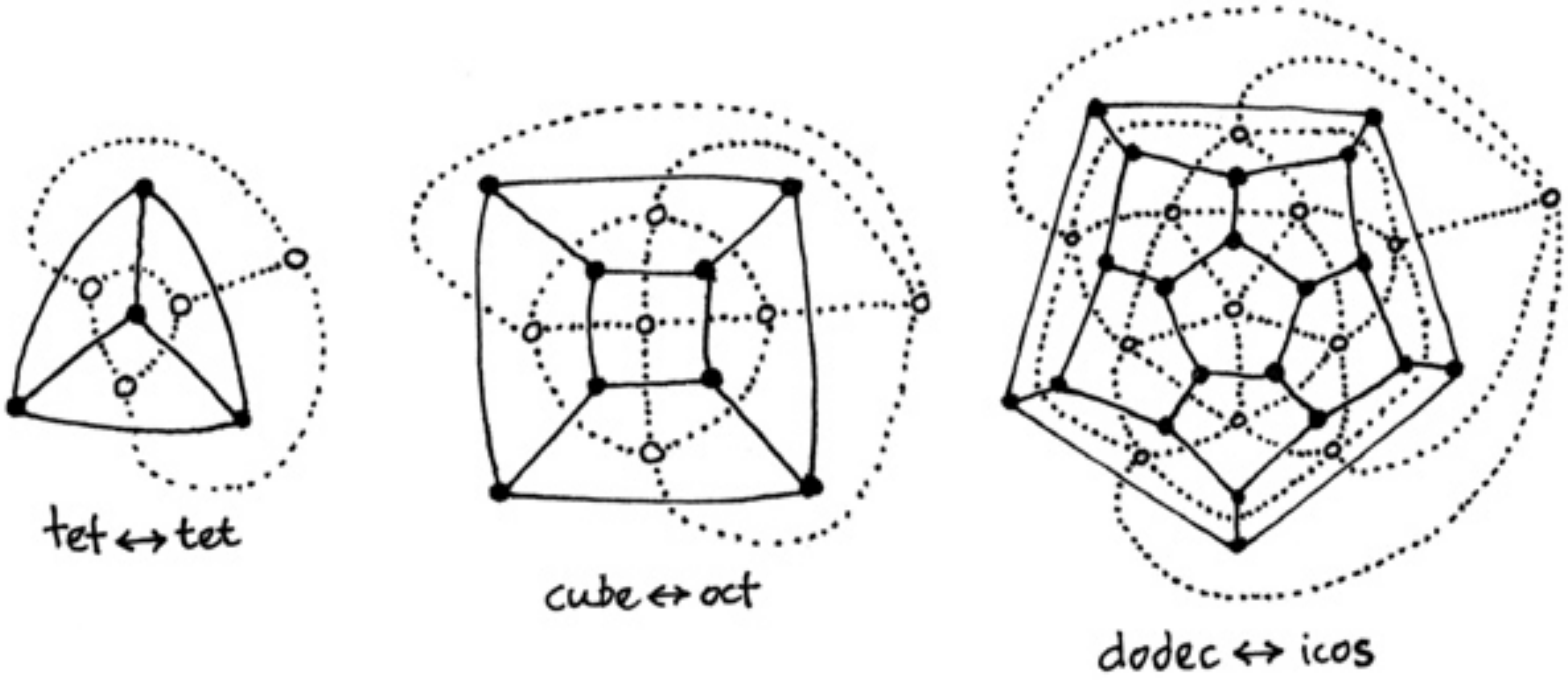
Iterating over Neighbors of a Vertex




```
HalfEdgeIter he = v->he;
do
{
   // do something with he->flip->vertex

   he = he->flip->next;
}
while( he != v->he );
```

