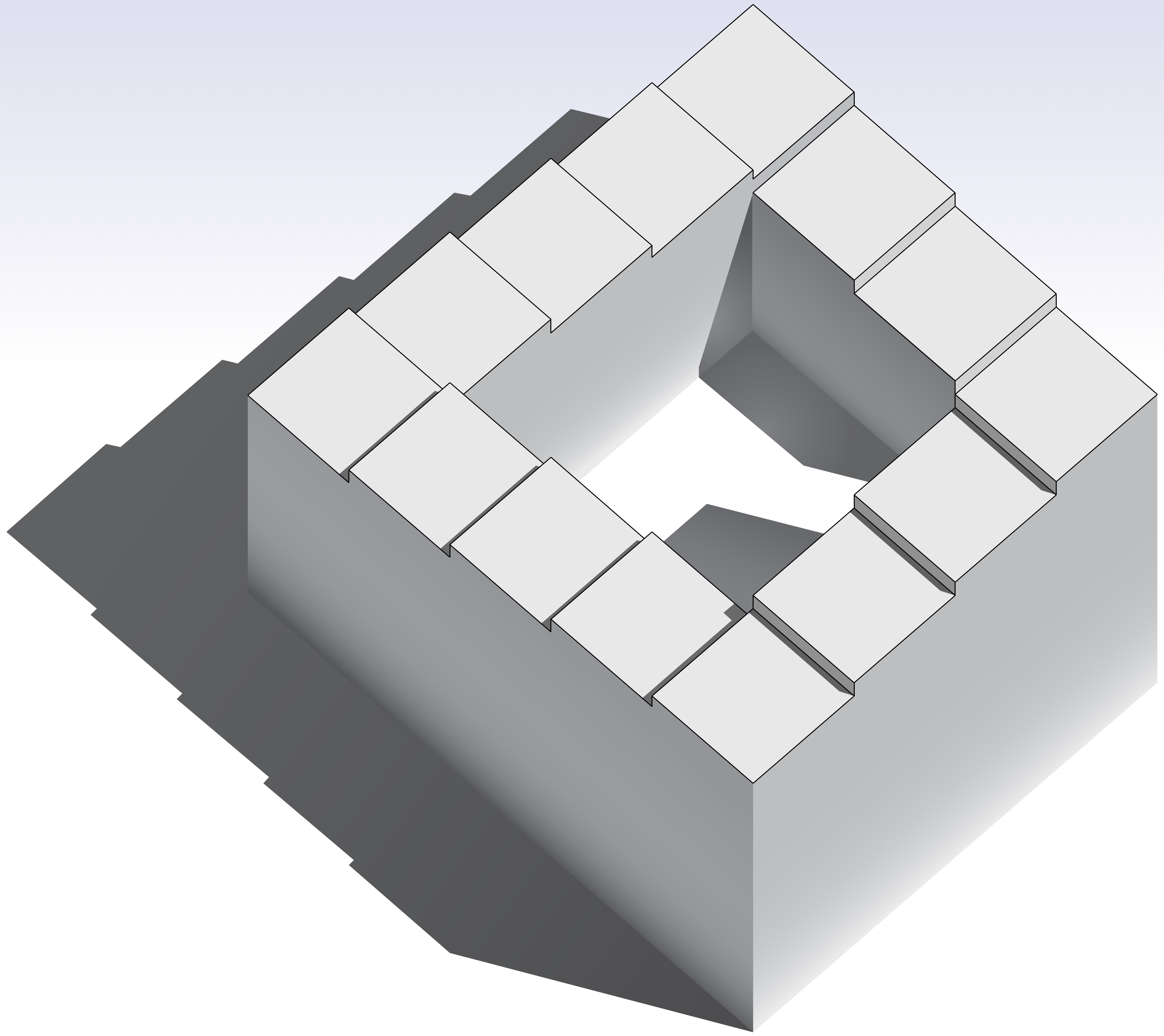
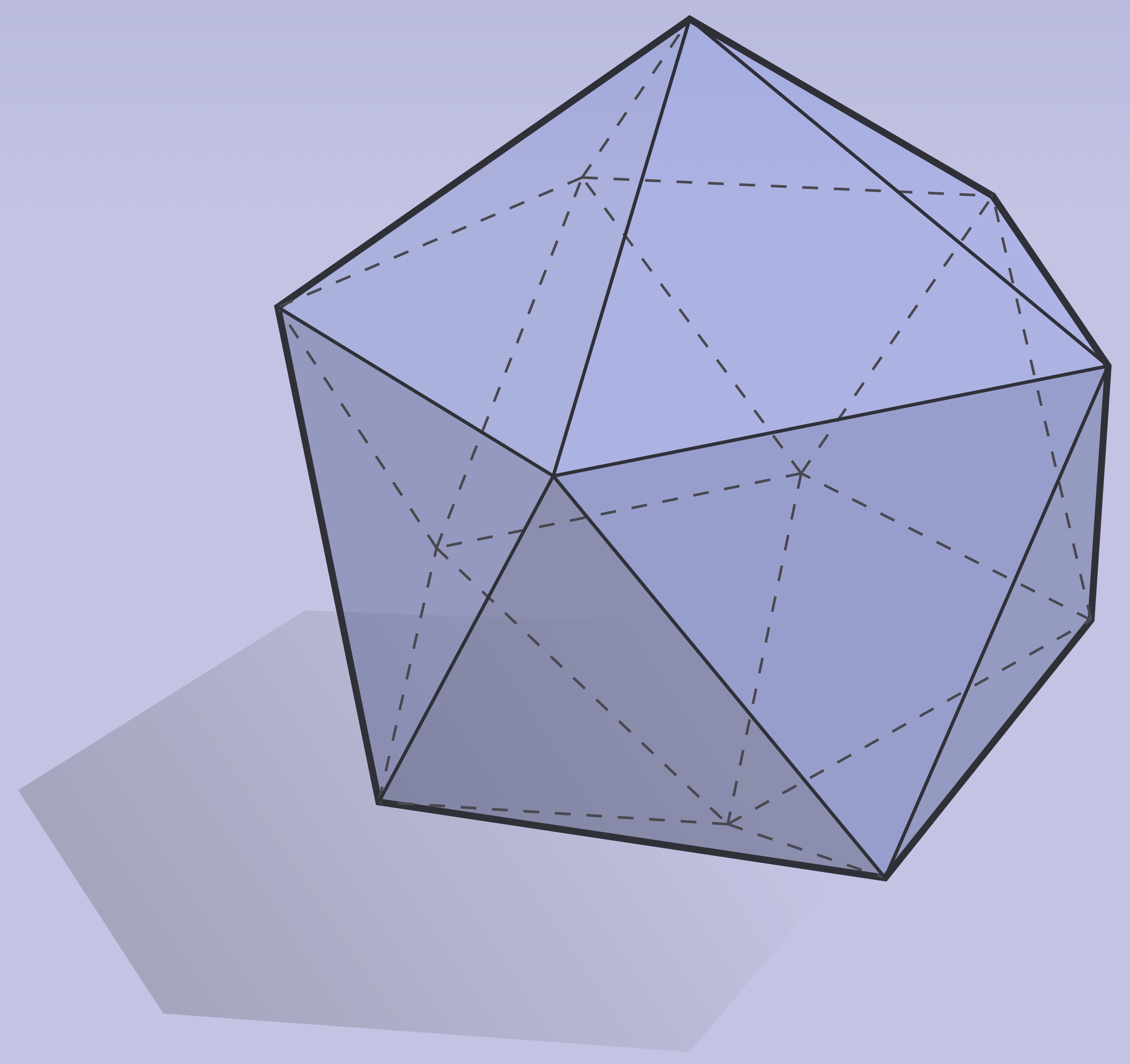


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B

LECTURE 18:

THE LAPLACE-BELTRAMI OPERATOR



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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Laplace Beltrami—Overview

- *Laplace-Beltrami operator*—or just "Laplacian"—generalizes ordinary Laplacian to curved domains
  - denote by capital delta (or nabla squared...)
- Shows up shockingly often in geometry & physics
- Discrete Laplacians ubiquitous in algorithms:
  - physical simulation
  - graph theory/networks
  - machine learning
  - **geometry processing**
  - ...
- Why? Reduces problems to *sparse linear algebra*
  - fast, lots of existing code/algorithms, ...

Δ **Laplacian**

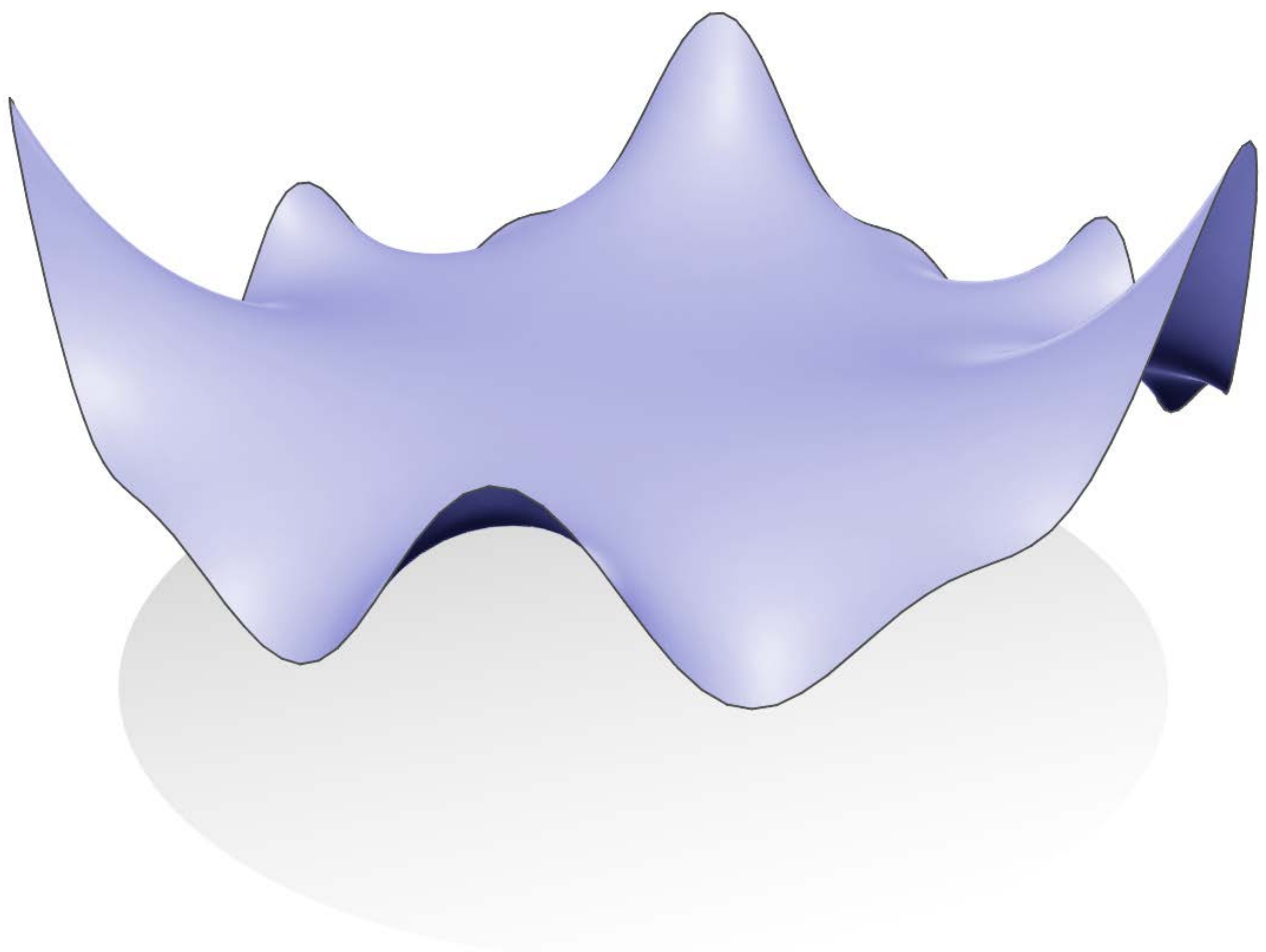
∇^2 **~~Laplacian~~** **Hessian**

Pierre-Simon Laplace

Laplacian in Physics

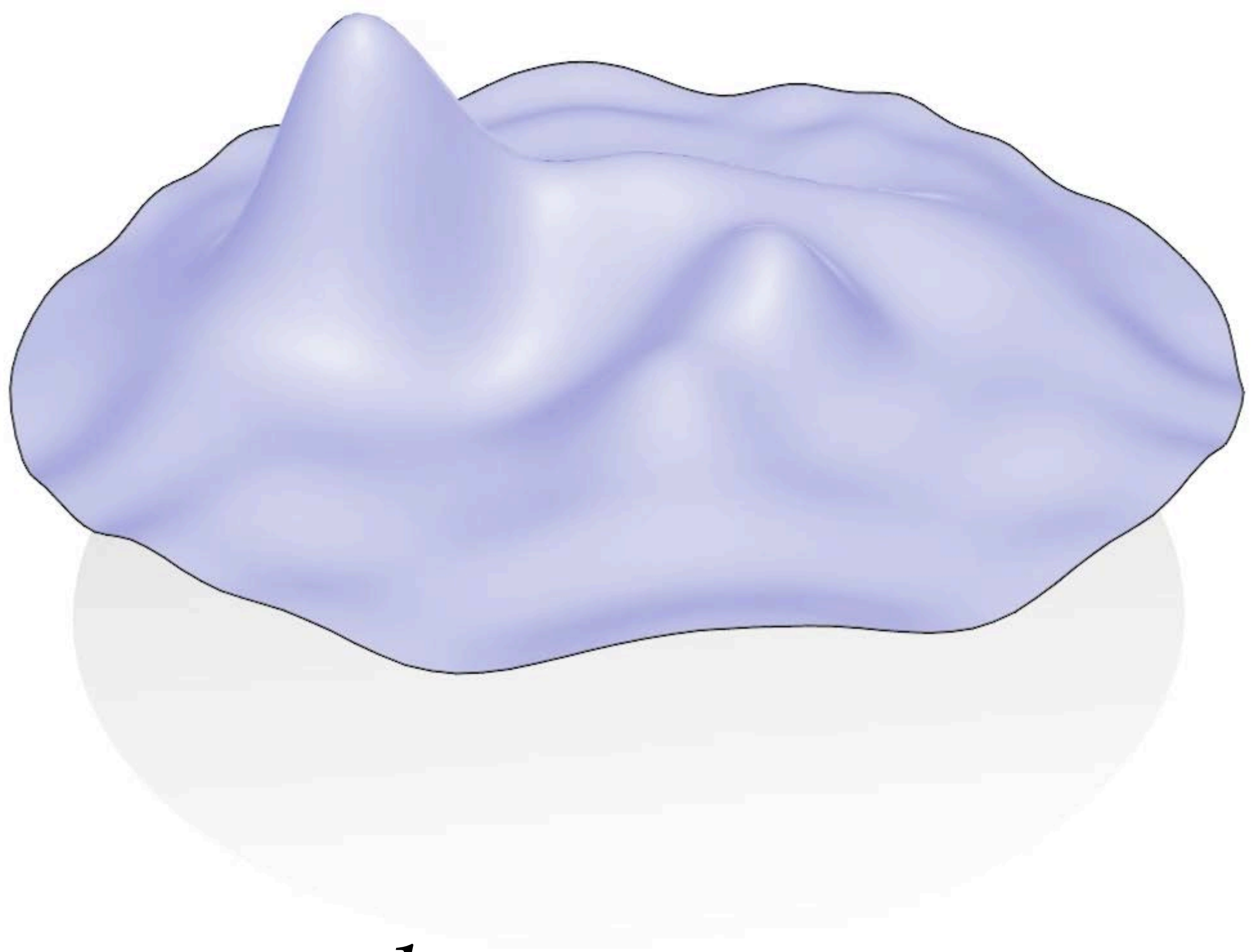
Provides basic model for wide variety of physical behavior:

Laplace equation



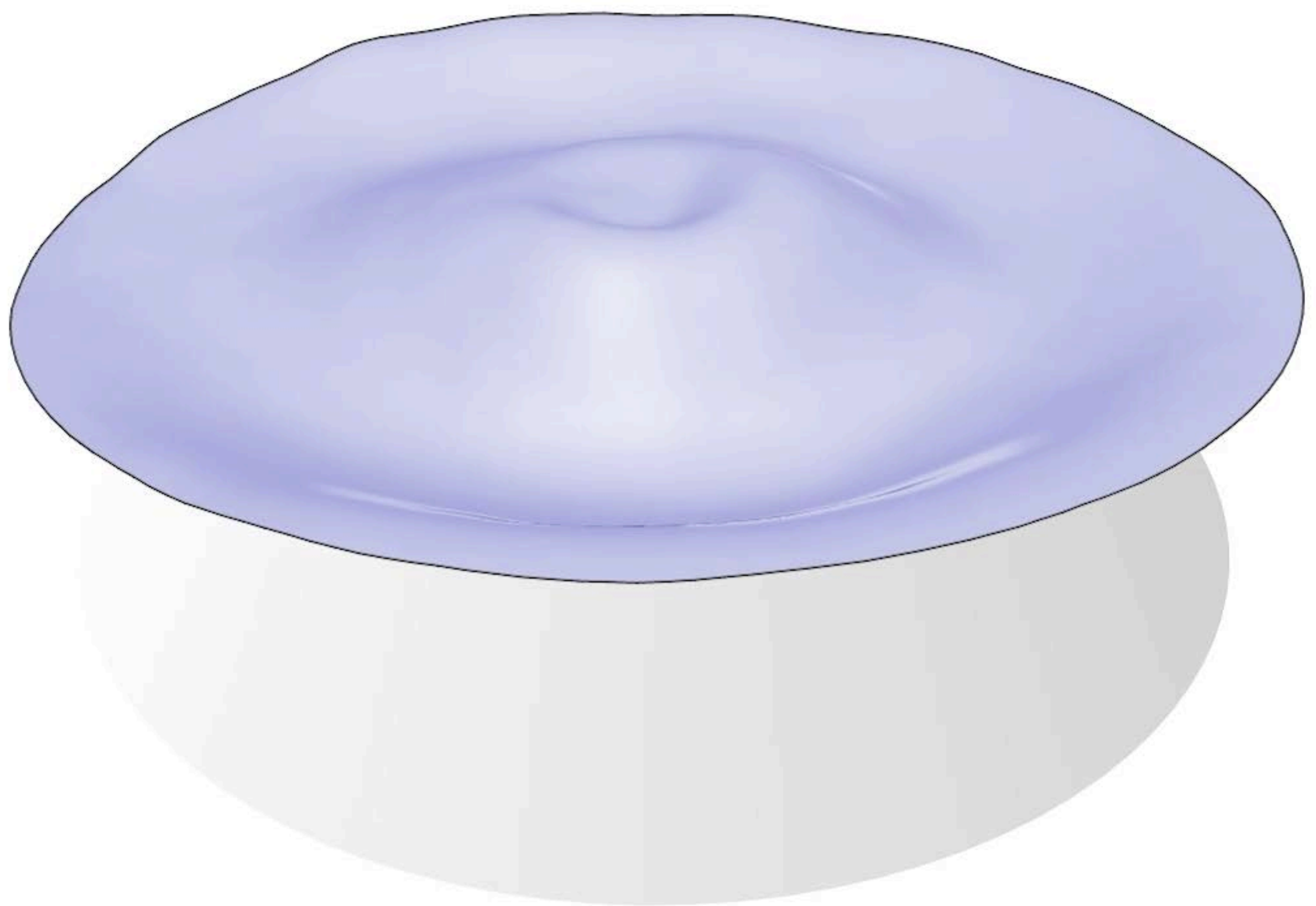
$$\Delta u = 0$$

heat equation



$$\frac{d}{dt}u = \Delta u$$

wave equation



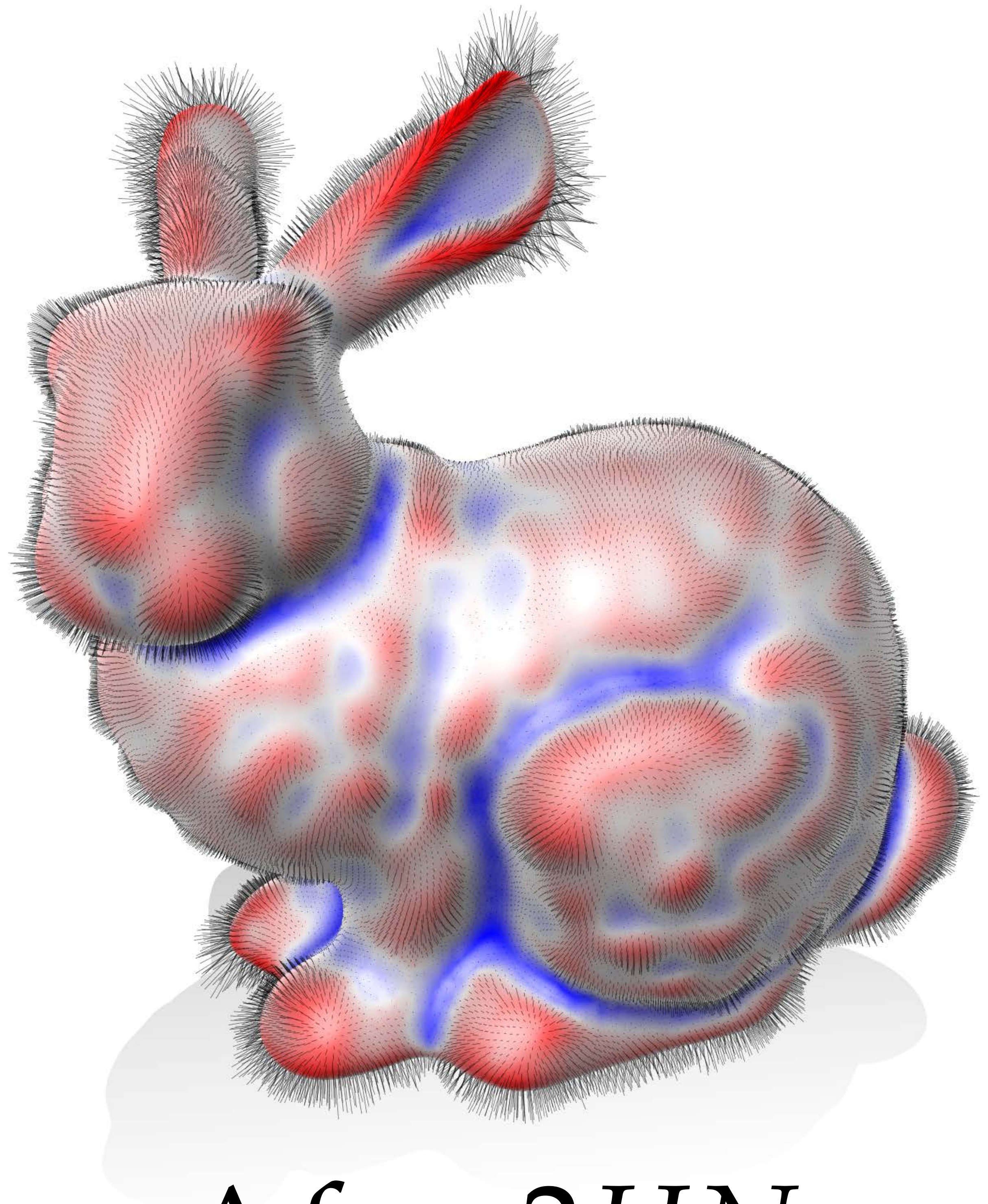
$$\frac{d^2}{dt^2}u = \Delta u$$

Build on top of basic equations to model many systems (elasticity, Schrödinger, ...)

Laplacian in Geometry

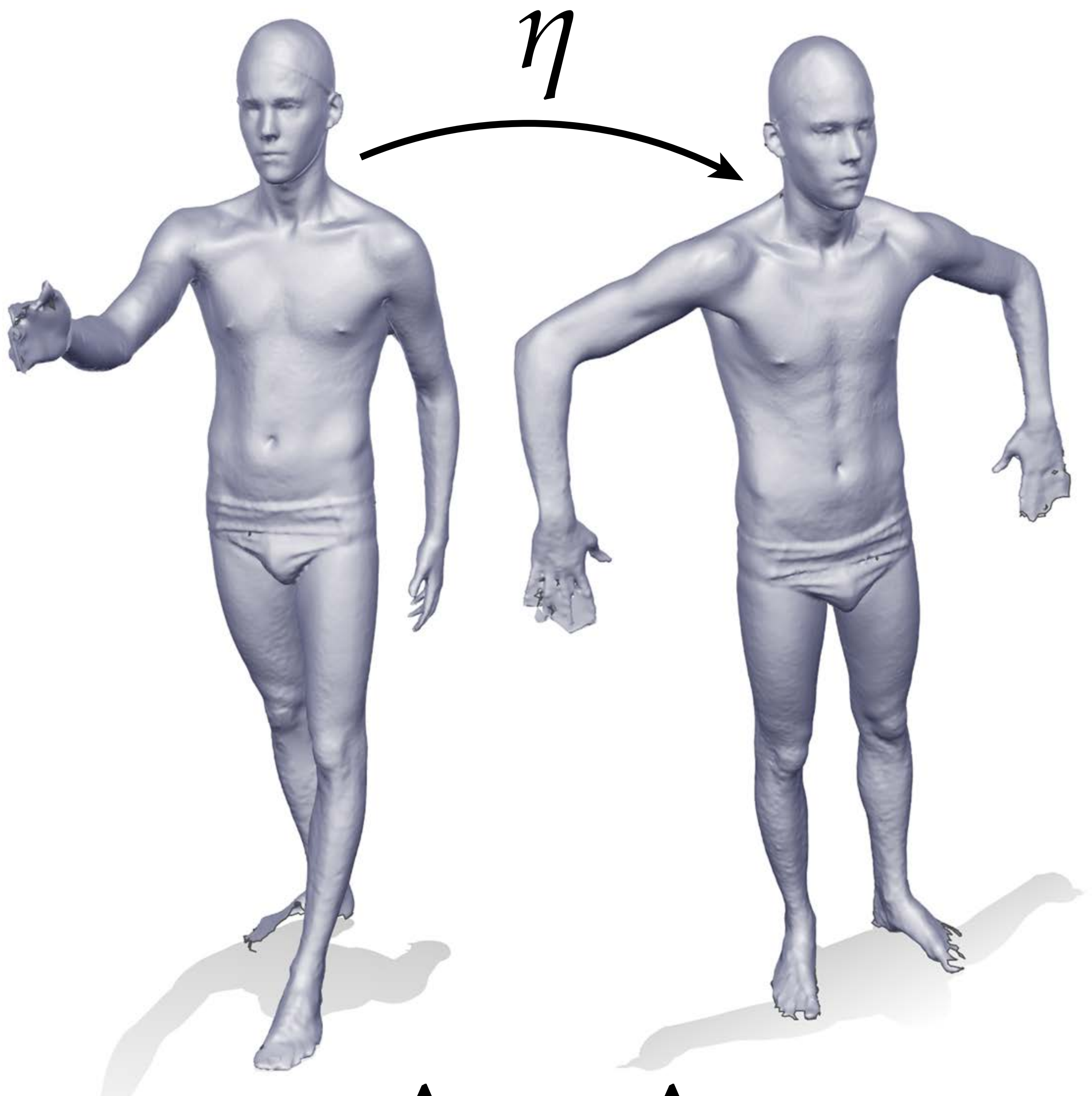
Also ubiquitous in differential geometry, mesh processing:

curvature



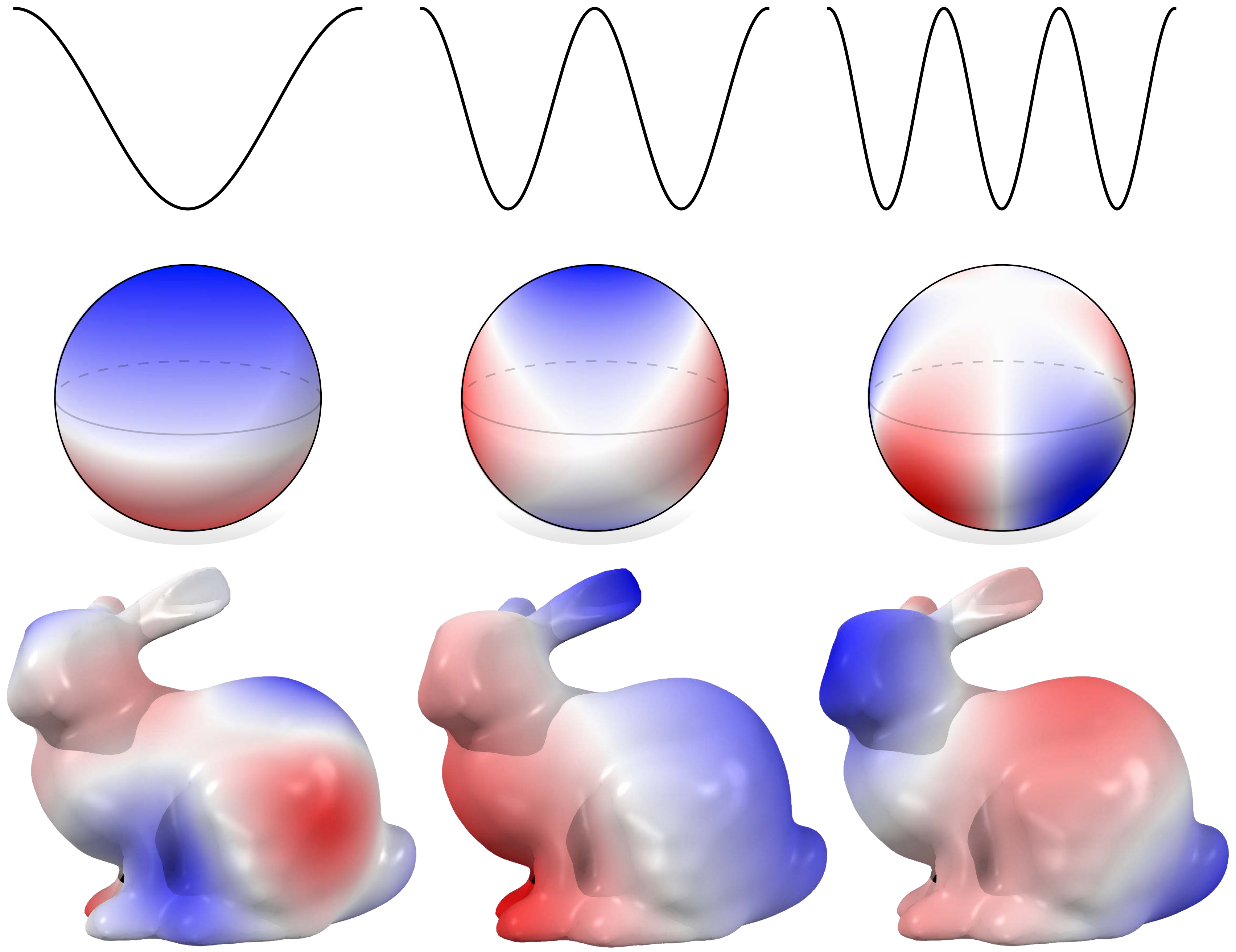
$$\Delta f = 2HN$$

isometry invariance



$$\eta \circ \Delta = \Delta \circ \eta$$

frequency decomposition



$$\Delta \phi = \lambda \phi$$

We will see *many* more properties/examples/applications as we progress...

