DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
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Discrete Surfaces
Two primary models of surfaces in discrete differential geometry:

- **Simplicial**
  - surfaces are simplicial 2-manifolds
  - natural fit with discrete exterior calculus

- **Nets**
  - surfaces are piecewise integer lattices
  - natural fit with discrete integrable systems

- Simplicial surfaces more common in applications; focus of our course
• Loosely speaking, a **simplicial surface** is “just a triangle mesh”

• But, being more careful about definitions will allow us to connect “triangle meshes” to concepts from differential geometry

• As with smooth surfaces, will also add some conditions that make life easier. E.g.,

  • mesh connectivity is **manifold**

  • vertex coordinates describe a **simplicial immersion**
An (abstract) simplicial surface is a manifold simplicial 2-complex

- highest-degree simplices are triangles
- every edge contained in two triangles (or one, along boundary)
- every vertex contained in a single edge-connected cycle of triangles (or path, along boundary)

- Will typically denote by $K=(V,E,F)$
- No “shape” — just connectivity
Simplicial Map

- How do we give a “shape” to an abstract simplicial surface?
- Assign coordinates $f_i$ to each vertex (discrete $\mathbb{R}^n$-valued 0-form)
- Linearly interpolate over each triangle via barycentric coordinates
- Image of each simplex in our abstract surface is now a simplex in $\mathbb{R}^n$
- Any map from simplices to simplices is called a simplicial map
Simplicial Map, continued

- What’s really going on here? E.g., what’s the domain of our map \( f \)?
- Abstract simplicial complex is just a set of subsets… How do we talk about points “inside” a simplex?
- Barycentric coordinates effectively associate each abstract simplex with a copy of the standard simplex
- Domain of \( f \) is then the (disjoint) union of all these simplices, “glued” together along shared edges*

*Formally: quotient space w.r.t. equivalence on barycentric coordinates

\[ K = \{ \{i,j,k\}, \{j,k,l\}, \{i,j\}, \{j,k\}, \{k,i\}, \ldots \} \]
Discrete Differential

- Map $f$ is given by a discrete, $R^n$-valued 0-form.
- **Discrete differential** $df$ is just discrete exterior derivative.
- What does it mean, geometrically?
- Recall that a discrete 1-form represents the integral of a smooth 1-form over a 1-simplex*:

  $$(df)_{ij} := \int_{\sigma_{ij}} df \left( \frac{\partial}{\partial s} \right) \, ds = \int_{\sigma_{ij}} df = \int_{\partial \sigma_{ij}} f = f_j - f_i$$

- In other words, **discrete differential is nothing more than the edge vectors**!
- Like any other 1-form, antisymmetric w.r.t. orientation: $df_{ji} = -df_{ij}$

*Here we can imagine $\sigma_{ij}$ is the standard 1-simplex.
Discrete Immersion

- In smooth setting, a map $f$ is an immersion if differential is nondegenerate, i.e., if it maps nonzero vectors to nonzero vectors.
- In discrete setting, a nondegenerate (discrete) differential just means no zero edge lengths.
- Doesn’t faithfully capture important features of smooth immersions! E.g., no branch points.
Simplicial Immersion

• In smooth setting, a map $f$ is an immersion if its differential $df$ is injective.

• In the discrete setting, a simplicial map $f$ is a discrete immersion if the map itself is locally injective.

• Fact. A simplicial map is locally injective if and only if every vertex star is embedded.

Note: “no degenerate elements/angles” is necessary but NOT sufficient!
Discrete Gauss Map
Discrete Gauss Map

- For a discrete immersion, the Gauss map is simply the triangle normals
- Most naturally viewed as a dual discrete $\mathbb{R}^3$-valued 0-form (vector per triangle)
- Visualize as points on the unit sphere
- Connecting adjacent normals by arcs corresponds to family of normals orthogonal to edge
Discrete Vertex Normal?

• Discrete Gauss map still doesn’t define normals at vertices (or edges)

• Can take ad-hoc approach, but may behave poorly

• E.g., uniformly averaging face normals yields results that depend on tessellation rather than geometry

• Better approach: start in the smooth setting & apply principled discretization
Discrete Vector Area

• Recall smooth vector area: \[ \int_{\Omega} N \, dA = \frac{1}{2} \int_{\Omega} df \wedge df = \frac{1}{2} \int_{\partial\Omega} f \times df \]

• Idea: Integrate \(NdA\) over dual cell to get normal at vertex \(p\)

\[
\frac{1}{3} \int_{\Omega} N \, dA = \frac{1}{6} \int_{\partial\Omega} f \times df = \]

\[
\frac{1}{6} \sum_{ij \in \partial\Omega} \int_{e_{ij}} f \times df = \]

\[
\frac{1}{6} \sum_{ij \in \partial\Omega} \frac{f_i + f_j}{2} \times (f_j - f_i) = \frac{1}{6} \sum_{ij \in \partial\Omega} f_i \times f_j
\]

