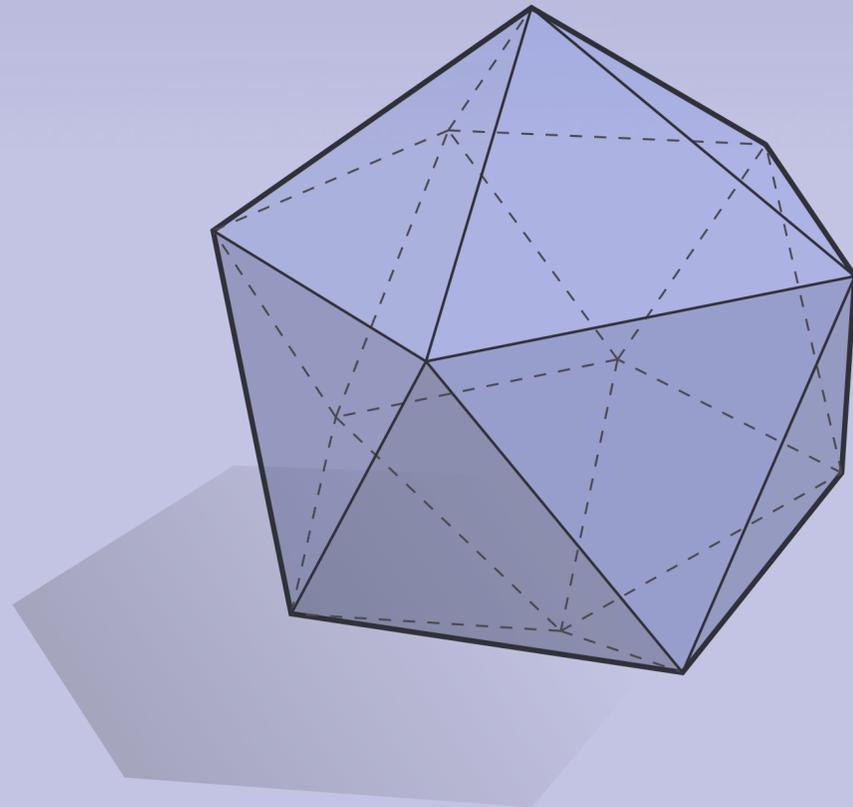


DISCRETE DIFFERENTIAL
GEOMETRY:
AN APPLIED INTRODUCTION
Keenan Crane • CMU 15-458/858

LECTURE 2:
COMBINATORIAL SURFACES

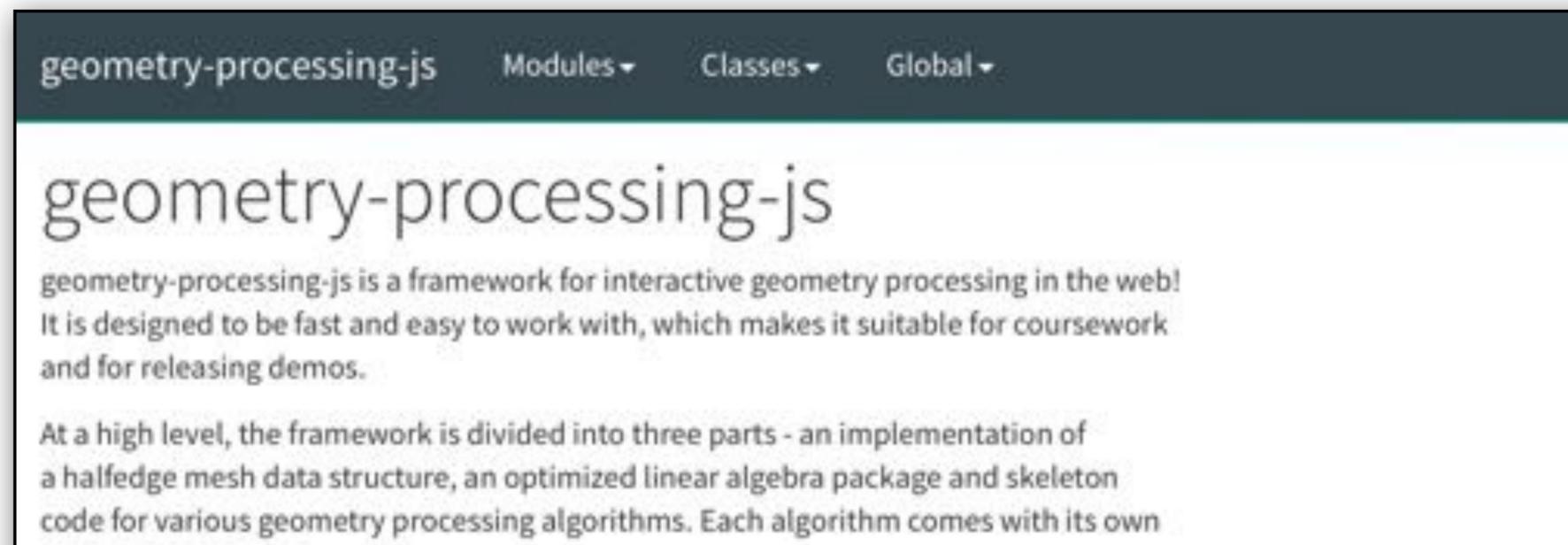


DISCRETE DIFFERENTIAL
GEOMETRY:
AN APPLIED INTRODUCTION

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Administrivia

- First reading assignment was due 10am today! **(Please use Andrew ID)**
- **First homework assignment (A1)** — coming soon!
 - Covers today's material in greater detail (combinatorial surfaces)
 - Written part out today, coding part out next week
- Special *recitation on how to use code framework*: **TBD**



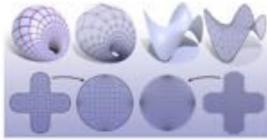
Reading: Overview of DDG

A Glimpse into Discrete Differential Geometry

Keenan Crane, Max Wardetzky

Communicated by Joel Hass

Note from Editor: The organizers of the 2018 Joint Mathematics Meetings Short Course on Discrete Differential Geometry have kindly agreed to provide this introduction to the subject. See p. XXV for more information on the JMM 2018 Short Course.



The emerging field of discrete differential geometry (DDG) studies discrete analogues of smooth geometric objects, providing an essential link between analytical descriptions and computation. In recent years it has unearthed a rich variety of new perspectives on applied problems in computational anatomy/biology, computational mechanics, industrial design, computational architecture, and digital geometry processing at large. The basic philosophy of discrete differential geometry is that a discrete object like a polyhedron is not merely an approximation of a smooth one, but rather a differential geometric object in its own right. In contrast to traditional numerical analysis which focuses on eliminating approximation error in the limit of refinement (e.g., by taking smaller and smaller finite differences), DDG places an emphasis on the so-called “mimetic” viewpoint, where key properties of a system are preserved exactly, independent of how large or small the elements of a mesh might be. Just as algorithms for simulating mechanical systems might seek to exactly preserve physical invariants such as total energy or momentum, structure-preserving models of discrete geometry seek to exactly preserve global geometric invariants such as total curvature. More broadly, DDG focuses on the discretization of objects that do not naturally fall under the umbrella of traditional numerical analysis. This article provides an overview of some of the themes in DDG.

The Game. Our article is organized around a “game” often played in discrete differential geometry in order to come up with a discrete analogue of a given smooth object or theory:

1. Write down several equivalent definitions in the smooth setting.
2. Apply each smooth definition to an object in the discrete setting.
3. Analyze trade-offs among the resulting discrete definitions, which are invariably inequivalent.

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Architectural Geometry

Helmut Pottmann^{1,2}, Michael Eigenatz³, Amir Vaxman³, Johannes Wallner⁴

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³École Normale Supérieure, 75005 Paris, France
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Abstract

Around 2005 it became apparent in the geometry processing community that freeform architecture contains many problems of a geometric nature to be solved, and many opportunities for optimization which however require geometric understanding. This area of research, which has been called architectural geometry, meanwhile contains a great wealth of individual contributions which are relevant in various fields. For mathematicians, the relation to discrete differential geometry is significant, in particular the integrable system viewpoint. Besides, new application contexts have become available for quite some old-established concepts. Regarding graphics and geometry processing, architectural geometry yields interesting new questions but also new objects, e.g. replacing meshes by other combinatorial arrangements. Numerical optimization plays a major role but in itself would be powerless without geometric understanding. Summing up, architectural geometry has become a rewarding field of study. We here survey the main directions which have been pursued, we show real projects where geometric considerations have played a role, and we outline open problems which we think are significant for the future development of both theory and practice of architectural geometry.

Keywords: Discrete differential geometry, architectural geometry, fabrication-aware design, paneling, double-curved surfaces, single-curved surfaces, support structures, polyhedral patterns, static-aware design, shading and lighting, interactive modeling

1. Introduction

Free forms constitute one of the major trends within contemporary architecture. In its earlier days a particularly important figure was Frank Gehry, with his design approach based on digital reconstruction of physical models, resulting in shapes which are not too far away from developable surfaces and thus ideally suited for his preferred characteristic metal cladding [94]. Nowadays we see an increasing number of landmark buildings involving geometrically complex freeform skins and structures (Fig. 1).



While the modeling of freeform geometry with current tools is well understood, the actual fabrication on the architectural scale is a challenge. One has to decompose the skins into manufacturable panels, provide appropriate support structures, meet structural constraints and last, but not least make sure that the cost does not become excessive. Many of these practically highly important problems are actually of a geometric nature and thus the architectural application attracted the attention of the geometric modeling and geometry processing community. This research area is now called Architectural Geometry. It is the purpose of the present survey to provide an overview of this field from the Computer Graphics perspective. We are not addressing here the many beautiful designs which have been realized by engineers with a clever way of using state of the art software, but we are focusing on research contributions which go well beyond the use of standard tools. This research direction has also been inspired by the work of the smart geometry group (www.smartgeometry.com), which promoted the use of parametric design and scripting for mastering geometric complexity in architecture.

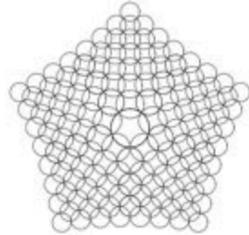
From a methodology perspective, it turned out that the probably two most important ingredients for the solution of Architectural Geometry problems are Discrete Differential Geometry (DDG) [84, 95] and Numerical Optimization. In order to keep this survey well within Graphics, we will be rather short in discussing the subject from the DDG perspective and only mention those insights which are essential for a successful implementation. It is a fact that understanding a problem from the DDG viewpoint is often equivalent to understanding how to successfully initialize and solve the numerical optimization problems

Proseptic submitted to Computers & Graphics December 28, 2014

arXiv:math/0504358v1 [math.DG] 18 Apr 2005

Discrete Differential Geometry. Consistency as Integrability

Alexander I. Bobenko, Yuri B. Suris



Supported by the DFG Forschergruppe “Polyhedral Surfaces” and the DFG Research Center MATHEON “Mathematics for key technologies” in Berlin.

“...I’m intimidated by the math...”

“...I’m intimidated by the coding...”

DDG is by its very nature interdisciplinary—*everyone* will feel a bit uncomfortable!

We are aware of this fact! Everyone will be ok. :-) Lots of *details*; focus on the **ideas**.

Assignment 1 — Written Out Later Today!

Written Assignment 1:
A First Look at Exterior Algebra and Exterior Calculus

CMU 15-458/858 (Fall 2017)

Due: September 26, 2017 at 5:59:59 PM Eastern

Submission Instructions. Please submit your solutions to the exercises (whether handwritten, LaTeX, etc.) as a single PDF file by email to Geometry@cs.cmu.edu. Scanned images/photographs can be converted to a PDF using applications like Preview (on Mac) or a variety of free websites (e.g., <http://www.pdfcrowd.com/>). Your submission email must include the string `DDGTYA1` in the subject line. Your graded submission will (hopefully) be returned to you at least one day before the due date of the next written assignment.

Grading. This assignment is worth 6.5% of your course grade. Please clearly show your work. Partial credit will be awarded for ideas toward the solution, so please submit your thoughts on an exercise even if you cannot find a full solution. If you don't know where to get started with some of these exercises, just ask! A great way to do this is to leave comments on the course webpage under this assignment; this way everyone can benefit from your questions. We are glad to provide further hints, suggestions, and guidance either here on the website, via email, or in person. Office hours are still TBD, but let us know if you'd like to arrange an individual meeting.

Late Days. Note that you have 5 no-penalty late days for the entire course, where a "day" runs from 6:00:00 PM Eastern to 5:59:59 PM Eastern the next day. No late submissions are allowed once all late days are exhausted. If you wish to claim one or more of your five late days on an assignment, please indicate which late day(s) you are using in your email submission. You must also draw Platonic solids corresponding to the late day(s) you are using (cube=1, tetrahedron=2, octahedron=3, dodecahedron=4, icosahedron=5). Use them wisely, as you cannot use the same polyhedron twice! If you are typesetting your homework on the computer, we have provided images that can be included for this purpose (in LaTeX these can be included with the `\usepackage{graphics}` command in the graphics package).



Collaboration and External Resources. You are strongly encouraged to discuss all course material with your peers, including the written and coding assignments. You are especially encouraged to seek out new friends from other disciplines (CS, Math, Engineering, etc.) whose experience might complement your own. However, your final work must be your own, i.e., direct collaboration on assignments is prohibited. You are allowed to refer to any external resources except for homework solutions from previous editions of this course (at CMU and other institutions). If you use an external resource, cite such help on your submission. If you are caught cheating, you will get a zero for the entire course.

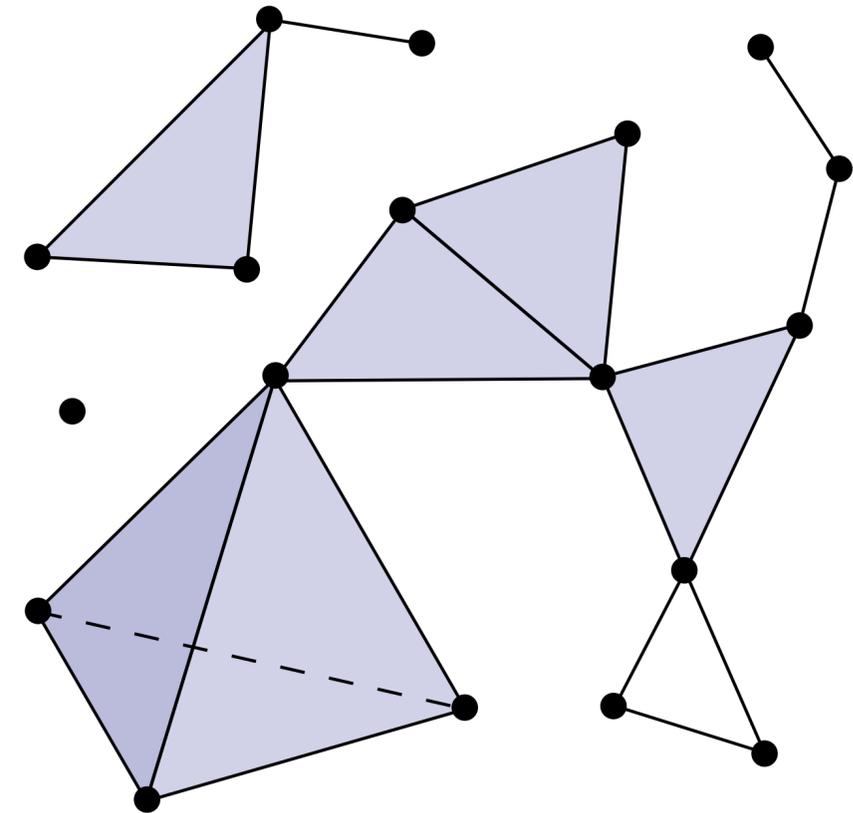
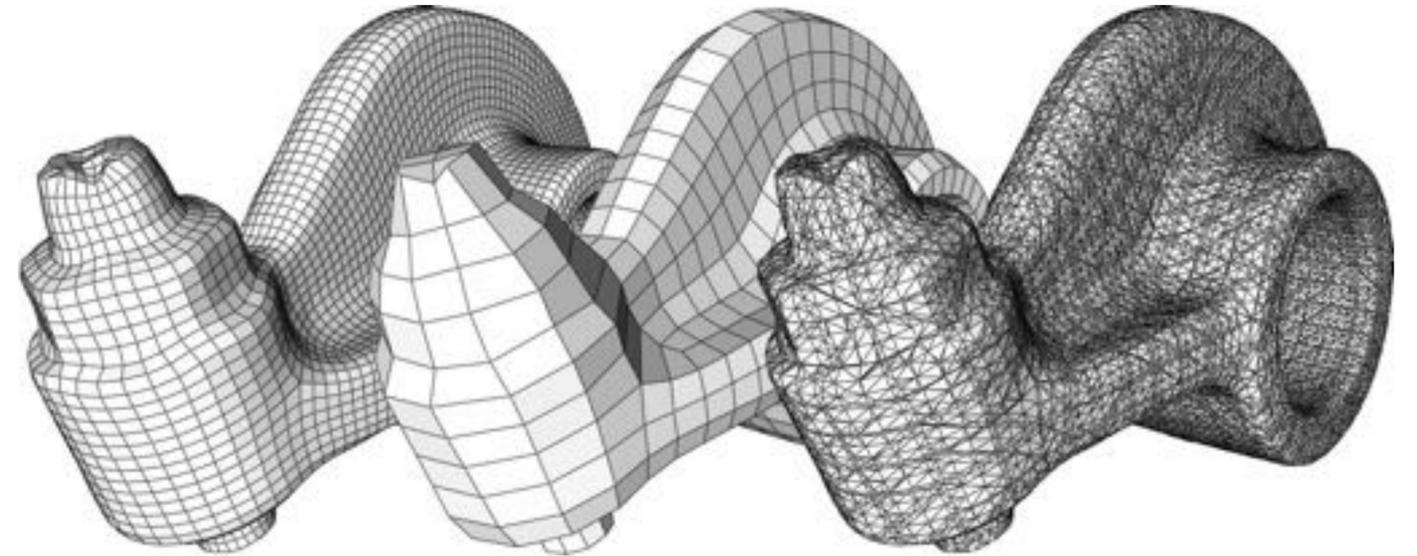
Warning! With probability 1, there are typos in this assignment. If anything in this handout does not make sense (or is blatantly wrong), let us know! We will be handing out extra credit for good catches. :-)

1

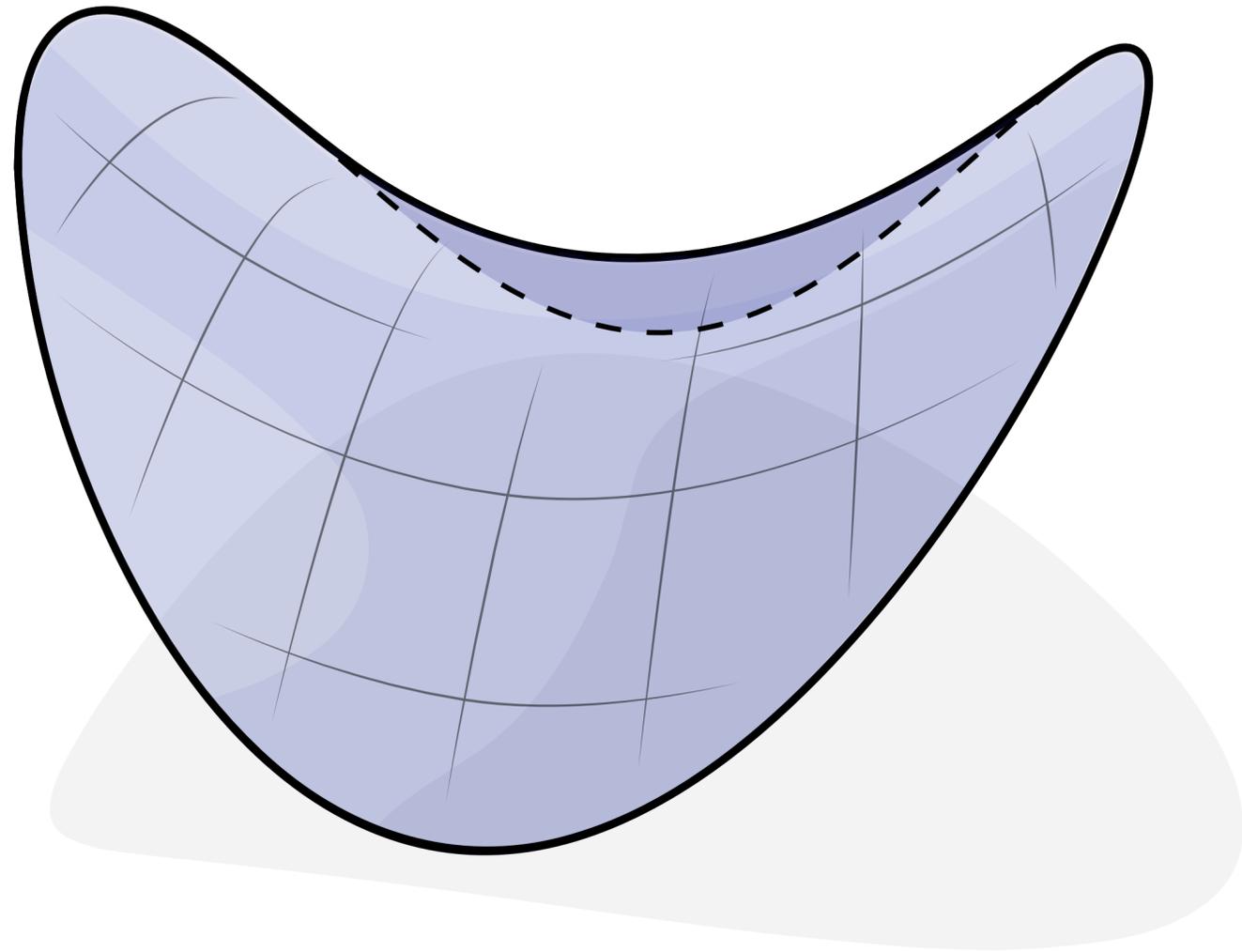
- First assignment
 - Written part
 - Coding part
- **Topic:** *combinatorial surfaces*
 - Basic tools, data structures used throughout semester
 - Can't skip this one!
- Goes along with next reading
 - Detailed background in our course notes
 - Good idea to get started now! (Read notes first.)
- All administrative details (handin, *etc.*) in assignment.

Today: What is a Mesh?

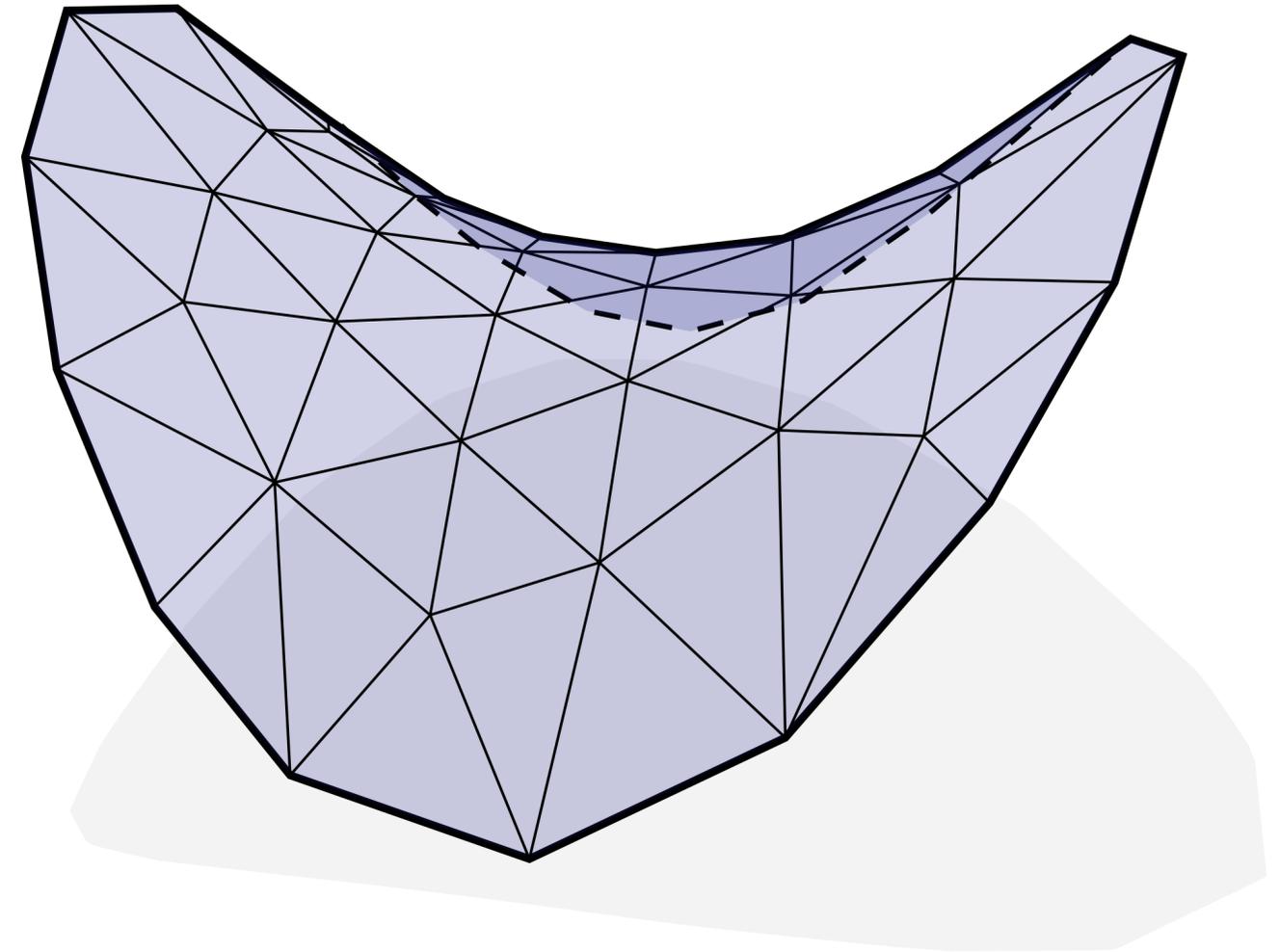
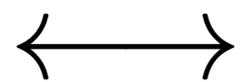
- Many possibilities...
- **Simplicial complex**
 - Abstract vs. geometric simplicial complex
 - Oriented, manifold simplicial complex
 - Application: *topological data analysis*
- **Cell complex**
 - Poincaré dual, discrete exterior calculus
- **Data structures:**
 - *adjacency list, incidence matrix, halfedge*



Connection to Differential Geometry?

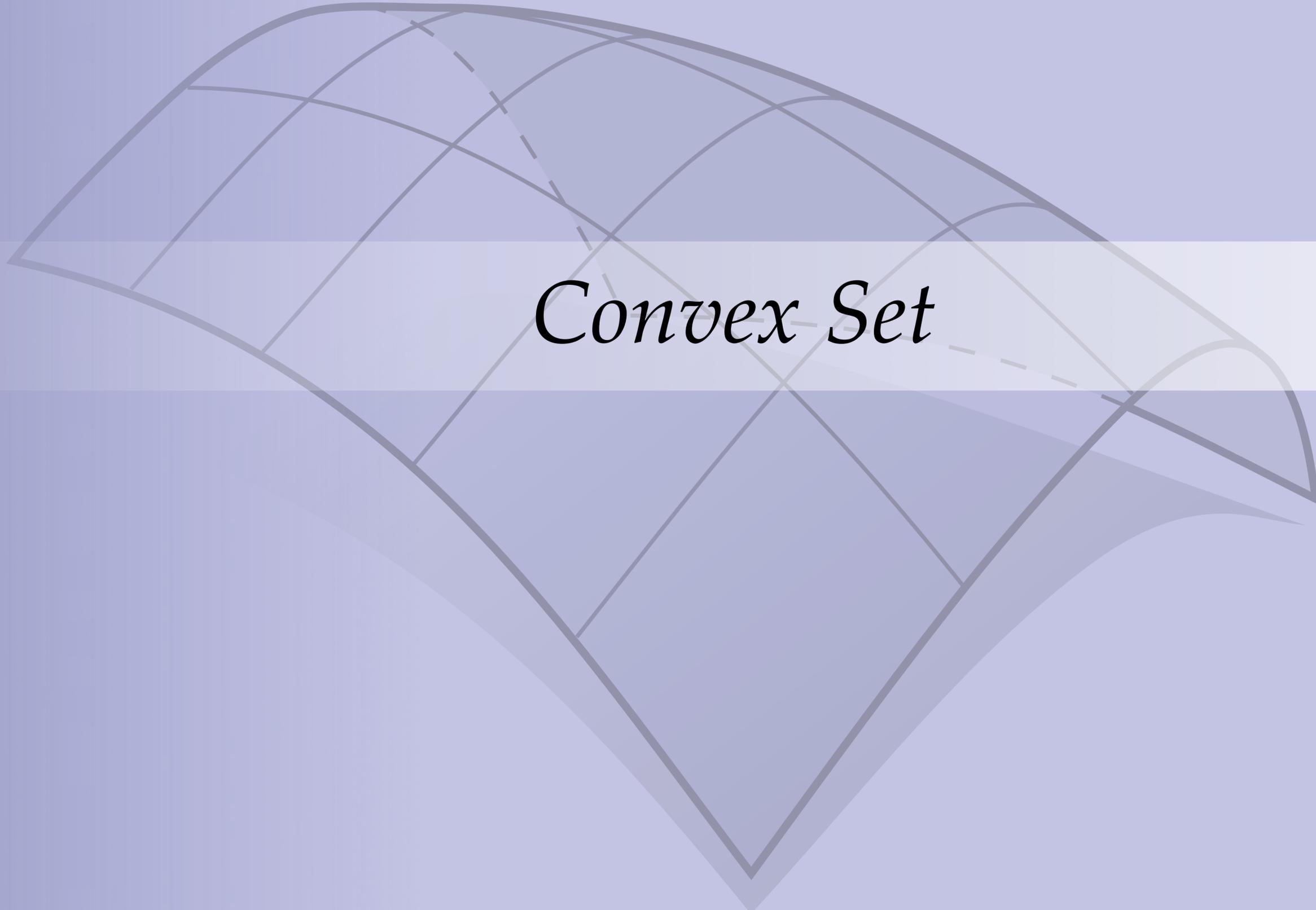


topological space*



abstract simplicial complex

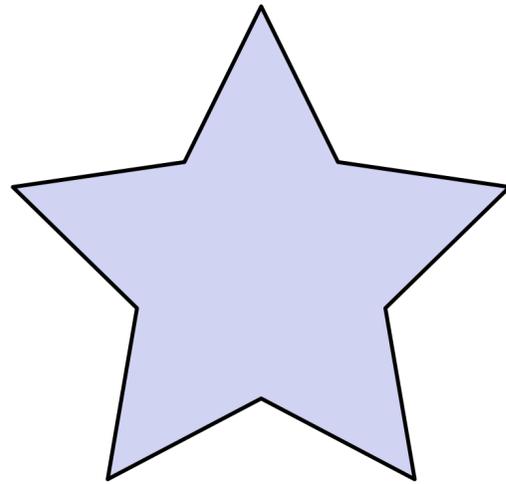
*We'll talk about this *later* in the course!



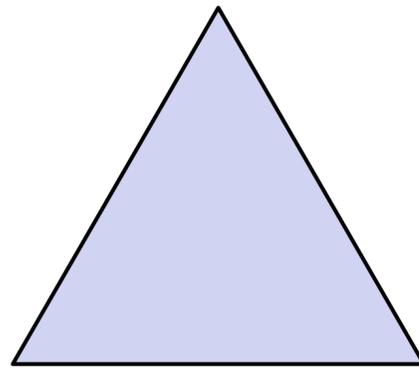
Convex Set

Convex Set—Examples

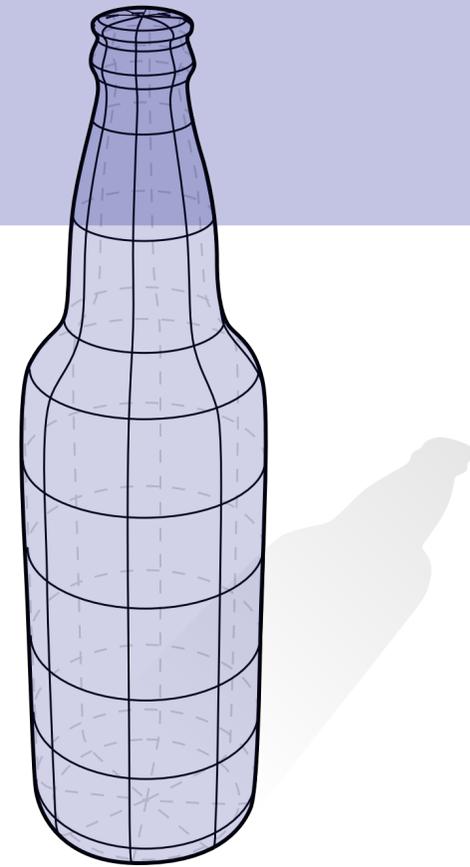
Which of the following sets are *convex*?