Review: Laplacian in R^n

$u : \mathbb{R}^n \to \mathbb{R}$ (twice differentiable)

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$$

<u>In 2D:</u>

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y)$$

Basic definition: Laplacian gives sum of 2nd derivatives along coordinate axes

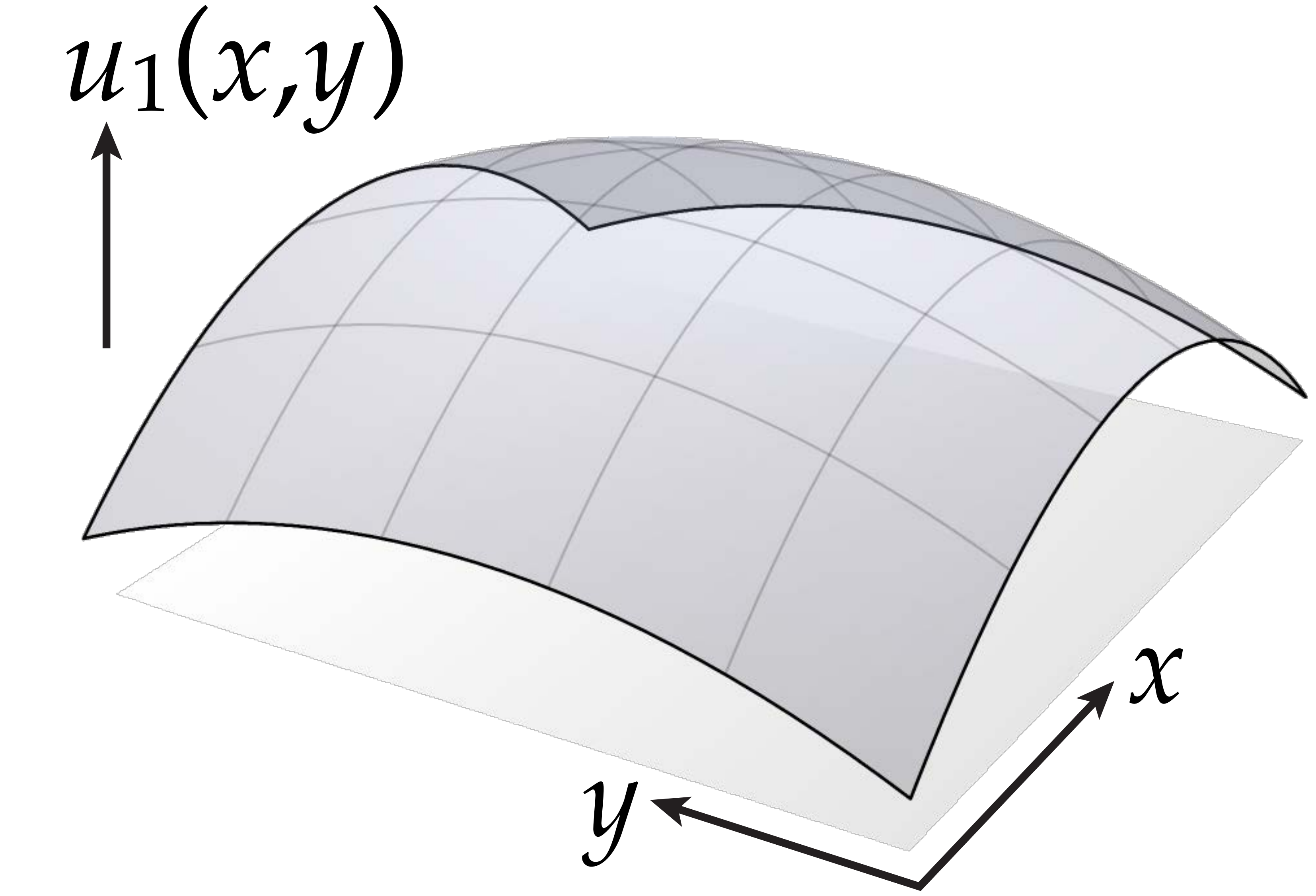
Laplacian in R^n—Examples

<u>**Example.**</u>

$$u_1(x,y) = -x^2 - 2y^2$$

$$\Delta u_1 = \frac{\partial^2}{\partial x^2}(-x^2 - 2y^2) + \frac{\partial^2}{\partial y^2}(-x^2 - 2y^2) =$$

$$-2 - 4 = -6$$

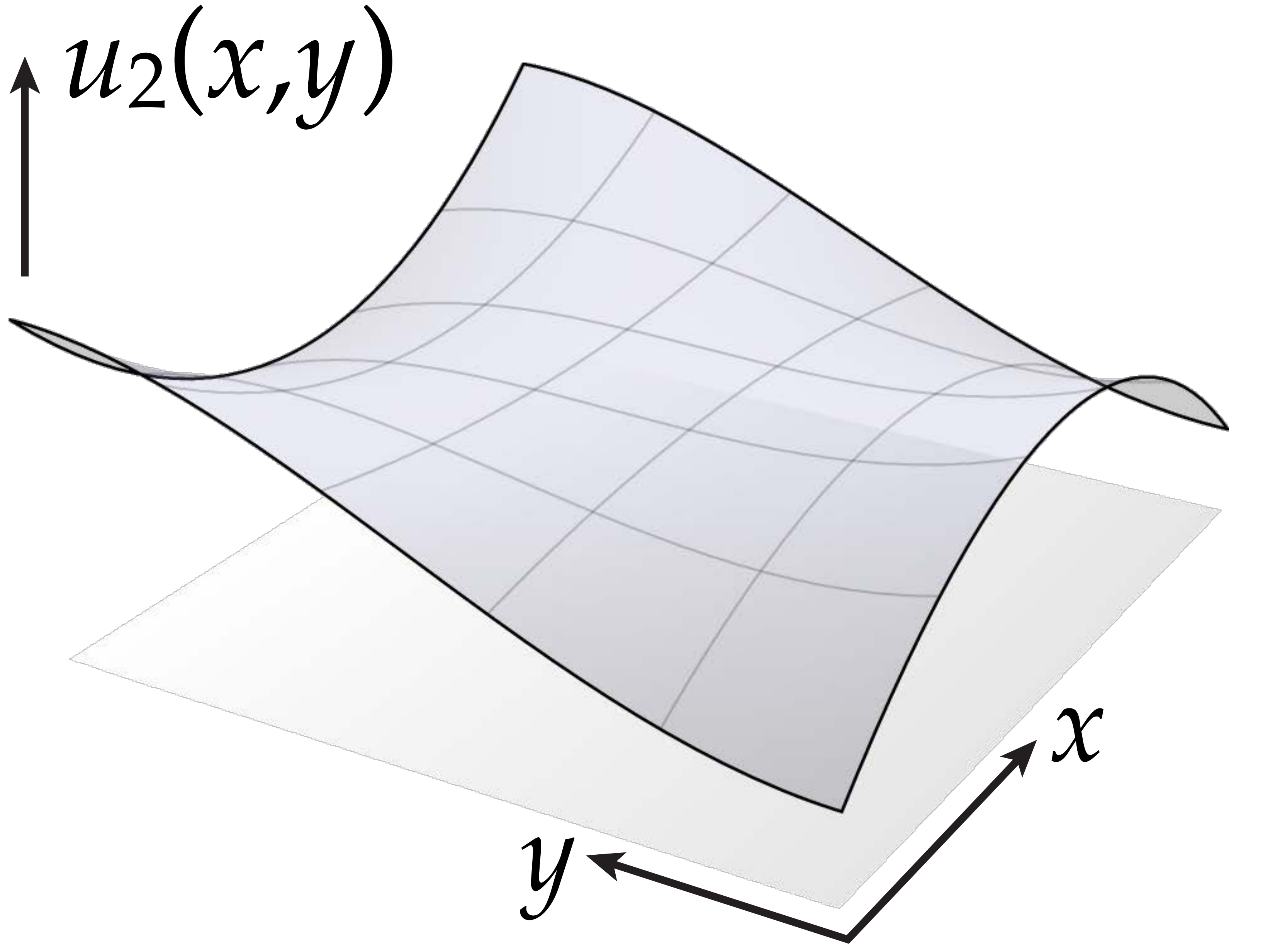


<u>**Example.**</u>

$$u_2(x,y) = x^3 - 3xy^2$$

$$\Delta u_2 = \frac{\partial^2}{\partial x^2}(x^3 - 3xy^2) + \frac{\partial^2}{\partial y^2}(x^3 - 3xy^2) =$$

$$6x - 6x = 0$$



Question: what does the Laplacian *mean?*

Second Derivative—Convexity

$$\Delta u(x) = \frac{d^2u}{dx^2}$$

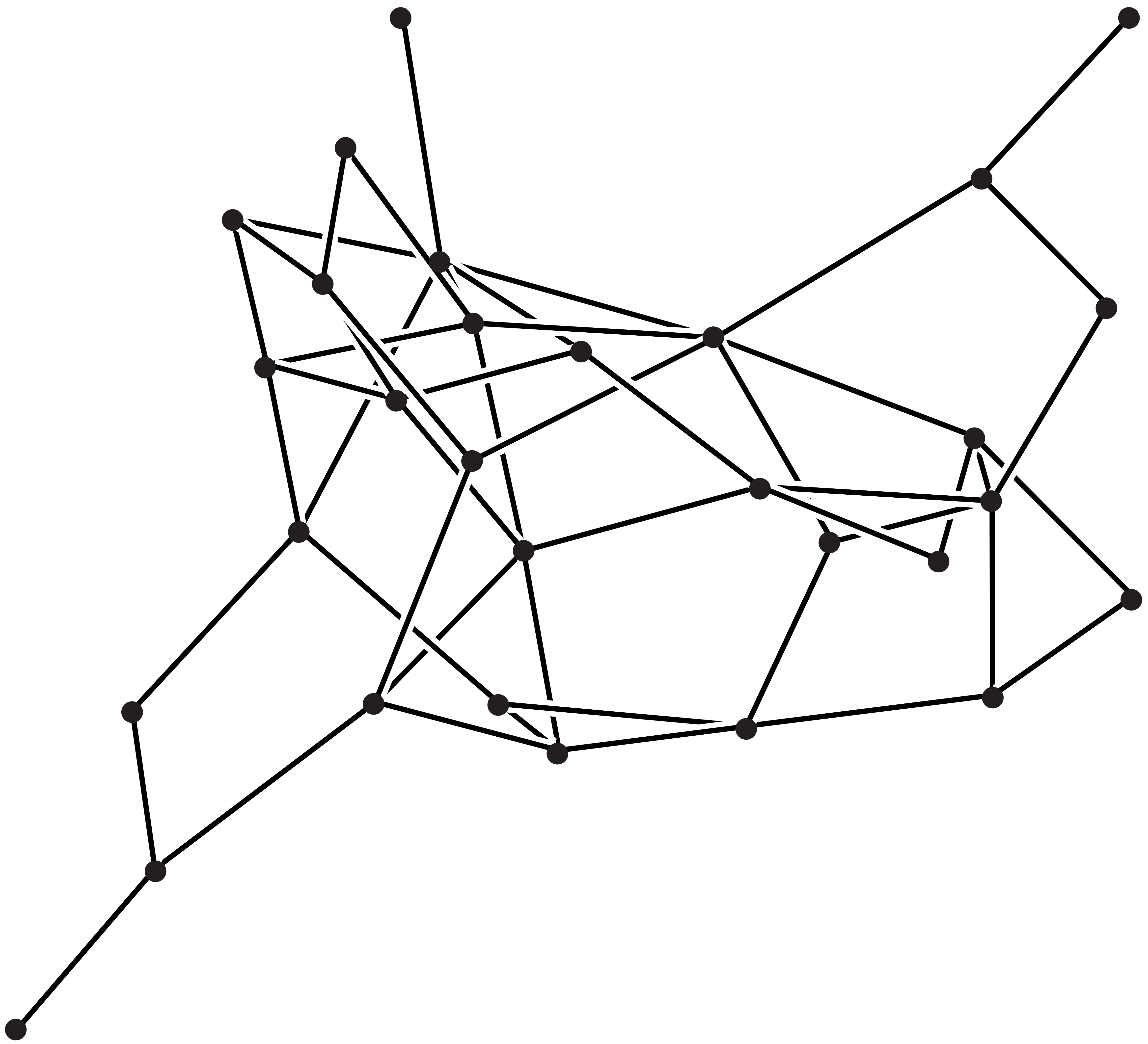
Second Derivative — Curvature

$$\kappa = u''/(1 + (u')^2)^{3/2}$$

$$\boxed{\Delta u(x) = \frac{d^2u}{dx^2}}$$

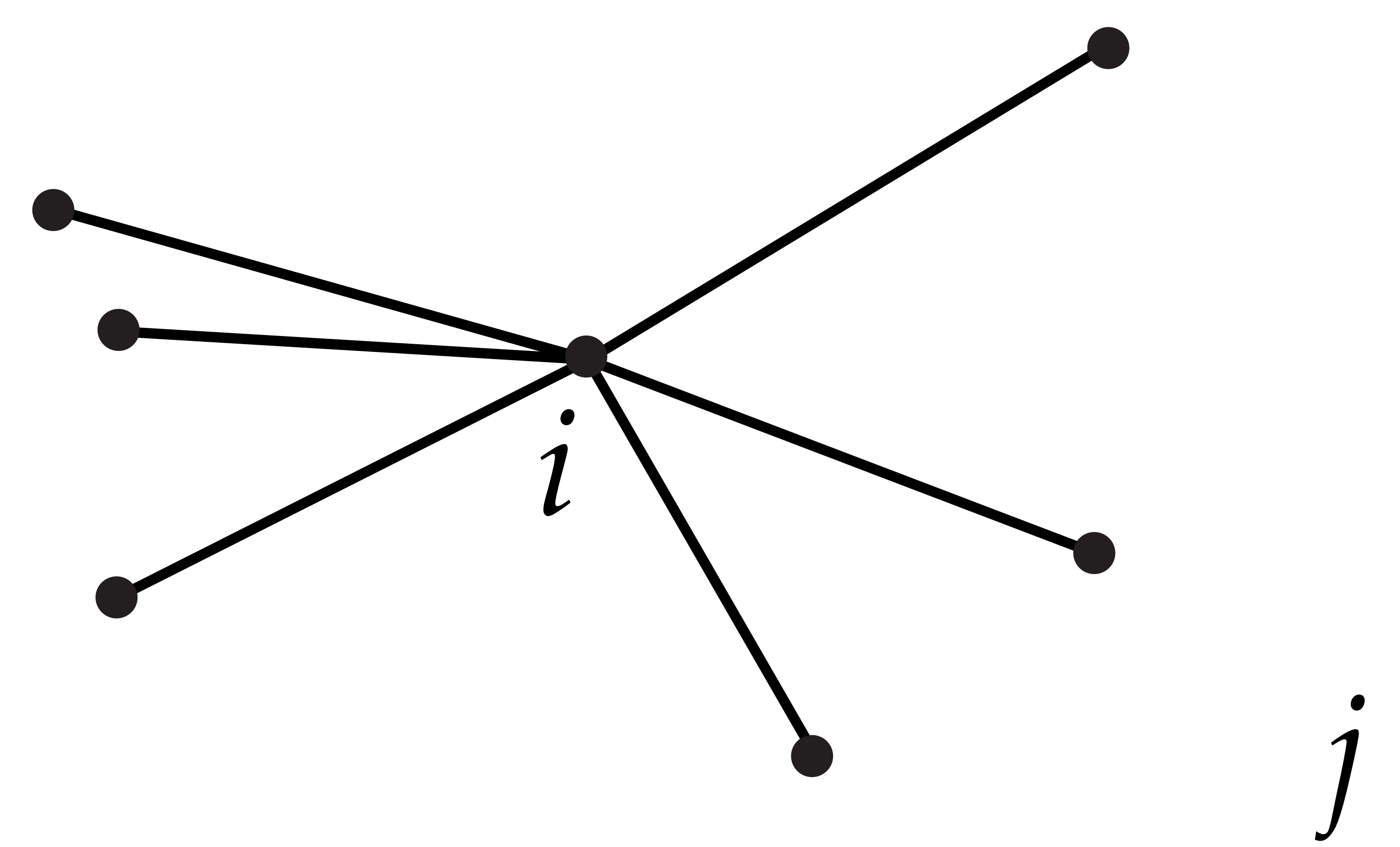
Review: Graph

- Useful discrete analogy for understanding Laplace-Beltrami: *graph Laplacian*
- Recall that a *graph* $G = (V,E)$ is a collection of *vertices* V connected by *edges* E
- **Example:** each vertex represents a person in a social network; two people are connected by an edge if they are friends.



Graph Laplacian

- Suppose we store a value u_i on each vertex i
- *Graph Laplacian L* gives deviation from average value of all neighbors j
- *E.g.*, if values encode the intelligence of each person in the network, then Laplacian says whether, on average, you're more or less intelligent than your friends.

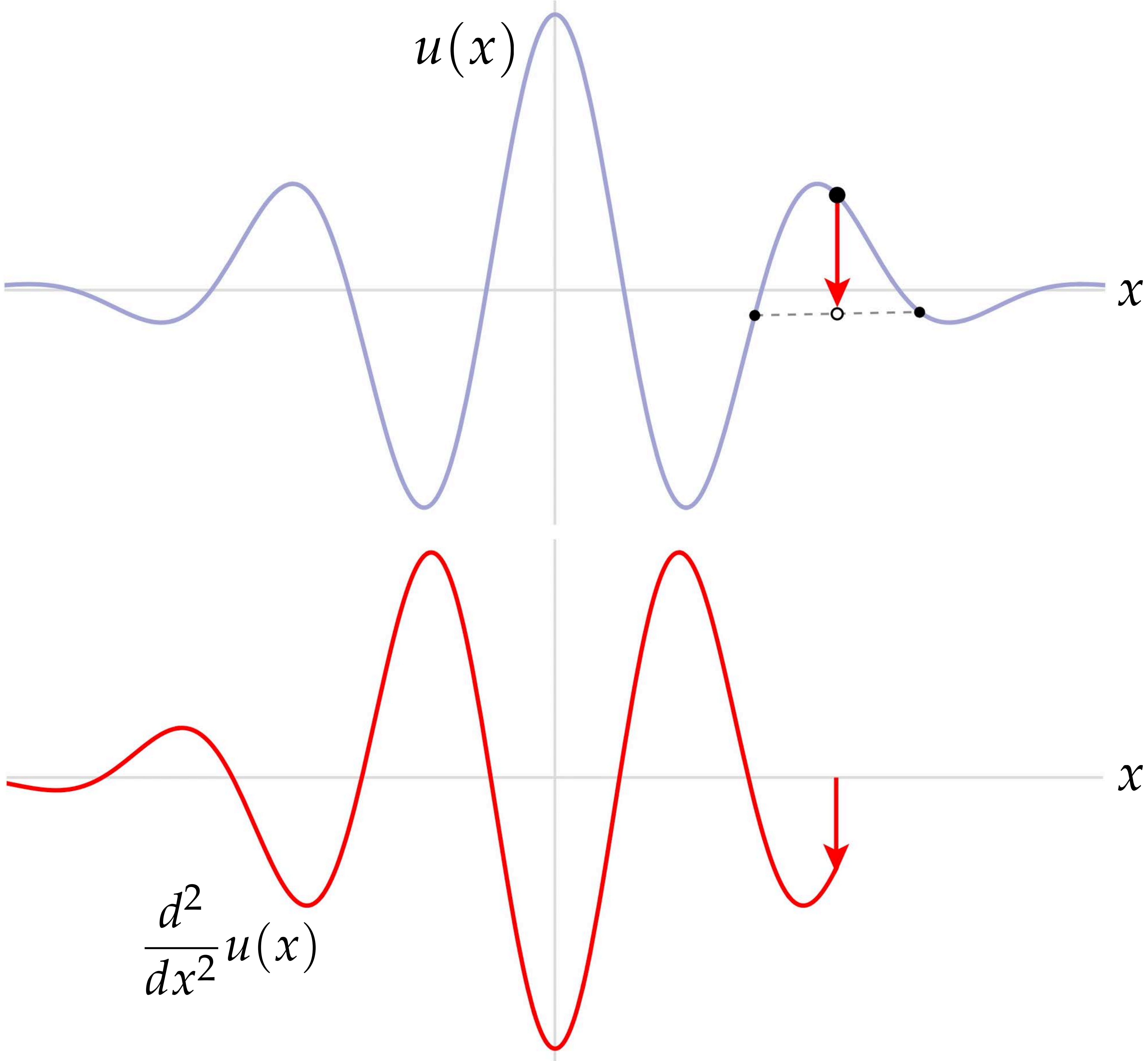


$$(Lu)_i := \left(\frac{1}{\deg(i)} \sum_{ij \in E} u_j \right) - u_i$$

Key idea: Laplacian is deviation from local average

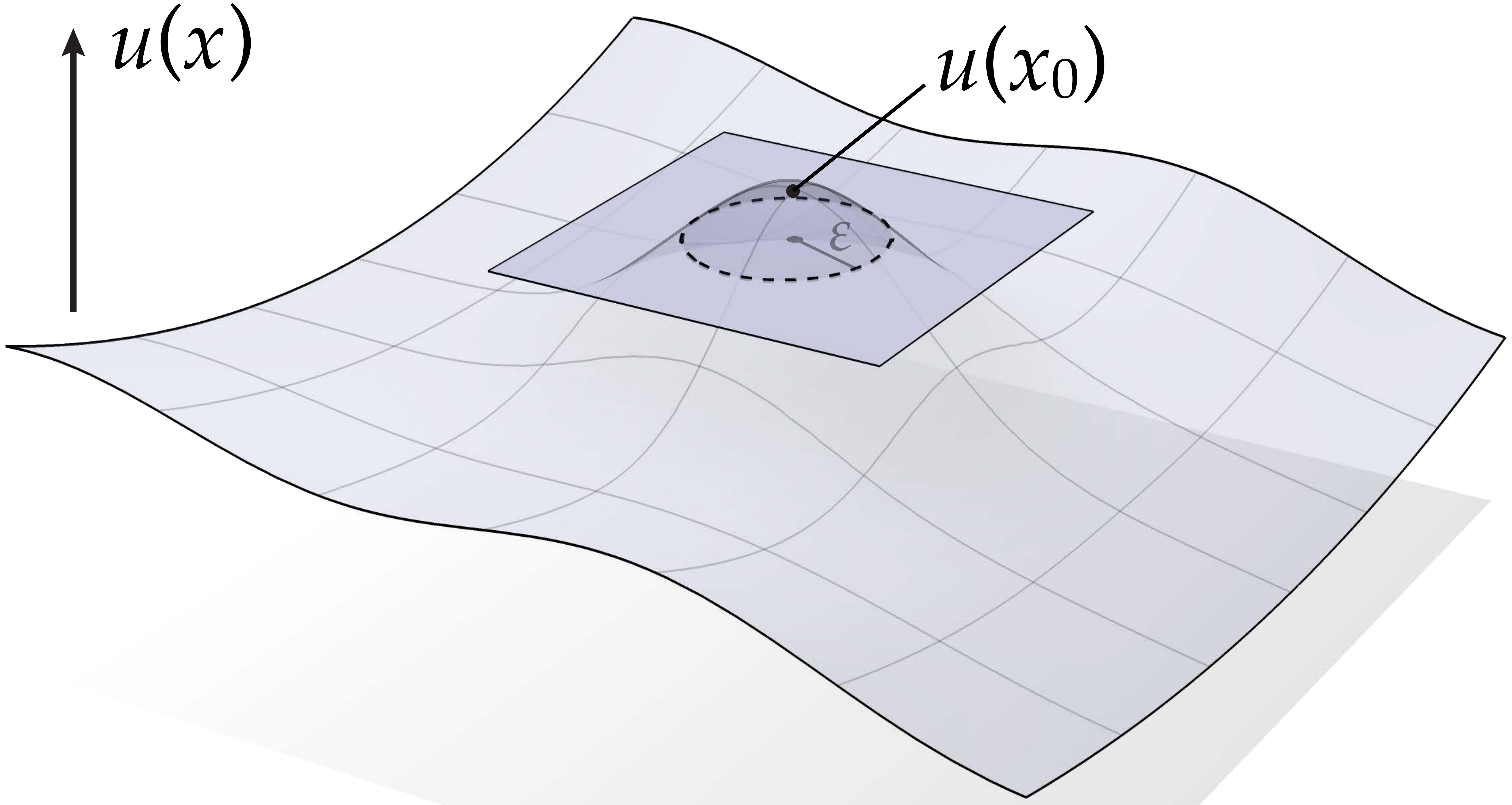
Second Derivative — Deviation from Average

Second Derivative—Deviation from Average



Laplacian — Deviation from Average

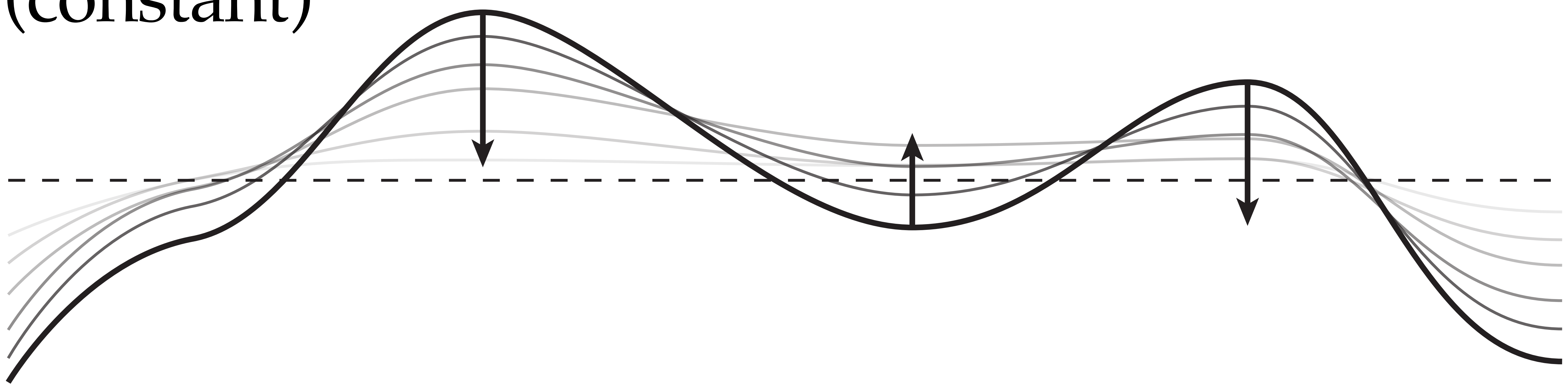
In general, can think of the Laplacian of a function u as difference between value at a point x_0, and the average value over a small sphere (or ball) around x_0.