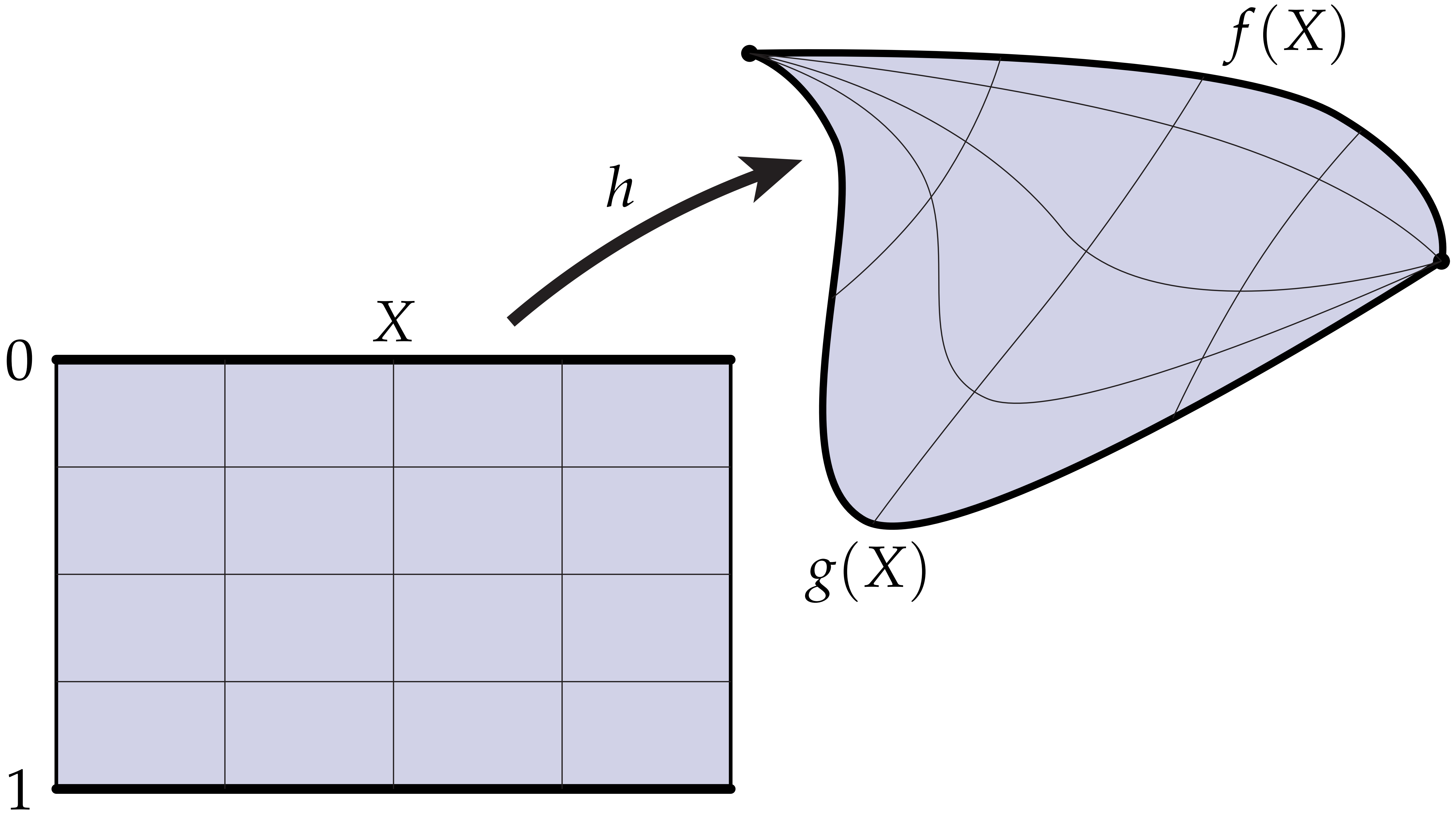
Data Structures—Quad Edge

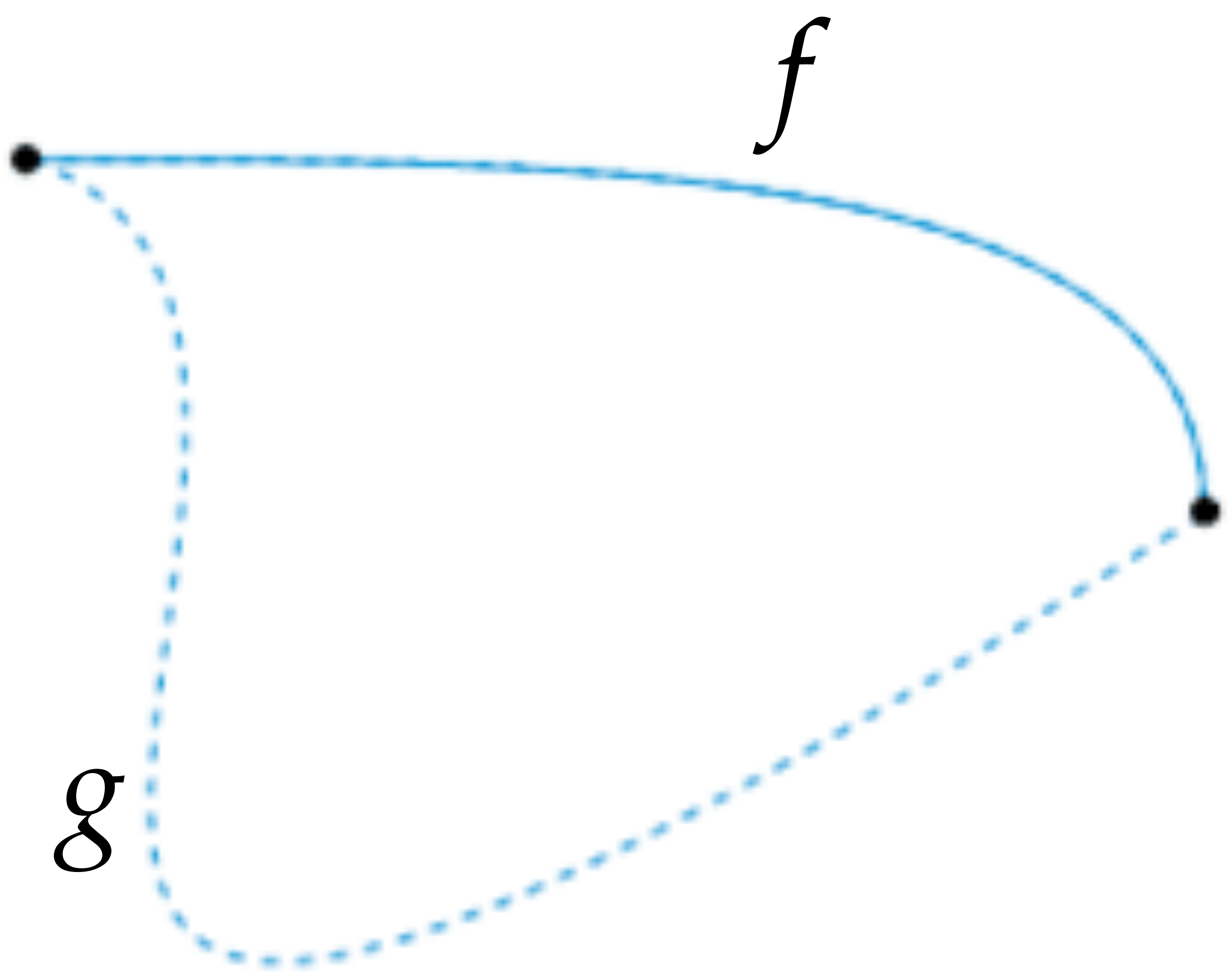


(L. Guibas & J. Stolfi, "Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams")

