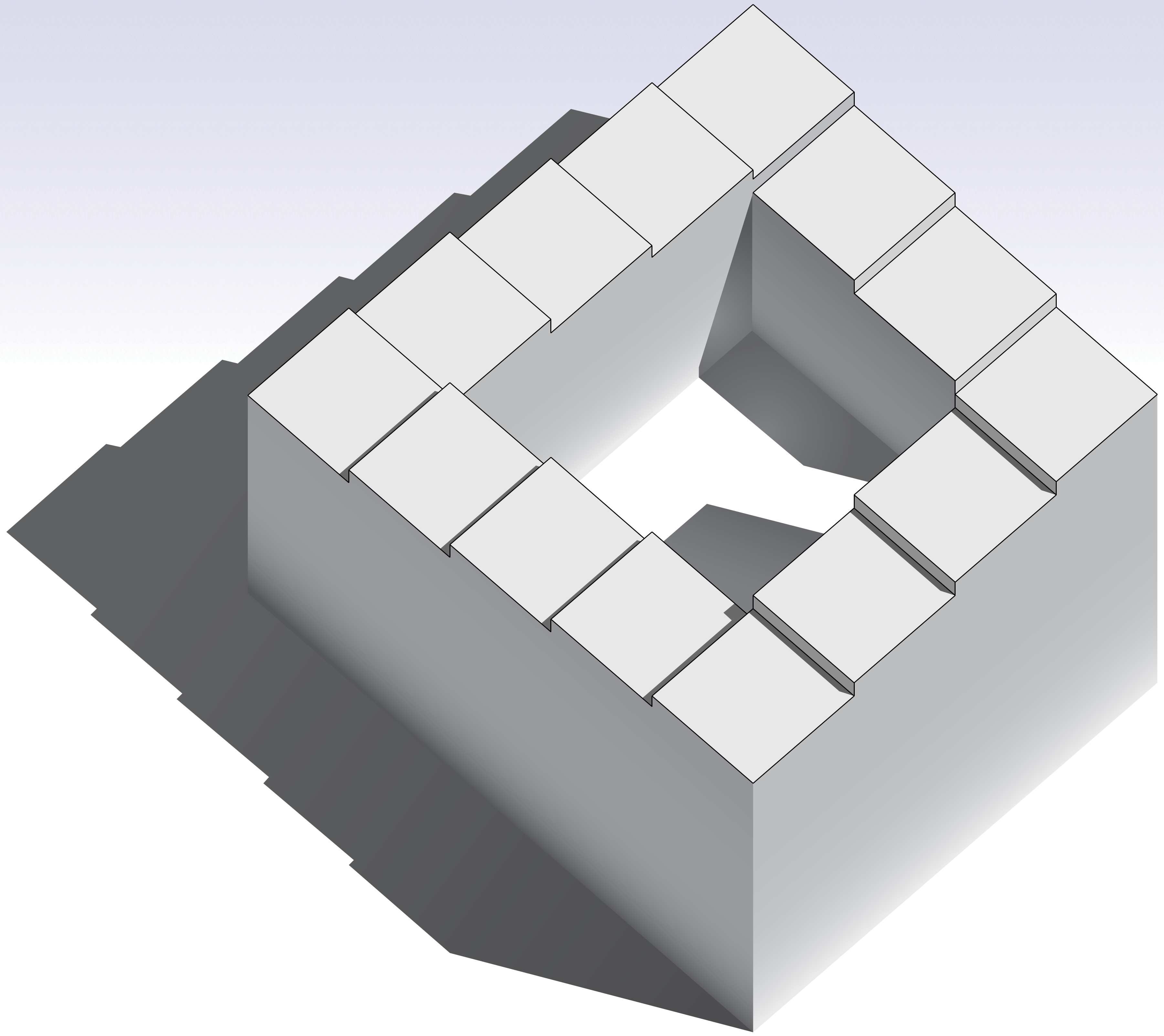
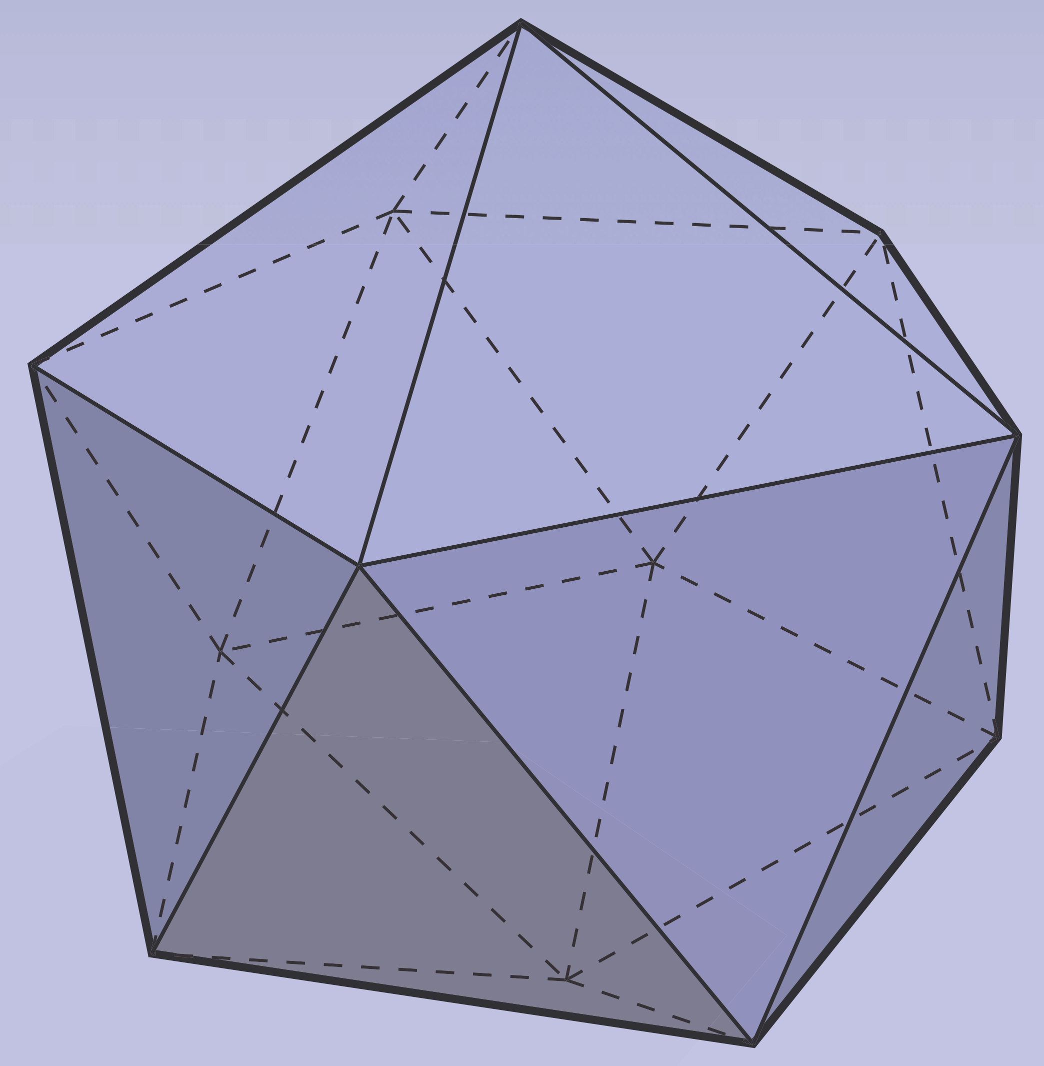


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858

LECTURE 4:

k-VECTORS AND k-FORMS



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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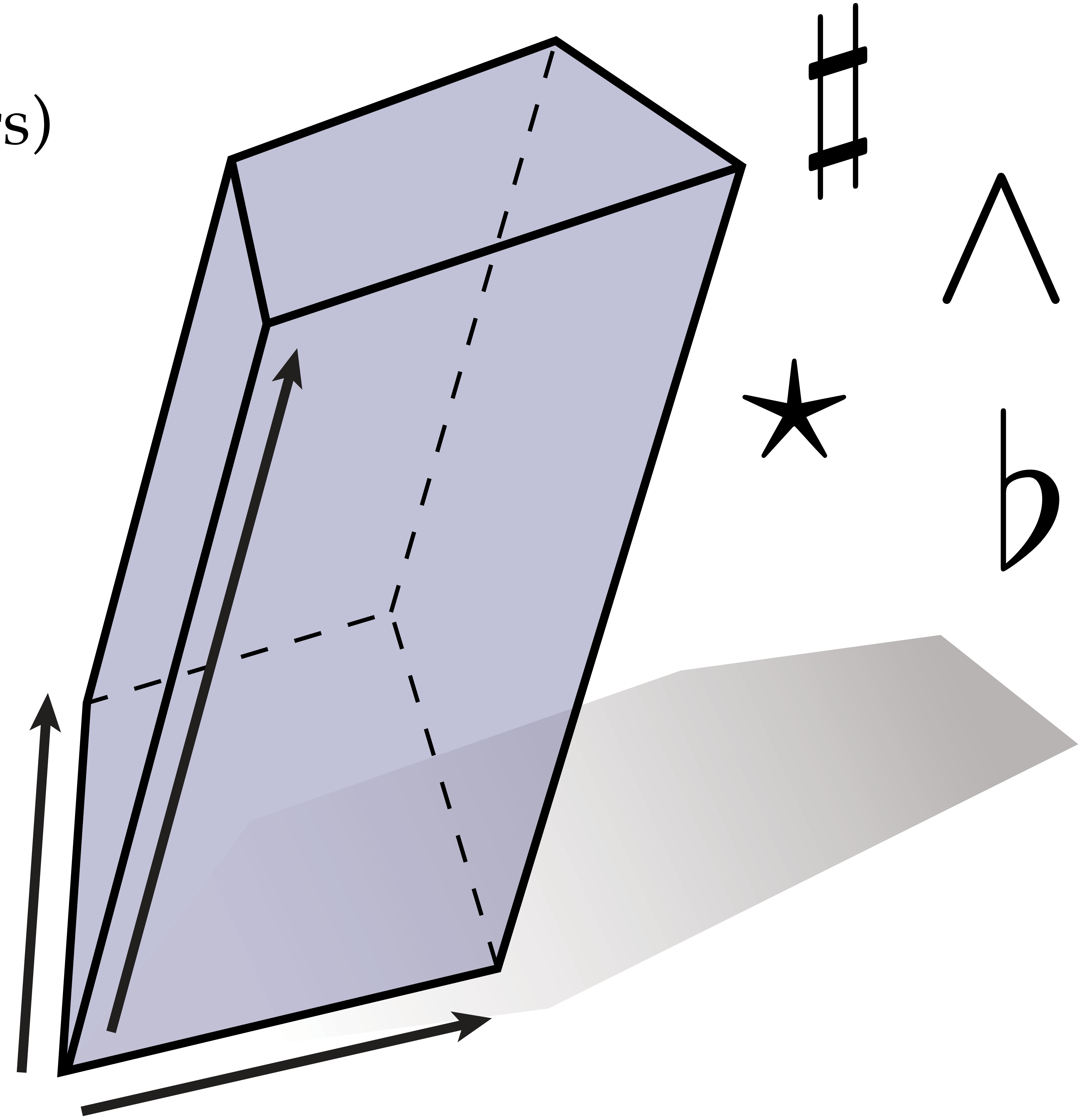
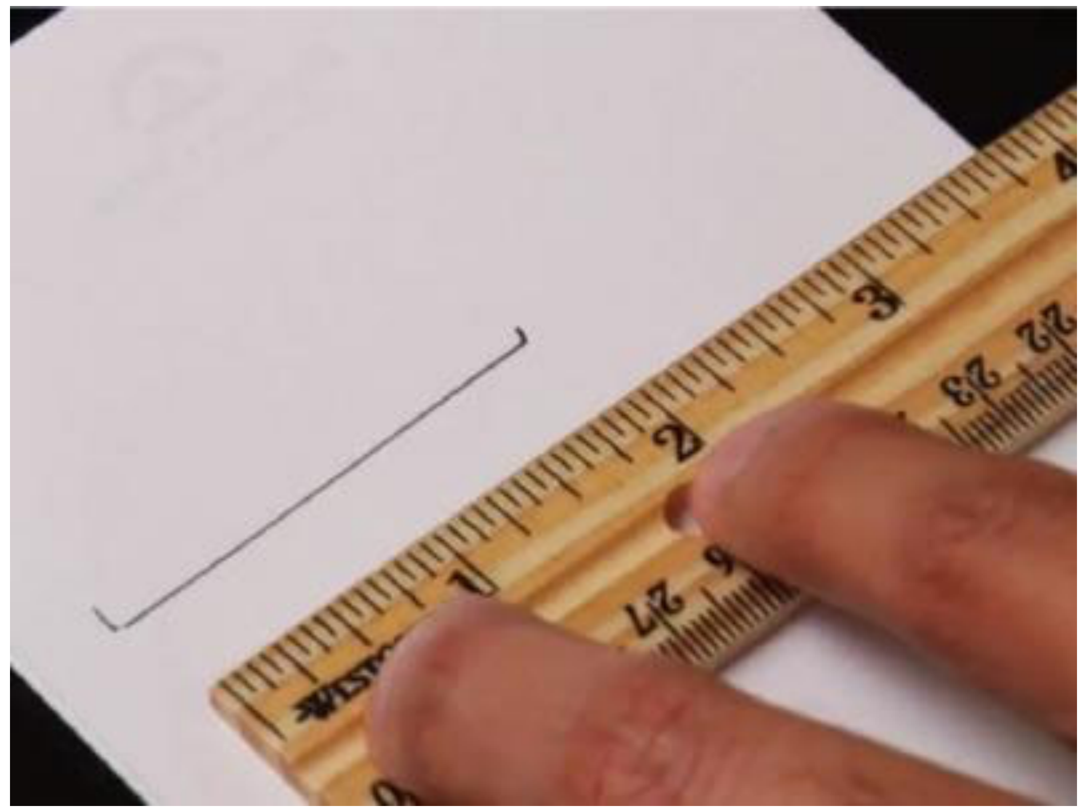
k-Vectors and k-Forms — Overview

- *Last time:*
  - **Exterior algebra**—"little volumes" (k-vectors)
- *Where we're headed:*
  - **Exterior calculus**—how do lengths, areas, volumes change over curved surfaces?
  - Essential tool for modeling nature!
- *Today:*
  - Focus on how to *measure* little volumes
  - Key idea: volumes are measured by other volumes!
  - Will call such volumes "*k*-forms"