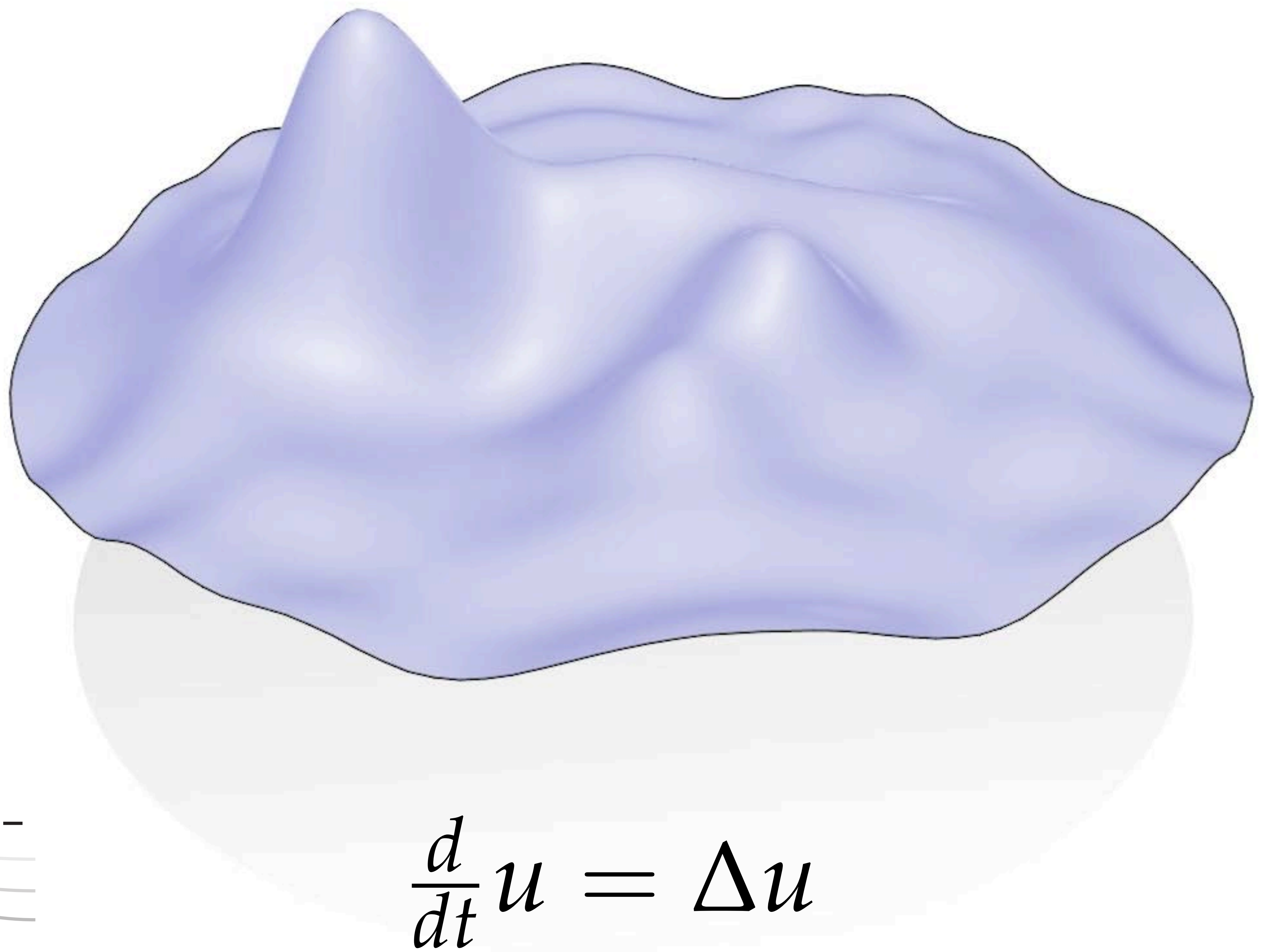
$$\Delta u(x_0) \propto \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\underbrace{\frac{1}{|S_\varepsilon(x_0)|}}_{\text{sphere area}} \underbrace{\int_{S_\varepsilon(x_0)} u(x)\, dx}_{\text{integral over sphere}} - \underbrace{u(x_0)}_{\text{value at center}} \right)$$

Heat Equation

- Averaging perspective provides intuition for basic physical equations
- E.g., *heat equation* says change in function value is equal to Laplacian of function
- Intuitively: at each point in time, value moves toward average of nearby values
- Eventually, all values become the same (constant)



heat equation

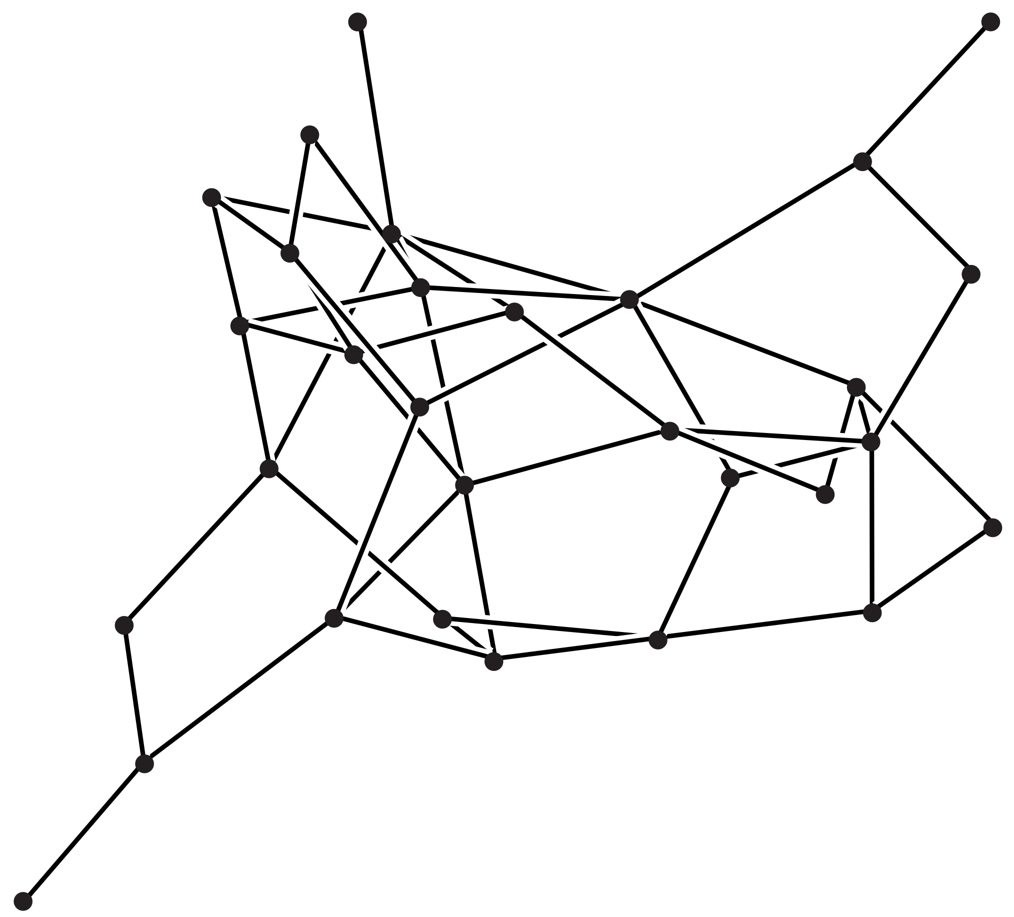


$$\frac{d}{dt}u = \Delta u$$

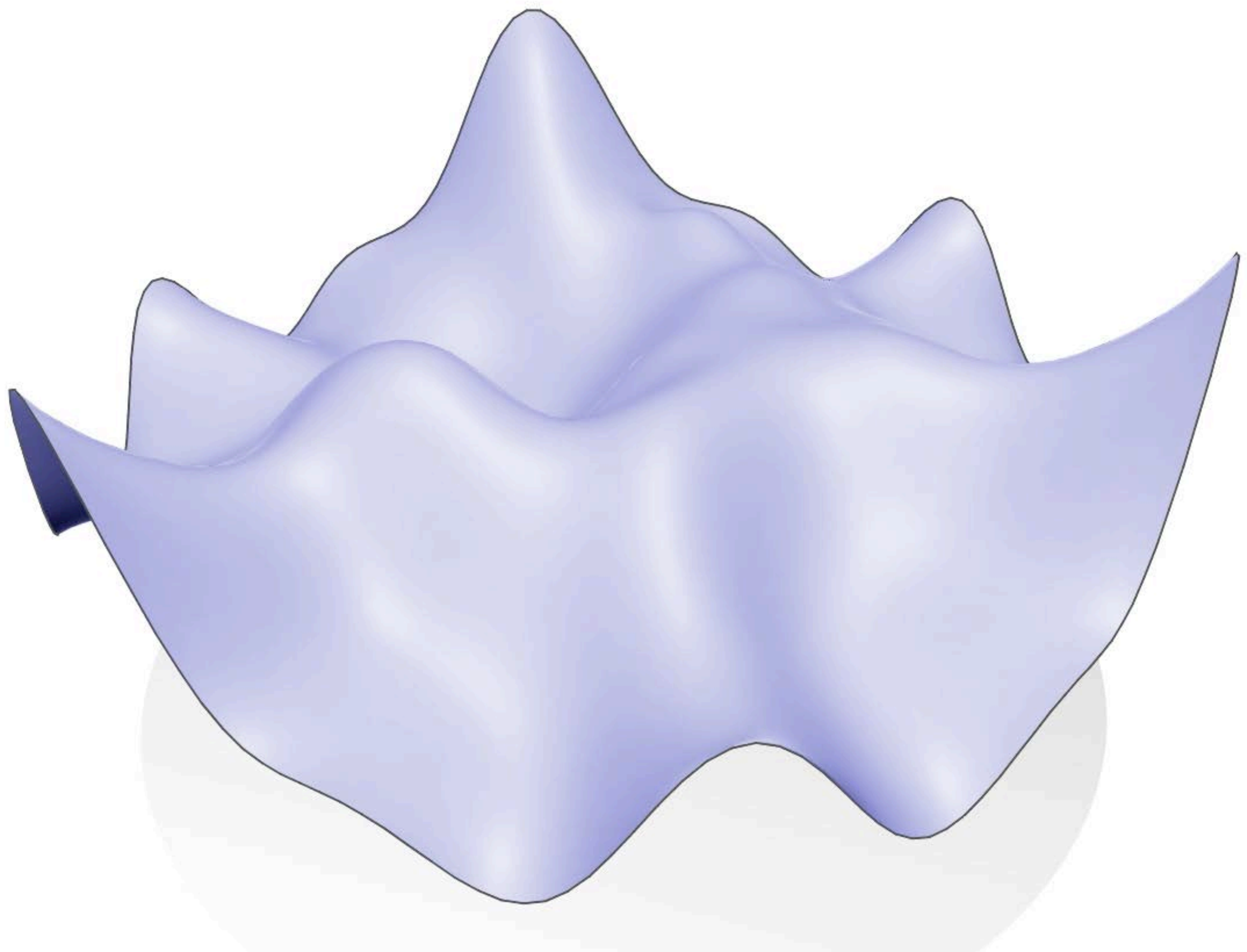
Key idea: concave bumps get "pushed down" / convex bumps get "pushed up"

Laplace equation

- Suppose we keep boundary values fixed, and run the heat equation for a *very* long time…
- Eventually, value at each point will *equal* the average value in a small neighborhood
- Resulting function is called "harmonic," solution to *Laplace equation*
- Graph analogy: everybody in a social network is, on average, just as intelligent as all their friends



$$\frac{d}{dt}u = \Delta u, \quad u|_{\partial\Omega} = g$$



$$\frac{d}{dt}u = 0 \quad \Longrightarrow \quad \boxed{\Delta u = 0}$$

Laplace equation

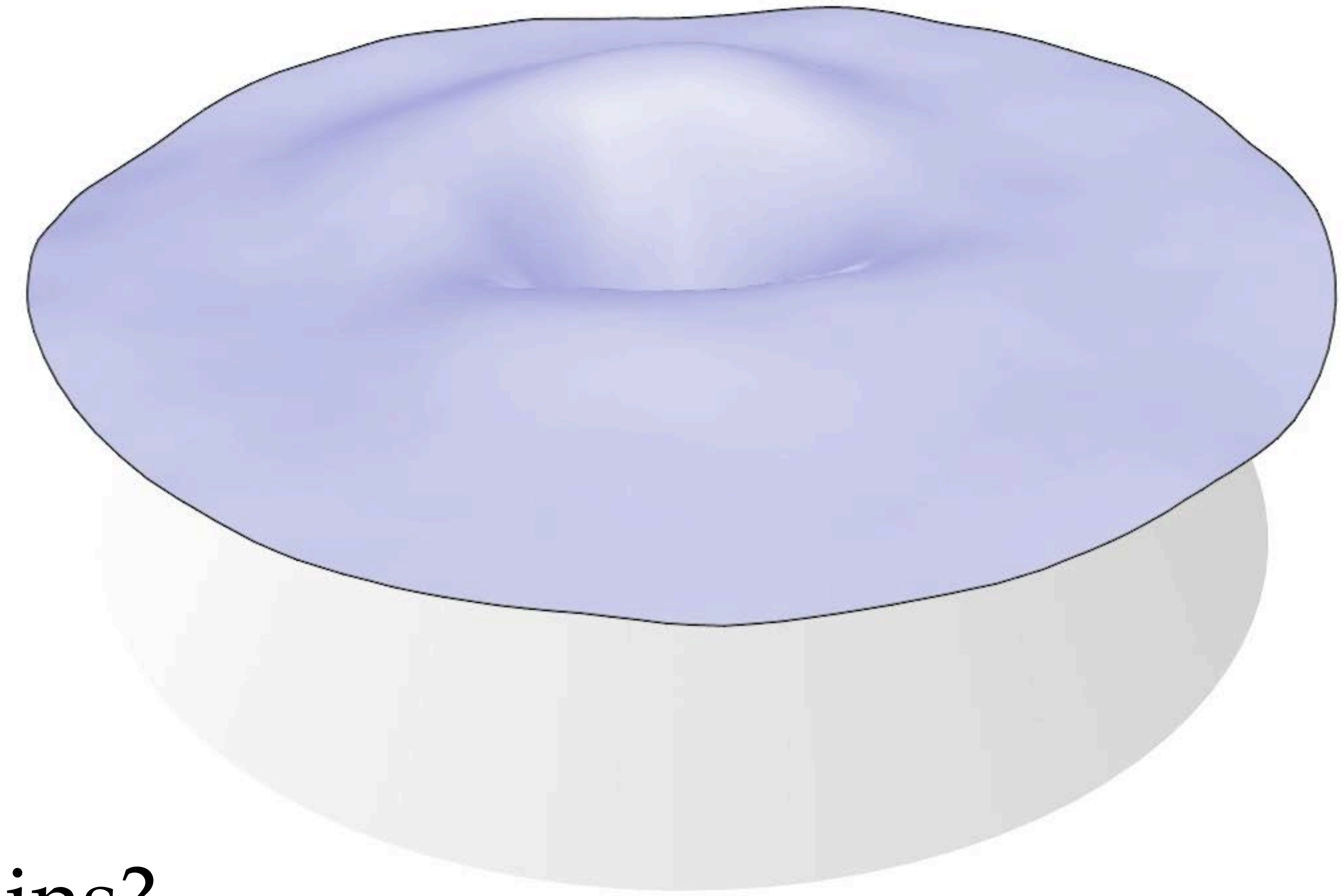
Key idea: each value is equal to average of its neighbors

Wave Equation

- Wave equation instead says that change in *velocity* is equal to Laplacian of displacement
- *I.e.*, if a point is above the local average height, it will experience a downward force; if below, an upward force

wave equation

$$\boxed{\frac{d^2}{dt^2}u = \Delta u}$$



Question: how can we generalize to curved domains?

Definitions & Basic Properties

Many Definitions

In the smooth setting, there are many equivalent ways to express the Laplacian:

$$\boxed{\Delta u = \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon^2} \left(u - \mathrm{mean}_{S_\varepsilon}(u)\right)}$$

deviation from local average

$$\boxed{\Delta u = \ast d \ast d u}$$

differential forms

$$\boxed{\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}}$$

sum of partial derivatives

$$\boxed{\Delta u = \mathrm{tr}(\nabla^2 u)}$$

trace of Hessian

$$\boxed{\Delta u = \delta^2 \int_M |\nabla u|^2 \, dV}$$

Hessian of Dirichlet energy

$$\boxed{\Delta u = \nabla \cdot \nabla u}$$

divergence of gradient

$$\boxed{\Delta u = \left.\tfrac{d}{dt}\right|_{t=0} \mathbb{E}[u(X_t)]}$$

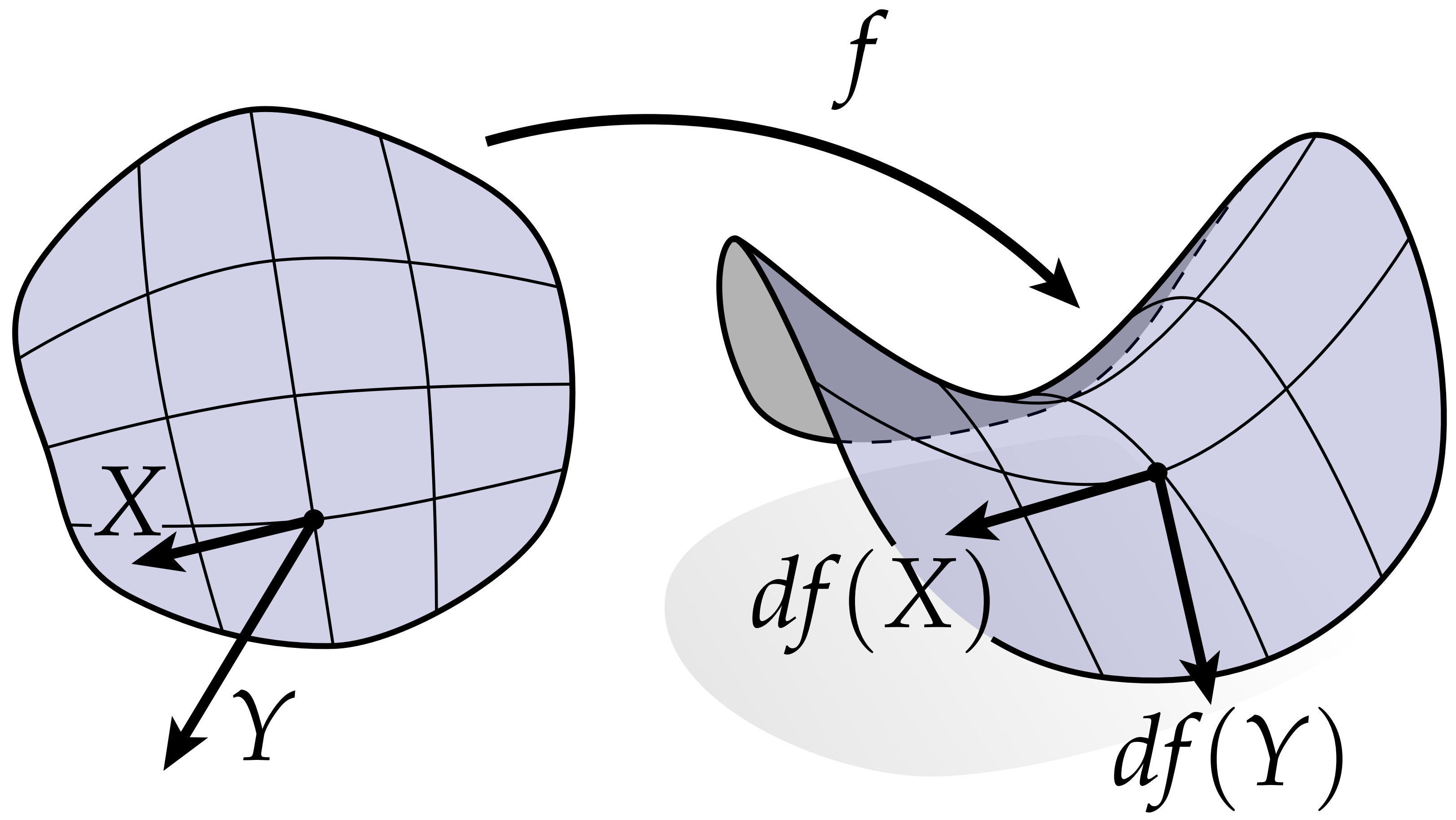
Brownian motion (random walks)

Most of these apply directly to curved domains (Laplace-Beltrami)…

Sum of Partial Derivatives

Riemannian metric

$$g : T_pM \times T_pM \to \mathbb{R}$$



$$g(X,Y) := \langle df(X), df(Y) \rangle$$

Laplace-Beltrami operator

$$\boxed{\Delta u = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} (g^{-1})_{ij} \frac{\partial}{\partial x_j} u \right)}$$

Euclidean case (2D):

$$g = g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \det(g) = 1$$

$$\implies \Delta u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$$

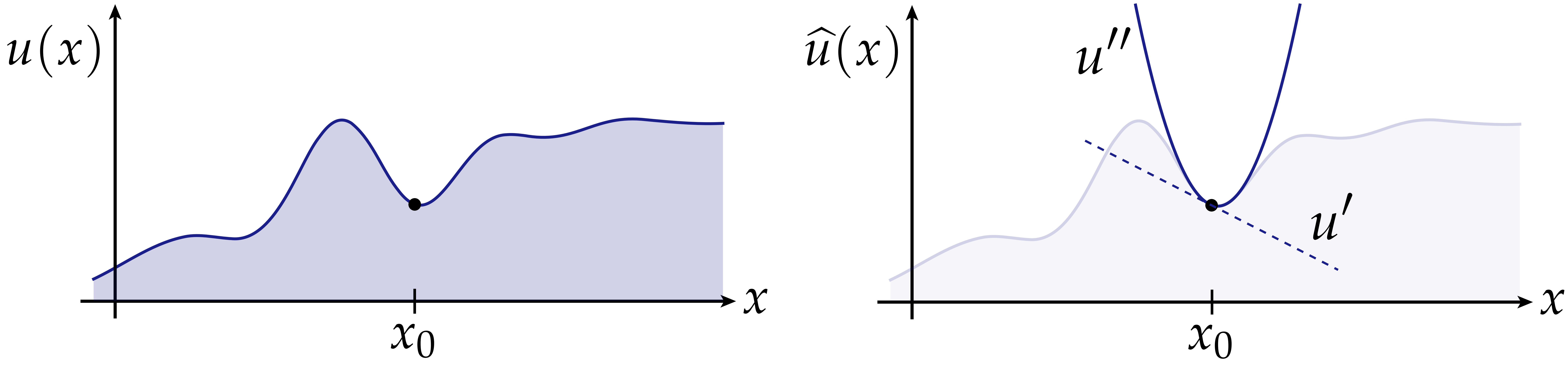
Note: rarely used as a starting point for numerics/algorithms…

Review: Hessian

- In R^n, Hessian $\nabla^2 u$ of a function u is matrix of second partial derivatives
- Provides "best quadratic approximation" of u around a point

$$u : \mathbb{R}^2 \to \mathbb{R}$$

$$\nabla^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix}$$



$$\hat{u}(x) \approx u(x_0) + (x - x_0)u'(x_0) + \tfrac{1}{2}(x - x_0)^2 u''(x_0)$$

Review: Hessian

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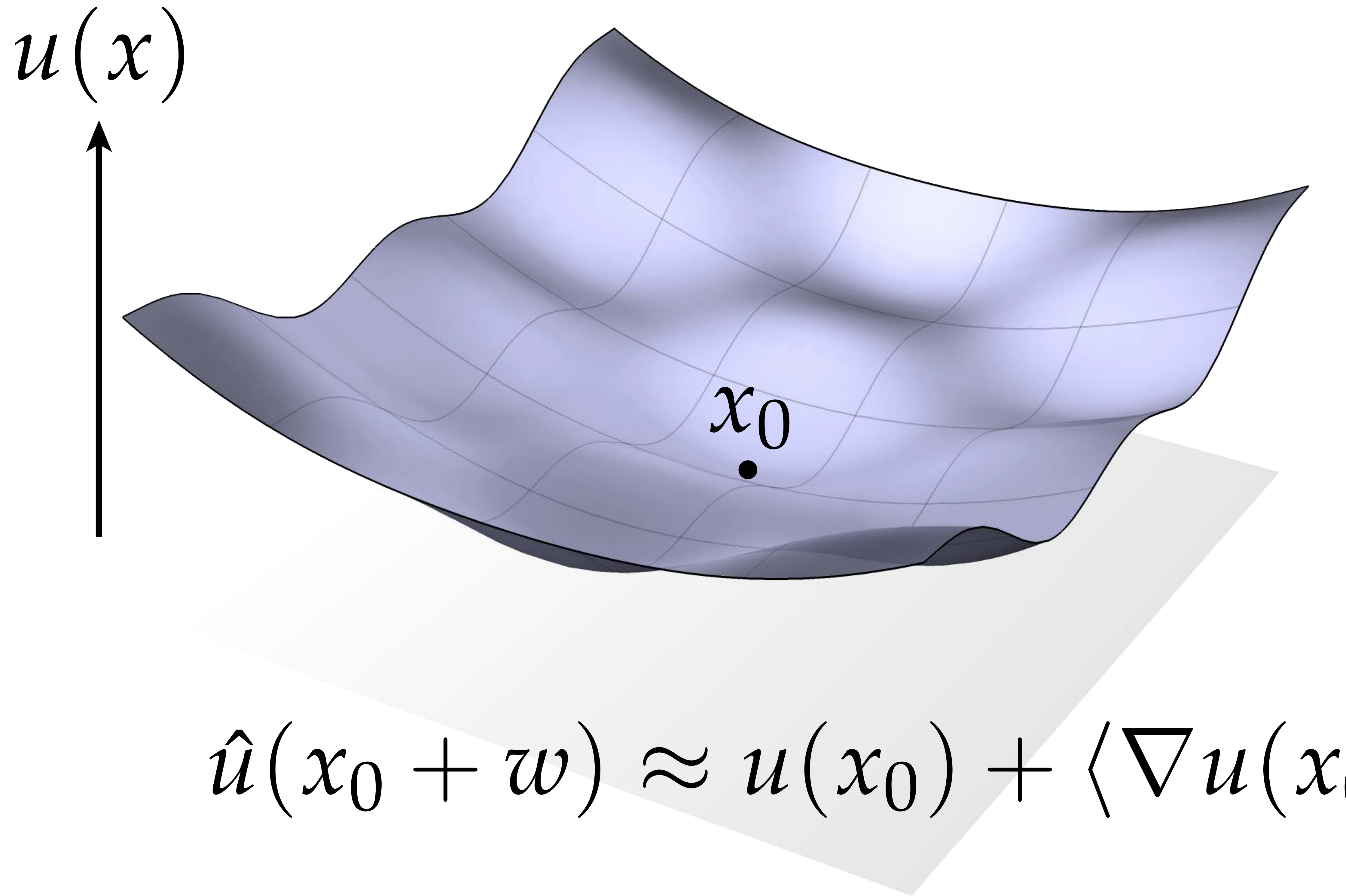
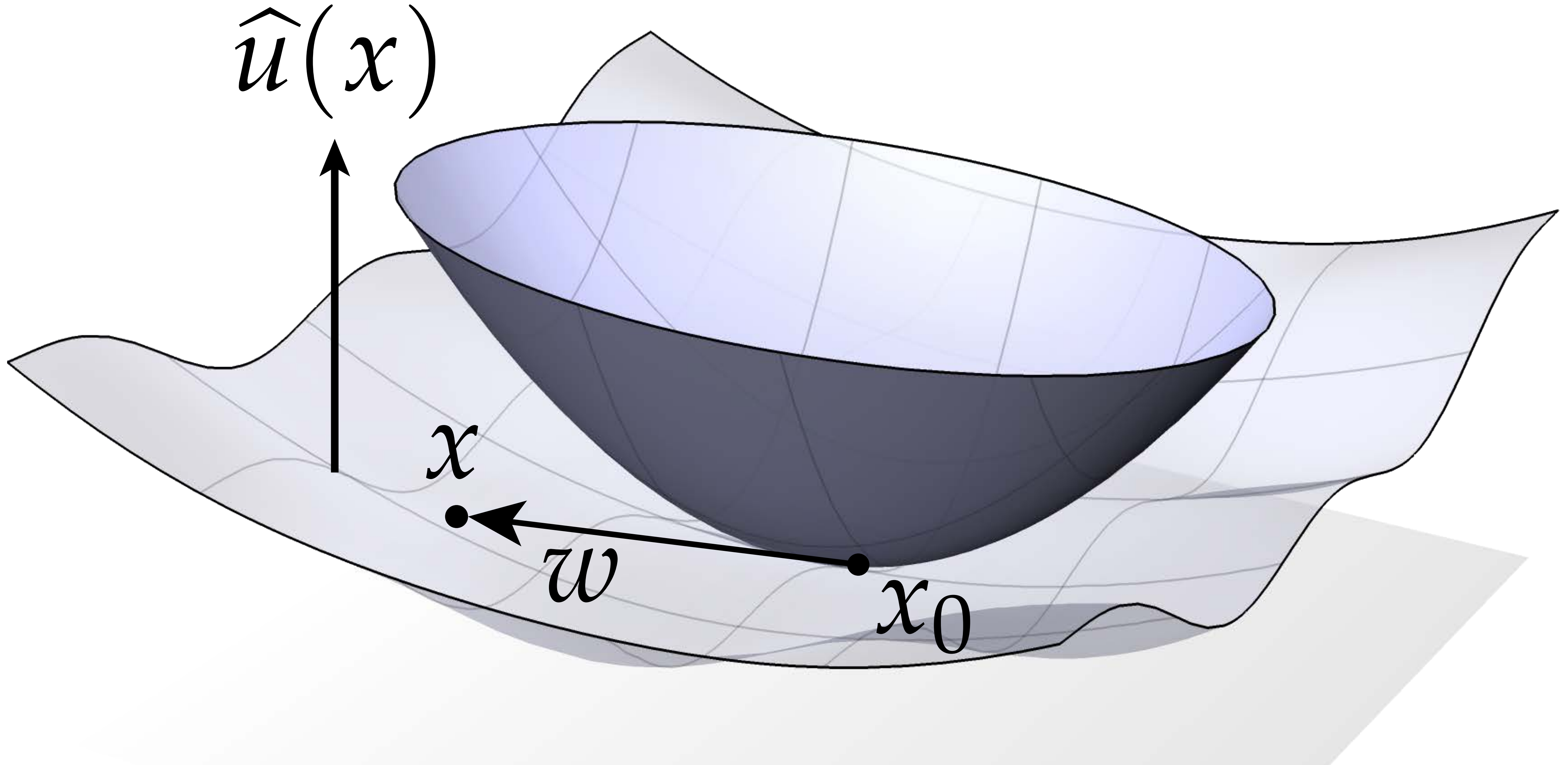




$$\hat{u}(x_0 + w) \approx u(x_0) + \langle \nabla u(x_0), w \rangle + \tfrac{1}{2} \langle \nabla^2 u(x_0) w, w \rangle$$

Laplacian via Hessian

- Laplacian is the *trace* of the Hessian
  - In R^n: just the sum of diagonal elements
- Can also express Hessian as directional derivative of gradient
- Similar idea on a curved surface:
  - first take the exterior derivative of the function (instead of the gradient)
  - then take the *covariant derivative** of the resulting 1-form to get the Hessian
  - Laplacian is again the trace of the Hessian

$$u : \mathbb{R}^2 \to \mathbb{R}$$

$$\mathrm{tr}(\nabla^2 u) = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$$

