DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
CMU 15-458/858 • Keenan Crane
Lecture 21: Geodesics

Discrete Differential Geometry: An Applied Introduction

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Geodesics — Overview

• Geodesics generalize the notion of a “line” to curved spaces

• Two basic features:
  1. **straightest** — no curvature/acceleration
  2. **shortest** — (locally) minimize length

• Can have very different behavior from Euclidean lines!
  • No parallel lines (spherical)
  • Multiple parallel lines through a point (hyperbolic)

• Part of the “origin story” of differential geometry…

• Also important in physics: all of life is motion along a geodesic!
Examples of Geodesics

• Many familiar examples of geodesics:
  • straight line in the plane
  • great arc on circle (airplane trajectory)
  • shortest path in maze (path planning)
  • shortest path in *thickened* graph
  • light paths (gravitational lensing)
Aside: Geodesics on Domains with Boundary

- On domains with boundary, shortest path will not always be along a “straight” curve.
- On the interior, path will still be both shortest & straightest.
- May also “hug” pieces of the boundary (curvature will match boundary curvature, acceleration will match boundary normal).
- (For simplicity, we will mainly consider domains without boundary.)
Isometry Invariance of Geodesics

- *Isometries* are special deformations of curves, surfaces, etc., that don’t change the “intrinsic” geometry, i.e., anything that can be measured using the Riemannian metric $g$.

- For instance, rolling or folding up a map doesn’t change the angle between tangent vectors pointing “north” and “south”.

- Geodesics are also intrinsic: for instance, the shortest path between two cities will not change just because we roll up the map.
• How can we approach a definition of *discrete* geodesics?

• Play “The Game” of DDG and consider different smooth starting points:
  – zero acceleration
  – locally shortest
  – no geodesic curvature
  – harmonic map from interval to manifold
  – gradient of distance function
  – …

• Each starting point will have different consequences

• E.g., for simplicial surfaces will see that *shortest* and *straightest* disagree
Shortest
Locally Shortest Paths

- A Euclidean line segment can be characterized as the shortest path between two distinct points.
- How can we characterize a whole Euclidean line?
- Say that it’s *locally shortest*: for any two “nearby” points on the path*, can’t find a shorter route.
- This description directly gives us one possible definition for (smooth) geodesics.
- Note that *locally shortest* doesn’t imply *globally shortest*! (But still critical points…)

* i.e., within the injectivity radius
Consider an arc-length parameterized planar curve $\gamma(s): [a,b] \rightarrow \mathbb{R}^2$. Its squared length is given by the Dirichlet energy

$$L^2(\gamma) = \int_a^b |d\gamma|^2$$

• We can get the shortest path between two points by minimizing this energy subject to fixed endpoints $\gamma(a) = p$ and $\gamma(b) = q$.

• For planar curves, “setting the derivative to zero” yields a simple 1D Poisson equation.

$$\frac{d^2}{ds^2} \gamma(s) = 0$$

s.t. $\gamma(a) = p$

$\gamma(b) = q$

Q: What’s the solution? Why does it make sense?
Shortest Geodesic—Variational Perspective

• In exactly the same way, we can characterize geodesics on curved manifolds as length-minimizing paths

• E.g., let $M$ be a surface with Riemannian metric $g$, and let $\gamma: [a,b] \rightarrow M$ be an arc-length parameterized curve. Its squared length is again given by the Dirichlet energy

$$L(\gamma) := \int_a^b |d\gamma|^2 = \int_a^b g(d\gamma(\frac{d}{ds}), d\gamma(\frac{d}{ds})) \, ds$$

• Geodesics are still critical points (harmonic)

• But when $M$ is curved, critical points no longer found by solving easy linear equations…

• In general, really need numerical algorithms!
How can we find a shortest path in the discrete case?

Dijkstra’s algorithm obviously comes to mind, but a shortest path in the edge graph is almost never geodesic (even if you refine the mesh!)

One can still start with a Dijkstra path and iteratively shorten local pieces until path is *locally* shortest

However, no reason local shortening should always give a *globally* shortest path…

Discrete Shortest Paths—Vertices

- Even *locally* straightest paths near vertices require some care—behave differently depending on angle defect $\Omega$
- **Flat** ($\Omega = 0$)
  Can lay out in plane; shortest path simply goes straight through vertex
- **Cone** ($\Omega > 0$)
  Always faster to go around one side or the other
- **Saddle** ($\Omega < 0$)
  Always faster to go through the vertex, but not unique!

$$\Omega_i = 2\pi - \sum_{ijk} \theta_{ijk}^i$$
Algorithms for *Shortest* Polyhedral Geodesics

- Algorithms for *shortest* polyhedral geodesics largely based on two closely related methods:
  
1. Mitchell, Mount, Papadimitrou (MMP)
   “*The Discrete Geodesic Problem*” (1986) — $O(n^2 \log n)$

2. Chen & Han (CH)
   “*Shortest Paths on a Polyhedron*” (1990) — $O(n^2)$

- Basic idea: track intervals or “windows” of common geodesic paths

- Great deal of work on improving efficiency by pruning windows, approximation, … though still fairly expensive.