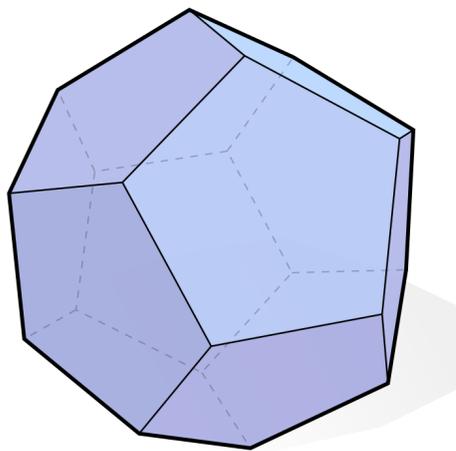
(A)



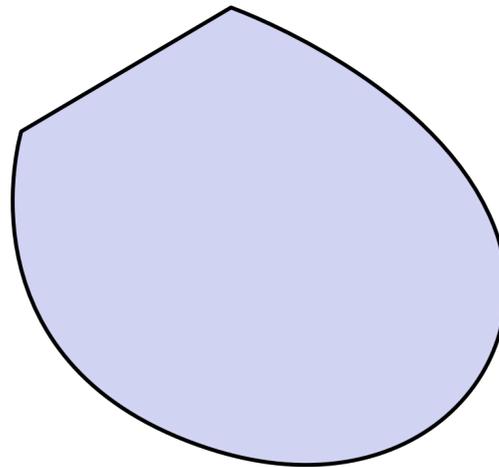
(B)



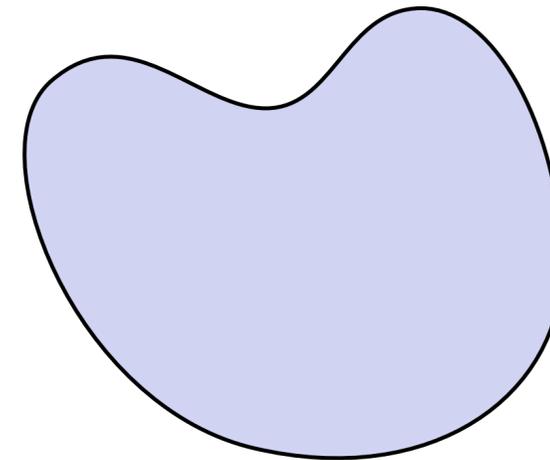
(C)



(D)



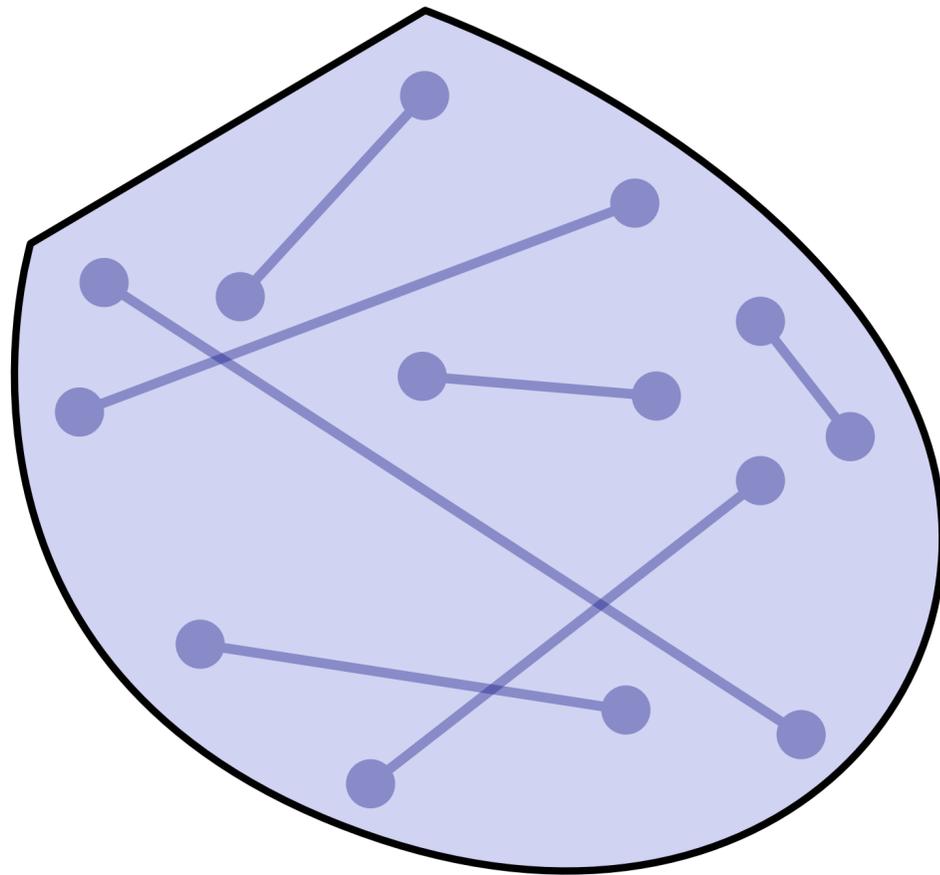
(E)



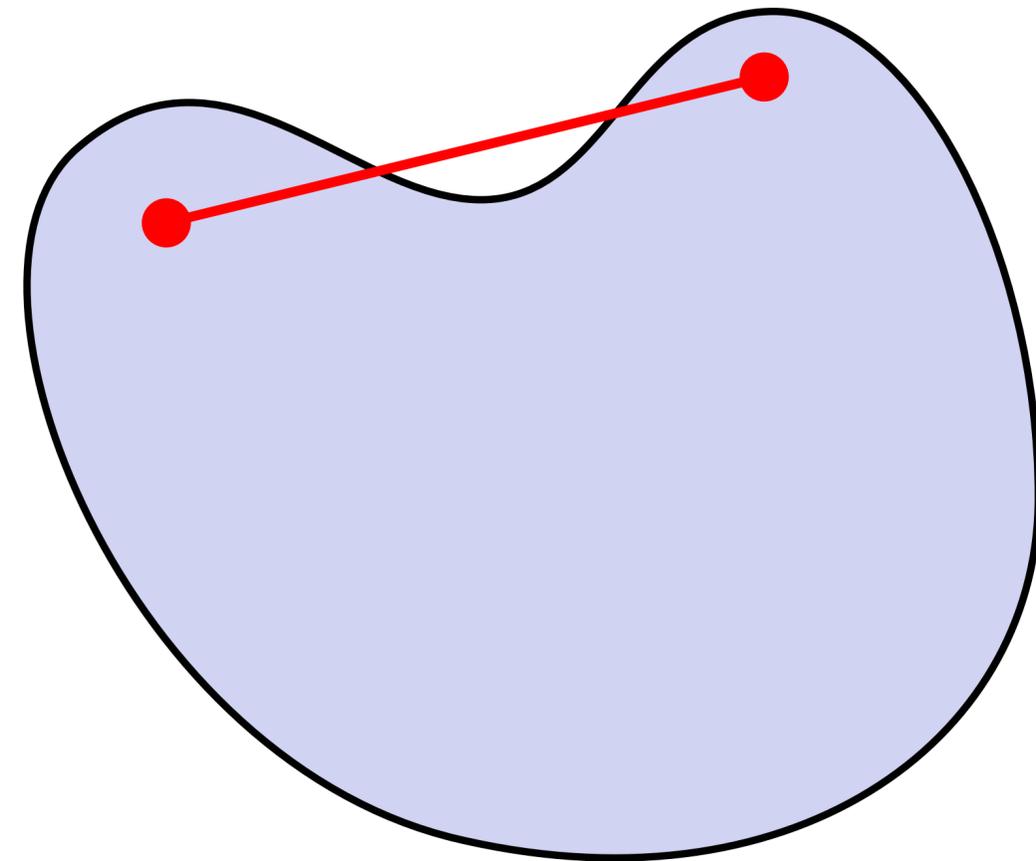
(F)

Convex Set

Definition. A subset $S \subset \mathbb{R}^n$ is *convex* if for every pair of points $p, q \in S$ the line segment between p and q is contained in S .

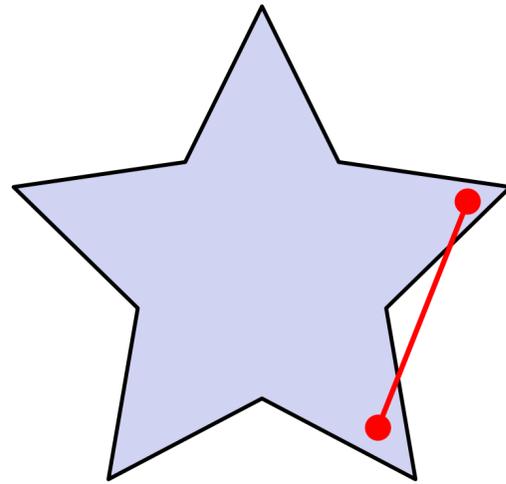


convex

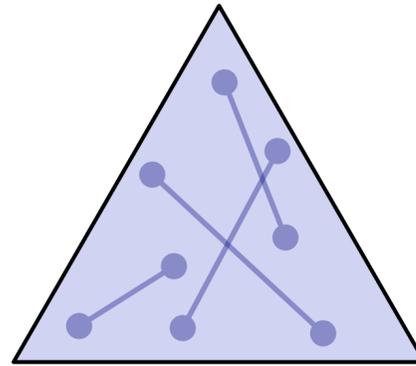


not convex

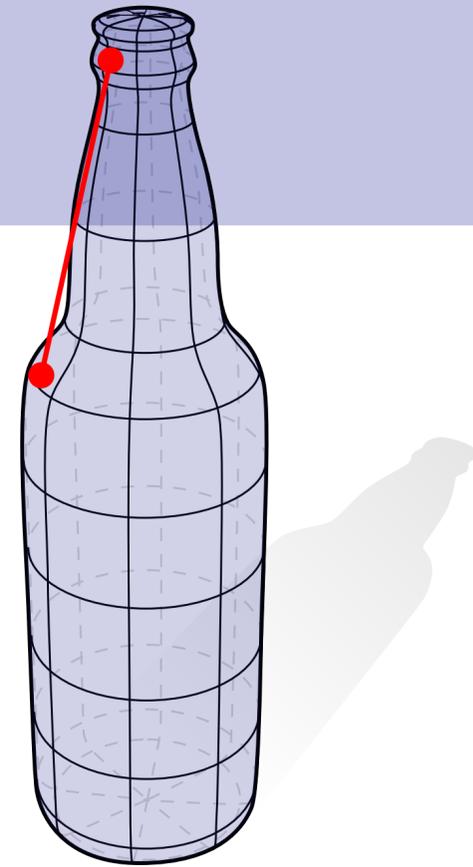
Convex Set—Examples



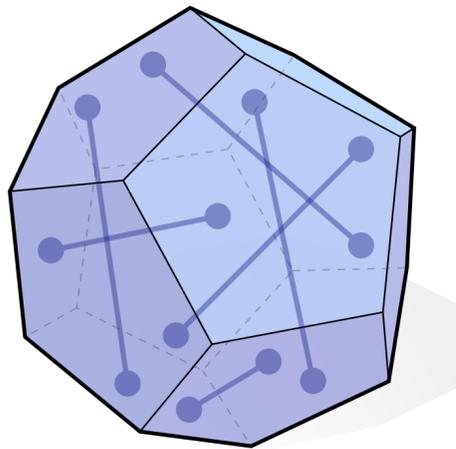
(A)



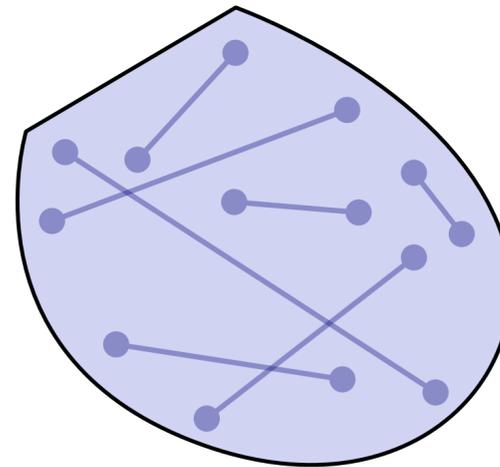
(B)



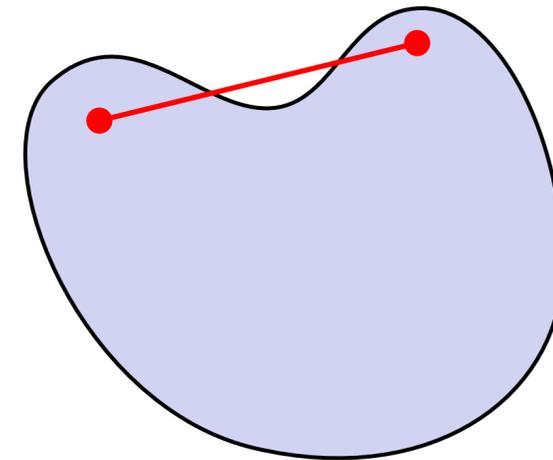
(C)



(D)



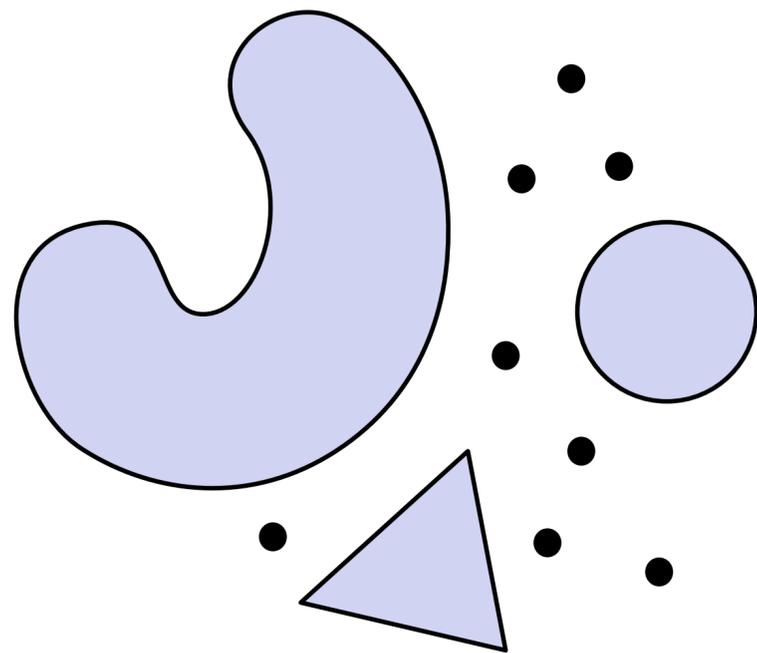
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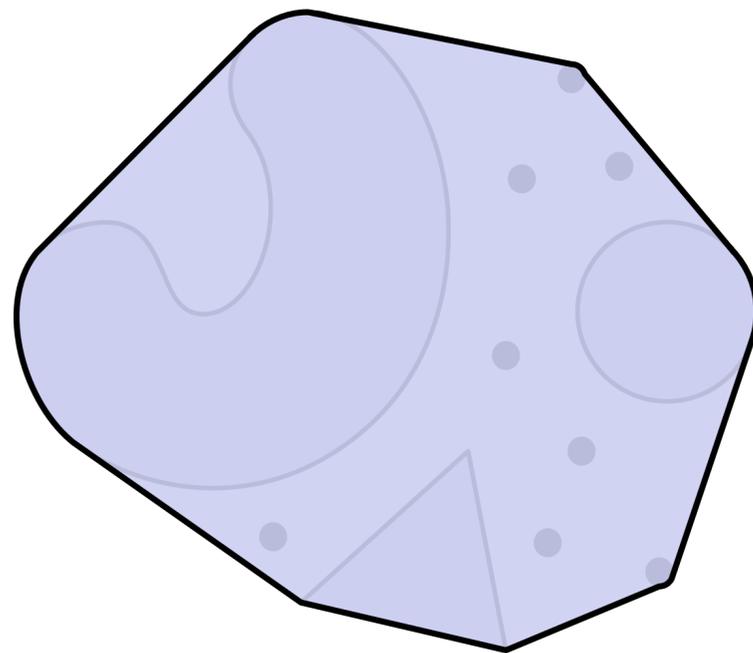
(F)

Convex Hull

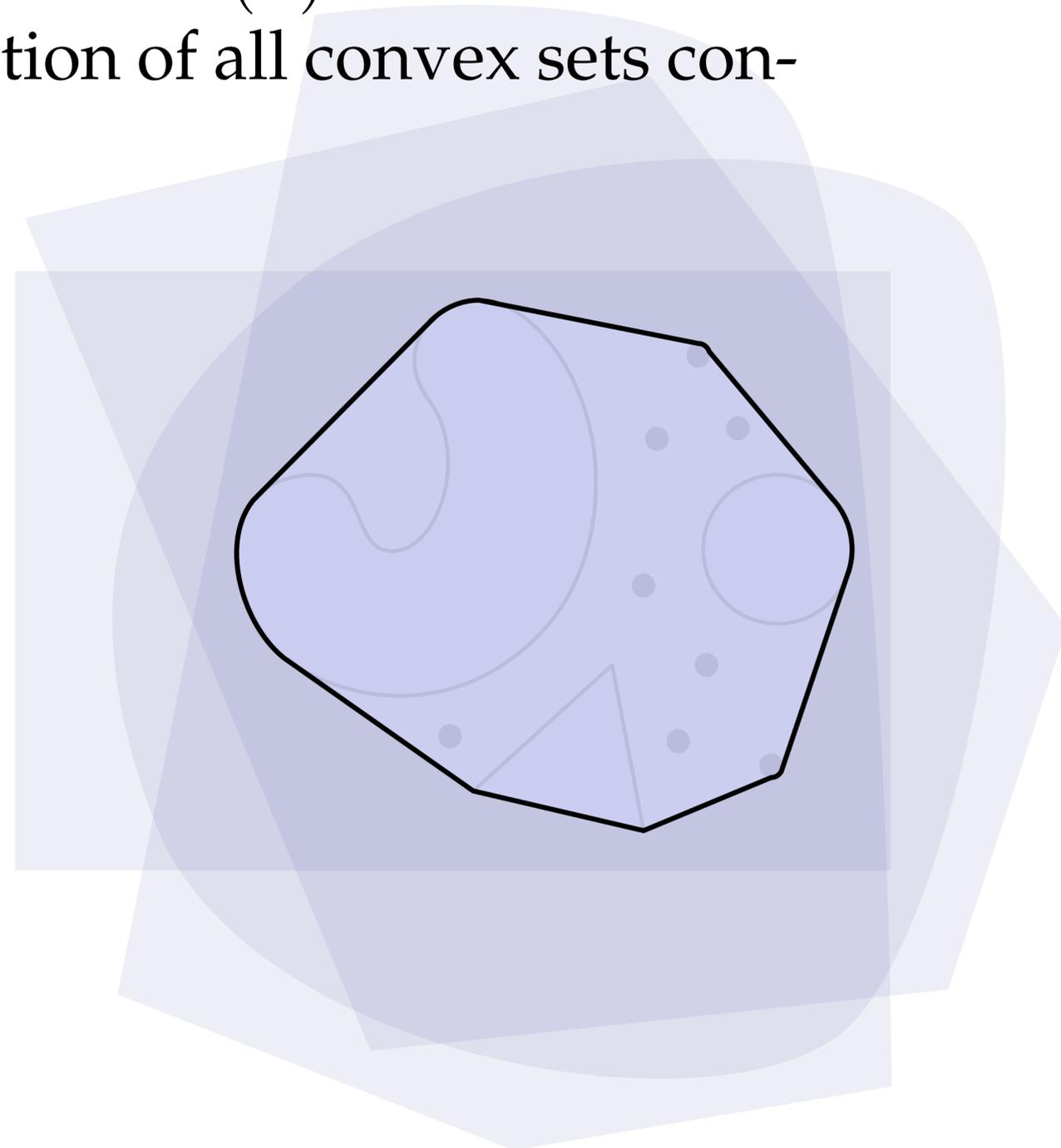
Definition. For any subset $S \subset \mathbb{R}^n$, its convex hull $\text{conv}(S)$ is the smallest convex set containing S , or equivalently, the intersection of all convex sets containing S .



S



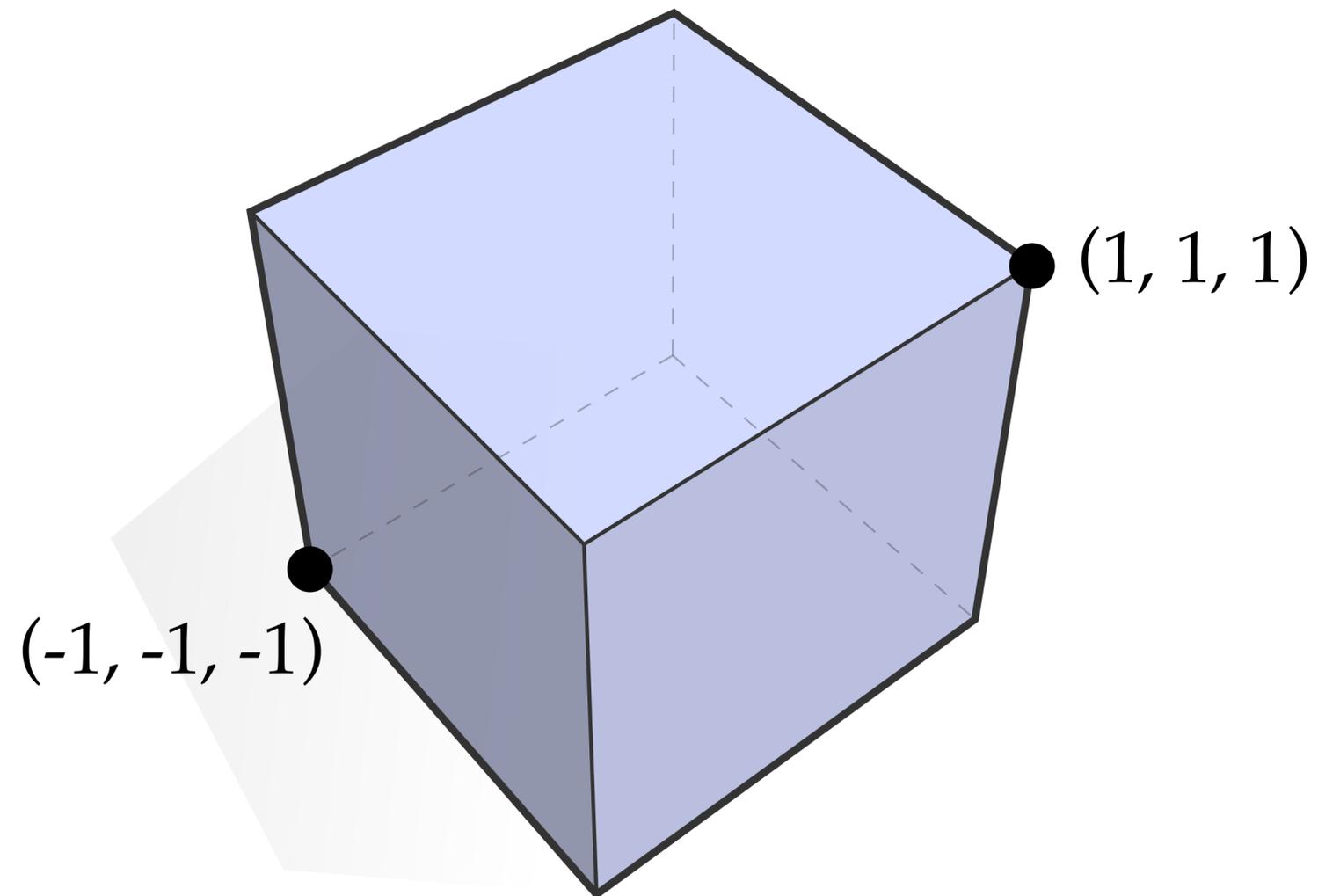
$\text{conv}(S)$

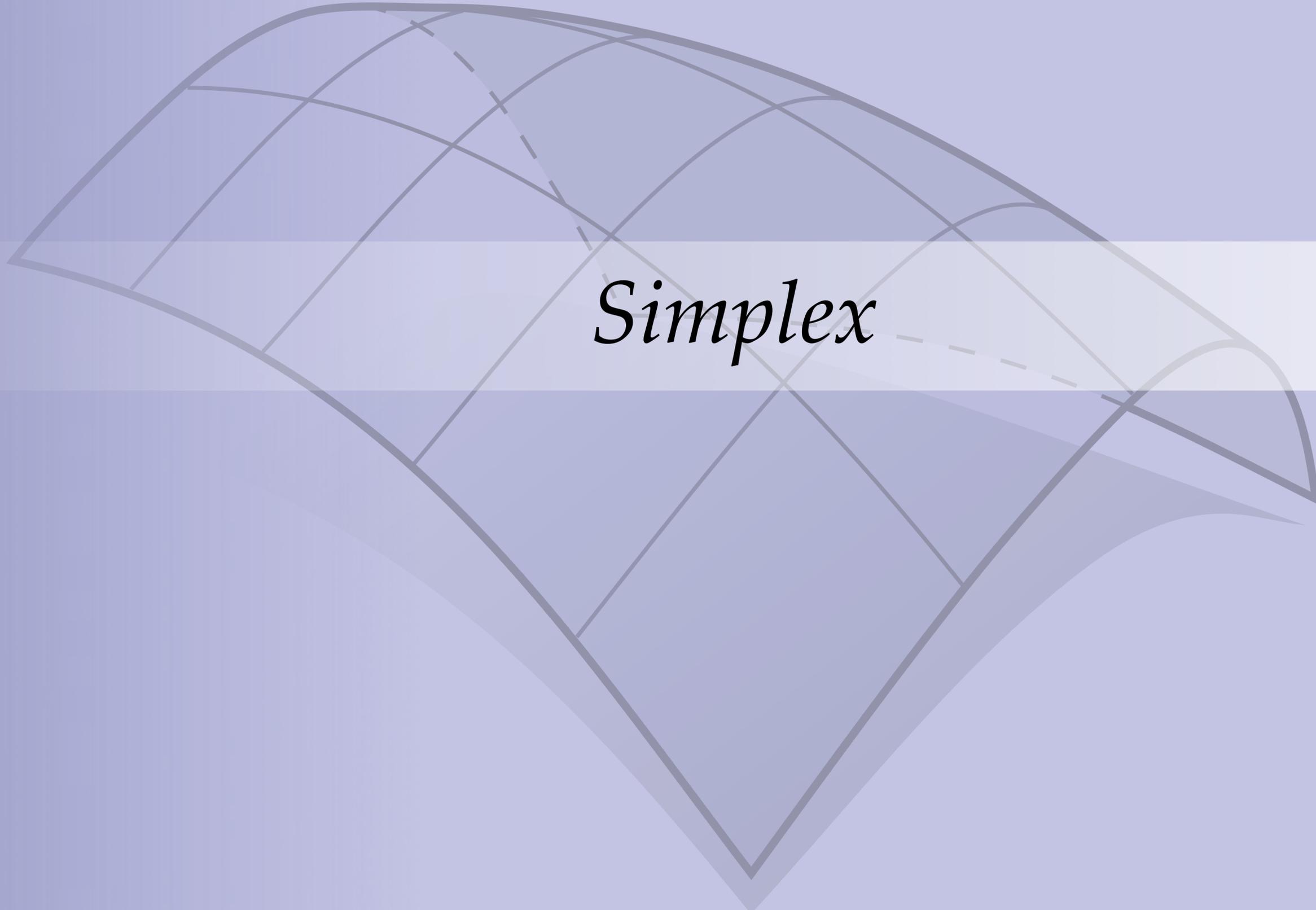


Convex Hull—Example

Q: What is the convex hull of the set $S := \{(\pm 1, \pm 1, \pm 1)\} \subset \mathbb{R}^3$?

A: A cube.

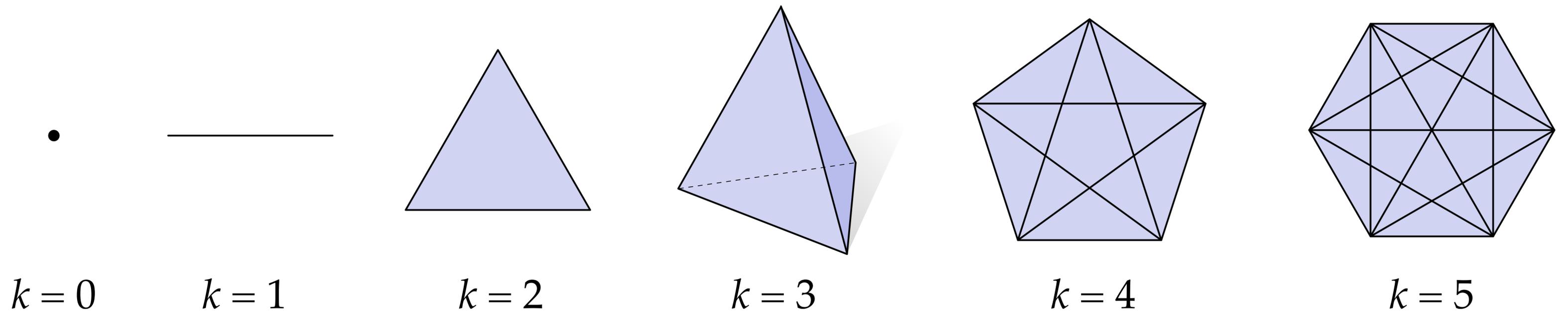




Simplex

Simplex — Basic Idea

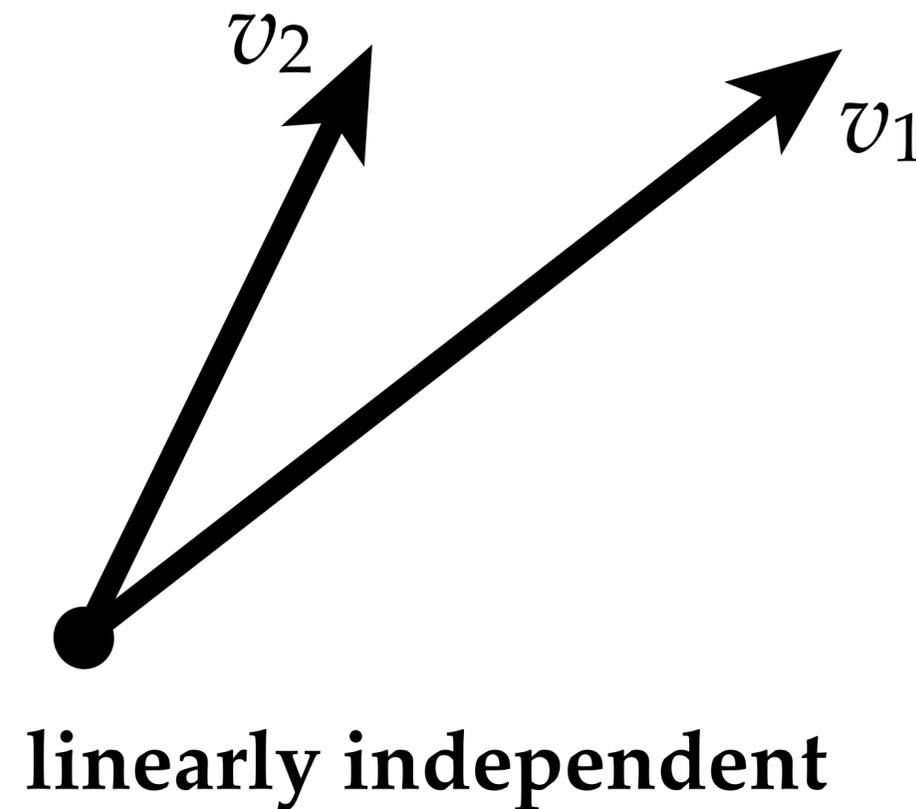
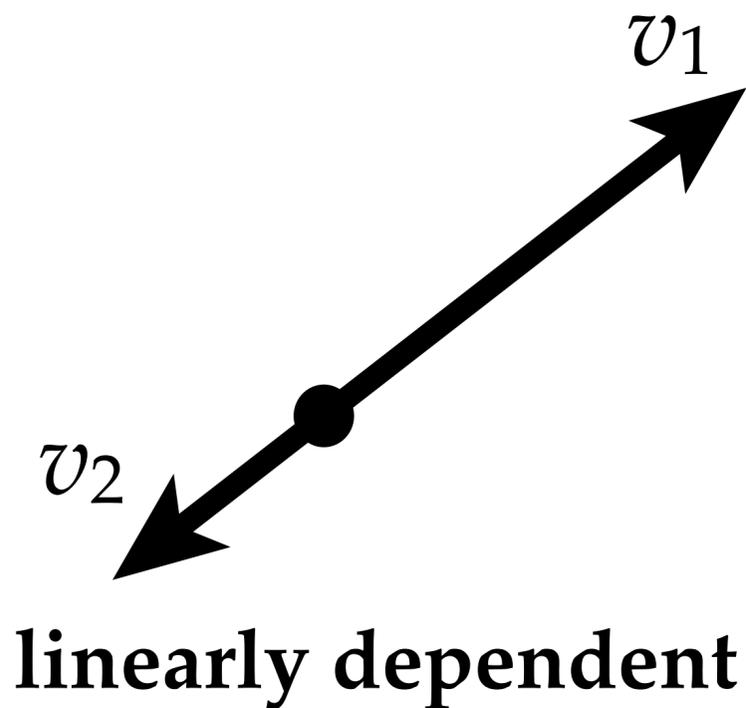
Roughly speaking, a k -simplex is a point, a line segment, a triangle, a tetrahedron...



...much harder to draw for large k !

Linear Independence

Definition. A collection of vectors v_1, \dots, v_n is *linearly independent* if no vector can be expressed as a linear combination of the others, *i.e.*, if there is no collection of coefficients $a_1, \dots, a_n \in \mathbb{R}$ such that $v_j = \sum_{i \neq j} a_i v_i$ (for any v_j).



Affine Independence

Definition. A collection of points p_0, \dots, p_k are *affinely independent* if the vectors $v_i := p_i - p_0$ are linearly independent.

(A)



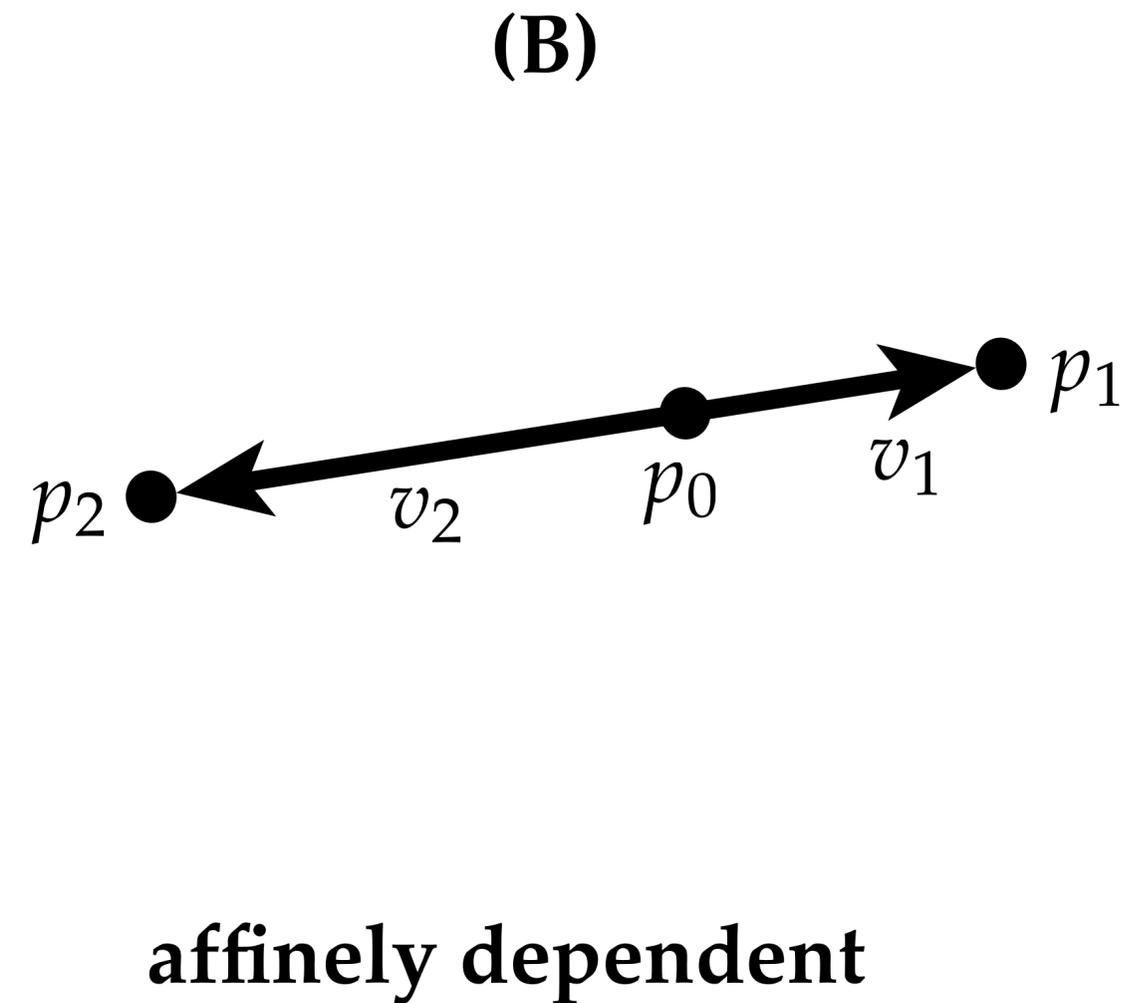
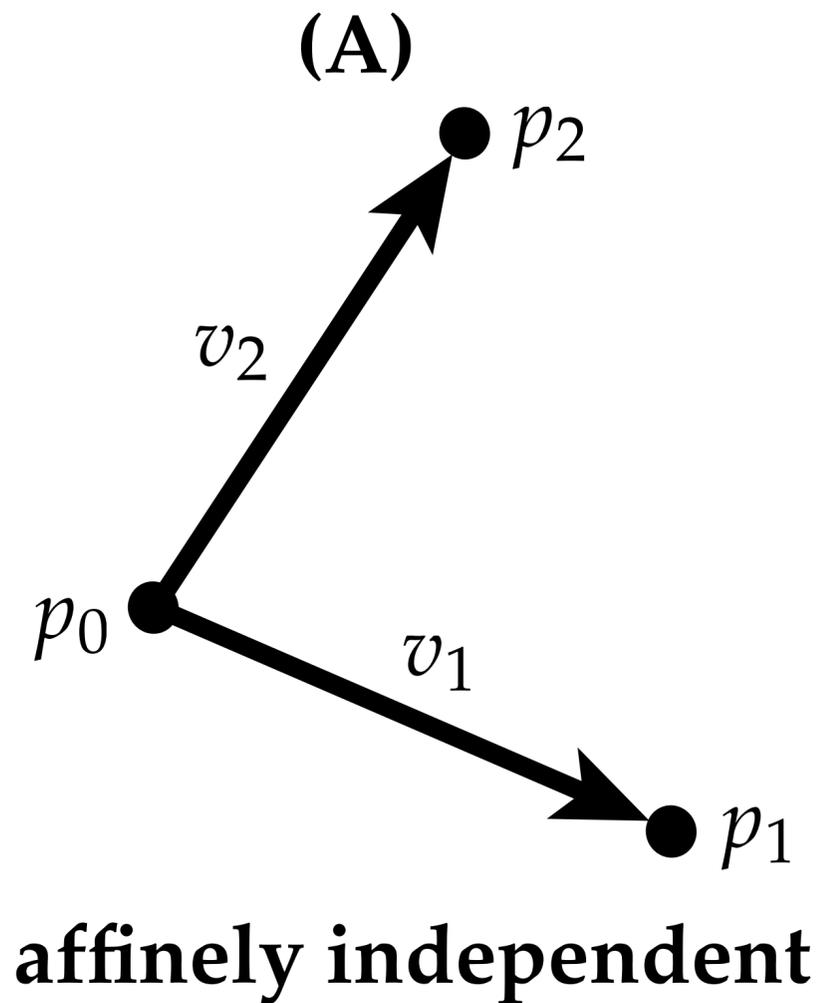
(B)



(Colloquially: might say points are in “*general position*”.)