<u>Euclidean:</u>

$$(\nabla^2 u)(X, Y) = \langle D_X \nabla u, Y \rangle$$

<u>Curved surface:</u>

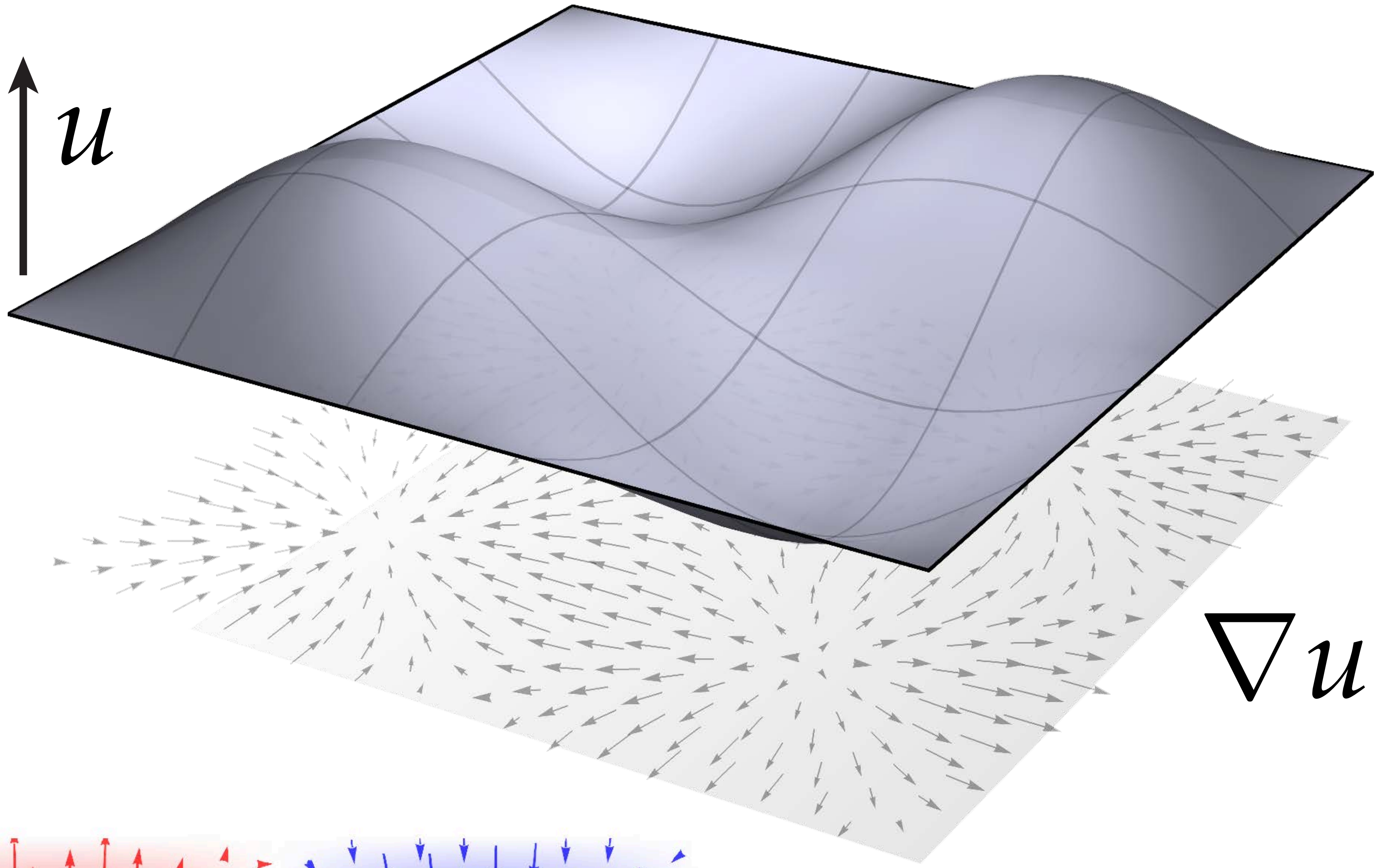
$$\nabla^2_{X,Y} u = (\nabla_X du)(Y)$$

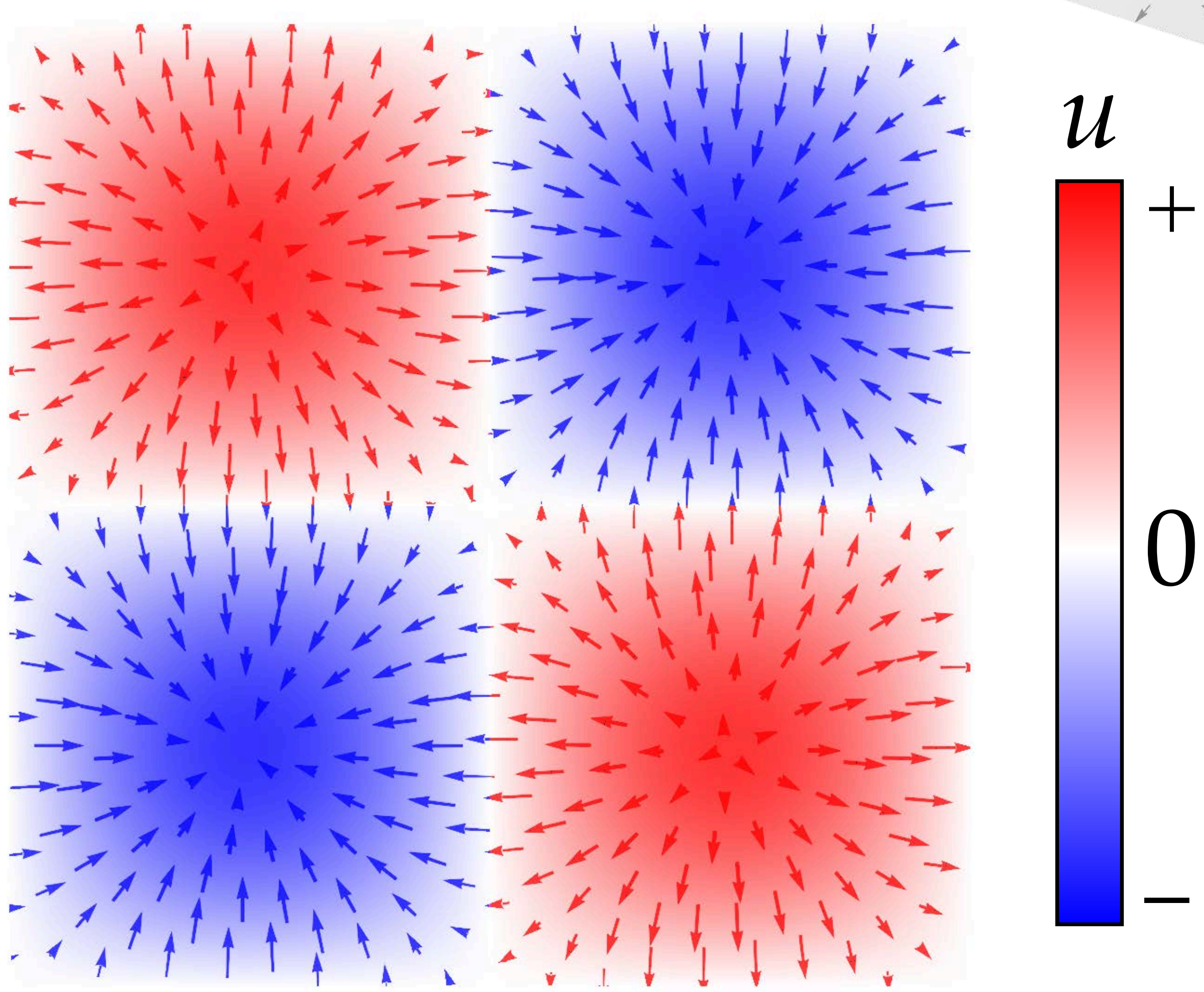
$$\Delta u = \mathrm{tr}(\nabla^2 u) = \mathrm{tr}(\nabla du)$$

*Will define covariant derivative later on…

Laplacian via Divergence of Gradient

- Another common way to express the Hessian: divergence of the gradient
- Gradient of any function u gives vector field that points in direction of "steepest ascent"
  - maxima become sinks; minima become sources
- Divergence of any vector field X measures how much it locally behaves like a sink/source
- Laplacian will therefore be positive near minima, negative near maxima
- Can generalize to manifolds using our grad/div operators for curved domains…





$$\nabla \cdot \nabla u = \Delta u$$

Laplacian via Exterior Calculus

- To express grad, div, and curl on curved domains, we used the exterior derivative d & Hodge star $\ast$
- By composing these operators and simplifying, we get another nice expression for the Laplacian
- For surfaces, nicely splits up geometric aspects of operator
- **Bonus:** easy to implement numerically via *discrete* exterior calculus

grad

$$\nabla u = (du)^\sharp$$

div

$$\nabla \cdot X = \ast d \ast X^\flat$$

$$\Delta u = \nabla \cdot \nabla u = \ast d \ast ((du)^\sharp)^\flat = \ast d \ast d\, u$$

Laplace-Beltrami

$$\boxed{\Delta = \ast d \ast d}$$

0-form Hodge star
(area form)

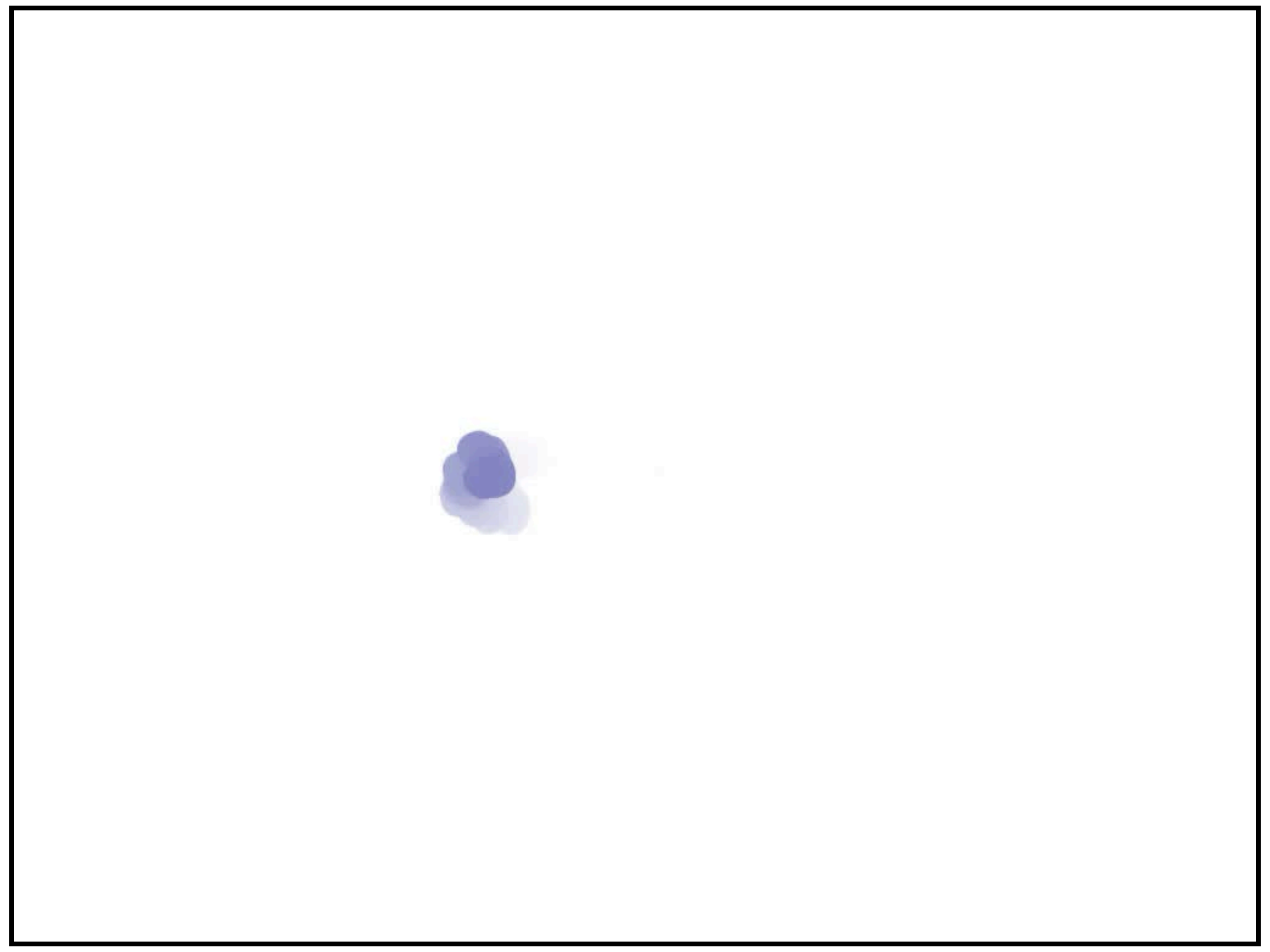
1-form Hodge star
(conformal structure)

Laplacian via Random Walks

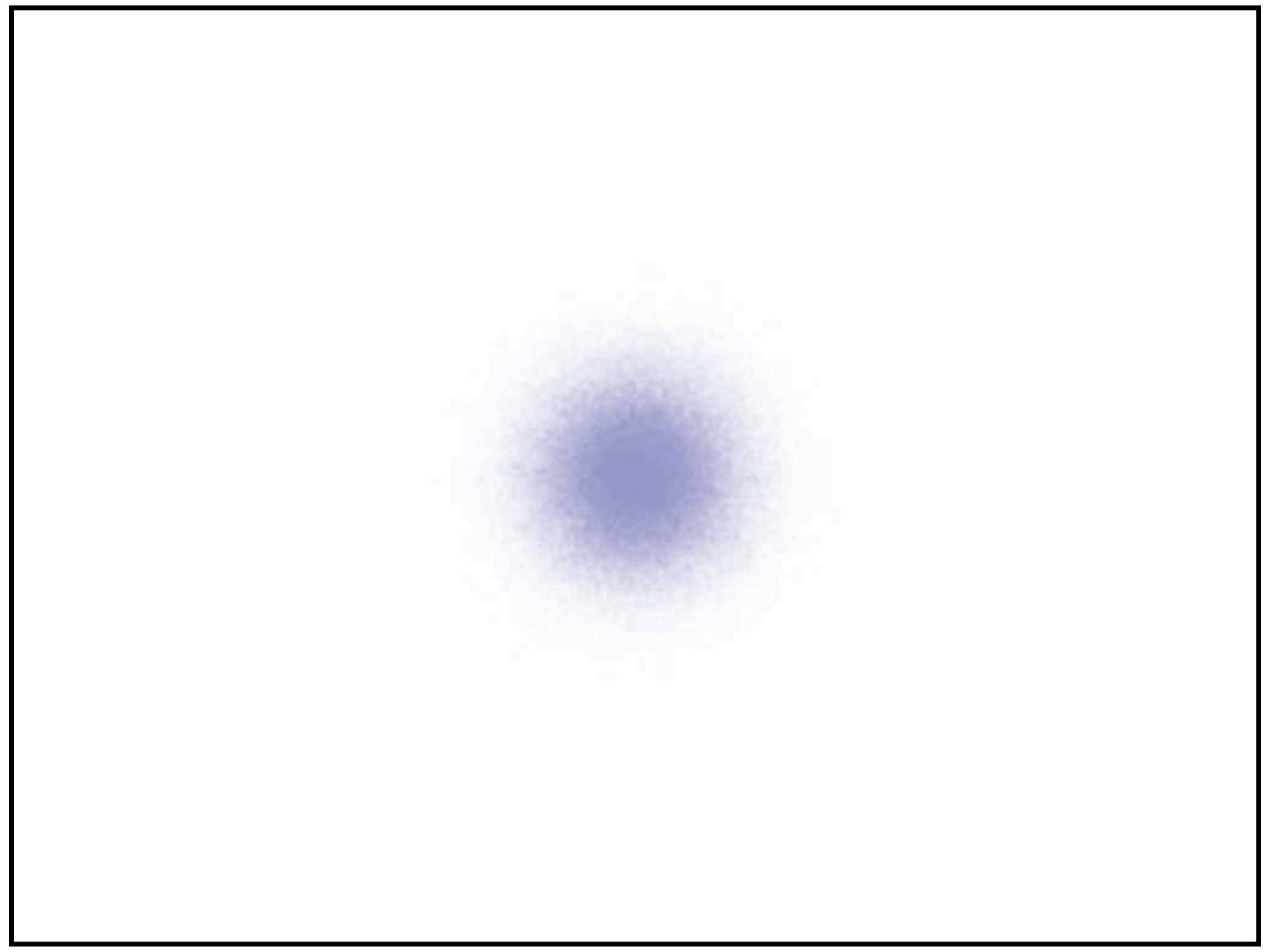
- Deep connection between Laplacian and random walks—formally: *Brownian motion* X_t
- Average location of many random walks approaches *heat kernel* $k_t(x,y)$
  - – heat diffused from x to y after time t
- Heat kernel is "fundamental solution" to heat equation
- Laplacian of function is hence change in average value seen by a random walker over time ("infinitesimal generator")

Intuition: Δu is difference between function and "blurred version" of function.

Brownian motion X_t



heat kernel $k_t(x,y)$



$$\begin{cases} \frac{d}{dt}u = \Delta u \\ u|_{t=0} = \phi \end{cases} \implies u(t,x) = \underbrace{\int_\Omega k_t(x,y)\phi(y)\,dy}_{\mathbb{E}[\phi(X_t)]}$$

$$\Delta\phi = \lim_{t\to 0} \tfrac{1}{t}(u(t) - u(0))$$

$$\boxed{\Delta\phi = \tfrac{d}{dt}|_{t=0}\mathbb{E}[\phi(X_t)]}$$

Laplacian via Dirichlet Energy

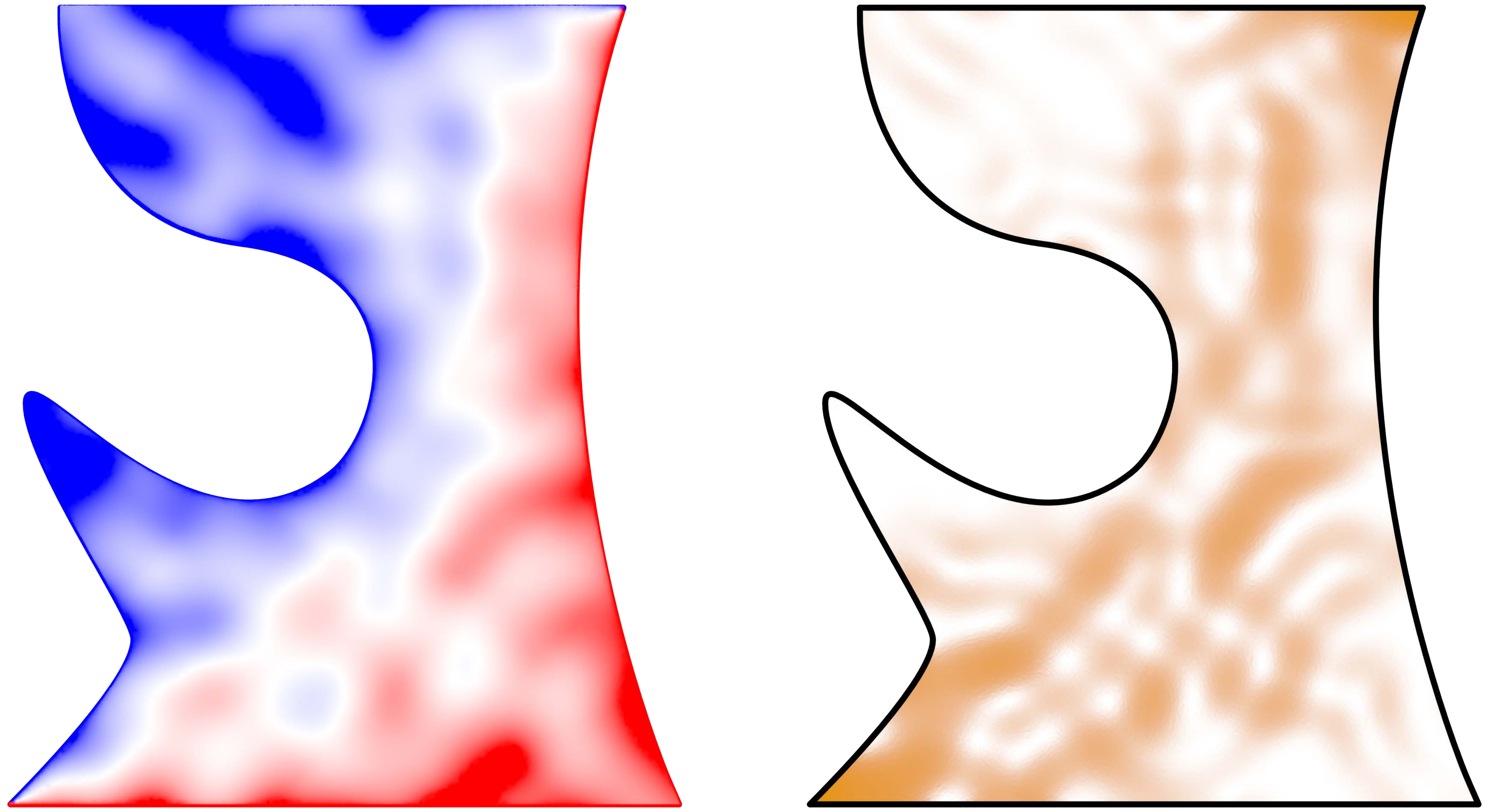
- Finally, can understand Laplacian in terms of the *Dirichlet energy*
- Common notion of regularity / "smoothness" arising in geometry, physics, & algorithms
- Natural starting point for discretization, e.g., *finite element methods*
- Can use Laplacian to express Dirichlet energy as a quadratic form:

$$\langle\langle \Delta u, u \rangle\rangle = \int_M u \Delta u \, dV$$

- Will take a closer look later, via a basic *interpolation problem*

Dirichlet energy

$$\boxed{\int_M |\nabla u|^2 \, dV}$$

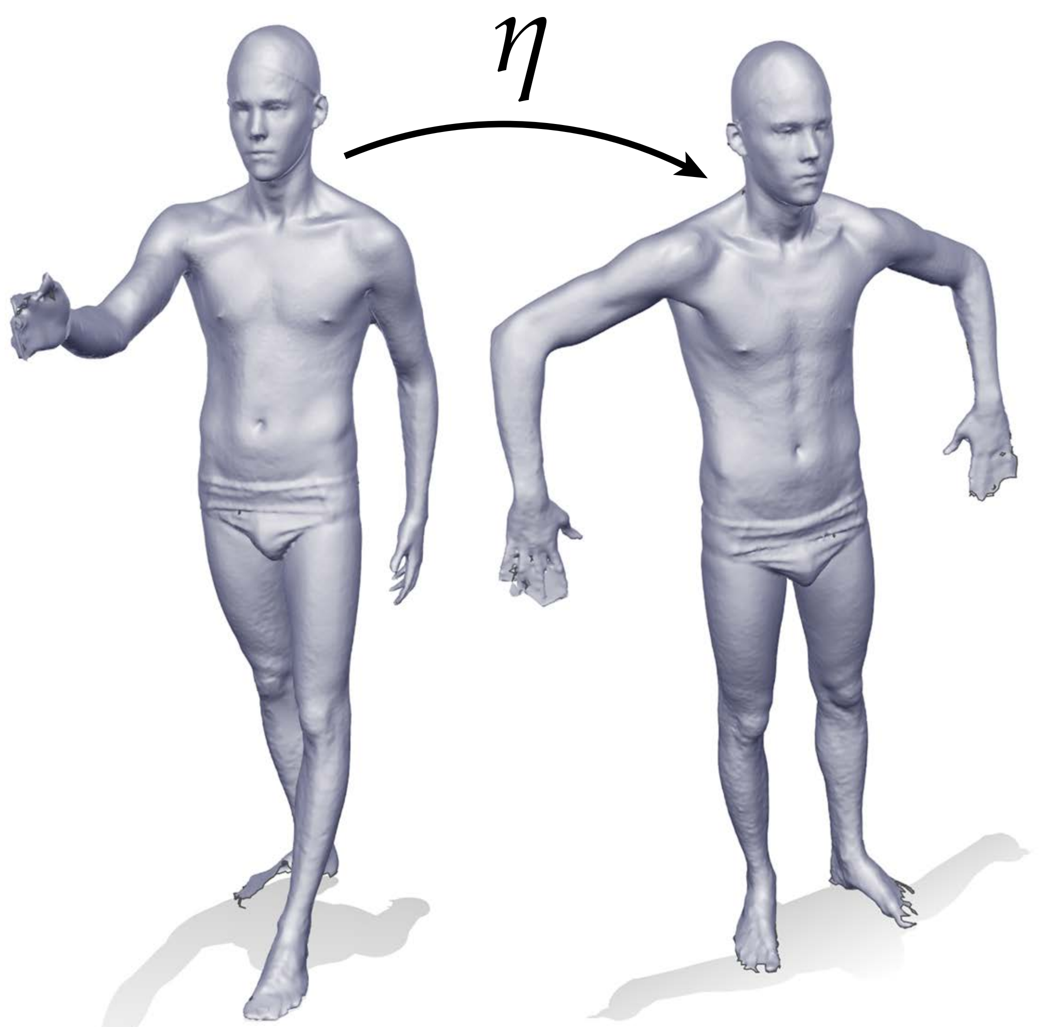


u $\qquad$ $|\nabla u|^2$

Some Basic Properties

- Constant functions in the kernel
  - in R^n: linear functions in the kernel
- Invariant to rigid motions
- In fact, invariant to *isometries*
  - *e.g.*, f and $\eta \circ f$ give same induced metric
- Self-adjoint (analogy: symmetric)
- Elliptic (loose analogy: positive definite)
  - both $x^{\mathsf{T}}Ax$ and -$\langle \Delta u, u \rangle$ are *convex* / *"bowl shaped"*
  - have a unique minimizer (up to constants)

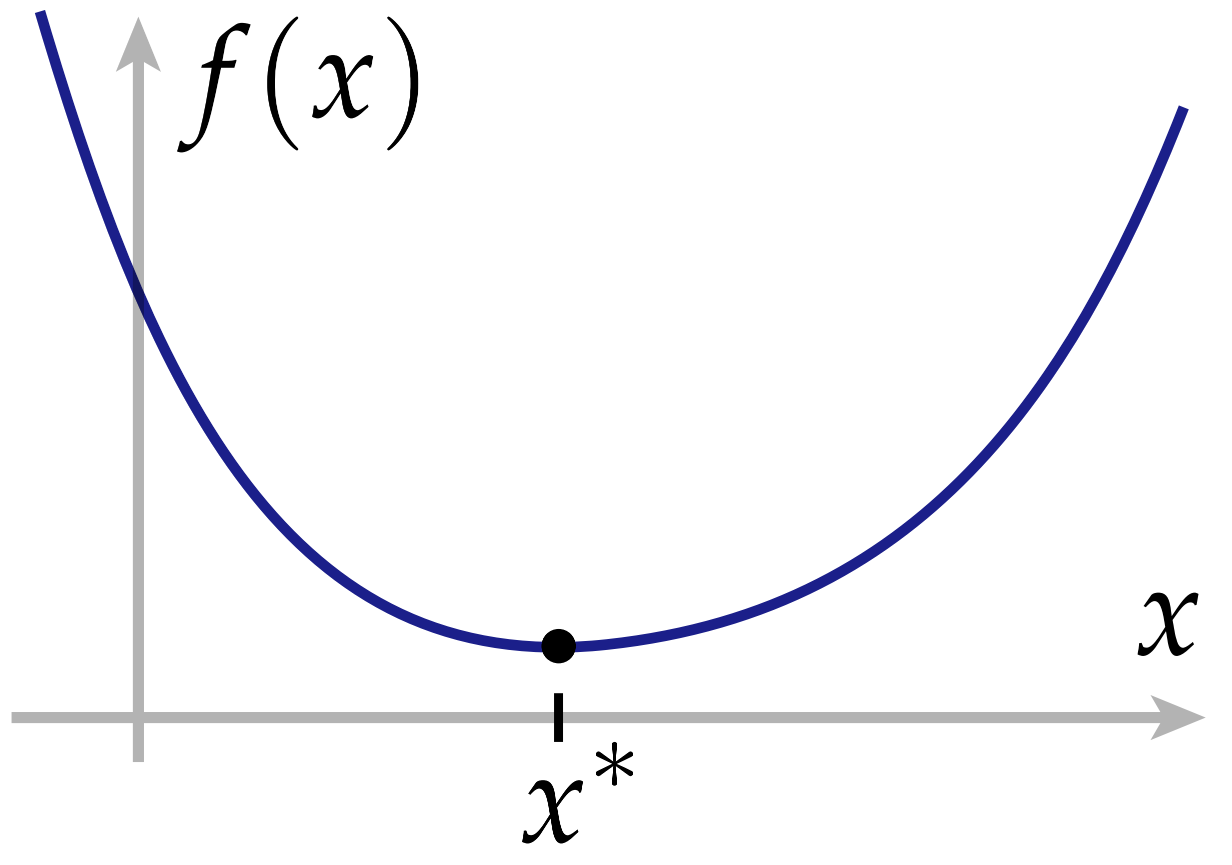
$$\Delta u = 0, \quad u(x) = c \in \mathbb{R}$$



$$\eta \circ \Delta = \Delta \circ \eta$$

<u>self-adjoint</u>

$$\int_M u \Delta v = \int_M v \Delta u$$



Key idea: Laplacian behaves like an (*almost* invertible) positive-semidefinite matrix.

