DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
Keenan Crane • CMU 15-458/858B • Fall 2017
A Unified Picture of Discrete Curvature

- By making some connections between smooth and discrete surfaces, we get a unified picture of many different discrete curvatures scattered throughout the literature.

- To tell the full story we’ll need a few pieces:
  - geometric derivatives
  - Steiner polynomials
  - sequence of curvature variations
  - assorted theorems (Gauss-Bonnet, Schläfli, $\Delta f = 2HN$)

- Start with integral viewpoint (1st lecture), then cover variational viewpoint (2nd lecture).
Discrete Geometric Derivatives
Discrete Geometric Derivatives

- Practical technique for calculating derivatives of discrete geometric quantities
- Basic question: how does one geometric quantity change with respect to another?
- E.g., what’s the gradient of triangle area with respect to the position of one of its vertices?
- Don’t just grind out partial derivatives!
- Do follow a simple geometric recipe:
  - First, in which **direction** does the quantity change quickest?
  - Second, what’s the **magnitude** of this change?
  - Together, direction & magnitude give us the gradient vector
Dangers of Partial Derivatives

- Why not just take derivatives “the usual way?”
- usually takes way more work!
- can lead to expressions that are
  - inefficient
  - numerically unstable
  - hard to interpret
- **Example:** gradient of angle between two segments \((b,a), (c,a)\) w.r.t. coordinates of point \(a\)
Geometric Derivation of Angle Derivative

• Instead of taking partial derivatives, let’s break this calculation into two pieces:

1. **(Direction)** What direction can we move the point $a$ to most quickly increase the angle $\theta$?

   **A:** Orthogonal to the segment $ab$.

2. **(Magnitude)** How much does the angle change if we move in this direction?

   **A:** Moving around a whole circle changes the angle by $2\pi$ over a distance $2\pi r$, where $r = |b-a|$. Hence, the instantaneous change is $1/|b-a|$.

• Multiplying the unit direction by the magnitude yields a final gradient expression.
Q: What’s the gradient of triangle area with respect to one of its vertices $p$?

A: Can express via its unit normal $N$ and vector $e$ along edge opposite $p$:

$$\nabla_p A = \frac{1}{2} N \times e$$
Geometric Derivation

- In general, can lead to some pretty slick expressions (give it a try!)

\[ \nabla_{p_3} \theta = \frac{|e|}{2A_1} N_1 \]

\[ \nabla_p A = \frac{1}{2} N \times e \]

\[ d_{f_i} N(X) = \frac{\langle N, X \rangle}{2A} e_i \times N \]

\[ du(v) = \frac{v - \langle v, b-a \rangle (b-a)}{|b-a|^3} \]
Aside: Automatic Differentiation

- Geometric approach to differentiation greatly simplifies “small pieces” (gradient of a particular, angle, length, area, volume, …)

- For larger expressions that combine many small pieces, approach of automatic differentiation is extremely useful*

- Basically does nothing more than automate repeated application of chain rule

- Simplest implementation: use pair that store both a value and its derivative; operations on these tuples apply operation & chain rule

*More recently known as backpropagation

Example.

```
// define how multiplication and sine // operate on (value,derivative) pairs // (usually done by an existing library)
(a,a')*(b,b') := (a*b,a*b'+b*a')
sin((a,a')) := (sin(a),a'*cos(a))

// to evaluate a function and its // derivative at a point, we first // construct a pair corresponding to the // identity function f(x) = x at the // desired evaluation point
x = (5,1) // derivative of x w.r.t x is 1

// now all we have to do is type a // function as usual, and it will yield // the correct value/derivative pair
fx = sin(x*x) // (-0.132352, 9.91203)
```
Schläfli Formula
Schläfli Formula

- Consider a closed polyhedron in $\mathbb{R}^3$ with edge lengths $l_{ij}$ and dihedral angles $\varphi_{ij}$. Then for any motion of the vertices,

\[
\sum_{ij \in E} l_{ij} \frac{d}{dt} \varphi_{ij} = 0
\]
Curvature Variations
Sequence of Variations (Smooth)

For a smooth surface $f: M \rightarrow \mathbb{R}^3$ (without boundary), let

\[
\text{volume}(f) := \frac{1}{3} \int_M N \cdot f \, dA \\
\text{mean}(f) := \int_M H \, dA \\
\text{area}(f) := \int_M dA \\
\text{Gauss}(f) := \int_M K \, dA = 2\pi \chi
\]

How can we move the surface so that each of these quantities changes as quickly as possible? Remarkably enough...

\[
\delta \text{volume}(f) = 2N \\
\delta \text{area}(f) = 2HN \\
\delta \text{mean}(f) = 2KN \\
\delta \text{Gauss}(f) = 0
\]

<table>
<thead>
<tr>
<th>volume</th>
<th>area</th>
<th>mean</th>
<th>Gauss</th>
<th>$\delta f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta f$</td>
<td>$\delta f$</td>
<td>$\delta f$</td>
<td>$\delta f$</td>
<td>0</td>
</tr>
</tbody>
</table>
Discrete Normal via Volume Variation

• Recall that we still don’t have a clear definition for discrete normals at vertices, where the surface is not differentiable.