Affine Independence

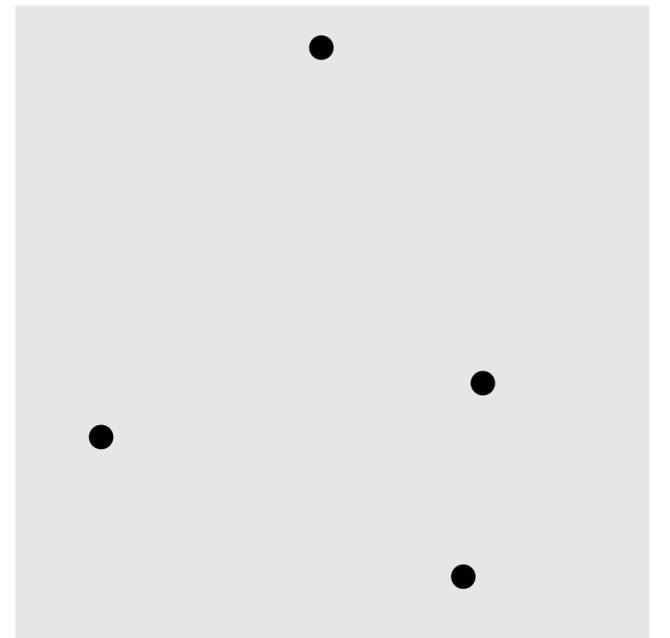
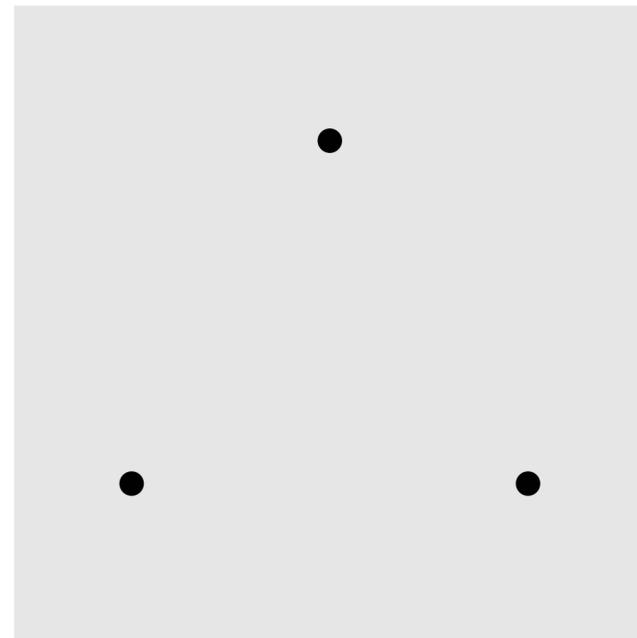
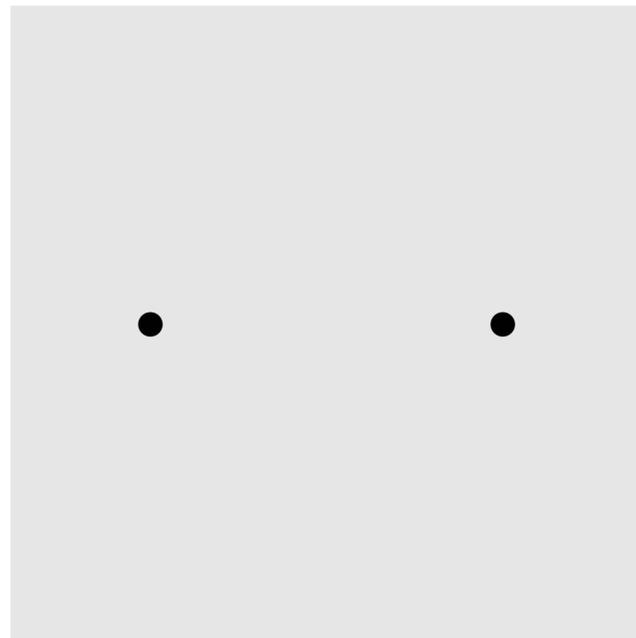
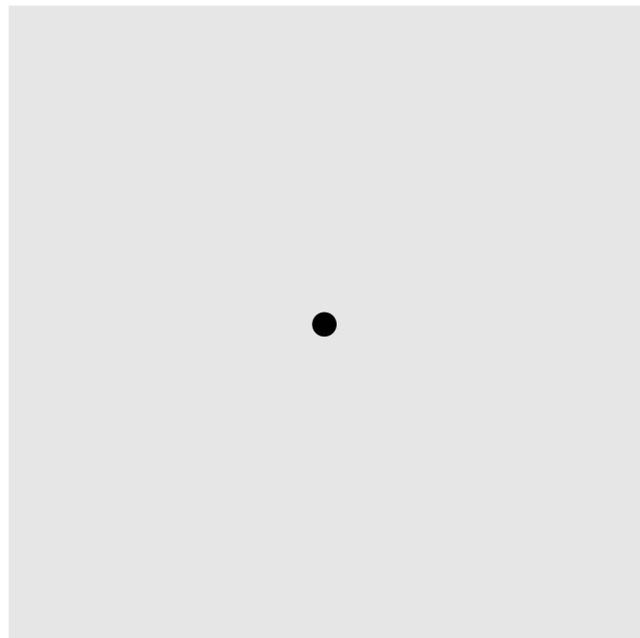
Definition. A collection of points p_0, \dots, p_k are *affinely independent* if the vectors $v_i := p_i - p_0$ are linearly independent.



(Colloquially: might say points are in “*general position*”.)

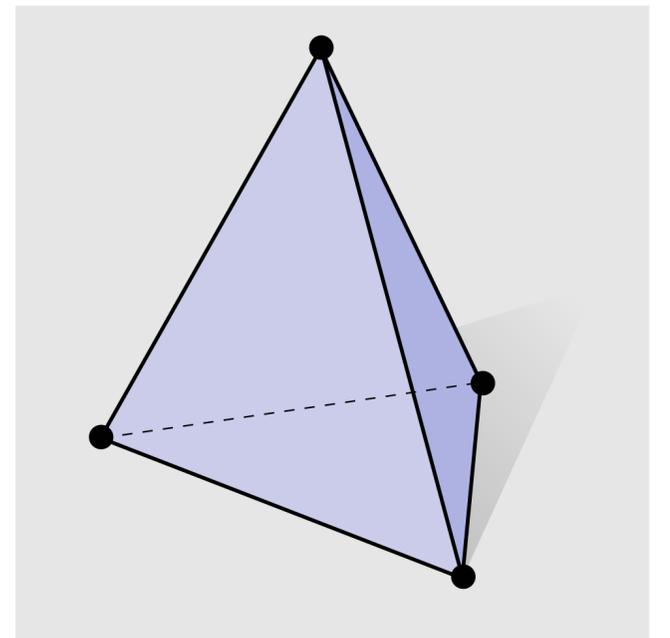
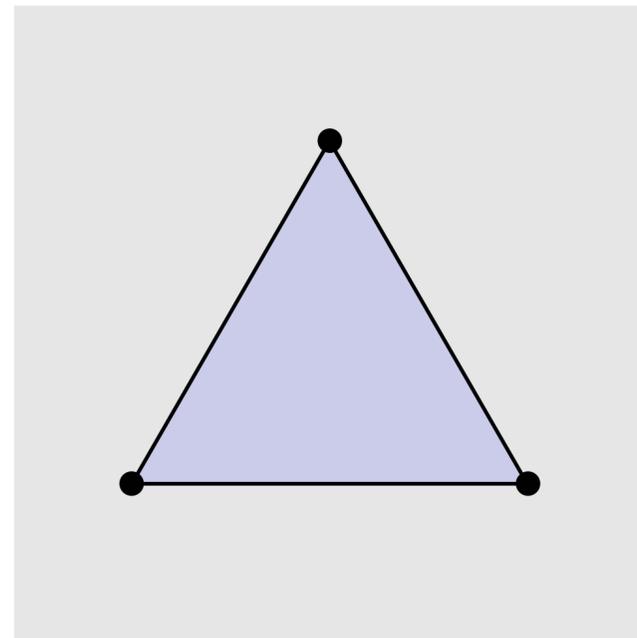
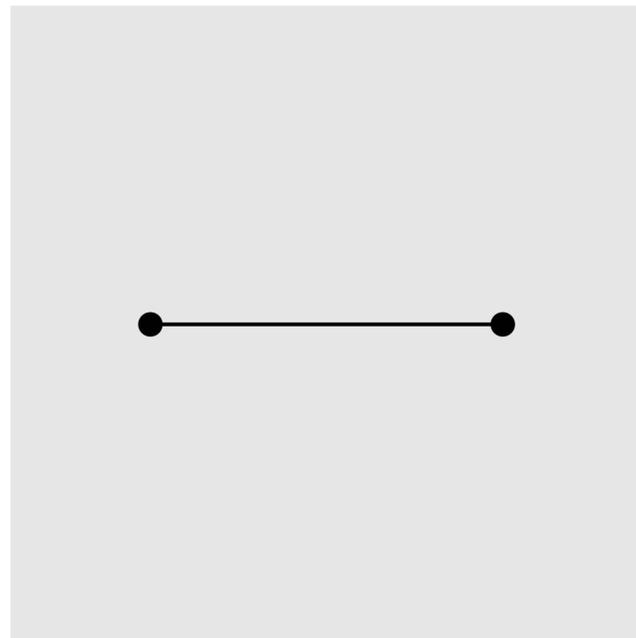
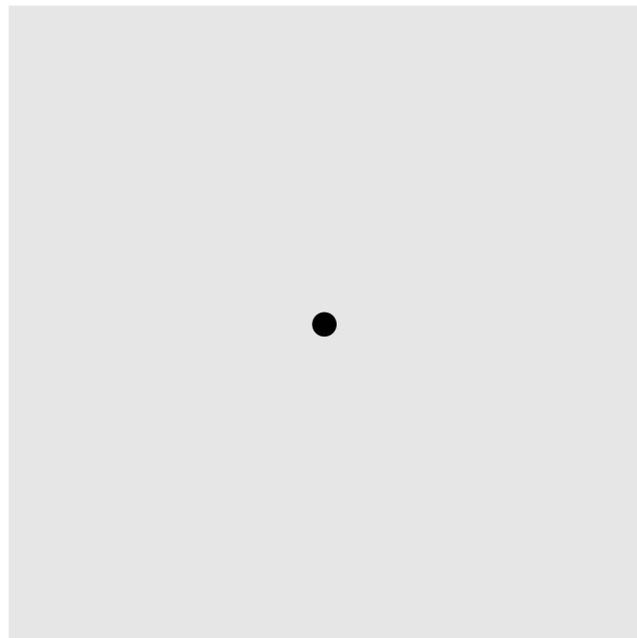
Simplex — Geometric Definition

Definition. A k -simplex is the convex hull of $k + 1$ affinely-independent points, which we call its *vertices*.



Simplex — Geometric Definition

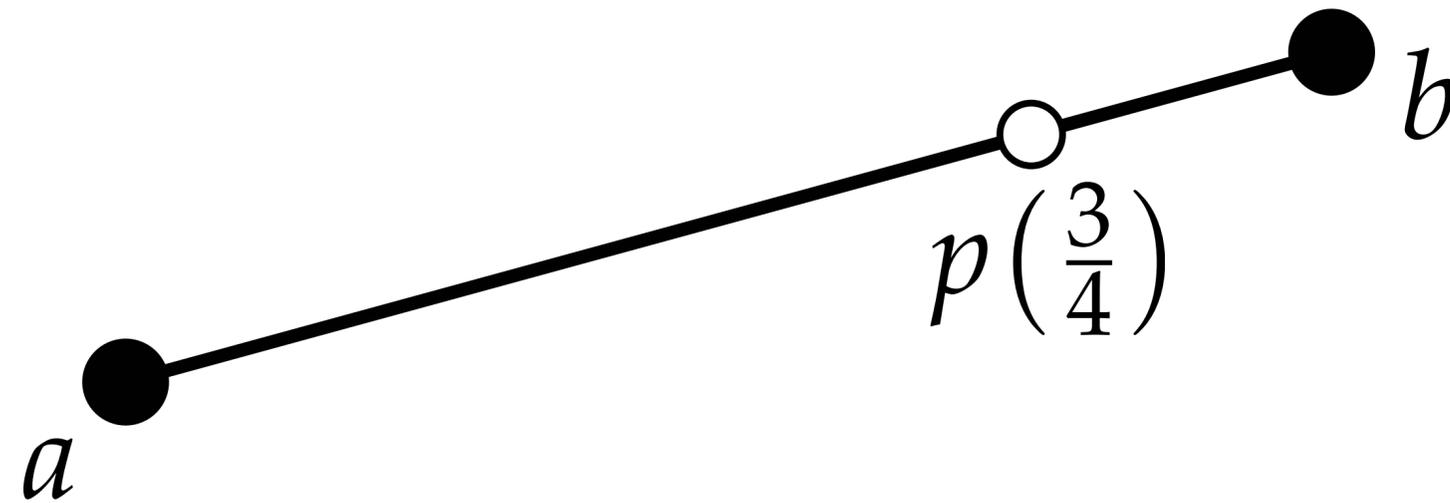
Definition. A k -simplex is the convex hull of $k + 1$ affinely-independent points, which we call its *vertices*.



Q: How many affinely-independent points can we have in n dimensions?

Barycentric Coordinates—1-Simplex

- We can describe a *simplex* more explicitly using barycentric coordinates.
- For instance, a 1-simplex is comprised of all weighted combinations of the two points where the weights sum to 1:

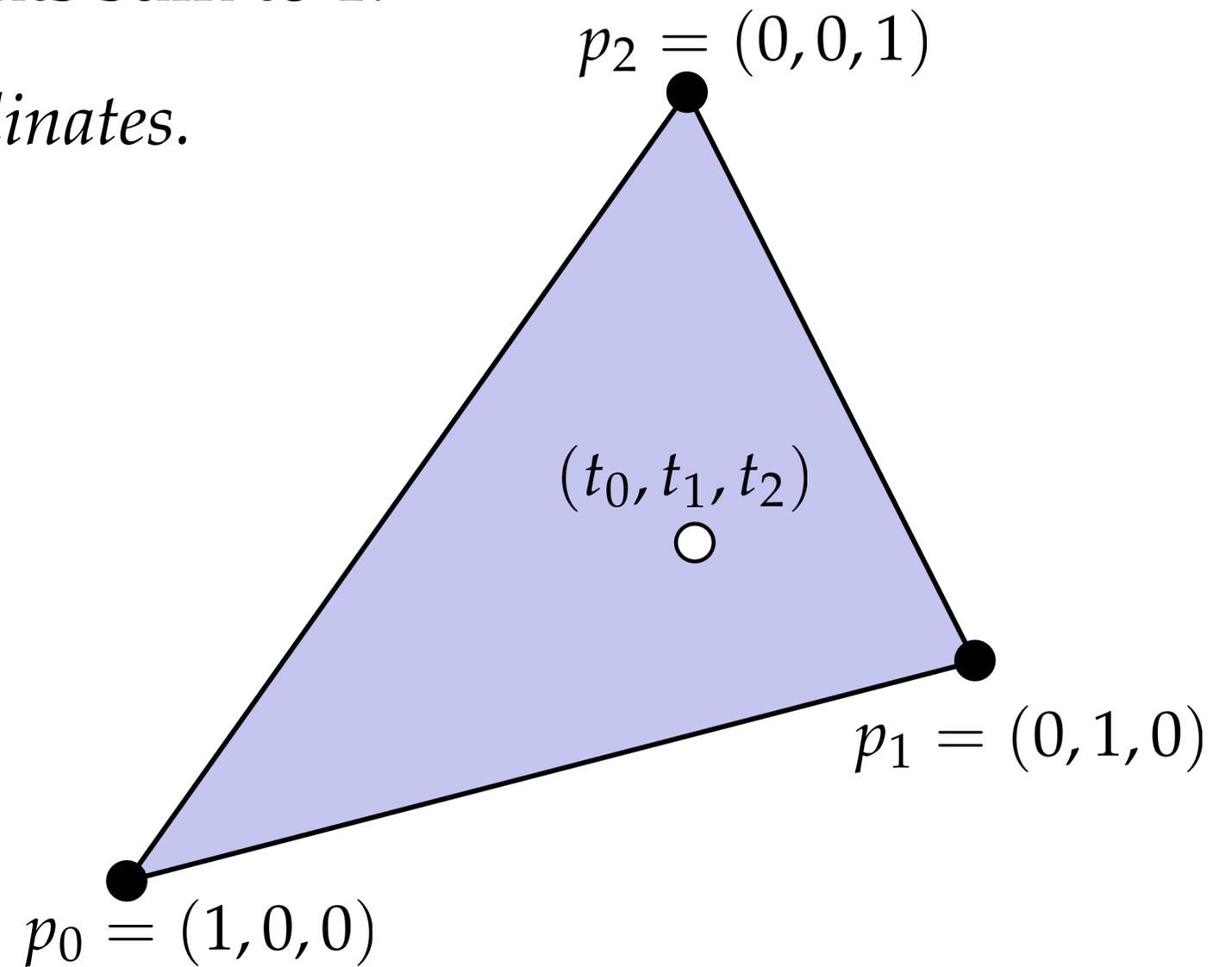


$$p(t) := (1 - t)a + tb, \quad t \in [0, 1]$$

Barycentric Coordinates — k -Simplex

- More generally, any point in a k -simplex can be expressed as a weighted combination of the vertices, where the weights sum to 1.
- The weights t_i are called the *barycentric coordinates*.

$$\sigma = \left\{ \sum_{i=0}^k t_i p_i \mid \sum_{i=0}^k t_i = 1, t_i \geq 0 \forall i \right\}$$

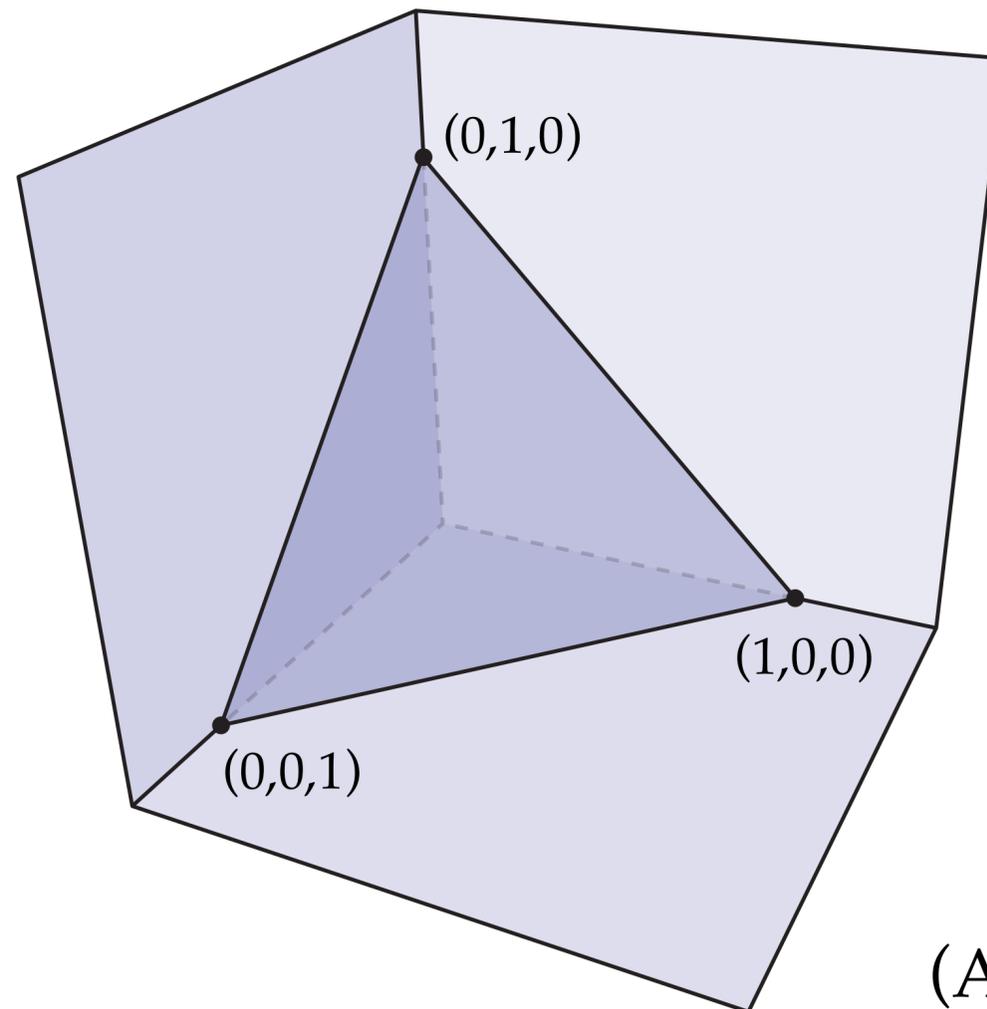


(Also called a “convex combination.”)

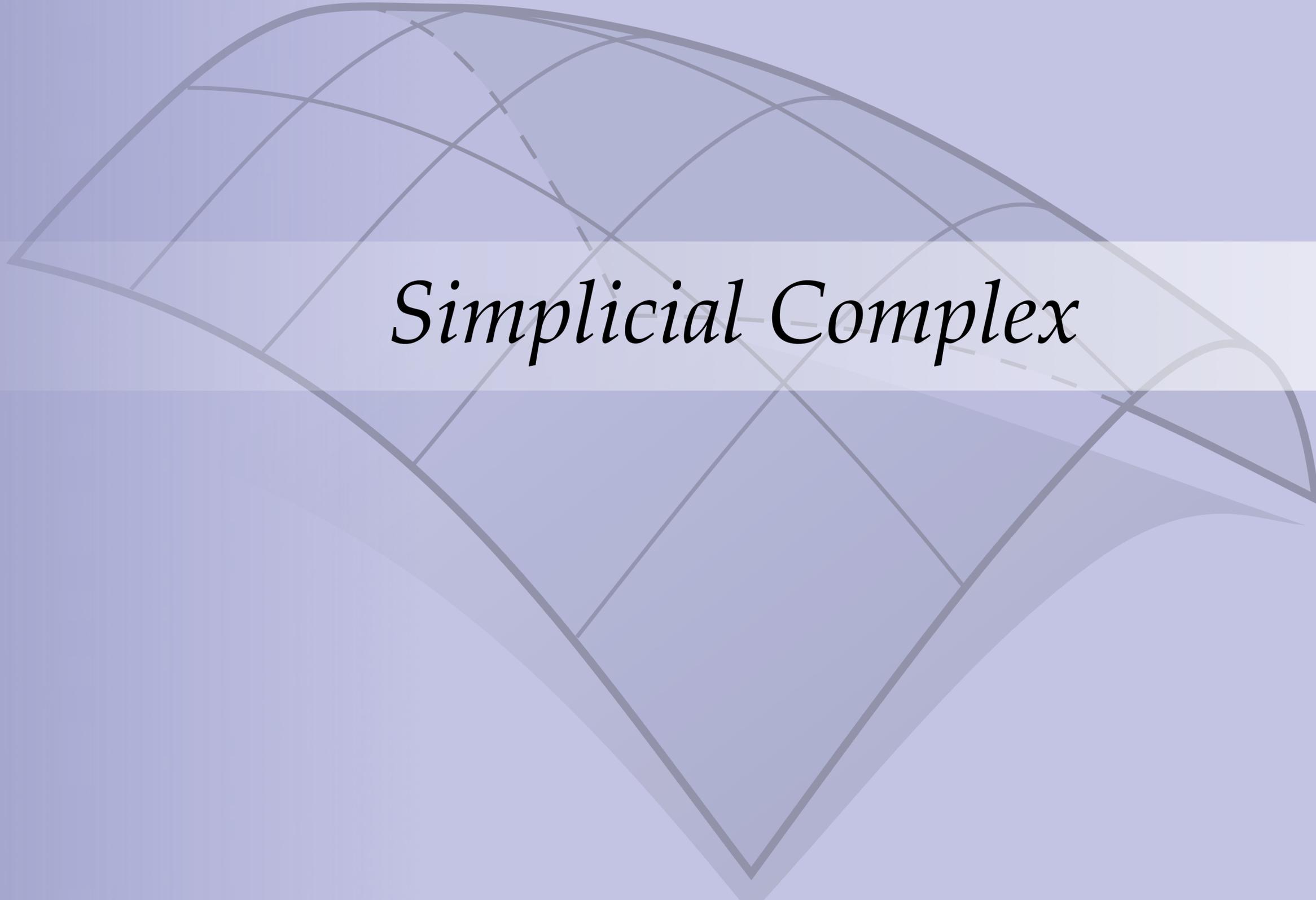
Simplex — Example

Definition. The *standard n -simplex* is the collection of points

$$\sigma := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1, x_i \geq 0 \forall i \right\}.$$



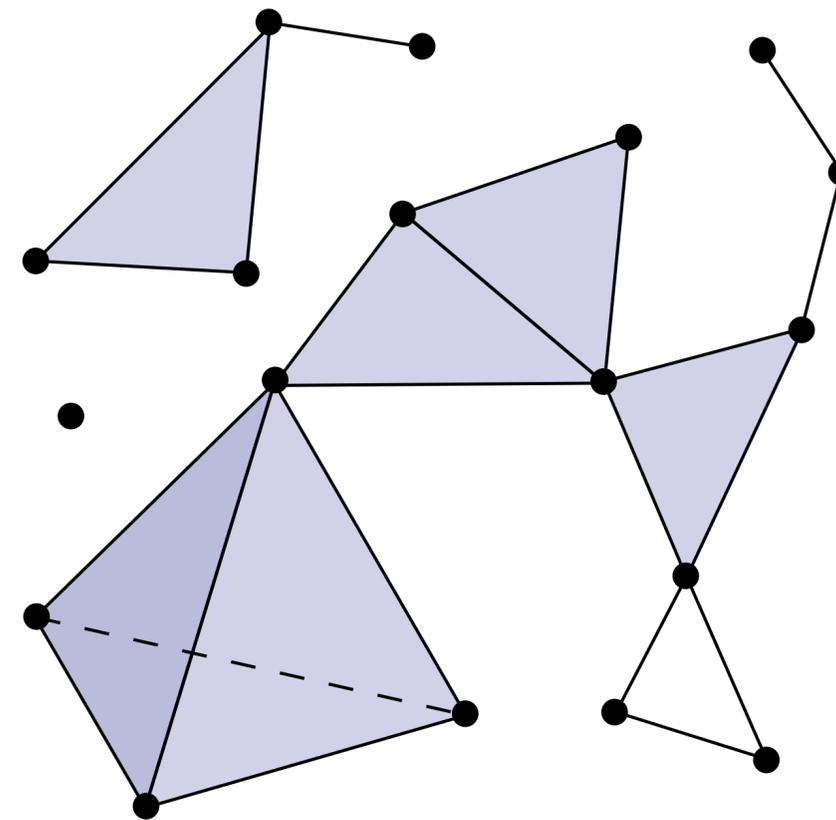
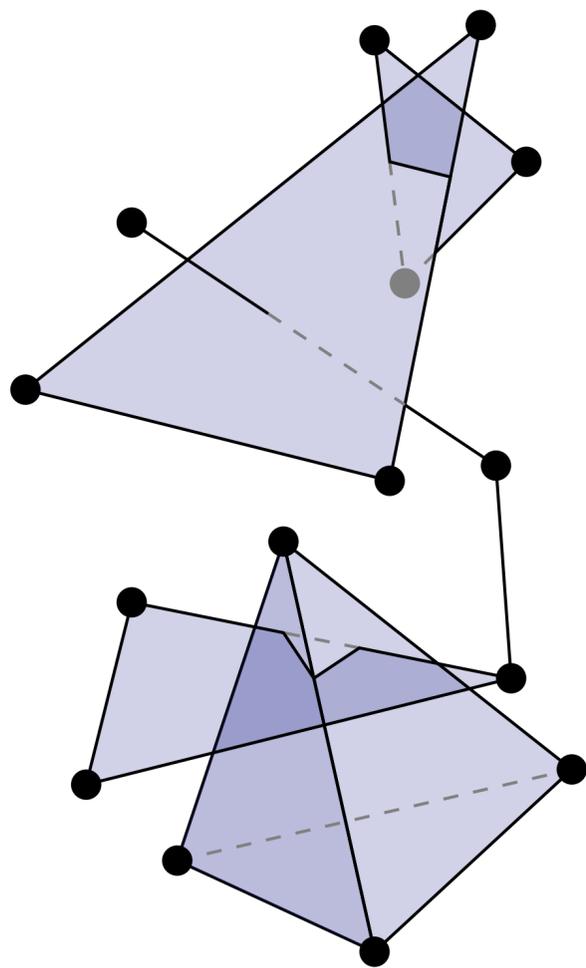
(Also known as the “probability simplex.”)



Simplicial Complex

Simplicial Complex—Rough Idea

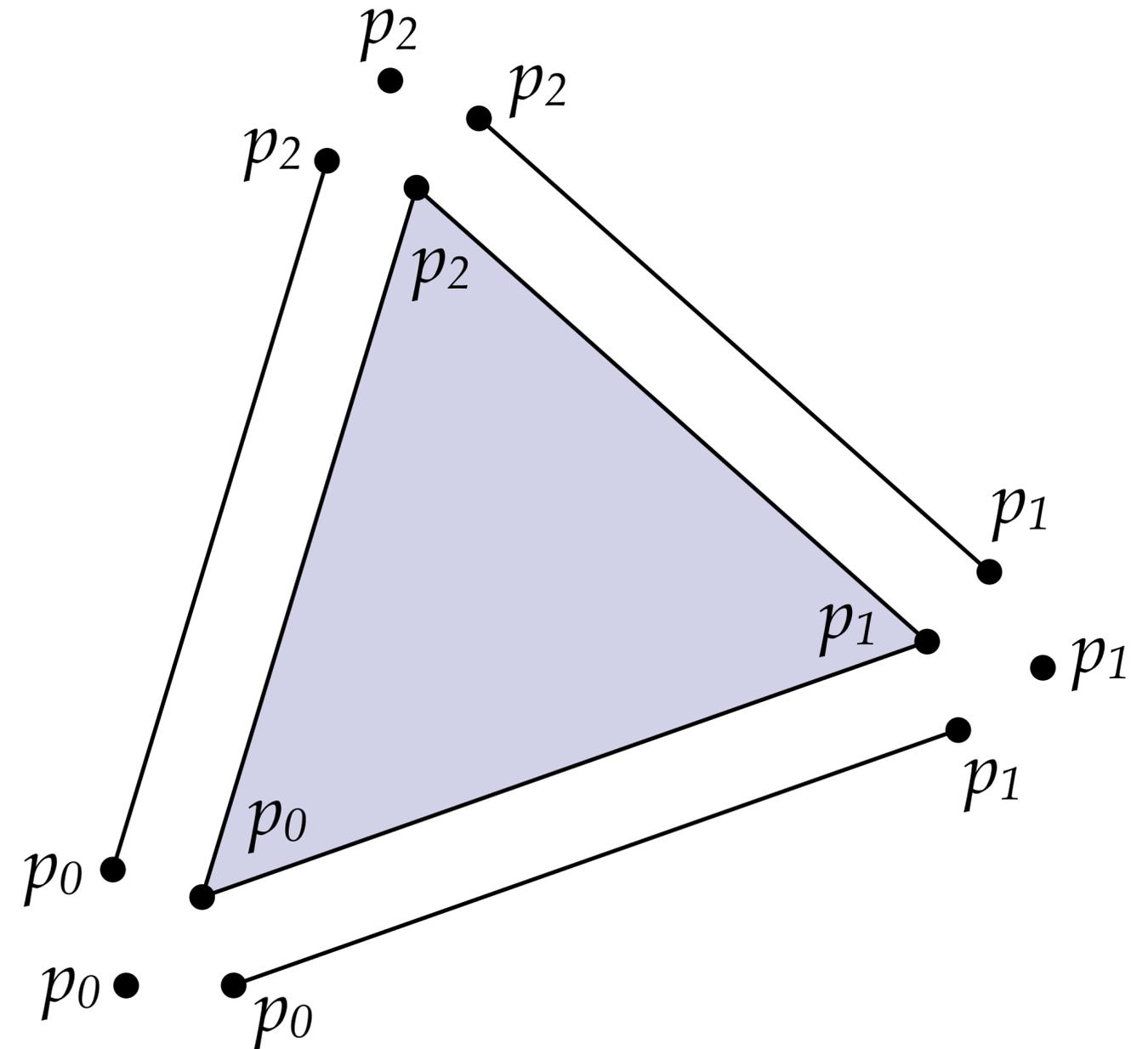
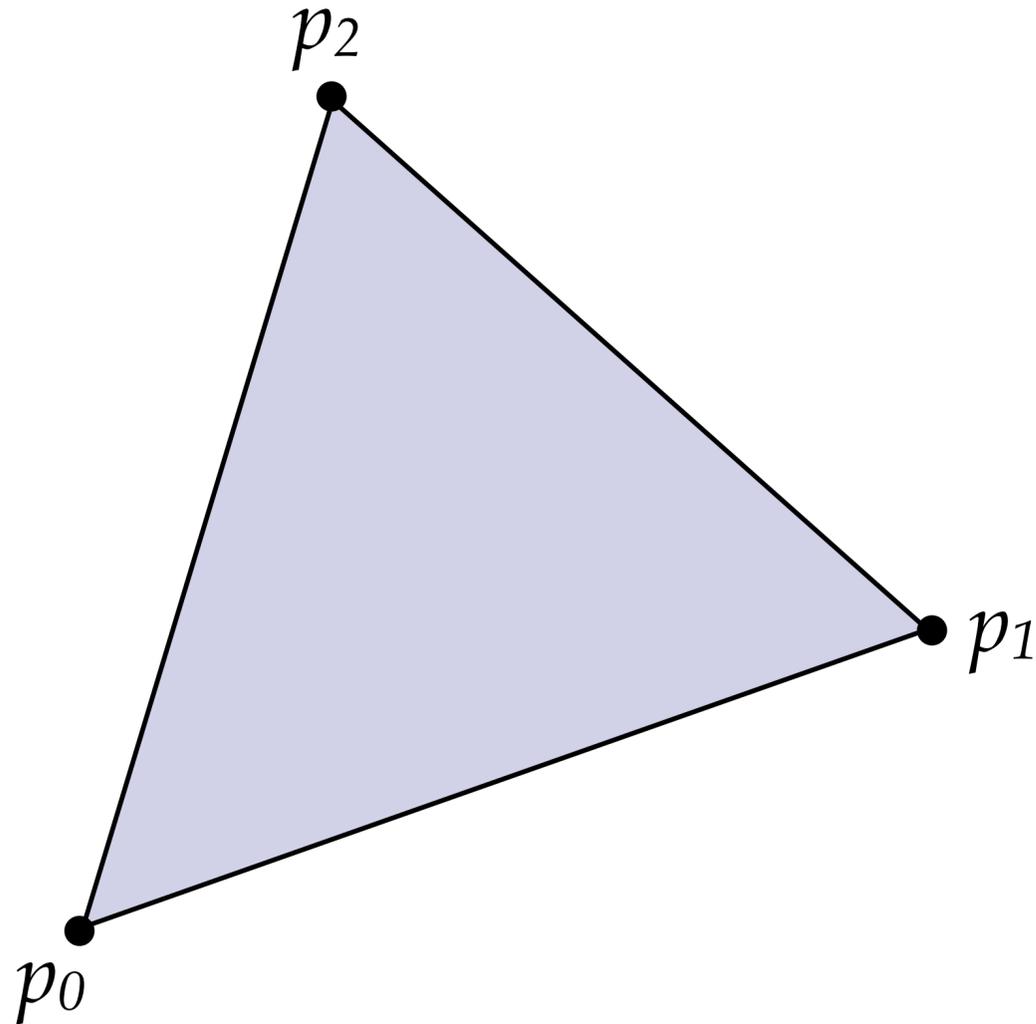
- Roughly speaking, a *simplicial complex* is “a bunch of simplices*”
 - ...but with some specific properties that make them easy to work with.
- Also have to resolve some basic questions—*e.g.*, how can simplices intersect?



Plural of simplex; not “simplexes.” Pronounced like *vertices* and *vortices*.

Face of a Simplex

Definition. A *face* of a simplex σ is any simplex whose vertices are a subset* of the vertices of σ .



Q: Anything missing from this picture?

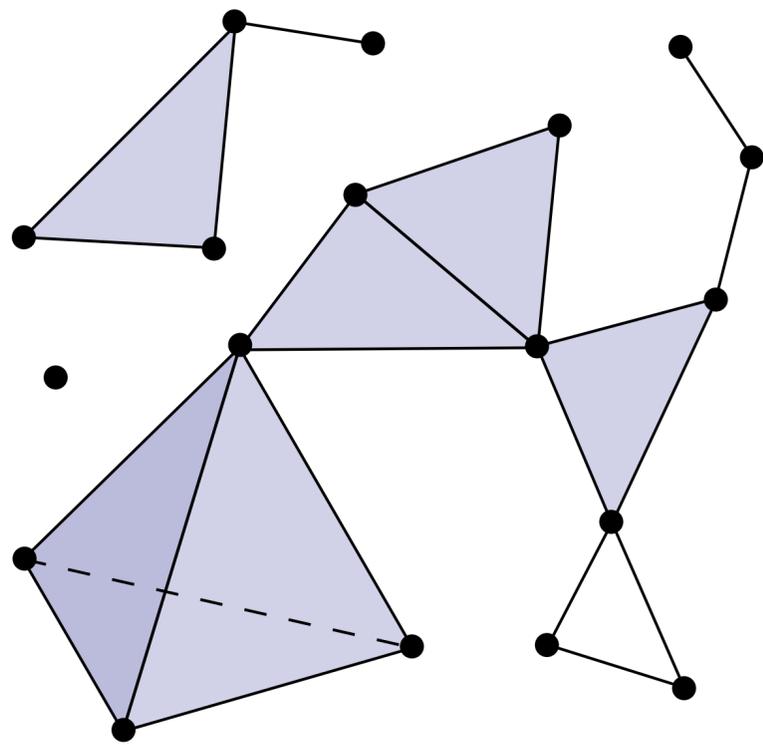
A: Yes—formally, the *empty set* \emptyset .