Spectral Properties

- *Review:* **spectral theorem.** Real symmetric matrix A has
  - real eigenvalues $\lambda_1, \ldots, \lambda_n$
  - orthogonal eigenvectors $e_1, \ldots, e_n$

$$\boxed{\begin{gathered} A^T = A \\ Ae_i = \lambda_i e_i \end{gathered}}$$

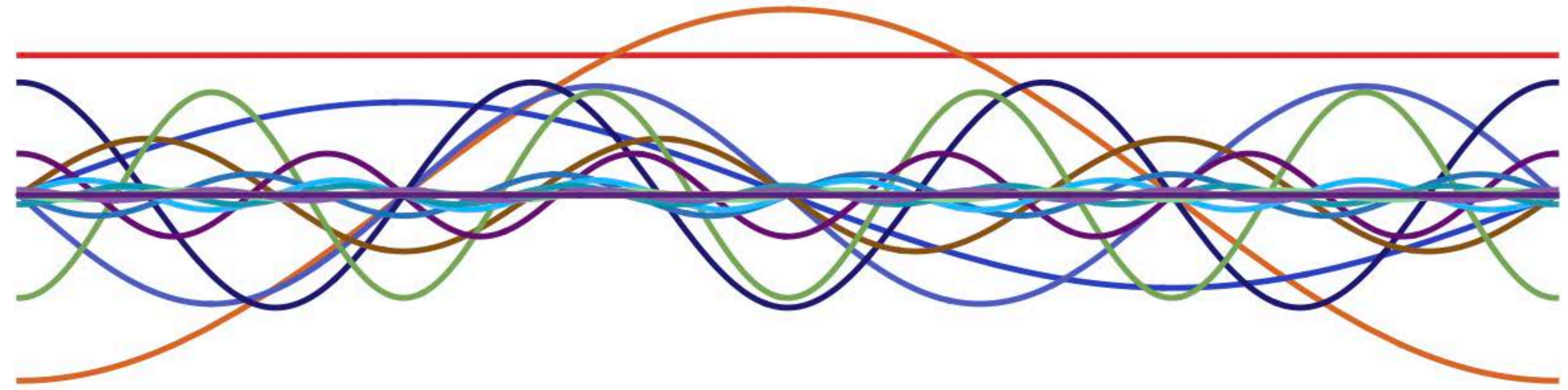
- Likewise, self-adjoint elliptic operator L on a compact domain has:
  - a discrete set of eigenvalues $\lambda_1, \lambda_2, \ldots$
  - orthogonal eigenfunctions $\phi_1, \phi_2, \ldots$
- *E.g.*, 2nd derivative operator on circle
  - basis for Fourier analysis/signal processing

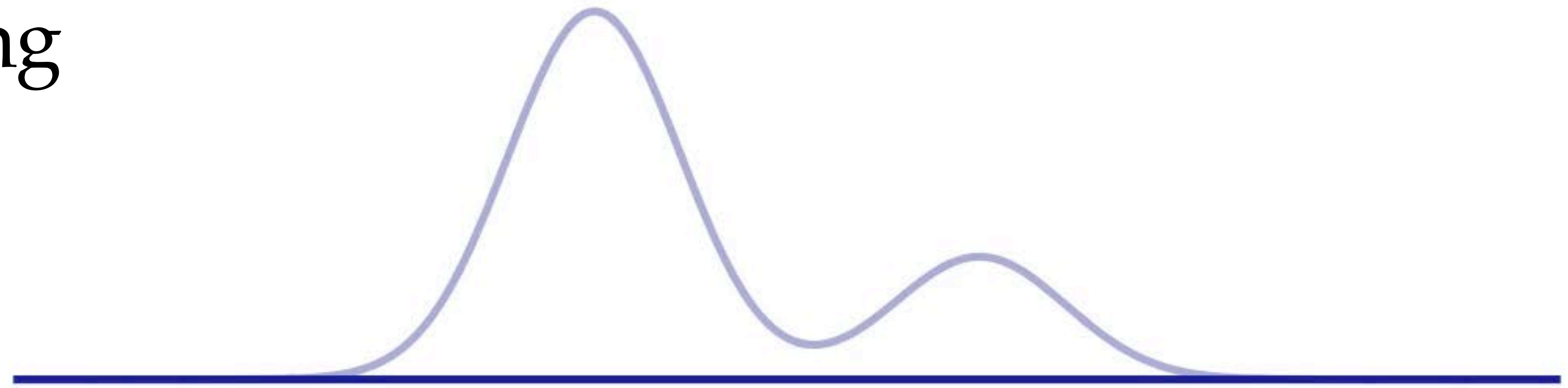
<u>**Example:**</u> 2nd derivative on $S^1=[0,2\pi)$

$$\int_{S^1} uv''\,dx = -\int_{S^1} u'v'\,dx = \int_{S^1} u''v\,dx$$

$$\frac{d^2}{dx^2}\cos(nx) = -n^2\cos(nx)$$

$$\frac{d^2}{dx^2}\sin(nx) = -n^2\sin(nx)$$





Spectral Properties

- *Review:* **spectral theorem.** Real symmetric matrix A has

$$\boxed{\begin{array}{c} A^T = A \\ Ae_i = \lambda_i e_i \end{array}}$$

  - real eigenvalues $\lambda_1, \ldots, \lambda_n$
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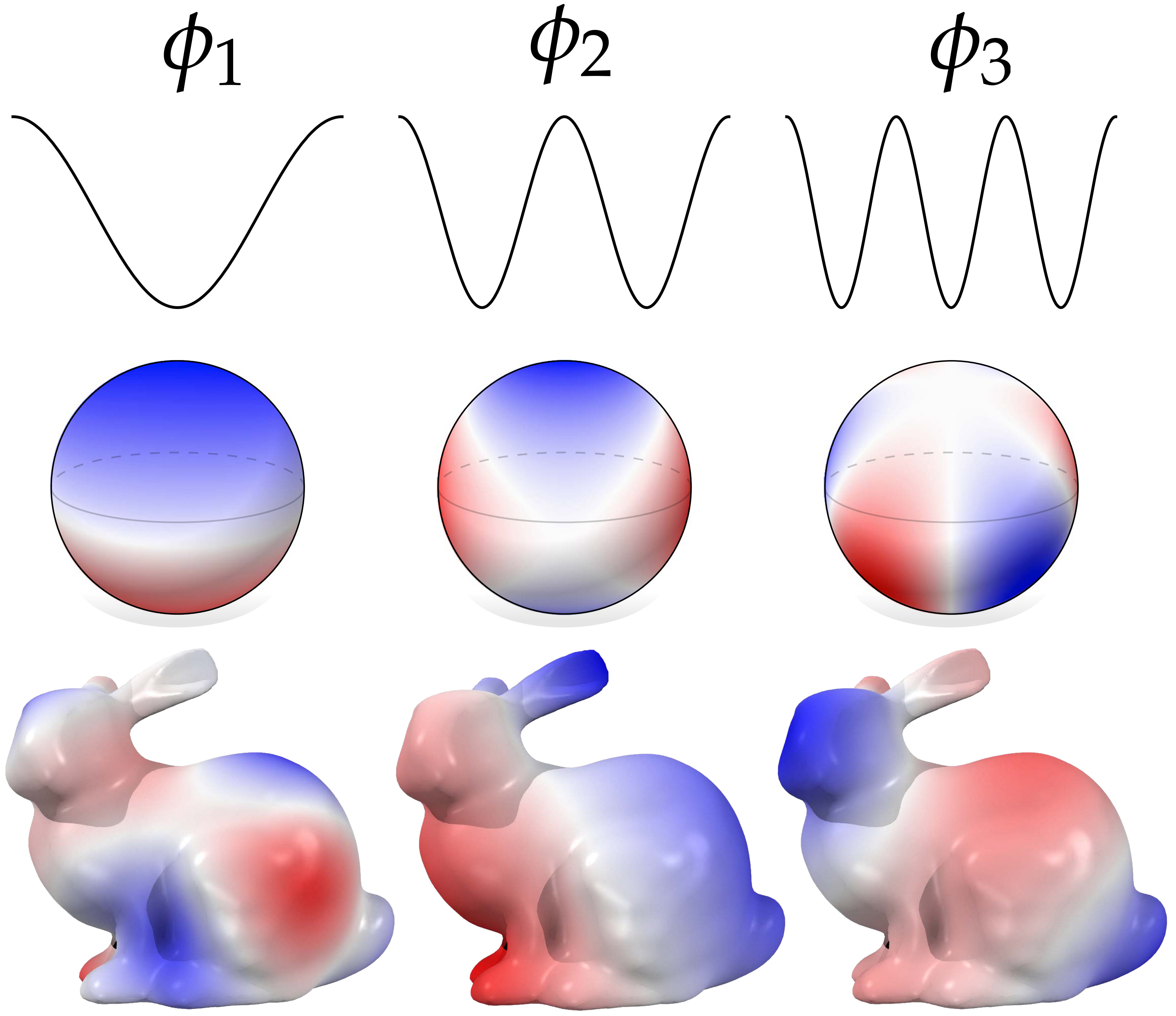
- Likewise, self-adjoint elliptic operator L on a compact domain has:
  - a discrete set of eigenvalues $\lambda_1, \lambda_2, \ldots$
  - orthogonal eigenfunctions $\phi_1, \phi_2, \ldots$
- *E.g.*, 2nd derivative operator on circle
  - basis for Fourier analysis/signal processing
- Laplacian: "frequencies" on any shape
  - Basis for *spectral geometry processing*

$$\Delta\phi_i = \lambda_i \phi_i$$



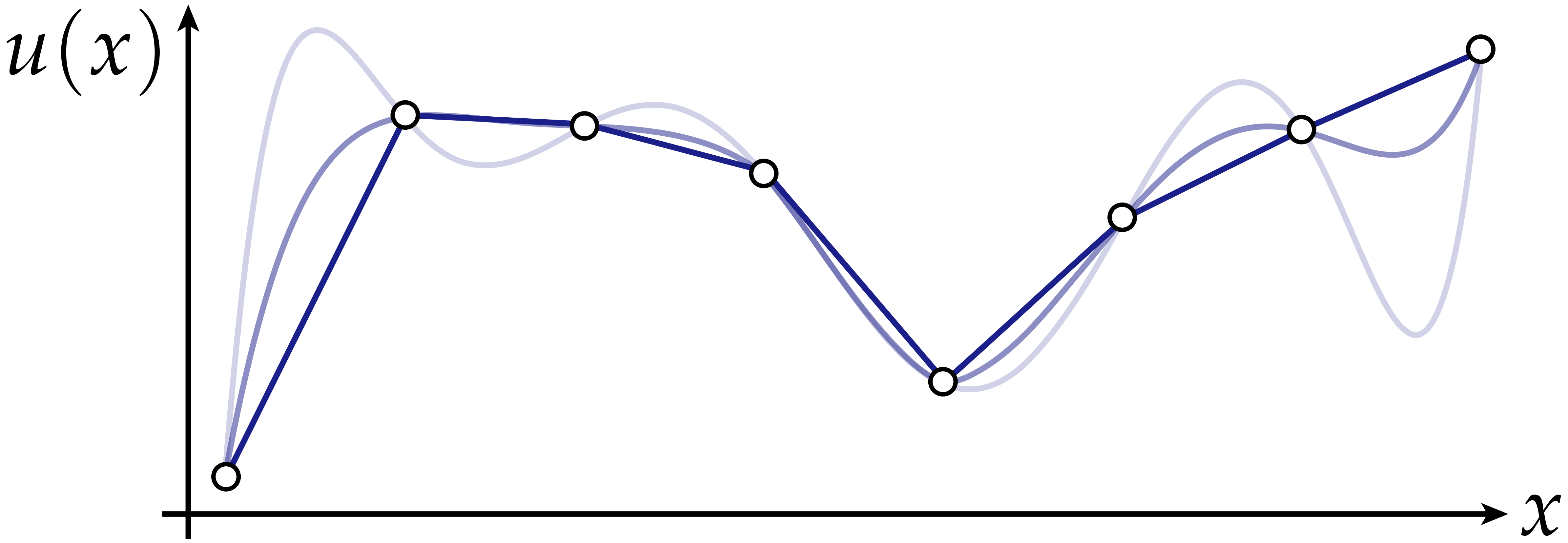
see: Lévy & Zhang, *"Spectral Mesh Processing"*

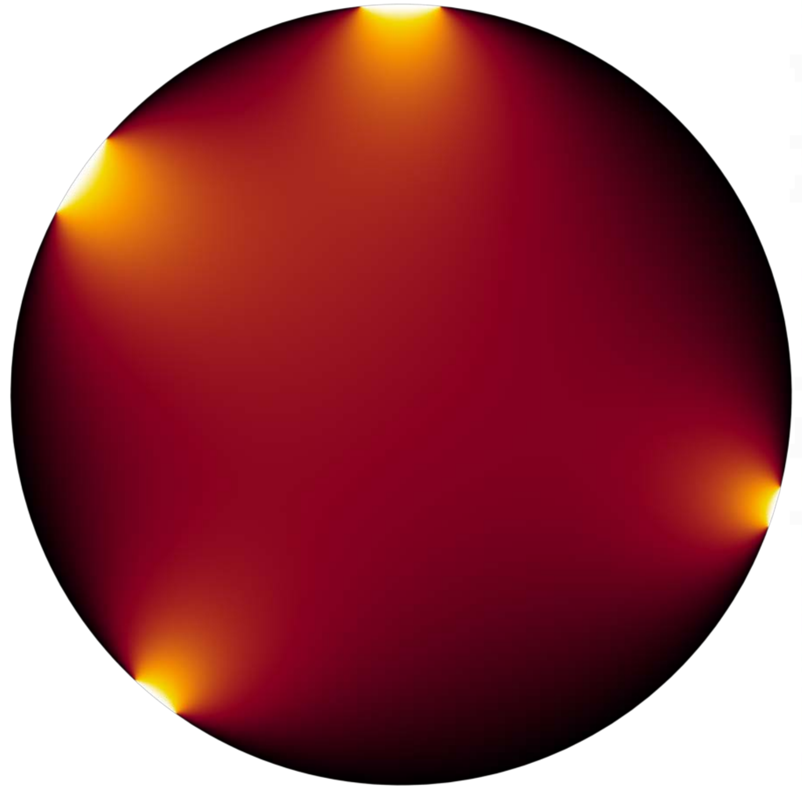
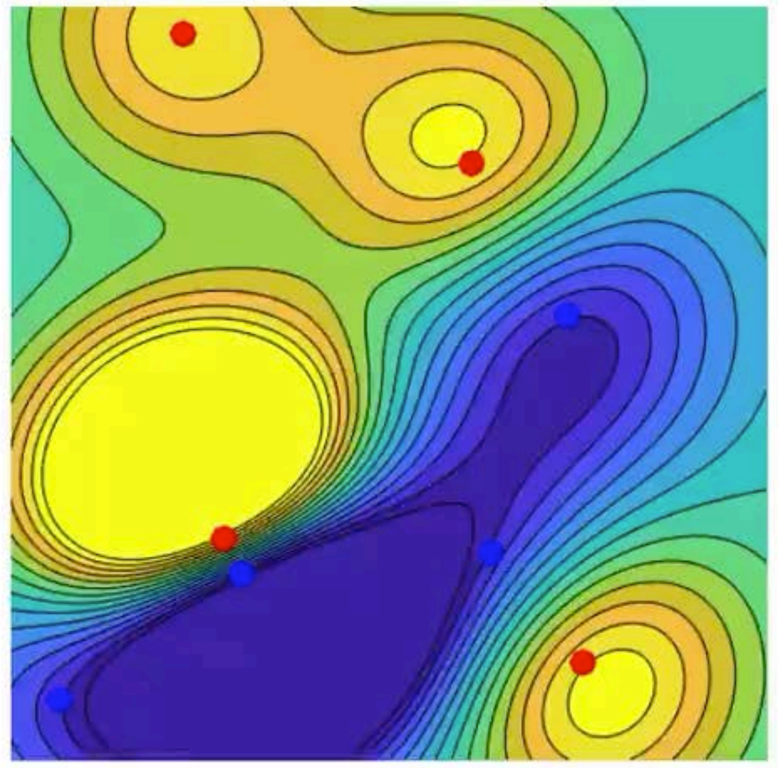
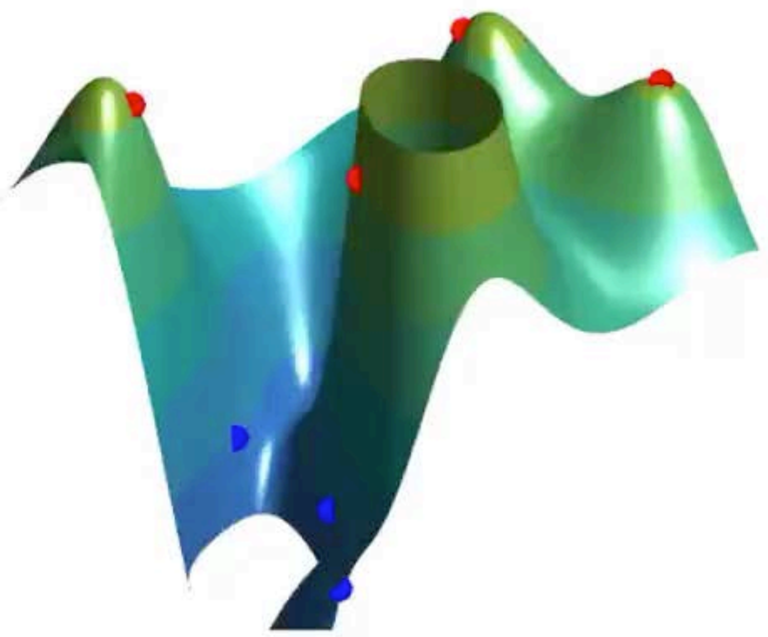
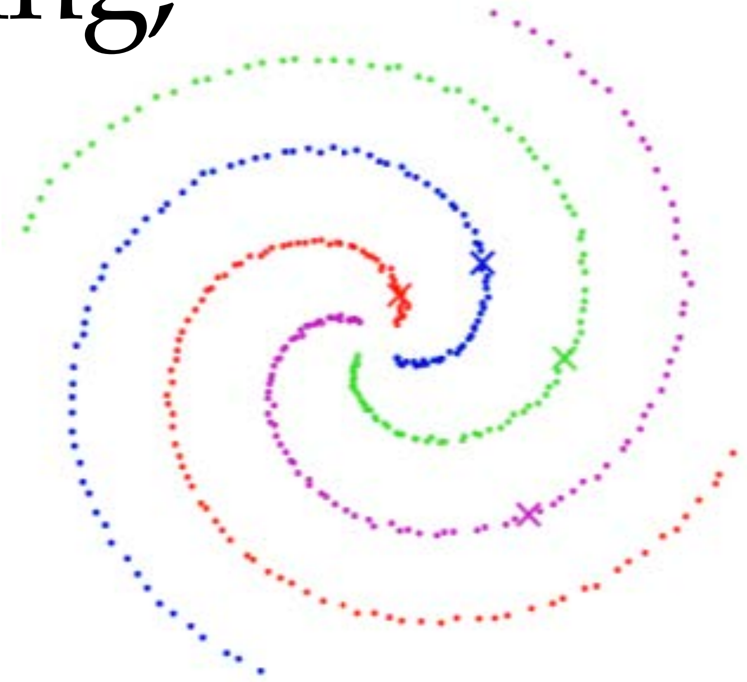
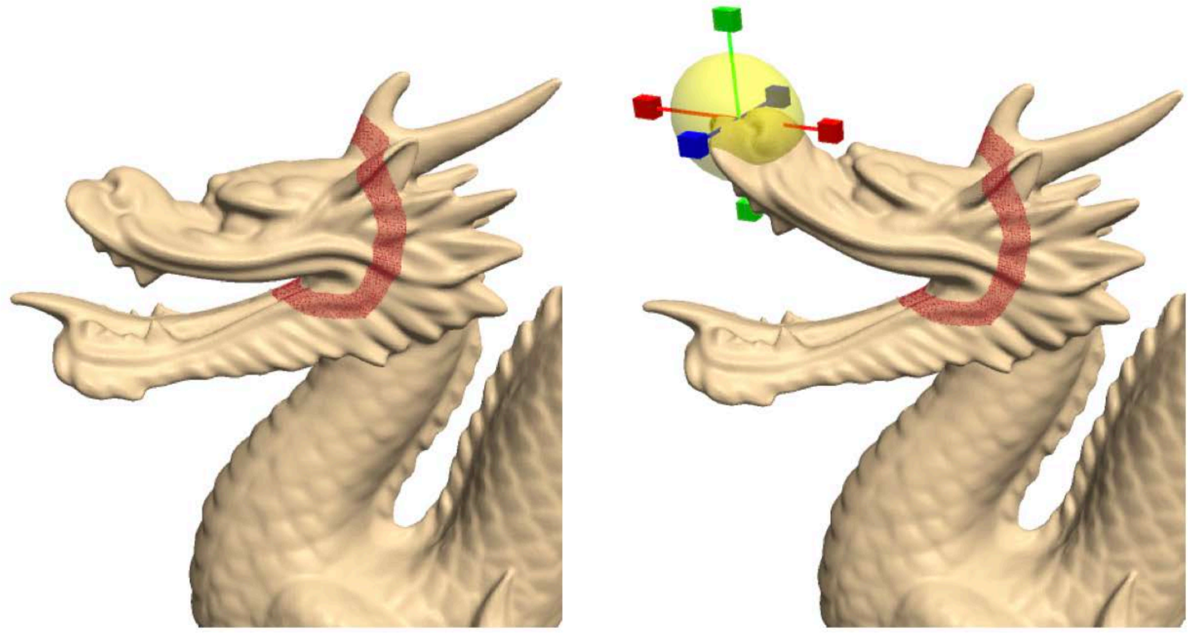
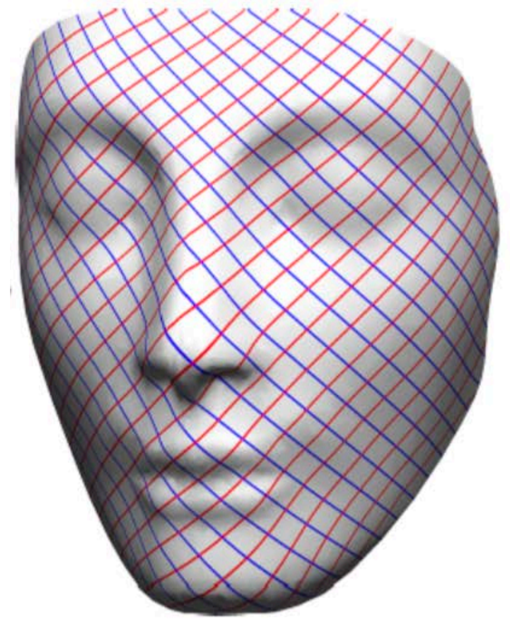
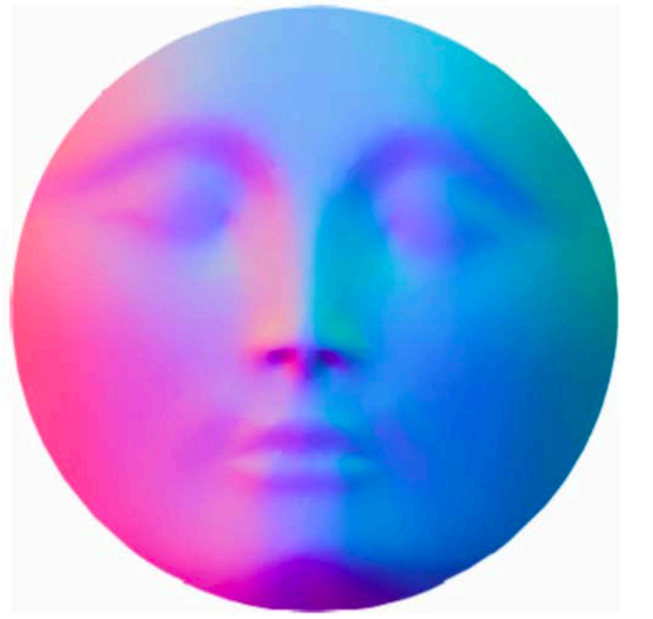
Dirichlet Energy & Harmonic Functions

adapted from: Crane, Solomon, Vouga, "Laplace-Beltrami: The Swiss Army Knife of Geometry Processing"

Interpolation

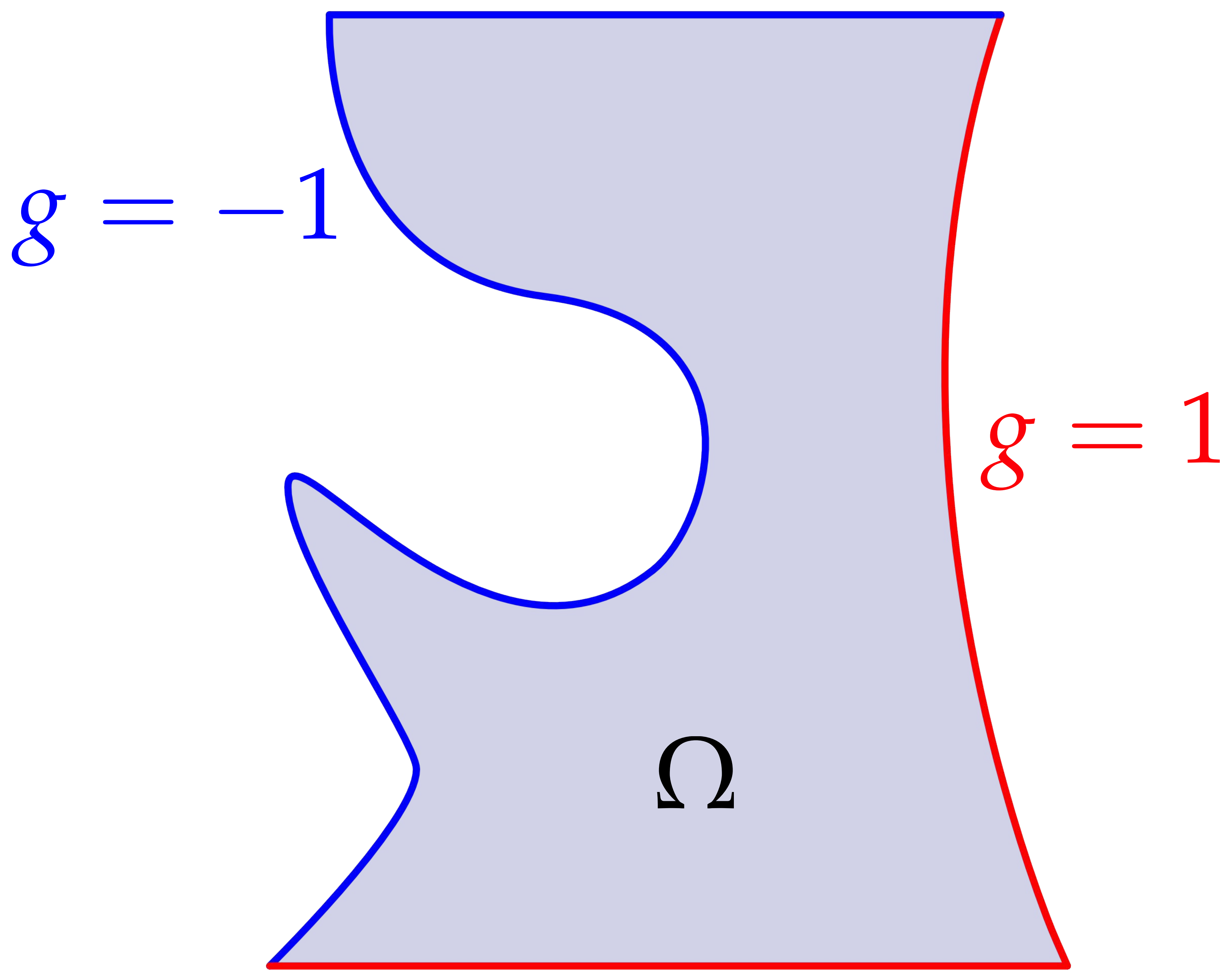
- Given a few data points, or values on the boundary, how should rest of the function look?
- *Statistics:* scattered data interpolation ("thin plate spline")
- *Machine learning:* semi-supervised learning ("Laplacian learning")
- *Physics:* steady-state solution (e.g., heat flow, elasticity, soap bubbles, …)
- *Geometry processing:* shape editing, surface parameterization, …
- …



Interpolation Problem

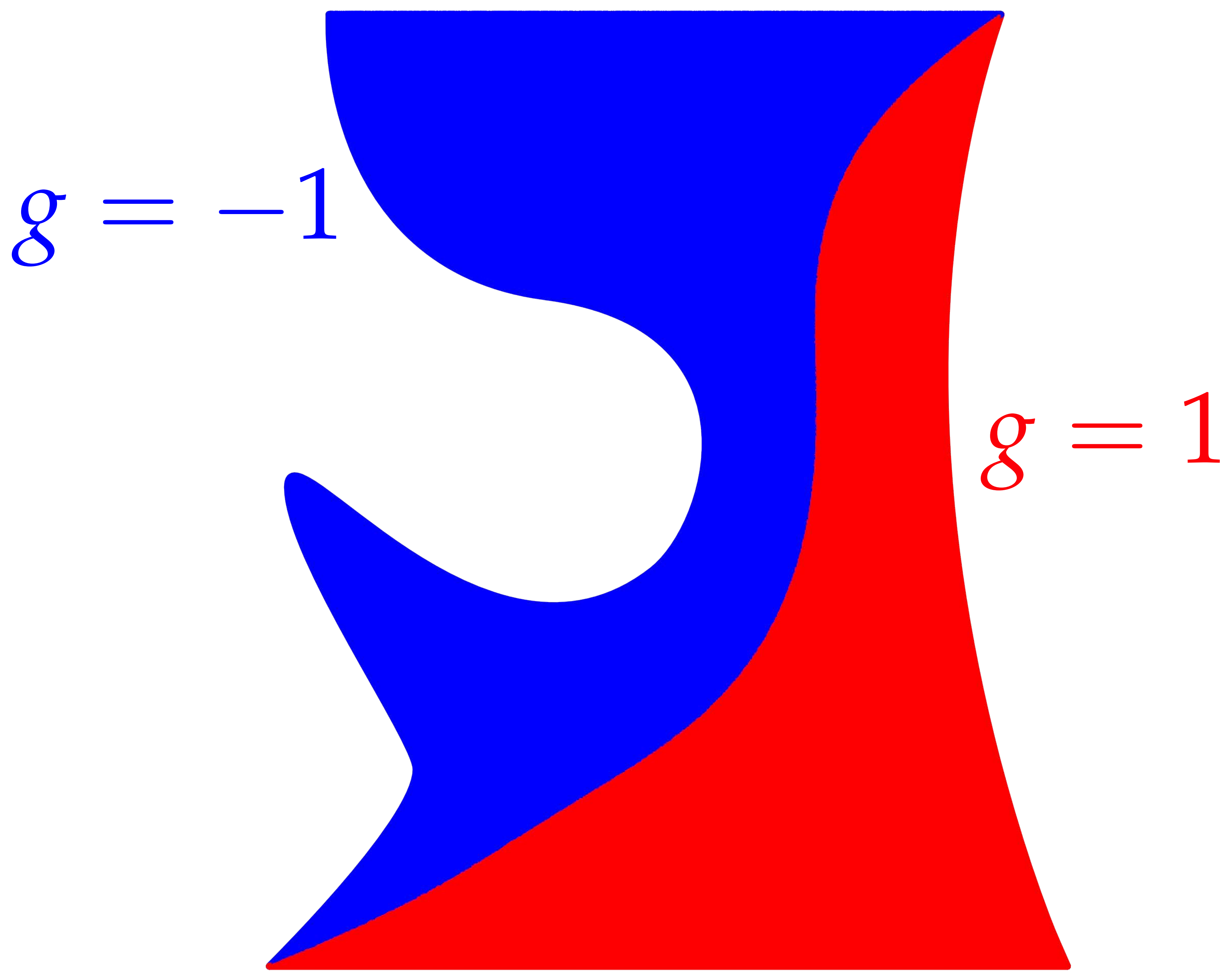
- **Given:**
  - a region $\Omega \subset \mathbb{R}^2$
  - boundary values $g : \partial\Omega \to \mathbb{R}$
- **Goal:** find a function u that
  - is equal to g on the boundary
  - fills in the interior *"as smoothly as possible"*



Question: what does *"as smoothly as possible"* mean?