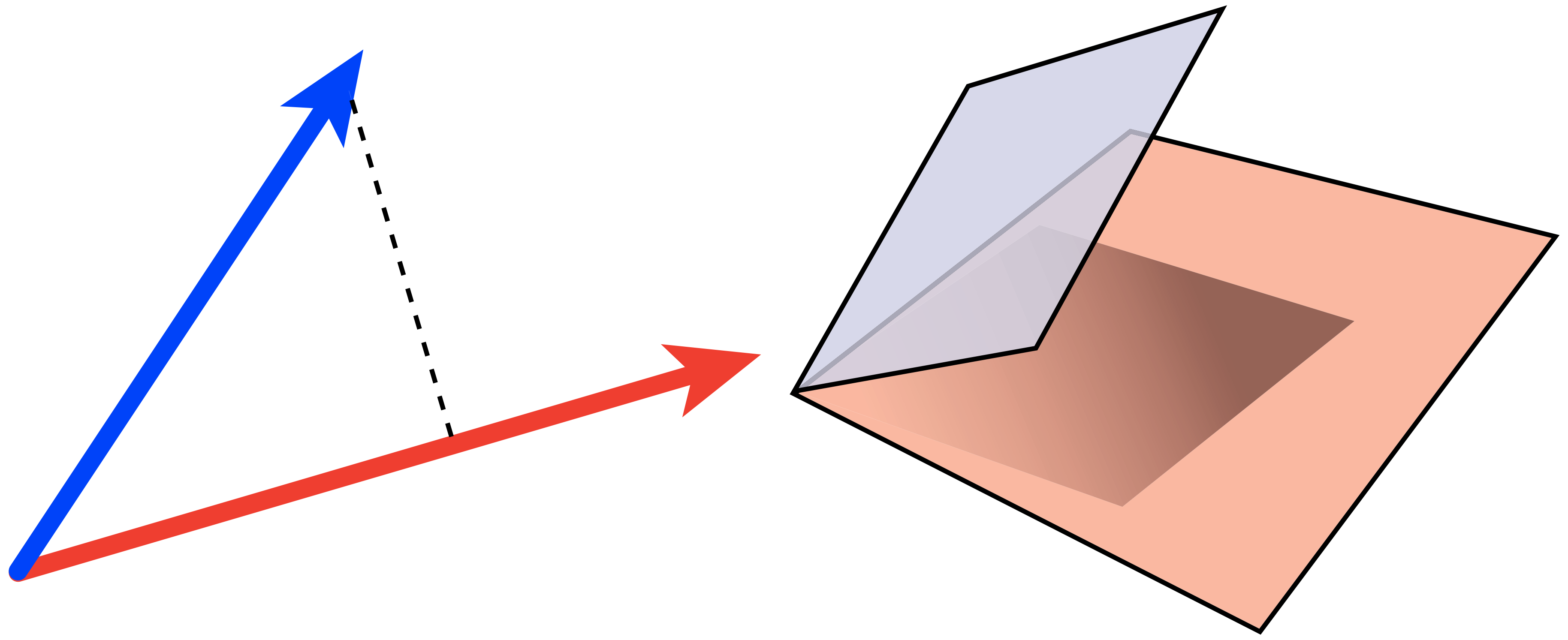


Measurement and Duality

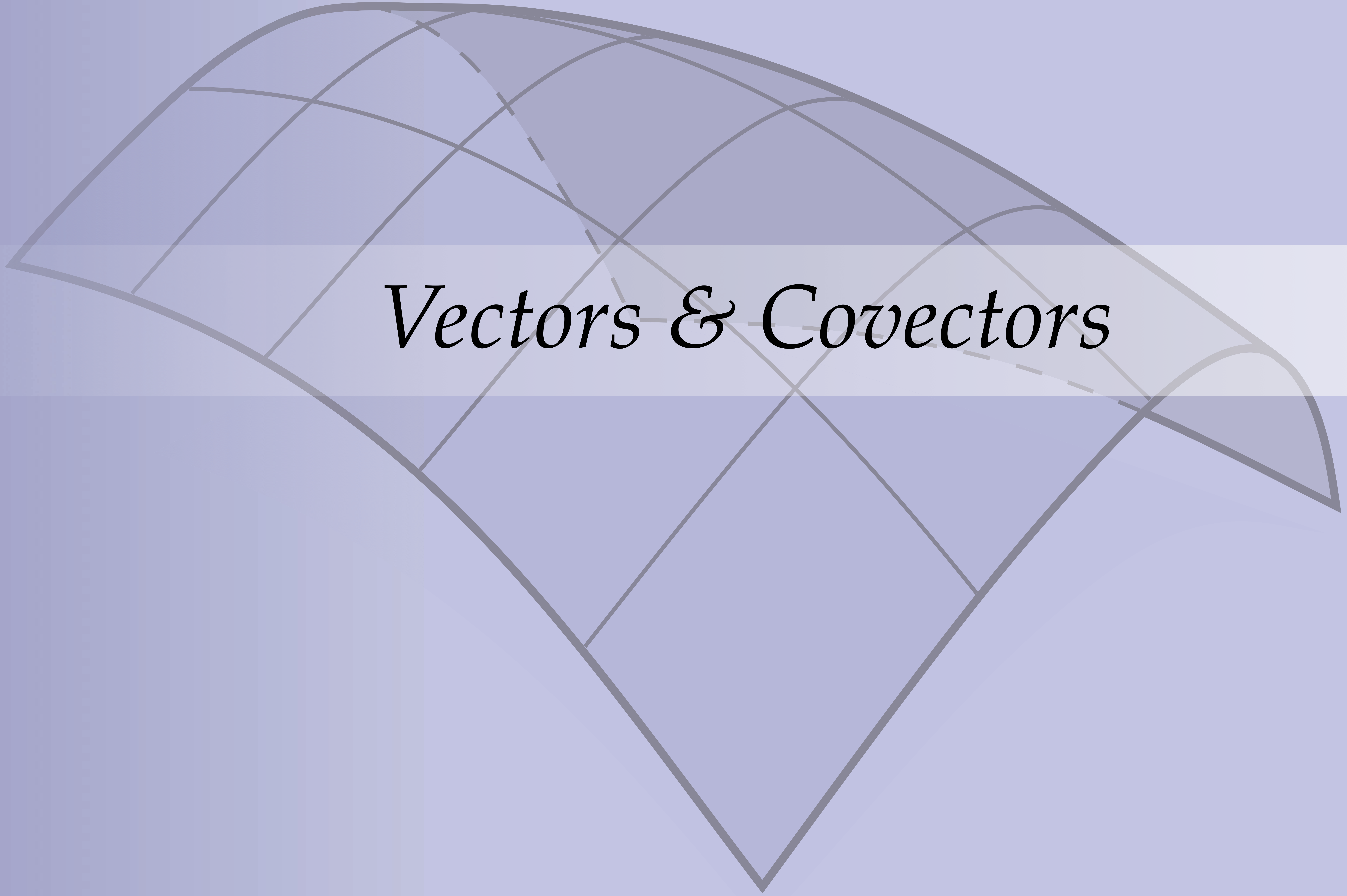


- Measurement devices have the same dimension as the thing they're measuring:
  - to measure length, use something one-dimensional (ruler, string, etc.)

  - to measure volume, use something three-dimensional (e.g., liquid measure)
  - etc.
- Same idea shows up in linear algebra, exterior calculus:
  - a vector can be "paired" with a *dual* vector to get a measurement
  - a *k*-dimensional volume can be "paired" with a *dual* *k*-dimensional volume to get a measurement

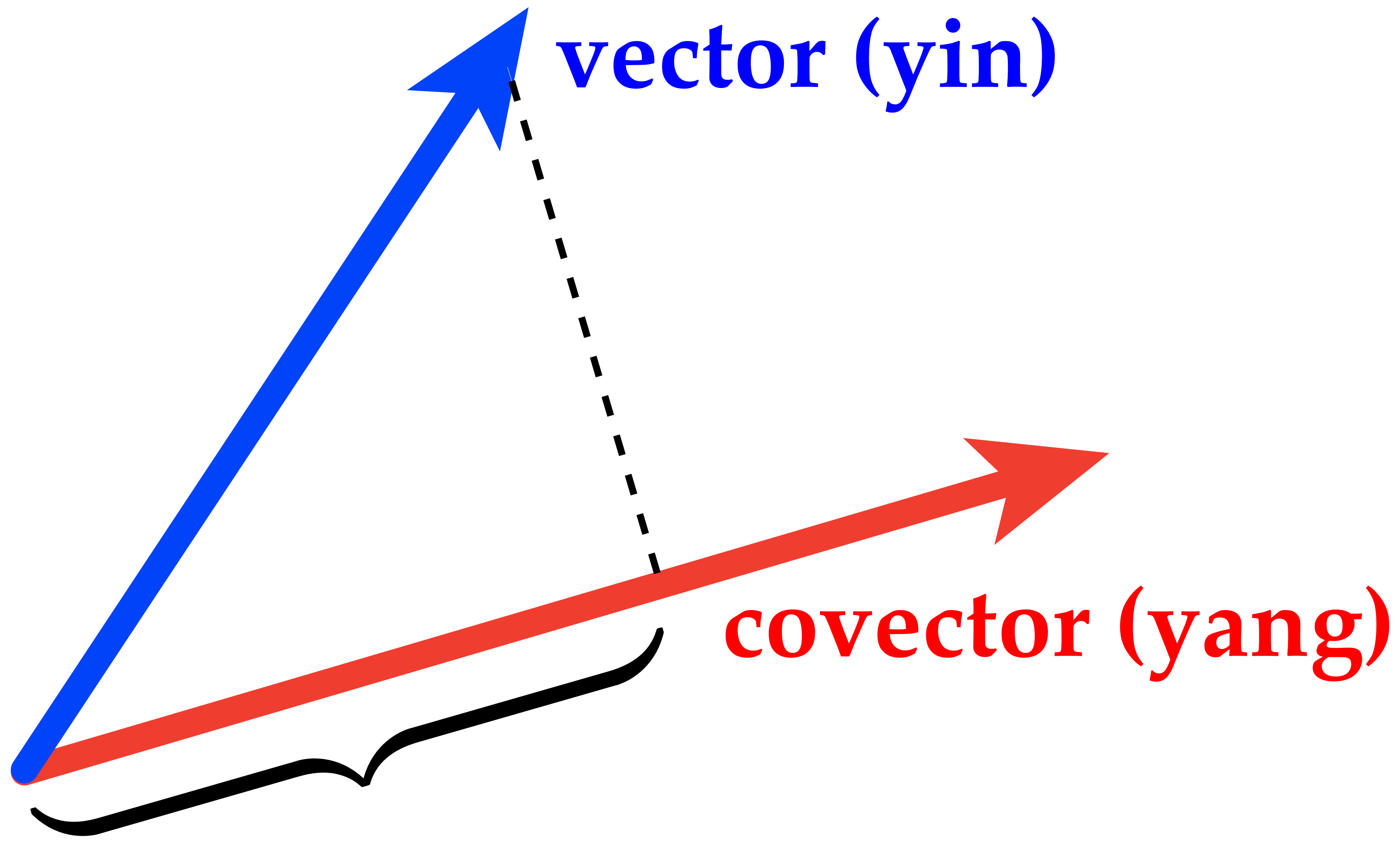


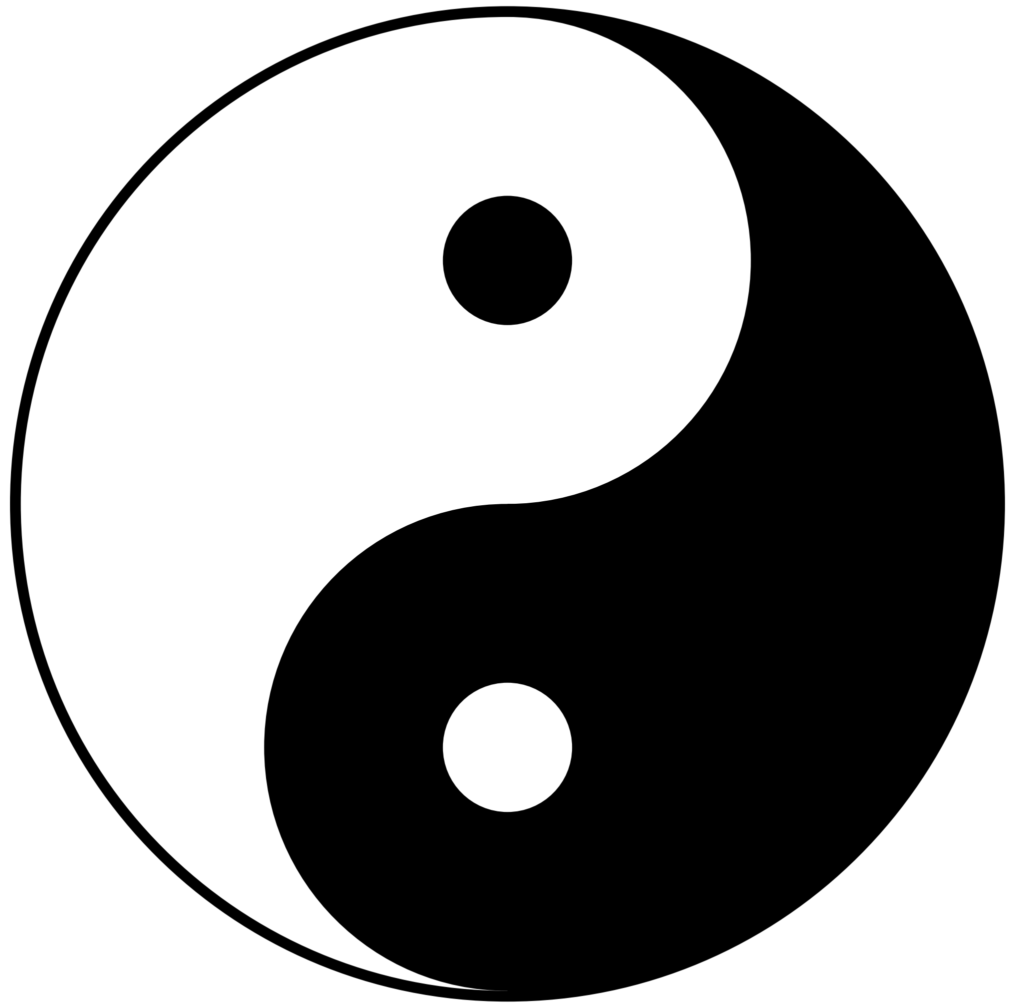
Vectors & Covectors



Vector-Covector Duality

- Much of the expressivity of exterior calculus comes from the relationship between *vectors* and *covectors*.
- Loosely speaking:
  - **covectors** are objects that *"measure"*
  - **vectors** are objects that *"get measured"*
- We say that vectors and covectors are "dual"—roughly speaking, the are two sides of the same coin (but with important differences…).
- Ultimately, differential forms will be built up by taking wedges of covectors (rather than vectors).





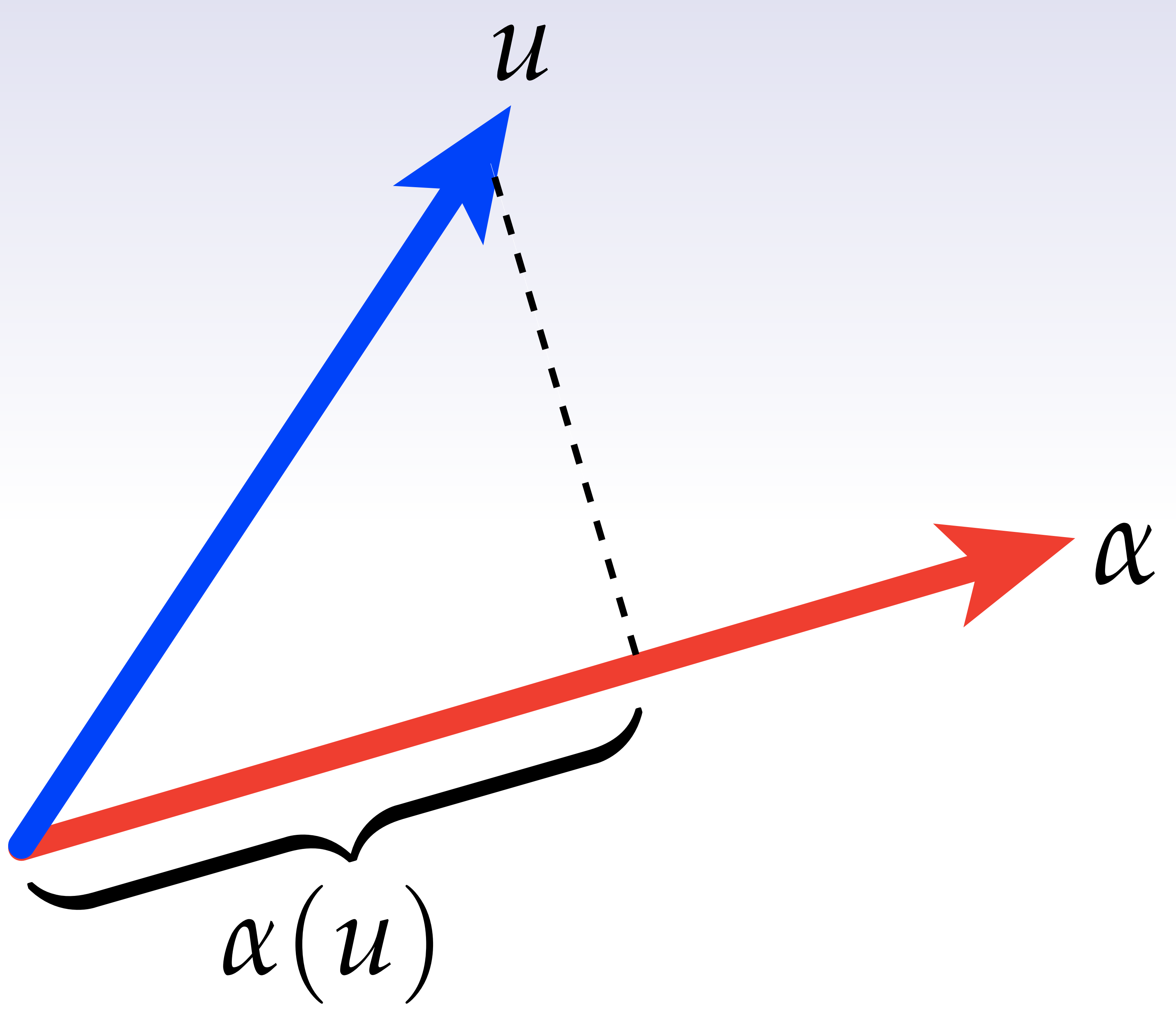
Analogy: Row & Column Vectors

In matrix algebra, we make a distinction between *row vectors* and *column vectors*:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Q: Why do we make the distinction? What does it mean geometrically? What does it mean as a linear map? *(Is this convention useful?)*

Vectors and Covectors



Analogy: *row vs. column vector*

Key idea: a covector *measures* length of vector along a particular direction

Dual Space & Covectors

Definition. Let V be any real vector space. Its *dual space* V^* is the collection of all linear functions $\alpha : V \to \mathbb{R}$ together with the operations of *addition*

$$(\alpha + \beta)(u) := \alpha(u) + \beta(u)$$

and *scalar multiplication*

$$(c\alpha)(u) := c(\alpha(u))$$

for all $\alpha, \beta \in V^*$, $u \in V$, and $c \in \mathbb{R}$.

Definition. An element of a dual vector space is called a *dual vector* or a *covector*.

(**Note:** unrelated to *Hodge dual*!)

Covectors — Example (R^3)

- As a concrete example, let's consider the vector space $V = R^3$
- Recall that a map f is *linear* if for all vectors **u**, **v** and scalars a, we have

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \qquad \textbf{and} \qquad f(a\mathbf{u}) = af(\mathbf{u})$$

- **Q:** What's an example of a *linear* map from R^3 to R?
  - Suppose we express our vectors in coordinates $\mathbf{u} = (x,y,z)$
- **A:** One of *many* possible examples: $f(x,y,z) = x + 2y + 3z$
- **Q:** What are *all* the possibilities?
- **A:** They all just look like $f(x,y,z) = ax + by + cz$ for constants a,b,c
- In other words in Euclidean R^3, a *covector* looks like just another 3-vector!