*Doesn't have to be a *proper* subset, *i.e.*, a simplex is its own face.

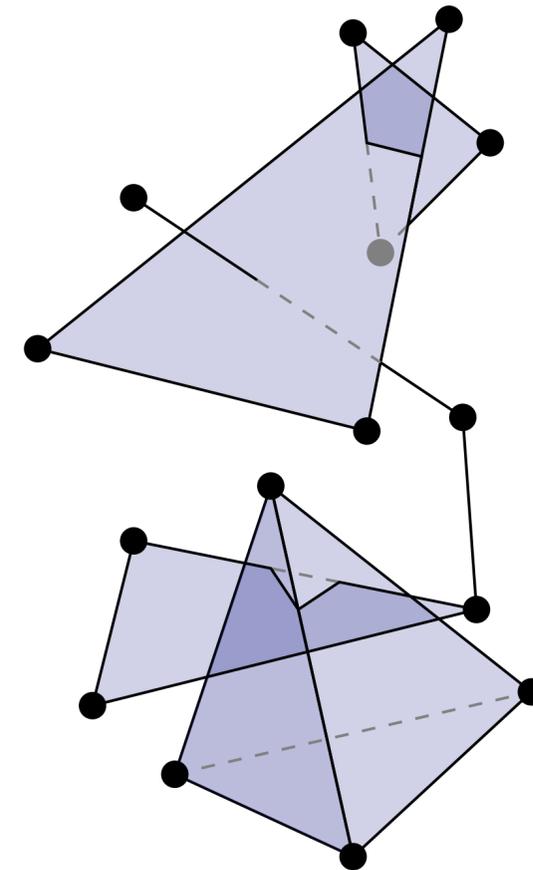
Simplicial Complex — Geometric Definition

Definition. A (geometric) simplicial complex is a collection of simplices where:

- the intersection of any two simplices is a simplex, and
- every face of every simplex in the complex is also in the complex.

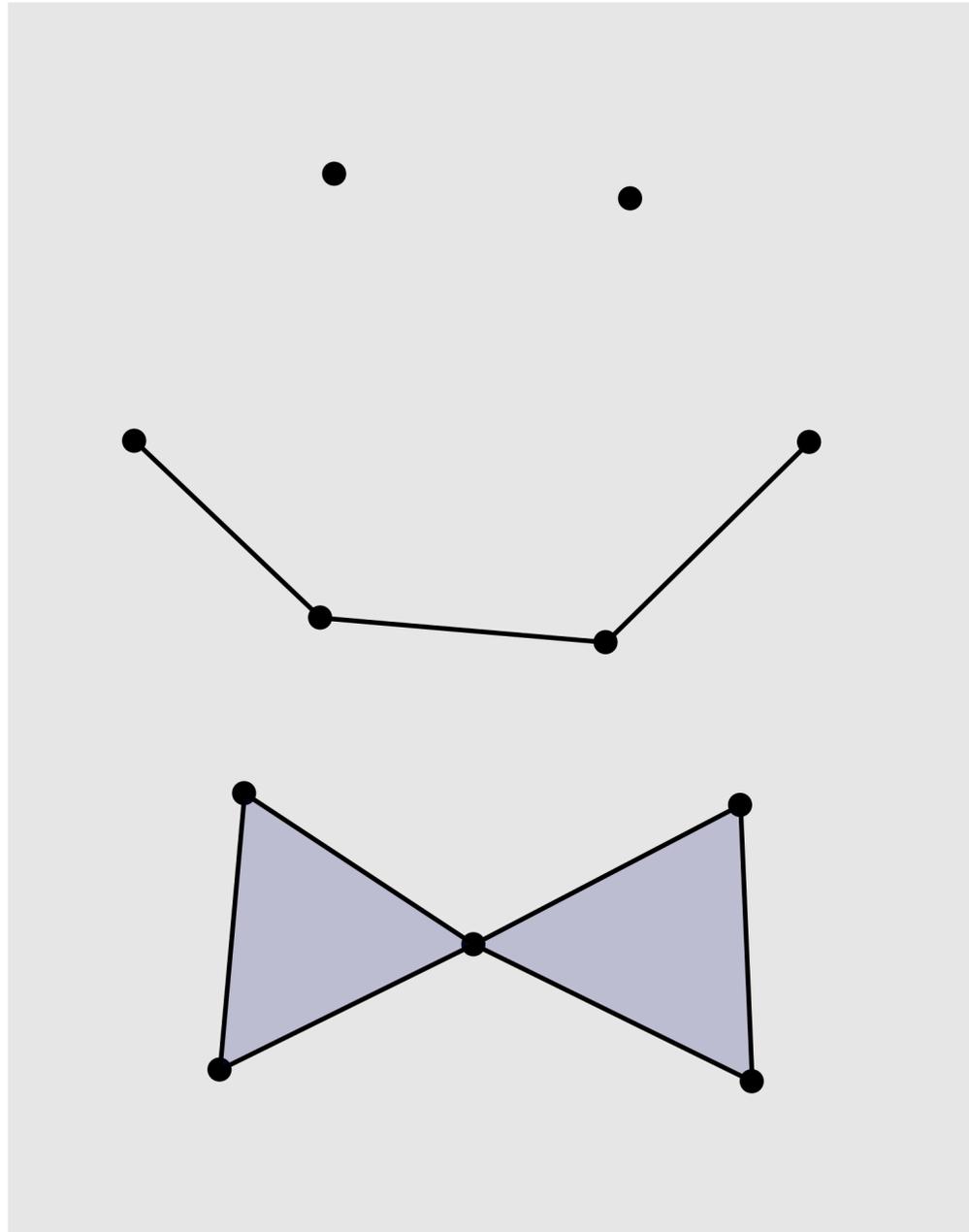


simplicial complex

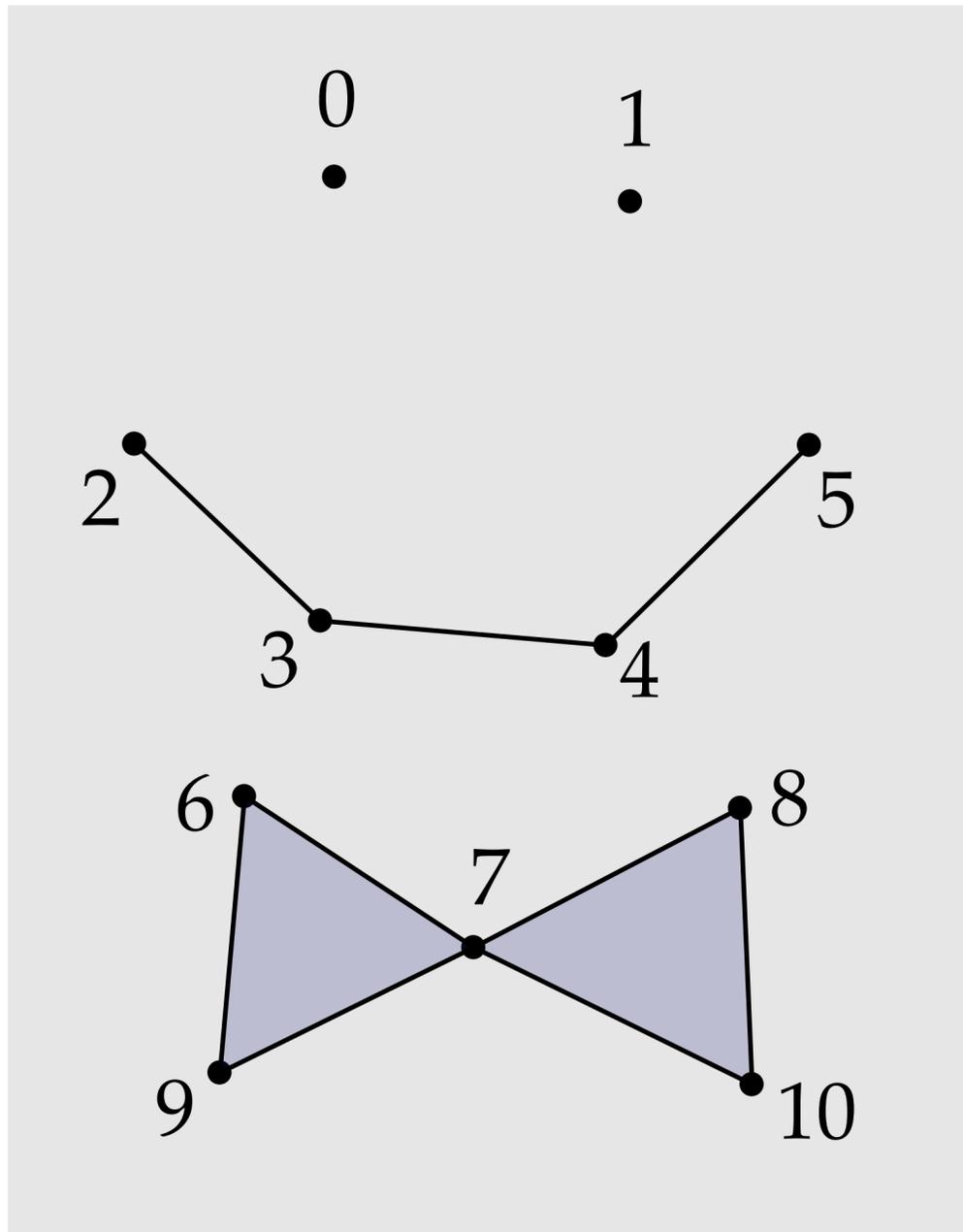


not a geometric simplicial complex...

Simplicial Complex — Example



Simplicial Complex — Example



Q: What are all the simplices?

A: {6,7,9} {7,10,8} {2,3} {3,4} {4,5} {0} {1}

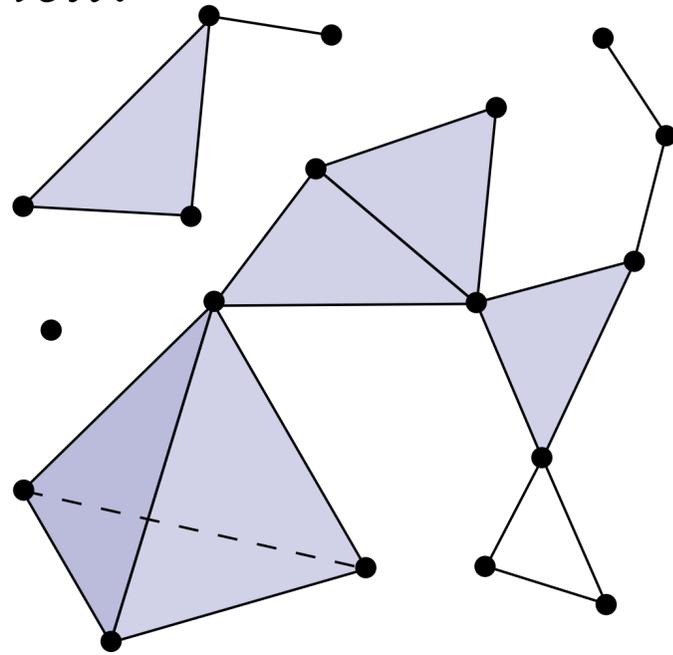
{6,7} {7,9} {9,6} {7,8} {8,10} {10,7} {2} {3} {4} {5}

{6} {7} {8} {9} {10}

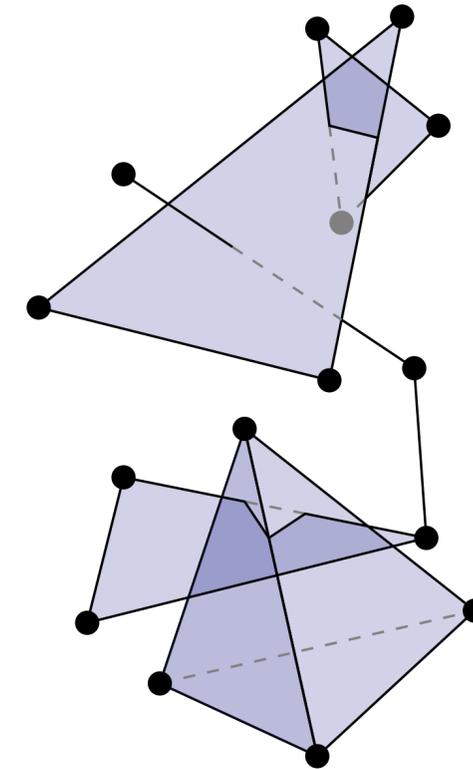
Didn't really use the geometry here...

Abstract Simplicial Complex

Definition. Let S be a collection of sets. If for each set $\sigma \in S$ all subsets of σ are contained in S , then S is an *abstract simplicial complex*. A set $\sigma \in S$ of size $k + 1$ is an (*abstract*) *simplex*.



geometric simplicial complex



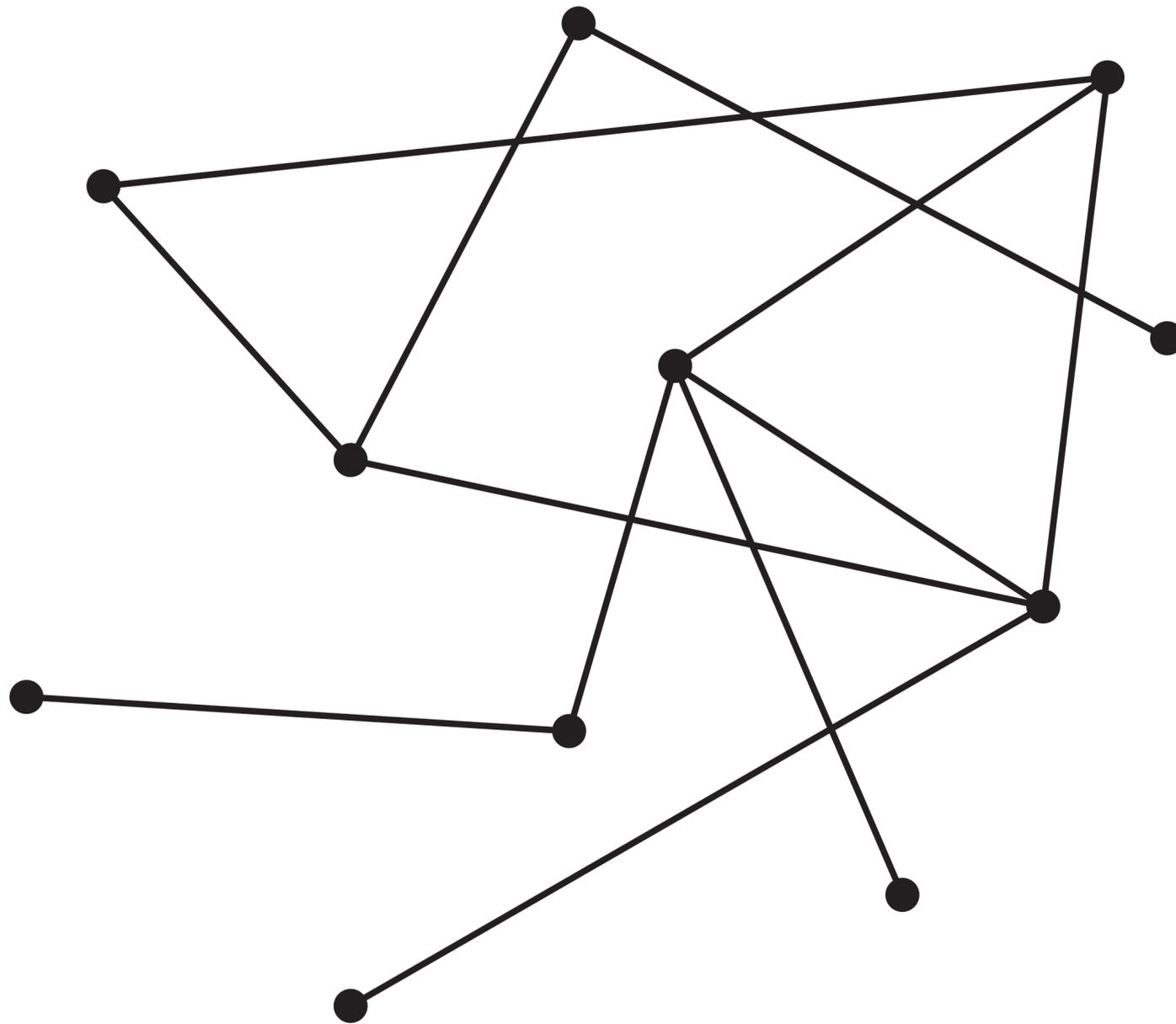
abstract simplicial complex*

- Only care about how things are *connected*, not how they are arranged geometrically.
- Serve as our discretization of a *topological space*

*...visualized by mapping it into R^3 .

Abstract Simplicial Complex — Graphs

- Any (*undirected*) graph $G = (V, E)$ is an abstract simplicial (1-)complex
- 0-simplices are vertices
- 1-simplices are edges



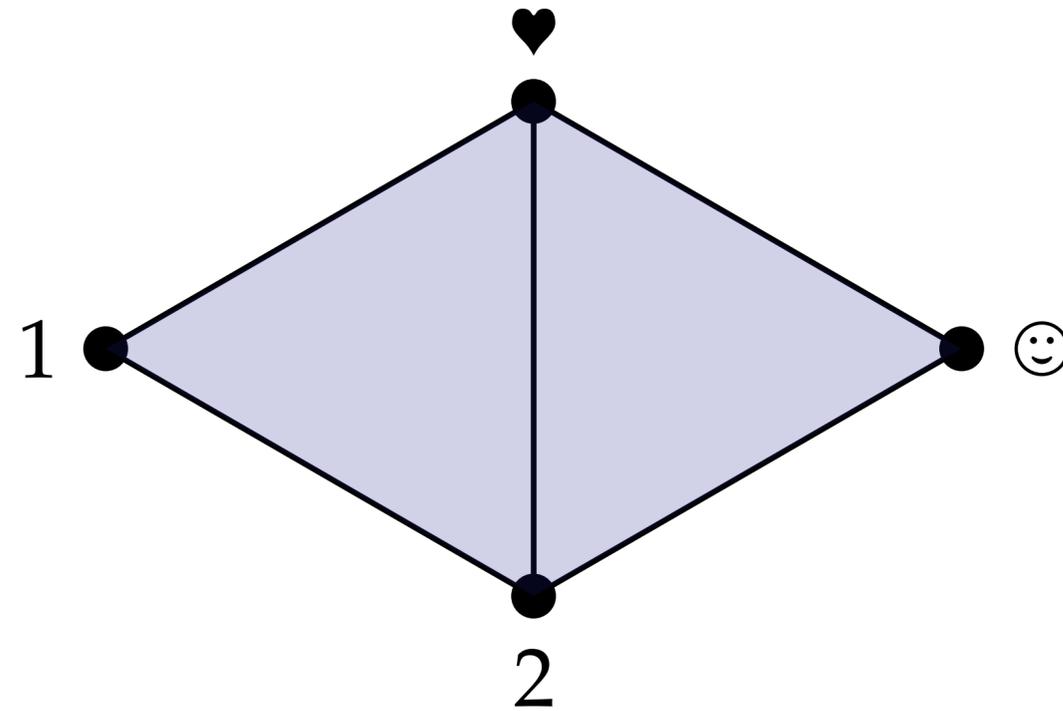
Abstract Simplicial Complex—Example

Example: Consider the set

$S := \{\{1, 2, \heartsuit\}, \{2, \heartsuit, \smile\}, \{1, 2\}, \{2, \heartsuit\}, \{\heartsuit, 1\}, \{2, \smile\}, \{\heartsuit, \smile\}, \{1\}, \{2\}, \{\heartsuit\}, \{\smile\}, \emptyset\}$

Q: Is this set an abstract simplicial complex? If so, what does it look like?

A: Yes—it's a pair of 2-simplices (triangles) sharing a single edge:

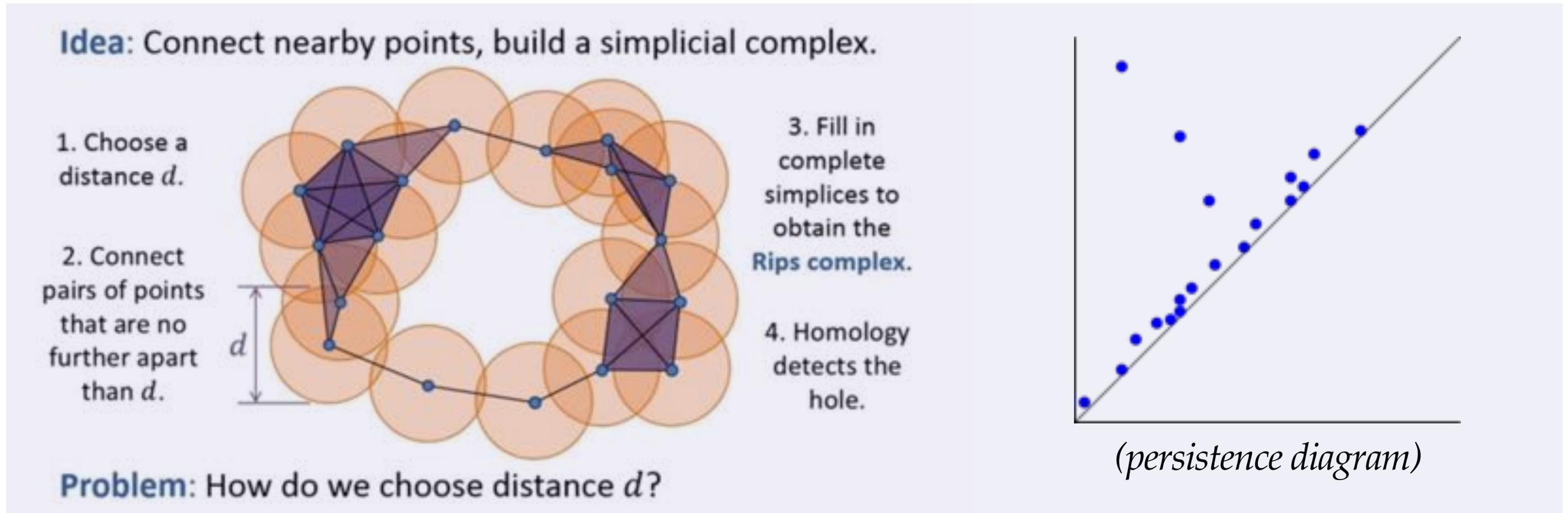


Vertices no longer have to be points in space; can represent anything at all.

Application: Topological Data Analysis

Forget (mostly) about geometry—try to understand data in terms of *connectivity*.

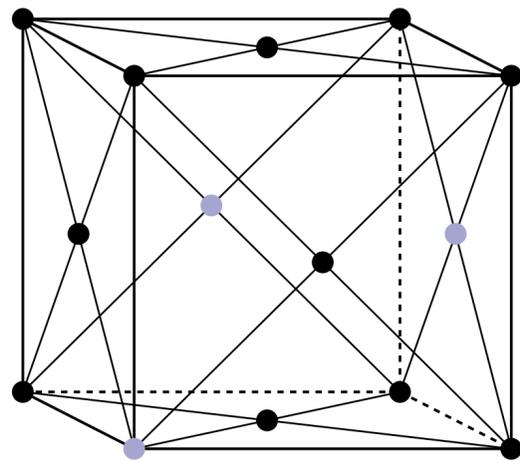
E.g., *persistent homology*:



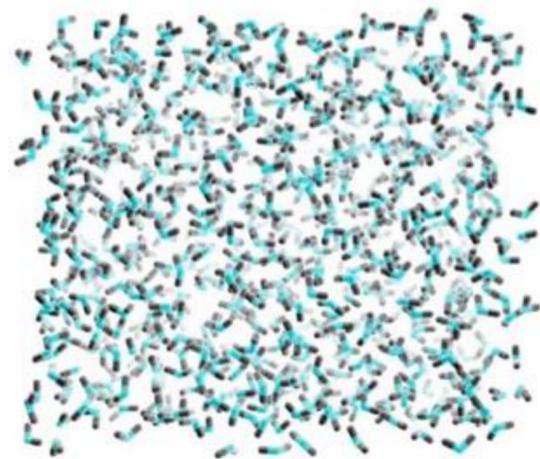
<https://youtu.be/h0bnG1Wavag>

Material Characterization via Persistence

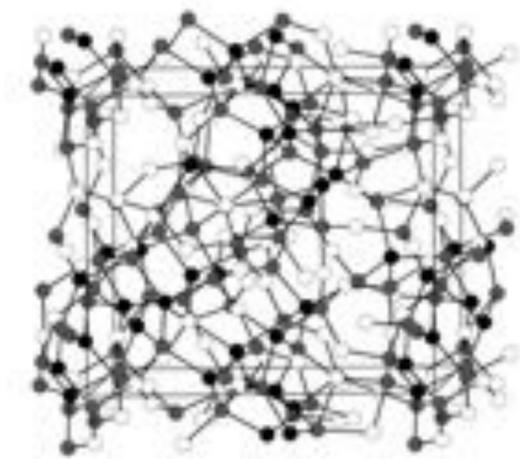
Regular



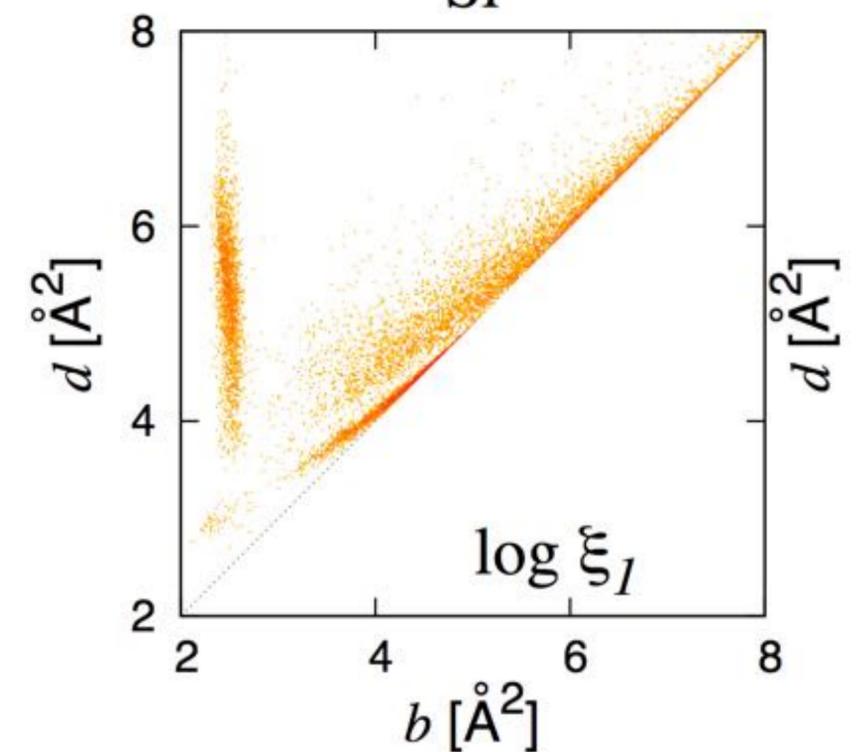
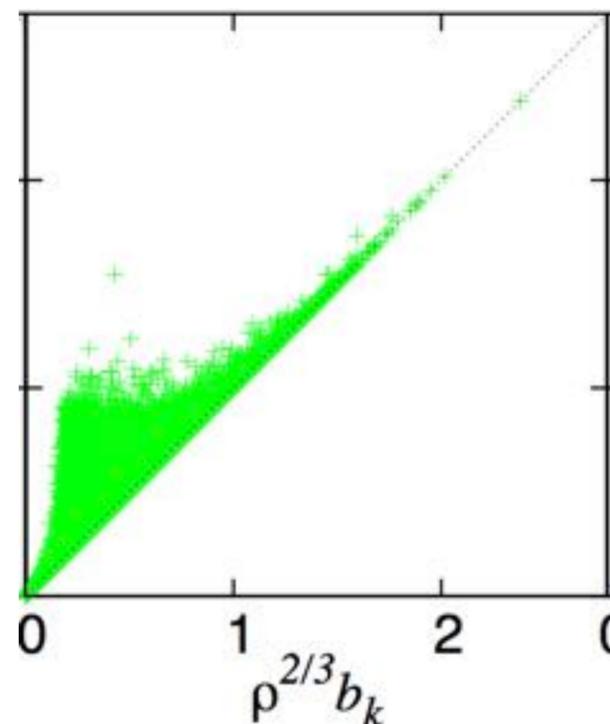
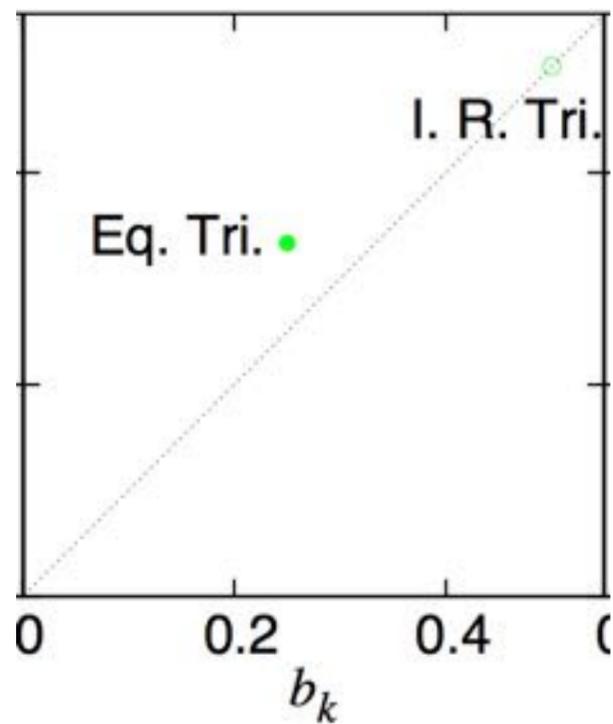
Random



Glass

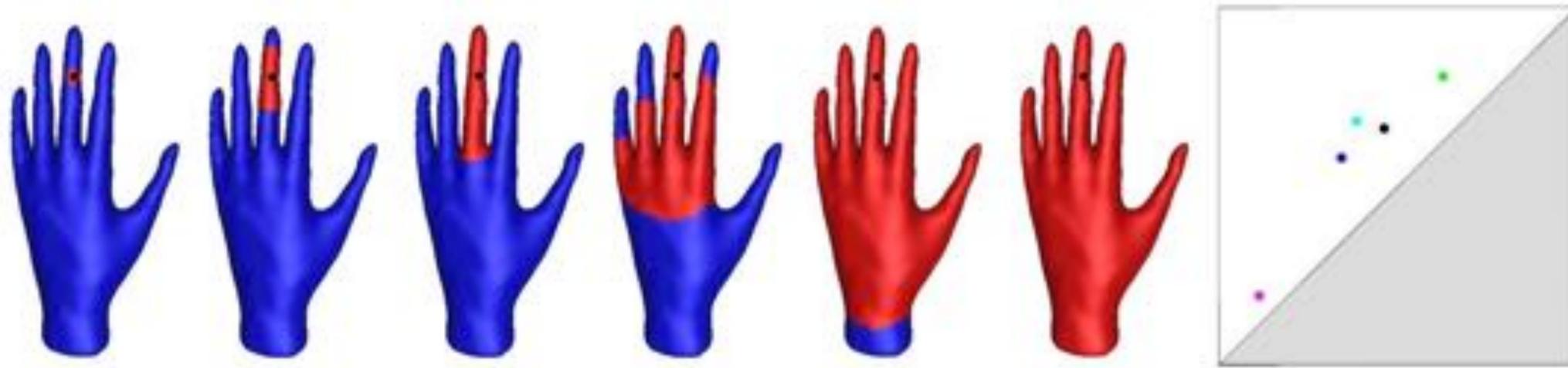


Si

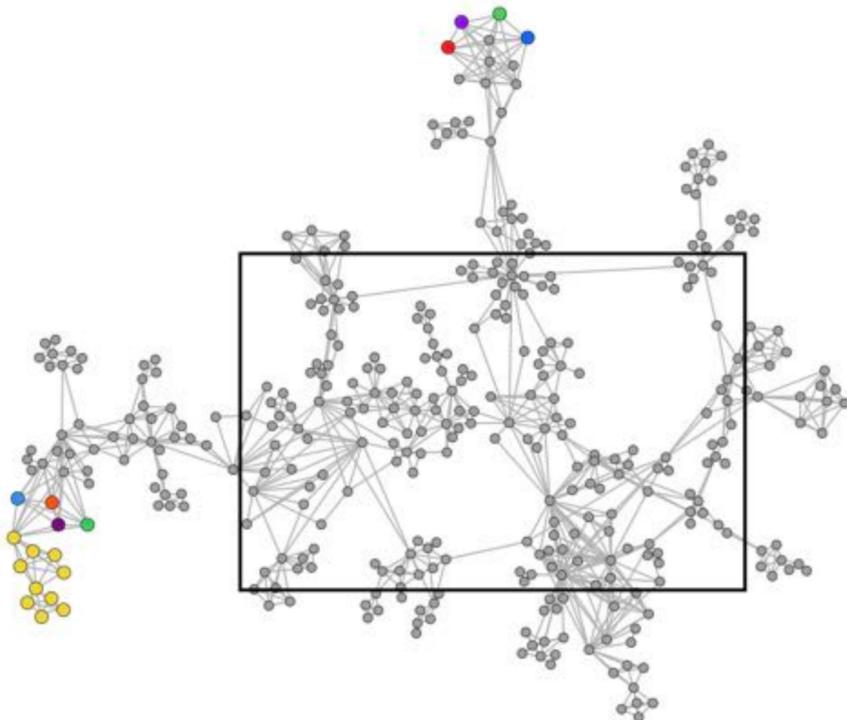


Persistent Homology—More Applications

M. Carrière, S. Oudot, M. Ovsjanikov, “Stable Topological Signatures for Points on 3D Shapes”



C. Carstens, K. Horadam,
“Persistent Homology of Collaboration Networks”



H. Lee, M. Chung, H. Kang, B. Kim, D. Lee
“Discriminative Persistent Homology of Brain Networks”

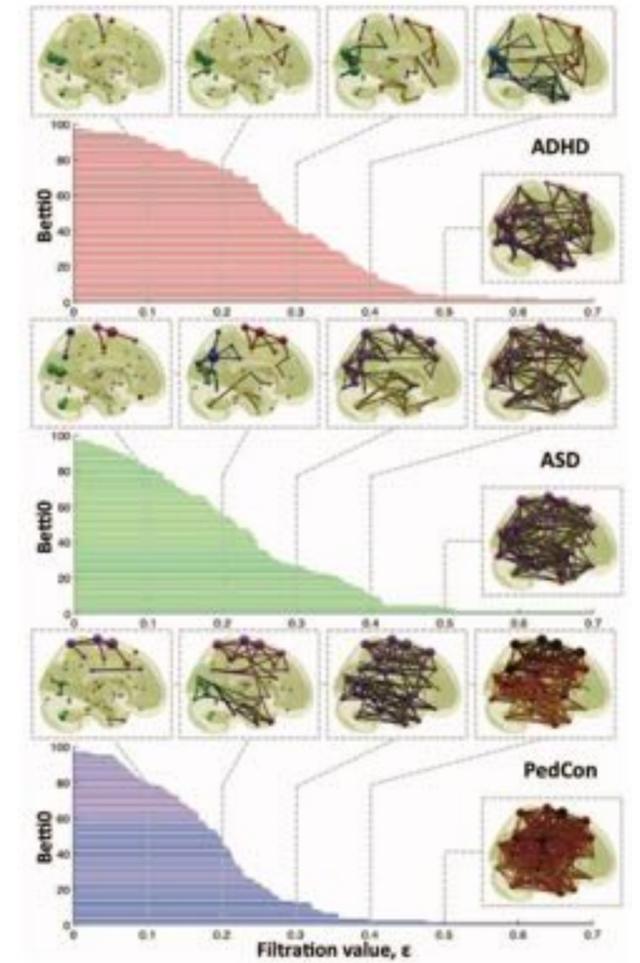
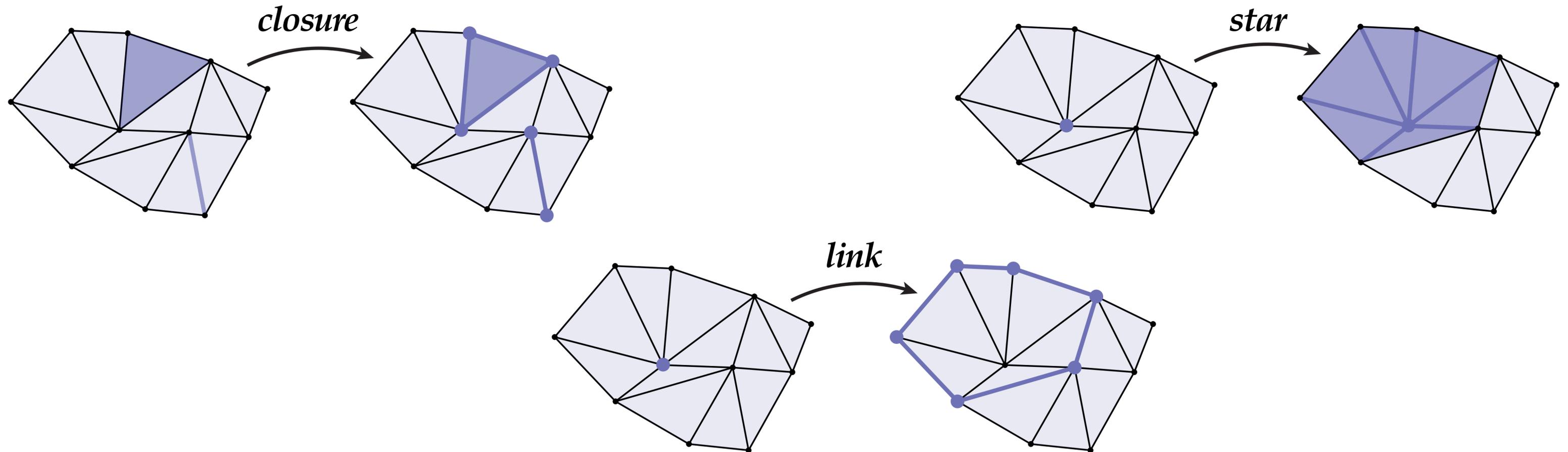


Fig. 4. Barcode of the 0-th Betti number.