Interpolation Problem—Piecewise Constant

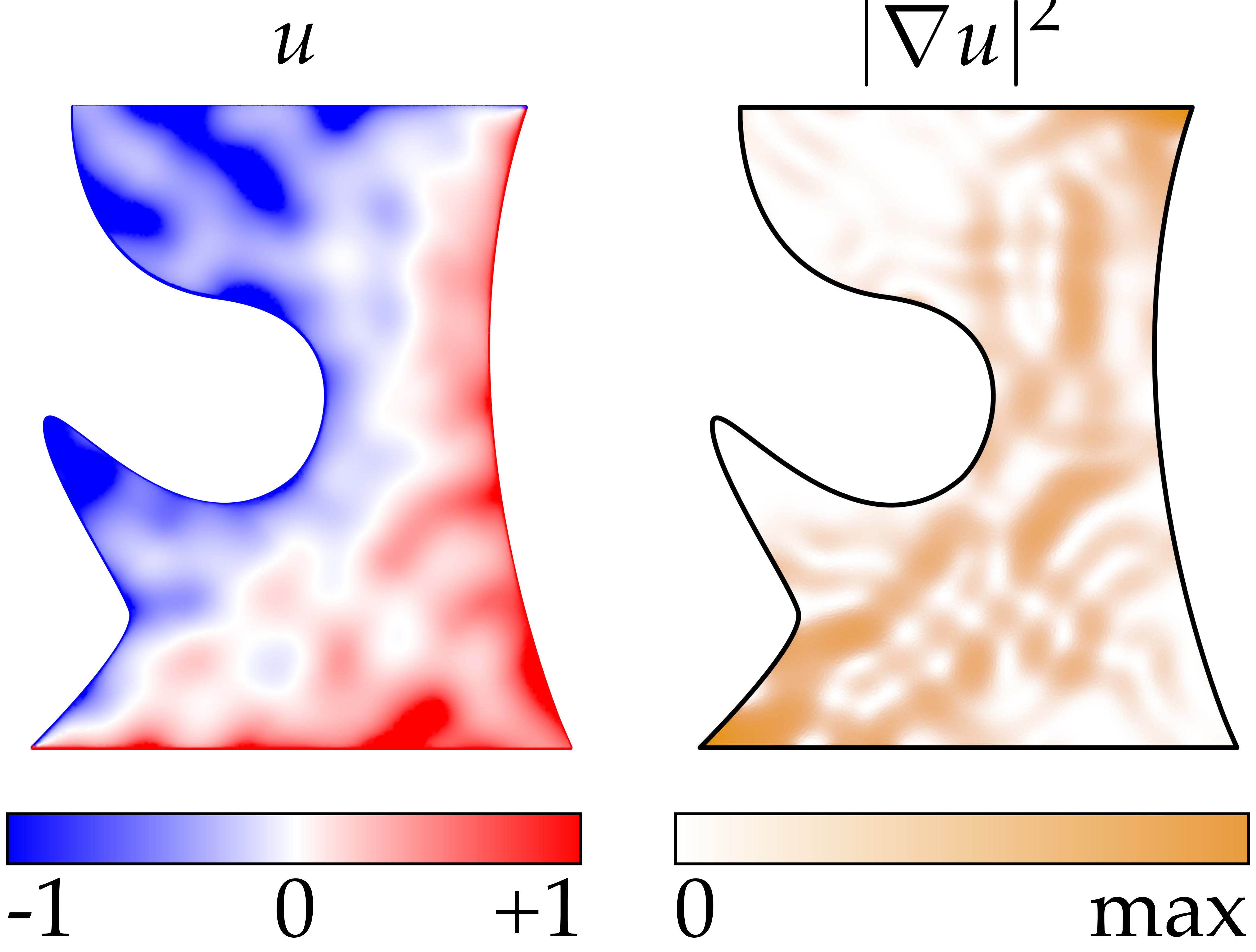
- Smoothest possible function, perhaps, is one that is *constant*
  - but no constant function $u(x) = c$ can interpolate *both* boundary values $g = +1$ and $g =$ -1.
  - *piecewise* constant function has big "jump"—not very smooth
- **Idea:** look for function that matches boundary data and is *"as close to constant as possible"*



Interpolation Problem — Dirichlet Energy

- *Dirichlet energy E_D* measures failure of a function to be constant
  - zero for constant functions
  - integrand will be large in regions with rapid change in value
- To find a good interpolating function, *minimize Dirichlet energy*
  - (among functions with given boundary data)

$$E_D(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dA$$



Dirichlet's Principle

- **Q:** How do we minimize $E_D(u)$?
- **A:** As with an ordinary function, find argument u^* for which 1st derivative (gradient) is equal to zero
  - will be a global minimizer because E_D is *convex*
- **Exercise:** show that

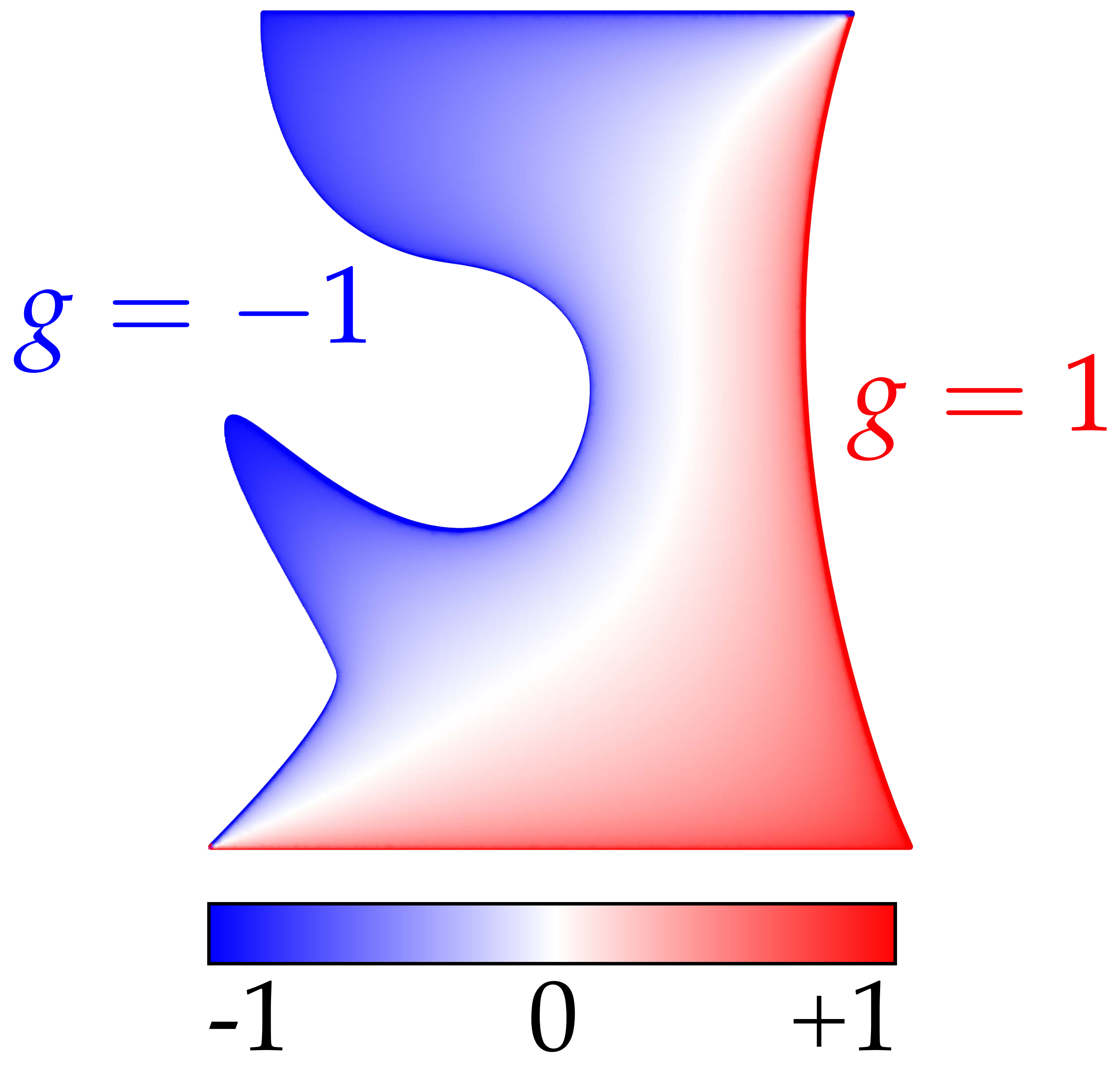
1. Dirichlet energy can be written in terms of Laplacian
2. Minimizing function has Laplacian equal to zero

$$\min_{u:\Omega\to\mathbb{R}} E_D(u)$$

$$\nabla E_D(u^*) = 0$$

$$E_D(u) = \int_\Omega u \Delta u \, dA$$

$$\boxed{\begin{array}{rcl} \Delta u & = & 0 \text{ on } \Omega \\ u & = & g \text{ on } \partial\Omega \end{array}}$$



A function minimizes Dirichlet energy if and only if it solves Laplace equation.

Aside: History of Dirichlet's Principle

The history of the Dirichlet principle is remarkable. Green, Dirichlet, Thomson, and others of their time regarded it as a completely sound method and used it freely. Then Riemann in his complex function theory showed it to be extraordinarily instrumental in leading to major results. All of these men were aware that the fundamental existence question was not settled, even before Weierstrass announced his critique in 1870, which discredited the method for several decades. The principle was then rescued by Hilbert and was used and extended in this [the 20th] century. Had the progress made with the use of the principle awaited Hilbert's work, a large segment of nineteenth-century work on potential theory and function theory would have been lost.

$$\min_u \int_\Omega |\nabla u|^2 \, dV$$

$$\updownarrow$$

$$\Delta u = 0$$

Green

Dirichlet

Riemann

Weierstrass

Hilbert

from Morris Kline, *"Mathematical thought from ancient to modern times"*, vol 3 (1972)

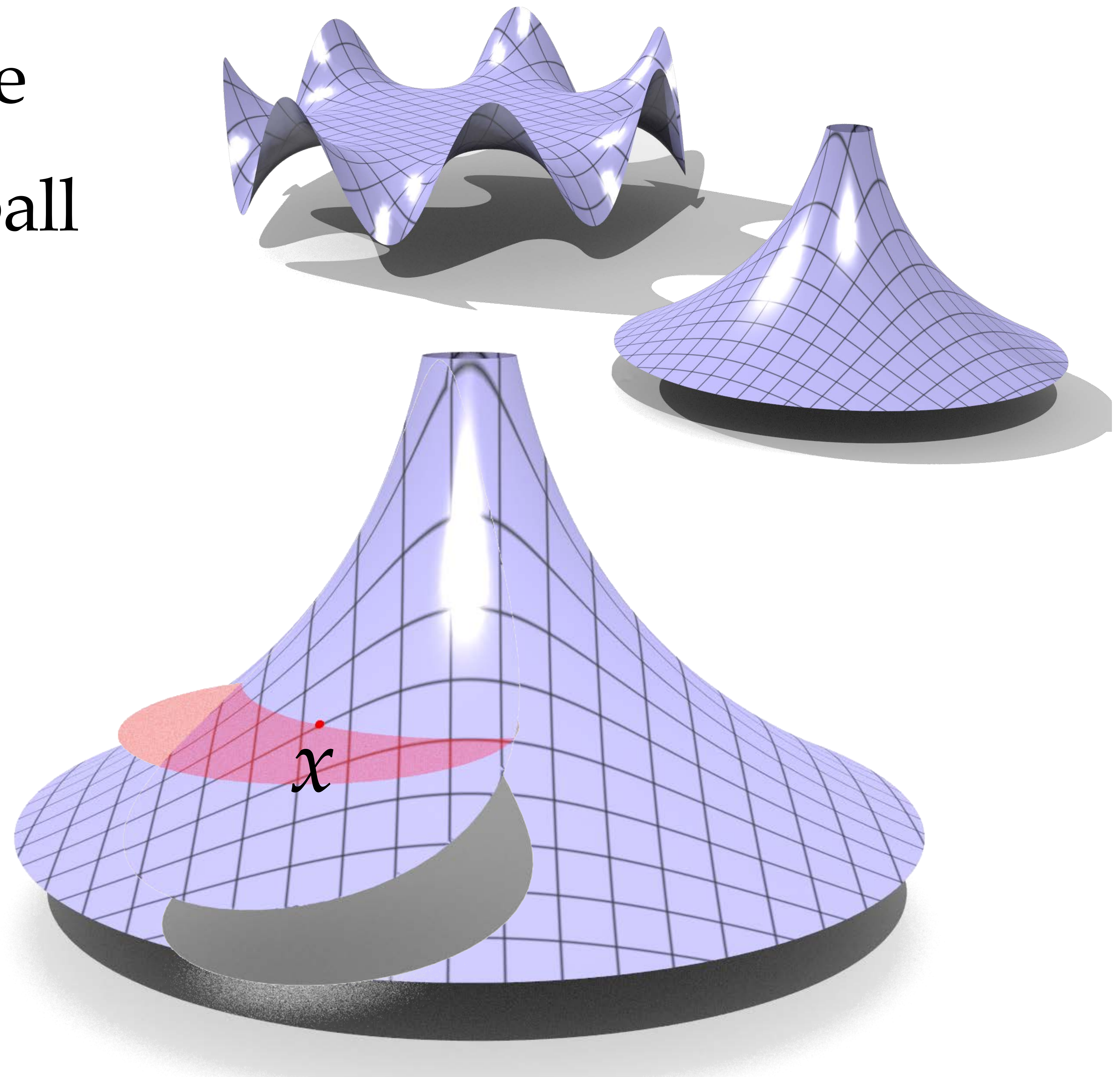
Harmonic Functions

- Minimizer of Dirichlet energy is a *harmonic function*
- Play a key role throughout geometry, physics, …
- Physical interpretation: temperature at steady state
- **Mean value property:** equal to average over *any* ball
- **Maximum principle:**
  - no extrema at interior points
  - max/min must be found on boundary

$$u(x) = \frac{1}{\pi\varepsilon^2}\int_{B_\varepsilon(x)} u\,dA = \frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x)} u\,d\ell$$

harmonic function

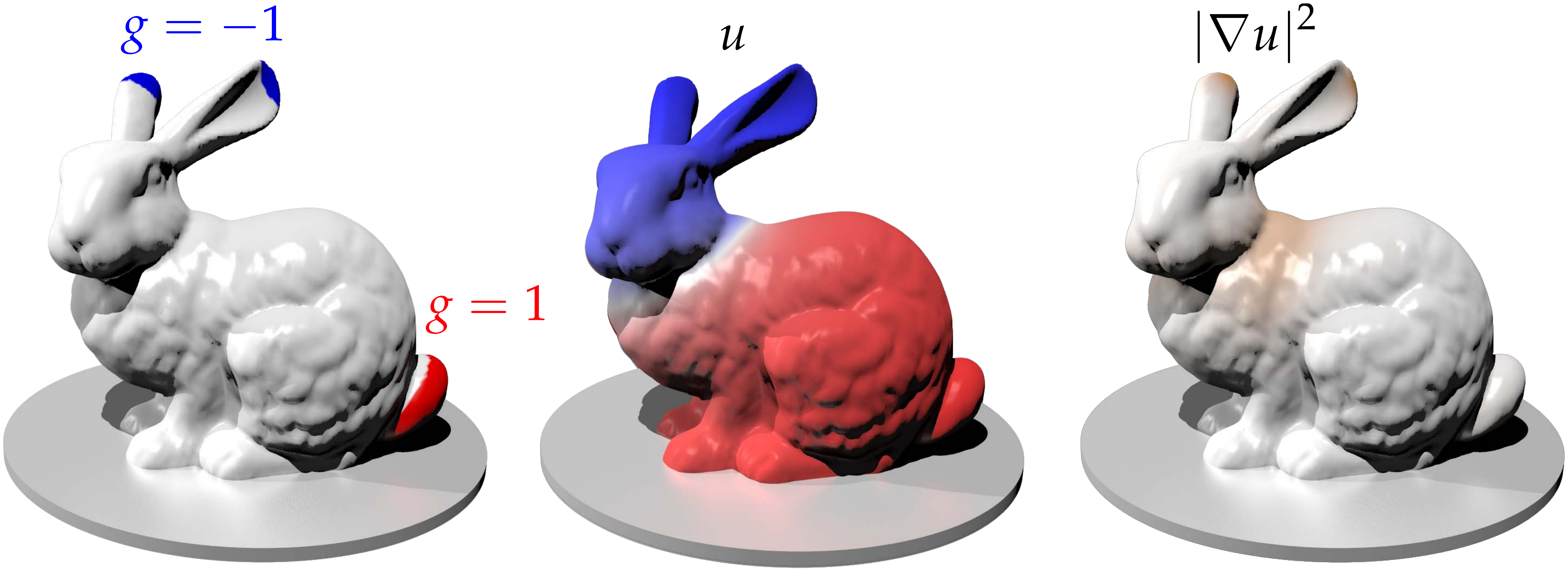
$$\boxed{\Delta u = 0}$$



Harmonic Functions on a Surface

- Analogous problem on curved domains:
  - fix function values on some region $A \subset M$
  - solve Laplace equation (now using *Laplace-Beltrami*)

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad M \setminus A \\ u &= g \quad \text{on} \quad A \end{aligned}$$



Poisson Equation

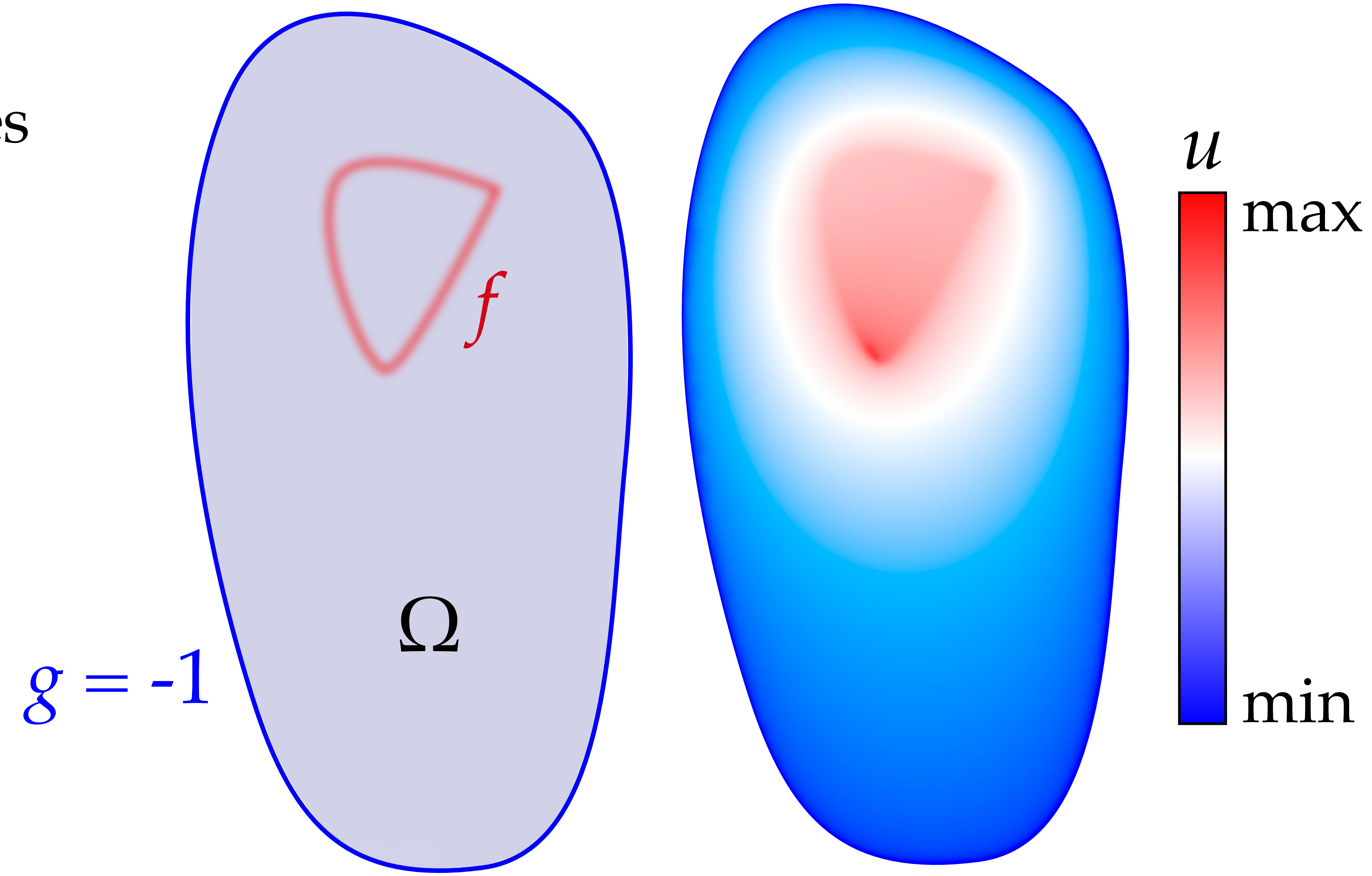
- Recall that Laplace equation is stationary solution to heat equation
- What if we have a heat source inside the domain?
  - and still have fixed boundary values (*e.g.*, heat sink)
- After a long time, get a stationary solution—*Poisson equation*

$$\begin{aligned} \tfrac{d}{dt}u &= \Delta u + f && \text{on } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

$(\lim_{t\to\infty})$

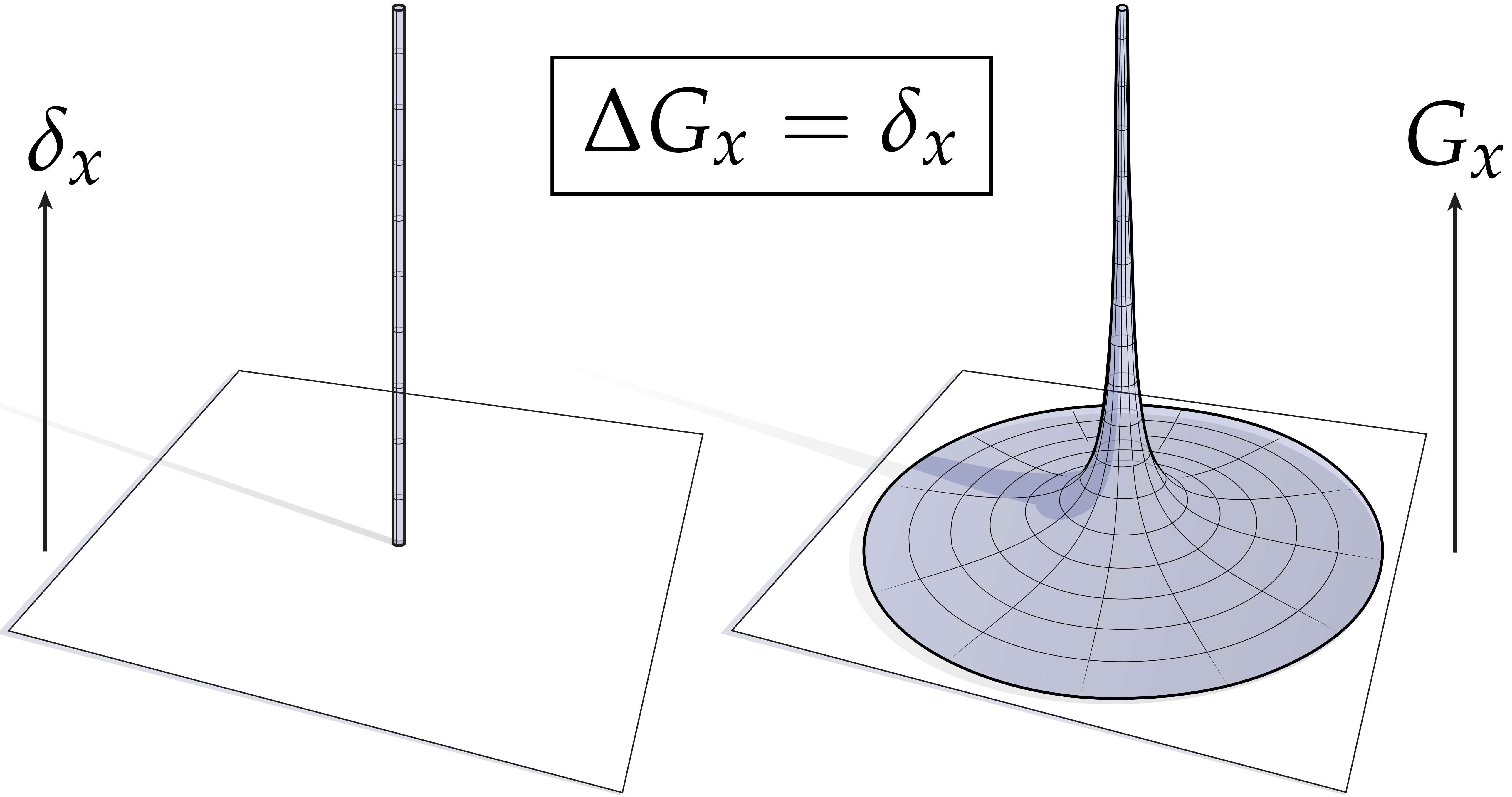
$$\boxed{\begin{aligned} \Delta u &= -f && \text{on } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}}$$

Poisson equation



Harmonic Green's Function

- Can also think about what Poisson equation does for a single "spike" on the right-hand side (Dirac delta)
- Solution falls off smoothly, called a *harmonic Green's function*
- Since equation is linear, get the solution for multiple spikes by summing Green's functions
- More generally, can *convolve* right-hand side with Green's function to get solution



$$\Delta u = \sum_i c_i \delta_{y_i} \qquad \Rightarrow u(x) = \sum_i c_i G_{y_i}(x)$$

$$\Delta u = f \qquad \Rightarrow u(x) = \int_M G_y(x) f(y)\, dy$$

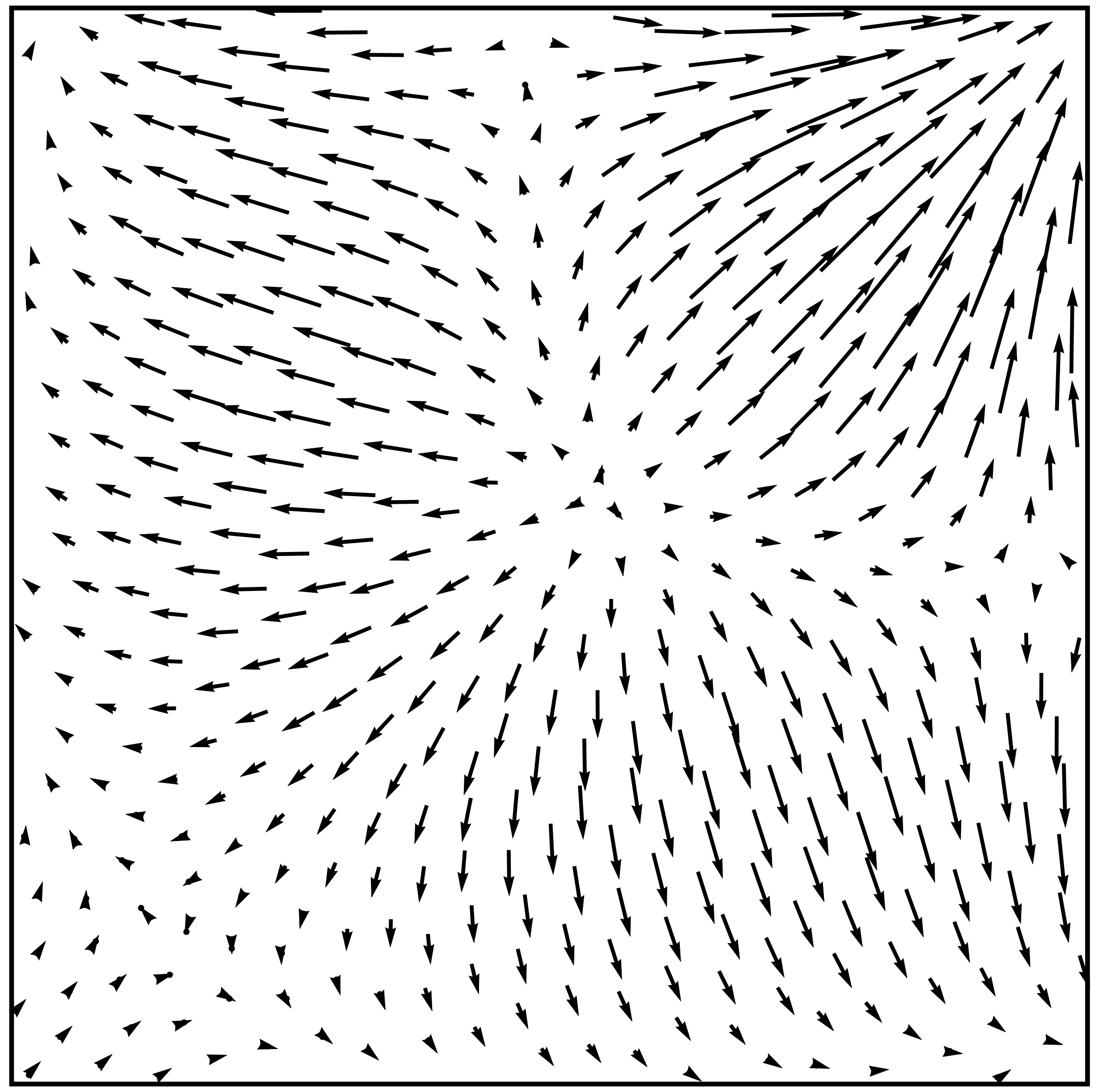
Key idea: solving a linear PDE is equivalent to convolving with its *fundamental solution*

