DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
Keenan Crane • CMU 15-458/858
Lecture 6: Exterior Calculus — Differentiation

Discrete Differential Geometry: An Applied Introduction

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### Exterior Calculus—Overview

**Previously:**
- **1-form**—linear measurement of a vector
- **$k$-form**—multilinear measurement of volume
- **differential $k$-form**—$k$-form at each point

**Today: exterior calculus**
- how do $k$-forms change?
- how do we integrate $k$-forms?
Integration and Differentiation

• Two big ideas in calculus:
  • differentiation
  • integration
  • linked by fundamental theorem of calculus

• Exterior calculus generalizes these ideas
  • differentiation of k-forms (exterior derivative)
  • integration of k-forms (measure volume)
  • linked by Stokes’ theorem

• Goal: integrate differential forms over meshes to get discrete exterior calculus (DEC)
Motivation for Exterior Calculus

- Why study these two very similar viewpoints? (I.e., vector vs. exterior calculus)
  - Hard to measure change in volumes using basic vector calculus
  - Duality clarifies the distinction between different concepts/quantities
  - **Topology**: notion of differentiation that does not require metric (e.g., cohomology)
  - **Geometry**: clear language for calculus on curved domains (Riemannian manifolds)
  - **Physics**: clear distinction between physical quantities (e.g., velocity vs. momentum)
  - **Computer Science**: Leads directly to discretization/computation!
Exterior Derivative
Derivative—Many Interpretations…

“best linear approximation”

“slope of the graph”/
“rise over run”

\[ f'(x) := \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \]

“difference in the limit”

“pushforward”
Vector Derivatives—Visualized

$\phi$

grad $\phi$

$X$

div $X$

$Y$

curl $Y$
How do we express grad, div, and curl in coordinates?

Consider a scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$ and a vector field

$$X = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

where $u, v, w : \mathbb{R}^n \to \mathbb{R}$ are coordinate functions that vary over the domain, and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are the standard basis vector fields.

**grad**

$$\nabla \phi = \frac{\partial \phi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial z}$$

**div**

$$\nabla \cdot X = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

**curl**

$$\nabla \times X =$$

$$\left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \frac{\partial}{\partial x} +$$

$$\left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \frac{\partial}{\partial y} +$$

$$\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial}{\partial z}$$
Exterior Derivative \( \Omega^k \) — space of all differential \( k \)-forms

Unique linear map \( d : \Omega^k \to \Omega^{k+1} \) such that

- **differential** for \( k = 0 \), \( d\phi(X) = D_X\phi \)
- **product rule** \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \)
- **exactness** \( d \circ d = 0 \)

Where do these rules come from? (What’s the geometric motivation?)
Exterior Derivative—Differential
The **directional derivative** of a scalar function at a point $p$ with respect to a vector $X$ is the rate at which that function increases as we walk away from $p$ with velocity $X$.

More precisely:

$$D_X \phi \bigg|_p := \lim_{\epsilon \to 0} \frac{\phi(p + \epsilon X) - \phi(p)}{\epsilon}$$

Alternatively, suppose that $X$ is a **vector field**, rather than just a vector at a single point. Then we can write just:

$$D_X \phi$$

The result is a **scalar function**, whose value at each point $p$ is the directional derivative along the vector $X(p)$. 
Let $\phi : \mathbb{R}^n \to \mathbb{R}$. What is the gradient of $\phi$?

**Geometric intuition.** “Uphill direction.”

**Coordinate approach.** In Euclidean $\mathbb{R}^n$, list of partials:

$$\nabla \phi = \frac{\partial \phi}{\partial x^1} \frac{\partial}{\partial x^1} + \cdots + \frac{\partial \phi}{\partial x^n} \frac{\partial}{\partial x^n} = \left[ \frac{\partial \phi}{\partial x^1} \cdots \frac{\partial \phi}{\partial x^n} \right]^T$$

**Coordinate-free approach.** $\langle \nabla \phi, X \rangle = D_X(\phi)$ for all $X$.

I.e., at each point the gradient is the unique vector* such that taking the inner product $\langle \cdot, \cdot \rangle$ with a given vector $X$ yields the directional derivative along $X$.

*Assuming it exists! I.e., assuming the function is differentiable.
Differential of a Function

- Recall that differential 0-forms are just ordinary scalar functions.
- Change in a scalar function can be measured via the differential.
- Two ways to define differential:
  1. As unique 1-form such that applying to any vector field gives directional derivative along those directions:
     \[ d\phi(X) = D_X\phi \]
  2. In coordinates:
     \[ d\phi(X) := \frac{\partial \phi}{\partial x^1} dx^1 + \cdots + \frac{\partial \phi}{\partial x^n} dx^n \]

...but wait, isn’t this just the same as the gradient?
Gradient vs. Differential

• Superficially, gradient and differential look quite similar (but not identical!):

\[ \langle \nabla \phi, X \rangle = D_X \phi \]

\[ d\phi(X) = D_X \phi \]

• Especially in \( \mathbb{R}^n \):

\[ \nabla \phi = \frac{\partial \phi}{\partial x^1} \frac{\partial}{\partial x^1} + \cdots + \frac{\partial \phi}{\partial x^n} \frac{\partial}{\partial x^n} \]

\[ d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \cdots + \frac{\partial \phi}{\partial x^n} dx^n \]

• So what’s the difference?