...and much much more (identifying patients with breast cancer, classifying players in basketball, new ways to compress images, *etc.*)

Anatomy of a Simplicial Complex

- **Closure:** smallest simplicial complex containing a given set of simplices
- **Star:** union of simplices containing a given subset of simplices
- **Link:** closure of the star minus the star of the closure



Vertices, Edges, and Faces

- Just a little note about notation:

- For simplicial **1-complexes** (graphs) we often write $G = (V, E)$

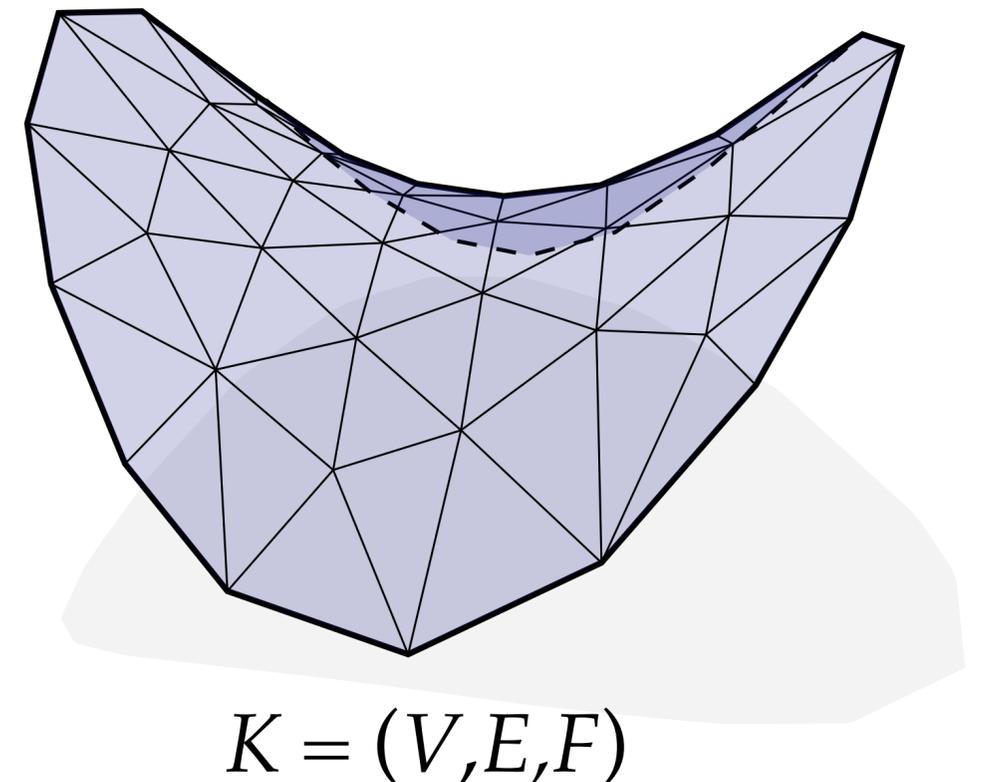
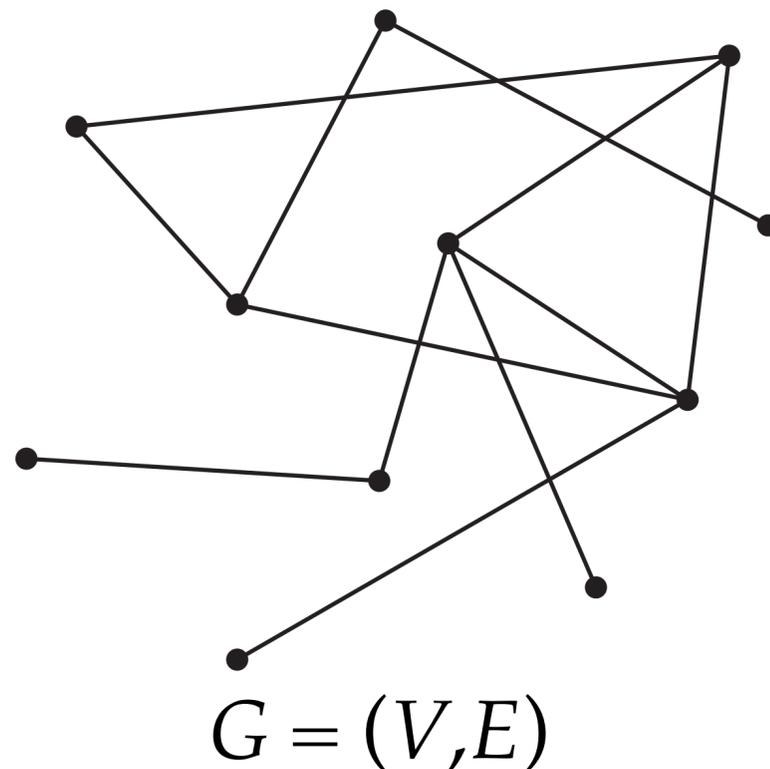
- Likewise, for simplicial **2-complexes** (triangle meshes) we write $K = (V, E, F)$

- Vertices

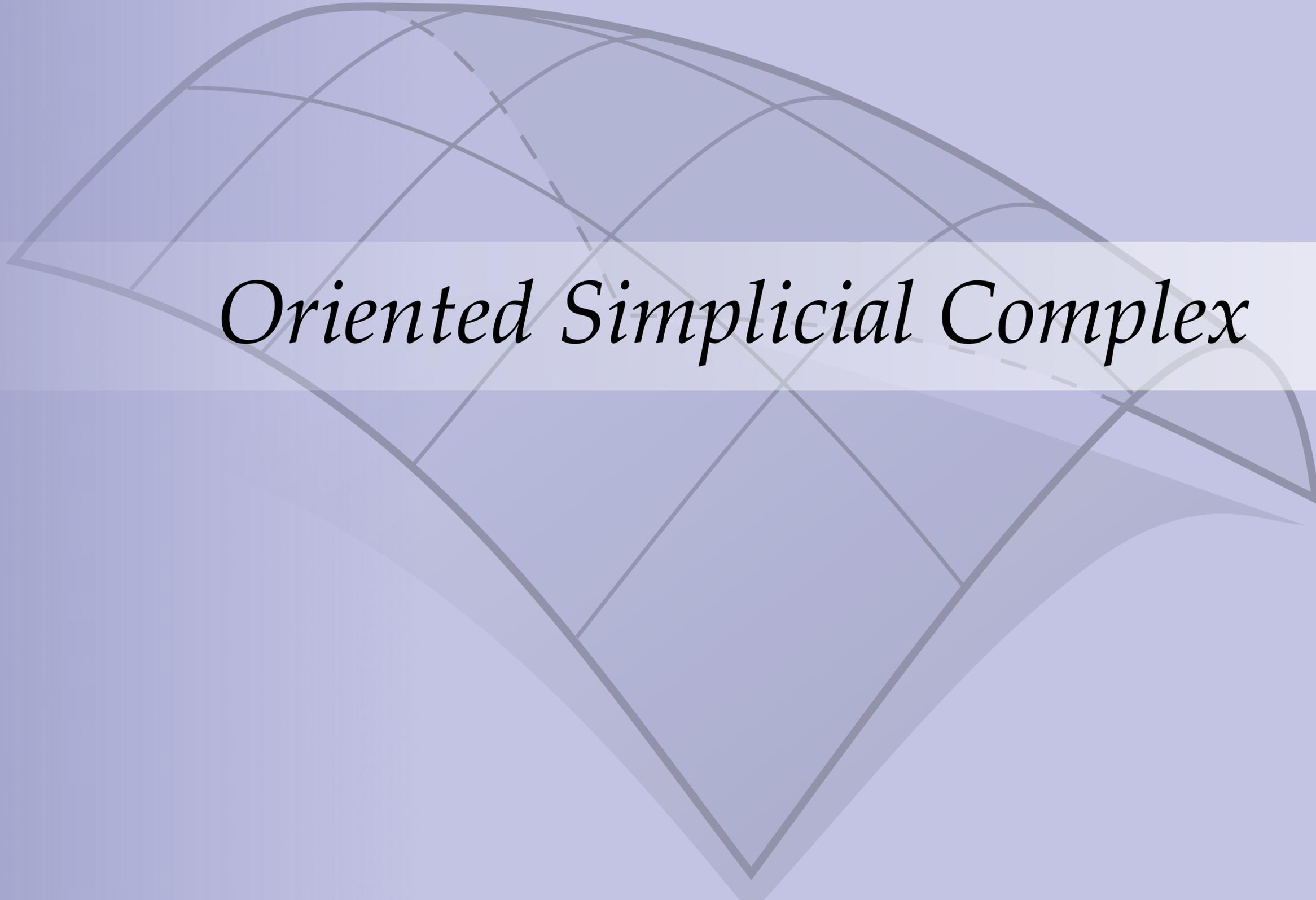
- Edges

- Faces*

- K is for “*Komplex!*”

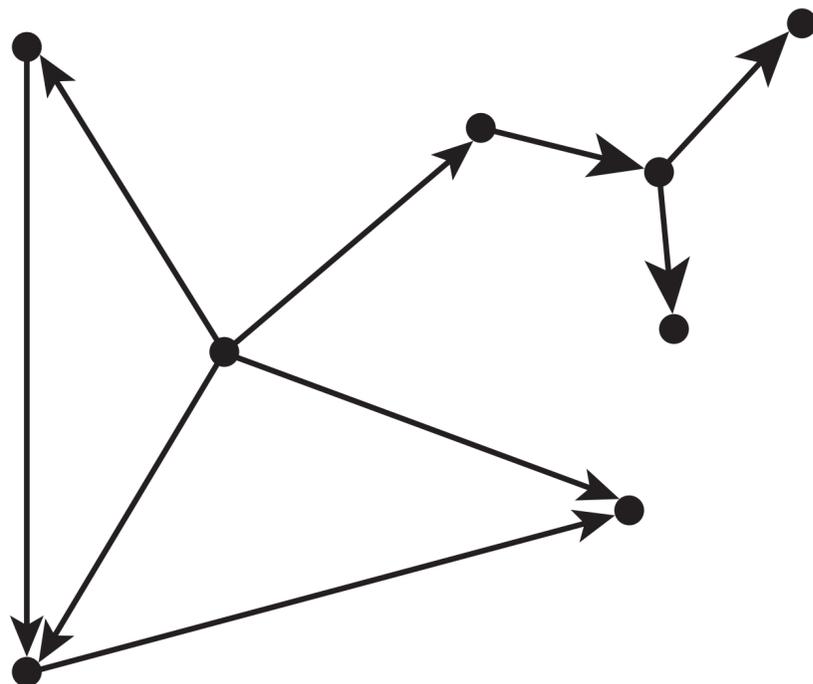
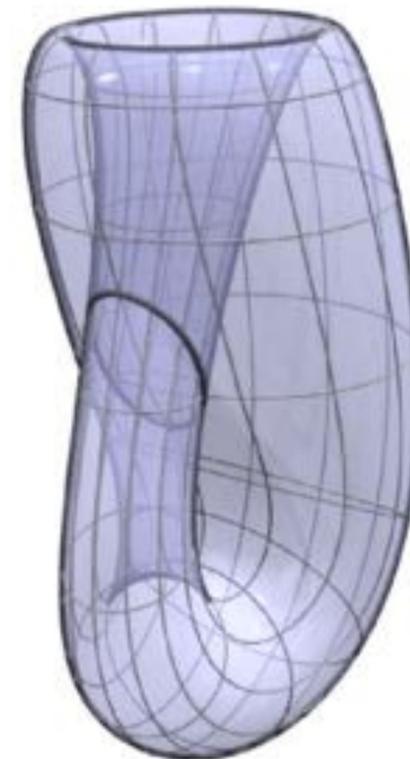
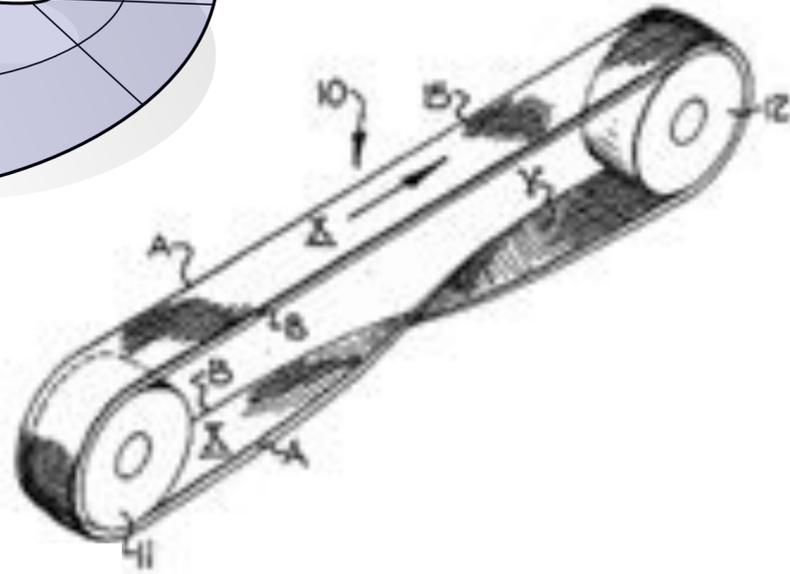
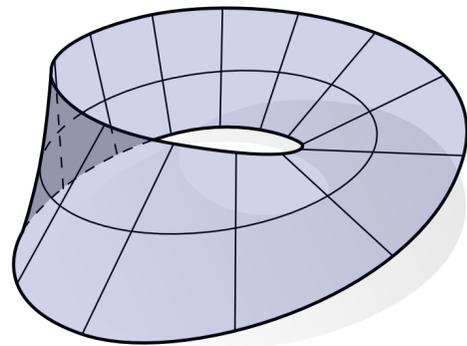
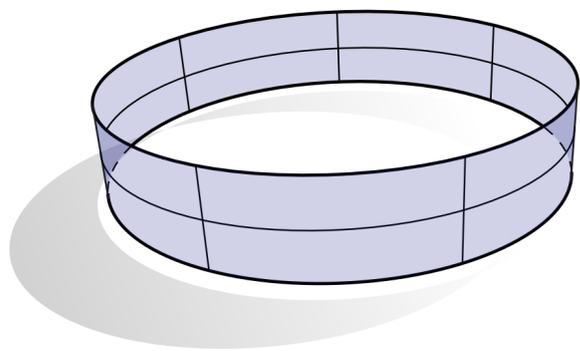


*Not to be confused with the generic *face* of a simplex...



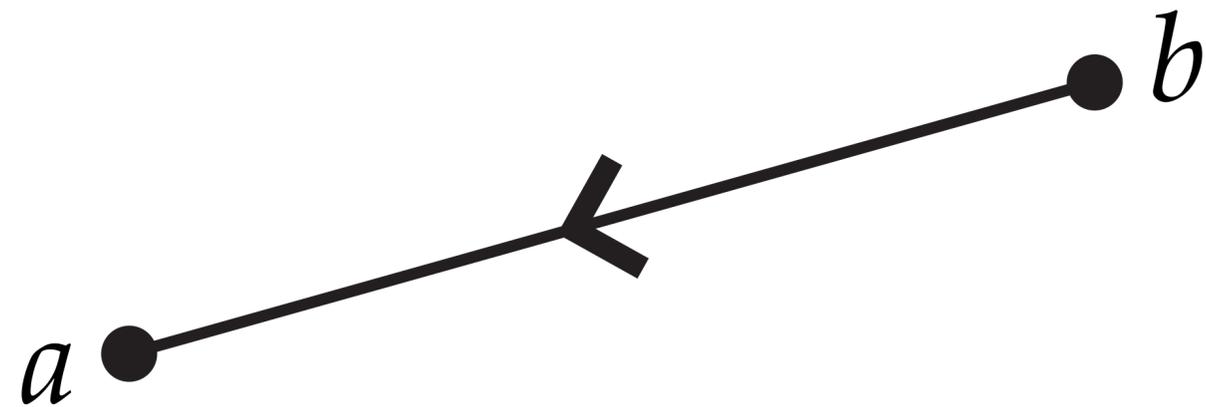
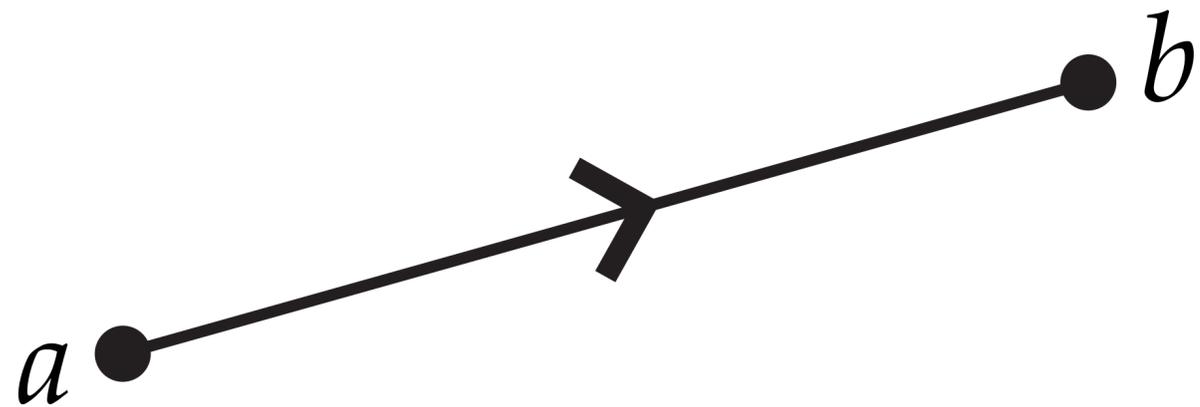
Oriented Simplicial Complex

Orientation—Visualized



Orientation of a 1-Simplex

- Basic idea: does a 1-simplex $\{a,b\}$ go from a to b or from b to a ?
- Instead of set $\{a,b\}$, now have *ordered tuple* (a,b) or (b,a)



- Why do we care? *Eventually* will have to do with integration...

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

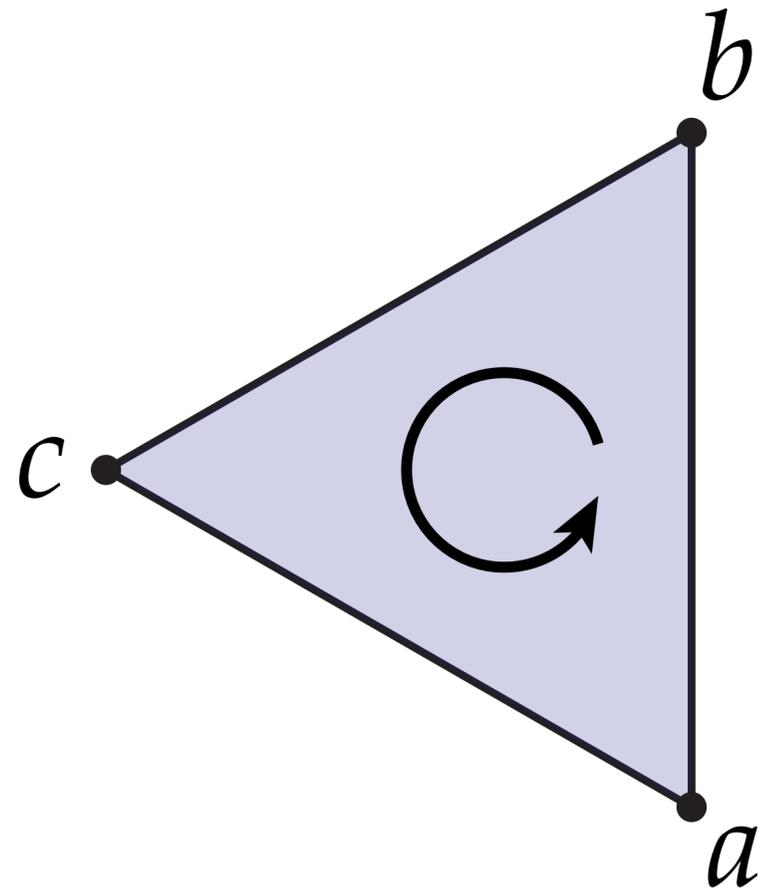
Orientation of a 2-Simplex

- For a 2-simplex, orientation given by “winding order” of vertices:

(a,b,c)

(b,c,a)

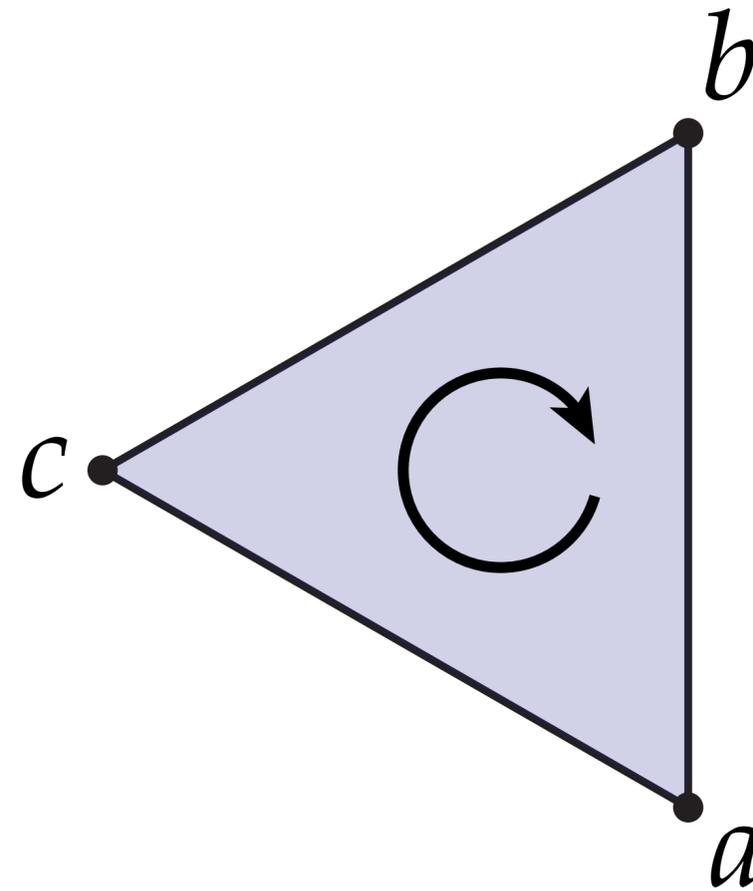
(c,a,b)



(a,c,b)

(c,b,a)

(b,a,c)

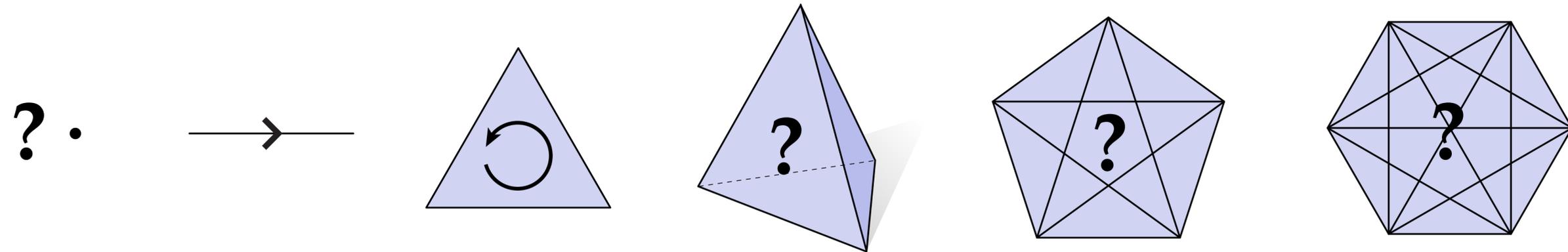


Q: How can we encode these *oriented 2-simplices*?

A: Oriented tuples, up to circular shift.

Oriented k -Simplex

How do we define orientation in general?



Similar idea to orientation for 2-simplex:

Definition. An oriented k -simplex is an ordered tuple, up to even permutation.

Hence, always* two orientations: *even* or *odd* permutations of vertices. Call even permutations of $(0, \dots, k)$ “**positive**”; otherwise “**negative**.”

Oriented 0-Simplex?

What's the orientation of a single vertex?



Only one permutation of vertices, so only one orientation! (Positive):

(a)

Oriented 3-Simplex

Hard to draw pictures as k gets large!

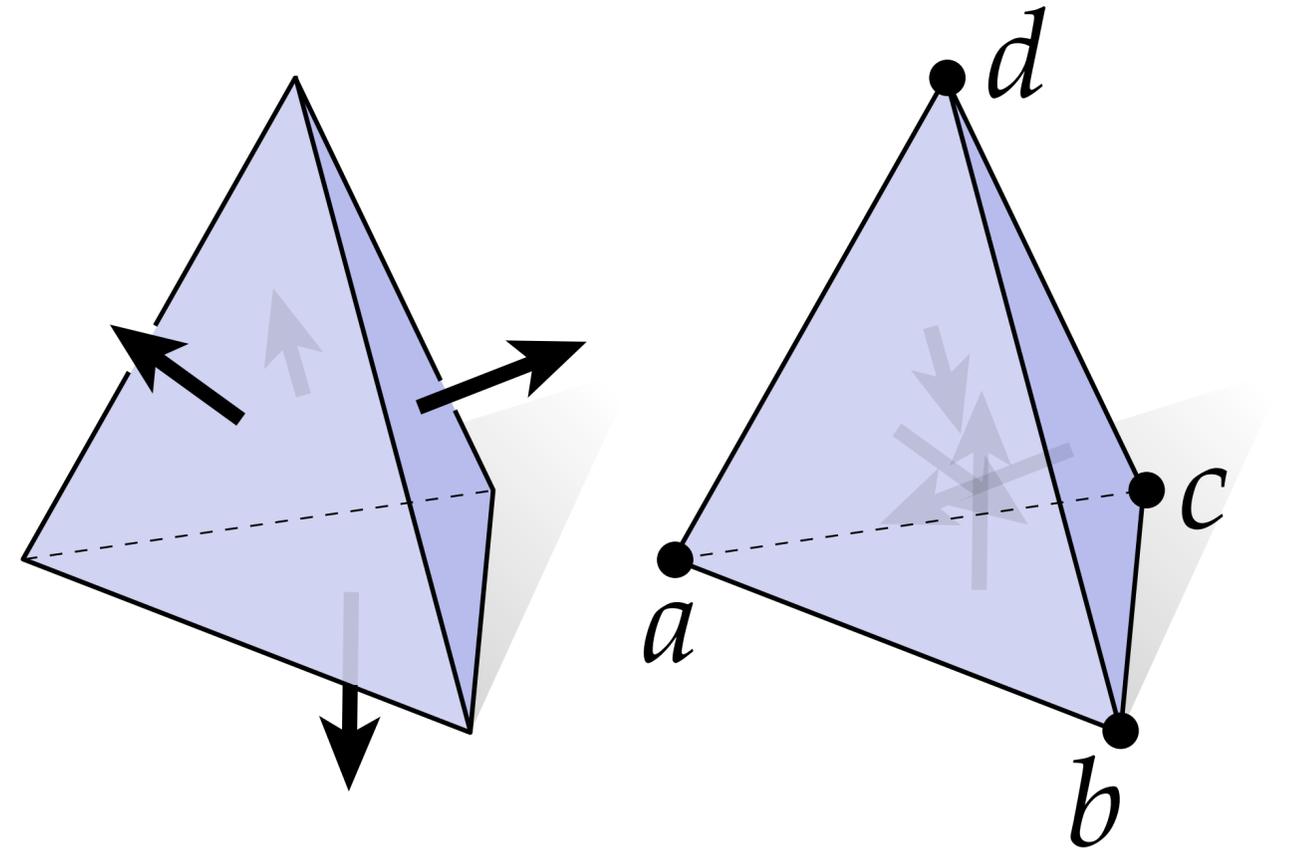
But still easy to apply definition:

even / positive

$(1, 2, 3, 4)$	$(3, 1, 2, 4)$
$(1, 3, 4, 2)$	$(3, 2, 4, 1)$
$(1, 4, 2, 3)$	$(3, 4, 1, 2)$
$(2, 1, 4, 3)$	$(4, 1, 3, 2)$
$(2, 3, 1, 4)$	$(4, 2, 1, 3)$
$(2, 4, 3, 1)$	$(4, 3, 2, 1)$

odd / negative

$(1, 2, 4, 3)$	$(3, 1, 4, 2)$
$(1, 3, 2, 4)$	$(3, 2, 1, 4)$
$(1, 4, 3, 2)$	$(3, 4, 2, 1)$
$(2, 1, 3, 4)$	$(4, 1, 2, 3)$
$(2, 3, 4, 1)$	$(4, 2, 3, 1)$
$(2, 4, 1, 3)$	$(4, 3, 1, 2)$



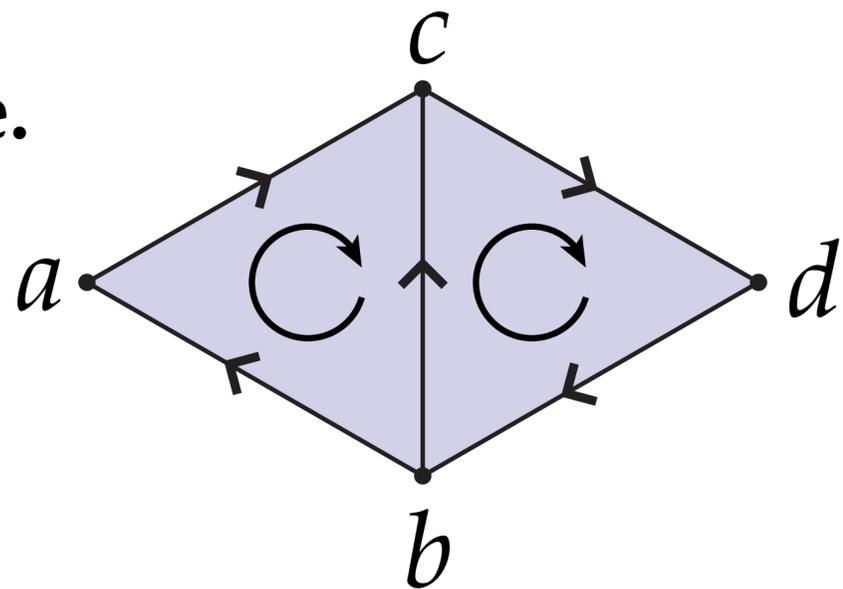
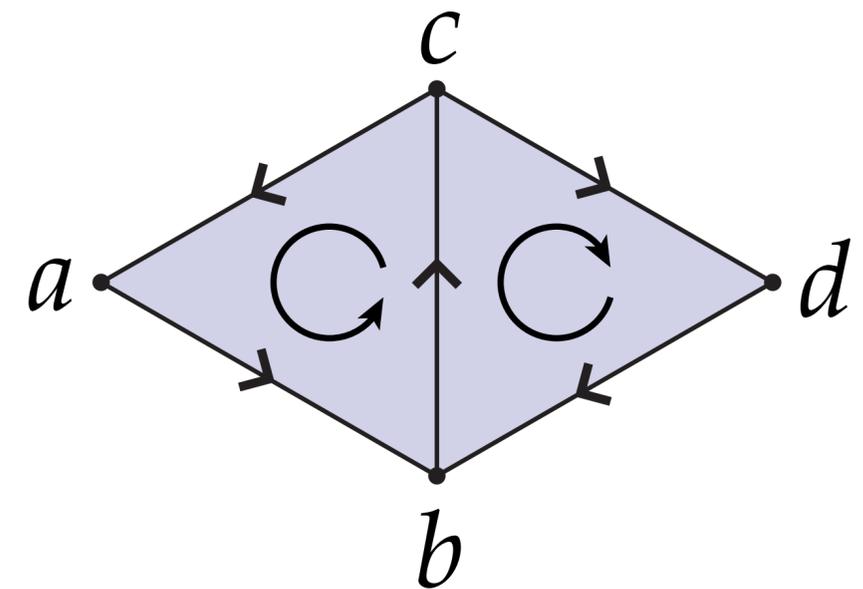
...much easier, of course, to just pick a single representative.

E.g., $+\sigma := (1, 2, 3, 4)$, and $-\sigma := (1, 2, 4, 3)$.

Oriented Simplicial Complex

Definition. An *orientation* of a simplex is an ordering of its vertices up to even permutation; one can specify an oriented simplex via one of its representative ordered tuples. An *oriented simplicial complex* is a simplicial complex where each simplex is given an ordering.

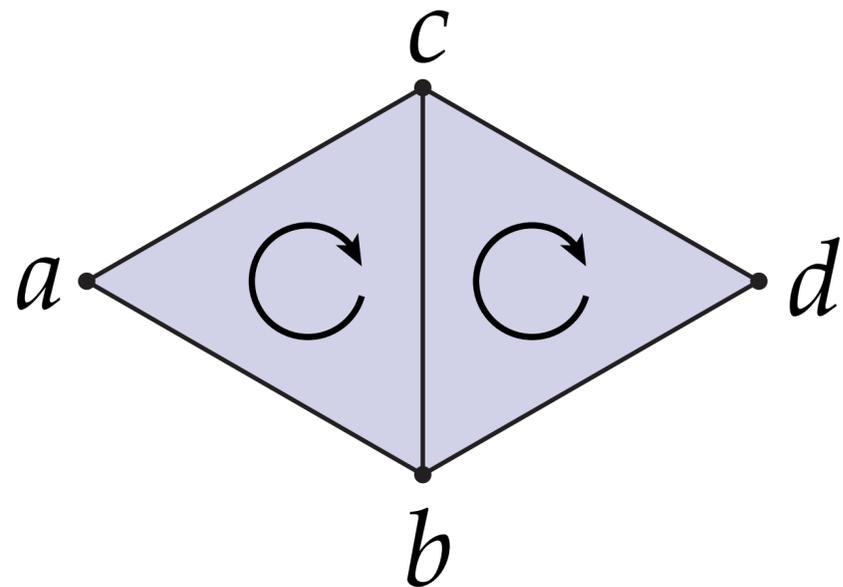
Example.


$$\{\emptyset, (a), (b), (c), (d),$$
$$(a, c), (b, a), (b, c), (c, d), (d, b),$$
$$(a, c, b), (b, c, d)\}$$

$$\{\emptyset, (a), (b), (c), (d),$$
$$(c, a), (a, b), (b, c), (c, d), (d, b),$$
$$(a, b, c), (b, c, d)\}$$

Relative Orientation

Definition. Two distinct oriented simplices have the same *relative orientation* if the two (maximal) faces in their intersection have **opposite** orientation.