- Good intro in Surazhsky et al.
  “*Fast Exact and Approximate Geodesics on Meshes*” (2005)
Shortest Geodesics—Smooth vs. Discrete

- **Smooth**: two minimal geodesics $\gamma_1, \gamma_2$ from a source $p$ to distinct points $p_1, p_2$ (resp.) intersect only if $\gamma_1 \subseteq \gamma_2$ or $\gamma_2 \subseteq \gamma_1$

- **Discrete**: many geodesics can coincide at saddle vertex ("pseudo-source")

N.B. Shortest polyhedral geodesics may not faithfully capture behavior of smooth ones!
Closed Geodesics

• **Theorem.** (Birkhoff 1917) Every smooth convex surface contains a simple closed geodesic, *i.e.*, a geodesic loop that does not cross itself ("Birkhoff equator")

• **Theorem.** (Luysternik & Shnirel’man 1929) Actually, there are at least three—and this result is sharp (*only* three on some smooth surfaces).

• **Theorem.** (Galperin 2002) Most convex polyhedra do not have simple closed geodesics (in the sense of discrete *shortest* geodesics).

• *Shortest* characterization of discrete geodesics again fails to capture properties from smooth setting…

A *shortest* discrete geodesic can’t pass through convex vertices; by discrete Gauss-Bonnet, has to partition vertices into two sets that each have total angle defect of exactly $2\pi$. 
Cut Locus

- Given a source point $p$ on a smooth surface $M$, the cut locus is the set of all points $q$ such that there is not a unique (globally) shortest geodesic between $p$ and $q$.

- E.g., on a sphere the cut locus of any point $+p$ is just the antipodal point $-p$.

- In general can be much more complicated…

Image credit: S. Markvorsen and P.G. Hjorth (The Cut Locus Project)
Discrete Cut Locus

- What does cut locus look like for polyhedral surfaces?

- Recall that it’s always shorter to go “around” a cone-like vertex (i.e., vertex with positive curvature $\Omega_i > 0$)

- Hence, polyhedral cut locus will contain every cone vertex in the entire surface

- Can look very different from smooth cut locus!

- E.g., sphere vs. polyhedral sphere?

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Image adapted from Itoh & Sinclair, “Thaw: A Tool for Approximating Cut Loci on a Triangulation of a Surface”
Medial Axis

• Similar to the cut locus, the *medial axis* of a curve or surface $M \subset \mathbb{R}^n$ is the set of all points $q$ that do not have a unique closest point on $M$

• A *medial ball* is a point on the medial axis, with radius given by the distance to the closest point

• Typically three branches (*why?*)

• Provides a “dual” representation: can recover original shape from
  – medial axis
  – radius function
**Discrete Medial Axis**

- What does the medial axis of a discrete domain look like?
- Let’s start with a square. (What did the medial axis for a circle look like?)
- What about a rectangle? (What did an ellipse look like?)
- How about a nonconvex polygon?
Discrete Medial Axis

- In general, medial axis touches *every* convex vertex
- May not look much like true (smooth) medial axis!
- One idea: “filter” using radius function…
  - still hard to say exactly which pieces should remain
  - lots of work on alternative “shape skeletons” for discrete curves & surfaces
Computing the Medial Axis

• Many algorithms for computing/approximating medial axis & other “shape skeletons”

• One line of thought: use Voronoi diagram as starting point:
  – densely sample boundary points
  – compute Voronoi diagram
  – keep “short” facets of tall/skinny cells

• Works in 2D, 3D, …

• Very similar algorithm gives surface reconstruction from points

Amenta et al, “A New Voronoi-Based Surface Reconstruction Algorithm”
Medial Axis—Applications

• Many applications of medial axis
  • shape skeletons
  • local feature size
  • fast collision detection
  • fluid particle re-seeding
  • …

(1) Giesen et al, “The Scale Axis Transform”
(2) Adams et al, “Adaptively Sampled Particle Fluids”
(3) Peters & Ledoux, “Robust approximation of the Medial Axis Transform of LiDAR point clouds”
(4) Bradshaw & Sullivan, “Adaptive Medial-Axis Approximation for Sphere-Tree Construction”
Straightest
Straightest Paths

• A Euclidean line can be characterized as a curve that is “as straight as possible”

• **Q:** How can we make this statement more precise?
  
  • geometrically: no curvature
  
  • dynamically: no acceleration

• How can we generalize to curves in manifolds?
  
  • geometrically: no geodesic curvature
  
  • dynamically: zero covariant derivative
Straightness—Geometric Perspective

• Consider a curve \( \gamma(s) \) with tangent \( T \) in a surface with normal \( N \), and let \( B := T \times N \).

• Can decompose “bending” into normal curvature \( \kappa_n \) and geodesic curvature \( \kappa_g \):

\[
\kappa_n := \langle N, \frac{d}{ds} T \rangle \\
\kappa_g := \langle B, \frac{d}{ds} T \rangle
\]

• Curve is “forced” to have normal curvature due to curvature of \( M \)

• Any additional bending beyond this minimal amount is geodesic curvature

• Geodesic is curve such that \( \kappa_g = 0 \)
Discrete Curves on Discrete Surfaces

• To understand straightest curves on discrete surfaces, first have to define what we mean by a discrete curve.