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

vector

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

covector

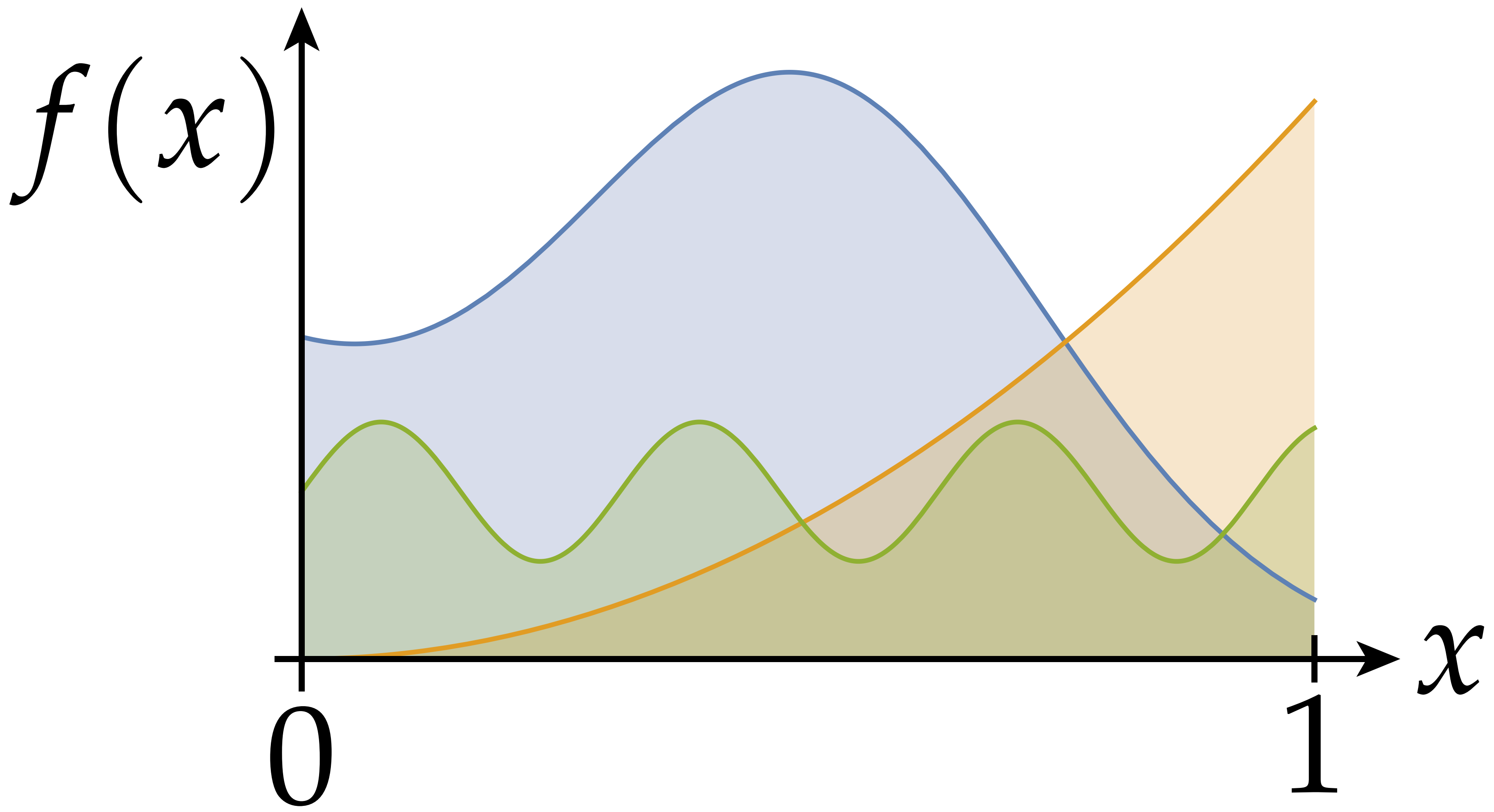
Covectors — Example (Functions)

- If covectors are just the same as vectors, why even bother?
- Here's a more interesting example:

Example. Let V be the set of integrable functions $f : [0,1] \to \mathbb{R}$, and consider maps

- $\phi : V \to \mathbb{R}; f \mapsto \int_0^1 f(x)\,dx$
- $\delta : V \to \mathbb{R}; f \mapsto f(0)$

Is V a vector space? Are ϕ and δ covectors?



Key idea: the difference between primal & dual vectors is not merely superficial!

Sharp and Flat

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xrightarrow{\mathsf{T}} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u, v \xrightarrow{\flat} u^\flat(v)$$

$$\alpha, \beta \xrightarrow{\sharp} \alpha(\beta^\sharp)$$

Analogy: *transpose*

(What's up with the musical symbols? Will see in a bit…)

Sharp and Flat w/ Inner Product

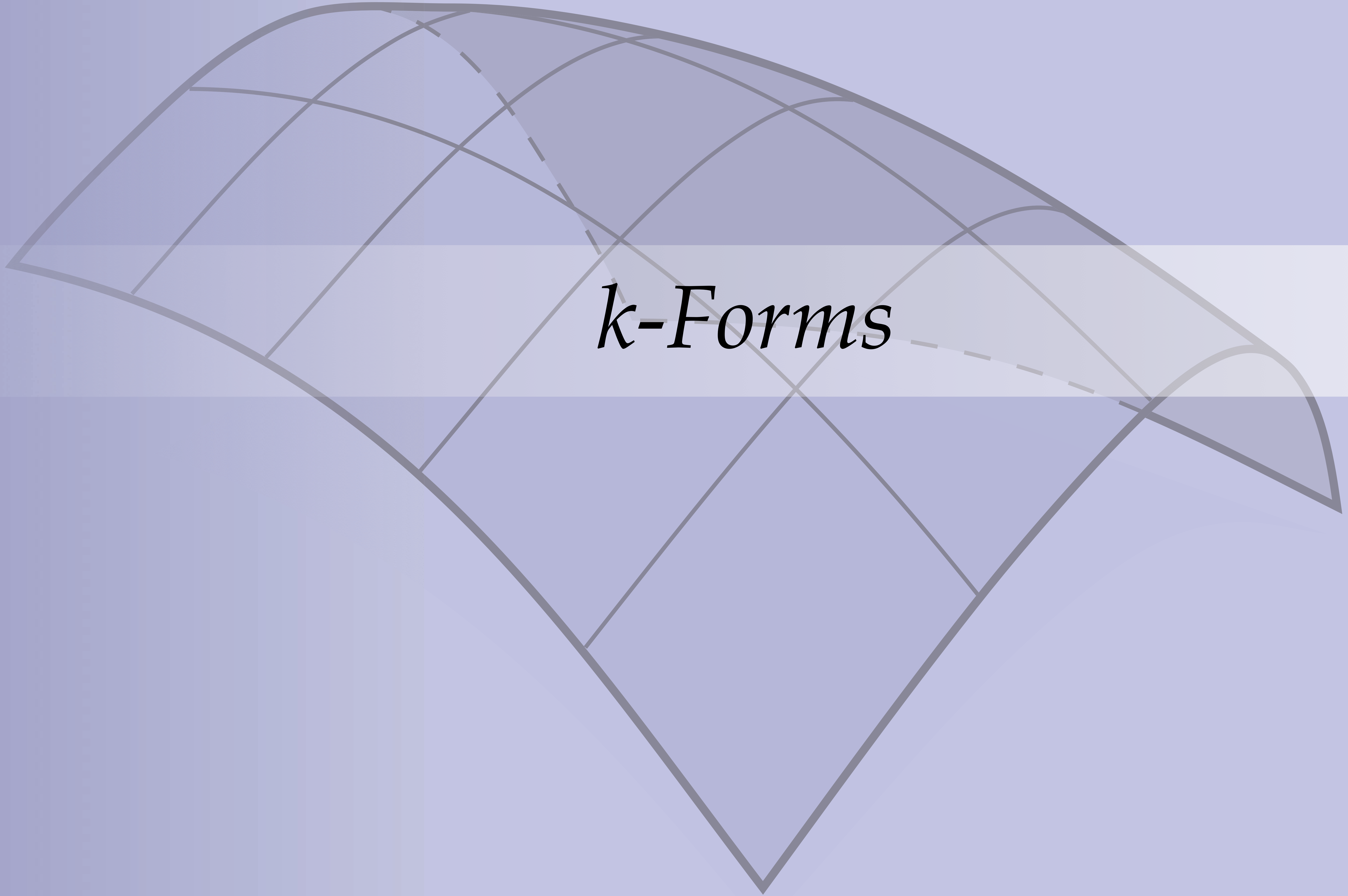
$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u^\flat(v) = u^\mathsf{T} M v \qquad \Longleftrightarrow \qquad u^\flat(\cdot) = \langle u, \cdot \rangle$$

$$\alpha(\beta^\sharp) = \alpha M^{-1} \beta^\mathsf{T} \qquad \Longleftrightarrow \qquad \langle \alpha^\sharp, \cdot \rangle = \alpha(\cdot)$$

Basic idea: applying the flat of a vector is the same as taking the inner product; taking the inner product w/ the sharp is same as applying the original 1-form.

k-Forms



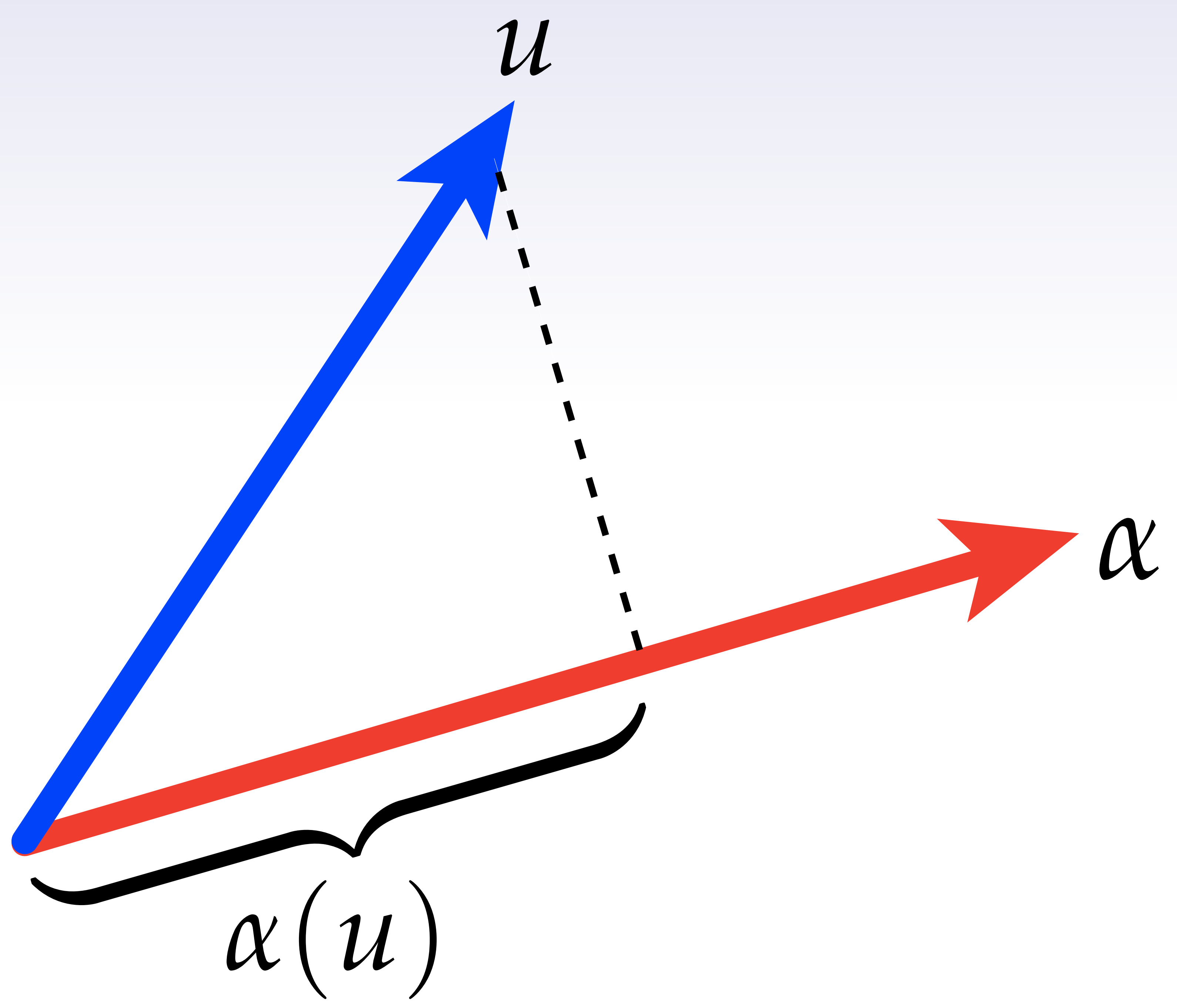
Covectors, Meet Exterior Algebra

- So far we've studied two distinct concepts:
  - **exterior algebra**—build up "volumes" from vectors
  - **covectors**—linear maps from vectors to scalars
- Combine to get an *exterior algebra of covectors*
  - Will call these objects *k-forms*
  - Just as a covector measures vectors…
  - …a *k*-form will *measure* *k*-vectors.
  - In particular, measurements will be **multilinear**, *i.e.*, linear in each 1-vector.

	primal	**dual**
vector space	vectors	covectors
exterior algebra	*k*-vectors	*k*-forms

Measurement of Vectors

Geometrically, what does it mean to take a **multilinear** measurement of a 1-vector?

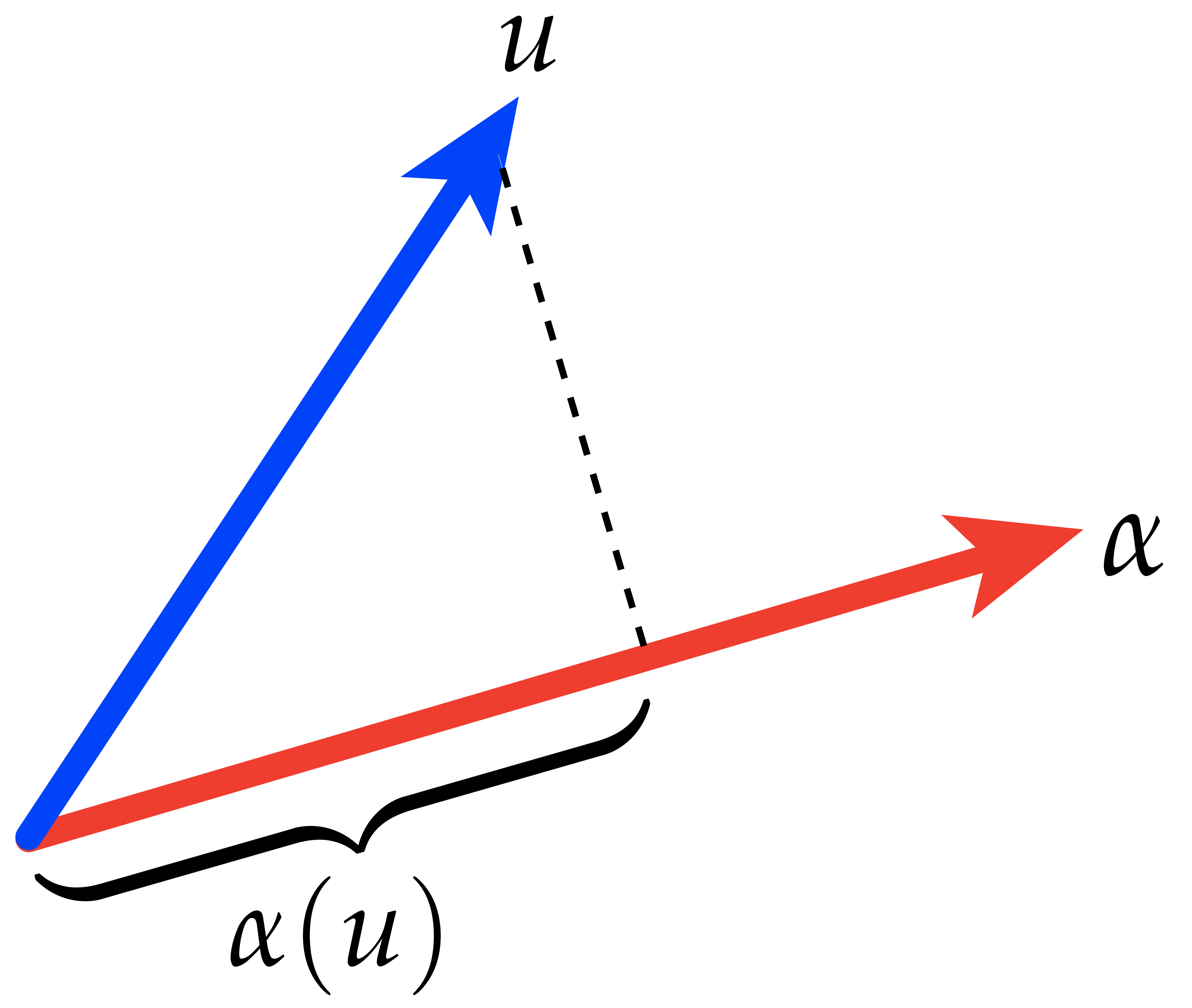


Observation: only thing we can do is measure extent along some other vector.