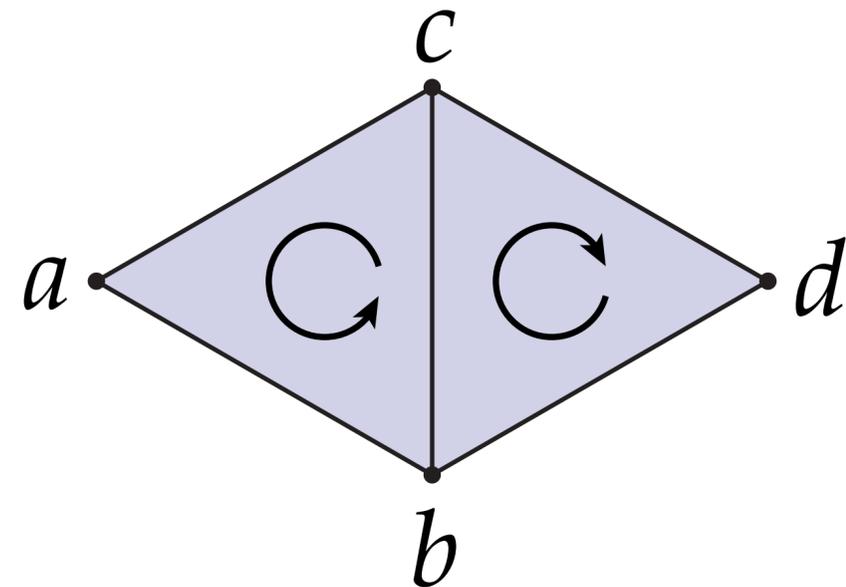
Example: Consider two triangles that intersect along an edge:



same relative orientation

$$\begin{aligned}(a, c, b) &\Rightarrow (c, b) \\ (b, c, d) &\Rightarrow (b, c)\end{aligned}$$

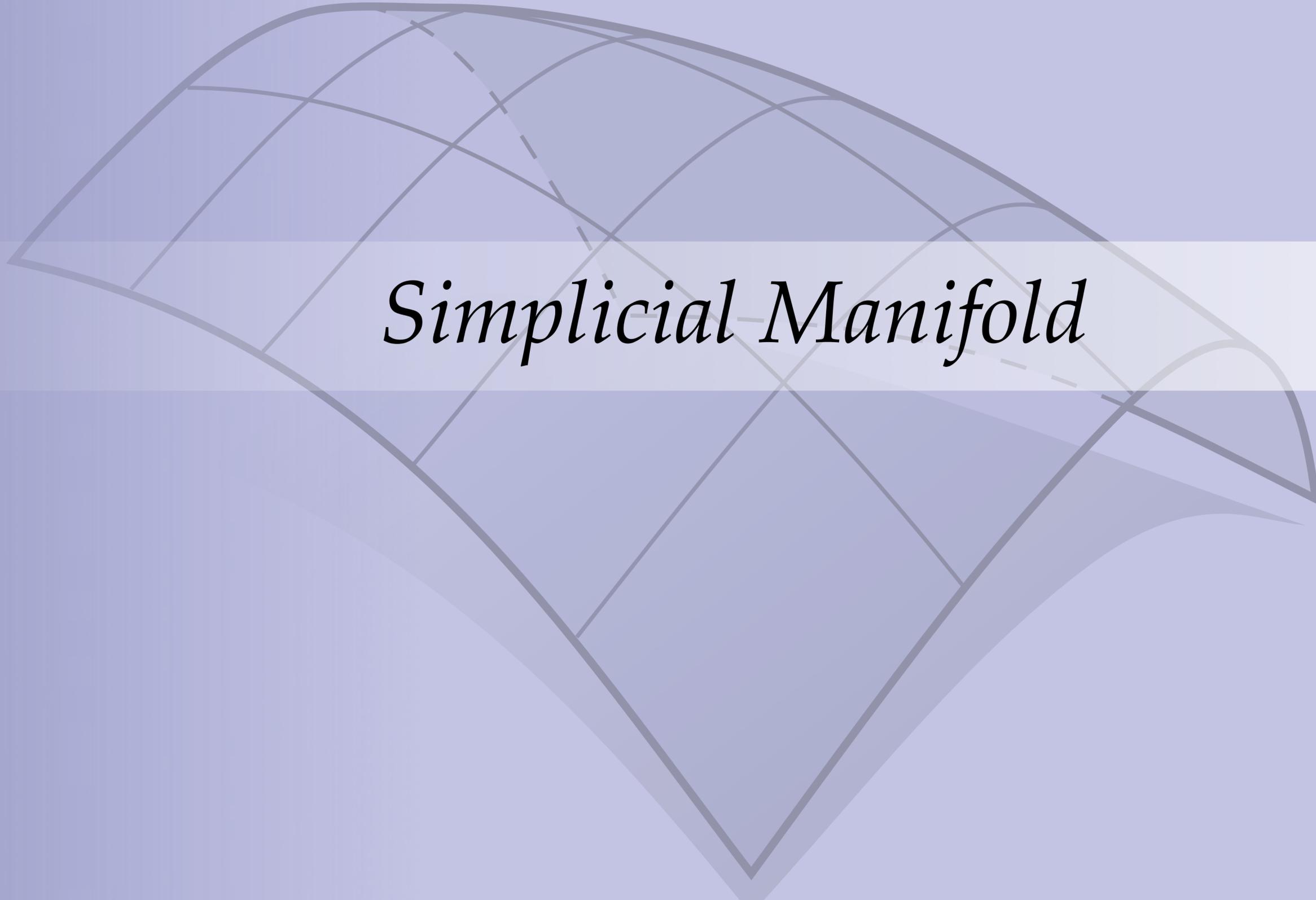
$$(c, b) = -(b, c)$$



different relative orientation

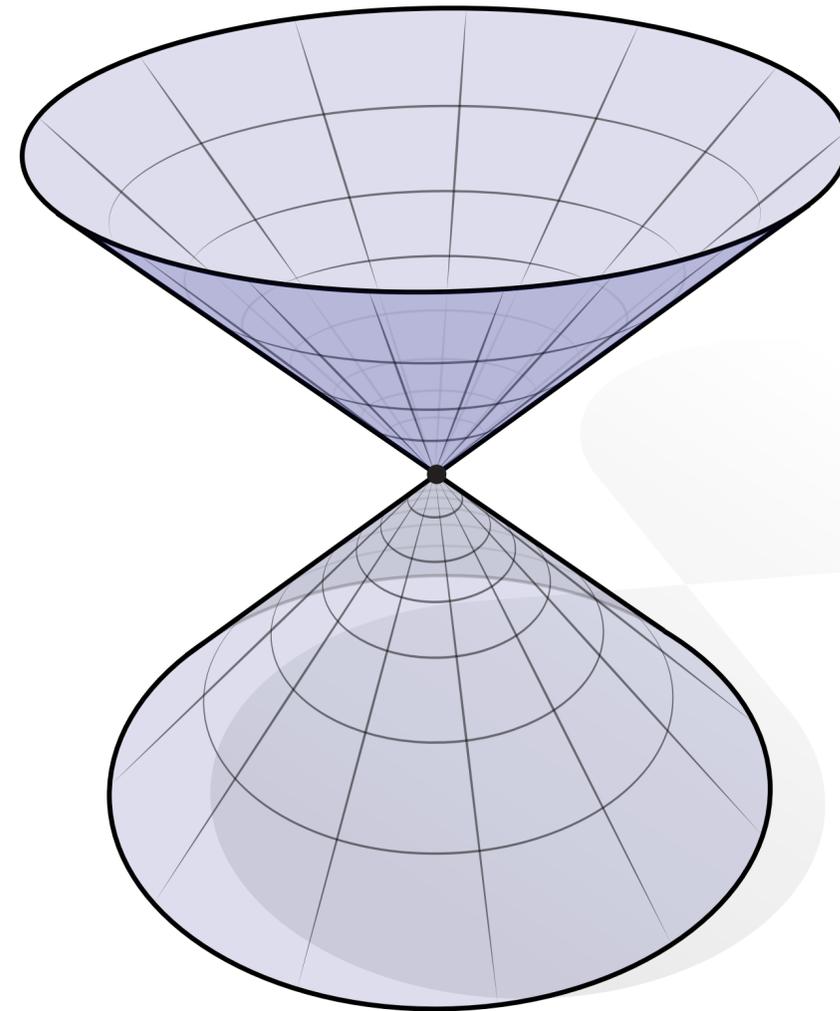
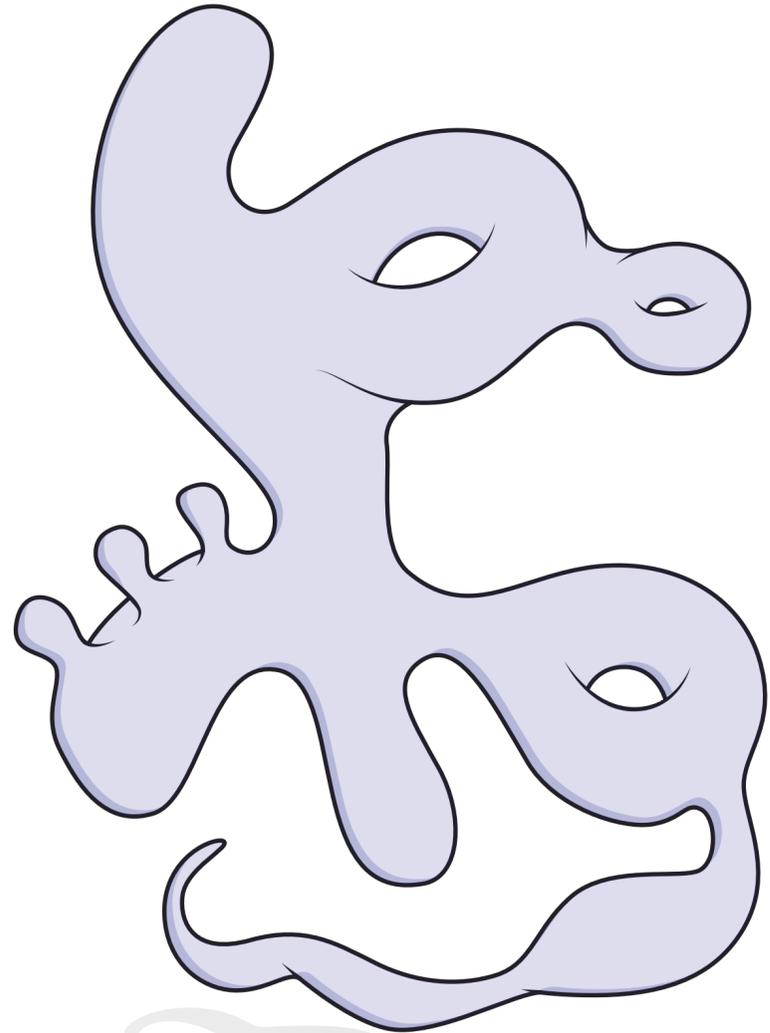
$$\begin{aligned}(a, b, c) &\Rightarrow (b, c) \\ (b, c, d) &\Rightarrow (b, c)\end{aligned}$$

$$(b, c) = +(b, c)$$



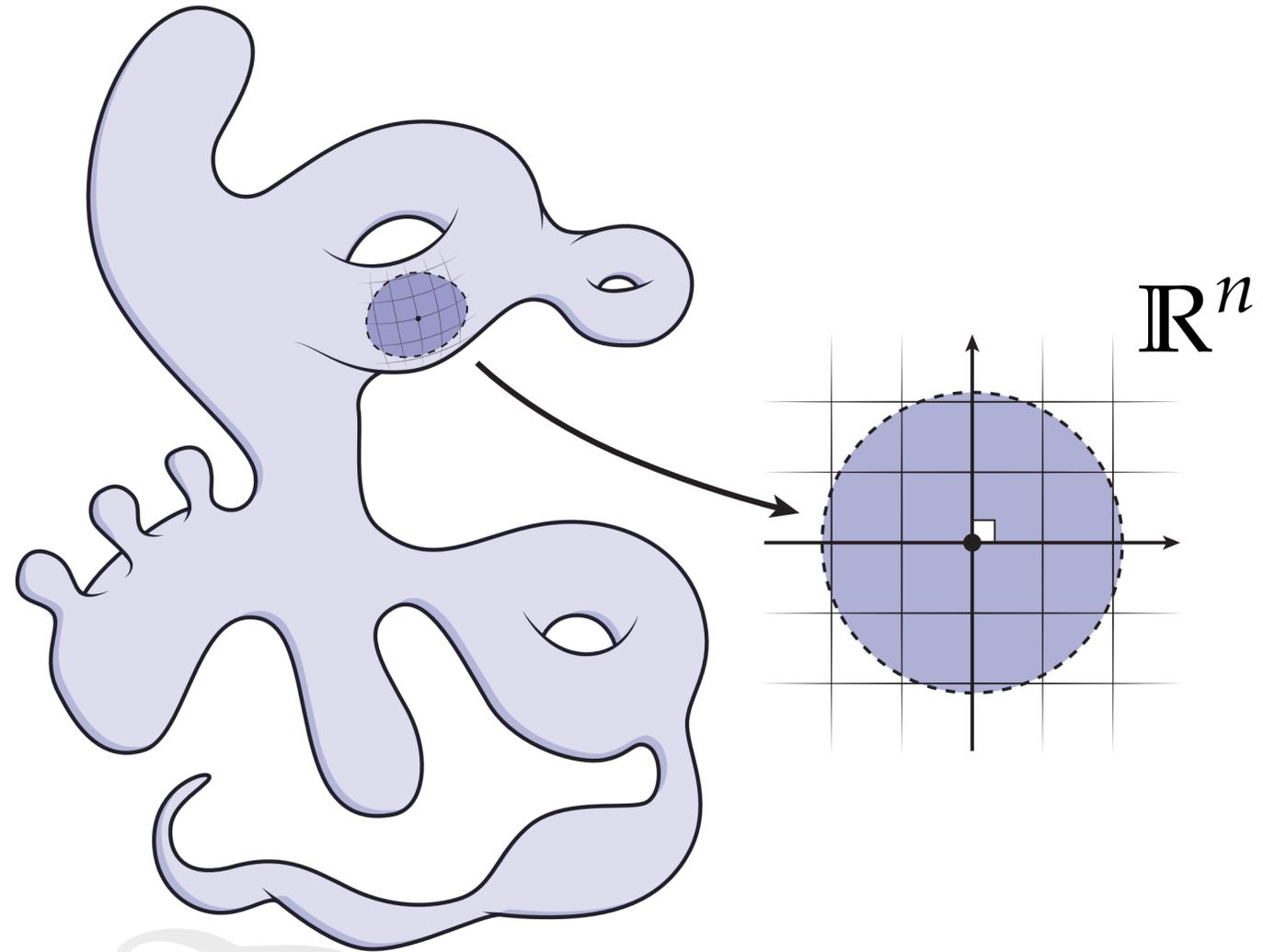
Simplicial Manifold

Manifold—First Glimpse

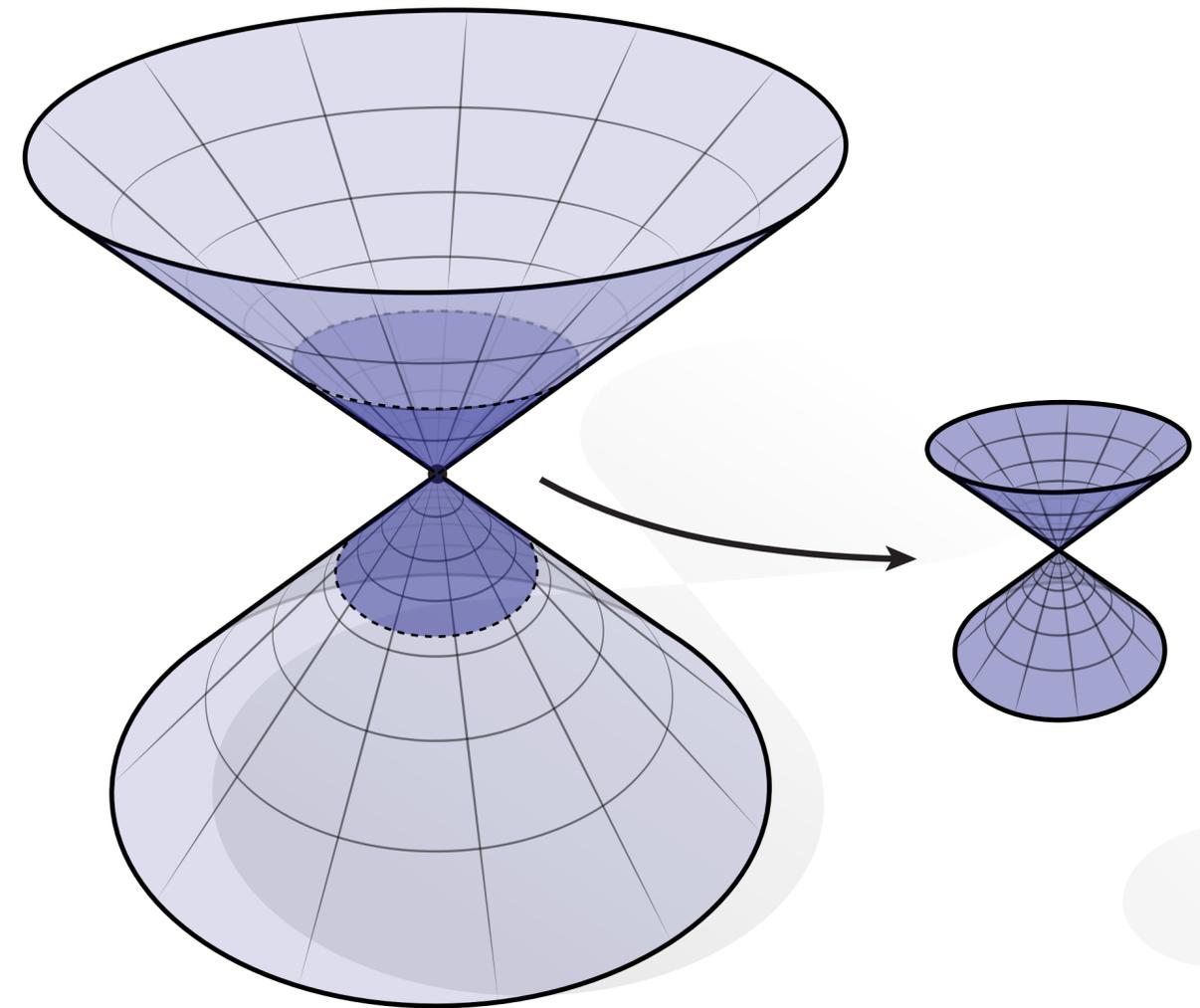


Manifold—First Glimpse

manifold



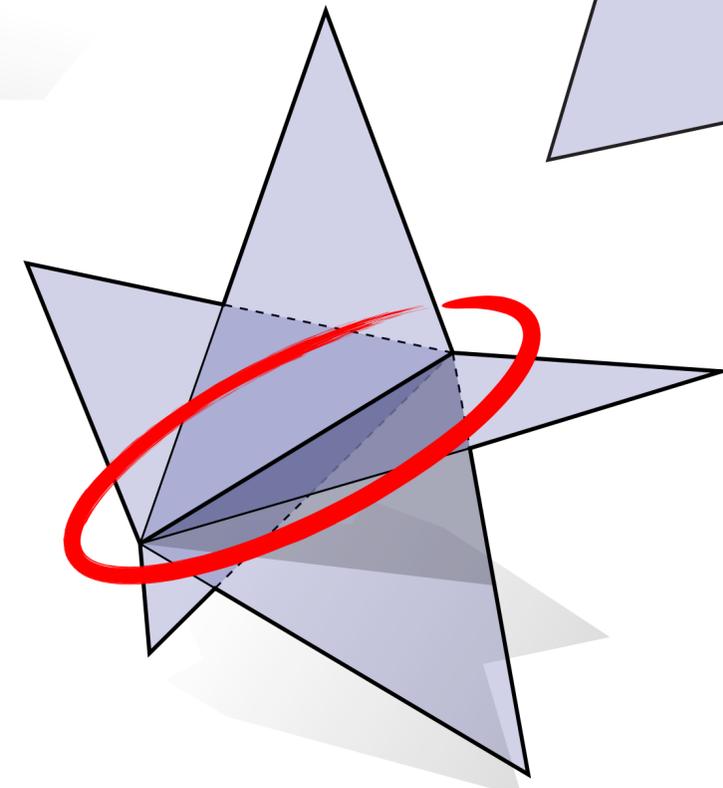
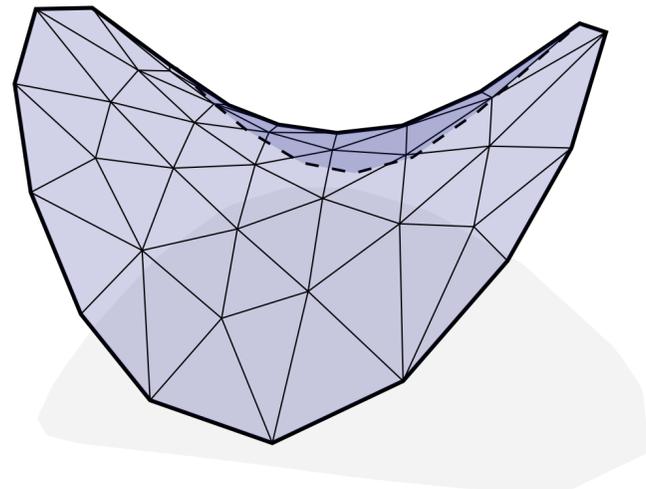
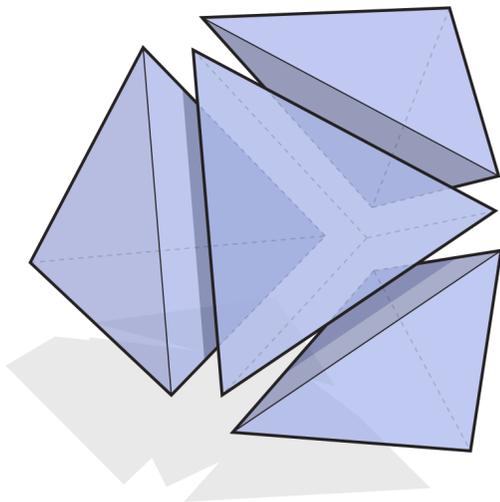
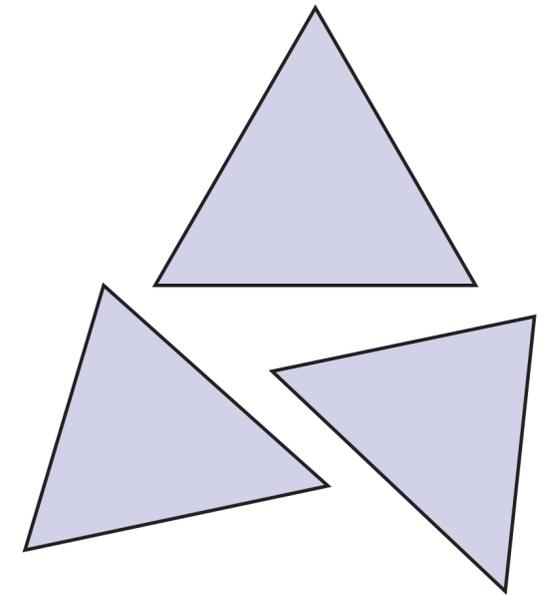
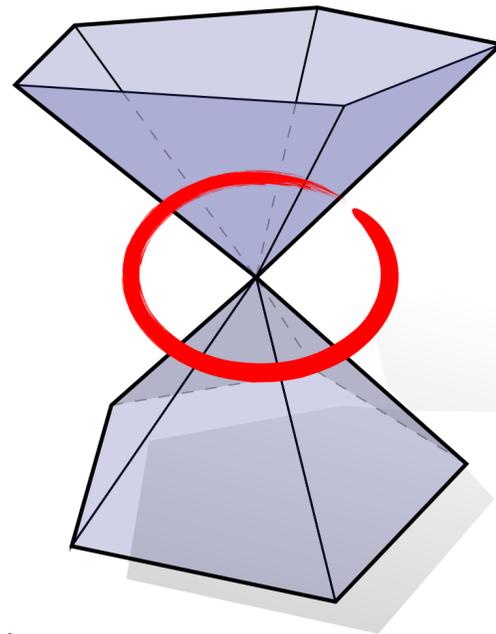
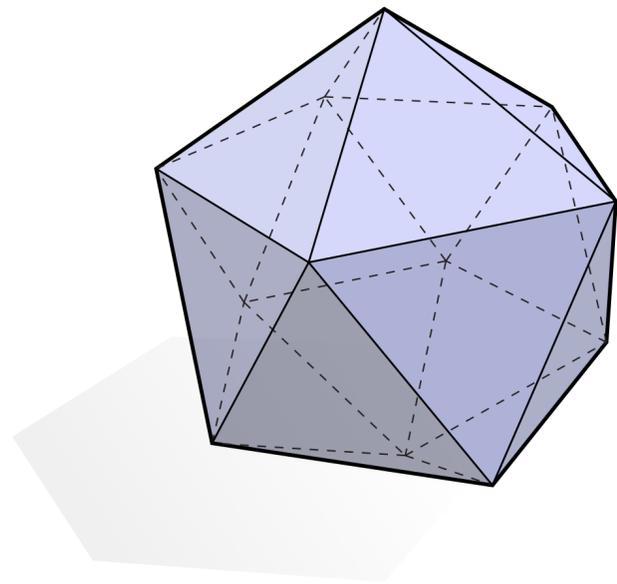
nonmanifold



Key idea: “looks like \mathbb{R}^n up close”

Simplicial Manifold—Visualized

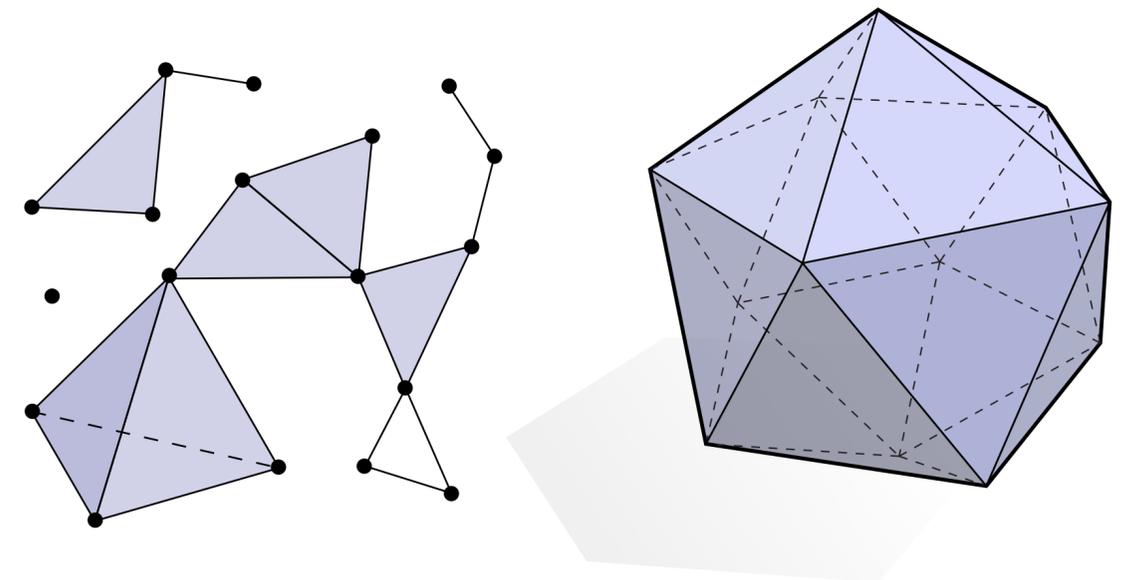
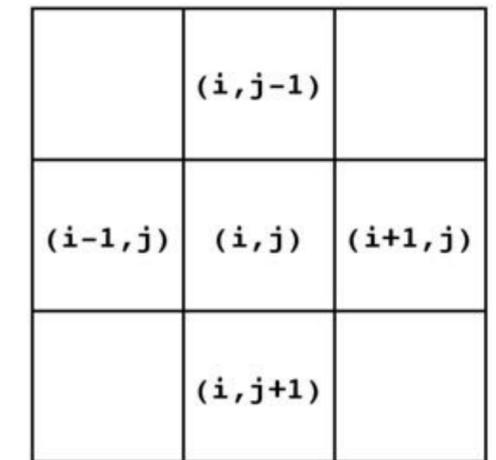
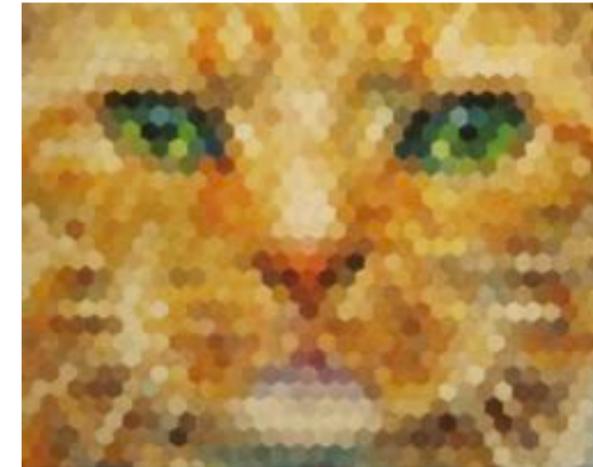
Which of these simplicial complexes look “*manifold*?”



(E.g., where might it be hard to put a little xy -coordinate system?)

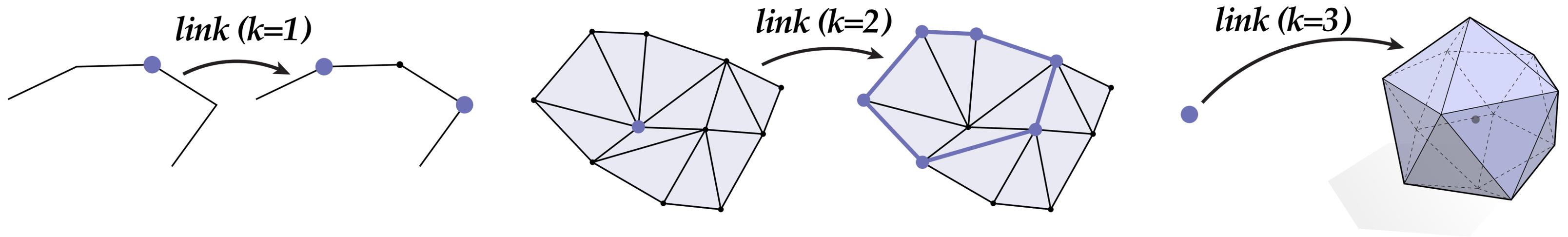
Manifold Meshes — Motivation

- Why might it be preferable to work with a *manifold* mesh?
- Analogy: 2D images
 - Lots of ways you *could* arrange pixels...
 - A regular grid does everything you need
 - And very simple (always have 4 neighbors)
- Same deal with manifold meshes
 - *Could* allow arbitrary meshes...
 - A manifold mesh is often good enough
 - And very simple (*e.g.*, regular neighborhoods)
 - *E.g.*, leads to nice **data structures** (later)



Simplicial Manifold—Definition

Definition. A simplicial k -complex is *manifold* if the **link** of every vertex looks like* an $(k - 1)$ -dimensional sphere.



Aside: How hard is it to check if a given simplicial complex is manifold?

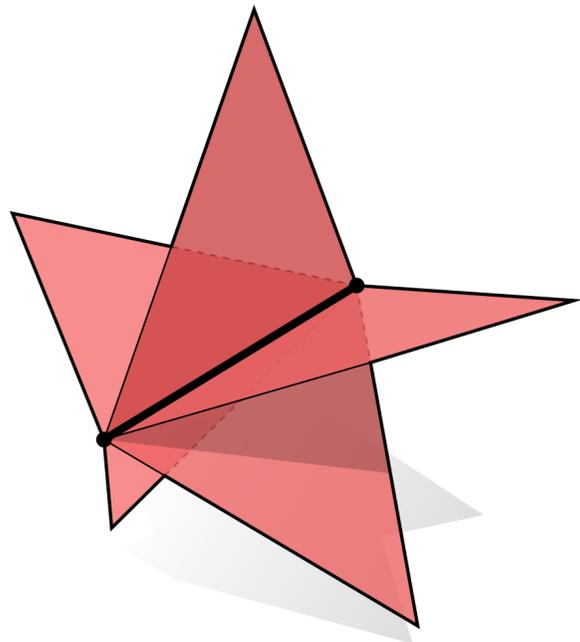
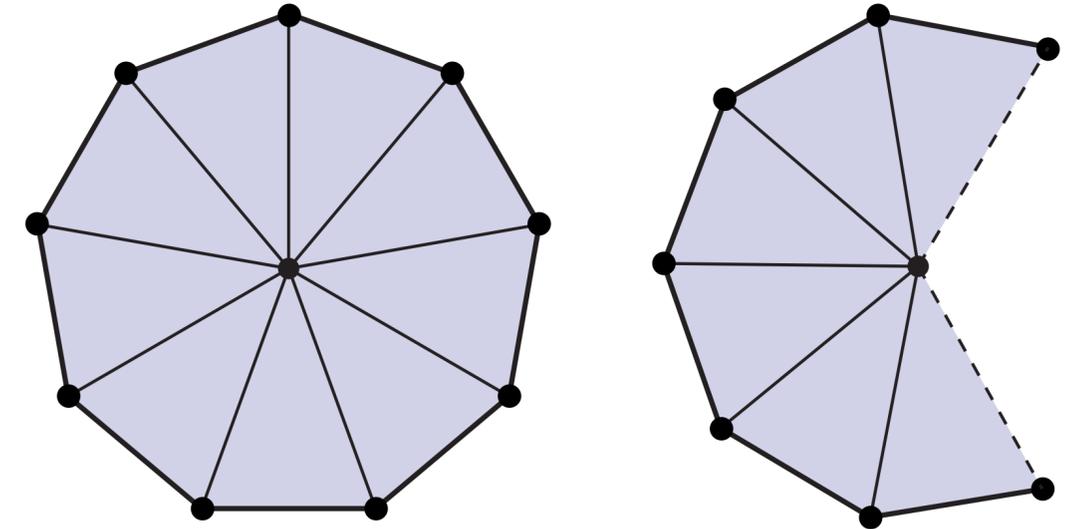
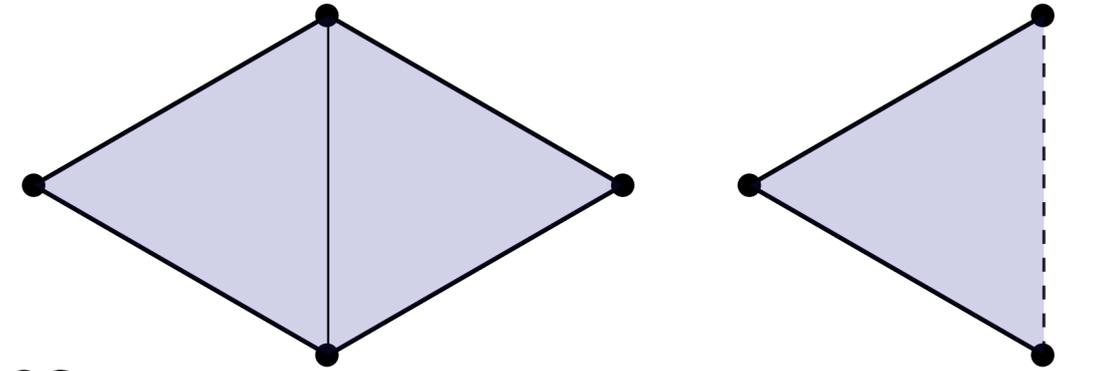
- ($k=1$) *trivial*—is it a loop?
- ($k=2$) *trivial*—is each link a loop?
- ($k=3$) is each link a 2-sphere? Just check if $V-E+F = 2$ (Euler's formula)
- ($k=4$) is each link a 3-sphere? ...Well, it's known to be in NP! [S. Schleimer 2004]

*I.e., is *homeomorphic* to.