Q: What kind of quantity is the final expression? Does that matter?
Other Natural Definitions

• area-weighted vertex normal
  • sum of triangle normals times triangle areas
  • smooth setting: volume variation gives \( N \, dA \)

• angle weighted vertex normal
  • sum of triangle normals times interior angles
  • gives same result, independent of triangulation
  • …Please, just anything but uniformly weighted!
Discrete Exterior Calculus on Curved Surfaces
Discrete Exterior Calculus on Curved Surfaces

• In the smooth setting, we first defined exterior calculus in $\mathbb{R}^n$, then saw how to augment it to work on curved surfaces.

• Key observation: only need to change the Hodge star, which encodes all the metric information (length, angle, area, …).

• For simplicial surfaces in $\mathbb{R}^3$, life is in a sense even easier since each simplex is already flat!

• Still need to think just a little about how to define the discrete Hodge star…
Diagonal Hodge Star on a Surface

- Recall that on a simplicial surface, we can discretize the Hodge star via diagonal matrices storing volume ratios (given by formulas below)

- **Q:** What happens if our mesh is no longer flat?

\[
\frac{1}{A_{ijk}} = \frac{1}{\sqrt{s - \ell_{ij}(s - \ell_{jk})(s - \ell_{ki})}}
\]

\[
s = \frac{1}{2} (\ell_{ij} + \ell_{jk} + \ell_{ki})
\]

\[
\frac{A_{\text{dual}}}{1} = \frac{1}{8} \sum_{ijk \in F} (\ell_{ij}^2 \cot \alpha_k^j + \ell_{ik}^2 \cot \alpha_i^k)
\]
Diagonal Hodge Star on a Curved Surface

• **A:** Nothing changes! As long we have a discrete immersion, we can still apply the same formulas—which depend only on *primal lengths* and *interior angles*

• In the case of the 1-form Hodge star, we are effectively taking a length ratio involving the dual distance “along” the surface

• Importantly, this means that our DEC operators are purely *intrinsic*: depends only on data that can be measured by an observer “crawling along the surface”
Discrete Laplace-Beltrami Operator

• As a result, we can immediately build discrete differential operators for curved surfaces by just composing our existing discrete exterior derivative and discrete Hodge star operators.

• For instance, the Laplacian on 0-forms now becomes something known as the Laplace-Beltrami operator (which we’ll talk much more about later!)

• Using our expressions for the discrete Hodge star, can write the discrete Laplace-Beltrami operator via the famous cotan formula:

\[
(\Delta u)_i = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)
\]
Recovery of Discrete Surfaces
Recovery of Discrete Surfaces

- In a variety of situations, we’ve seen that shape can be recovered (up to rigid motions) via “indirect” measurements (curvatures, etc.)
- **Plane curves** can be recovered from their curvature (exterior angle)
- **Space curves** can be recovered from their curvature and torsion
- **Smooth surfaces** can be recovered from 1st & 2nd fundamental form
- **Convex surfaces** can be recovered from Riemannian metric…

Q: What data is sufficient to describe a *discrete* surface?
Surface Recovery from Discrete Gauss Map

• **Q:** Given only discrete Gauss map, can we recover the immersion? (i.e., given only triangle normals, can we get vertex positions?)

• **A:** Yes! Basic recipe:
  - Cross product of normals gives edge directions
  - Dot product of edges gives interior angles
  - Angles + normals give triangles up to scale; normals give orientation
  - Build triangles one-by-one and “glue” together

• **Q:** Does this recipe always work?
Q: Is it strange that we can recover a discrete surface from Gauss map? Can we do something similar in the smooth setting?

Consider a simpler case: Gauss map on a curve

$N(s) := (\cos(s), \sin(s))$

Problem: unless we know curve is arc-length parameterized, $N$ is the Gauss map of any convex curve! Need additional data (parameterization)

Similar story for convex discrete curves, or convex smooth surfaces

So why don’t we need additional data for a discrete surface?
Recovery from Metric

• Theorem. (Cohn-Vossen) Smooth convex surface is uniquely determined (up to rigid motions) by its Riemannian metric.

• Theorem. (Alexandrov-Connelly) A convex polyhedron is uniquely determined by its edge lengths.

• Not always true in nonconvex case:
Recovery From Discrete Metric
Algorithm: Shape from Metric

- Recent algorithm (approximately!) recovers surface from edge lengths
- Chern et al, “Shape from Metric” (2018)
- Nice read if you want to get deeper into discrete surfaces: discrete immersion, discrete spin structure...

http://page.math.tu-berlin.de/~chern/projects/ShapeFromMetric/