Computing the Projected Length

- Concretely, how do we compute projected length of one vector along another?
  - If α has unit norm then, we can just take the usual *dot product*
  - Since we think of u as the vector *"getting measured"* and α as the vector *"doing the measurement"*, we'll write this as a function $\alpha(u)$:

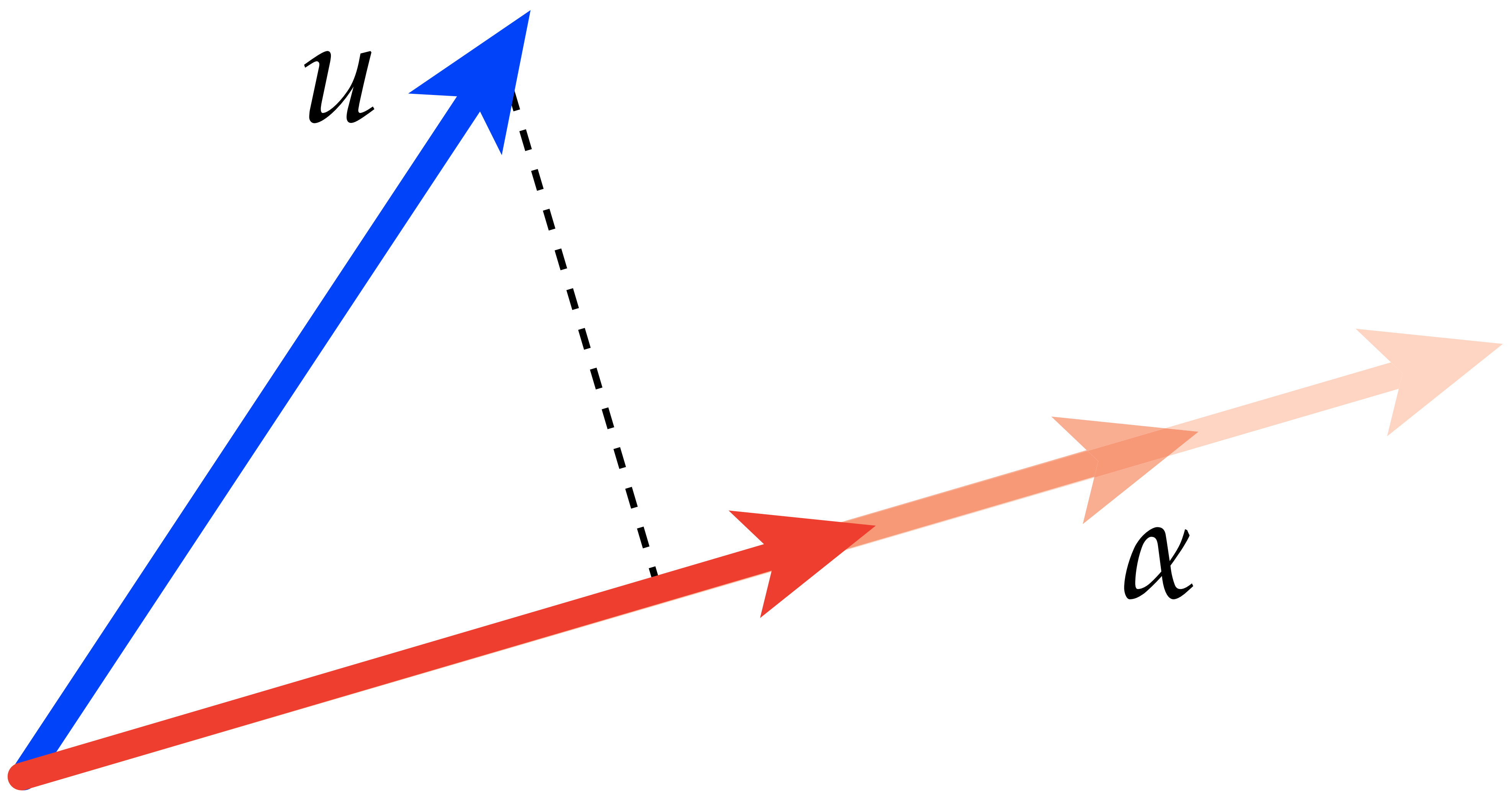
$$\alpha(u) = \sum_{i=1}^{n} \alpha_i u^i$$



1-form

We can of course apply this same expression when α does not have unit length:

$$\alpha(u) := \sum_i \alpha_i u^i$$

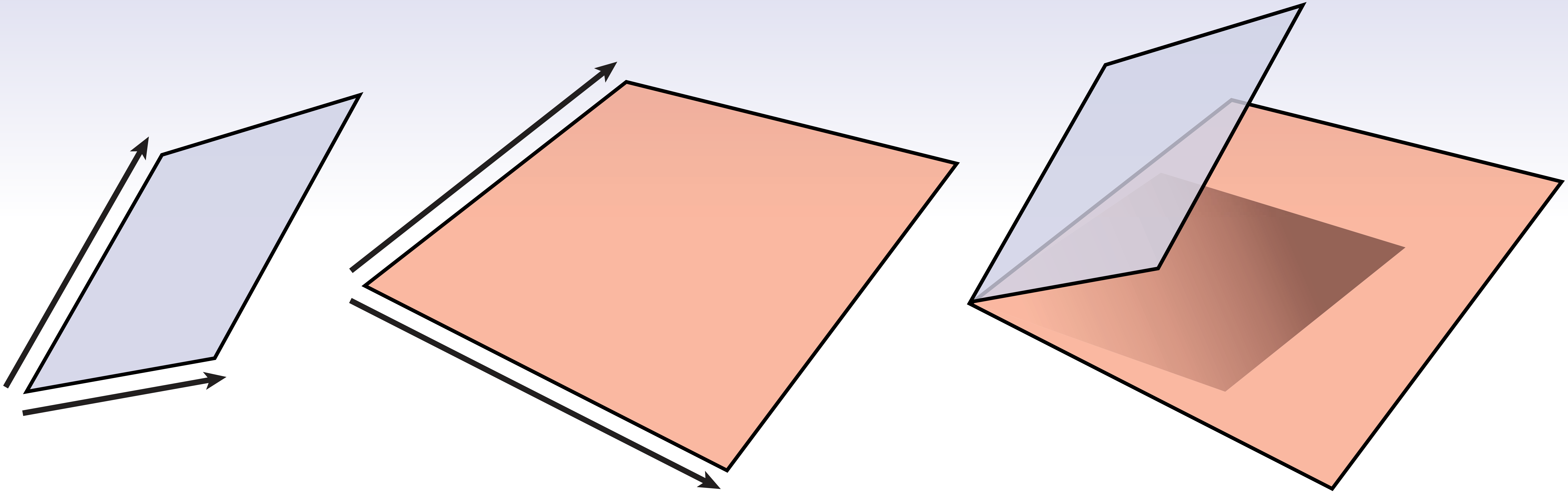


Interpretation?

Projected length gets scaled by magnitude of α.

Measurement of 2-Vectors

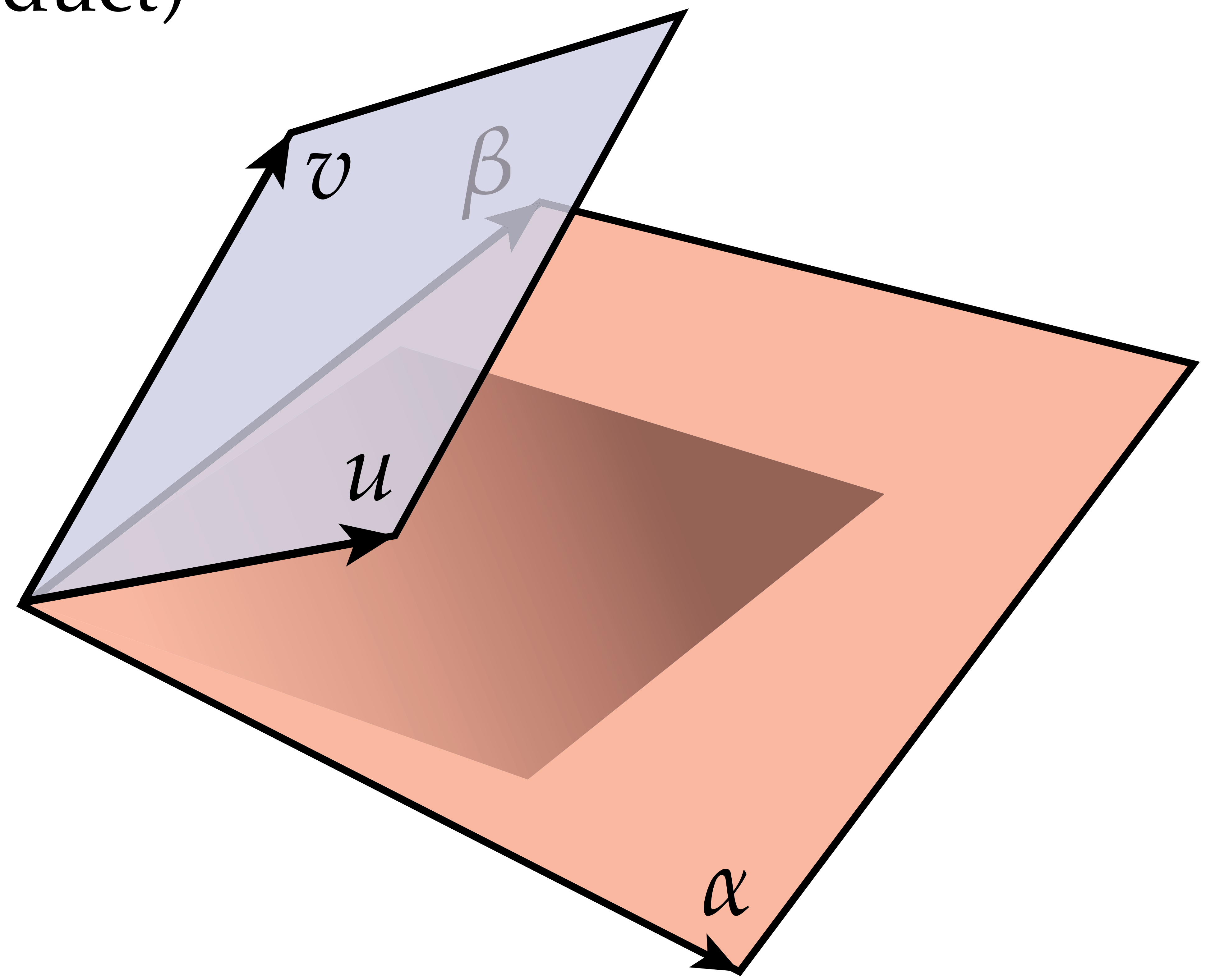
Geometrically, what does it mean to take a **multilinear** measurement of a 2-vector?



Intuition: size of “shadow” of one parallelogram on another.

Computing the Projected Area

- Concretely, how do we compute projected area of a parallelogram onto a plane?
  - First, project vectors defining parallelogram (u,v)
  - Then apply standard formula for area (cross product)
- Suppose for instance α,β are orthonormal:



Projection

$$u \mapsto (\alpha(u), \beta(u))$$

$$v \mapsto (\alpha(v), \beta(v))$$

Area

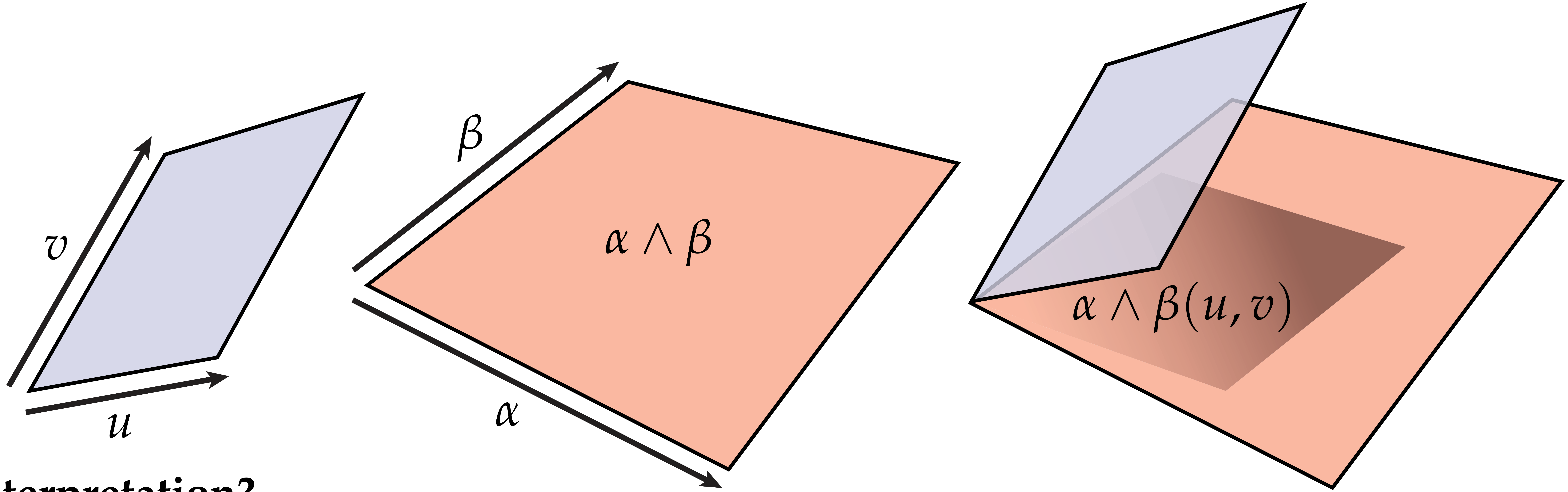
$$\alpha(u)\beta(v) - \alpha(v)\beta(u)$$

(Notice that in the projection we are treating α,β as 1-forms.)

2-form

We can of course apply this same expression when α,β are not orthonormal:

$$(\alpha \wedge \beta)(u, v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)$$



Interpretation?

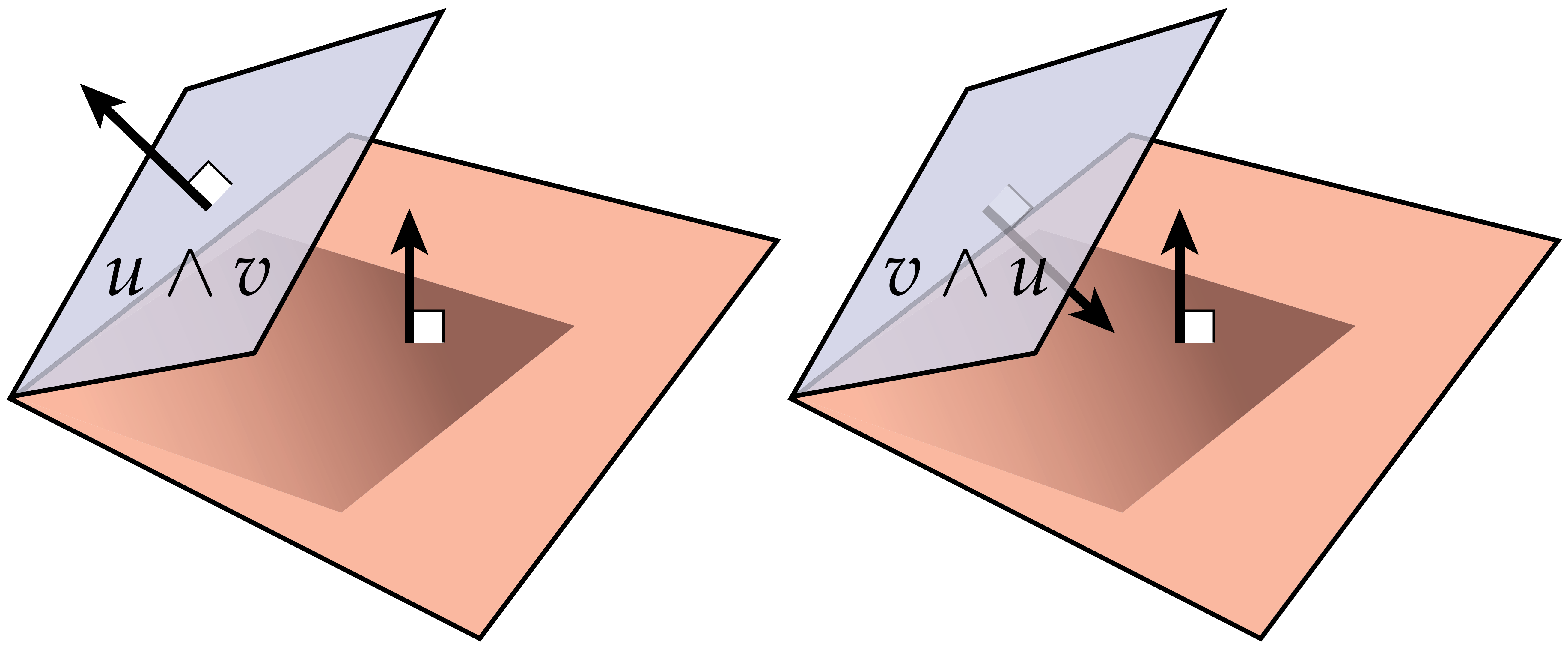
Projected area of u,v gets scaled by area of parallelogram with edges α,β.

Antisymmetry of 2-Forms

Notice that exchanging the arguments of a 2-form reverses sign:

$$\begin{aligned}(\alpha \wedge \beta)(v,u) &= \alpha(v)\beta(u) - \alpha(u)\beta(v) \\ &= -(\alpha(u)\beta(v) - \alpha(v)\beta(u)) \\ &= -(\alpha \wedge \beta)(u,v)\end{aligned}$$

Q: What does this *antisymmetry* mean geometrically?