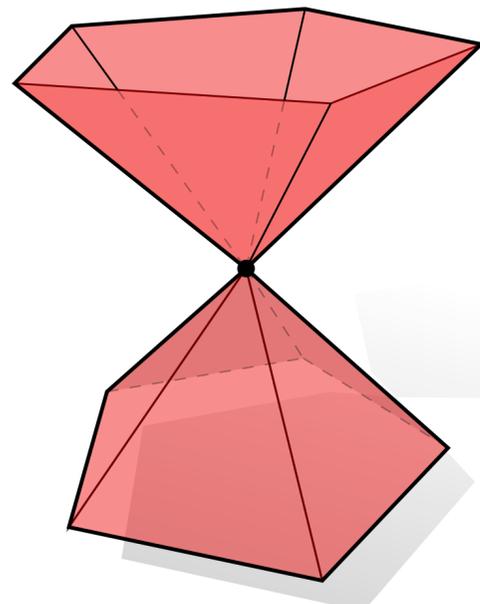
Manifold Triangle Mesh

Key example: For a triangle mesh ($k=2$):

- every edge is contained in exactly two triangles
 - ...or just one along the boundary
- every vertex is contained in a single “loop” of triangles
 - ...or a single “fan” along the boundary

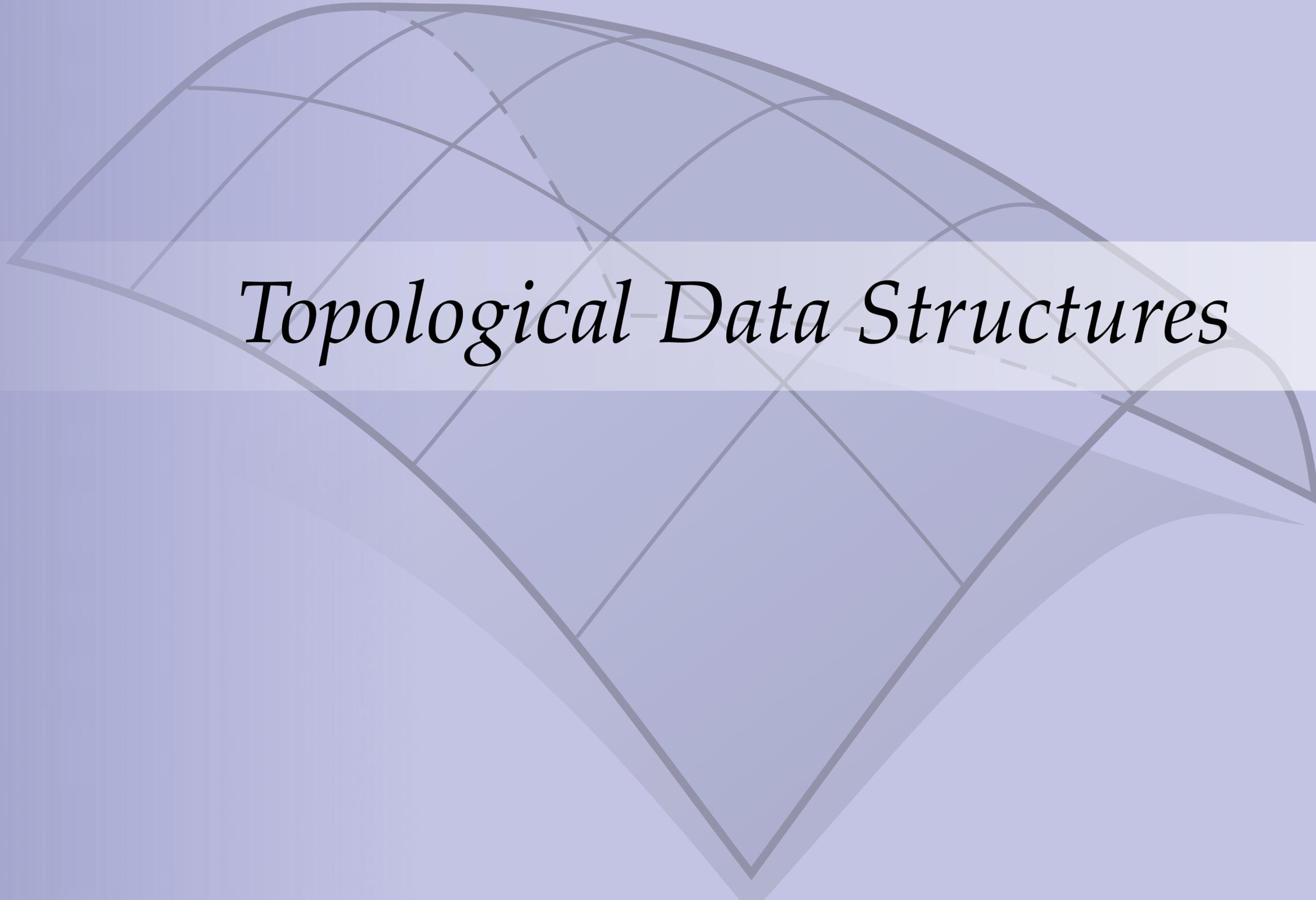


nonmanifold edge



nonmanifold vertex

Why? One reason: data structures...



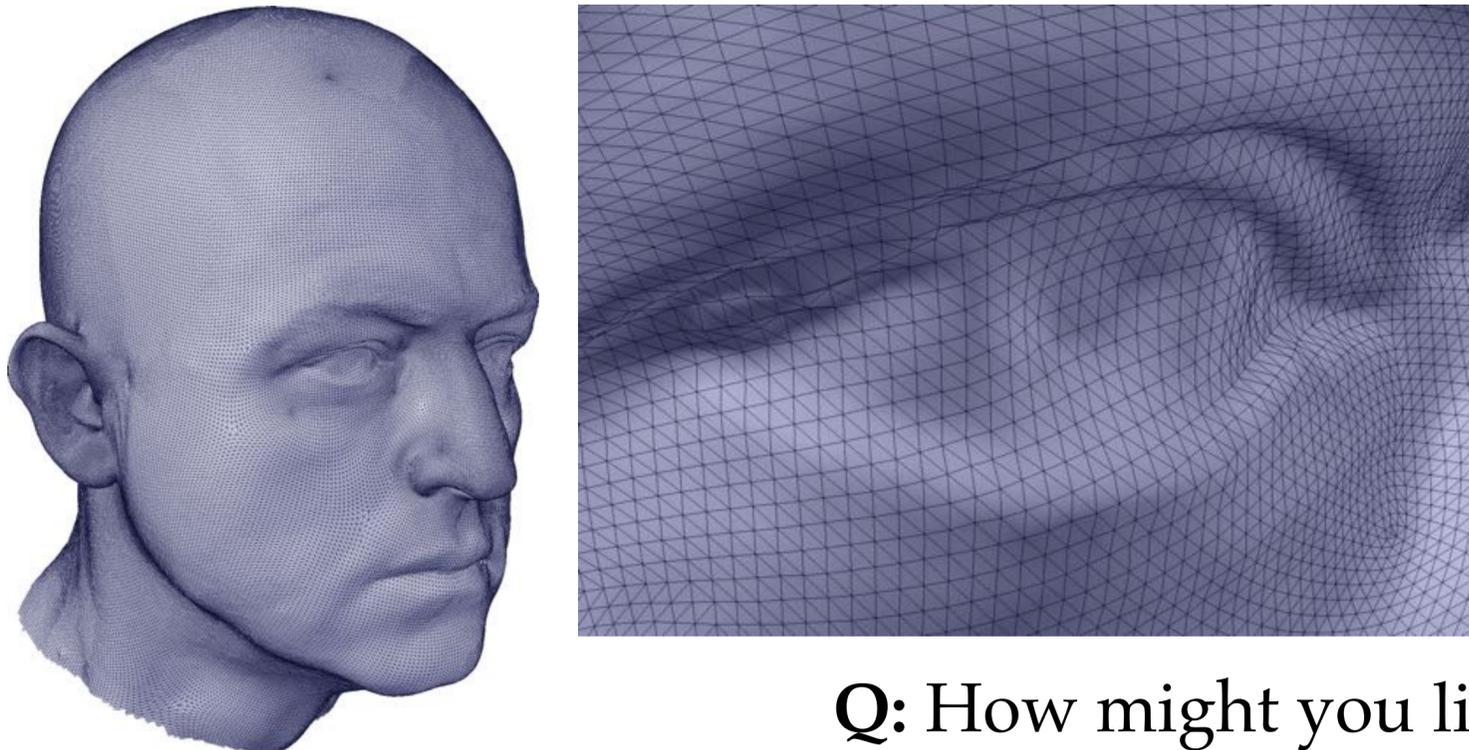
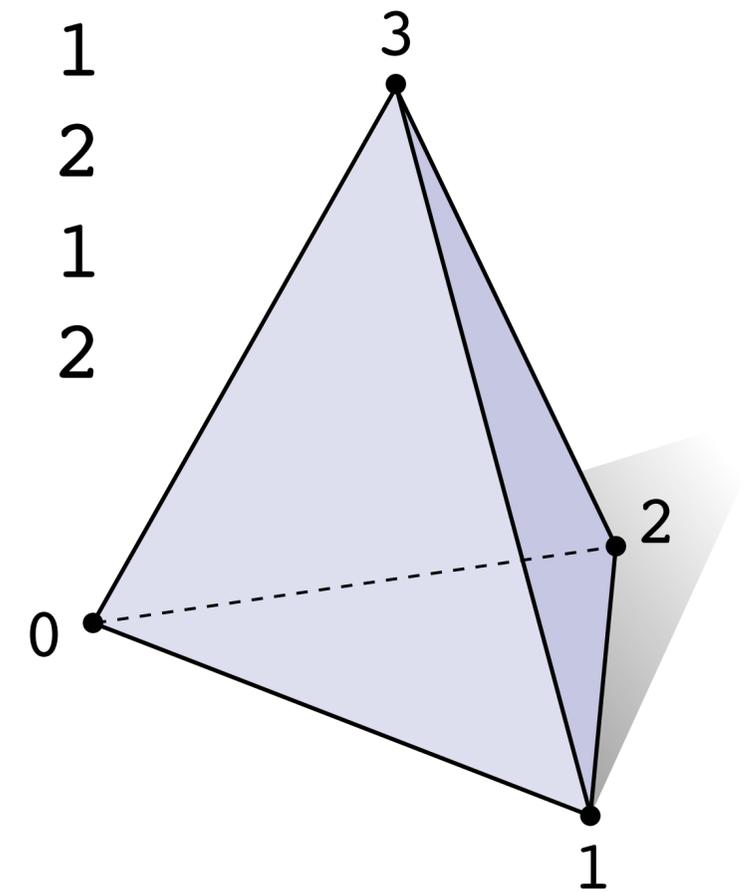
Topological Data Structures

Topological Data Structures — Adjacency List

- Store only top-dimensional simplices
- Implicitly includes all facets
- Pros: simple, small storage cost
- Cons: hard to access neighbors

Example.

0	2	1
0	3	2
3	0	1
3	1	2



Q: How might you list all edges touching a given vertex? *What's the cost?*

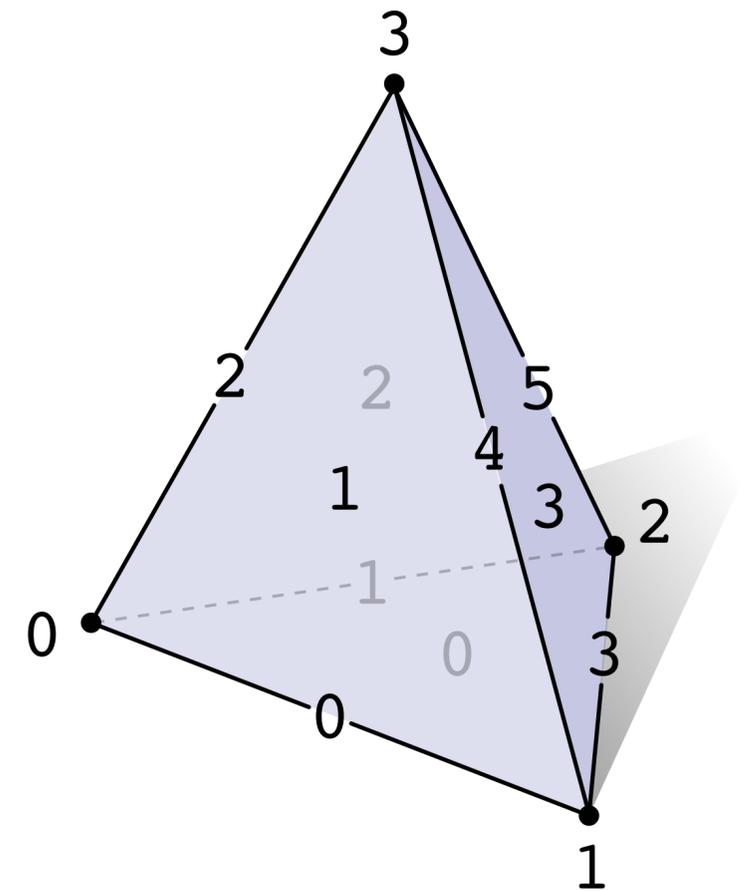
Topological Data Structures—Incidence Matrix

Definition. Let K be a simplicial complex, let n_k denote the number of k -simplices in K , and suppose that for each k we give the k -simplices a canonical ordering so that they can be specified via indices $1, \dots, n_k$. The k th *incidence matrix* is then a $n_{k+1} \times n_k$ matrix E^k with entries $E_{ij}^k = 1$ if the j th k -simplex is contained in the i th $(k+1)$ -simplex, and $E_{ij}^k = 0$ otherwise.

Example.

$$E^0 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$E^1 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

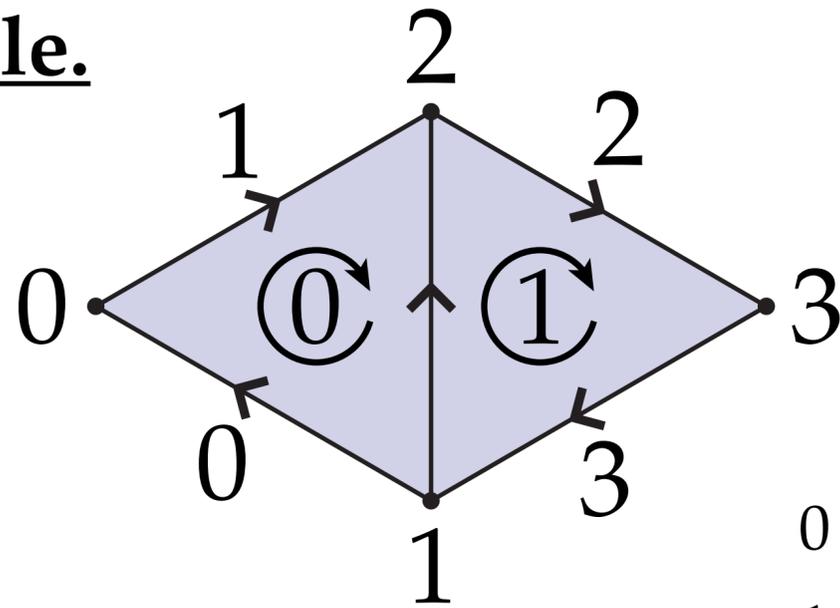


Q: Now what's the cost of finding edges incident on a given vertex?

Data Structures—Signed Incidence Matrix

A *signed incidence matrix* is an incidence matrix where the sign of each nonzero entry is determined by the relative orientation of the two simplices corresponding to that row / column.

Example.



$$E^0 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

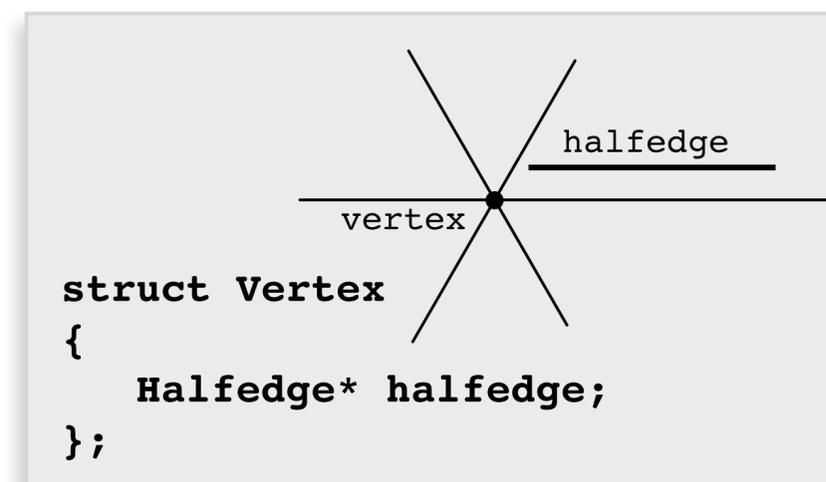
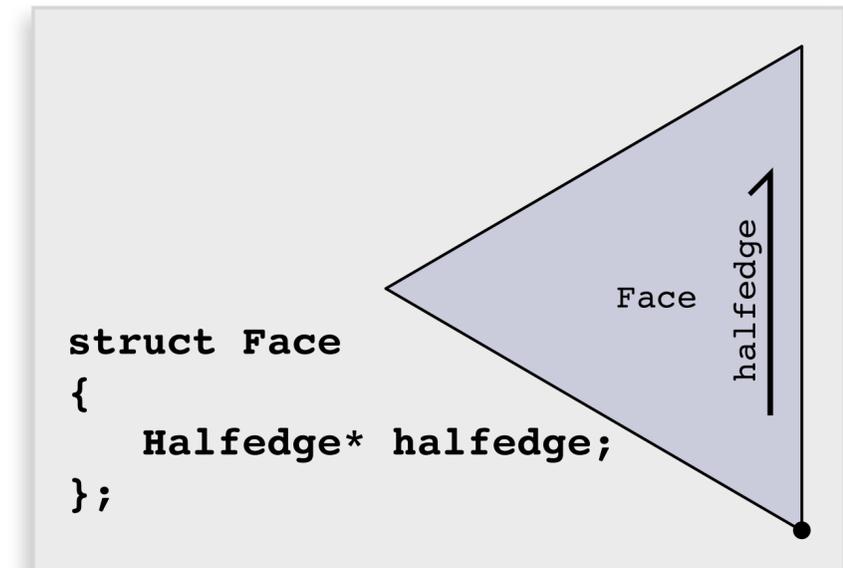
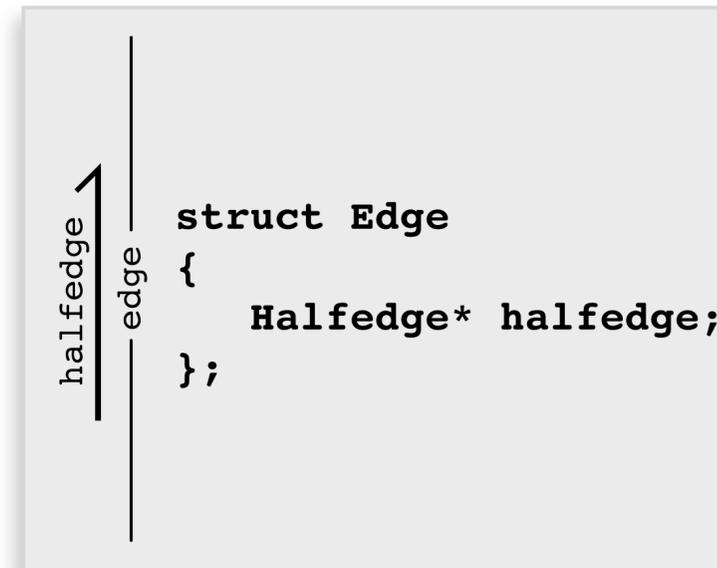
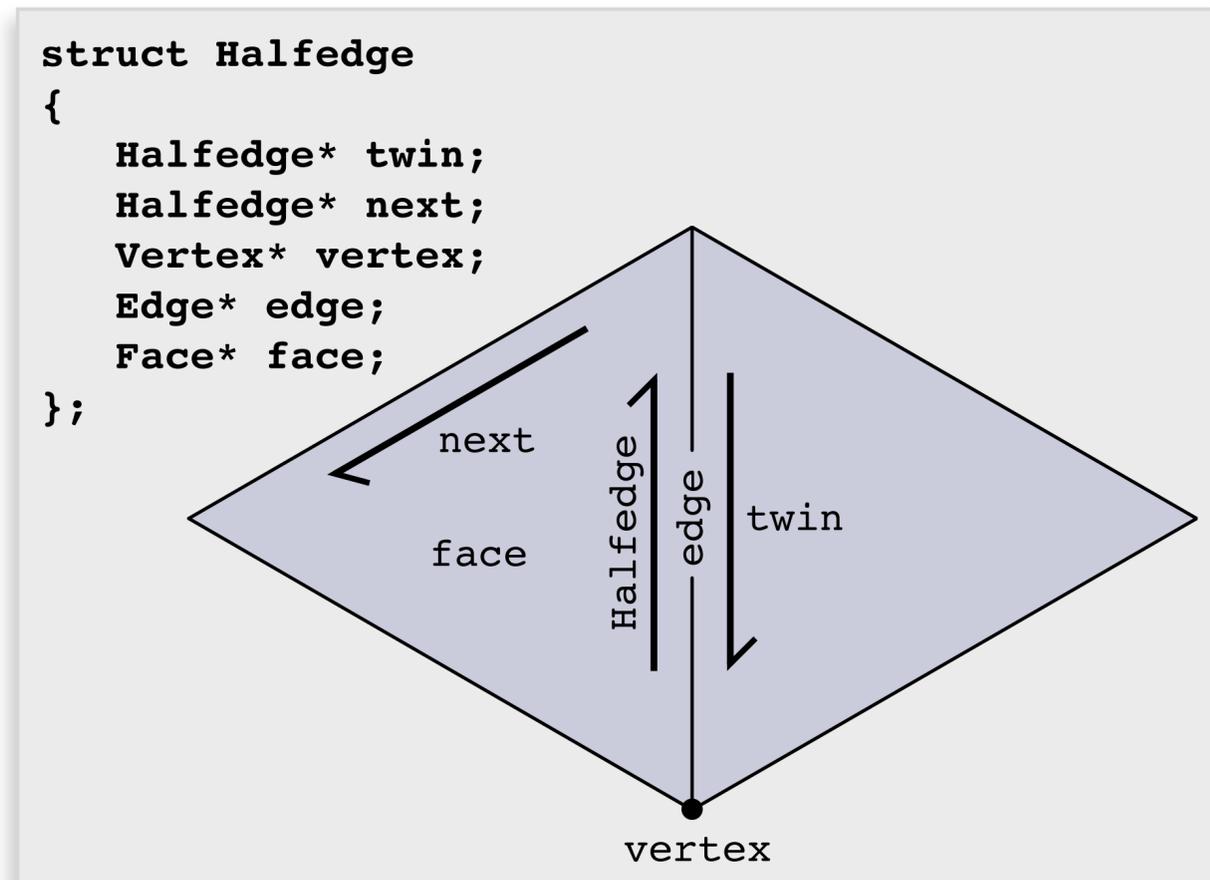
$$E^1 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

(Closely related to *discrete exterior calculus*.)

Topological Data Structures—Half Edge Mesh

Basic idea: each edge gets split into two *half edges*.

- Half edges act as “glue” between mesh elements.
- All other elements know only about a single half edge.



(You'll use this one in your assignments!)

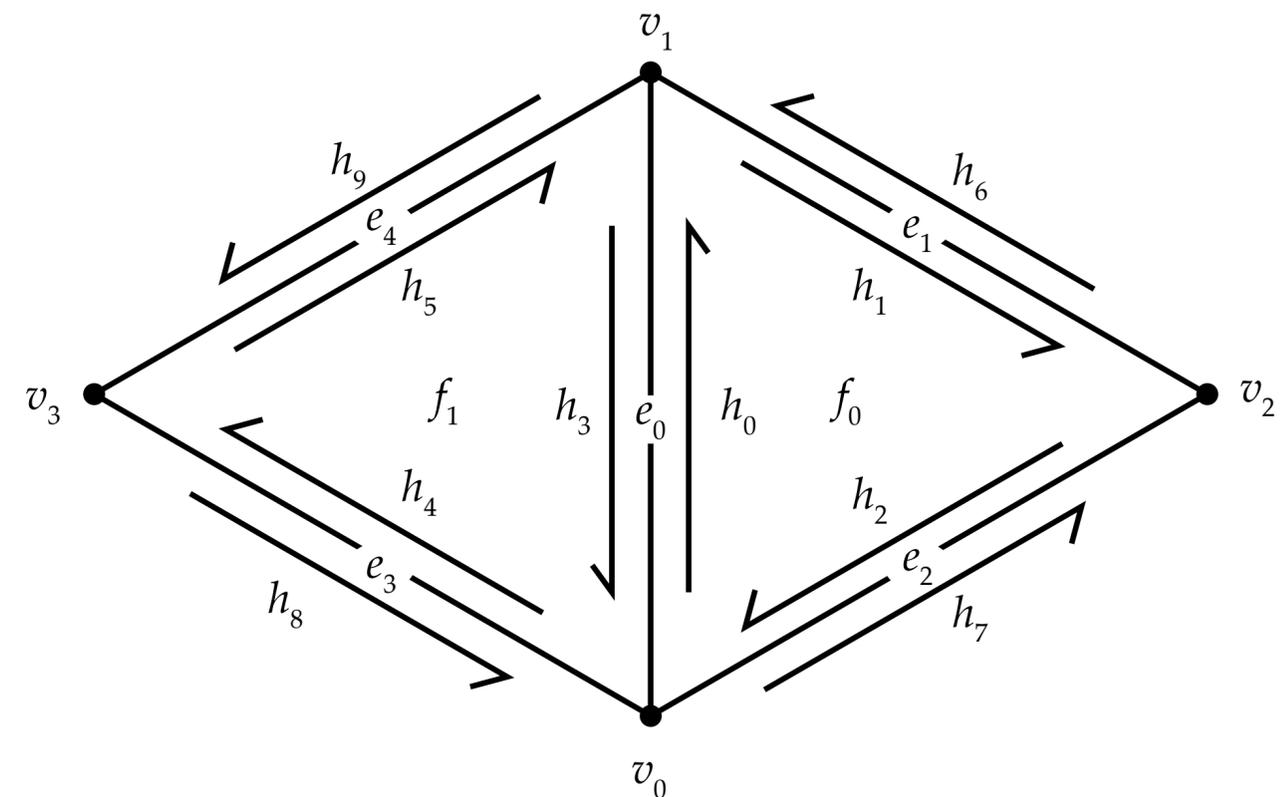
Half Edge — Algebraic Definition

Definition. Let H be any set with an even number of elements, let $\rho : H \rightarrow H$ be any permutation of H , and let $\eta : H \rightarrow H$ be an involution without any fixed points, i.e., $\eta \circ \eta = \text{id}$ and $\eta(h) \neq h$ for any $h \in H$. Then (H, ρ, η) is a *half edge mesh*, the elements of H are called *half edges*, the orbits of η are *edges*, the orbits of ρ are *faces*, and the orbits of $\eta \circ \rho$ are *vertices*.

Fact. Every half edge mesh describes a compact oriented topological surface (without boundary).

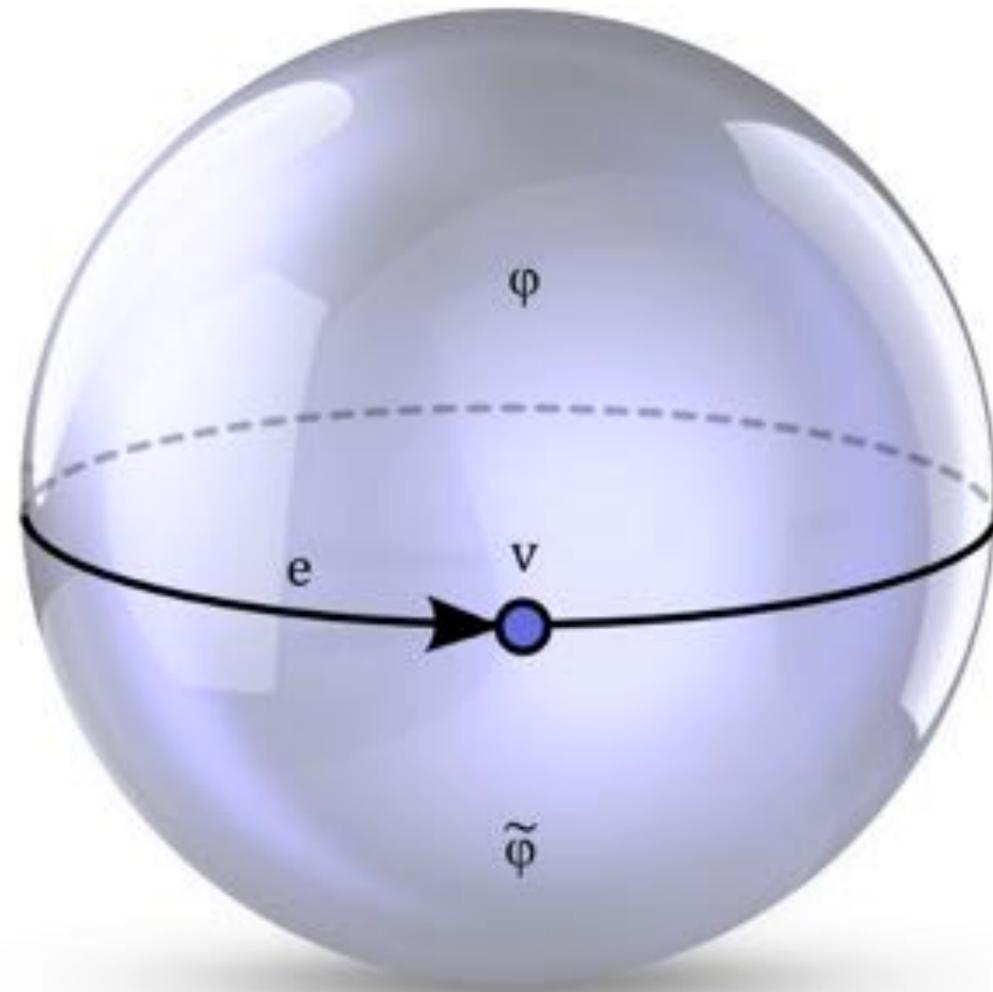
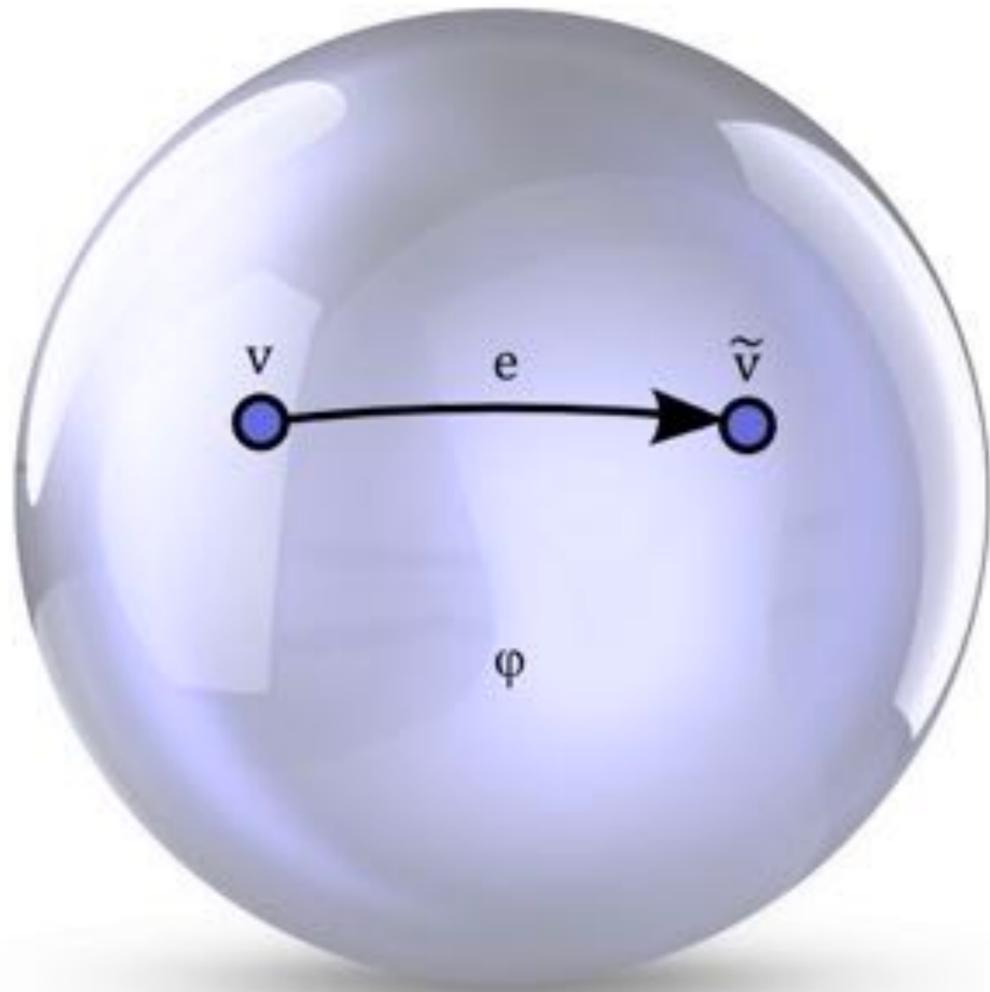
$$(h_0, \dots, h_9) \xrightarrow[\text{"next"}]{\rho} (h_1, h_2, h_0, h_4, h_5, h_3, h_9, h_6, h_7, h_8)$$

$$(h_0, \dots, h_9) \xrightarrow[\text{"twin"}]{\eta} (h_3, h_6, h_7, h_0, h_8, h_9, h_1, h_2, h_4, h_5)$$

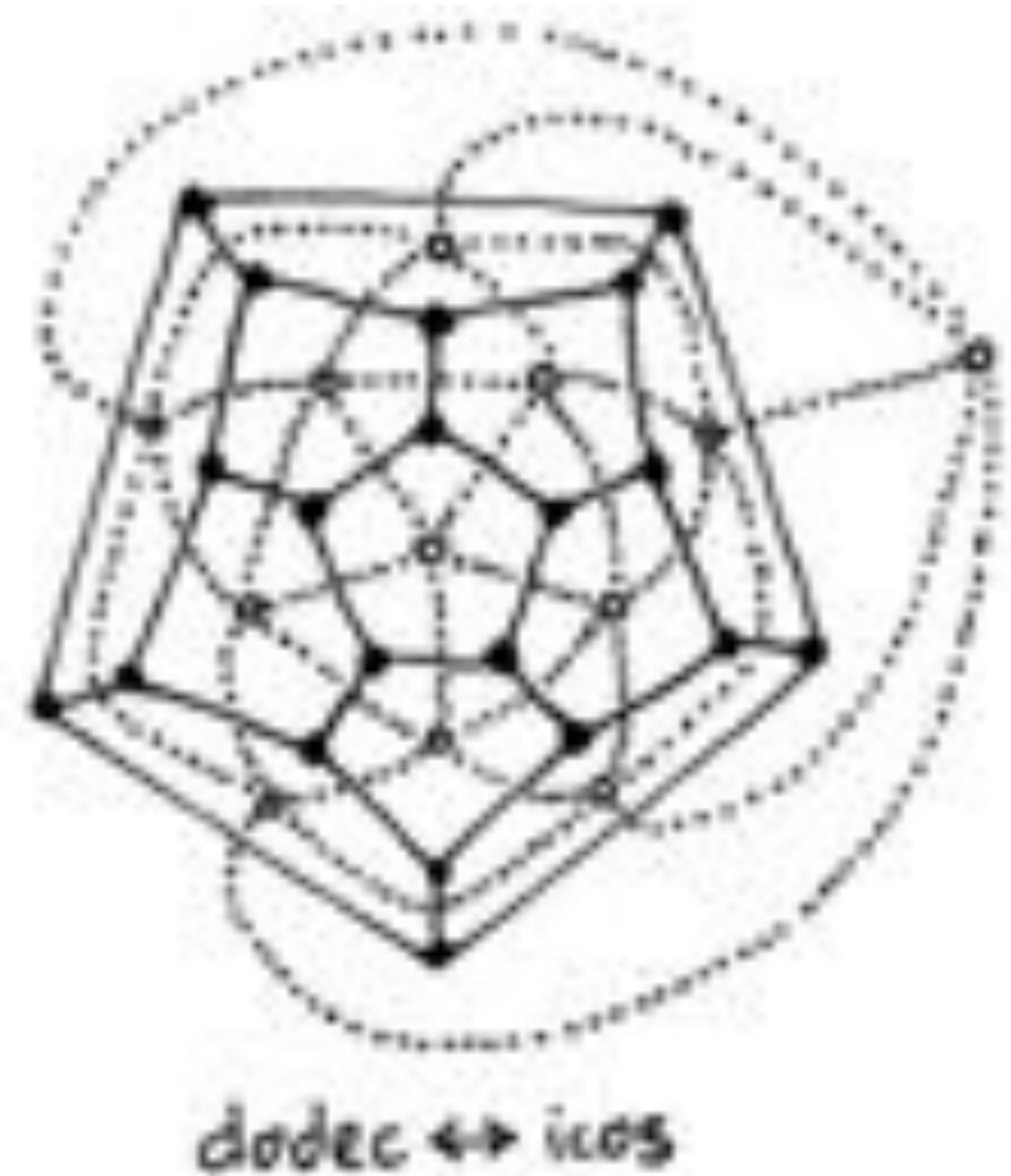
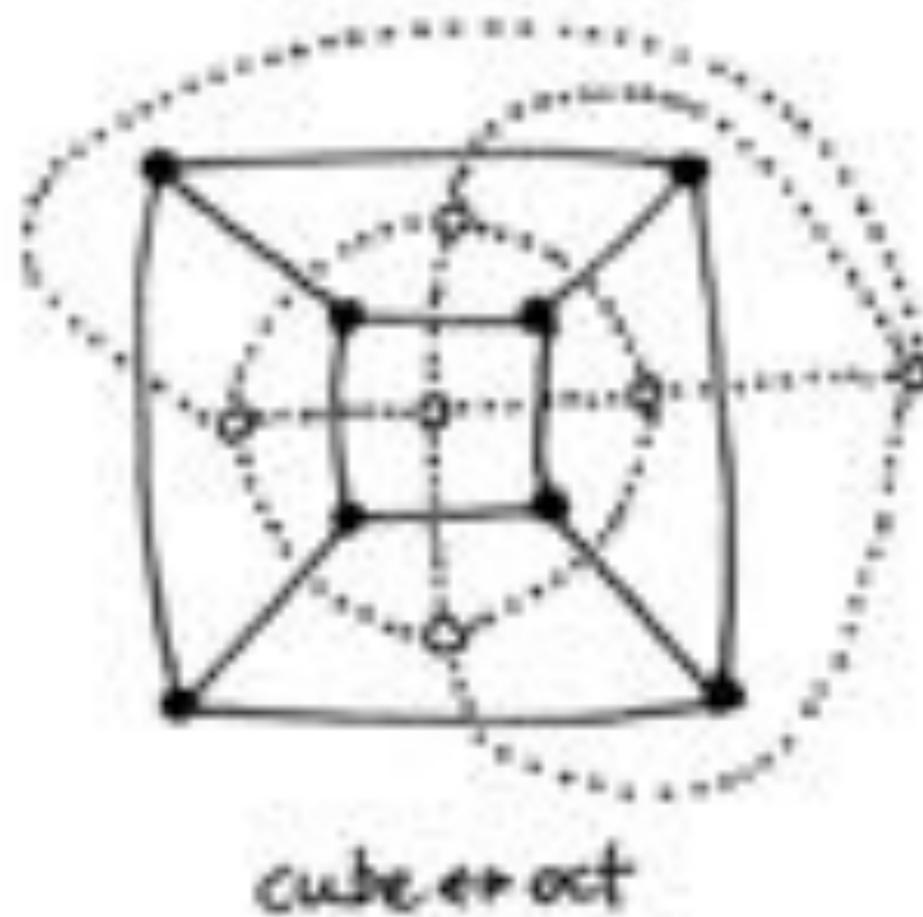