  • For one thing, one is a vector field; the other is a differential 1-form
  
  • More importantly, gradient depends on inner product; differential doesn’t

\[ (d\phi)^\flat = \nabla \phi \iff \begin{align*}
  d\phi(\cdot) &= \langle \nabla \phi, \cdot \rangle \\
  (\nabla \phi)^b &= d\phi
\end{align*} \]

Makes a big difference when it comes to curved geometry, numerical optimization, …
Exterior Derivative—Product Rule
Exterior Derivative

Unique \textit{linear} map $d : \Omega^k \rightarrow \Omega^{k+1}$ such that

\begin{align*}
\text{differential} & \quad \text{for } k = 0, \quad d\phi(X) = D_X\phi \\
\text{product rule} & \quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \\
\text{exactness} & \quad d \circ d = 0
\end{align*}
Review: Product of Numbers

Q: Why is it true that $ab = ba$ for any two real numbers $a, b$?

Q: What’s the geometric interpretation of the statement “$4 \times 3 = 12$”? How about “$3 \times 4 = 12$”?
Reminder: For any differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, $(fg)' = f'g + fg'$.  

Q: Why? What’s the geometric interpretation?

$$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
Let $\alpha$ be a $k$-form and let $\beta$ be an $\ell$-form. Then

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

Q: Geometric intuition?

(Does this cartoon depict the exterior derivative? Or a directional derivative?)

$$\alpha \wedge \beta + \alpha' \wedge \beta + \alpha \wedge \beta'$$
Example. Let $\alpha := u \, dx$, $\beta := v \, dy$, and $\gamma := w \, dz$ be differential 1-forms on $\mathbb{R}^n$, where $u, v, w : \mathbb{R}^n \rightarrow \mathbb{R}$ are 0-forms, i.e., scalar functions. Also, let $\omega := \alpha \wedge \beta$. Then

$$d(\omega \wedge \gamma) = (d\omega) \wedge \gamma + (-1)^2 \omega \wedge (d\gamma).$$

We can then “recursively” evaluate derivatives that appear on the right-hand side:

$$d\omega = (d\alpha) \wedge \beta + (-1)^1 \alpha \wedge (d\beta),$$

$$d\alpha = (du) \wedge dx + (-1)^0 u(dx),$$

$$d\beta = (dv) \wedge dy + (-1)^0 v(dy),$$

$$d\gamma = (dw) \wedge dz + (-1)^0 w(dz).$$

Key idea: The “base case” is the 0-forms, i.e., computing the final result boils down to taking the differential of ordinary scalar functions.
Exterior Derivative—Examples

Example. Let $\phi(x,y) := \frac{1}{2} e^{-(x^2+y^2)}$. Then $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$

$$= -2\phi(xdx + ydy)$$

Example. Let $\alpha(x,y) = xdx + ydy$. Then $d\alpha =$

$$(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy) \wedge dx + (\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy) \wedge dy$$

$$= dx \wedge dx + dy \wedge dy = 0 + 0 = 0.$$ 

Example. Again let $\alpha(x,y) = xdx + ydy$. Then $d \wedge \alpha = d(x \wedge dx + y \wedge dy)$

$$= d(xdy - ydx)$$

$$= dx \wedge dy - dy \wedge dx$$

$$= 2dx \wedge dy.$$
Exterior Derivative—Exactness
Exterior Derivative

Unique linear map $d : \Omega^k \rightarrow \Omega^{k+1}$ such that

<table>
<thead>
<tr>
<th>differential</th>
<th>for $k = 0$, $d\phi(X) = D_X\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>product rule</td>
<td>$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$</td>
</tr>
<tr>
<td>exactness</td>
<td>$d \circ d = 0$</td>
</tr>
</tbody>
</table>

Why?
Key idea: exterior derivative should capture a similar idea.
Q: Consider a 1-form $\alpha = udx + vdy + wdz$, where the coefficients $u, v, w$ are each scalar functions $\mathbb{R}^3 \to \mathbb{R}$. What is the exterior derivative $d\alpha$ in coordinates $x, y, z$?