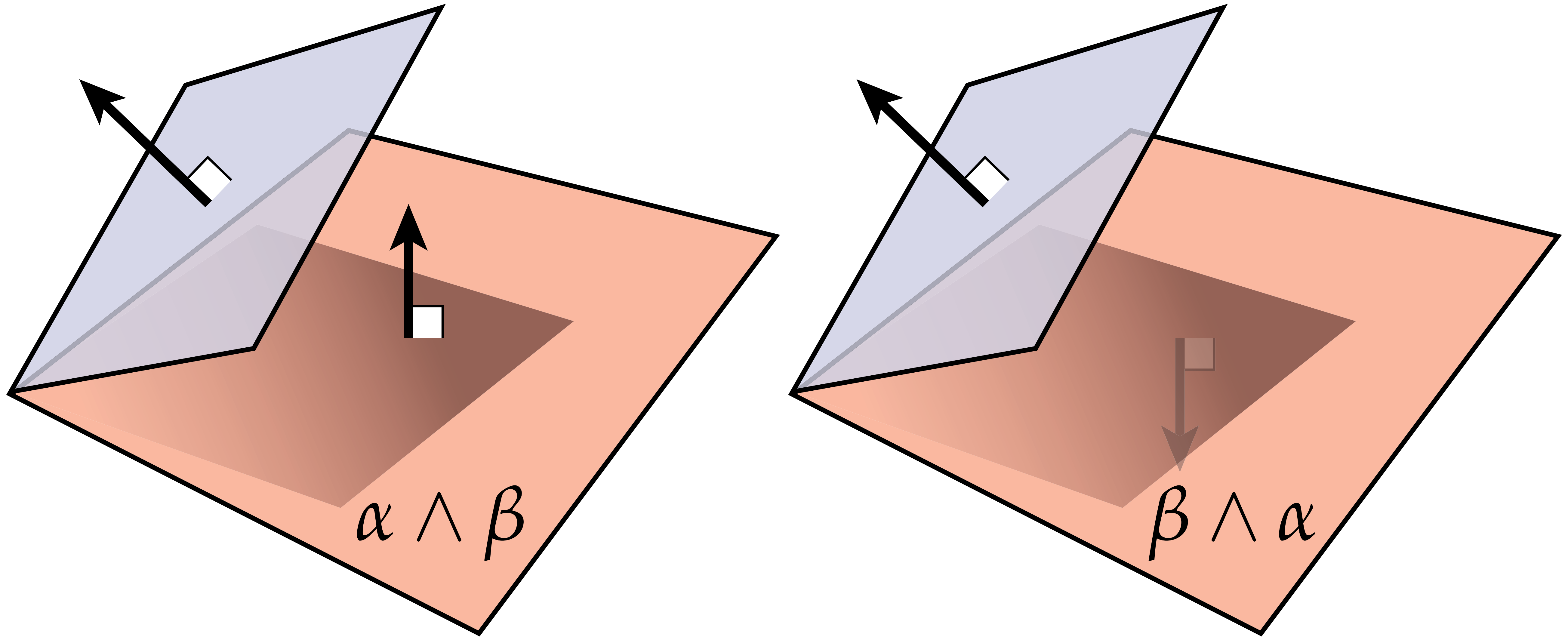
A: Opposite *orientations* of argument 2-vector.

Antisymmetry of 2-Forms

Remember that exchanging the arguments to a wedge product *also* reverses sign:

$$\begin{aligned}(\beta \wedge \alpha)(u, v) &= \beta(u)\alpha(v) - \beta(v)\alpha(u) \\ &= -(\alpha(u)\beta(v) - \alpha(v)\beta(u)) \\ &= -(\alpha \wedge \beta)(u, v)\end{aligned}$$

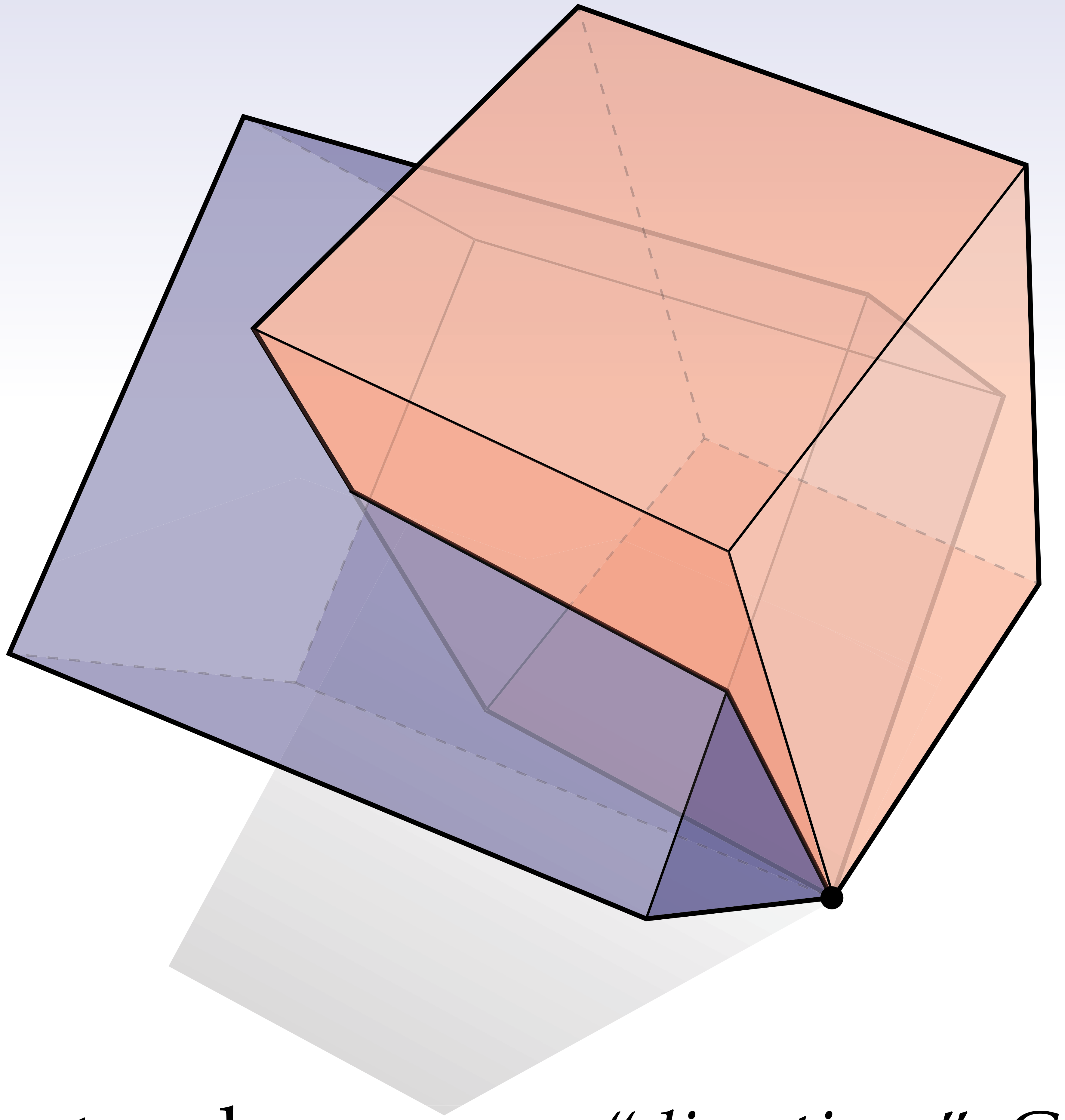
Q: What does this other kind of antisymmetry mean geometrically?



A: Opposite *orientations* of argument 2-vector.

Measurement of 3-Vectors

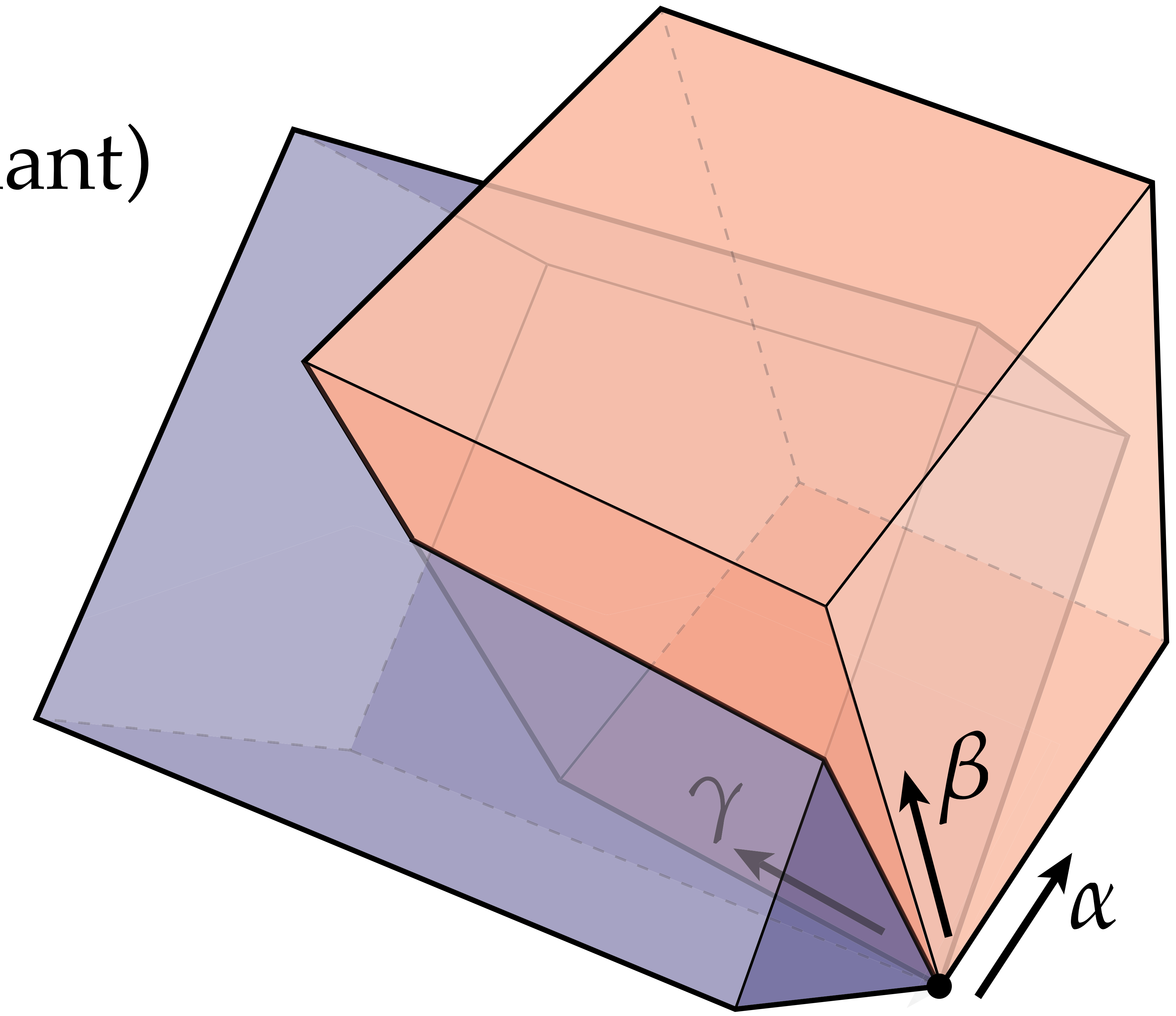
Geometrically, what does it mean to take a **multilinear** measurement of a 3-vector?



Observation: in R^3, all 3-vectors have same *"direction."* Can only measure *magnitude*.

Computing the Projected Volume

- Concretely, how do we compute the volume of a parallelepiped w/ edges u,v,w?
  - Suppose (α, β, γ) is an orthonormal basis
  - Project vectors u,v,w onto this basis
  - Then apply standard formula for volume (determinant)



Projection

$$\begin{aligned} u &\mapsto (\alpha(u), \beta(u), \gamma(u)) \\ v &\mapsto (\alpha(v), \beta(v), \gamma(v)) \\ w &\mapsto (\alpha(w), \beta(w), \gamma(w)) \end{aligned}$$

Volume

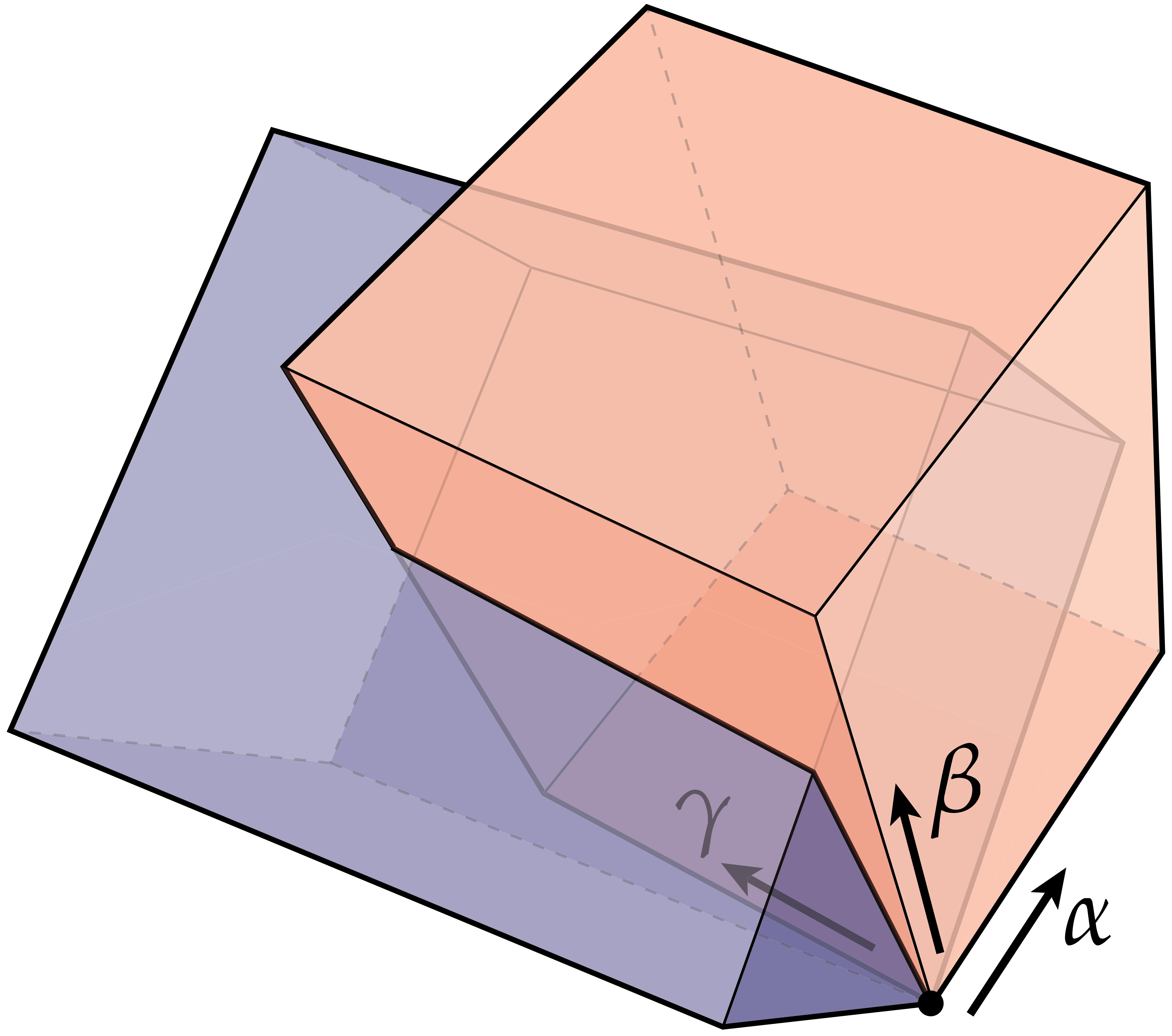
$$\begin{vmatrix} \alpha(u) & \alpha(v) & \alpha(w) \\ \beta(u) & \beta(v) & \beta(w) \\ \gamma(u) & \gamma(v) & \gamma(w) \end{vmatrix}$$

$$= \alpha(u)\beta(v)\gamma(w) + \alpha(v)\beta(w)\gamma(u) + \alpha(w)\beta(u)\gamma(v) \\ - \alpha(u)\beta(w)\gamma(v) - \alpha(w)\beta(v)\gamma(u) - \alpha(v)\beta(u)\gamma(w)$$

3-form

We can of course apply this same expression when α, β, γ are not orthonormal:

$$\begin{aligned}(\alpha \wedge \beta \wedge \gamma)(u, v, w) := \alpha(u)\beta(v)\gamma(w) + \alpha(v)\beta(w)\gamma(u) + \alpha(w)\beta(u)\gamma(v) \\ - \alpha(u)\beta(w)\gamma(v) - \alpha(w)\beta(v)\gamma(u) - \alpha(v)\beta(u)\gamma(w)\end{aligned}$$



Interpretation (in R^3)?