Half Edge — Example

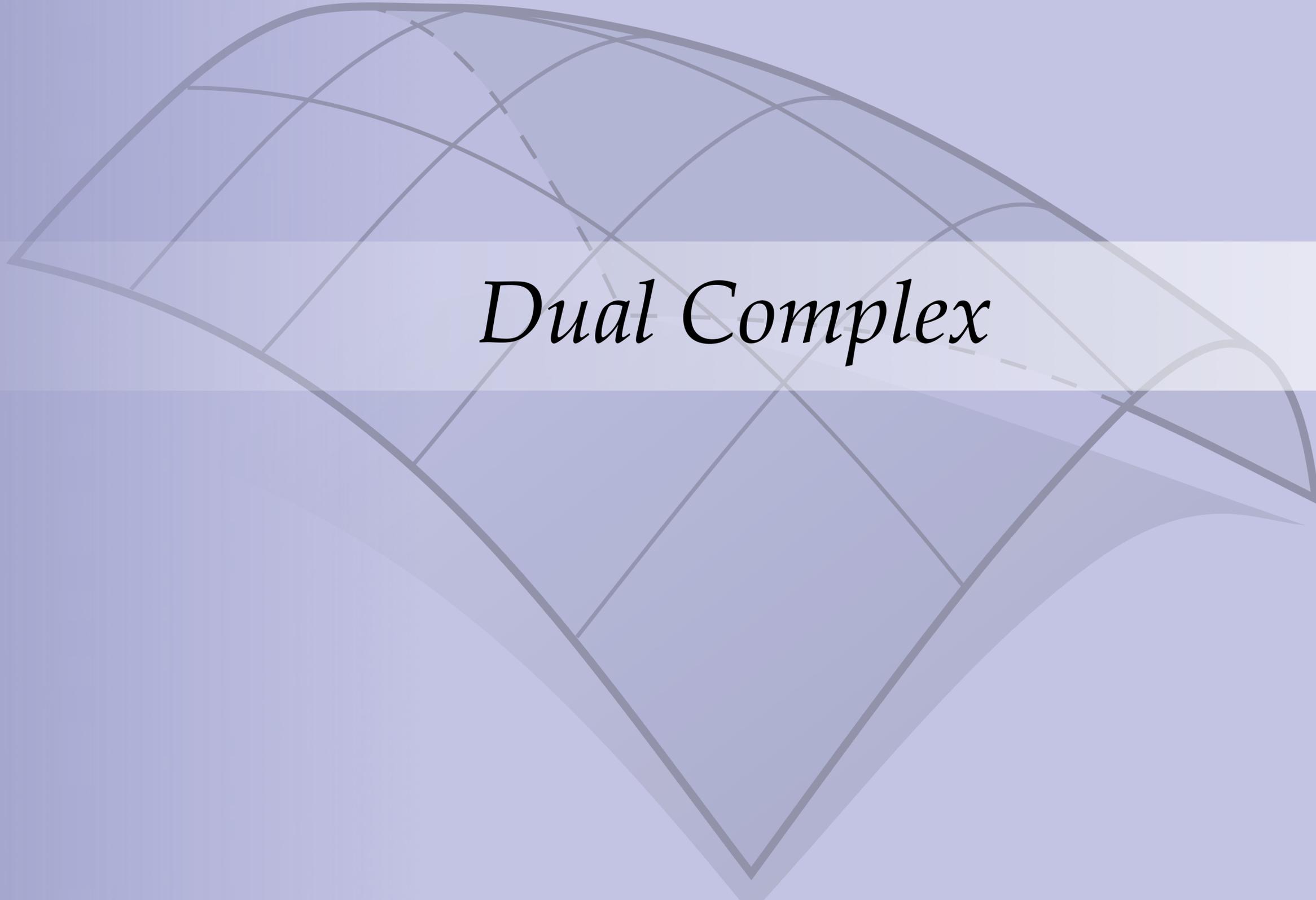
Smallest examples (two half edges):



Data Structures—Quad Edge

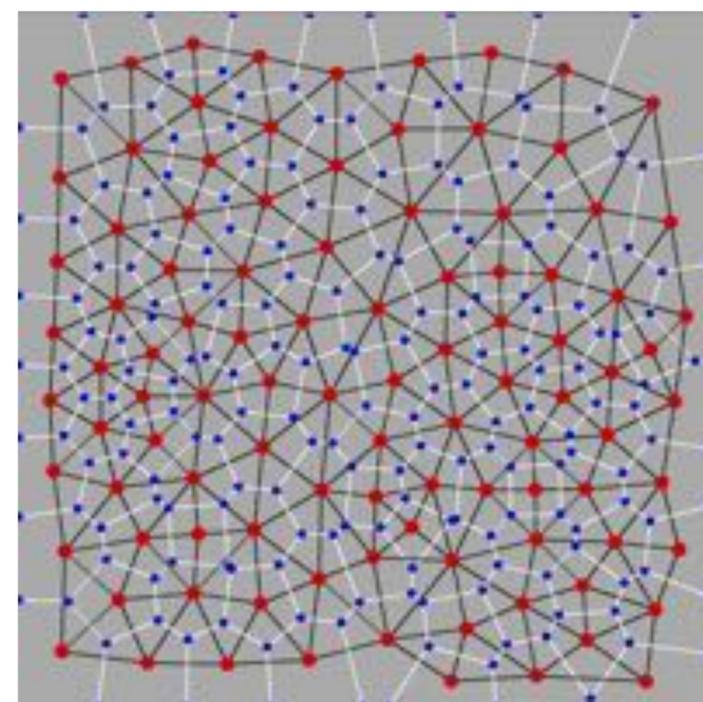
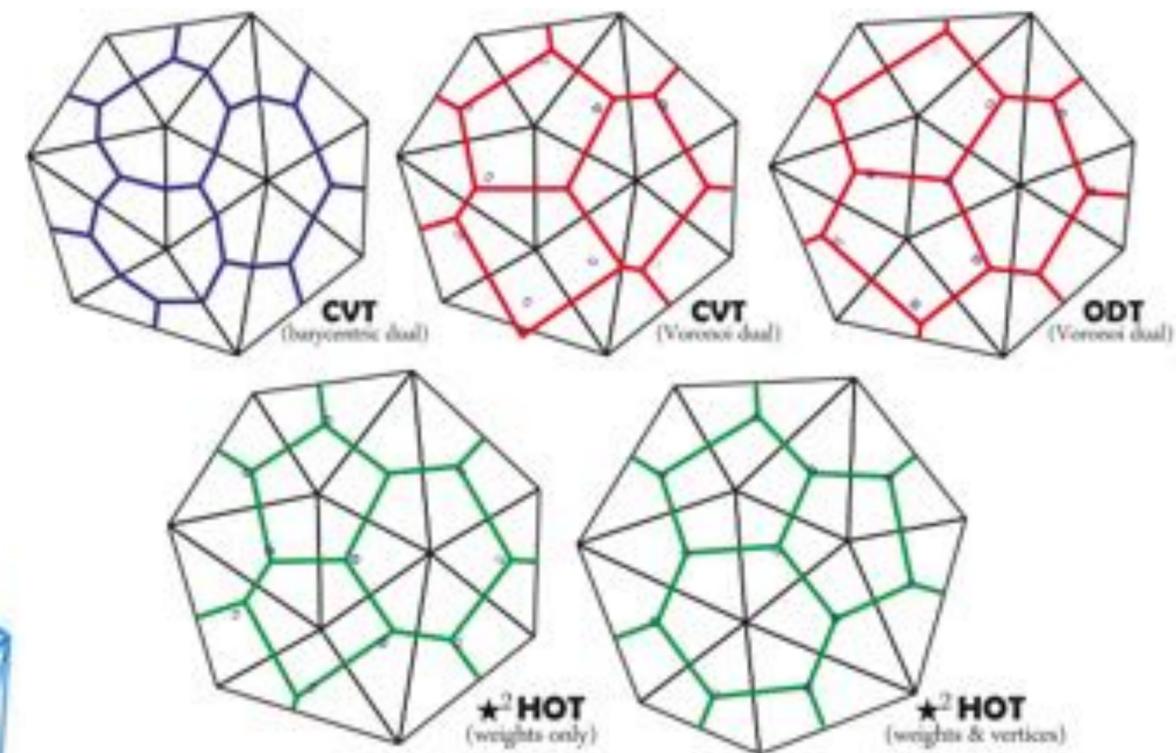
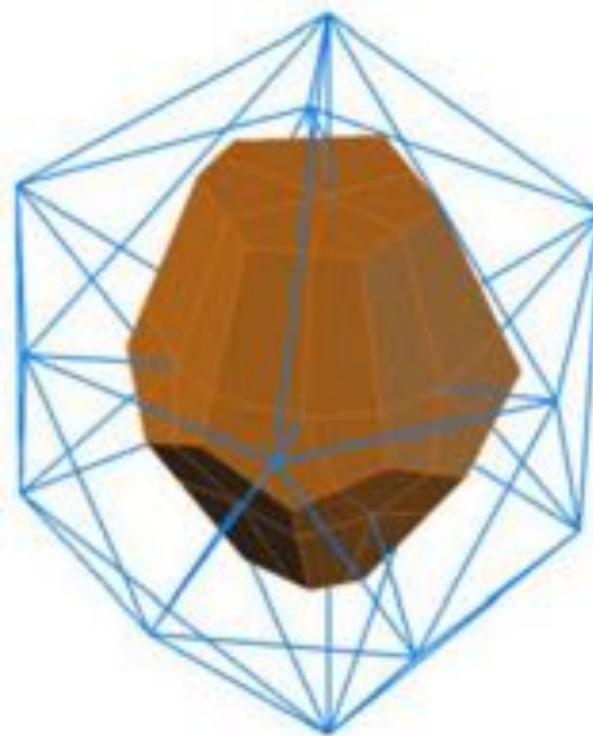
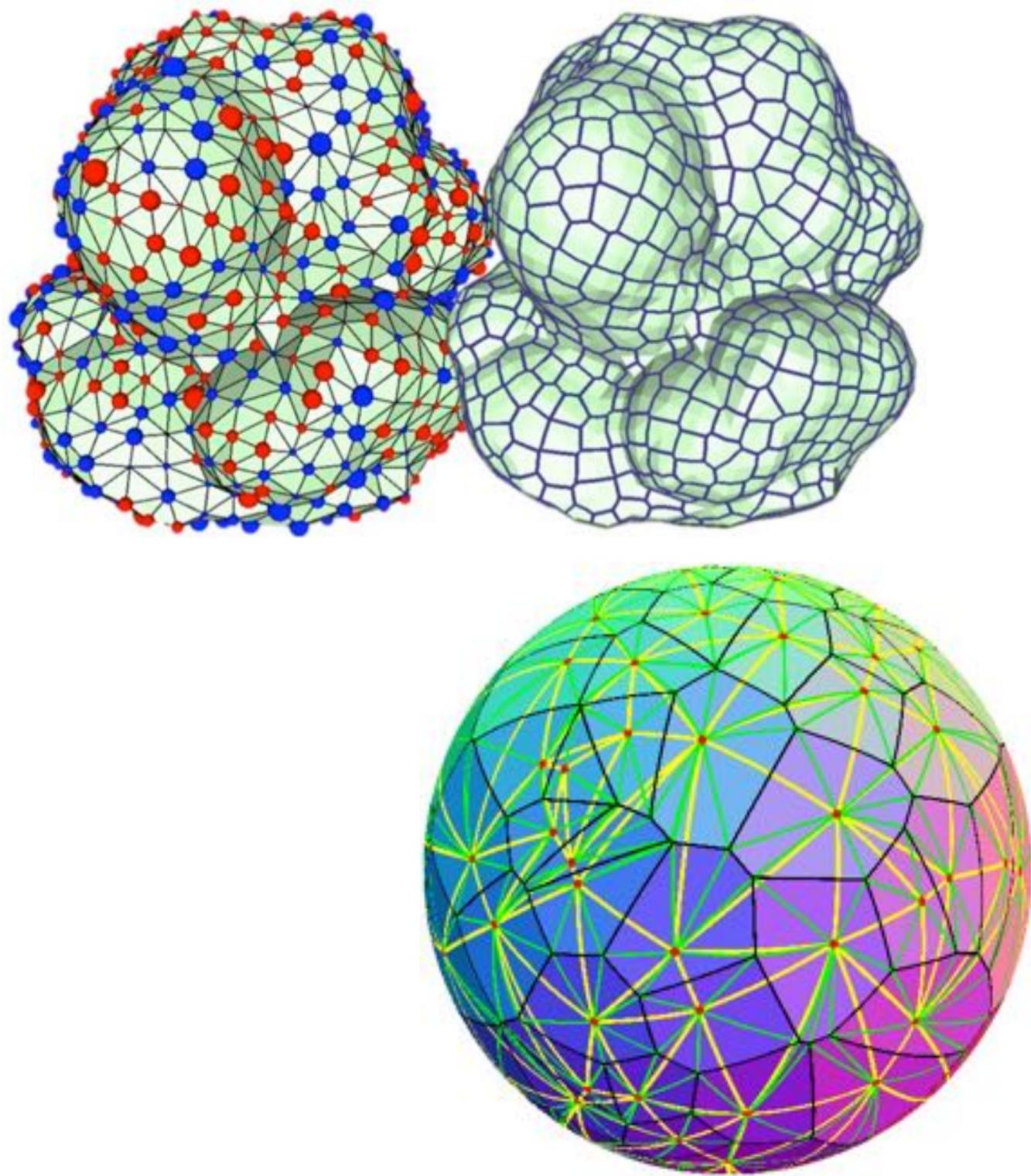


(L. Guibas & J. Stolfi, "Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams")

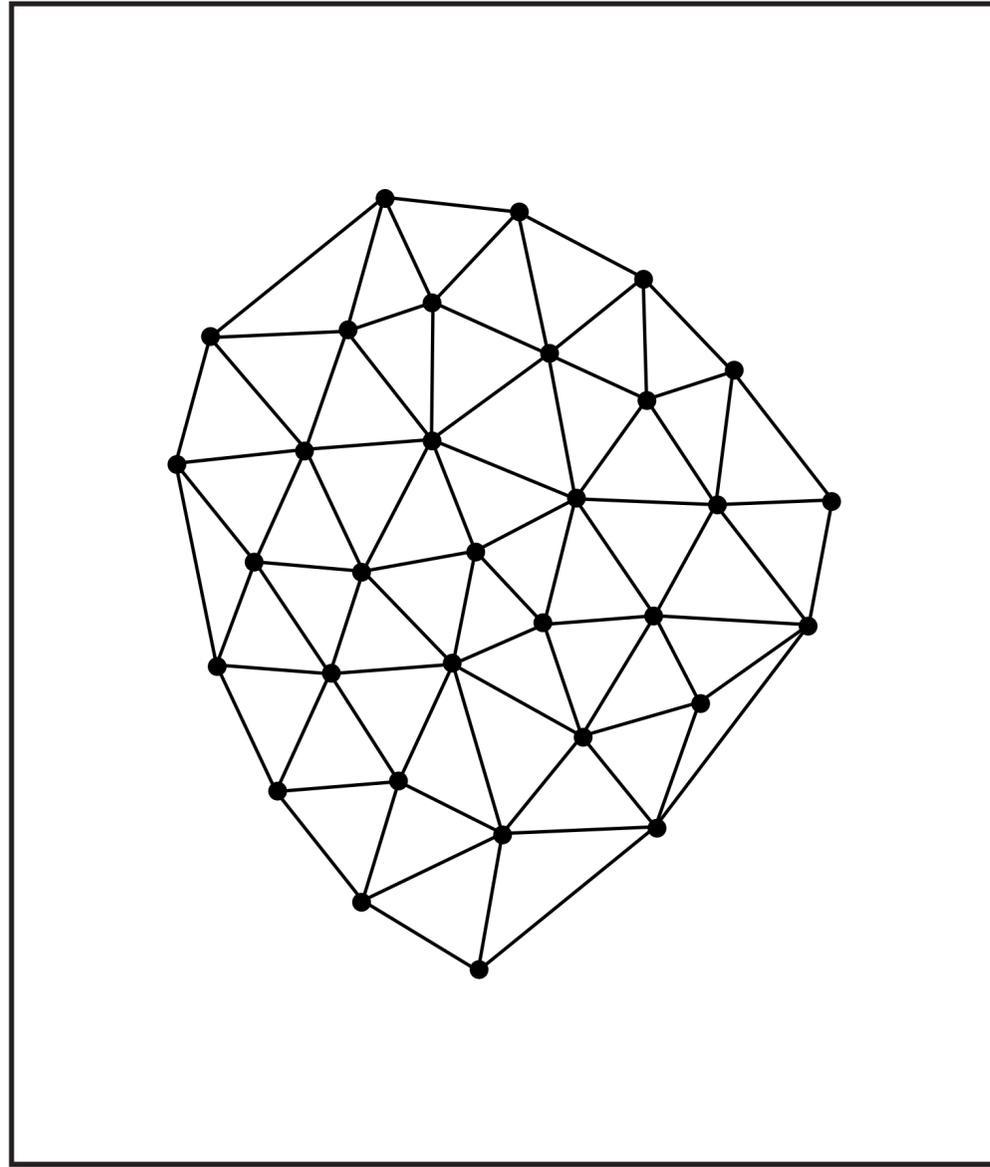


Dual Complex

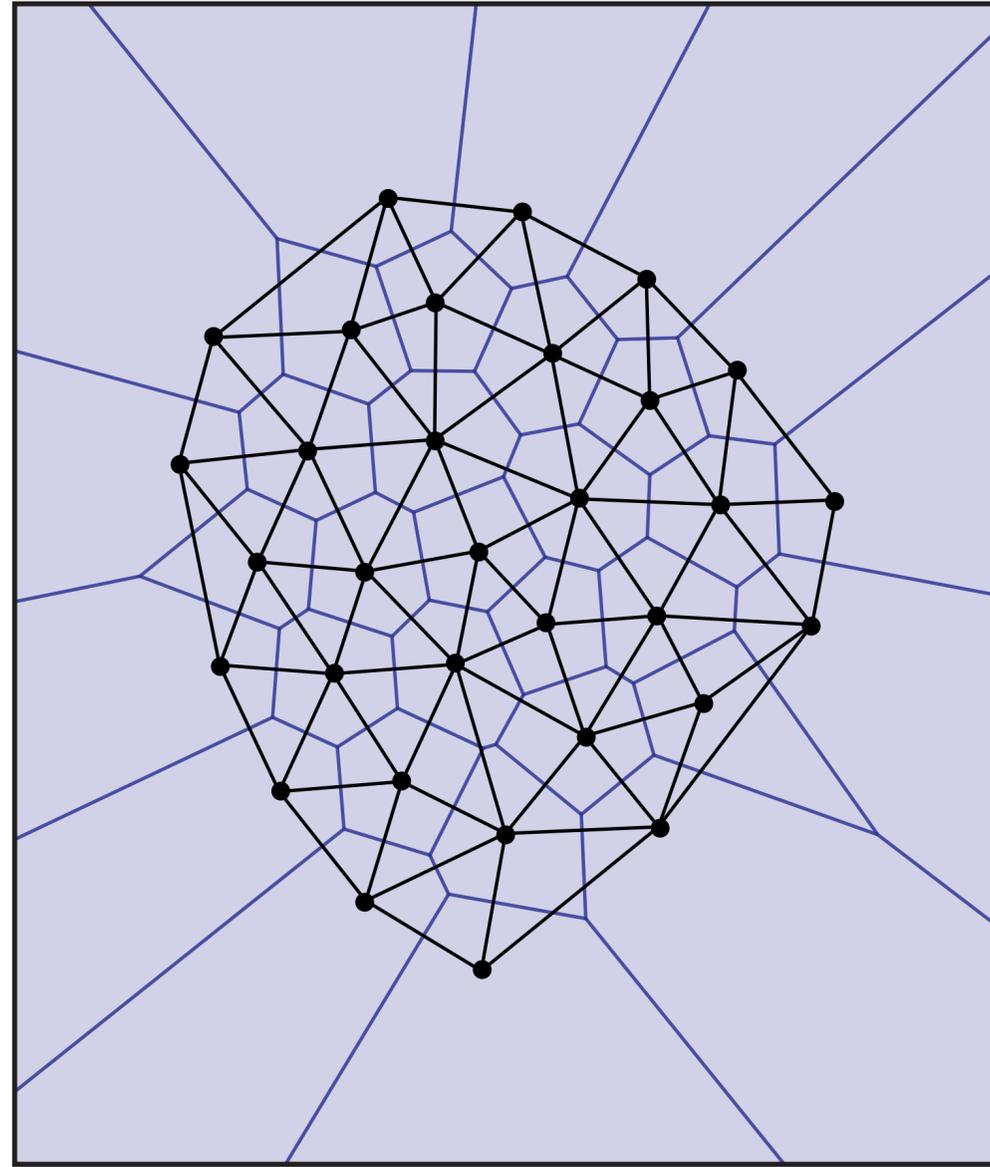
Dual Mesh—Visualized



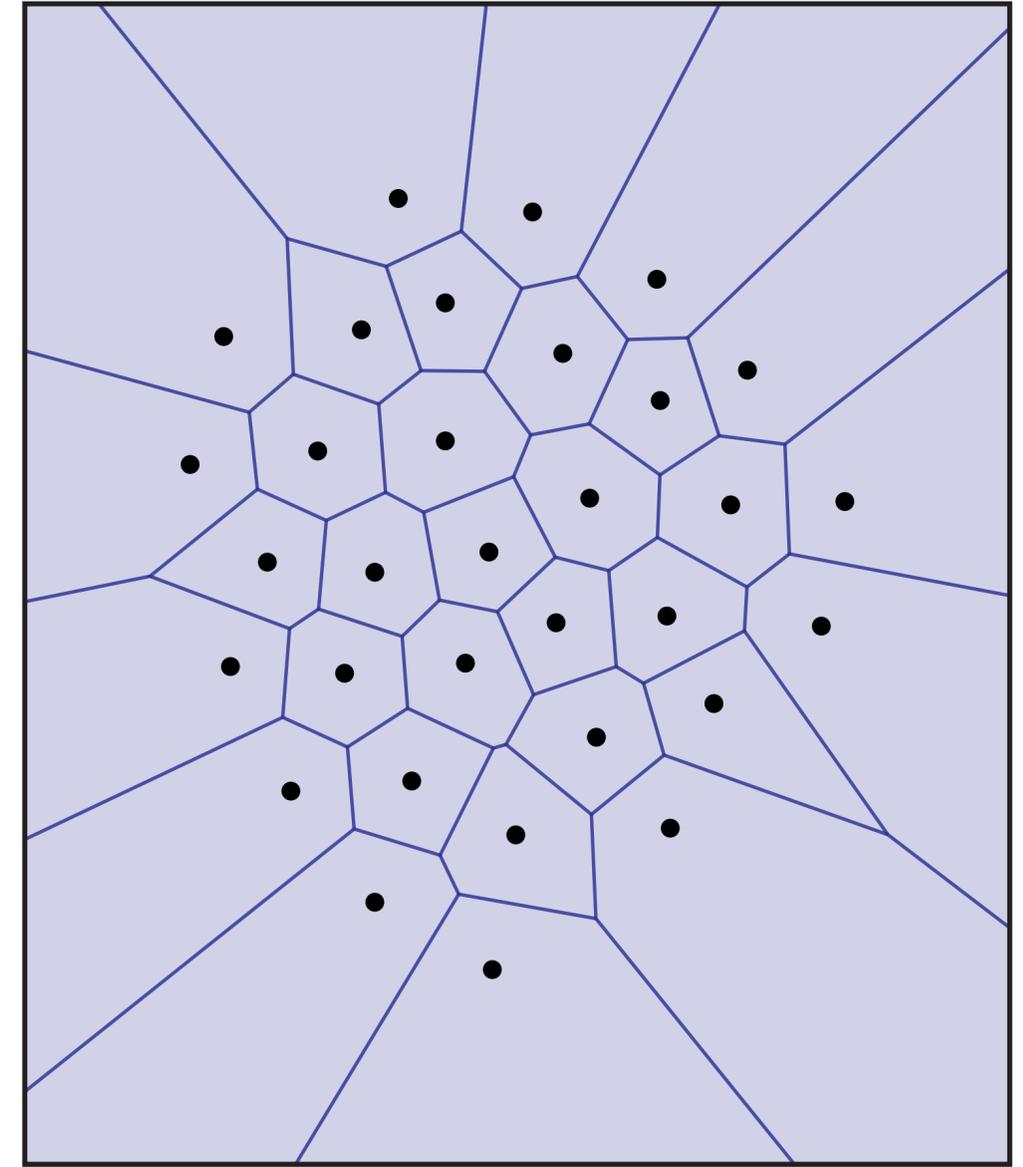
Poincaré Duality



simplicial complex

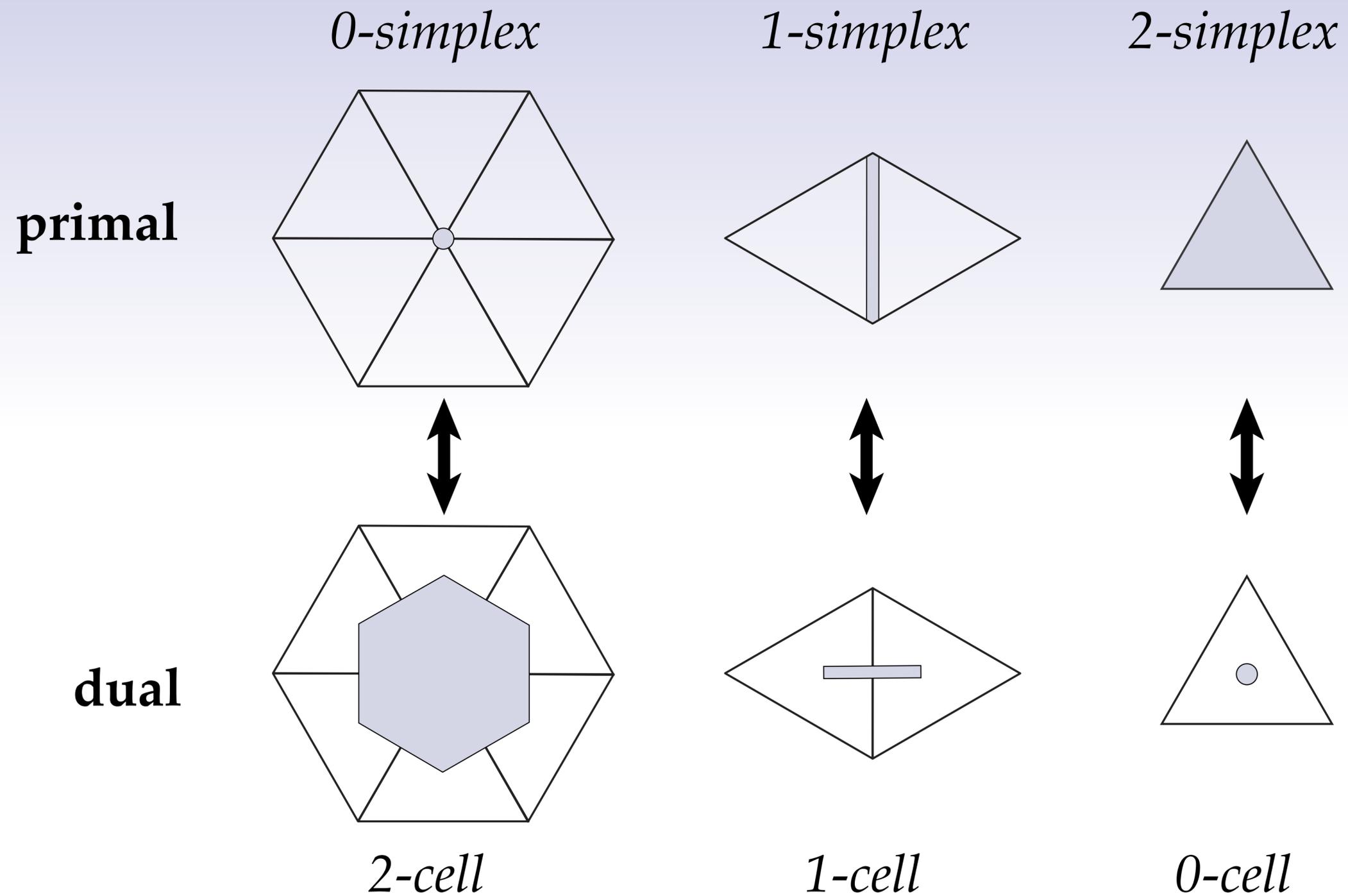


(Poincaré Duality)



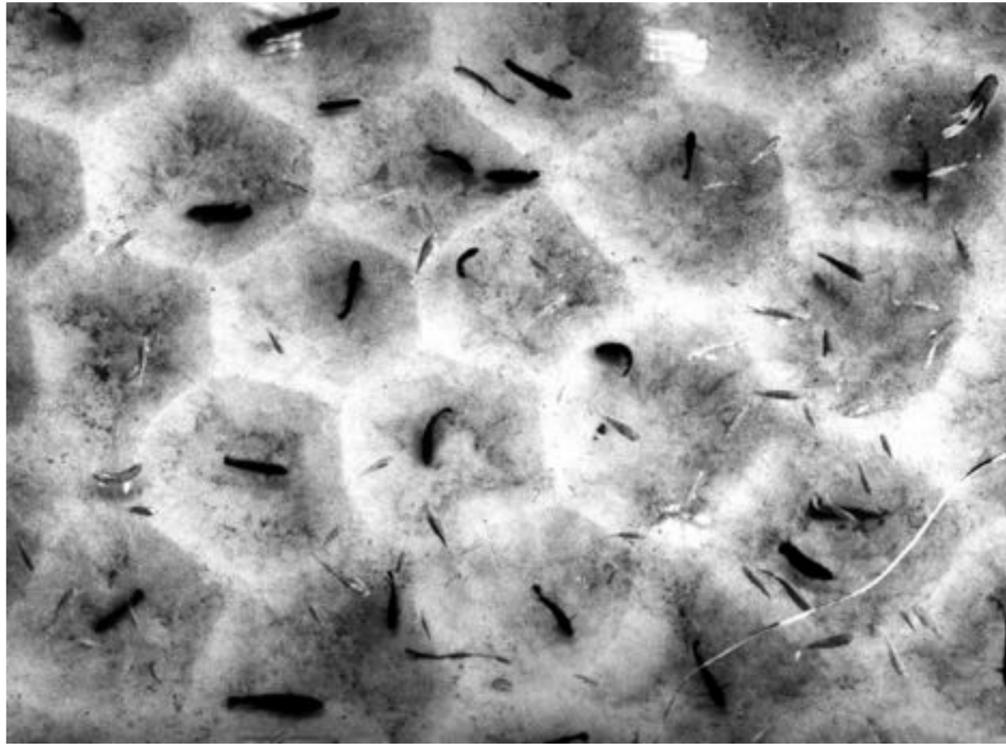
cell complex

Primal vs. Dual

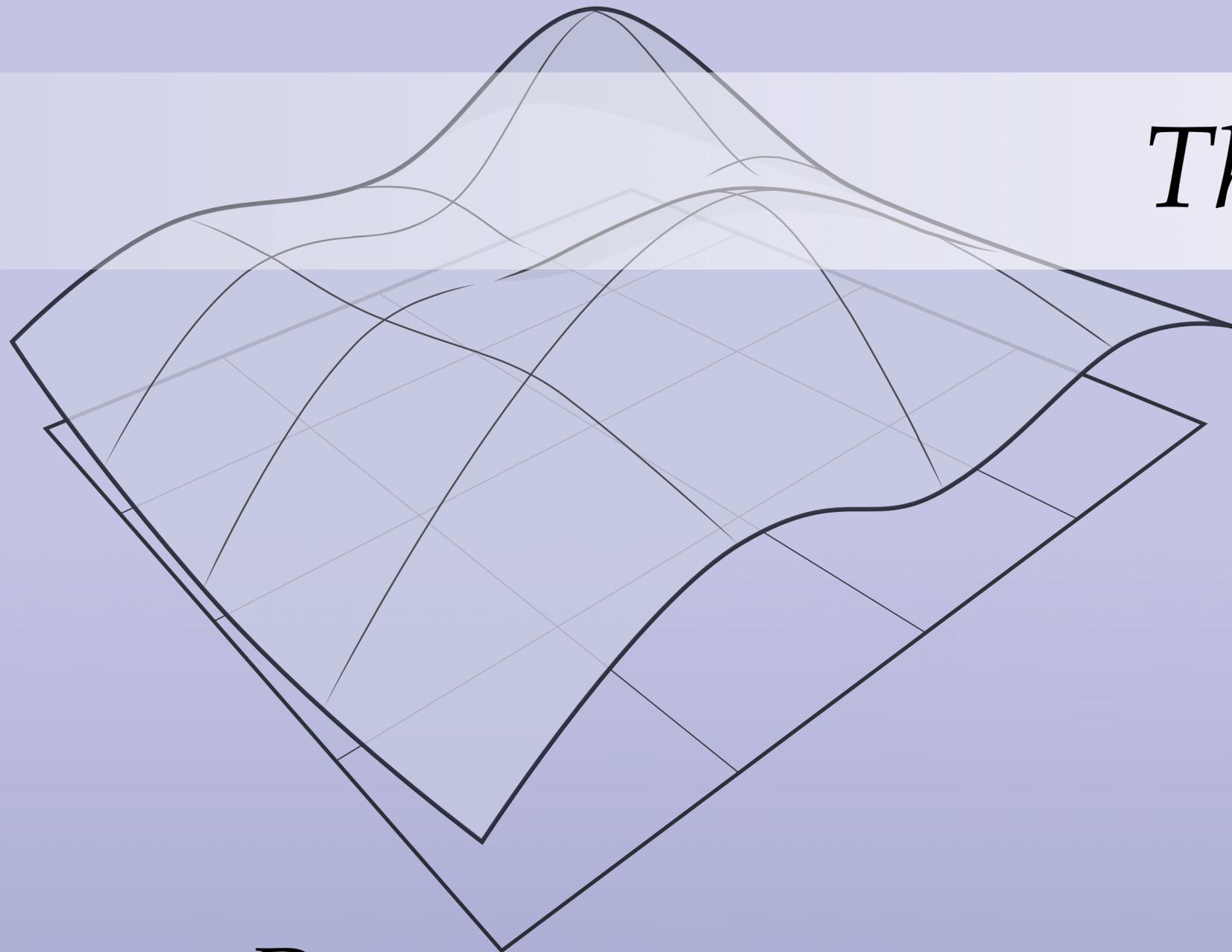


(Will say more when we talk about discrete exterior calculus!)

Poincaré Duality in Nature



Thanks!



DISCRETE DIFFERENTIAL

GEOMETRY:

AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858