Homotopy

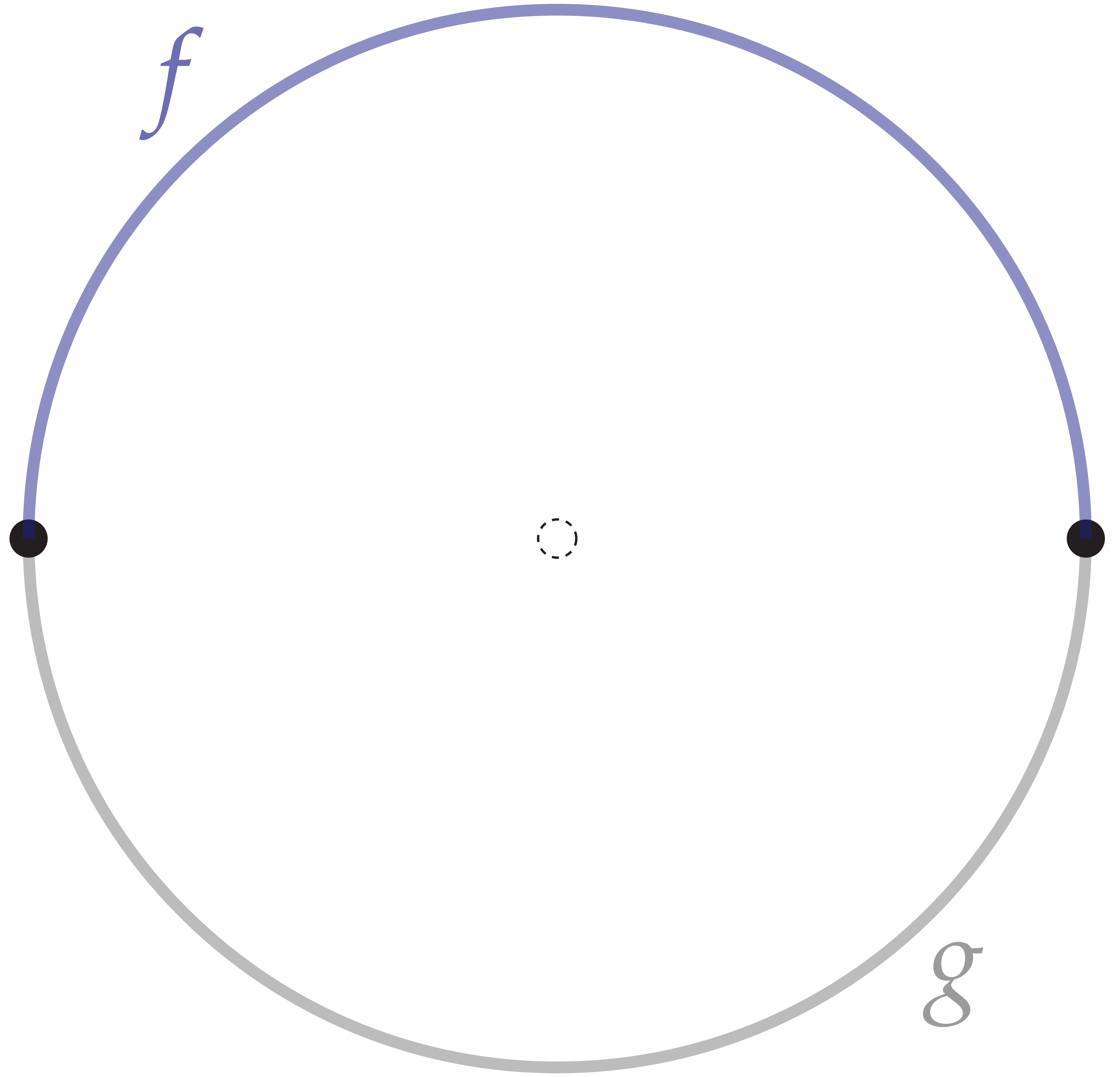
Definition. Let X, Y be topological spaces. Two maps $f, g : X \to Y$ are *homotopic* if there exists a continuous map $h : X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. If A is a subset of X then a h is a homotopy *relative to* A if $h(a,t) = f(a) = g(a)$ for all $a \in A$ and $t \in [0,1]$.