Volume of u,v,w gets scaled by volume of α, β, γ.

k-Form

- More generally, *k*-form is a *fully antisymmetric, multilinear* measurement of a *k*-vector.
- Typically think of this as a map from *k* vectors to a scalar:

$$\alpha : \underbrace{V \times \cdots V}_{k \text{ times}} \to \mathbb{R}$$

- *Multilinear* means "linear in each argument." *E.g.,* for a 2-form:

$$\left.\begin{array}{rcl} \alpha(au+bv,w) & = & a\alpha(u,w)+b\alpha(v,w) \\ \alpha(u,av+bw) & = & a\alpha(u,v)+b\alpha(u,w) \end{array}\right. , \quad \forall u,v,w \in V, a,b \in \mathbb{R}$$

- *Fully antisymmetric* means exchanging two arguments reverses sign. E.g., 3-form:

$$\alpha(u,v,w) = \alpha(v,w,u) = \alpha(w,u,v) =$$
$$-\alpha(u,w,v) = -\alpha(w,v,u) = -\alpha(v,u,w)$$

A Note on Notation

- A k-form effectively measures a k-vector
- For whatever reason, *nobody* writes the argument k-vector using a wedge
- Instead, the convention is to write a list of vectors:

$$(\alpha \wedge \beta)(u \wedge v)$$

$$(\alpha \wedge \beta)(u, v)$$

k-Forms and Determinants

- For 3-forms, saw that we could express a k-form via a *determinant*
- Captures the fact that k-forms are measurements of *volume*
- How does this work more generally?
  - **Conceptually**: "project" onto k-dimensional space and take determinant there

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(u_1, \ldots, u_k) := \begin{vmatrix} \alpha_1(u_1) & \cdots & \alpha_1(u_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(u_1) & \cdots & \alpha_k(u_k) \end{vmatrix}$$

k=1:

$$\det\left(\left[\ \alpha_1(u_1)\ \right]\right) = \alpha_1(u_1)$$

(Determinant of a 1x1 matrix is just the entry of that matrix!)

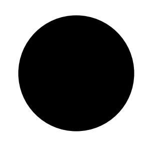
k=2:

$$\det\left(\begin{bmatrix} \alpha_1(u_1) & \alpha_1(u_2) \\ \alpha_2(u_1) & \alpha_2(u_2) \end{bmatrix}\right)$$

$$= \alpha_1(u_1)\alpha_2(u_2) - \alpha_1(u_2)\alpha_2(u_1)$$

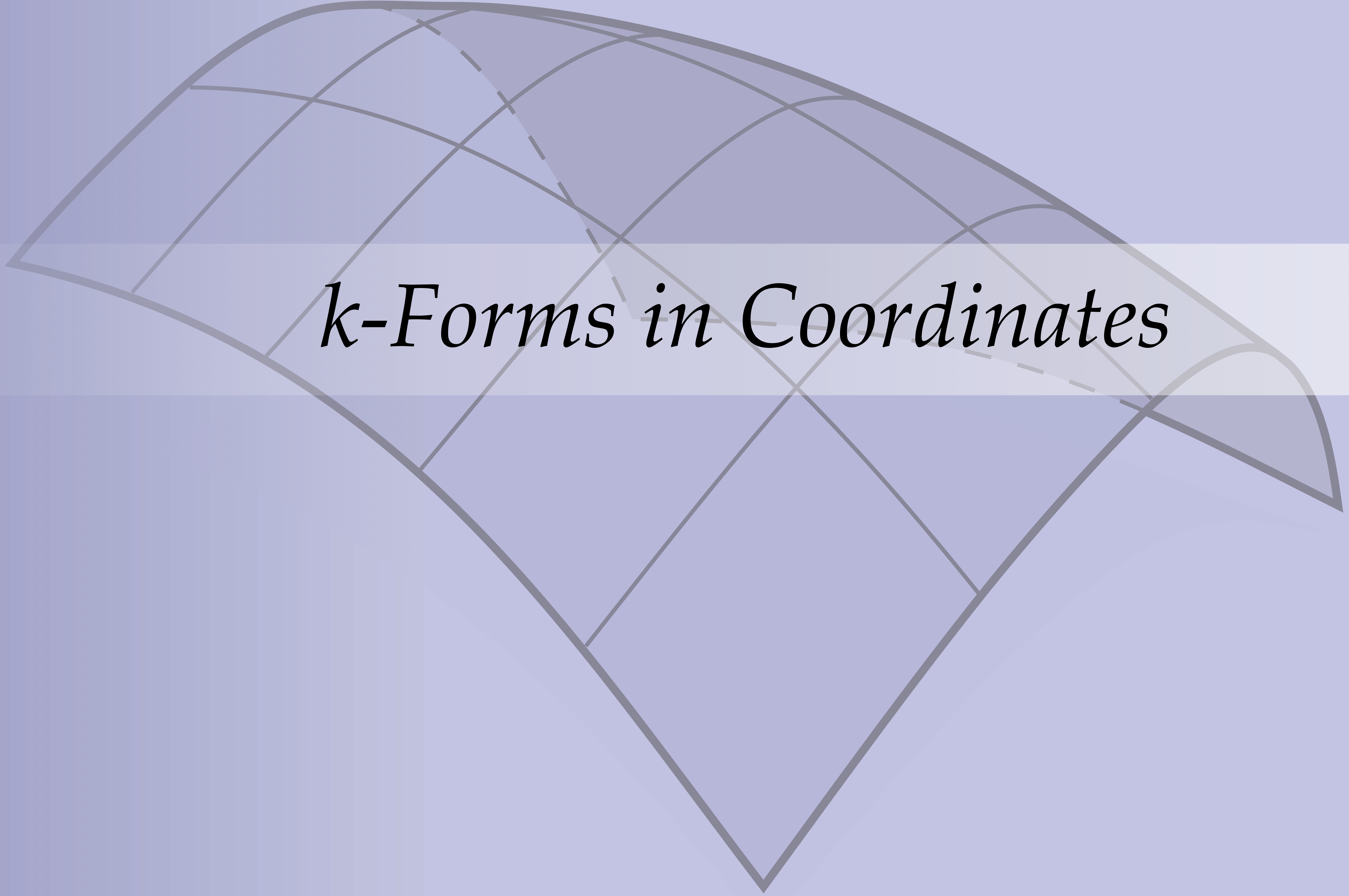
0-Forms

- What's a 0-form?
  - In general, a *k*-form takes *k* vectors and produces a scalar
  - So a 0-form must take 0 vectors and produce a scalar
  - I.e., ***a 0-form is a scalar!***
- Basically looks like this:



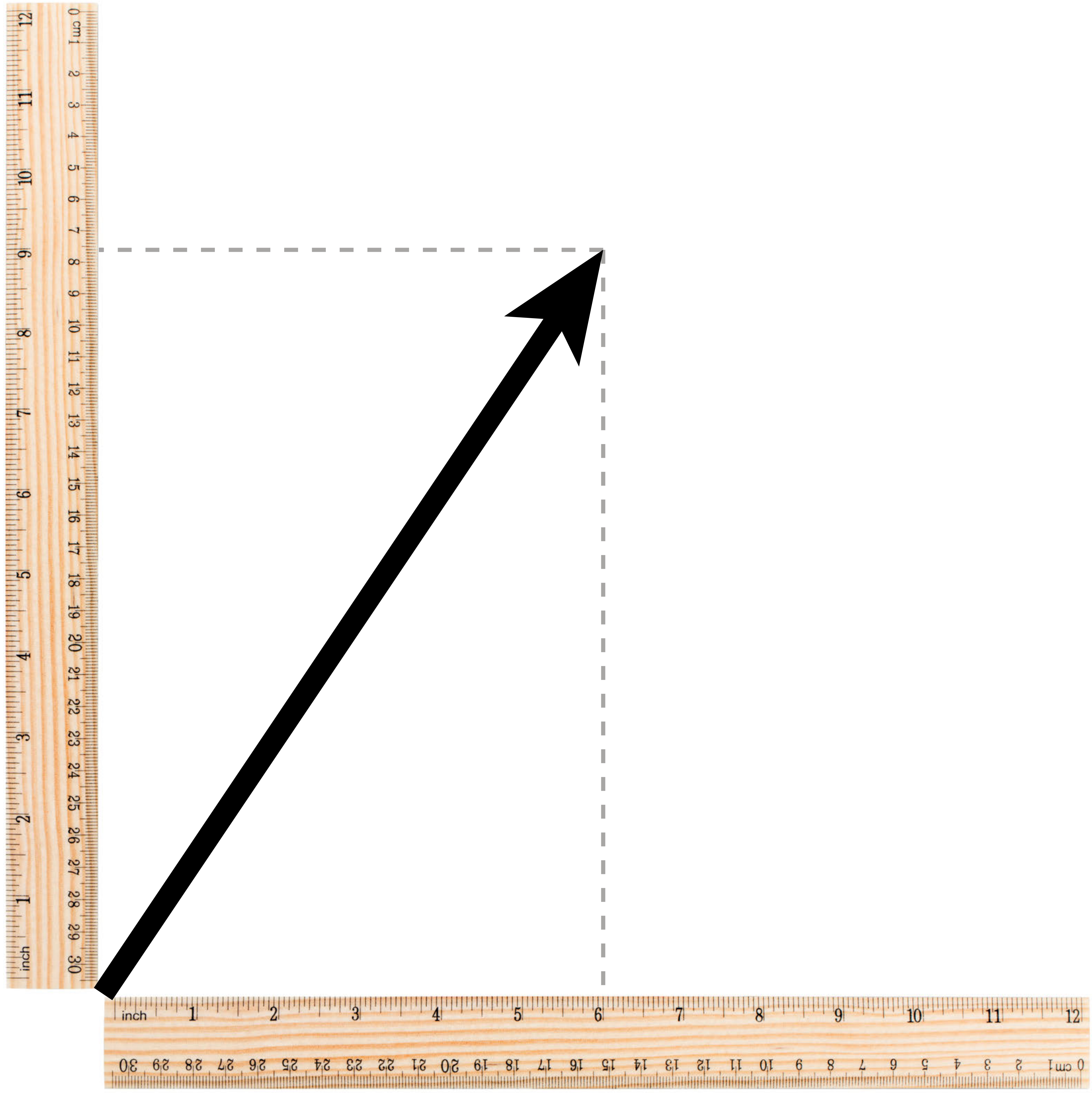
Note: still has *magnitude*, even though it has only one possible "direction."

k-Forms in Coordinates



Measurement in Coordinates

- Idea of measurement becomes very concrete once you have a coordinate system
- E.g., for a (1)-vector:
  - just measure along each coordinate axis
  - take a weighted linear combination



Let's see how this works for *k*-forms…

Dual Basis

In an n-dimensional vector space V, can express vectors v in a basis $e_1, \ldots, e_n$:

$$v = v^1 e_1 + \cdots + v^n e_n$$

The scalar values v^i are the *coordinates* of v.

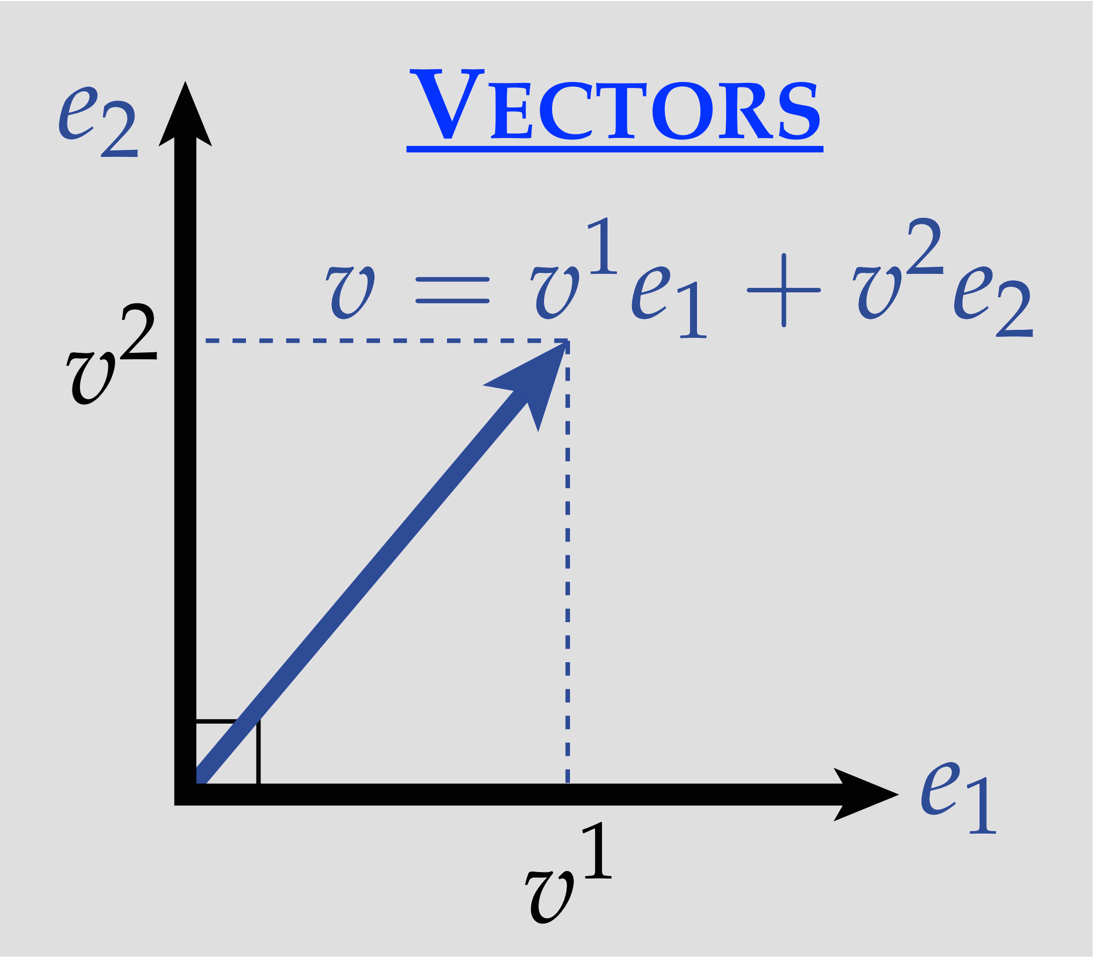
We can also write *covectors* α in a so-called *dual basis* $e^1, \ldots, e^n$:

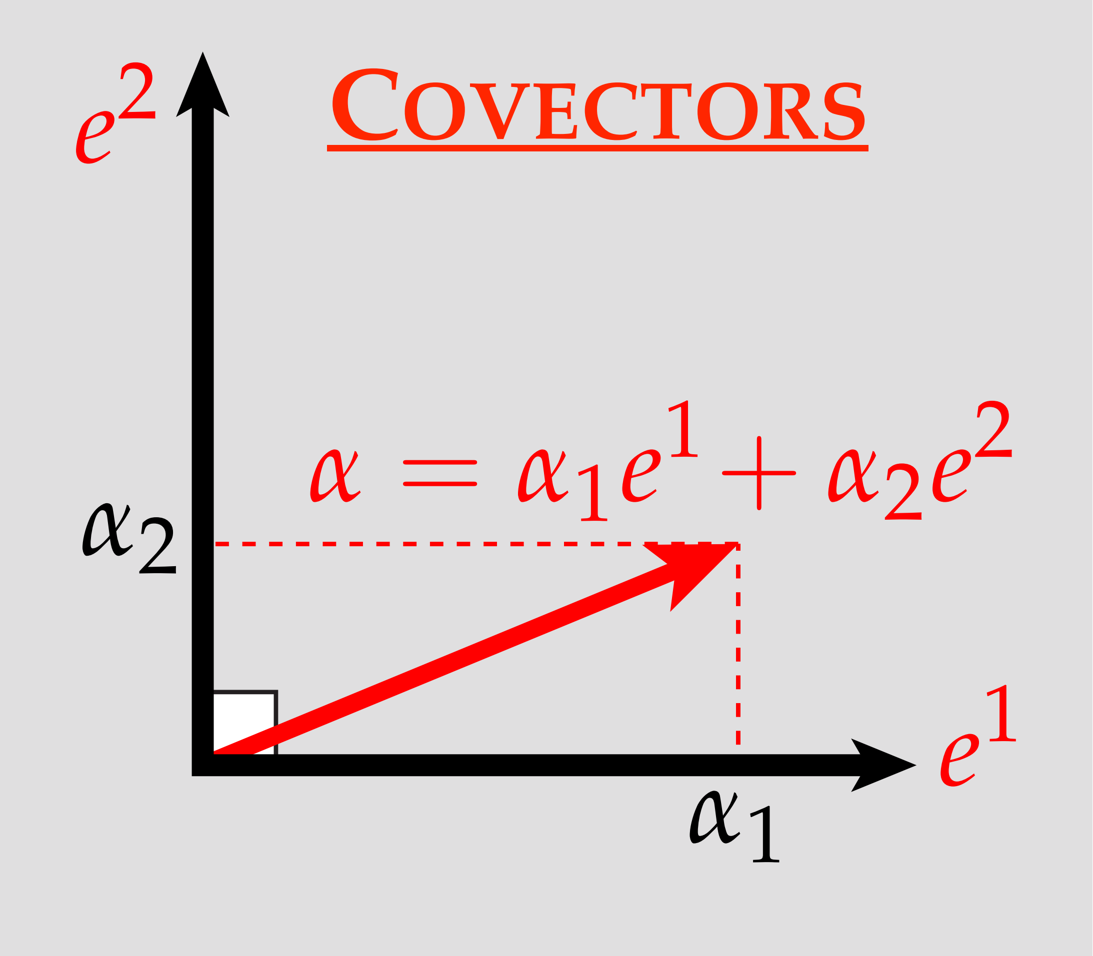
$$\alpha = \alpha_1 e^1 + \cdots + \alpha_n e^n$$

These bases have a special relationship, namely:

$$e^i(e_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

(**Q:** What does e^i mean, geometrically?)