Poisson Equation—Variational Perspective

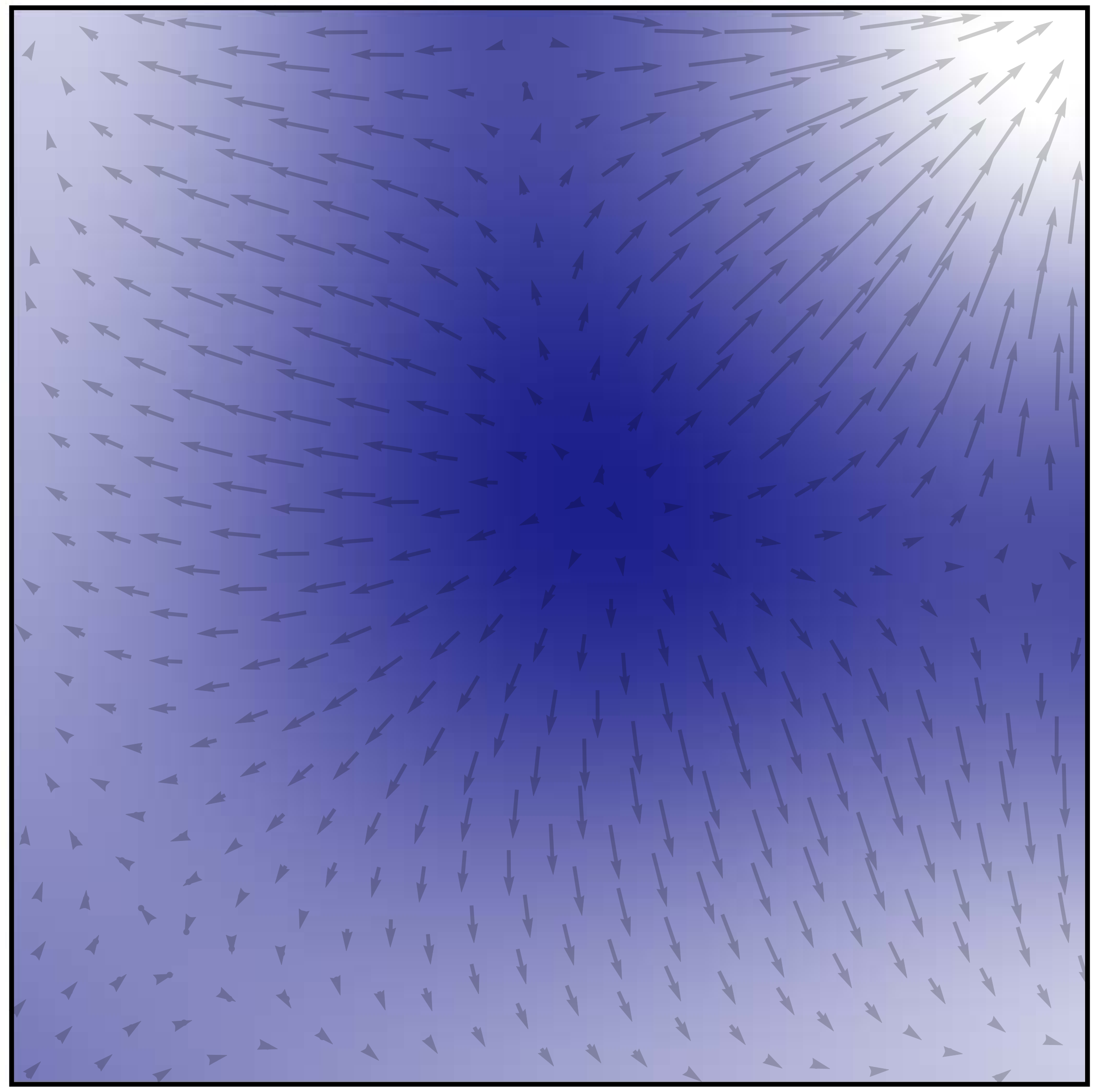
- Like Laplace equation, Poisson equations also arise naturally from energy minimization
- **<u>Example.</u>** Given vector field X, find scalar potential u that "best explains" X
- If X actually comes from the gradient of a function u, Poisson equation will recover this function

$$\min_u \int_M |\nabla u - X|^2$$

$$\implies \Delta u = \nabla \cdot X$$



X



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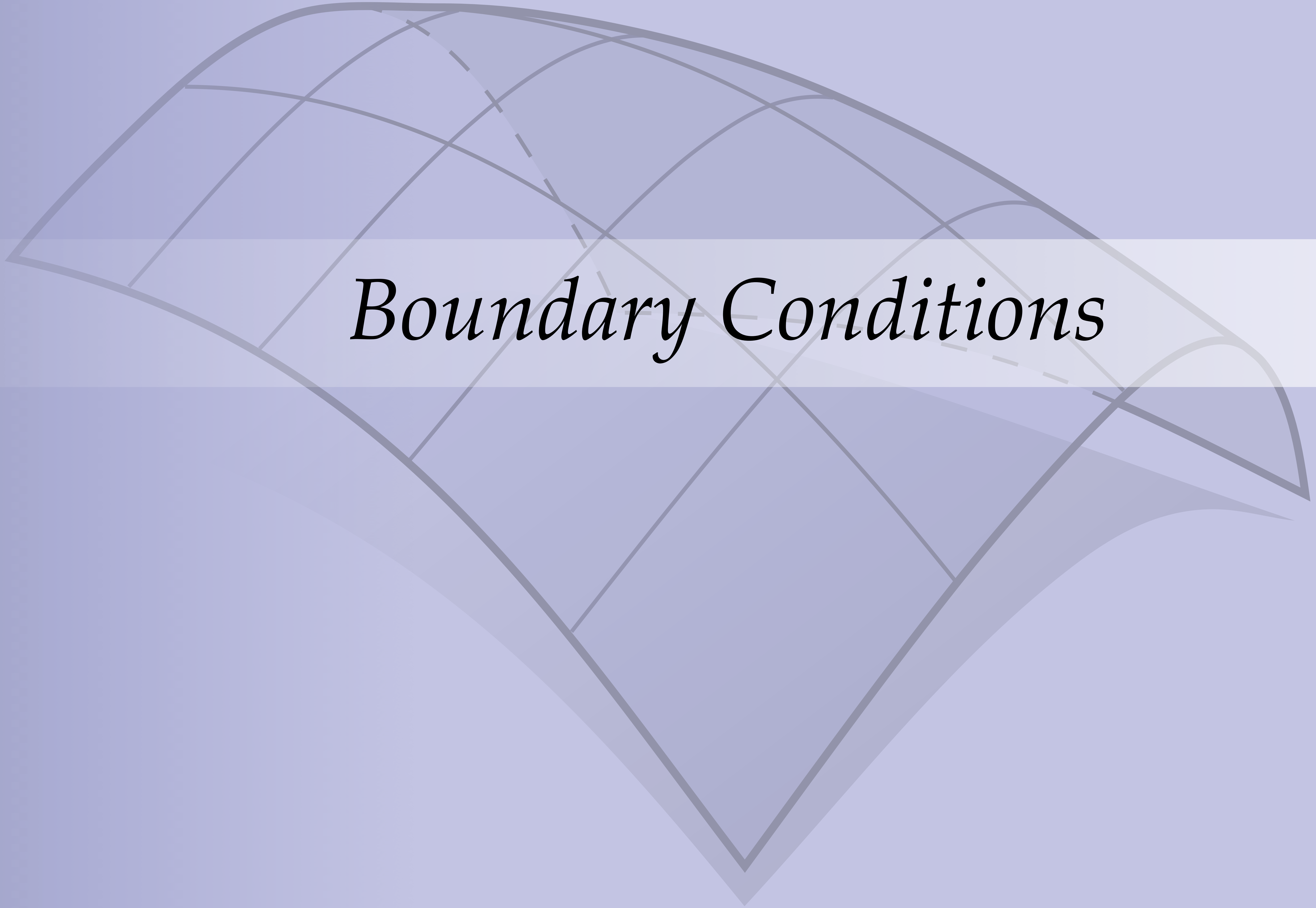
Key idea: Poisson equation can be used to "integrate" a vector field.

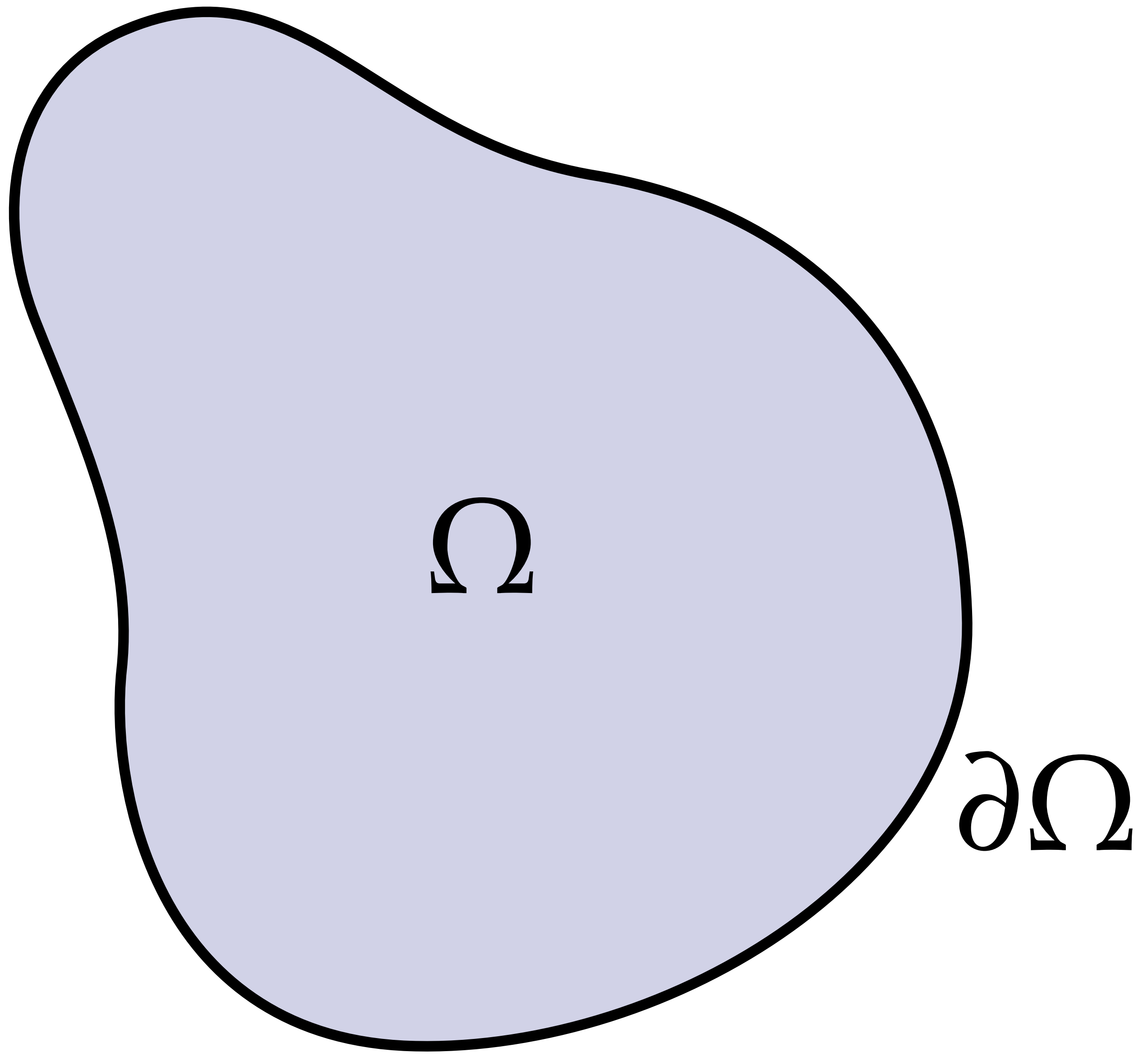
Boundary Conditions

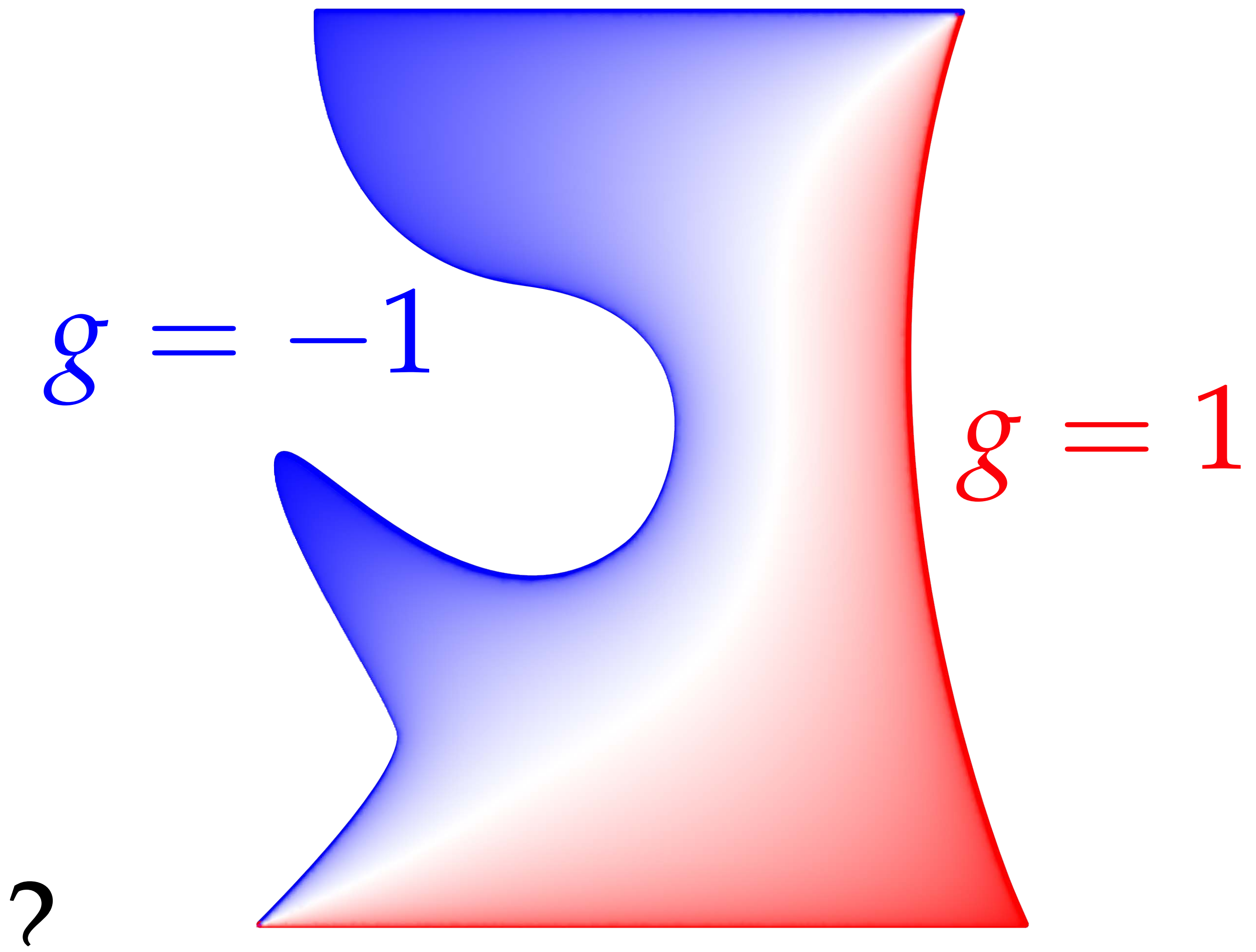


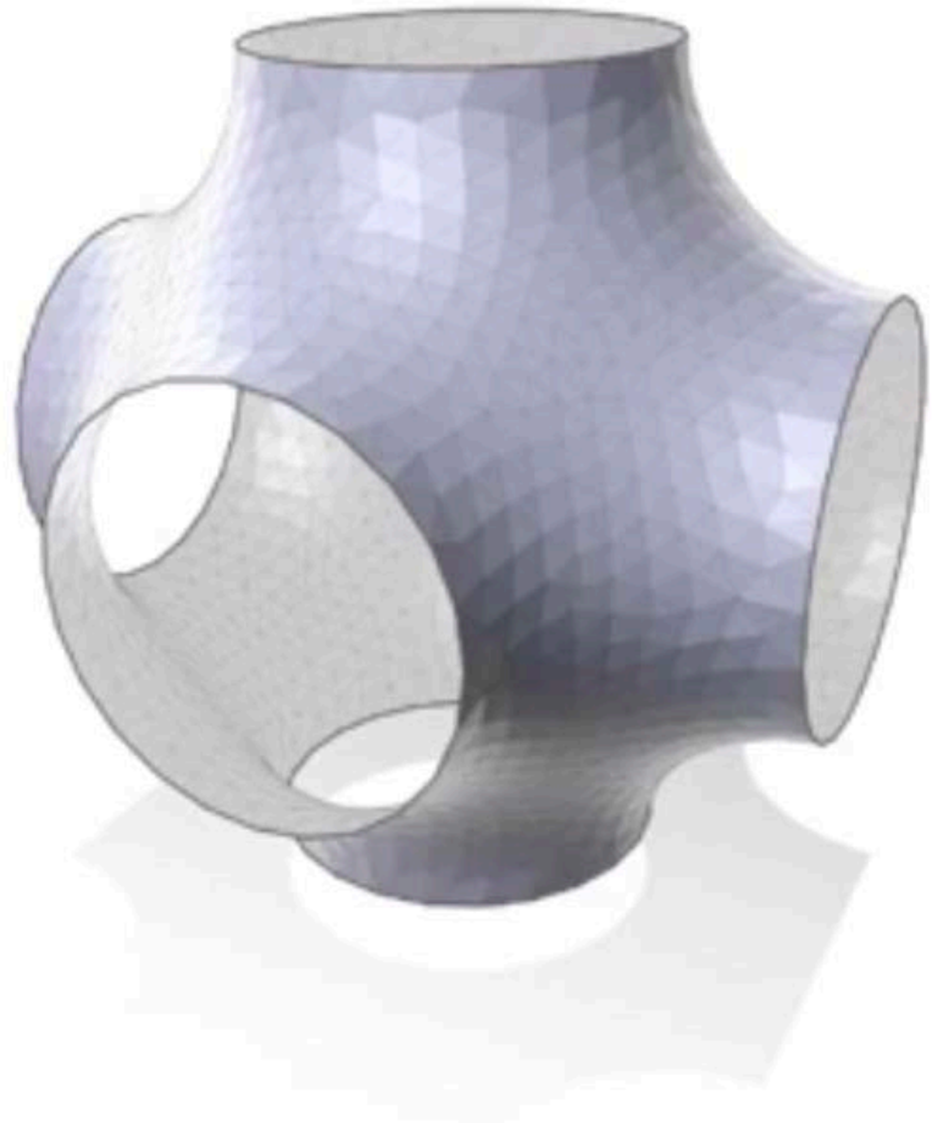
Boundary Conditions

- Get a sense from examples that boundary conditions are very important
  - *e.g.*, for harmonic functions, minimal surfaces, completely determine solution
- Often trickiest/most painful part—easy to get wrong!
- What kinds of boundary conditions can we have?
  - *Dirichlet* — fixed values
  - *Neumann* — fixed derivatives
  - *Robin* — mix of values & derivatives
  - …

Q: When can boundary conditions be satisfied?

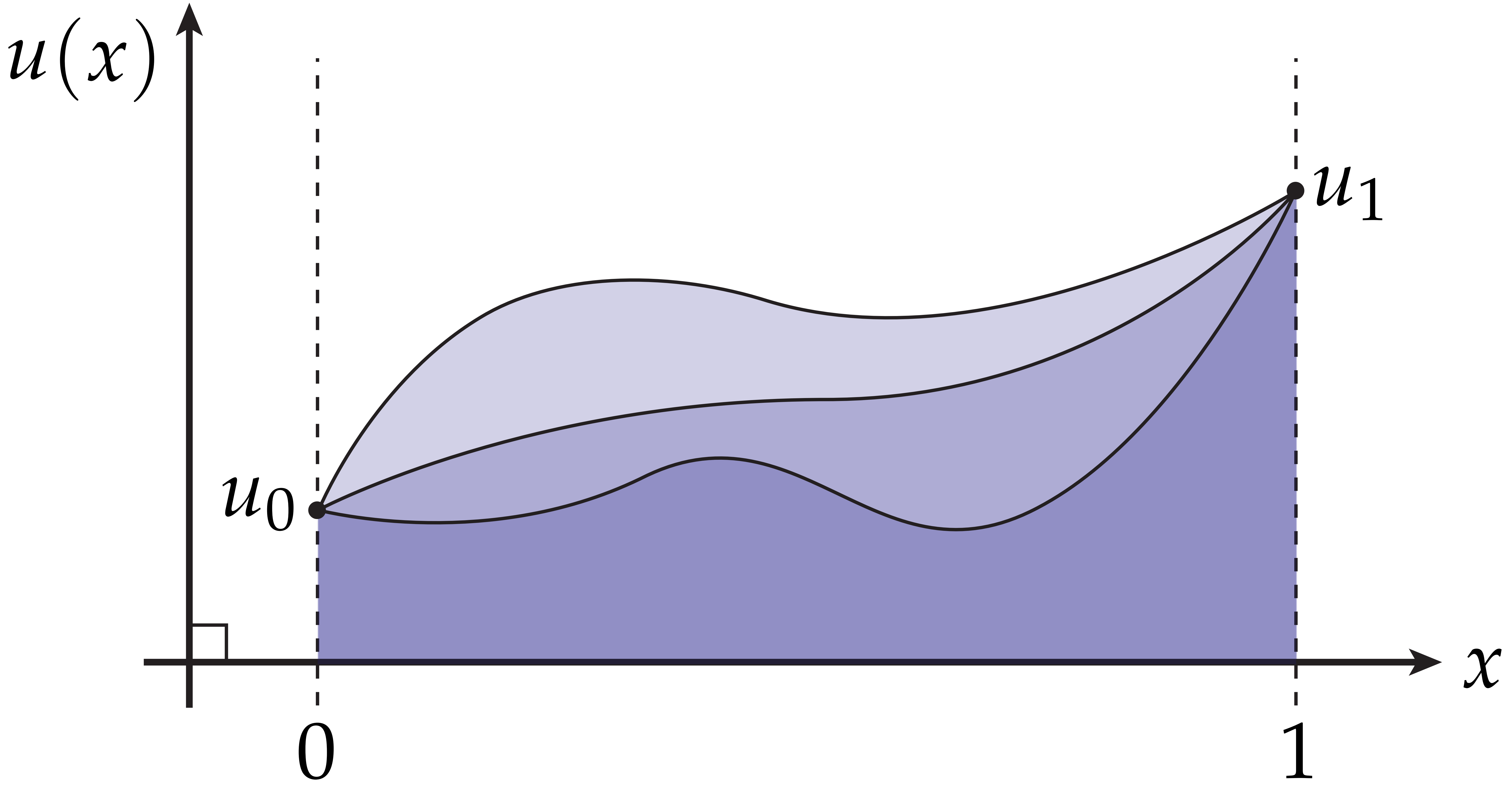






Dirichlet Boundary Conditions

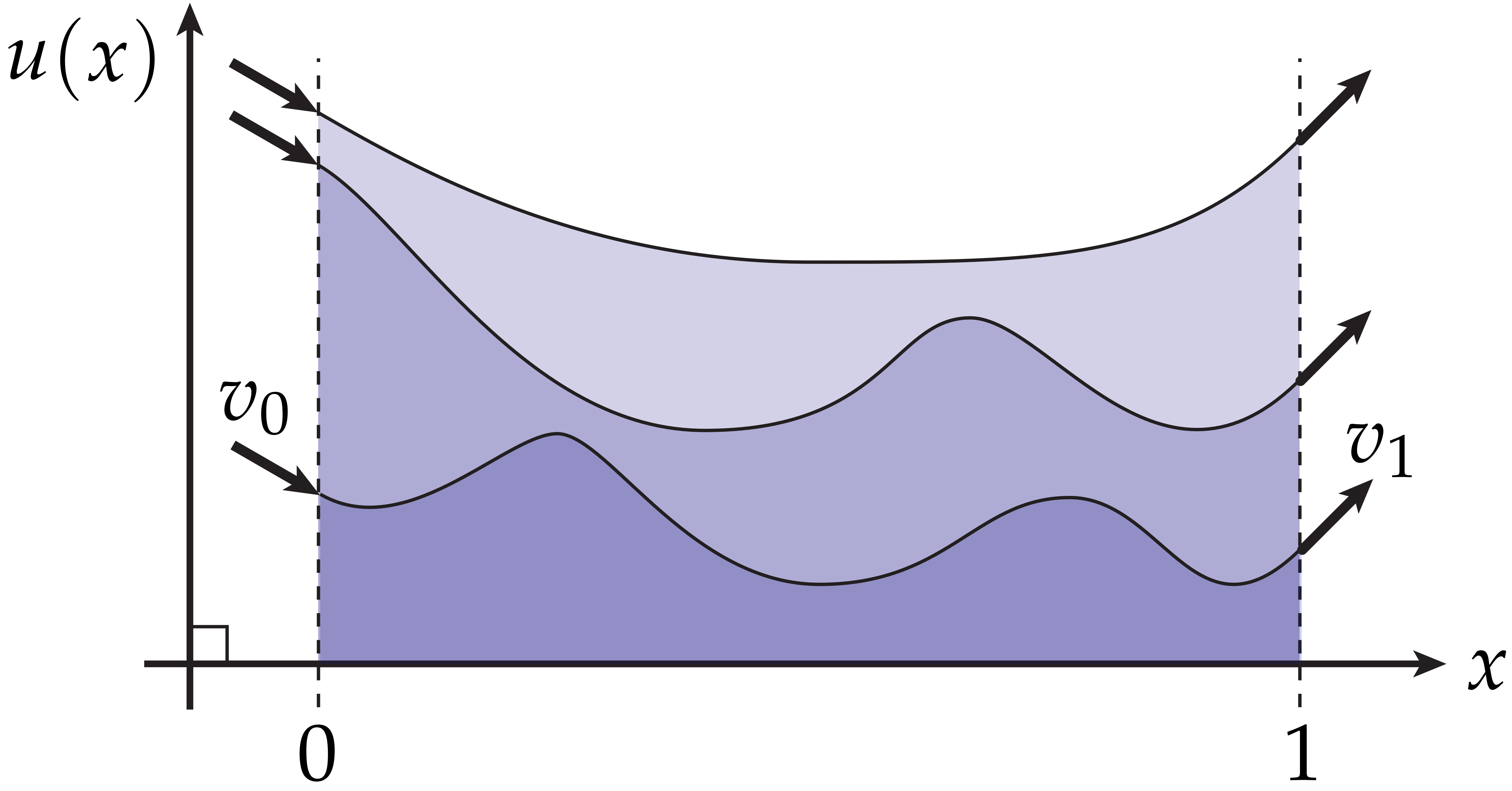
On interval [0,1], many possible functions w/ values u_0, u_1 at endpoints:



Key idea: "Dirichlet" just means boundary values are fixed.