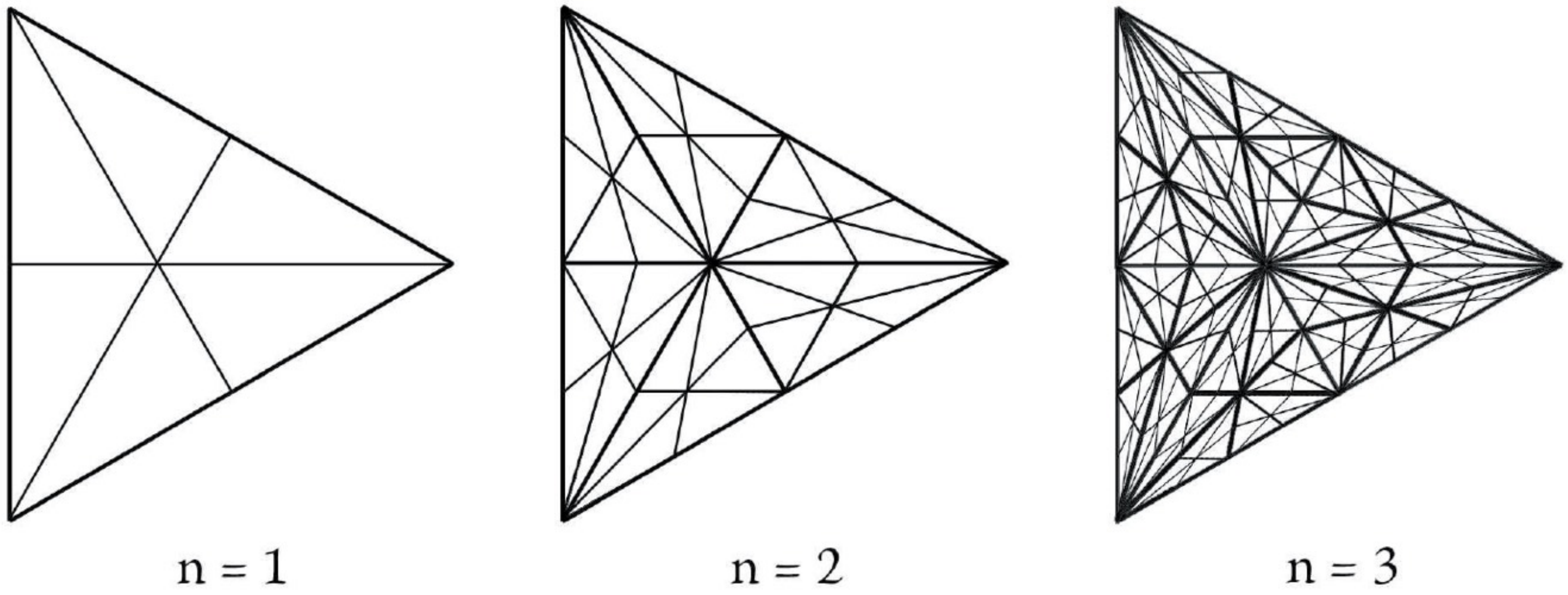
Homotopy — Example

Example. Let $X := \mathbb{R}^2 \setminus \{0\}$, *i.e.*, the plane minus the origin. Then the maps $f : [0, \pi] \to X; s \mapsto (\cos(s), \sin(s))$ and $g : [0, \pi] \to X; s \mapsto (\cos(s), -\sin(s))$ are not homotopic relative to their endpoints, *i.e.*, relative to the subset $A := \{(1, 0), (-1, 0)\} \subset \mathbb{R}^2$.