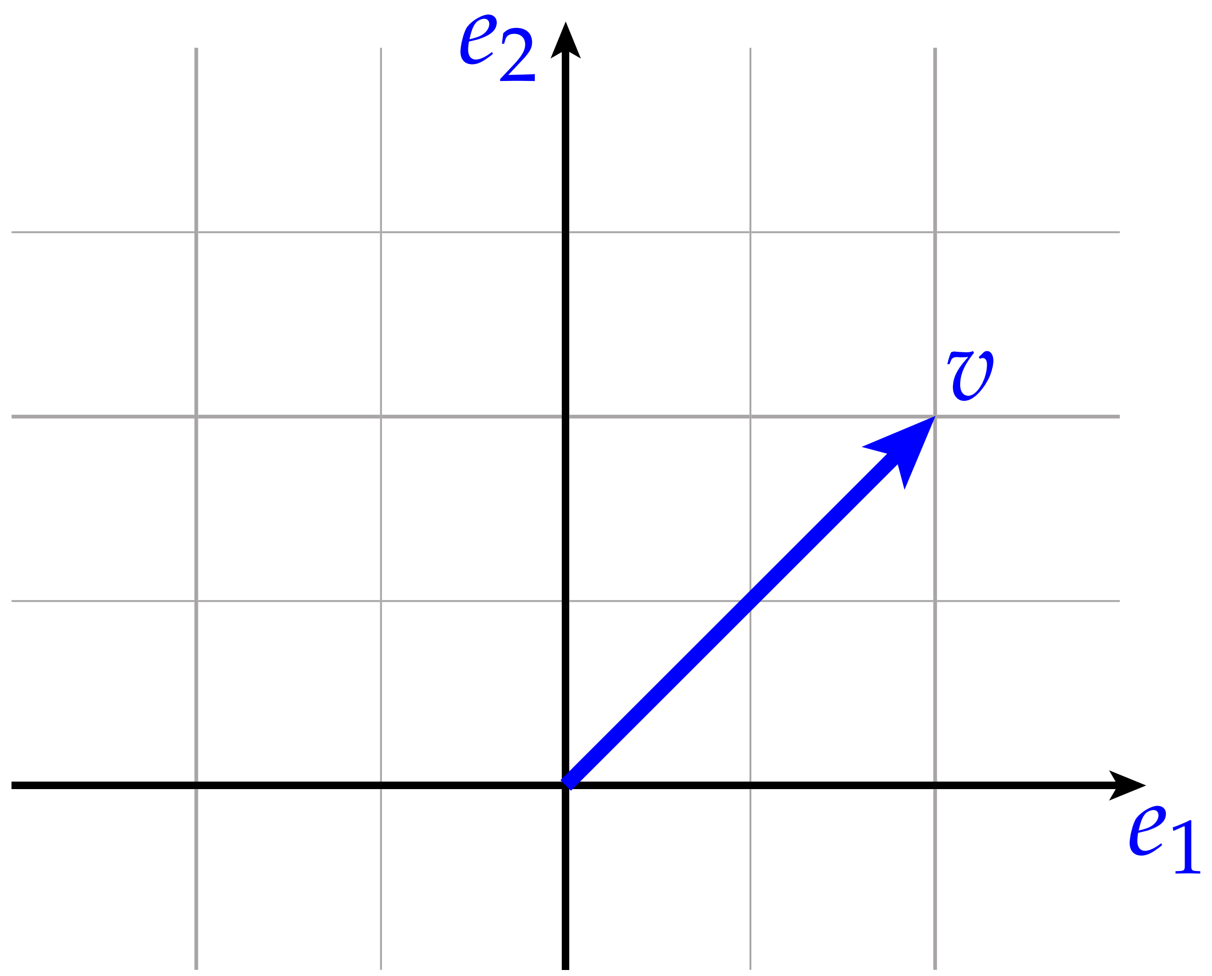


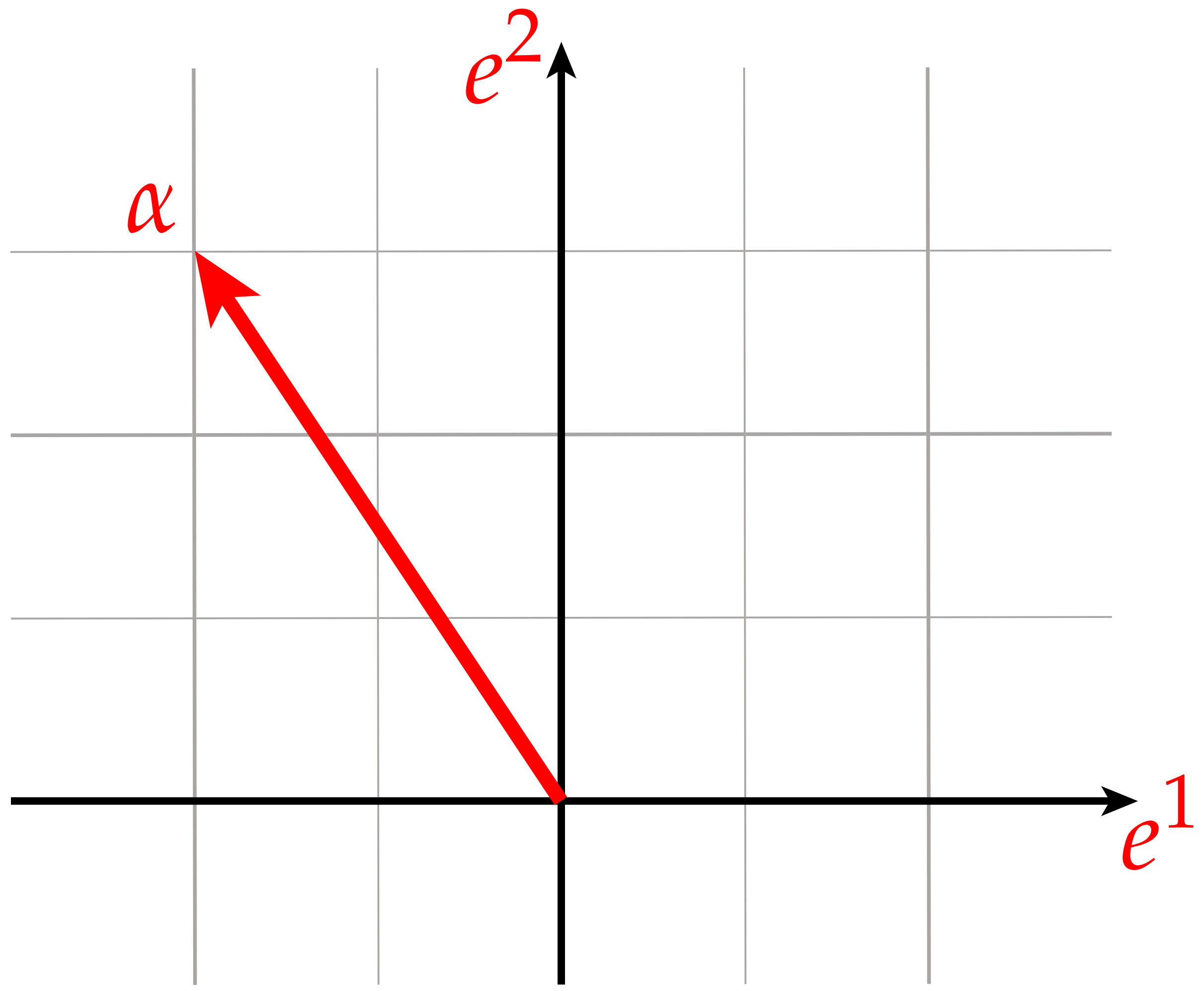
1-form — Example in Coordinates

- Some simple calculations in coordinates help to solidify understanding of k-forms.
- Let's start with a vector v and a 1-form α in the plane:

$$v = 2e_1 + 2e_2$$

$$\alpha = -2e^1 + 3e^2$$





$$\begin{aligned}
\alpha(v) &= (-2e^1 + 3e^2)(2e_1 + 2e_2) \\
&= -2e^1(2e_1 + 2e_2) + 3e^2(2e_1 + 2e_2) \\
&= -4\underbrace{e^1(e_1)}_{1} - 4\underbrace{e^1(e_2)}_{0} + 6\underbrace{e^2(e_1)}_{0} + 6\underbrace{e^2(e_2)}_{1} \\
&= -4 + 6 \quad \text{(Just like a \textit{dot product!})} \\
&= 2.
\end{aligned}$$

2-form — Example in Coordinates

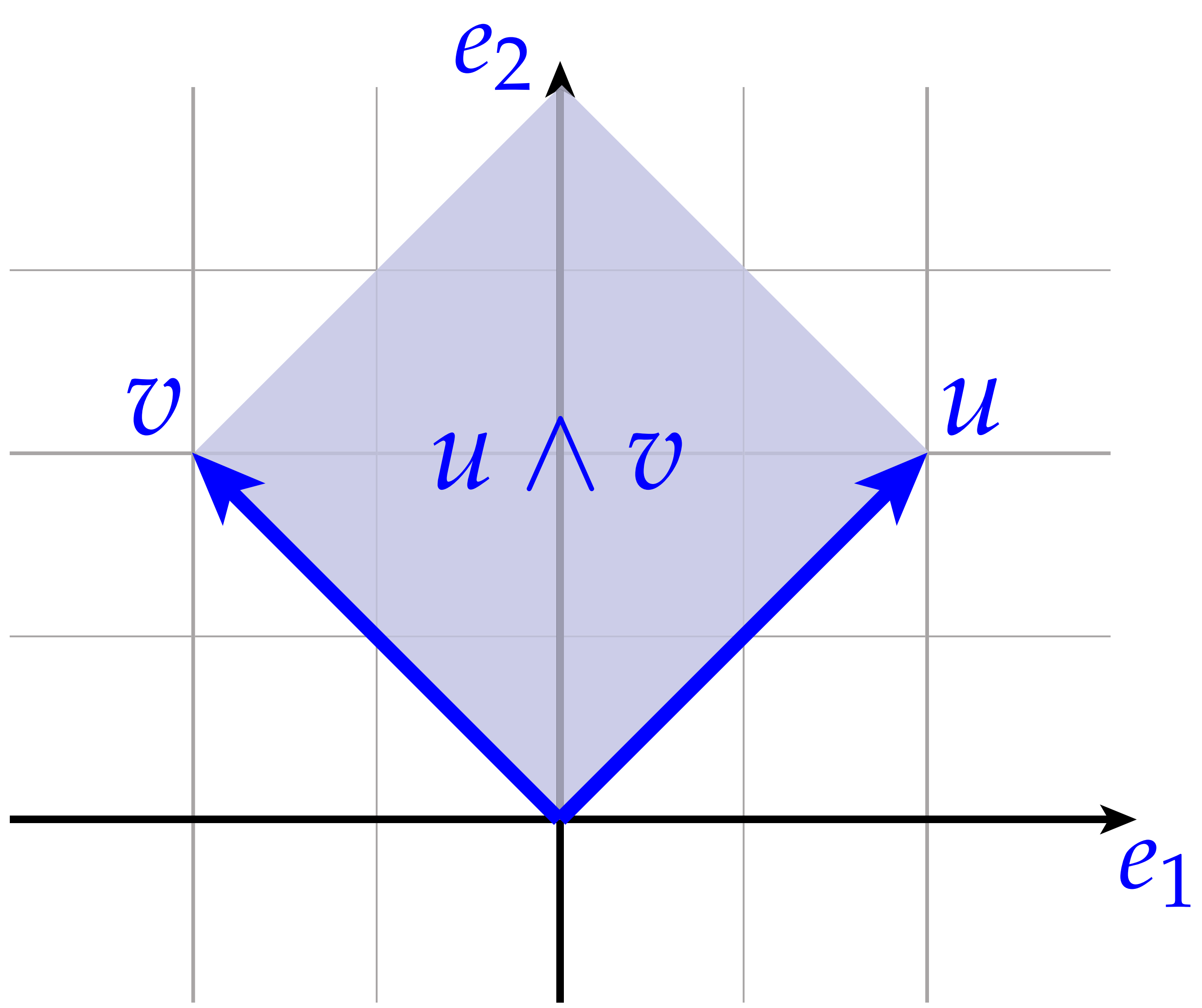
Consider the following vectors and covectors:

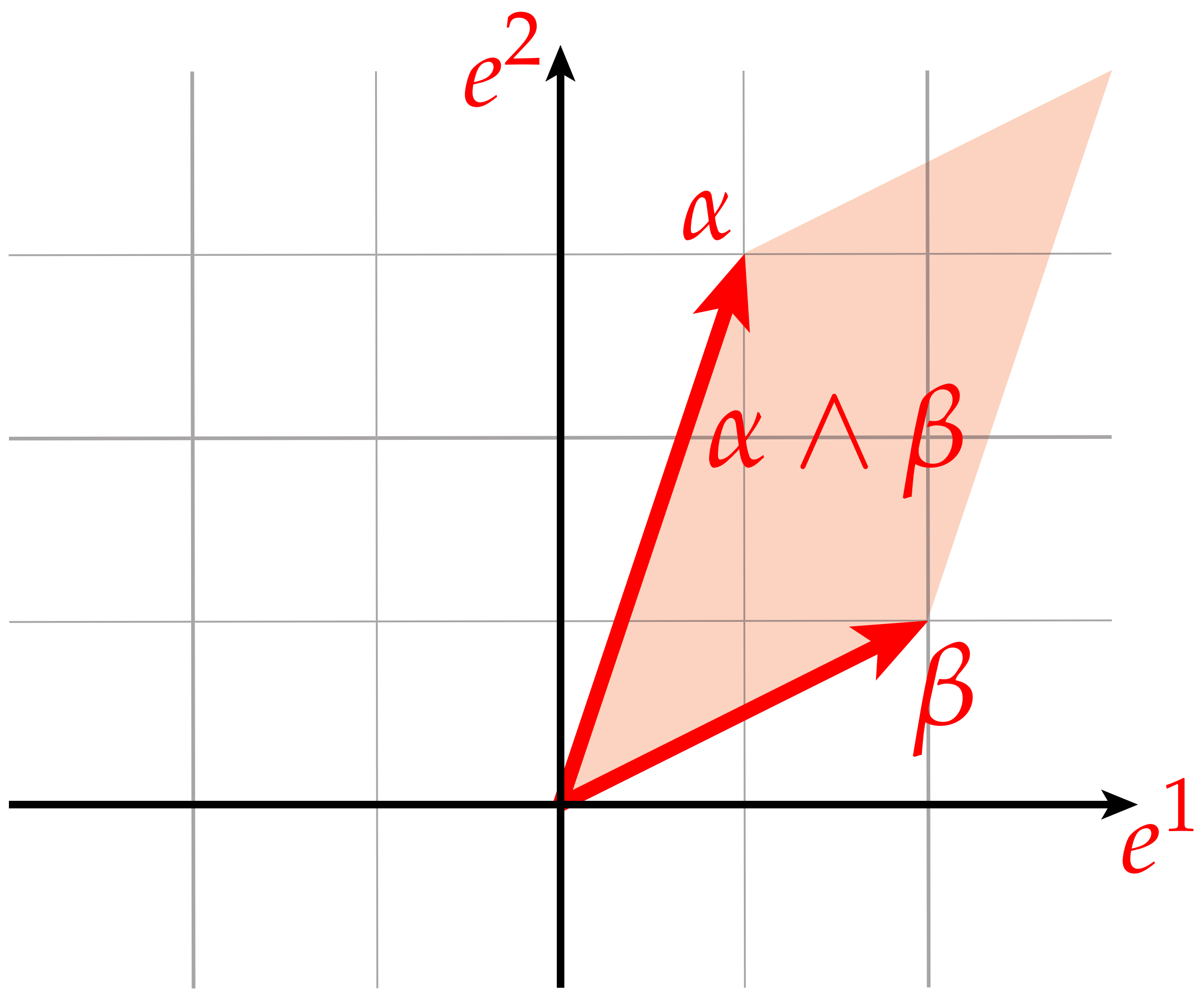
$$u = 2e_1 + 2e_2 \qquad \alpha = e^1 + 3e^2$$
$$v = -2e_1 + 2e_2 \qquad \beta = 2e^1 + e^2$$

We then have:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$
$$\alpha(u) = 1 \cdot 2 + 3 \cdot 2 = 8$$
$$\beta(v) = \cdots = -2$$
$$\alpha(v) = \cdots = 4$$
$$\beta(u) = \cdots = 6$$
$$\Rightarrow (\alpha \wedge \beta)(u, v) = 8 \cdot (-2) - 4 \cdot 6 = -40.$$

Q: What does this value mean, geometrically? Why is it *negative*?