• One definition: a discrete curve in a simplicial surface $M$ is any continuous curve $\gamma$ that is piecewise linear in each simplex.

• Doesn’t have to be a path of edges: could pass through faces, have multiple vertices in one face, …

• Practical encoding: sequence of $k$-simplices (not all same dimension), and barycentric coordinates for each simplex.

<table>
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<th>barycentric coordinates</th>
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<td>$ilj$</td>
<td>(.1, .7, .2)</td>
</tr>
<tr>
<td>$ij$</td>
<td>(.45, .55)</td>
</tr>
<tr>
<td>$ijk$</td>
<td>(.40, .15, .45)</td>
</tr>
<tr>
<td>$k$</td>
<td>(1)</td>
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</table>
Discrete Geodesic Curvature

• For planar curve, one definition of discrete curvature was **exterior angle** (or \(\pi\)-interior)

• Since most points of a simplicial surface are **intrinsically** flat, can adopt this same definition for discrete geodesic curvature

• **Faces**: just measure angle between segments

• **Edges**: “unfold” and measure angle

• **Vertices**: not as simple—can’t unfold!

• Recall trouble w/ **shortest** geodesics…
Discrete Straightest Geodesics

- In the smooth setting, characterized geodesics as curves with zero geodesic curvature.
- In the discrete setting, have a hard time defining geodesic curvature at vertices.
- Alternative smooth characterization: just have the same angle on either side of the curve.
- Translates naturally to the discrete setting: equal angle sum on either side of the curve.
- Provides definition of discrete *straightest* geodesics (Polthier & Schmies 1998).

Image adapted from Polthier & Schmies, “Straightest Geodesics on Polyhedral Surfaces”
Exponential Map

- At a point $p$ of a smooth surface $M$, the exponential map $\exp_p: T_pM \to M$ takes a tangent vector $X$ to the point reached by walking along a geodesic in the direction $X/|X|$ for distance $|X|$.

- Can also view as a map “wrapping” the tangent plane around the surface.

- Q: Is this map surjective? Injective?

- Injectivity radius at $p$ is radius of largest ball where $\exp_p$ is injective.
Discrete Exponential Map

• Not so hard to evaluate exponential map on discrete surface

• Given point and tangent vector, start walking along vector

• “walking” amounts to 2D ray tracing

• At vertices, straightest definition tells us how to continue

• (Still have to think about what it means to start at a vertex—what are tangent vectors?)

• Q: How big is the injectivity radius?

• A: Just the distance to the closest vertex!

• Q: Is the discrete exponential map surjective?

• A: No! Consider a saddle vertex…
Straightness—Dynamic Perspective

- Dynamically, geodesic has zero **tangential acceleration**
- How exactly do we define “tangential acceleration”?
- Consider curve $\gamma(t) : [a,b] \rightarrow M$ *(not necessary arc-length parameterized)*
- Tangential *velocity* is simply the tangent to the curve
- Tangential acceleration should be something like the “change in the tangent,” but:
  - **extrinsically**, change in tangent is not a tangent vector
  - **intrinsically**, tangents belong to different vector spaces
- So, how do we measure acceleration?
Covariant Derivative

• Since geodesics are intrinsic, can define “straightness” using only the metric $g$
• Covariant derivative $\nabla$ measures the change of one tangent vector field along another.
• For any function $\phi$, tangent vector fields $X, Y, Z$, operator $\nabla$ uniquely determined by

\[
\begin{align*}
\nabla_Z (X + Y) &= \nabla_Z X + \nabla_Z Y \\
\nabla_{X+Y} Z &= \nabla_X Z + \nabla_Y Z \\
\nabla_{fX} Y &= f \nabla_X Y \\
\nabla_X (fY) &= df(X)Y + f \nabla_X Y
\end{align*}
\]

\[
\nabla_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\]

Can really “solve” these equations for $\nabla$ in terms of $g$ (Christoffel symbols). We won’t!
Covariant derivative provides another, quite classic characterization of geodesics:

\[ \nabla \gamma \dot{\gamma} = 0 \]

“tangent doesn’t turn”

Q: Does this characterization suggest another approach to discrete geodesics?
A: Maybe—though to go down that road we’ll need discrete connections (later…)
Geodesics—Shortest vs. Straightest, Smooth vs. Discrete

- In smooth setting, several equivalent characterizations:
  - shortest (harmonic)
  - straightest (zero curvature, zero acceleration)
- In discrete setting, characterizations no longer agree!
  - **shortest** natural for boundary value problem
  - **straightest** natural for initial value problem
  - **convex**: shortest paths are straightest (but not vice versa)
  - **nonconvex**: shortest may not even be straightest! (saddles)
- **Neither** definition faithfully captures all smooth behavior:
  - (shortest) cut locus/medial axis touches *every* convex vertex
  - (straightest) exponential map is not surjective
- Use the right tool for the job (*and look for other definitions!*)

Thanks!

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