A: 

$$d\alpha = d(udx + vdy + wdz) = du \wedge dx + udx \wedge dx + dv \wedge dy + vdy \wedge dy + dw \wedge dz + wdz \wedge dz$$

$$= (\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz) \wedge dx + (\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz) \wedge dy + (\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz) \wedge dz$$

$$= -\frac{\partial u}{\partial y} dx \wedge dy + \frac{\partial u}{\partial z} dz \wedge dx + \frac{\partial v}{\partial x} dx \wedge dy - \frac{\partial v}{\partial z} dz \wedge dx + \frac{\partial v}{\partial y} dy \wedge dz - \frac{\partial w}{\partial x} dz \wedge dx + \frac{\partial w}{\partial y} dy \wedge dz$$

$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})dy \wedge dz + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz \wedge dx + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx \wedge dy.$$
Exterior Derivative and Curl

Suppose we have a vector field

\[ X := u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \]

Its curl is then

\[ \nabla \times X = (\partial w/\partial y - \partial v/\partial z) \frac{\partial}{\partial x} + \]
\[ (\partial u/\partial z - \partial w/\partial x) \frac{\partial}{\partial y} + \]
\[ (\partial v/\partial x - \partial u/\partial y) \frac{\partial}{\partial z} \]

Looks an awful lot like...

\[ d\alpha = (\partial w/\partial y - \partial v/\partial z) \ dy \wedge dz + \]
\[ (\partial u/\partial z - \partial w/\partial x) \ dz \wedge dx + \]
\[ (\partial v/\partial x - \partial u/\partial y) \ dx \wedge dy \]

Especially if we then apply the Hodge star.

\[ \nabla \times X \iff \star d\alpha \]
\[ \nabla \times X = (\star dX^b)^\# \]
$d \circ d = 0$

**Intuition:** in $\mathbb{R}^n$, first $d$ behaves just like gradient; second $d$ behaves just like curl.
Exterior Derivative in 3D (1-forms)

Q: How about $d \star \alpha$? (Still for $\alpha = udx + vdy + wdz$.)

A: $d \star \alpha = d(\star(udx + vdy + wdz))$

$$= d(udy \wedge dz + vdz \wedge dx + wdx \wedge dy)$$

$$= du \wedge dy \wedge dz + dv \wedge dz \wedge dx + dw \wedge dx \wedge dy$$

$$= \frac{\partial u}{\partial x} dx \wedge dy \wedge dz + \frac{\partial v}{\partial y} dy \wedge dz \wedge dx + \frac{\partial w}{\partial z} dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dx \wedge dy \wedge dz$$

Q: Does this operation remind you of anything (perhaps from vector calculus)?
Exterior Derivative and Divergence

Suppose we have a vector field

\[ X := u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \]

Its divergence is then

\[ \nabla \cdot X = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \]

Looks an awful lot like...

\[ d \star \alpha = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx \wedge dy \wedge dz \]

Especially if we then apply the Hodge star.

\[ \nabla \cdot X \iff \star d \star \alpha \]

\[ \nabla \cdot X = \star d \star X^b \]
Exterior Derivative - Divergence

\[ \nabla \cdot X = \ast d(\ast X^b) \] 

(codifferential: \( \delta := \ast d \ast \))
Exterior vs. Vector Derivatives — Summary

\begin{align*}
\text{grad } \phi & = (d\phi)^\# \\
\text{div } X & = \star d(\star X^b) \\
\text{curl } Y & = (\star (dX^b))^\#
\end{align*}
Exterior Derivative

Unique linear map \( d : \Omega^k \rightarrow \Omega^{k+1} \) such that

- **differential** for \( k = 0 \), \( d\phi(X) = D_X\phi \)
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- **exactness** \( d \circ d = 0 \)
Exterior Derivative—Summary
Taking a step back, we can draw many of the operators seen so far as diagrams:

\( \Omega_0 \xrightarrow{d} \Omega_1 \xleftarrow{\delta} \quad \Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{\delta} \Omega_2 \xleftarrow{\ast} \Omega_1 \xleftarrow{\delta} \Omega_0 \)

\( \mathbb{R}^1 / \text{curves} \)

\( \Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{\delta} \Omega_2 \xrightarrow{d} \Omega_3 \xleftarrow{\delta} \Omega_1 \xleftarrow{d} \Omega_0 \xleftarrow{\ast} \Omega_2 \xleftarrow{\delta} \Omega_3 \xleftarrow{\delta} \Omega_2 \)

\( \mathbb{R}^2 / \text{surfaces} \)

\( \Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{\delta} \Omega_2 \xleftarrow{\delta} \Omega_3 \xleftarrow{\ast} \Omega_2 \)

\( \mathbb{R}^3 / \text{volumes} \)

\( \Omega_k \) — differential \( k \)-forms
Laplacian

• Can now compose operators to get other operators
• E.g., Laplacian from vector calculus:
  \[ \Delta := \text{div} \circ \text{grad} \]
• Can express exact same operator via exterior calculus:
  \[ \Delta = \star d \star d \]
• …except that this expression easily generalizes to curved domains.
• Can also generalize to \( k \)-forms:
  \[ \Delta := \star d \star d + d \star d \star \]
• Will have much more to say about the Laplacian later on!
Exterior Derivative - Summary

- Exterior derivative $d$ used to differentiate k-forms
  - 0-form: “gradient”
  - 1-form: “curl”
  - 2-form: “divergence” (codifferential $\delta$)
  - and more...

- Natural product rule

- $d$ of $d$ is zero
  - Analogy: curl of gradient
  - More general picture (soon!) via Stokes’ theorem
 Thanks!

**DISCRETE DIFFERENTIAL GEOMETRY**

AN APPLIED INTRODUCTION