Homotopy—More Examples

Barycentric Subdivision



(image courtesy B. Hough)

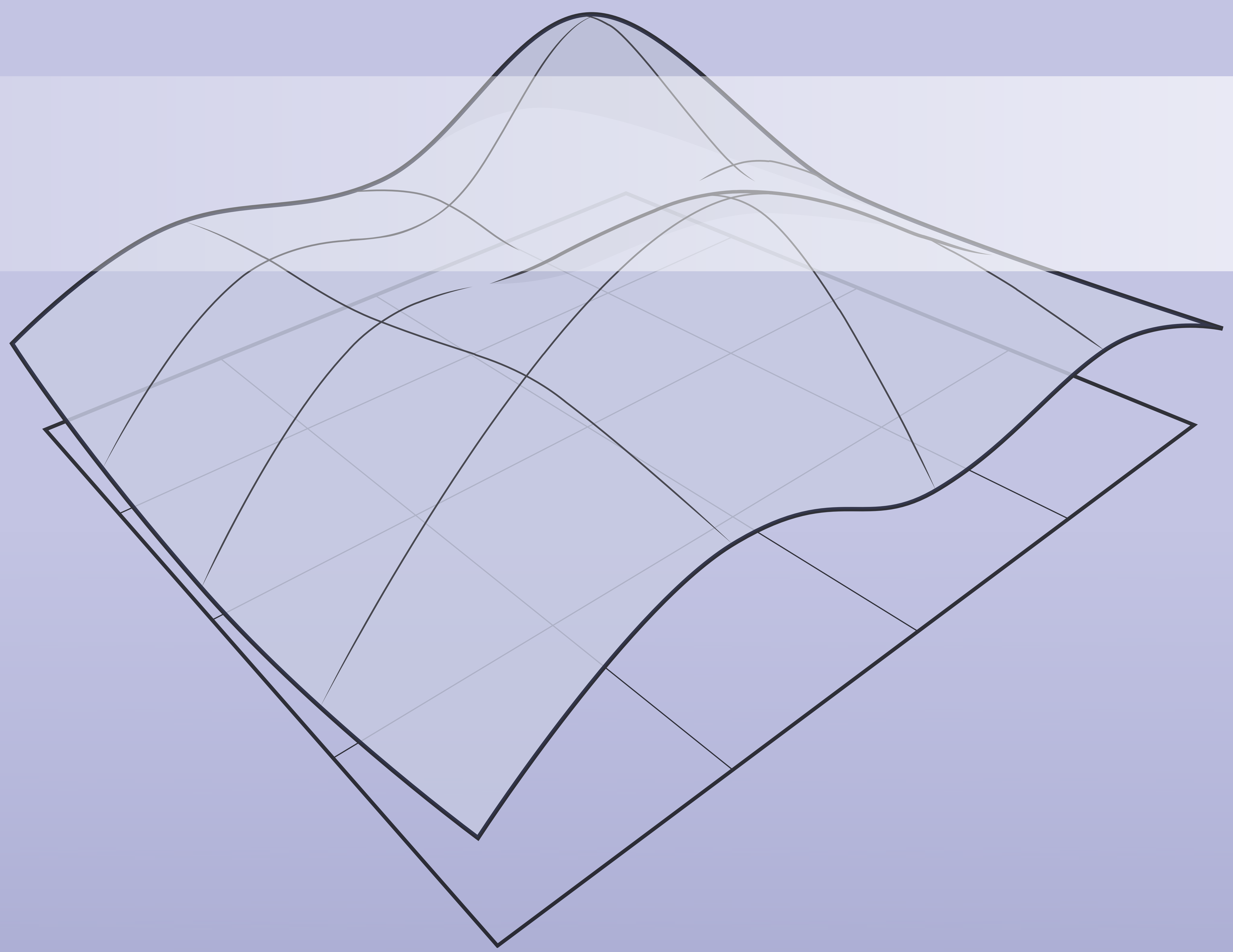
Summary—Topological Manifolds

CONTINUOUS	DISCRETE
topological space	simplicial complex
continuous map	simplicial map
homeomorphism	bijective simplicial map
topological manifold	simplicial manifold
compact	finite
…	…

Fact. For $n \leq 3$, an n-manifold can always be triangulated.

Fact. Every continuous map between simplicial complexes is homotopic to a *simplicial* map on a sufficiently fine (but finite) barycentric subdivision.

*Hence, **nothing** is lost by discretization!*



End

DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-869(J) • Spring 2016