• However, in the smooth setting we know that the normal is equal to (half) the volume gradient.

• **Idea:** Since volume is perfectly well-defined for a discrete surface, why not use volume gradient to define vertex normals?

• Now just need to calculate the gradient of volume with respect to motion of one of the vertices, which we can do using our “geometric approach”…
Volume Enclosed by a Smooth Surface

What’s the volume enclosed by a smooth surface $f$?

One way: pick any point $p$, integrate volume of “infinitesimal pyramids” over the surface.

For a pyramid with base area $b$ and height $h$, the volume is $V = bh/3$ (no matter what shape the base is).

For our infinitesimal pyramid, the height is the distance from the surface $f$ to the point $p$ along the normal direction: $h = (f - p) \cdot N$.

The area of the base is just the infinitesimal surface area $dA$. Now we just integrate…

\[
\frac{1}{3} \int_M (f - p) \cdot N \, dA = \frac{1}{3} \int_M f \cdot N \, dA - p \cdot \int_M N \, dA^0 = \frac{1}{3} \int_M f \cdot N \, dA
\]

Notice: doesn’t depend on choice of point $p$!
Volume Enclosed by a Discrete Surface

• What’s the volume enclosed by a discrete surface?

• Simply apply our smooth formula to a discrete $f$!

• **Exercise.** Show that the volume enclosed by a simplicial surface can be expressed as

$$\text{volume}(f) = \frac{1}{6} \sum_{ijk \in F} f_i \cdot (f_j \times f_k)$$
Discrete Volume Gradient

- Taking the gradient of enclosed volume with respect to the position $f_i$ of some vertex $i$ should now give us a notion of vertex normal:

\[
\nabla_{f_i} \text{ volume}(f) = \frac{1}{6} \nabla_{f_i} \sum_{ijk \in F} f_i \cdot (f_j \times f_k) = \frac{1}{6} \sum_{ijk \in F} f_j \times f_k
\]

- But wait—this expression is the same as the discrete area vector!

- In other words: taking the gradient of discrete volume gave us exactly the same thing as integrating the normal over the dual cell.

- Agrees with the first expression in our sequence of variations:

\[
\delta \text{ volume}(f) = N
\]
Total Area of a Discrete Surface

• Total area of a discrete surface is simply the sum of the triangle areas:

\[
\text{area}(f) := \sum_{ijk \in F} A_{ijk}
\]

Q: Suppose \( f \) is not a discrete immersion. Is area well-defined? Differentiable?
Discrete Area Gradient

• Recall that the gradient of triangle area with respect to position $p$ of a vertex is just half the normal cross the opposite edge:

$$\nabla_p A = \frac{1}{2} N \times e$$

• By summing contribution of all triangles touching a given vertex, can show that gradient of total surface area with respect to vertex coordinate $f_i$ is

$$\nabla_{f_i} \text{area}(f) = \sum_{ij} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$

• Agrees with second expression in our sequence:

$$\delta \text{area}(f) = HN = \frac{1}{2} \Delta f$$
Total Mean Curvature of a Discrete Surface

- From our Steiner polynomial, we know the total mean curvature of a discrete surface is

\[
\text{mean}(f) = \frac{1}{2} \sum_{ij \in E} \ell_{ij} \varphi_{ij}
\]

(In fact, total volume and area used for the previous two calculations also agree with Steiner polynomial...
Discrete Mean Curvature Gradient

- What's the gradient of total mean curvature with respect to a particular vertex position $f_i$?

$$\nabla_{f_i} \text{mean}(f) = \frac{1}{2} \sum_{ij \in E} \nabla_{f_i} (\ell_{ij} \varphi_{ij}) =$$

$$= \frac{1}{2} \sum_{ij \in E} (\nabla_{f_i} \ell_{ij}) \varphi_{ij} + \ell_{ij} (\nabla_{f_i} \varphi_{ij}) = 0 \text{ (Schläfli)}$$

$$= \frac{1}{2} \sum_{ij \in E} \frac{\varphi_{ij}}{\ell_{ij}} (f_i - f_j)$$

- Agrees with third expression in our sequence:

$$\delta \text{mean}(f) = KN$$
Total Gauss Curvature

• Total Gauss curvature of a discrete surface is sum of angle defects:

\[
\text{Gauss}(f) = \sum_{i \in V} \left( 2\pi - \sum_{ijk} \theta^j_i \right)
\]

• From (discrete) Gauss-Bonnet theorem, we know this sum is always equal to just \(2\pi \chi = 2\pi (V-E+F)\)

• Gradient with respect to motion of any vertex is therefore zero—sequence ends here!
Thanks!

Discrete Differential Geometry: An Applied Introduction
Keenan Crane • CMU 15-458/858B • Fall 2017