Dirichlet Boundary Conditions

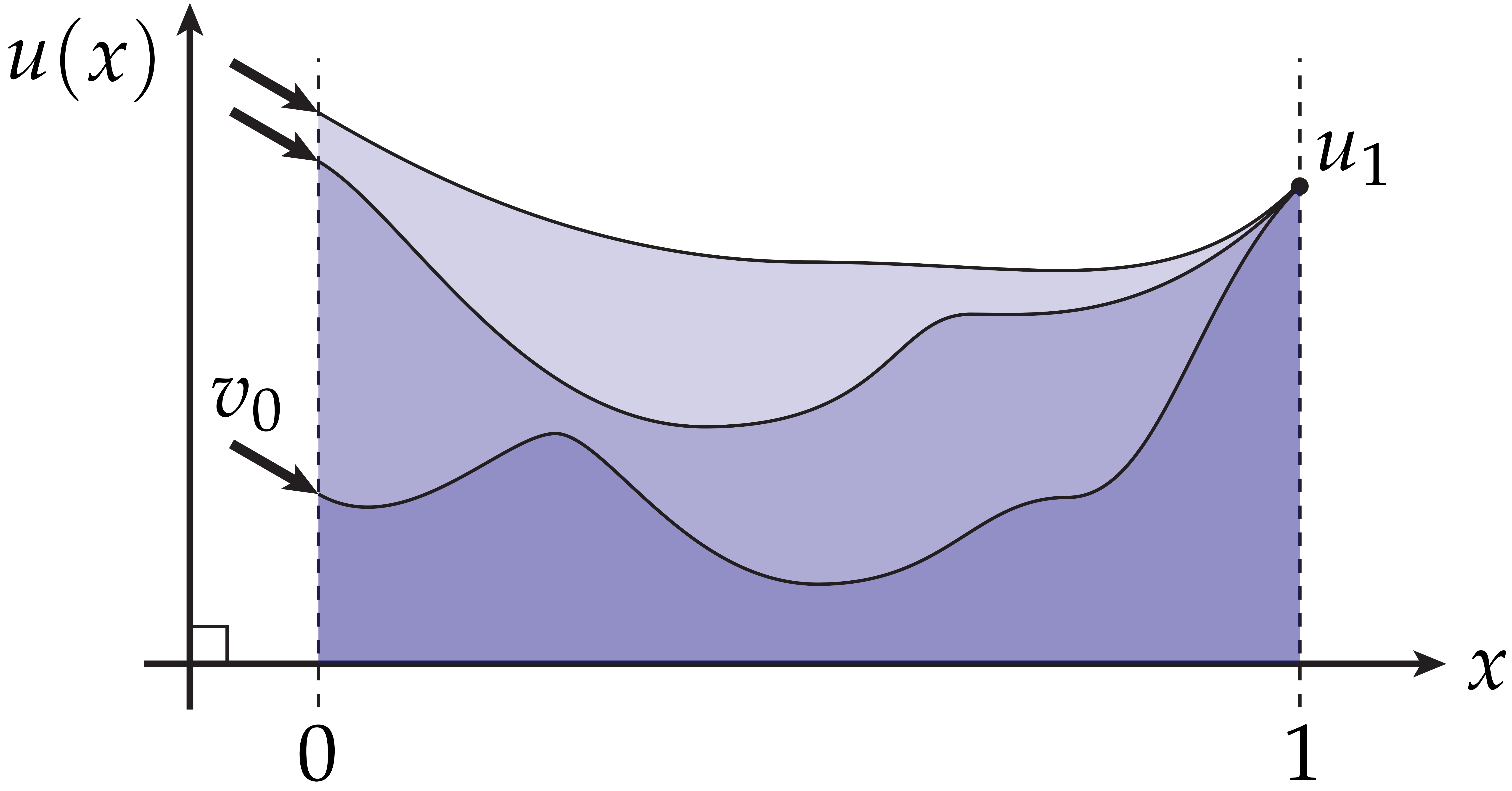
Likewise many possible functions w/ slope v_0, v_1 at endpoints:



Key idea: "Neumann" just means boundary <u>derivatives</u> are fixed.

Mixed Dirichlet & Neumann

Can also prescribe some values, some derivatives:



But what if we also have conditions on the interior?

Laplace w/ Dirichlet Boundary Conditions (1D)

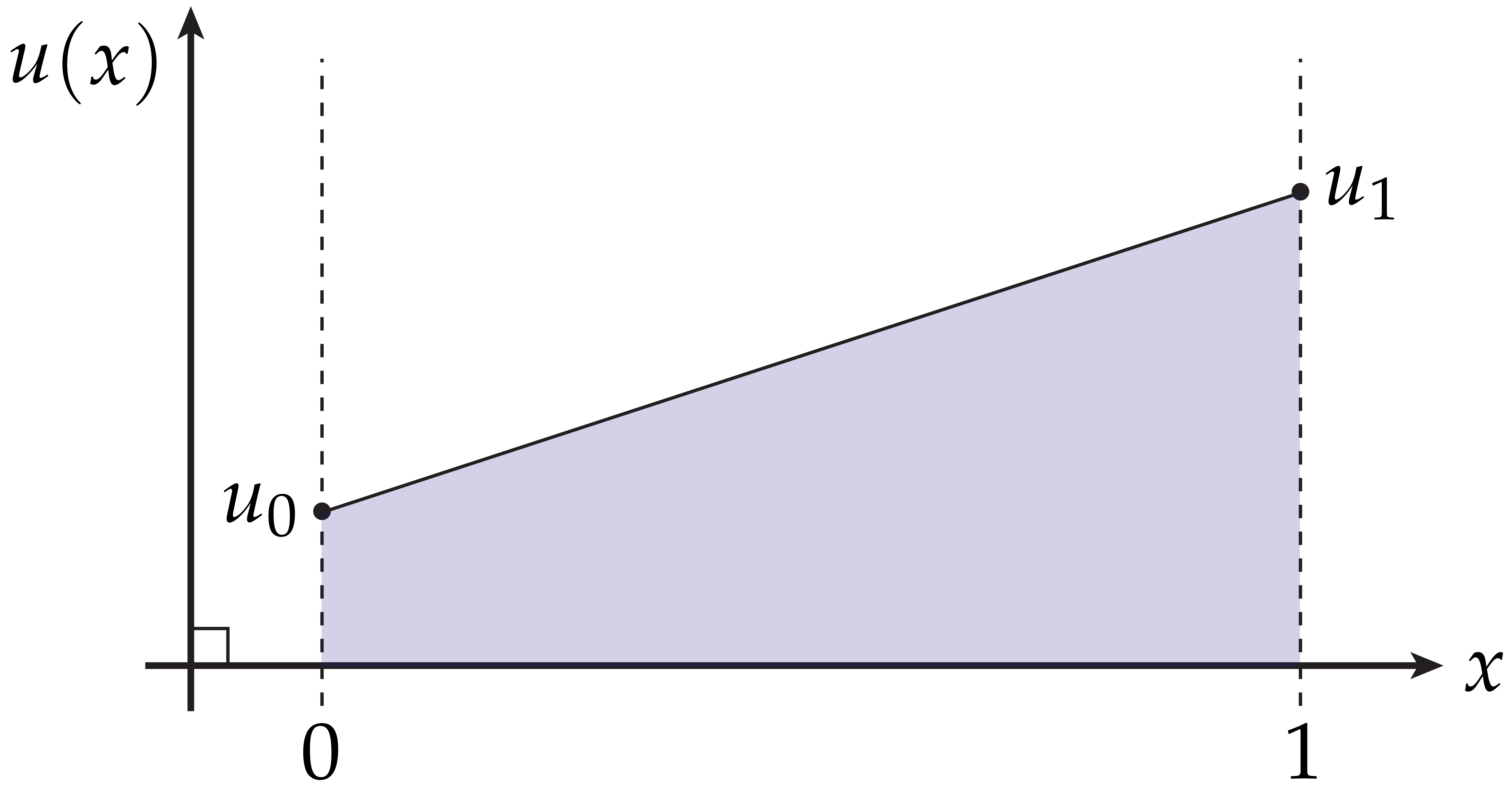
For a 1D Laplace equation, can we always satisfy Dirichlet conditions?

1D Laplace:

$$\frac{\partial^2 u}{\partial^2 x} = 0$$

Solutions:

$$u(x) = ax + b$$



Yes: a line can interpolate any two points.

Laplace w/ Neumann Boundary Conditions (1D)

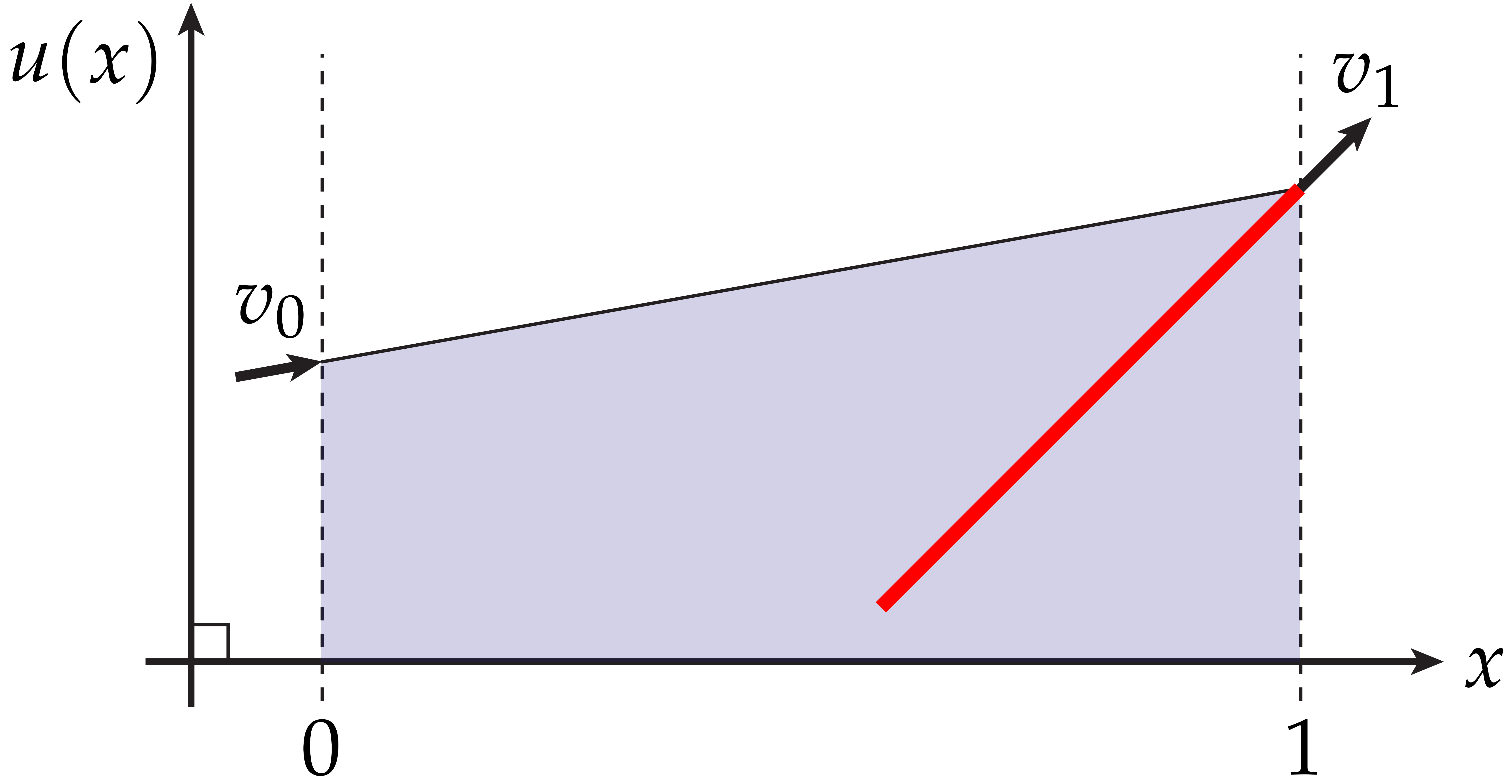
What about Neumann—can we prescribe the *derivative* at both ends?

1D Laplace:

$$\frac{\partial^2 u}{\partial^2 x} = 0$$

Solutions:

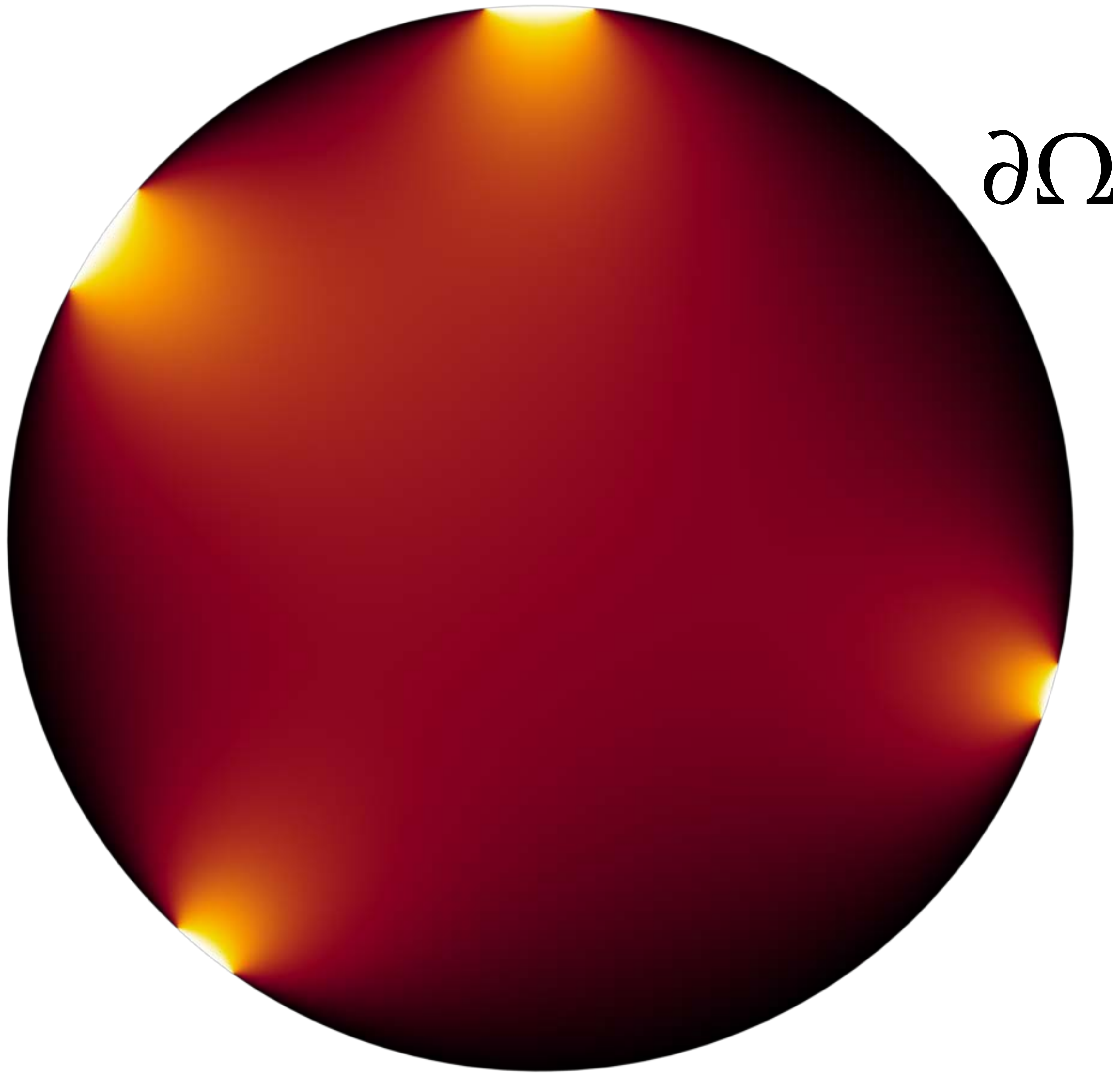
$$u(x) = ax + b$$



No! A line can have only one slope.

Laplace w/ Dirichlet Boundary Conditions (2D)

- Let's now consider a Laplace equation in 2D
- Can we always satisfy Dirichlet boundary conditions?
- Yes*: Laplace is steady-state solution to heat flow—just let it run for a long time…
  - Dirichlet data is "heat" along boundary



$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

*Subject to very mild/reasonable conditions on boundary geometry, boundary data

Laplace w/ Neumann Boundary Conditions (2D)

- Suppose instead we prescribe the *normal derivative* along the boundary
- Can we always find a solution to the Laplace equation?
- Well, consider the divergence theorem —*"what goes in, must come out!"*

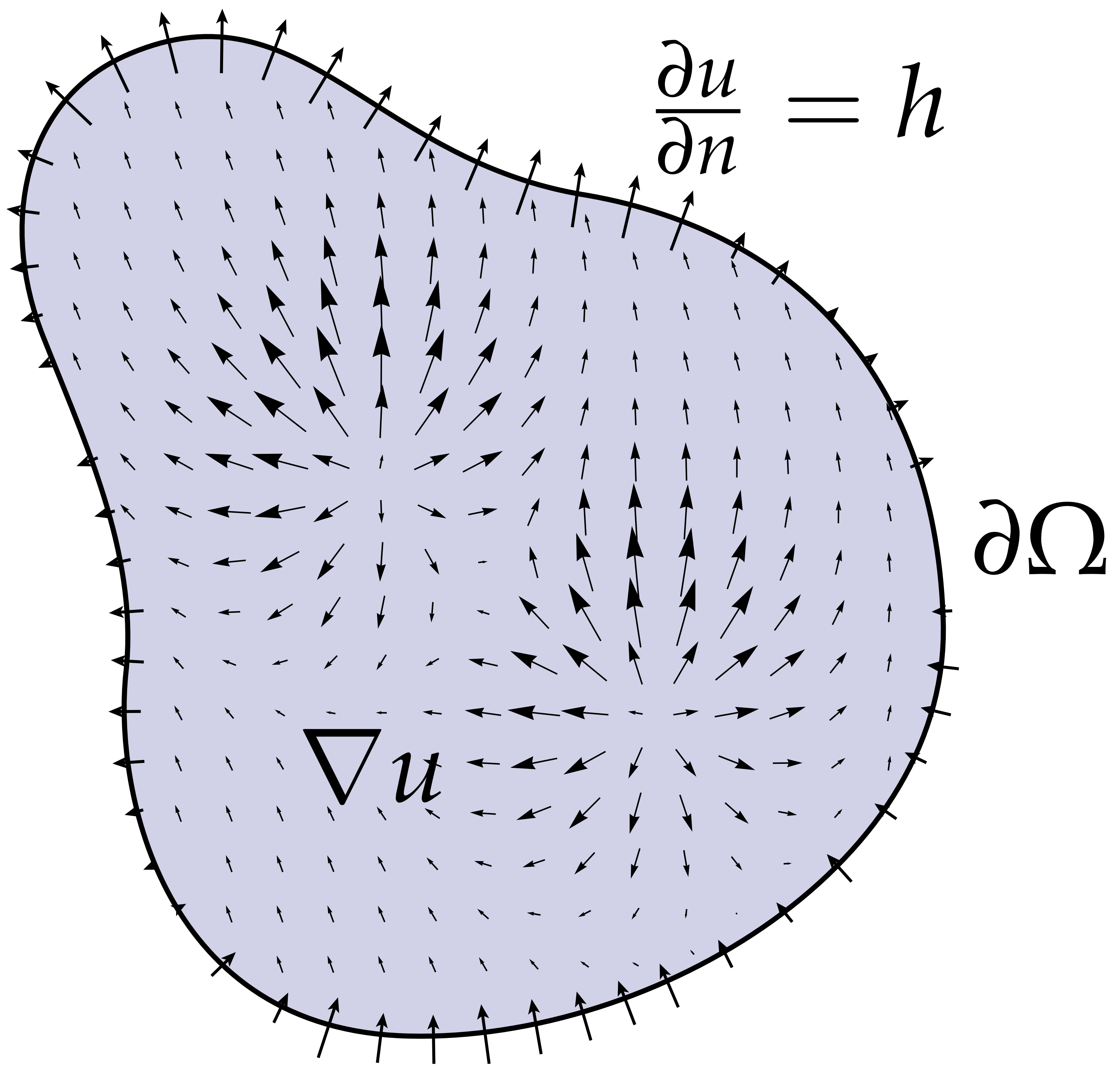
$$\int_\Omega 0\,dA \overset{!}{=} \int_\Omega \Delta u\,dA = \int_\Omega \nabla\cdot\nabla u\,dA = \int_{\partial\Omega} \underbrace{n\cdot\nabla u}_{\partial u/\partial n}\,dA$$

- Can only solve if Neumann data h integrates to zero over the boundary

Important: in general, a PDE <u>may not have solutions</u> for given boundary conditions

$$\begin{aligned}\Delta u &= 0 \quad \text{on} \quad \Omega\\ \frac{\partial u}{\partial n} &= h \quad \text{on} \quad \partial\Omega\end{aligned}$$

$$(h : \partial\Omega \to \mathbb{R})$$



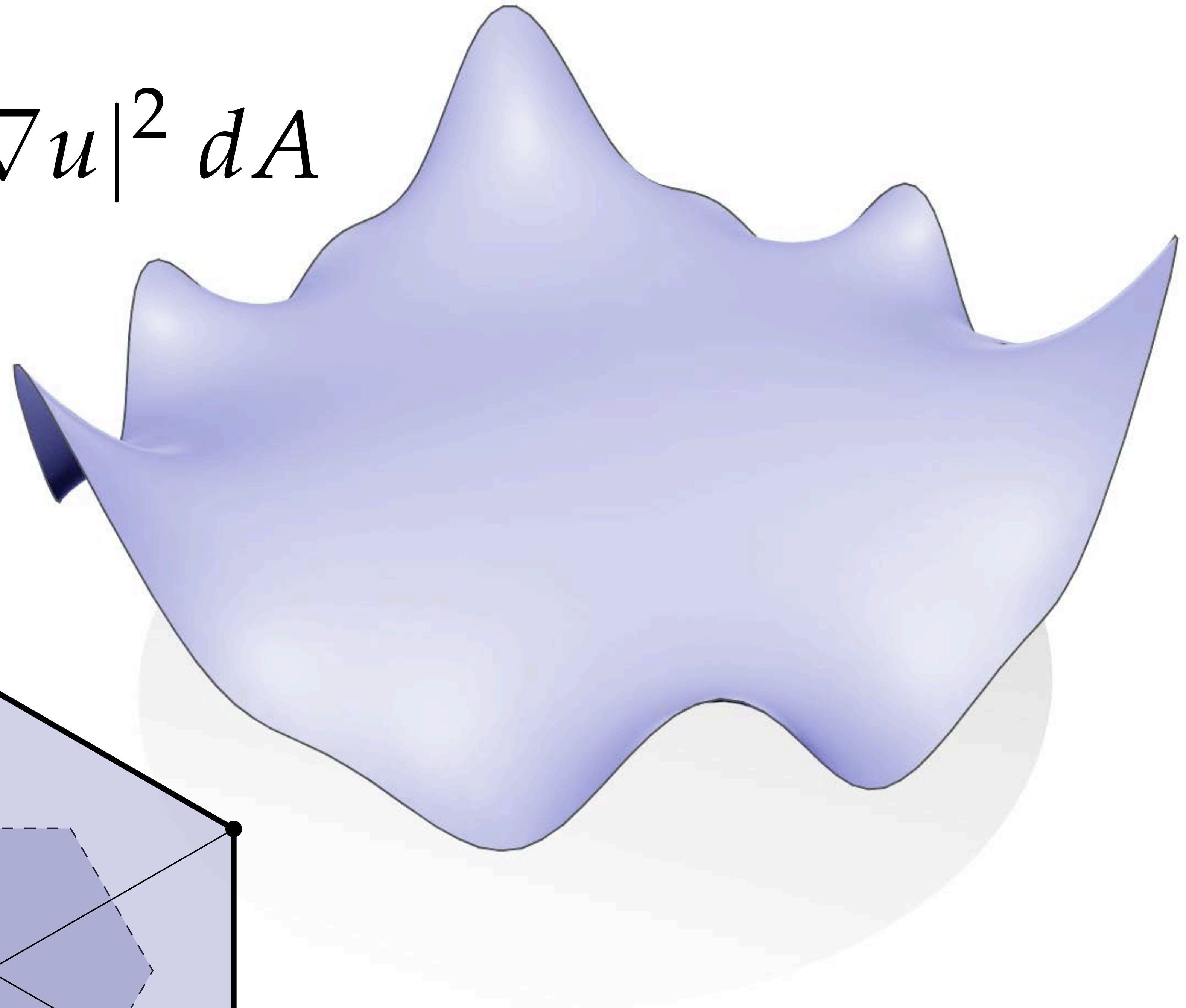
Summary

Laplace-Beltrami—Summary

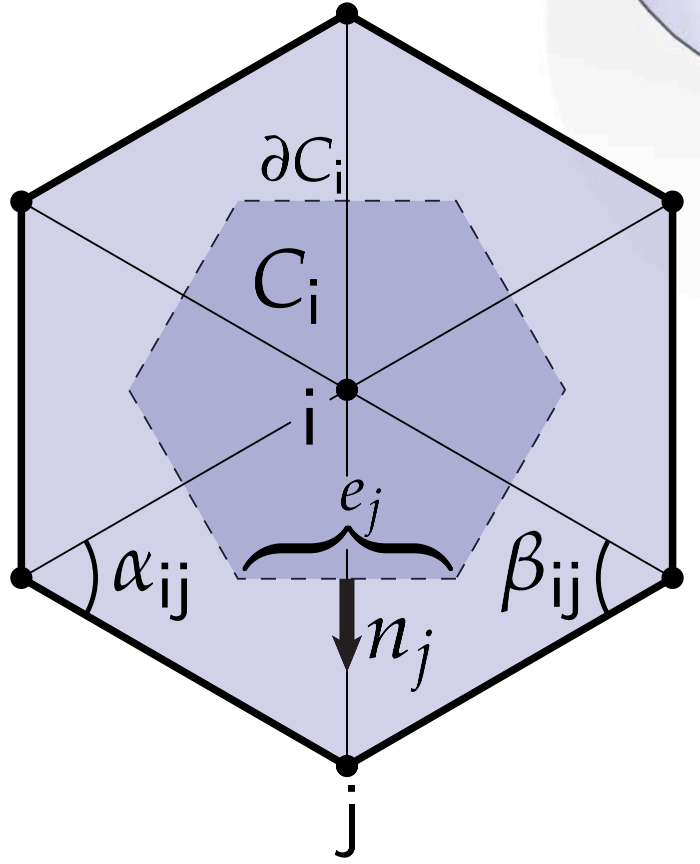
$$\boxed{\mathrm{tr}(\nabla^2 u)} \quad \boxed{\nabla \cdot \nabla u} \quad \boxed{\tfrac{d}{dt}\mathbb{E}[u(X_t)]} \quad \boxed{*\,d*d\,u}$$

- Fundamental object throughout geometry, physics, computer science
- Many different definitions in smooth setting
- Most basic idea: measures deviation from average
- Also closely connected to *Dirichlet energy*—measurement of "smoothness"
  - minimized by *harmonic function* (long-time heat flow)

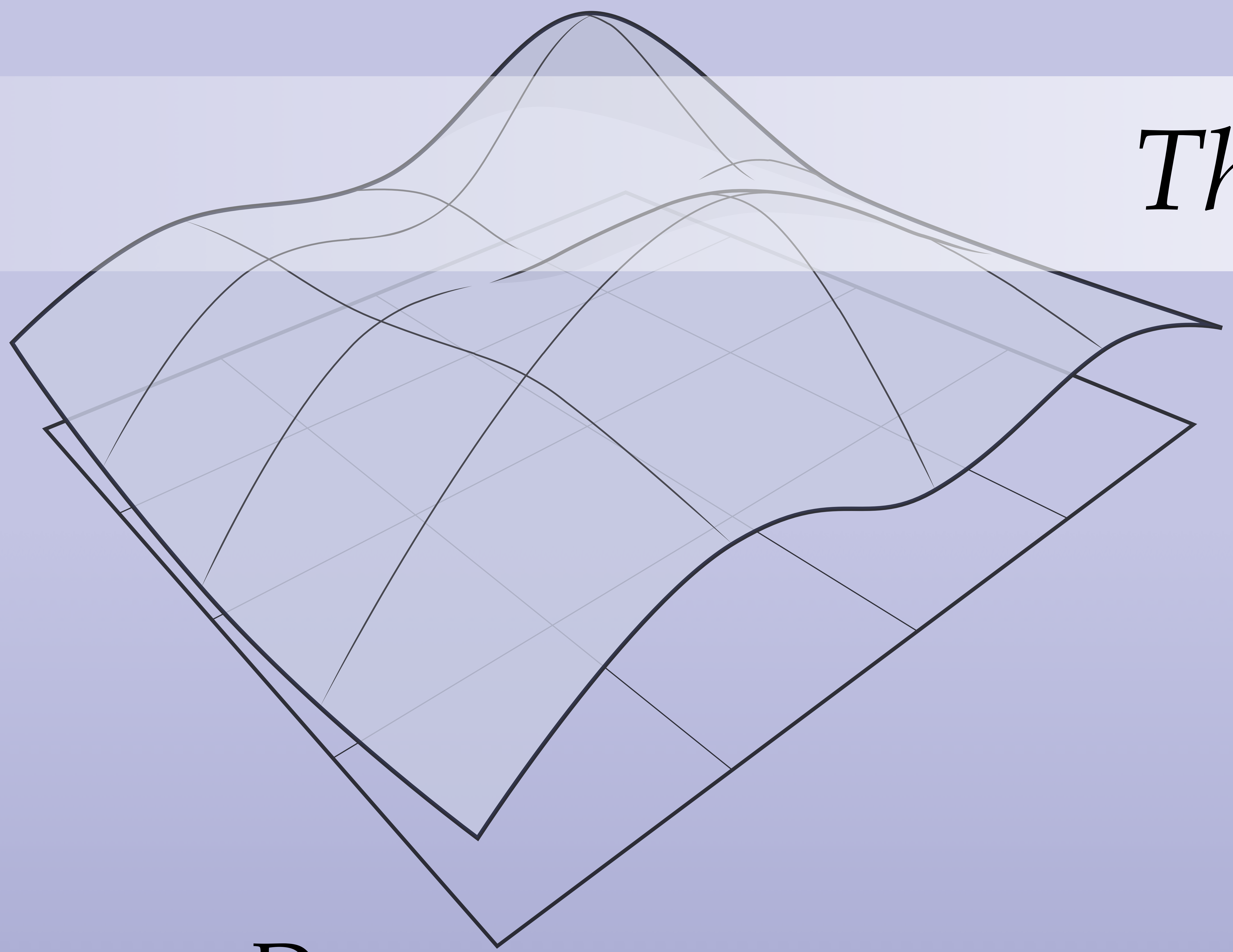
$$\int_M |\nabla u|^2 \, dA$$



- Must think carefully about boundary conditions—*solution will not always exist!*
  - major source of mistakes/bugs…
- **Next time:** *discretize!*



Thanks!



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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