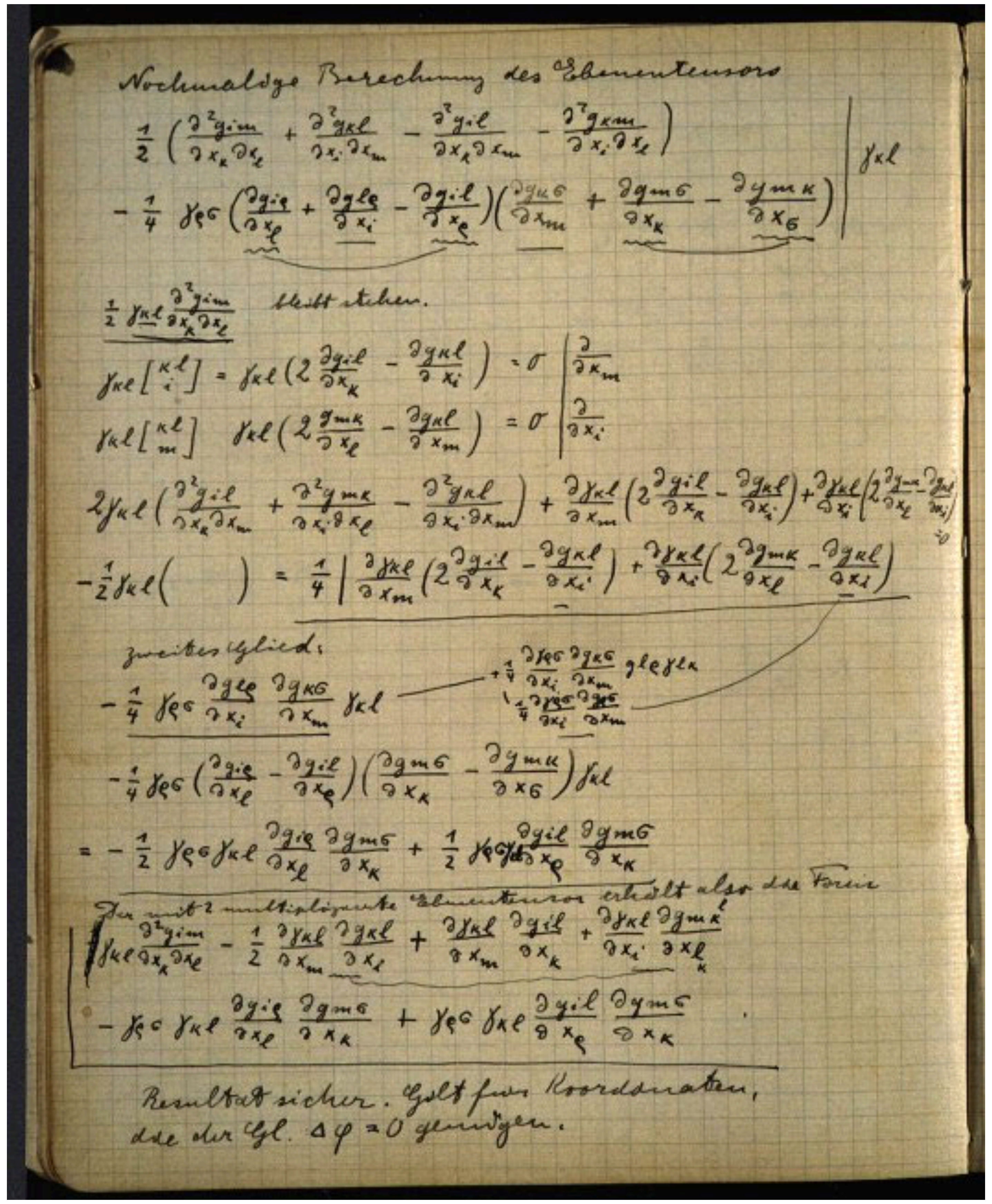


Einstein Summation Notation

Why are some indices *"up"* and others *"down"*?

Bemerkung zur Vereinfachung der Schreibweise der Ausdrücke.

Ein Blick auf die Gleichungen dieses Paragraphen zeigt, daß über Indizes, die zweimal unter einem Summenzeichen auftreten [z. B. der Index ν in (5)], stets summiert wird, und zwar *nur* über zweimal auftretende Indizes. Es ist deshalb möglich, ohne die Klarheit zu beeinträchtigen, die Summenzeichen wegzulassen. Dafür führen wir die Vorschrift ein: Tritt ein Index in einem Term eines Ausdruckes zweimal auf, so ist über ihn stets zu summieren, wenn nicht ausdrücklich das Gegenteil bemerkt ist.

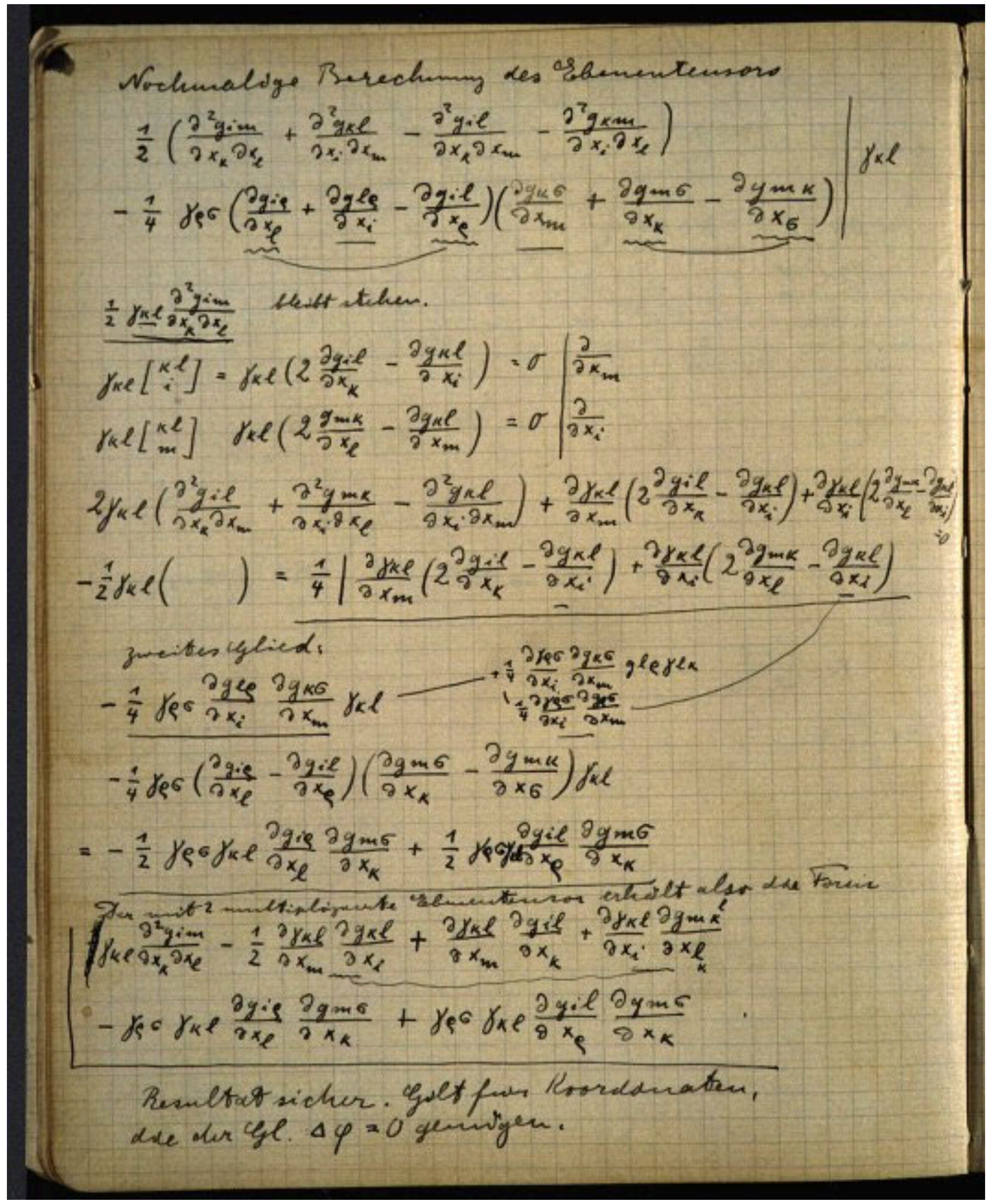


— *Einstein, "Die Grundlage der allgemeinen Relativitätstheorie"* (1916)

Einstein Summation Notation

Key idea: sum over repeated indices.

$$x^i y_i := \sum_{i=1}^{n} x^i y_i$$



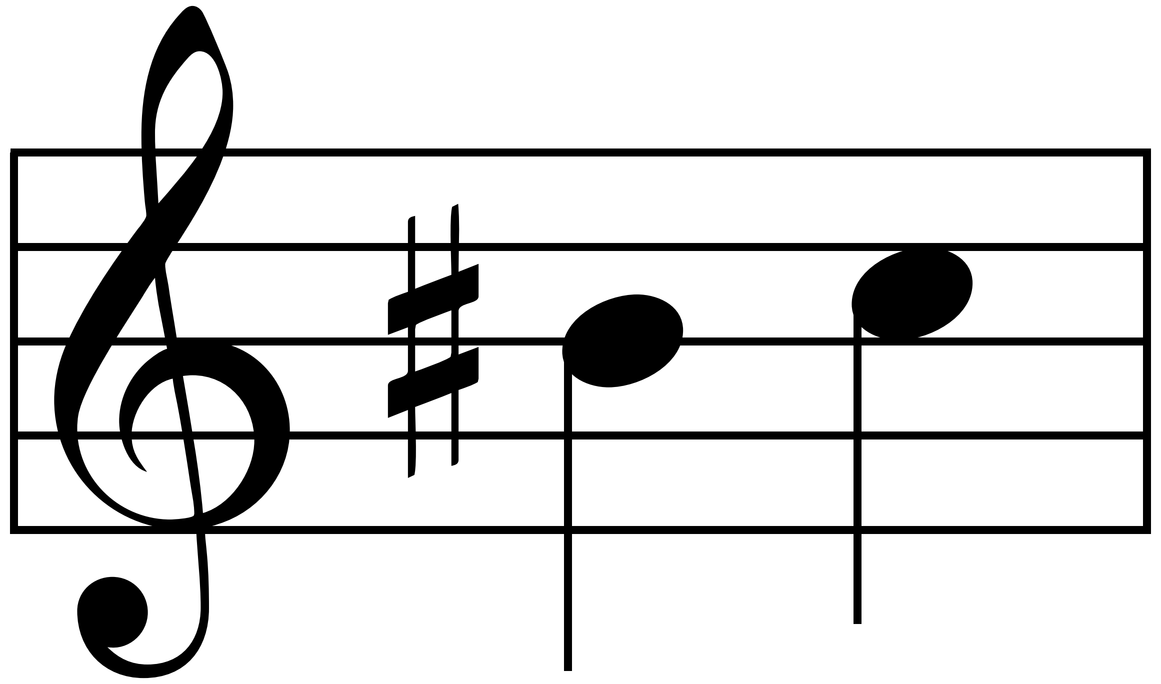
NOTE ON A SIMPLIFIED WAY OF WRITING EXPRESSIONS

A look at the equations of this paragraph show that there is always a summation over indices which occur twice, and only for twice-repeated indices. It is therefore possible, without detracting from clarity, to omit the sum sign. For this we introduce a rule: if an index in an expression appears twice, then a sum is implicitly taken over this index, unless specifically noted to the contrary.

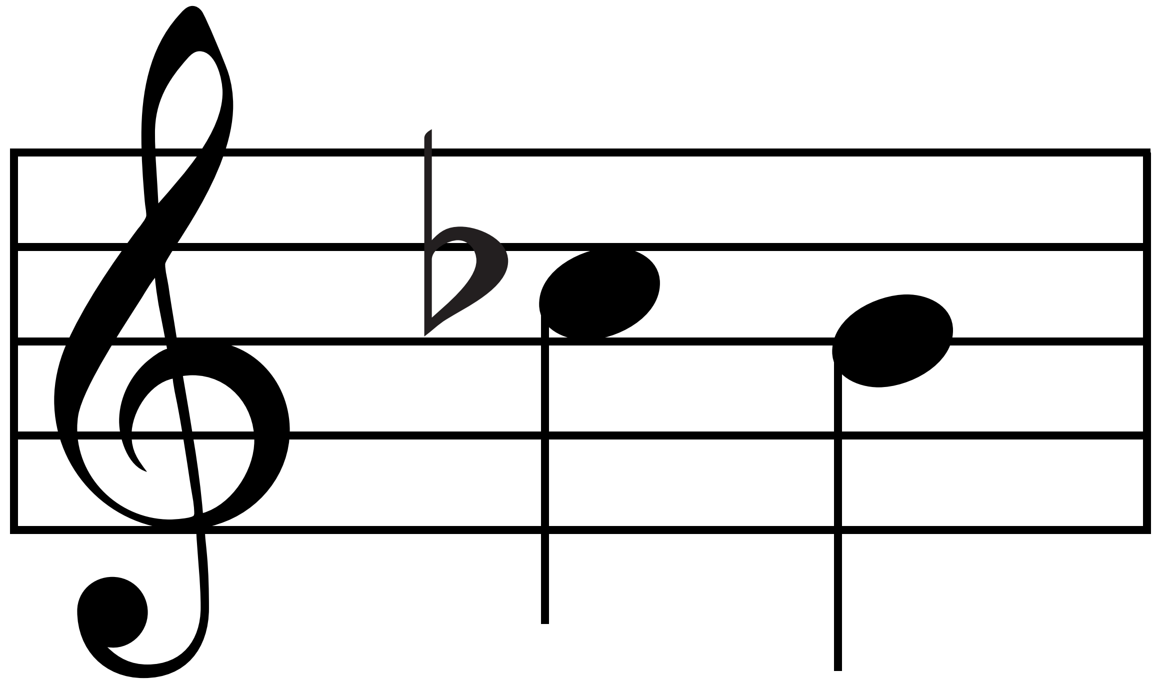
— Einstein, "Die Grundlage der allgemeinen Relativitätstheorie" (1916)

Sharp and Flat in Coordinates

Q: What do sharp and flat do on a musical staff?



(raise pitch)



(lower pitch)

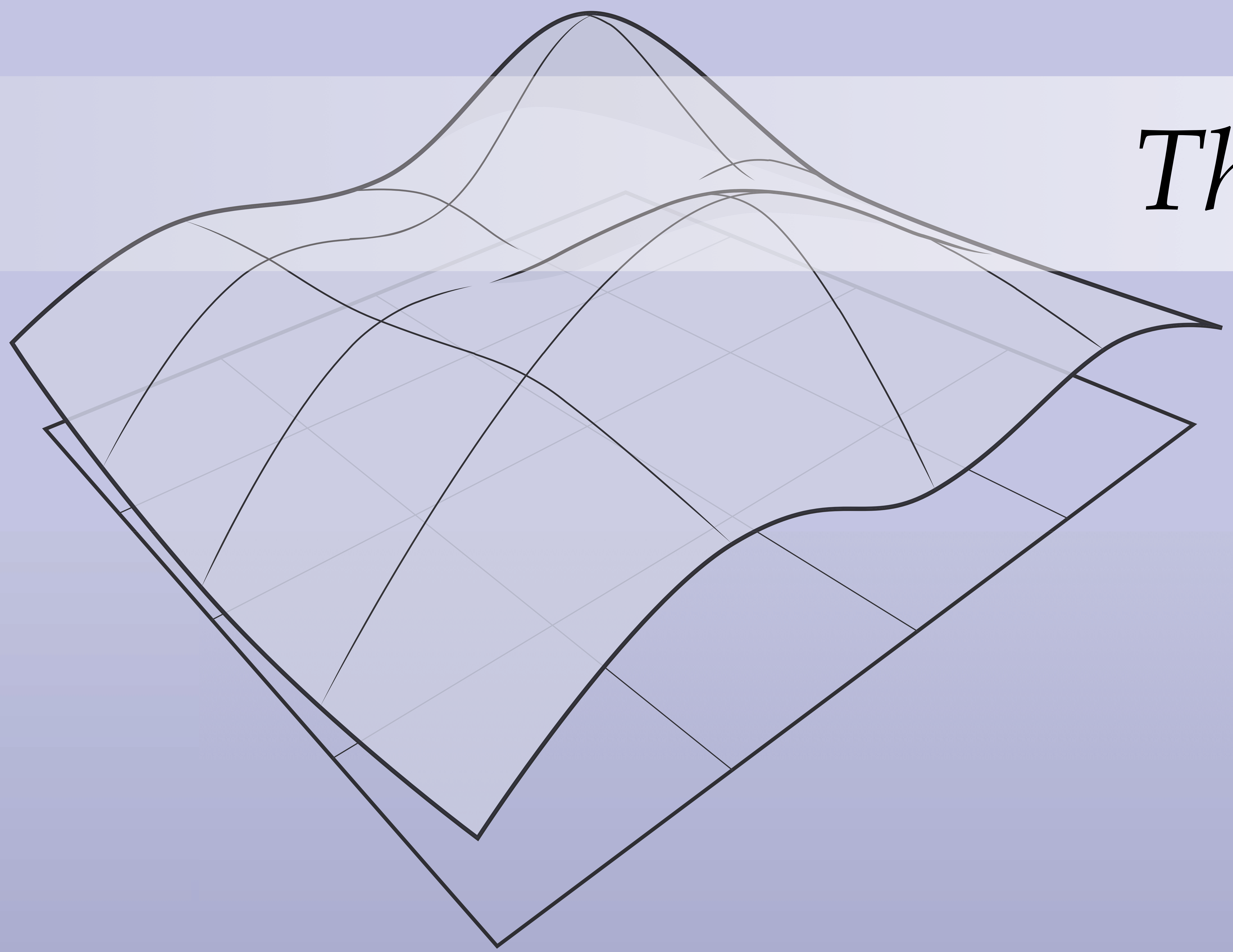
Likewise, sharp and flat *raise* and *lower* indices of coefficients for 1-forms/vectors.

Suppose for instance that $\alpha(v) = \langle u, v \rangle$ for all $v \in V$. Then

$$\alpha = \alpha_1 e^1 + \cdots + \alpha_n e^n \quad \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \quad u = u^1 e_1 + \cdots + u^n e_n$$

(Sometimes called the *musical isomorphisms.)*

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION