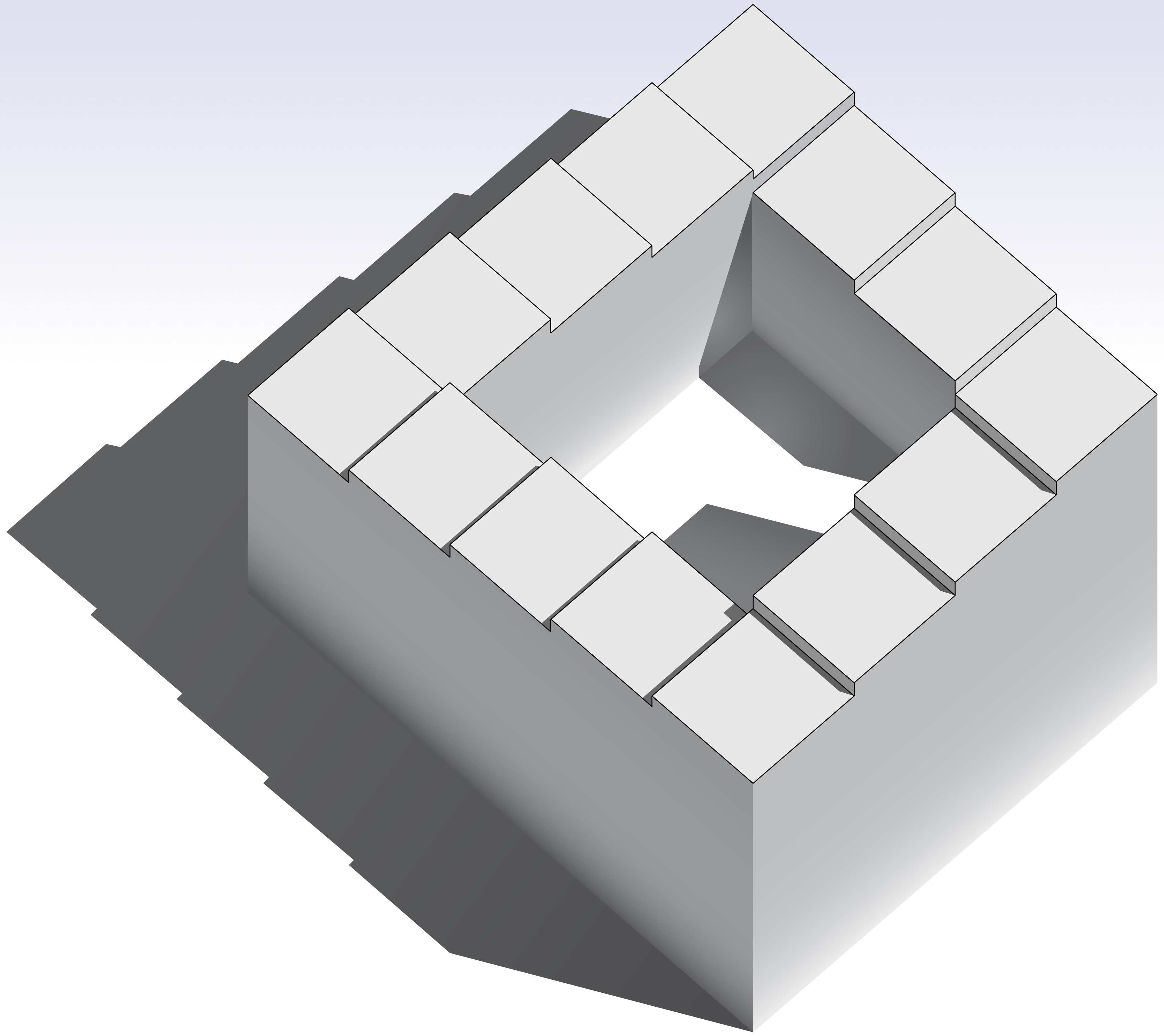


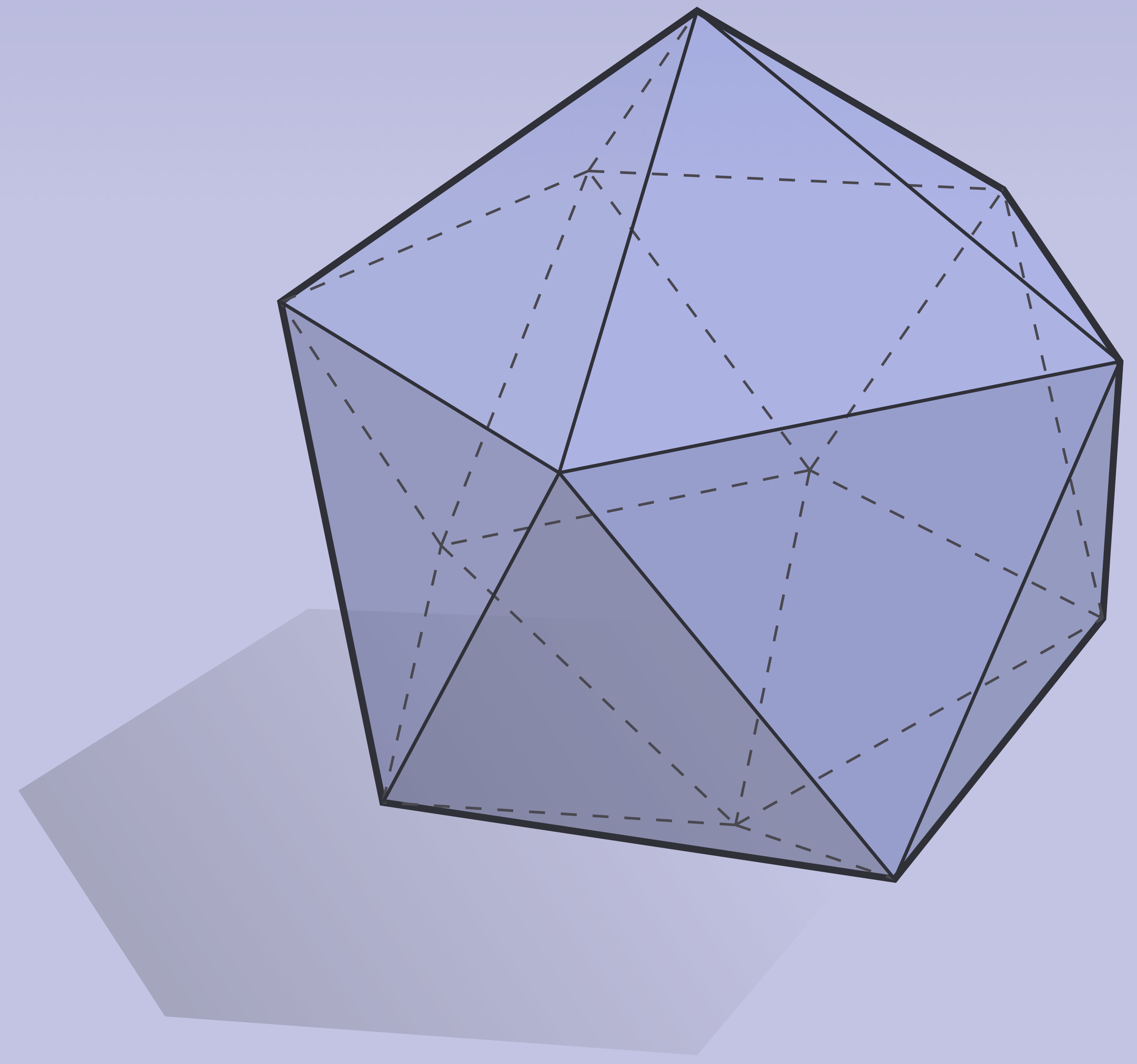
Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 17:

DISCRETE CURVATURE II

(VARIATIONAL VIEWPOINT)

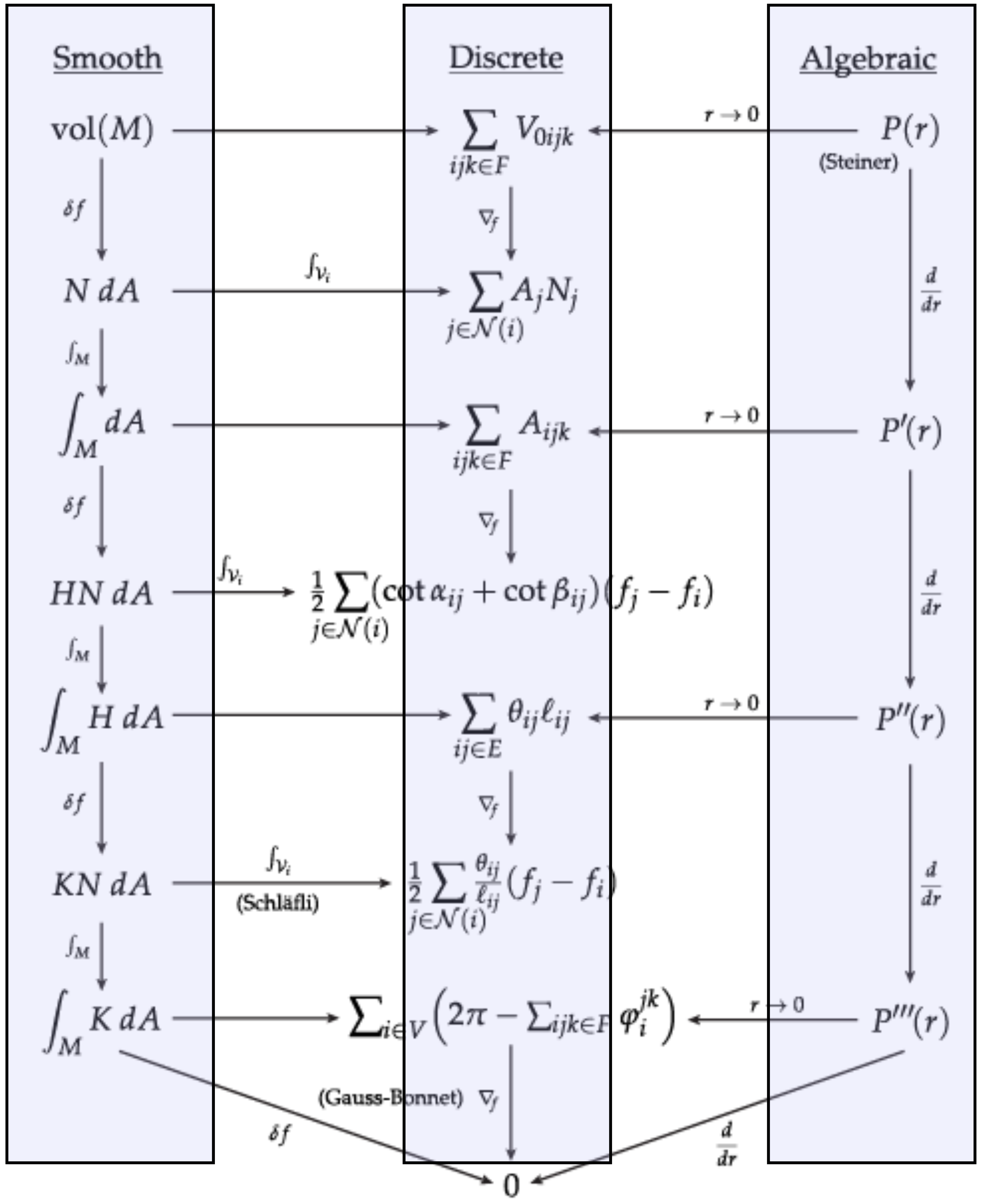


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858

A Unified Picture of Discrete Curvature

- Goal: obtain a unified picture of many different perspectives on discrete curvature by connecting smooth & discrete pictures
- Last time, took <u>integral</u> approach:
  - **vector-valued quantities**—<u>integrate</u> "curvature normals" over vertex neighborhood
  - **scalar quantities**—<u>integrate</u> curvatures on smoothed or "mollified" surface
- This time, take *variational* approach (derivatives)
  - Will see that our vector quantities actually just describe the change in our scalar quantities!

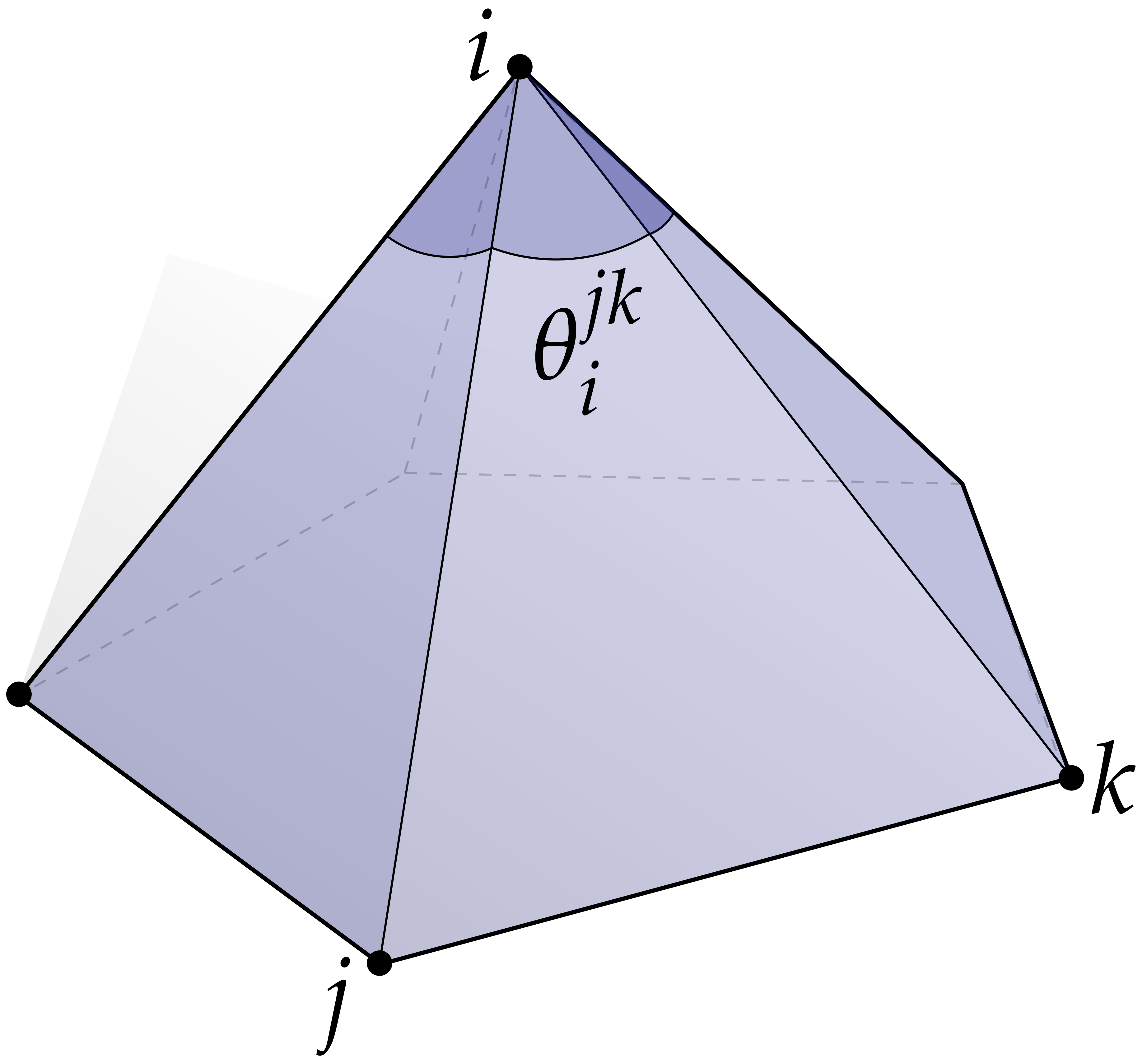


Recap: Vector Curvatures

	area (NdA)	**mean** ($HNdA$)	**Gauss** ($KNdA$)
smooth	$\frac{1}{2}\, df \wedge df$	$\frac{1}{2}\, df \wedge dN$	$\frac{1}{2}\, dN \wedge dN$
discrete	$\frac{1}{6} \sum_{ijk \in \mathrm{St}(i)} f_j \times f_k$	$\frac{1}{2} \sum_{ij \in \mathrm{St}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$	$\frac{1}{2} \sum_{ij \in \mathrm{St}(i)} \frac{\varphi_{ij}}{\ell_{ij}} (f_j - f_i)$

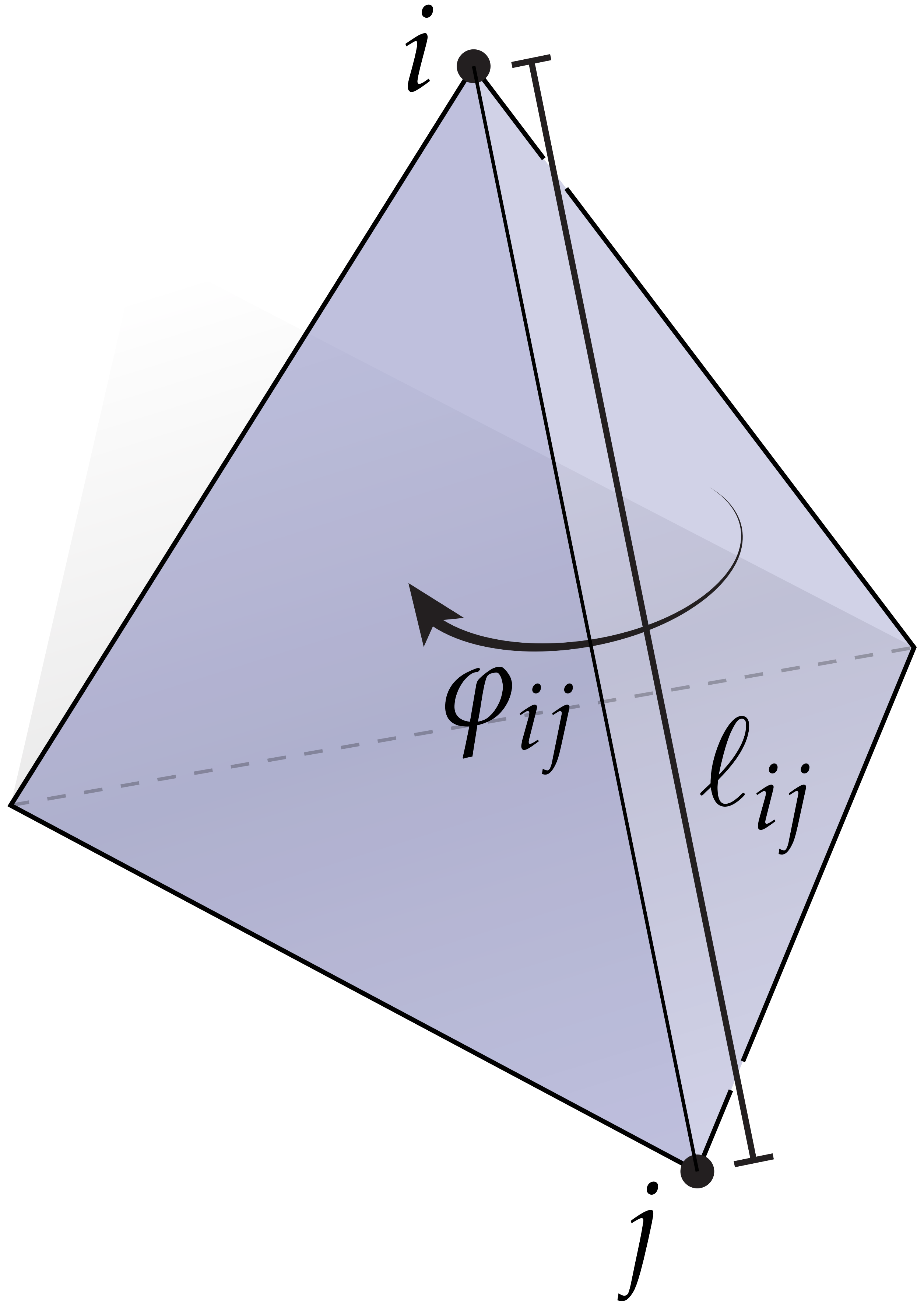
Recap: Scalar Curvatures

Gaussian curvature



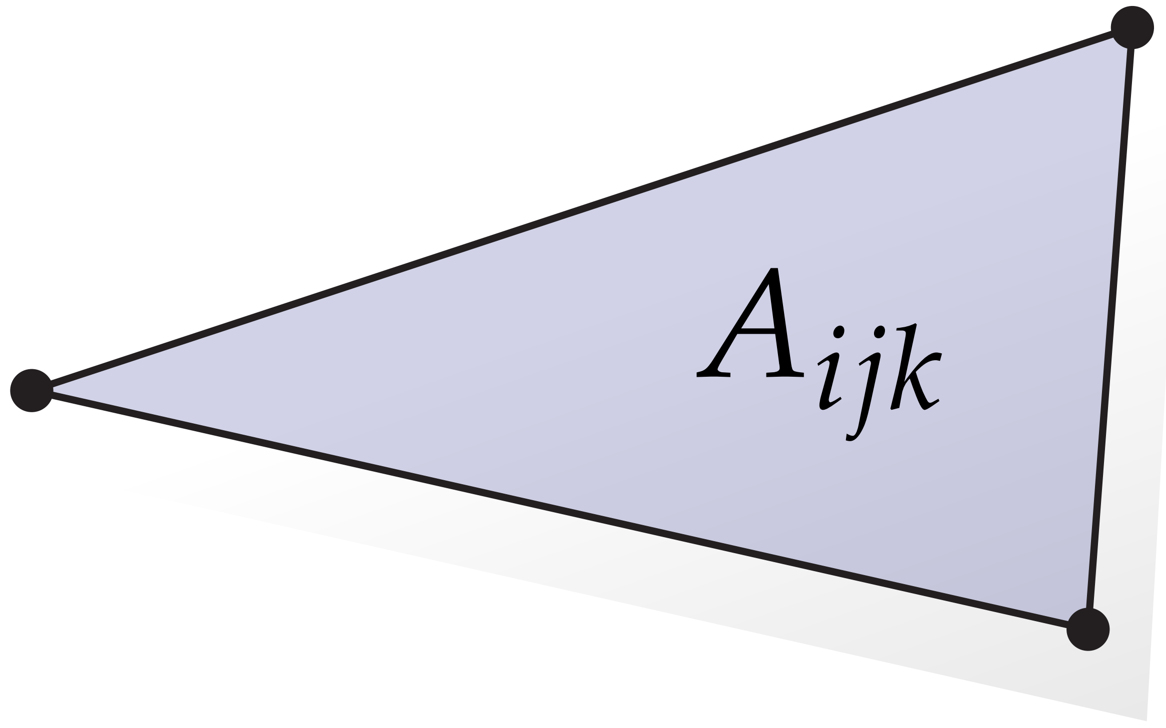
$$\Omega_i := 2\pi - \sum_{ijk} \theta_i^{jk}$$

mean curvature

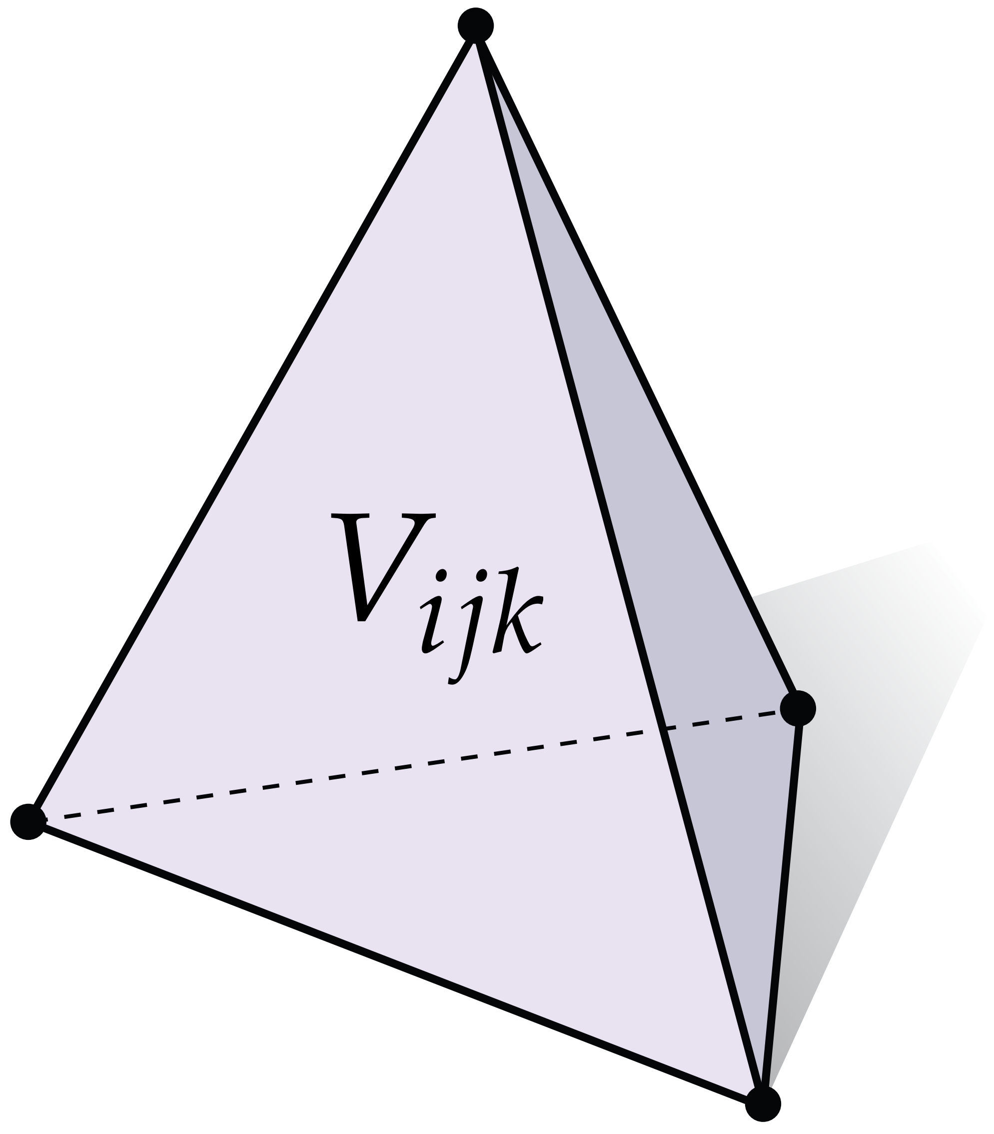


$$H_{ij} := \tfrac{1}{2}\ell_{ij}\varphi_{ij}$$

area



volume



Aside: Principal Curvatures

Gaussian: $K = \kappa_1 \kappa_2$

mean: $H = \dfrac{\kappa_1 + \kappa_2}{2}$

principal:

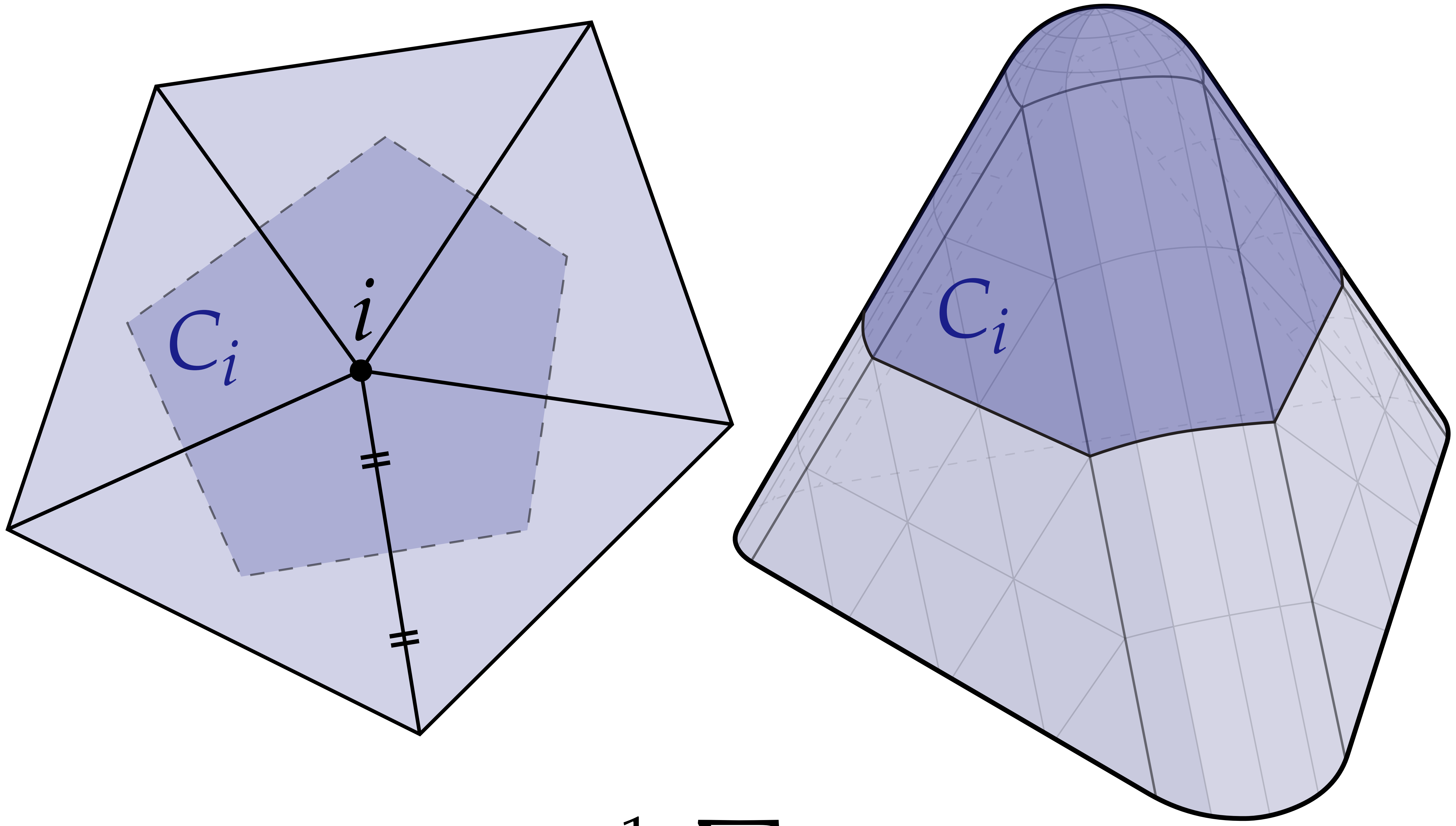
$$\kappa_1 = H - \sqrt{H^2 - K}$$
$$\kappa_2 = H + \sqrt{H^2 - K}$$

discrete principal curvatures:

$$\frac{H_i}{A_i} \pm \sqrt{\left(\frac{H_i}{A_i}\right)^2 - \frac{K_i}{A_i}}$$

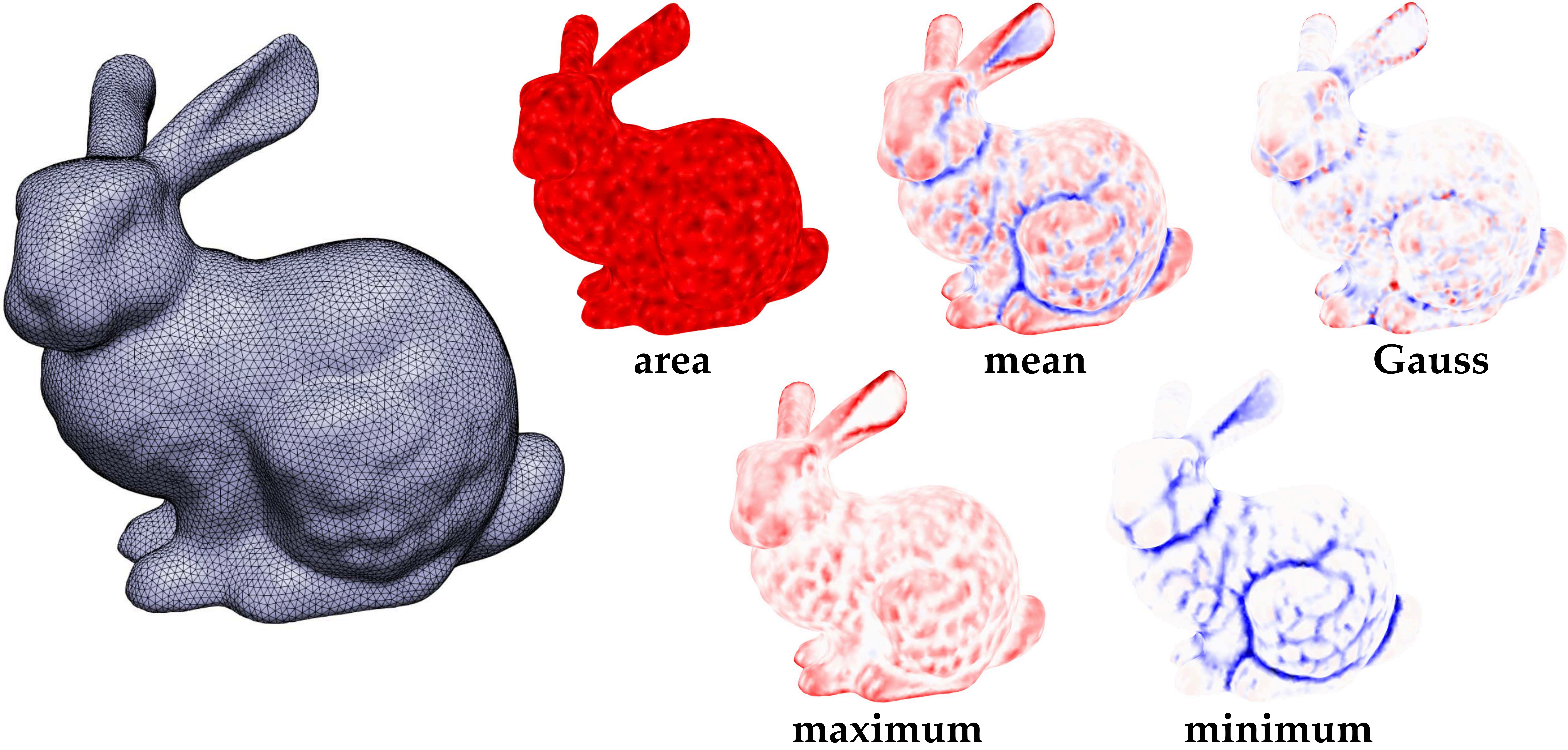
$$A_i := |C_i|$$

vertex mean curvature

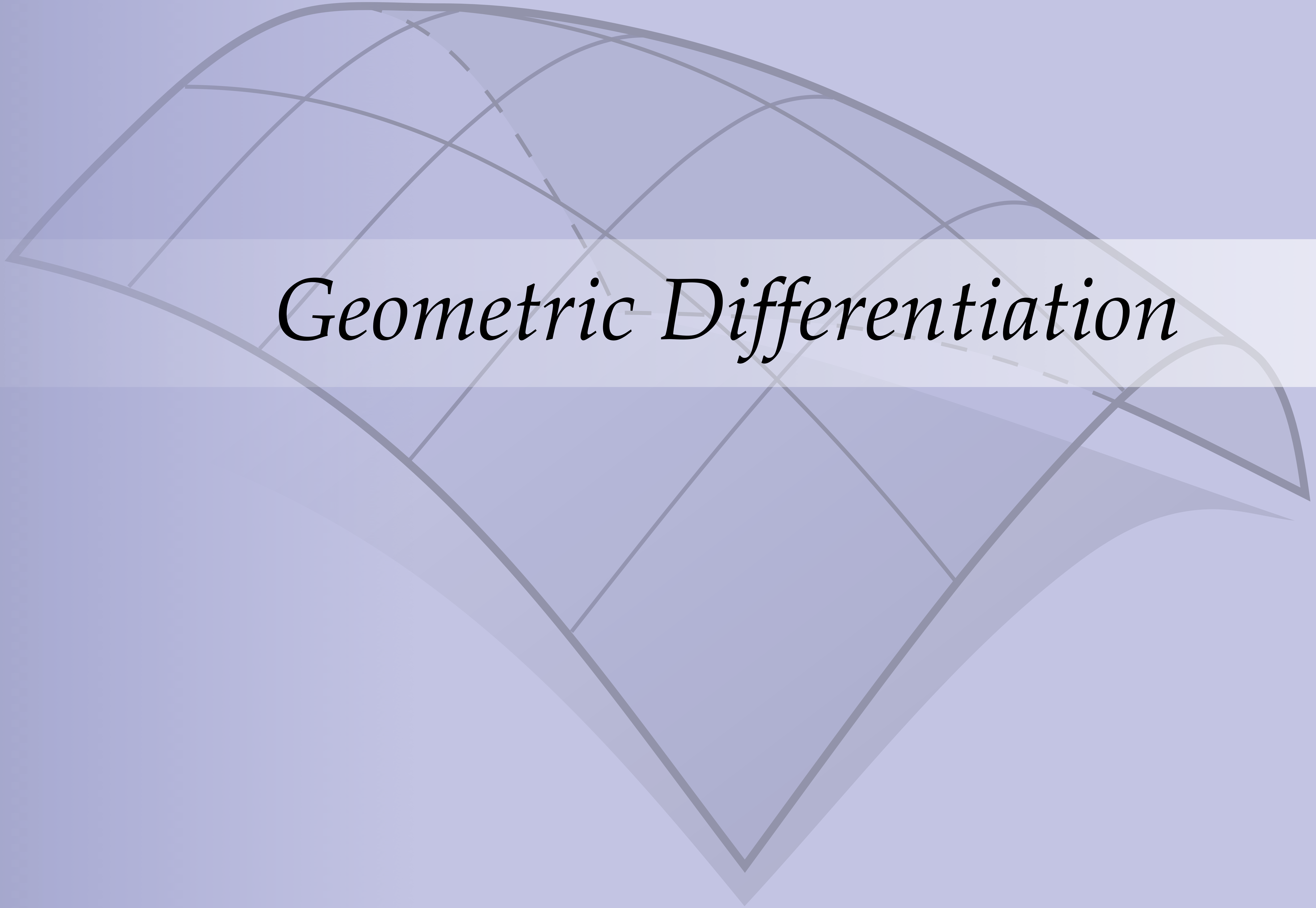


$$H_i := \frac{1}{4} \sum_{ij \in E} \ell_{ij} \varphi_{ij}$$

Scalar Curvatures — Visualized

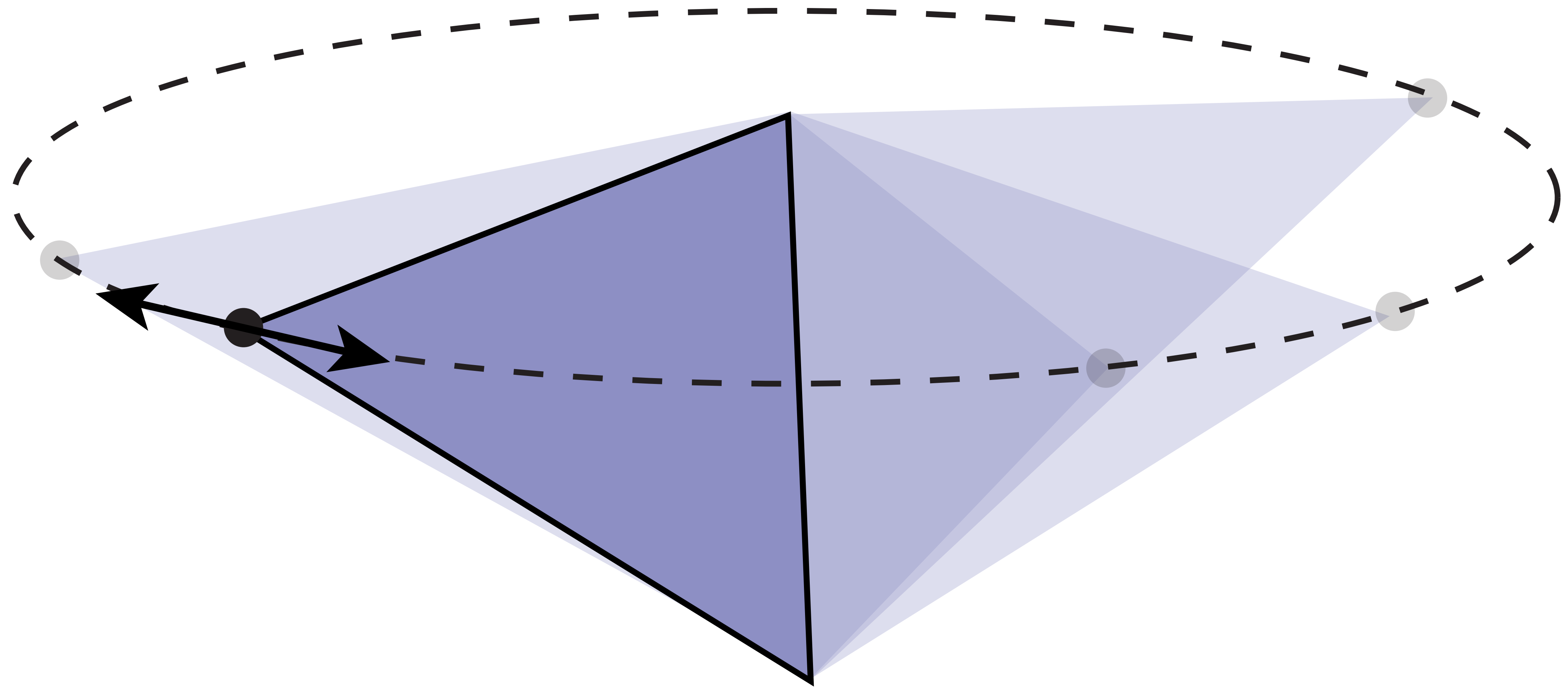


Geometric Differentiation



Geometric Differentiation

- Many geometric problems/algorithms involve taking derivatives of functions involving lengths, angles, areas, …
- *E.g.*, how does the area of a triangle change as we move one of its vertices?
- More generally: *how does one geometric quantity change with respect to another?*
- <u>**Don't**</u> just grind out partial derivatives!
- <u>**Do**</u> follow a simple geometric recipe:



1. First, in which **direction** does the quantity change quickest?
2. Second, what's the **magnitude** of this change?
3. Together, direction & magnitude give us the gradient vector

Dangers of Naïve Differentiation

- Why not just take derivatives *"the usual way?"*
- Usually takes way more work!
- can lead to expressions that are
  - – inefficient
  - – numerically unstable
  - – hard to understand
- **Example:** gradient of angle between two segments (*b*,*a*), (*c*,*a*) w.r.t. coordinates of point *a*

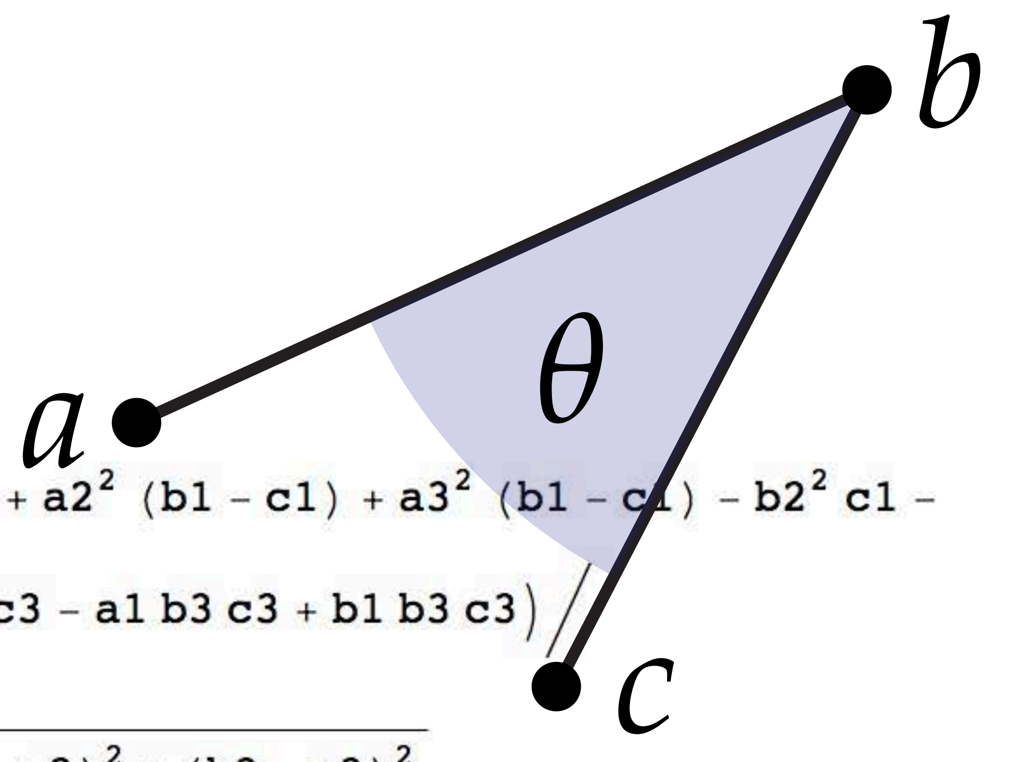
In[58]:=
```
a = {a1, a2, a3};
b = {b1, b2, b3};
c = {c1, c2, c3};
θ = ArcCos[(a - b).(c - b) / (√((a - b).(a - b)) √((c - b).(c - b)))];
FullSimplify[{∂a1 θ, ∂a2 θ, ∂a3 θ}]
```

Out[62]=

$$\left\{\left(a1\, b2^2 + a1\, b3^2 - a2\, b2\, (a1 + b1 - 2\, c1) - a3\, b3\, (a1 + b1 - 2\, c1) + a2^2\, (b1 - c1) + a3^2\, (b1 - c1) - b2^2\, c1 - b3^2\, c1 + a2\, (a1 - b1)\, c2 - a1\, b2\, c2 + b1\, b2\, c2 + a3\, (a1 - b1)\, c3 - a1\, b3\, c3 + b1\, b3\, c3\right) \Big/ \left(\left((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2\right)^{3/2} \sqrt{(b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2}\, \sqrt{1 - \frac{((a1 - b1)\,(-b1 + c1) + (a2 - b2)\,(-b2 + c2) + (a3 - b3)\,(-b3 + c3))^2}{((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2)\,((b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2)}}\right),\right.$$

$$\left(a3^2\, b2 - a3\, b2\, b3 + b1\, b2\, c1 + a1^2\, (b2 - c2) - a3^2\, c2 - b1^2\, c2 + 2\, a3\, b3\, c2 - b3^2\, c2 - a1\, (a2\, (b1 - c1) + b2\, (b1 + c1) - 2\, b1\, c2) + a2\, (b1\, (b1 - c1) - (a3 - b3)\, (b3 - c3)) - a3\, b2\, c3 + b2\, b3\, c3\right) \Big/ \left(\left((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2\right)^{3/2} \sqrt{(b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2}\, \sqrt{1 - \frac{((a1 - b1)\,(-b1 + c1) + (a2 - b2)\,(-b2 + c2) + (a3 - b3)\,(-b3 + c3))^2}{((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2)\,((b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2)}}\right),$$

$$\left.\left(b3\, (b1\, c1 + (a2 - b2)\, (a2 - c2)) + a3\, (b1\, (b1 - c1) - (a2 - b2)\, (b2 - c2)) + a1^2\, (b3 - c3) - (b1^2 + (a2 - b2)^2)\, c3 - a1\, (a3\, (b1 - c1) + b3\, (b1 + c1) - 2\, b1\, c3)\right) \Big/ \left(\left((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2\right)^{3/2} \sqrt{(b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2}\, \sqrt{1 - \frac{((a1 - b1)\,(-b1 + c1) + (a2 - b2)\,(-b2 + c2) + (a3 - b3)\,(-b3 + c3))^2}{((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2)\,((b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2)}}\right)\right\}$$



Geometric Derivation of Angle Derivative

- Instead of taking partial derivatives, let's break this calculation into two pieces:

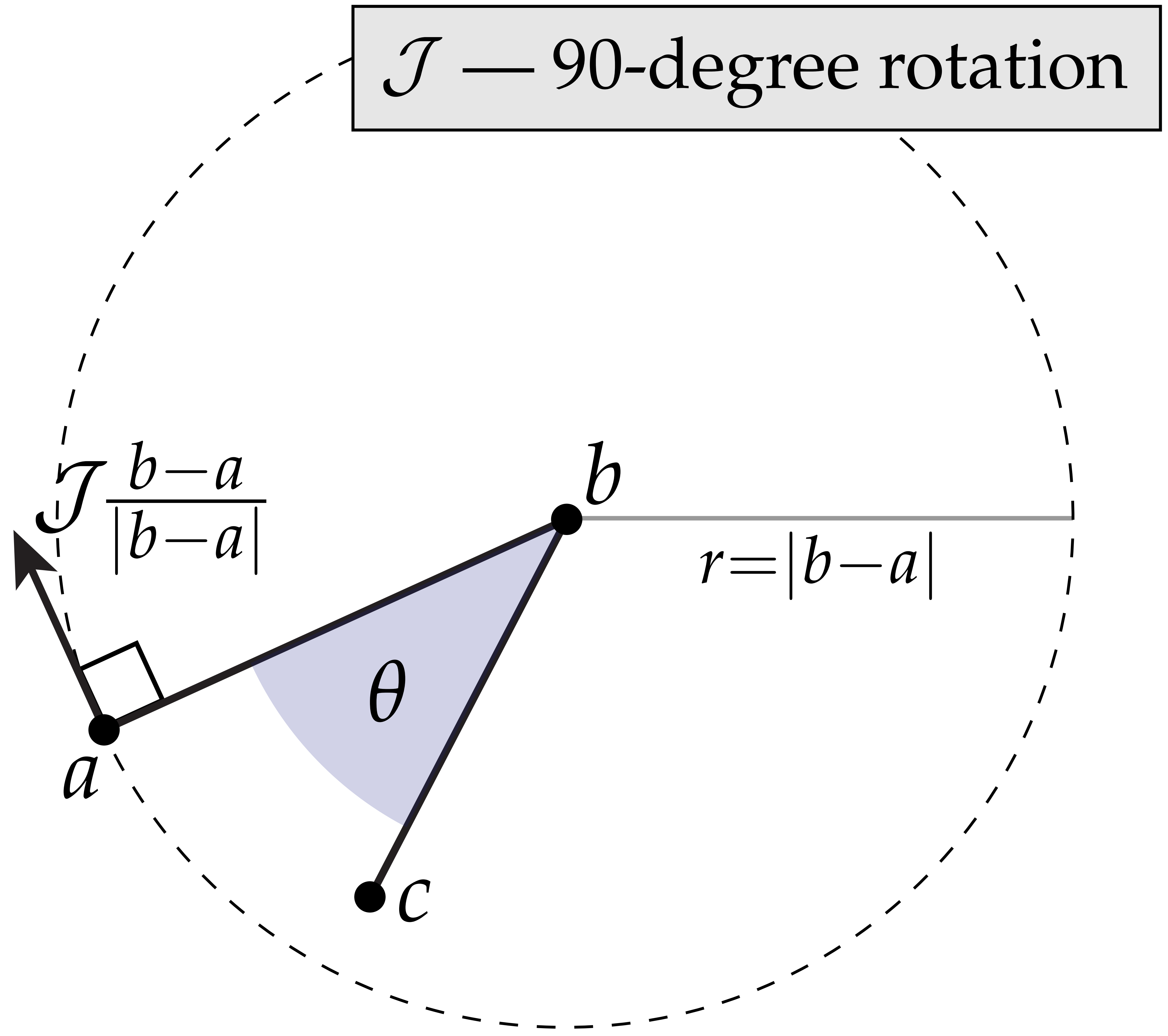
1. (**Direction**) What direction can we move the point a to most quickly increase the angle θ?

   A: *Orthogonal to the segment ab.*

2. (**Magnitude**) How much does the angle change if we move in this direction?

   A: *Moving around a whole circle changes the angle by* 2π *over a distance* $2\pi r$*. Hence, the instantaneous change is* $1/|b\text{-}a|$.

3. Multiplying the unit direction by the magnitude yields the final gradient expression.

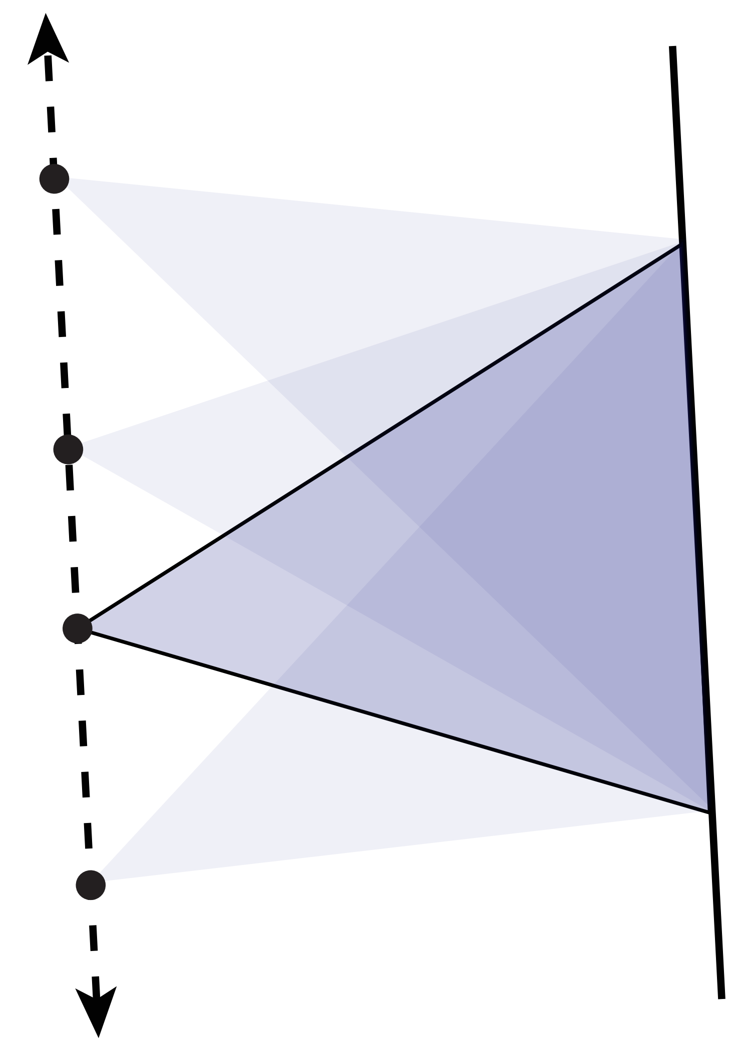


$$\boxed{\nabla_a \theta = \mathcal{J}\frac{b-a}{|b-a|^2}}$$

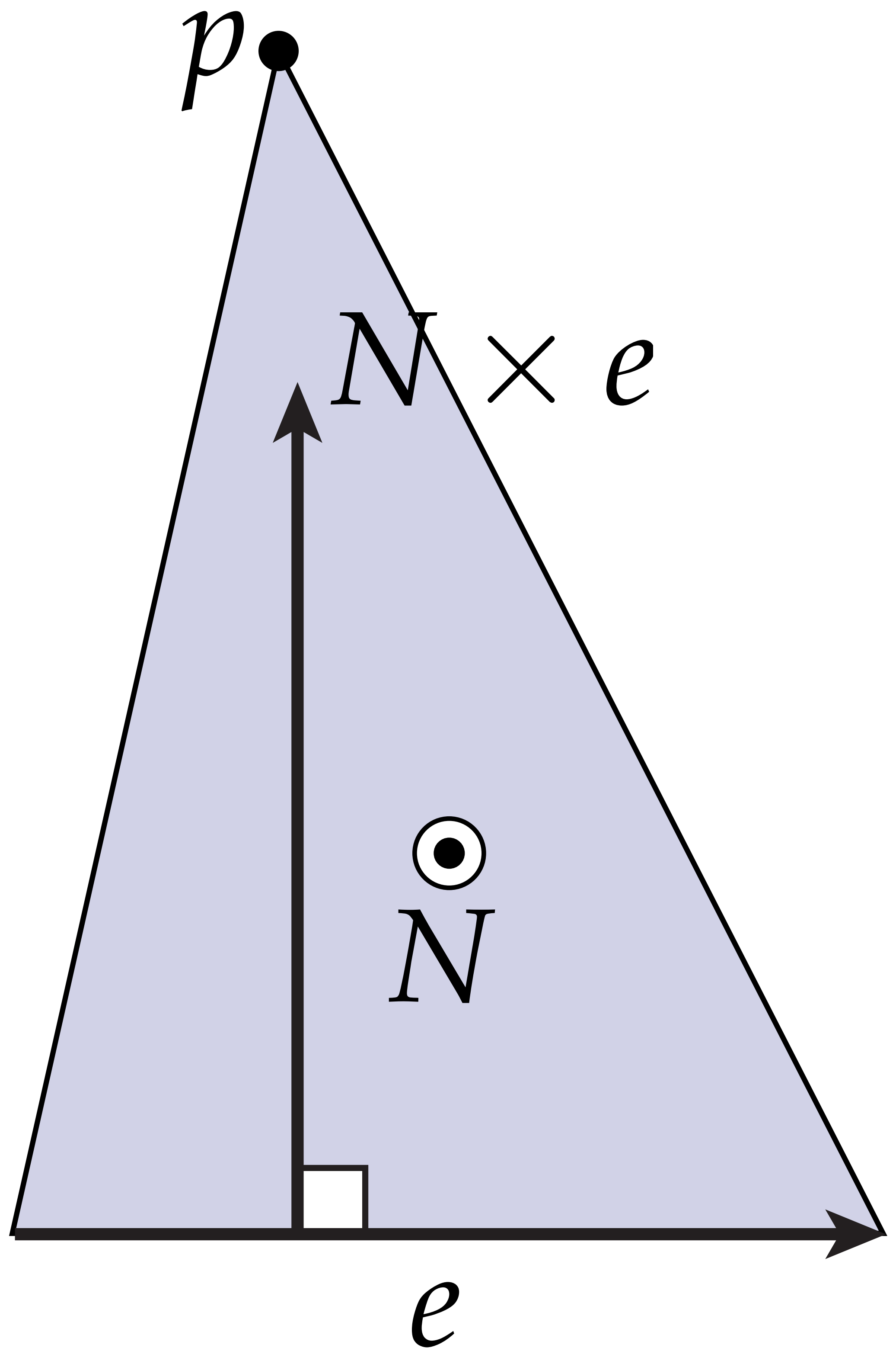
Gradient of Triangle Area

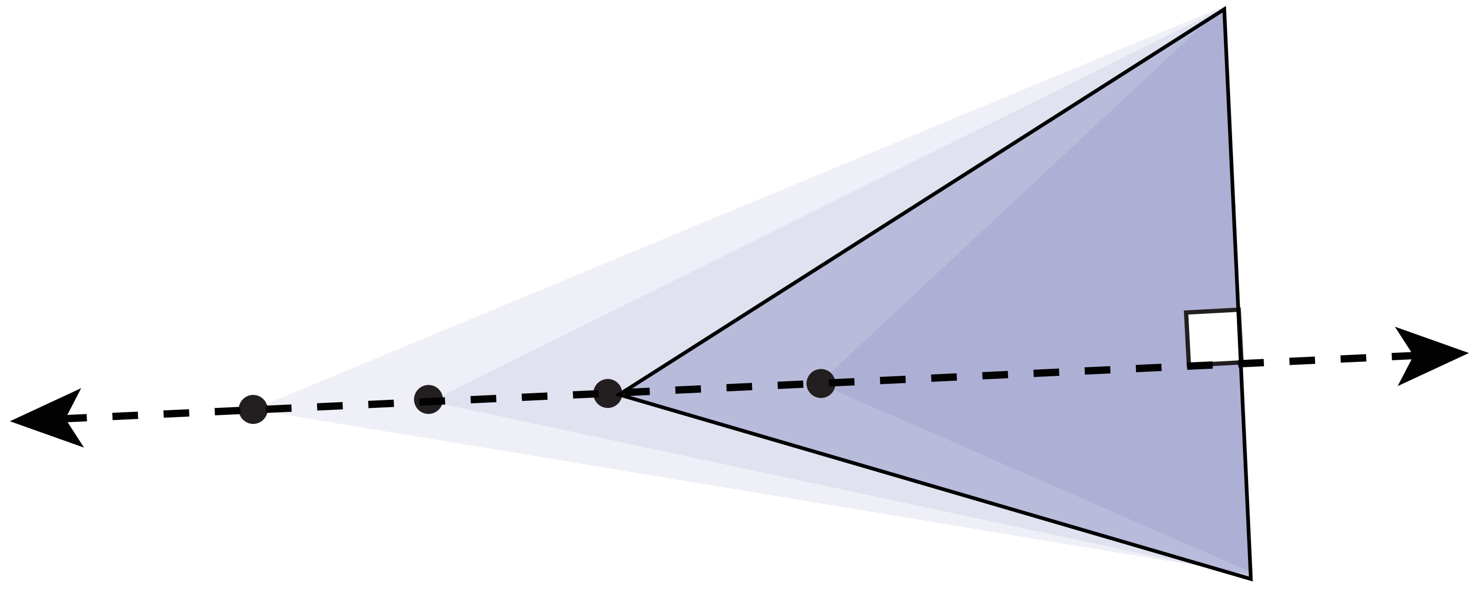
Q: What's the gradient of triangle area with respect to one of its vertices p?

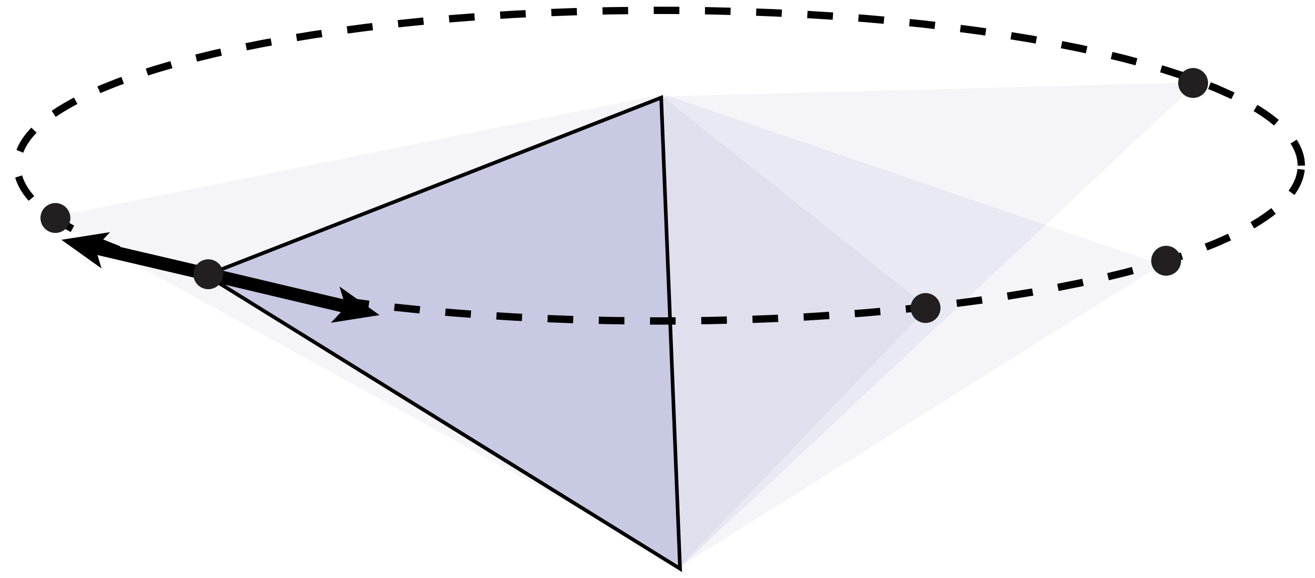
A: Can express via its unit normal N and vector e along edge opposite p:



$$\nabla_p A = \frac{1}{2} N \times e$$

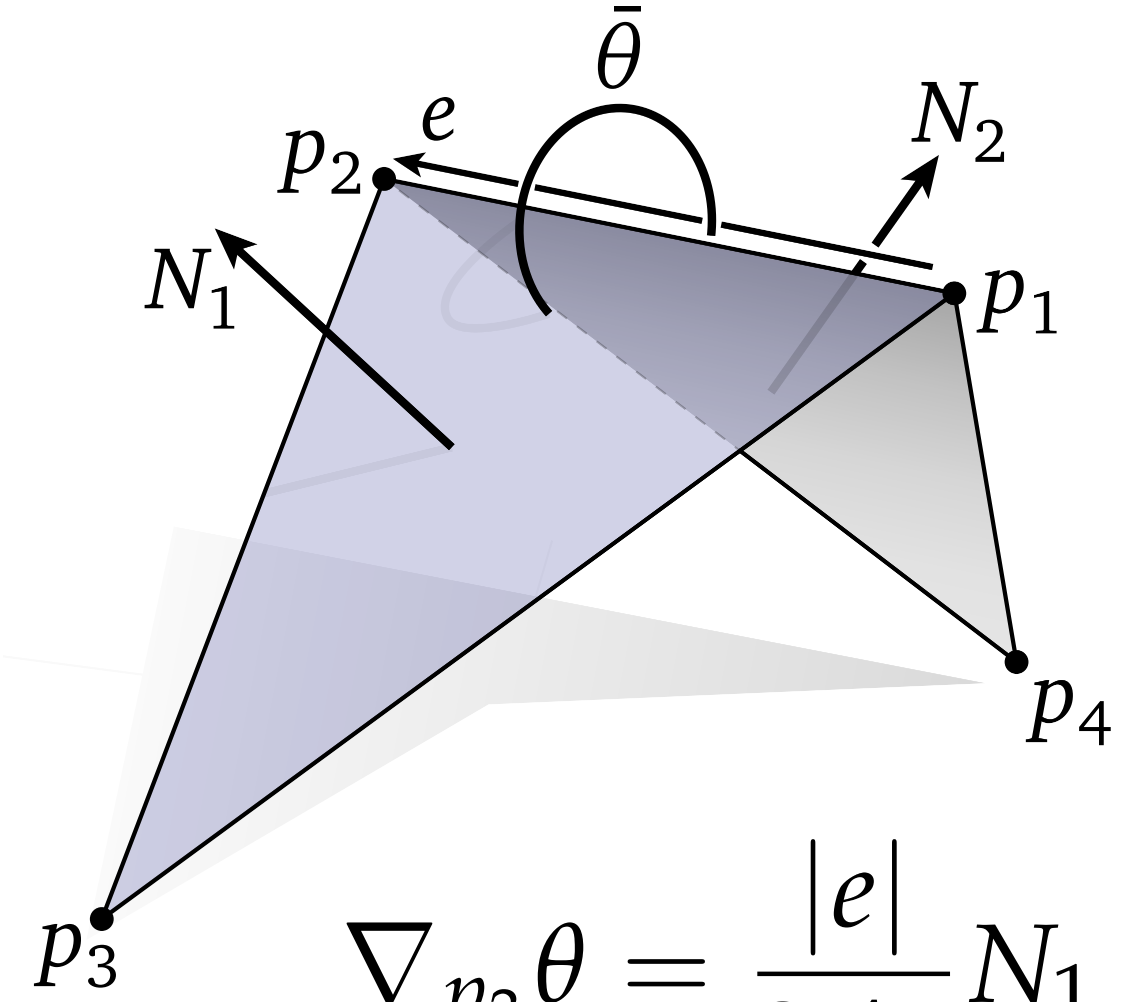




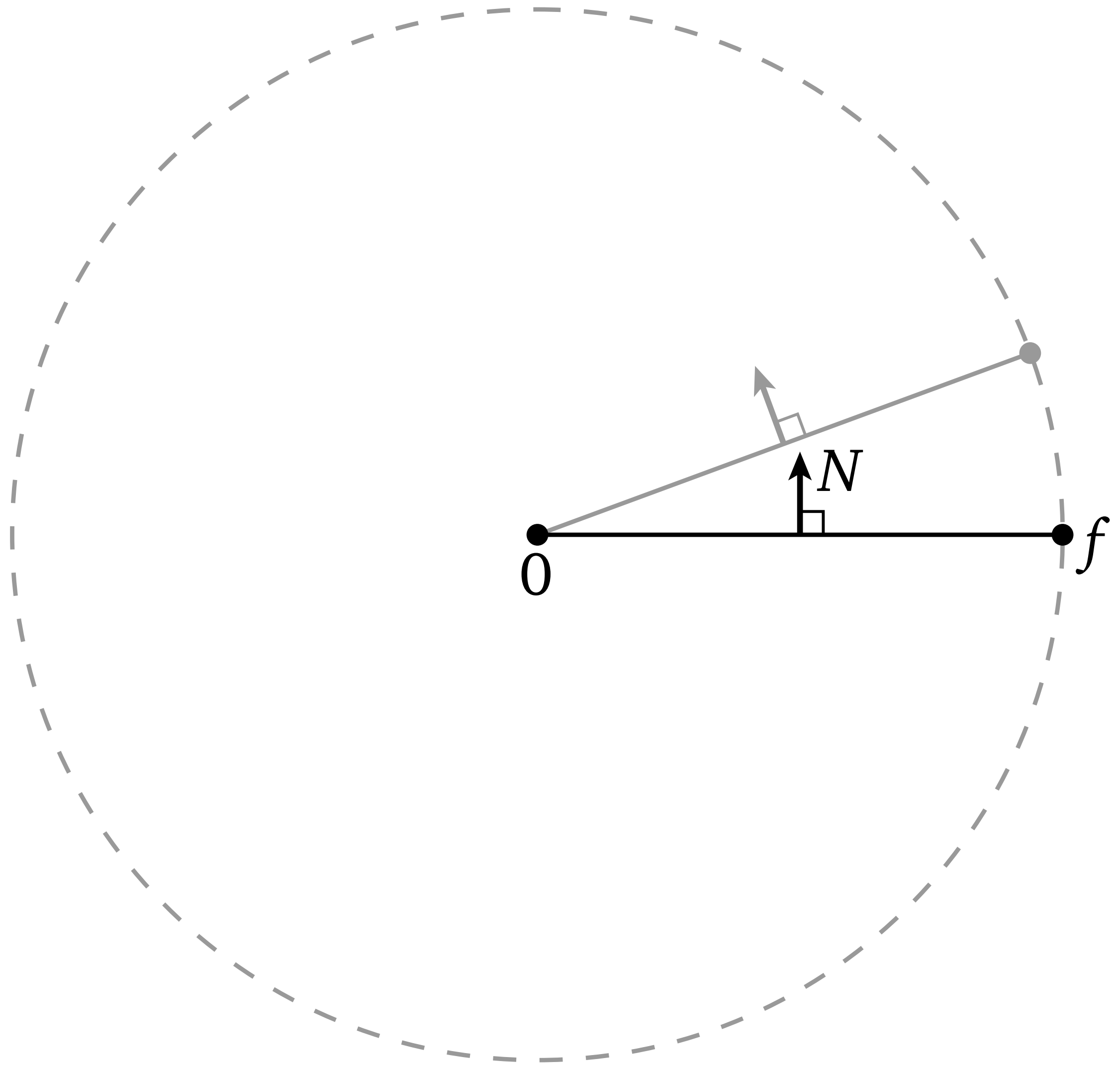


Geometric Derivation

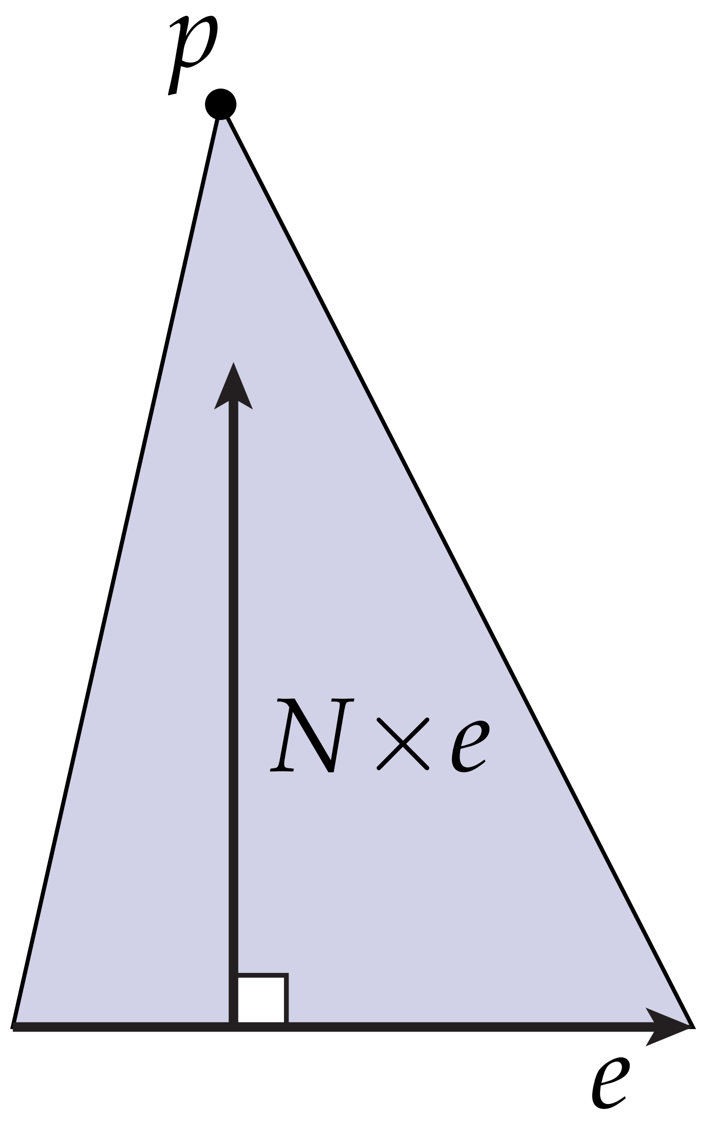
- In general, can lead to some pretty nice expressions (give it a try!)



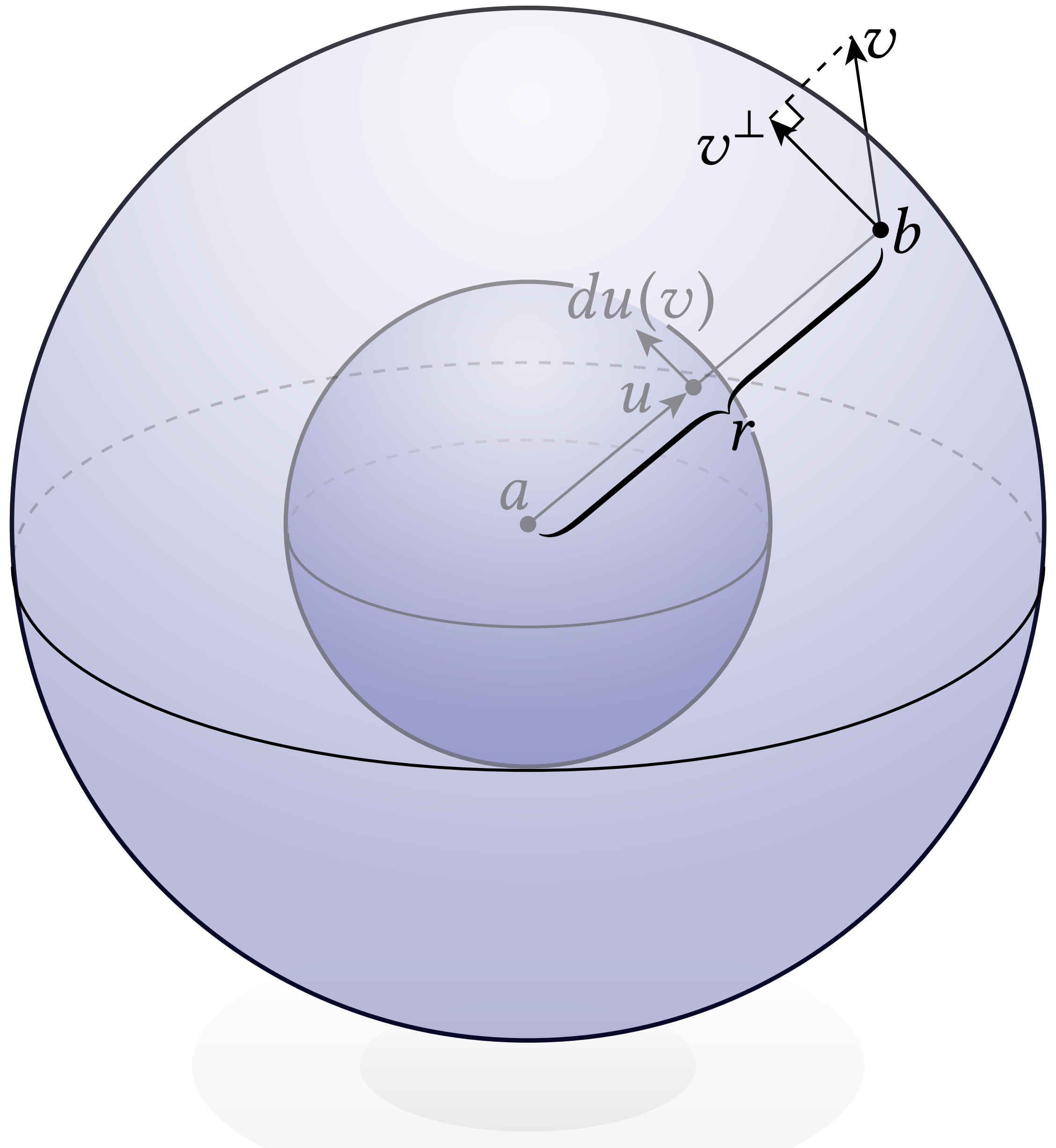
$$\nabla_{p_3}\theta = \frac{|e|}{2A_1}N_1$$



$$d_{f_i}N(X) = \frac{\langle N, X\rangle}{2A} e_i \times N$$



$$\nabla_p A = \tfrac{1}{2} N \times e$$



$$du(v) = \frac{v - \langle v, b-a\rangle(b-a)}{|b-a|^3}$$

(See also Appendix A of the course notes.)

Differentiation Strategies

Often have to differentiate complicated function built up from these "little pieces"—several common strategies for automating this process:

closed-form differentiation
Work it out by hand, write custom code
Pros: final code is fast and accurate
Cons: very time consuming, hard to change energy, easy to make mistakes

numerical differentiation
perturb each input by ε, measure change in energy
Pros: works directly with existing code / "black box" routines
Cons: expensive, inaccurate, hard to pick ε

automatic differentiation
differentiate each line of code; use chain rule to obtain overall derivative (*"backpropagation"*)
Pros: accurate, almost as fast as closed-form, no work "by hand"
Cons: must modify existing code / doesn't work in "black box" scenario

symbolic differentiation
perform transformation of symbolic expression tree
Pros: accurate, only have to take derivative once
Cons: must modify existing code, can lead to (very) large expressions

Also: no use of domain-specific knowledge.

Curvature Variations

Sequence of Variations (Smooth)

For a smooth surface $f : M \longrightarrow R^3$ (without boundary), let

$$\text{volume}(f) := \tfrac{1}{3}\int_M N \cdot f \, dA \qquad \text{mean}(f) := \int_M H \, dA$$

$$\text{area}(f) := \int_M dA \qquad \text{Gauss}(f) := \int_M K \, dA = 2\pi\chi$$

<u>Q.</u> What motion of the surface changes each of these quantities as quickly as possible?

<u>A.</u> Remarkably enough…

$$\delta\,\text{volume}(f) = 2N$$
$$\delta\,\text{area}(f) = 2HN$$
$$\delta\,\text{mean}(f) = 2KN$$
$$\delta\,\text{Gauss}(f) = 0$$

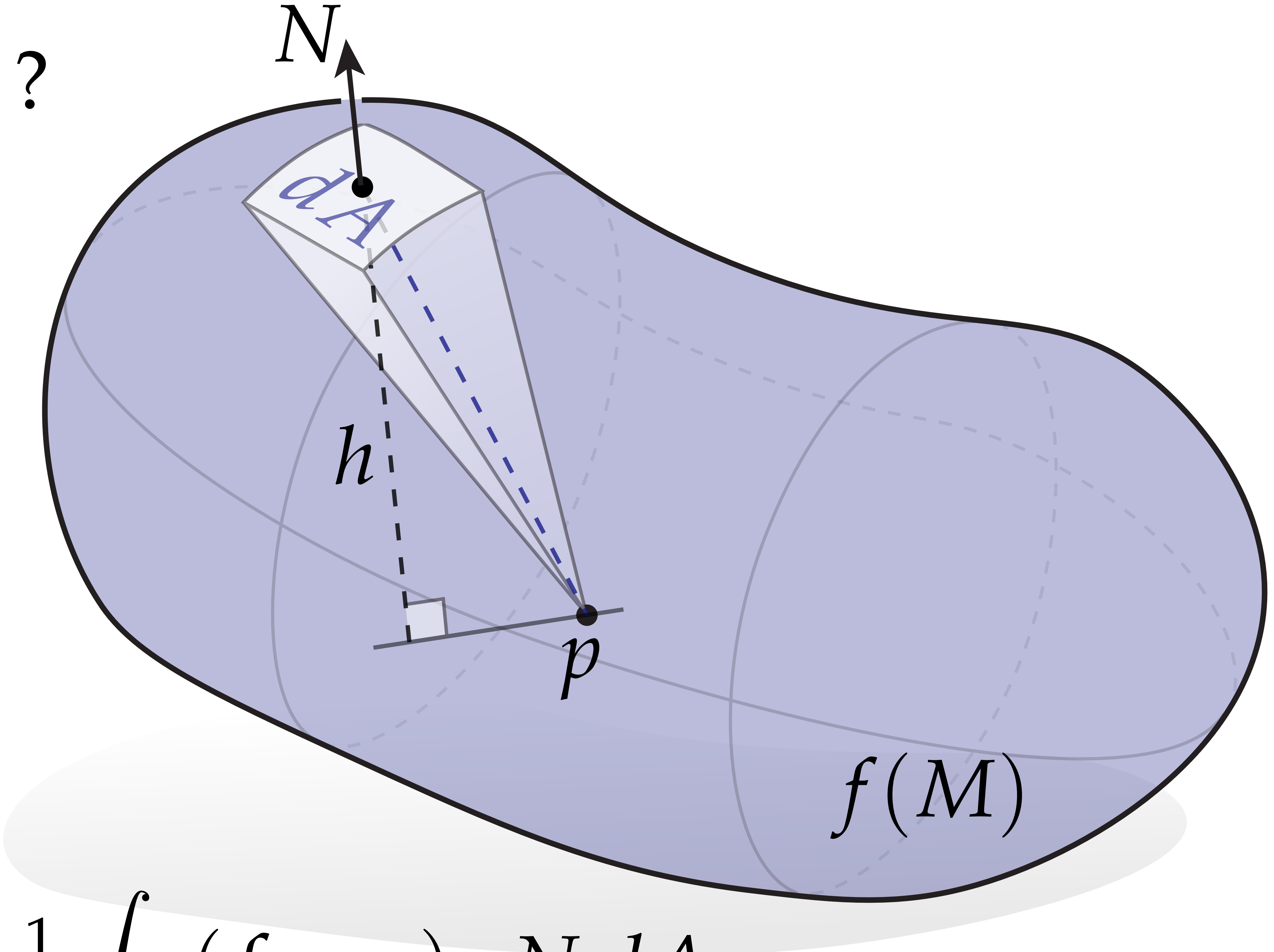
$$\boxed{\text{volume} \xrightarrow{\delta f} \text{area} \xrightarrow{\delta f} \text{mean} \xrightarrow{\delta f} \text{Gauss} \xrightarrow{\delta f} 0}$$

Volume Enclosed by a Smooth Surface

- What's the volume enclosed by a *smooth* surface $f(M)$?
- One way: pick any point p, integrate volume of "infinitesimal pyramids" over the surface
- For a pyramid with base area b and height h, the volume is $V = bh/3$ (for a base of any shape)
- For our infinitesimal pyramid, the height h is the distance from the surface f to the point p along the normal direction:

$$h = (f - p) \cdot N$$

- The area of the base is just the infinitesimal surface area dA. Now we just integrate…



$$\frac{1}{3}\int_M (f - p) \cdot N \, dA =$$

$$\frac{1}{3}\int_M f \cdot N \, dA - \frac{1}{3} p \cdot \cancelto{0}{\int_M N \, dA} =$$

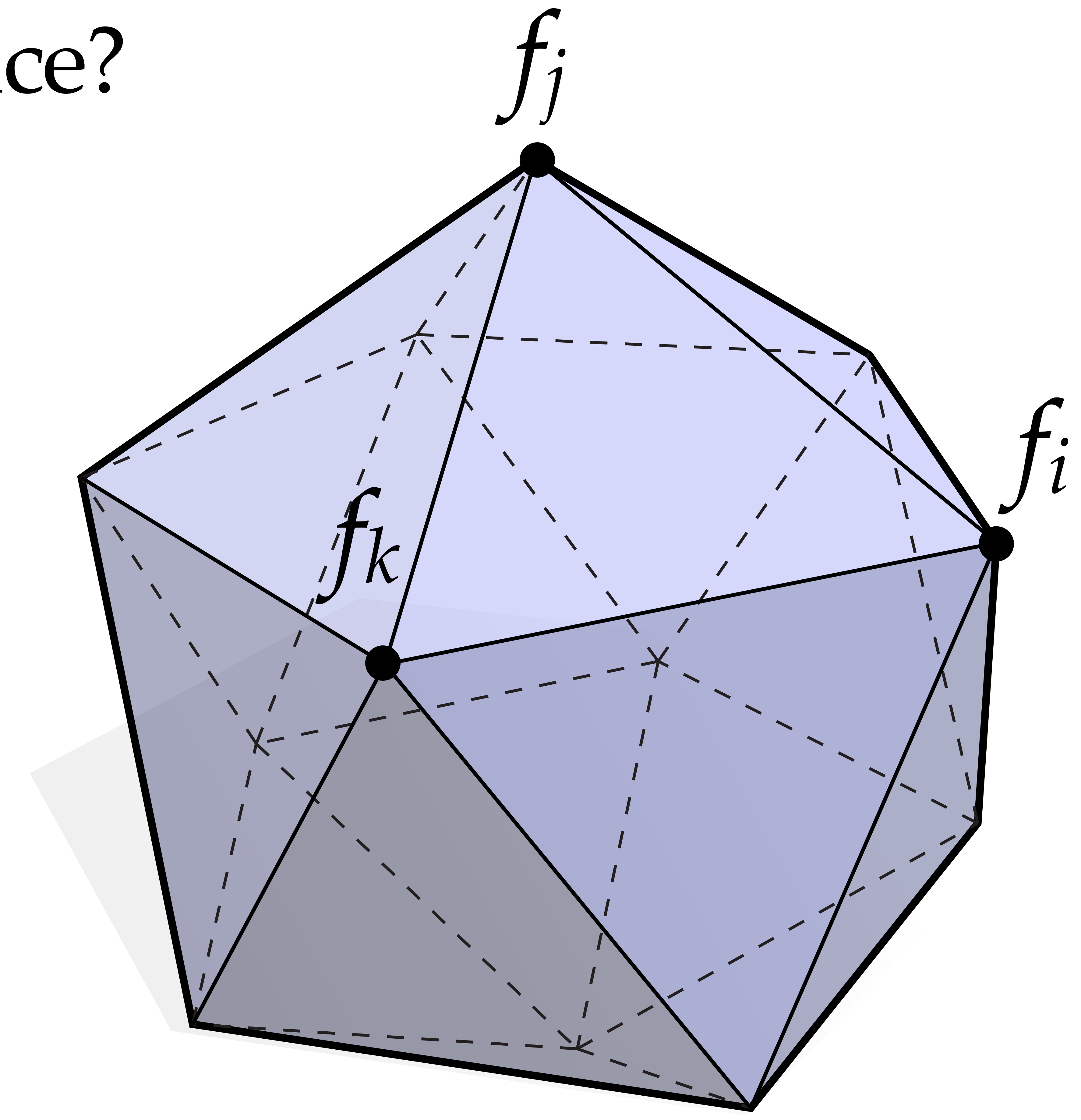
$$\boxed{\frac{1}{3}\int_M f \cdot N \, dA}$$

Notice: final expression doesn't depend on choice of point p!

Volume Enclosed by a Discrete Surface

- What's the volume enclosed by a *discrete* surface?
- Simply apply the smooth formula!
  - integrate $f \cdot N$ over each triangle
- **<u>Exercise.</u>** Show that the volume enclosed by a simplicial surface can be expressed as

$$\boxed{\mathrm{volume}(f) = \frac{1}{6} \sum_{ijk \in F} f_i \cdot (f_j \times f_k)}$$

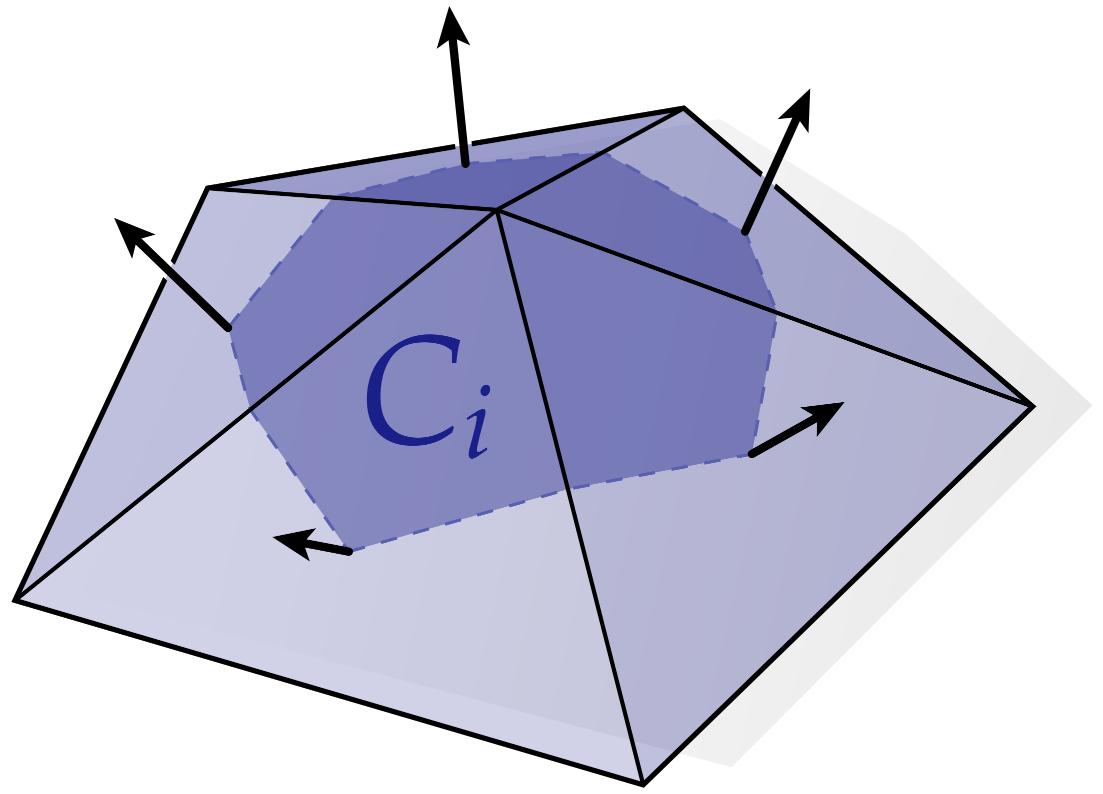


Discrete Volume Gradient

- Taking the gradient of enclosed volume with respect to the position f_i of some vertex i should now give us a notion of vertex normal:

$$\nabla_{f_i}\text{volume}(f) = \tfrac{1}{6}\nabla_{f_i} \sum_{ijk\in F} f_i \cdot (f_j \times f_k) = \frac{1}{6} \sum_{ijk\in F} f_j \times f_k = \int_{C_i} N\, dA$$

- But wait—this is the discrete vector area!



- **Key observation:** the gradient of discrete volume gives exactly the same thing as integrating the normal

- Captures the first expression in our sequence of variations:

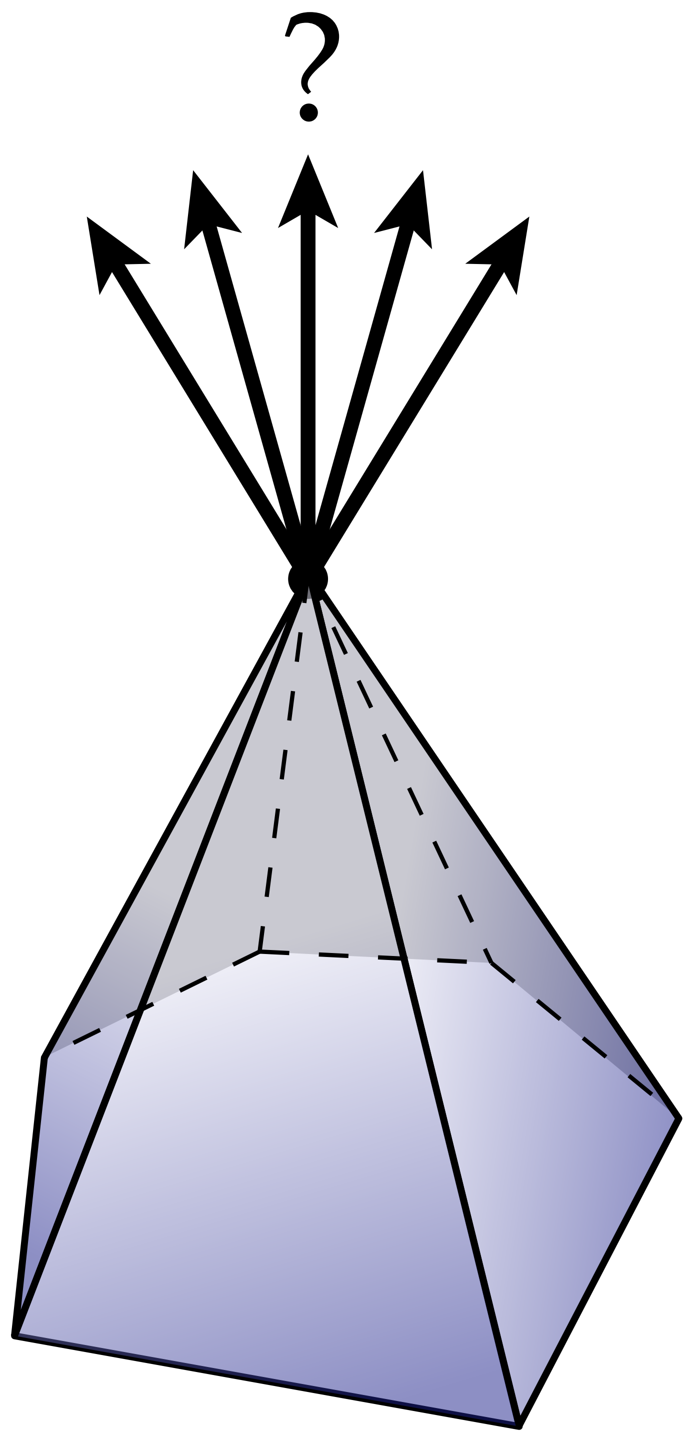
$$\delta\text{volume}(f) = N$$

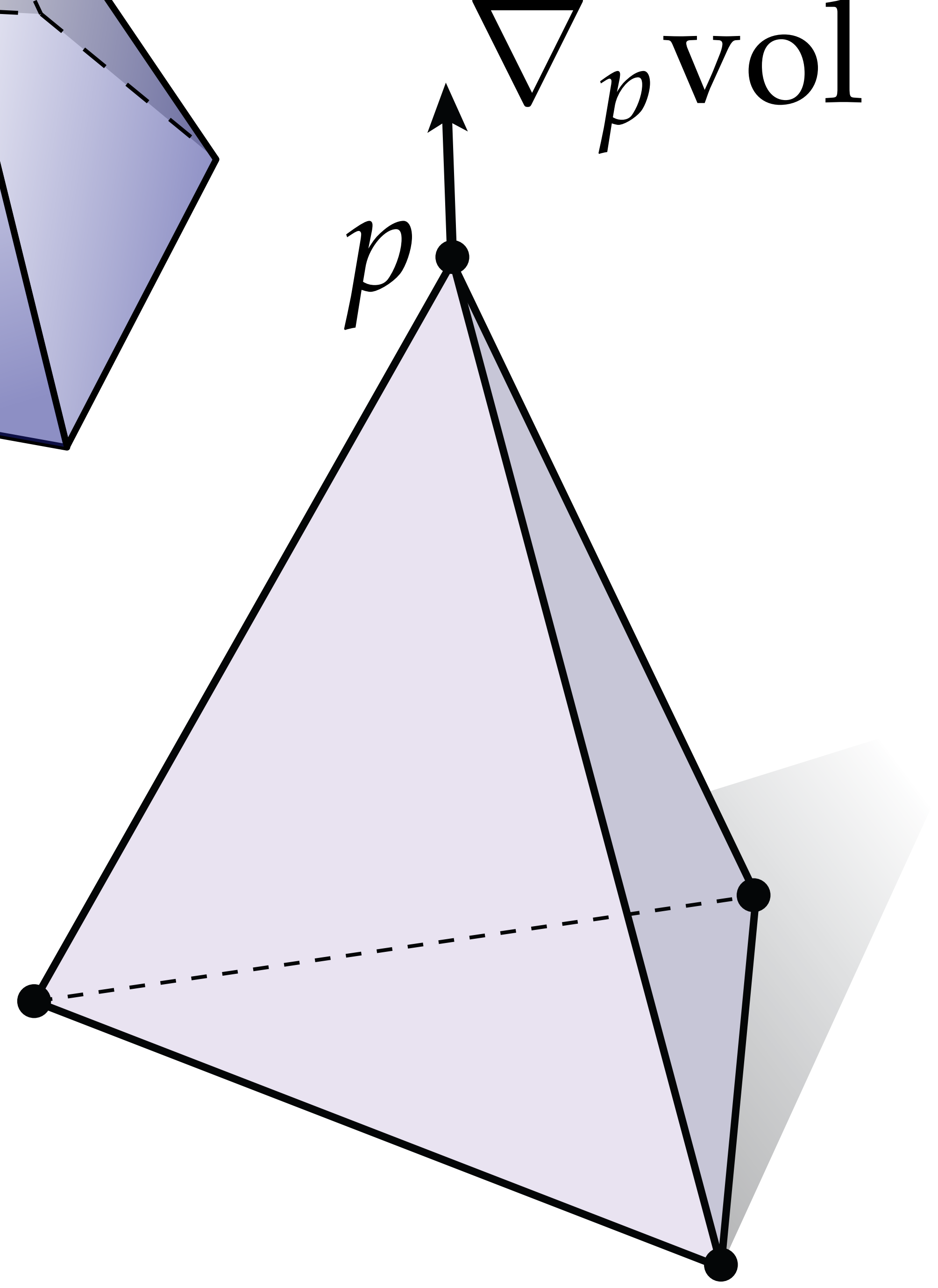
Vertex Normals via Volume Variation

- The relationship $\delta\text{volume} = N$ justifies our use of the area vector as (one possible) definition for vertex normals.

- Another way to derive this formula (exercise):
  - write down volume of discrete surface as sum of signed tetrahedron volumes
  - use geometric reasoning to derive an expression for tet volume gradient

- In this case, all paths lead to the *same* expression

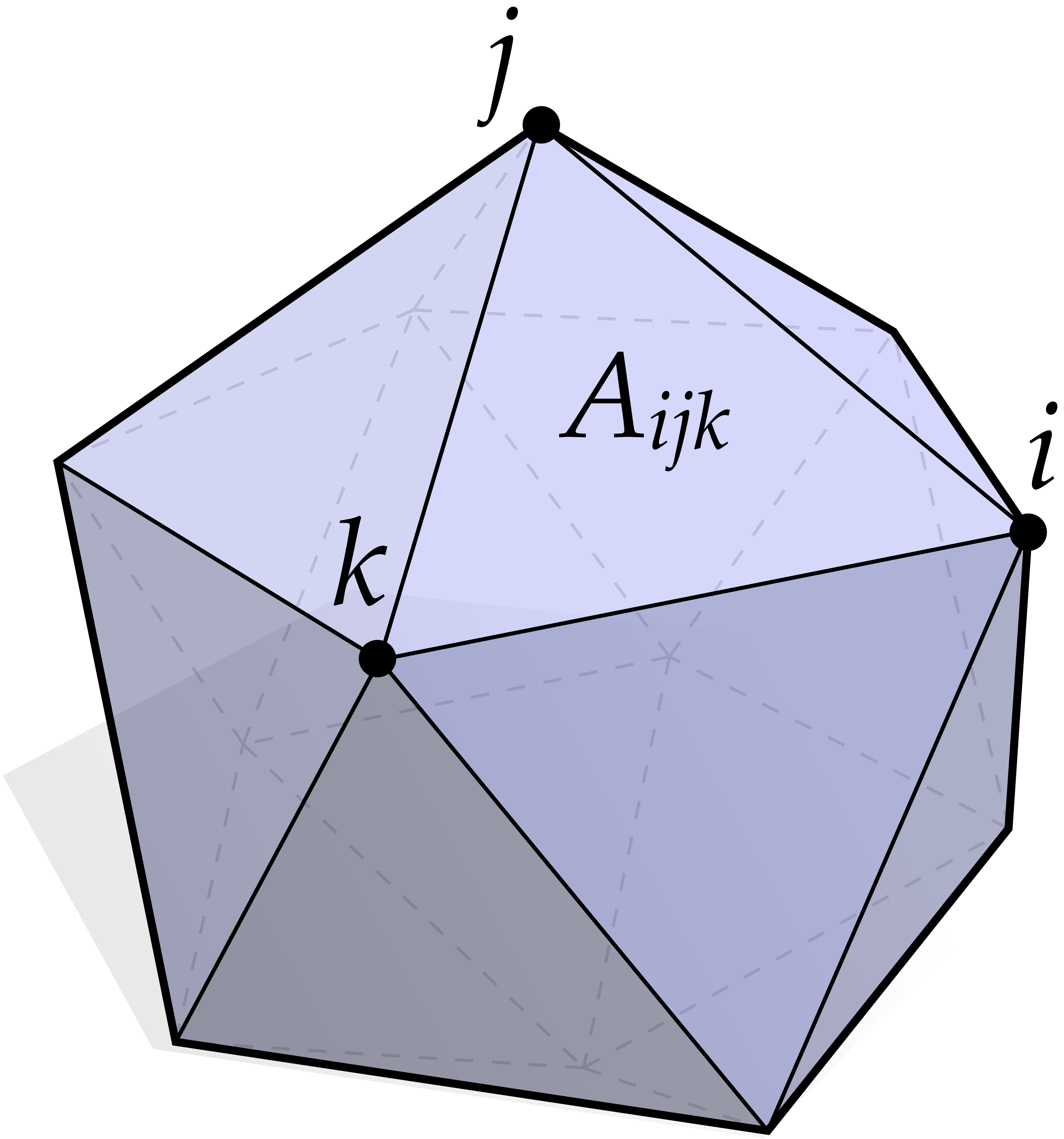




Total Area of a Discrete Surface

- Total area of a discrete surface is simply the sum of the triangle areas:

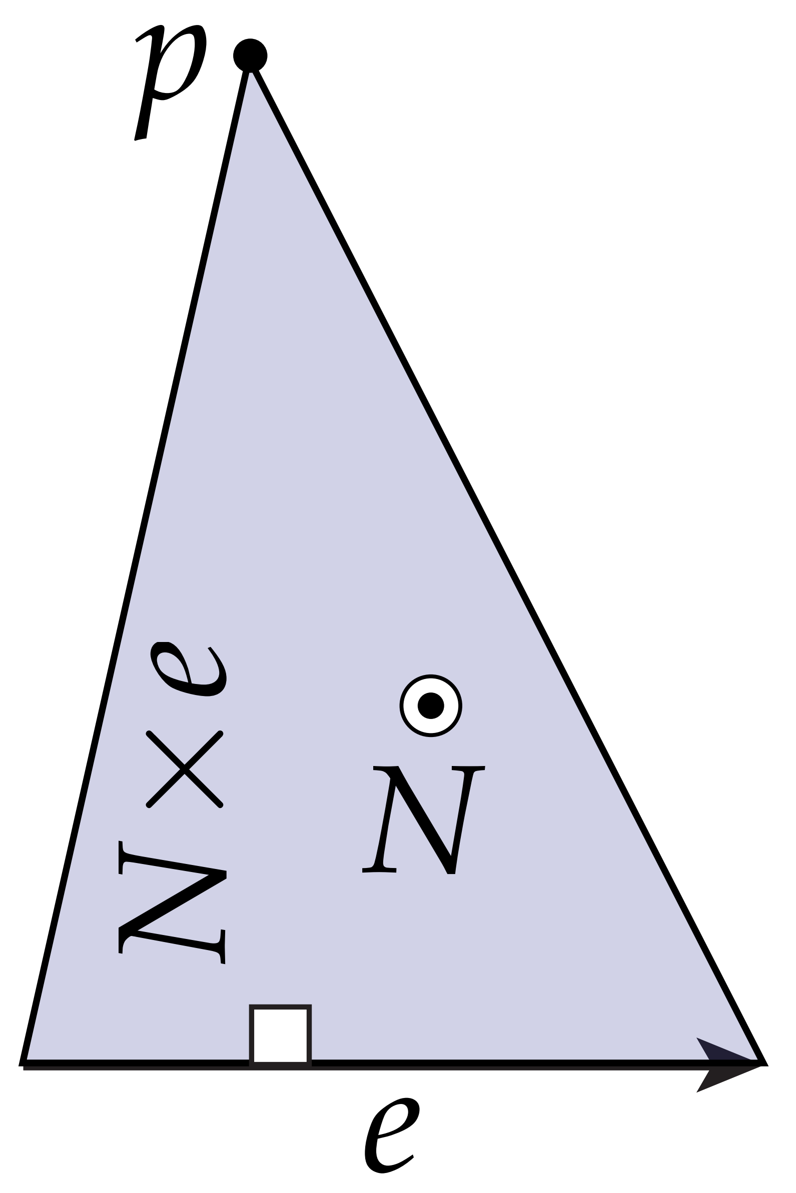
$$\boxed{\mathrm{area}(f) := \sum_{ijk \in F} A_{ijk}}$$



Discrete Area Gradient

- Recall that the gradient of triangle area with respect to position p of a vertex is just half the normal cross the opposite edge:

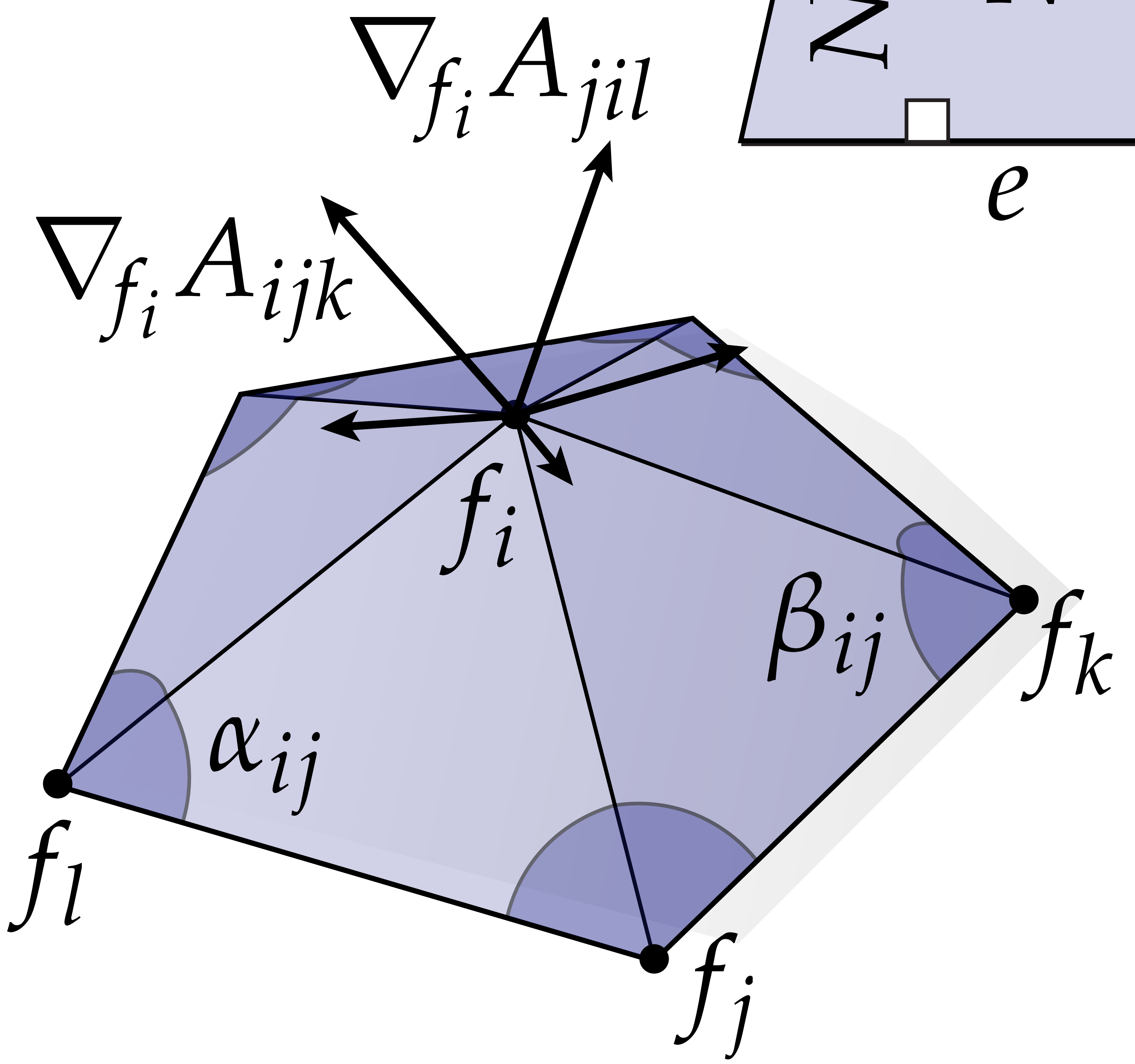
$$\nabla_p A = \tfrac{1}{2} N \times e$$



- Gradient of surface area with respect to position f_i of vertex i is sum of these per-triangle gradients

- Can write this sum via the *cotan formula*

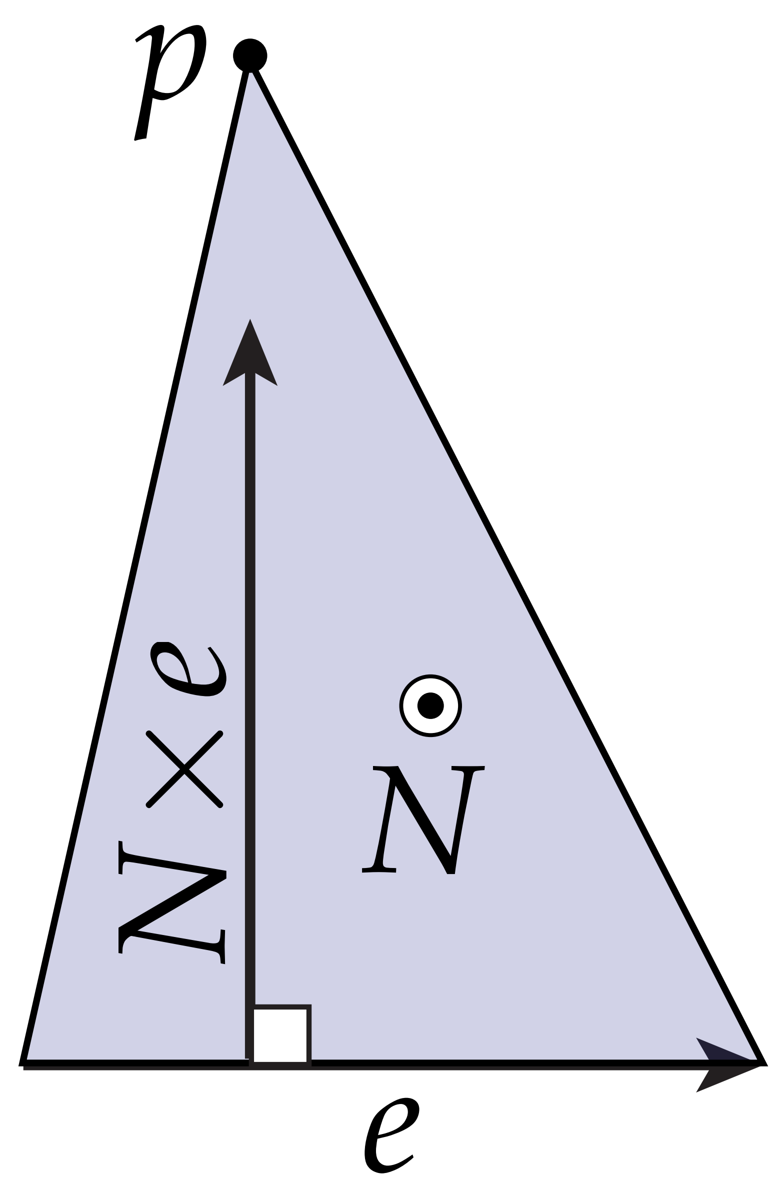
$$\nabla_{f_i} \text{area}(f) = \sum_{ij \in E} \tfrac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$



Discrete Area Gradient

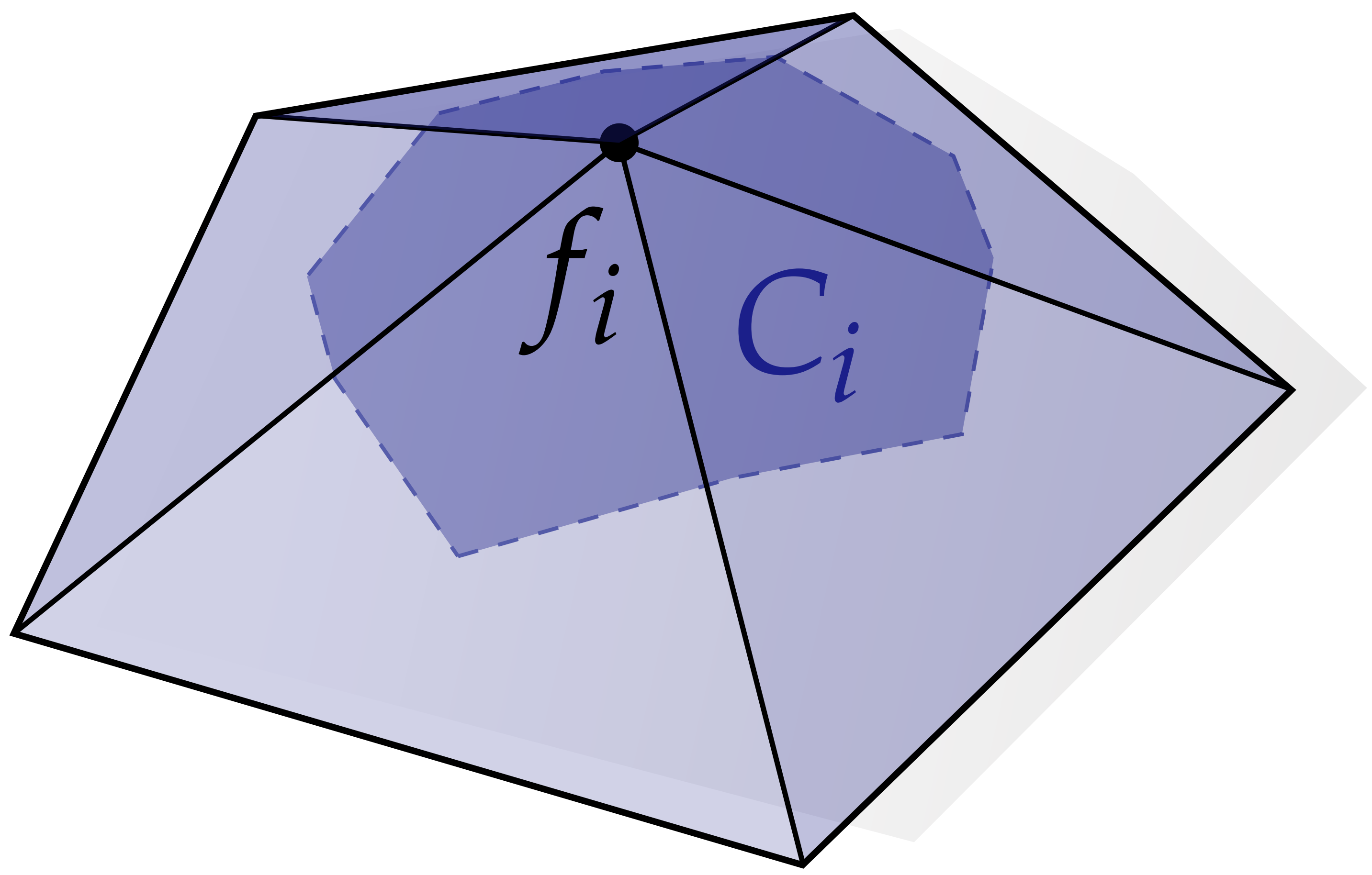
- Recall that the gradient of triangle area with respect to position p of a vertex is just half the normal cross the opposite edge:

$$\nabla_p A = \frac{1}{2} N \times e$$



- Gradient of surface area with respect to position f_i of vertex i is sum of these per-triangle gradients
- Can write this sum via the *cotan formula*

$$\nabla_{f_i} \text{area}(f) = \int_{C_i} HN \, dA$$



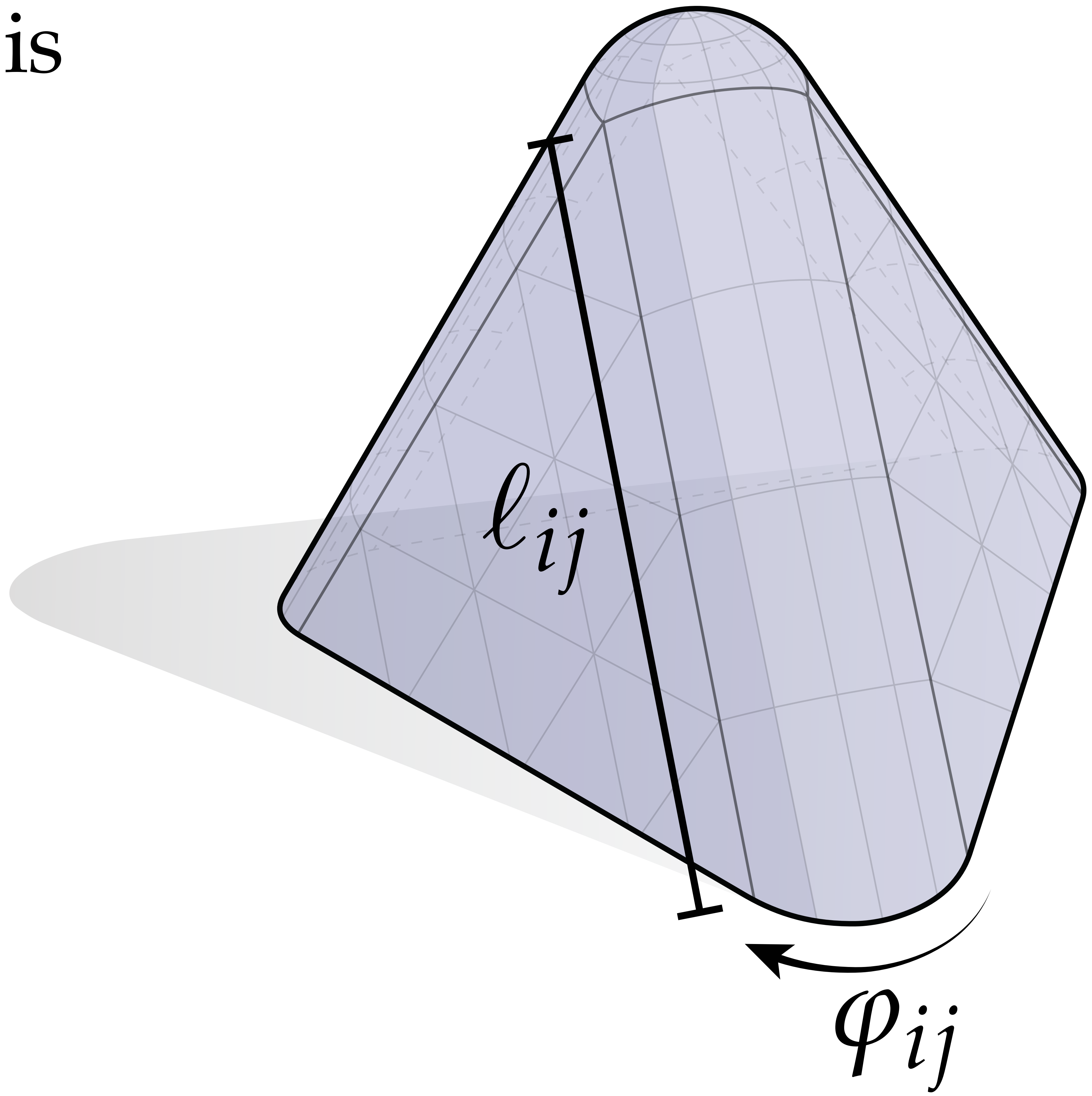
- Agrees with second expression in our sequence:

$$\delta \text{area}(f) = HN$$

Total Mean Curvature of a Discrete Surface

- According to our Steiner expansion, we know the total mean curvature of a discrete surface is

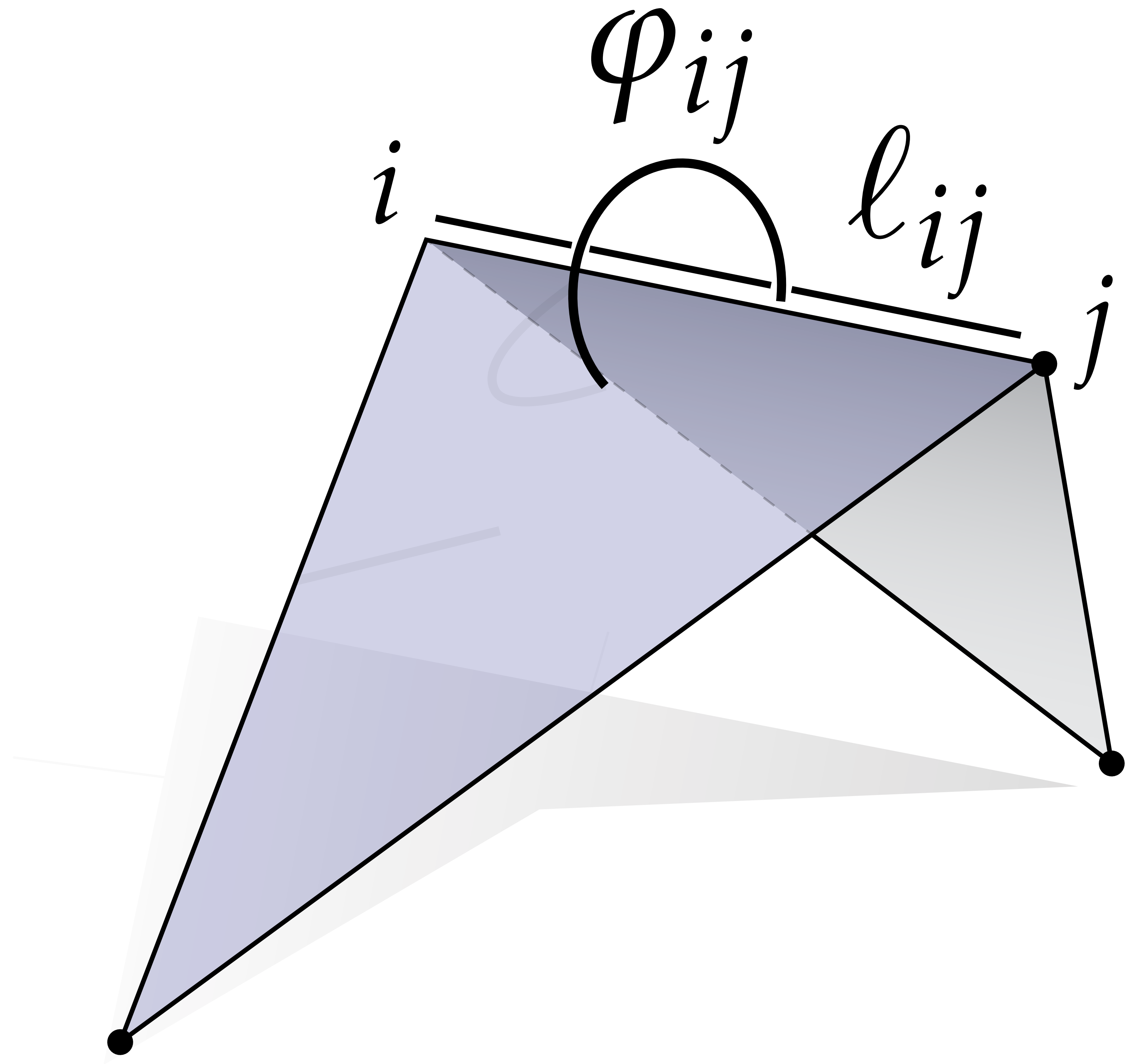
$$\boxed{\mathrm{mean}(f) = \frac{1}{2} \sum_{ij \in E} \ell_{ij} \varphi_{ij}}$$



Schläfli Formula

Theorem. Consider a closed polyhedron in R^3 with edge lengths l_{ij} and dihedral angles φ_{ij}. Then for *any* motion of the vertices,

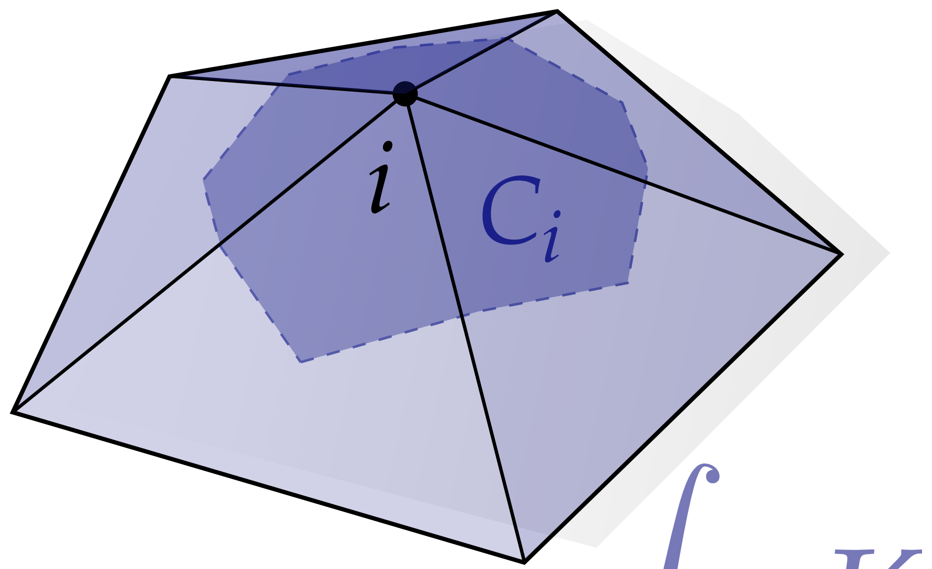
$$\boxed{\sum_{ij\in E} \ell_{ij} \tfrac{d}{dt} \varphi_{ij} = 0}$$



Discrete Mean Curvature Gradient

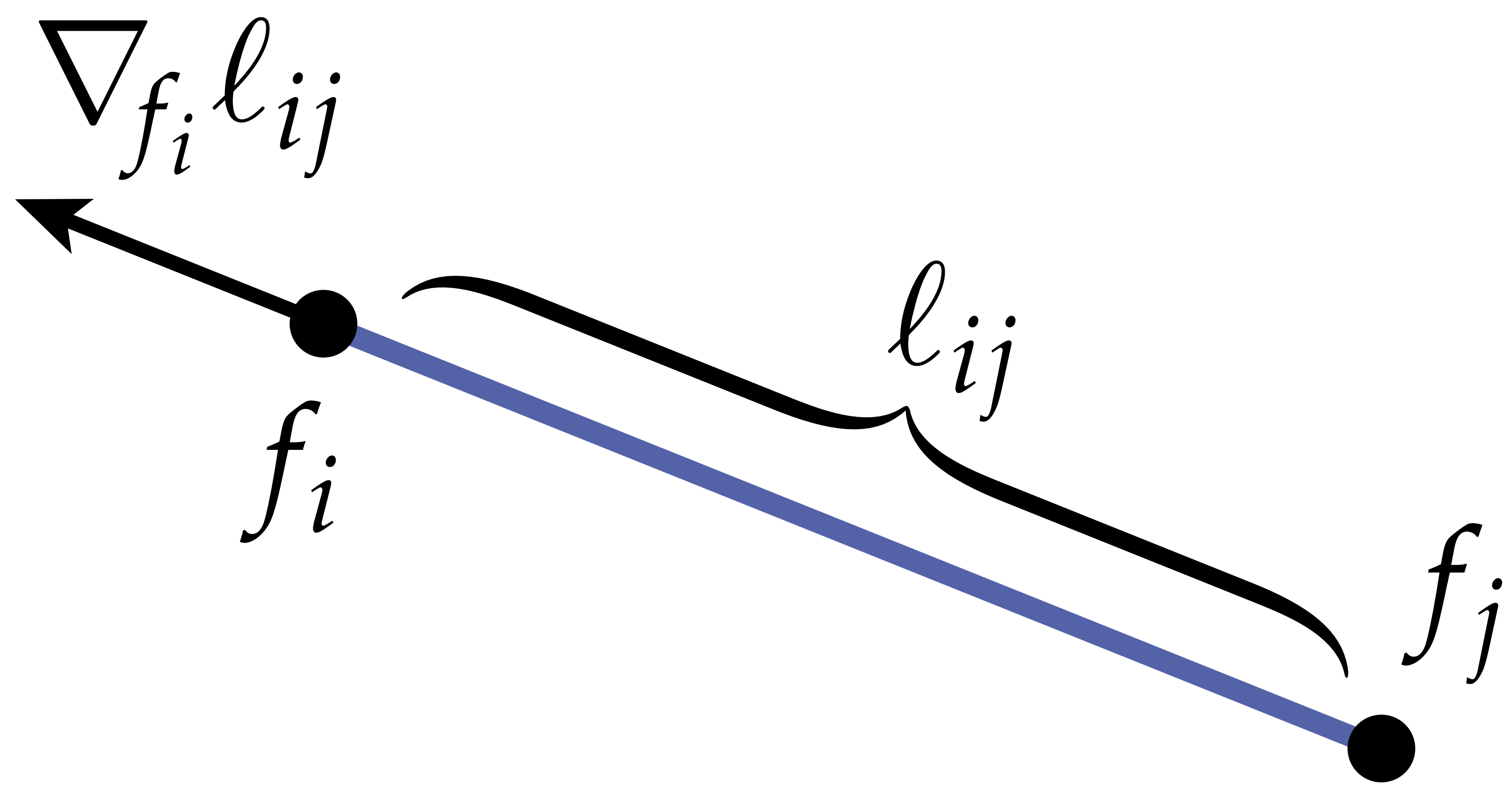
- What's the gradient of total mean curvature with respect to the location f_i of vertex i?

$$\nabla_{f_i}\mathrm{mean}(f) = \frac{1}{2}\sum_{ij\in E}\nabla_{f_i}(\ell_{ij}\varphi_{ij}) =$$



$$\frac{1}{2}\sum_{ij\in E}(\nabla_{f_i}\ell_{ij})\varphi_{ij} + \cancelto{0\ \text{(Schläfli)}}{\ell_{ij}(\nabla_{f_i}\varphi_{ij})} =$$

$$\int_{C_i} KN\,dA = \frac{1}{2}\sum_{ij\in E}\frac{\varphi_{ij}}{\ell_{ij}}(f_i - f_j)$$



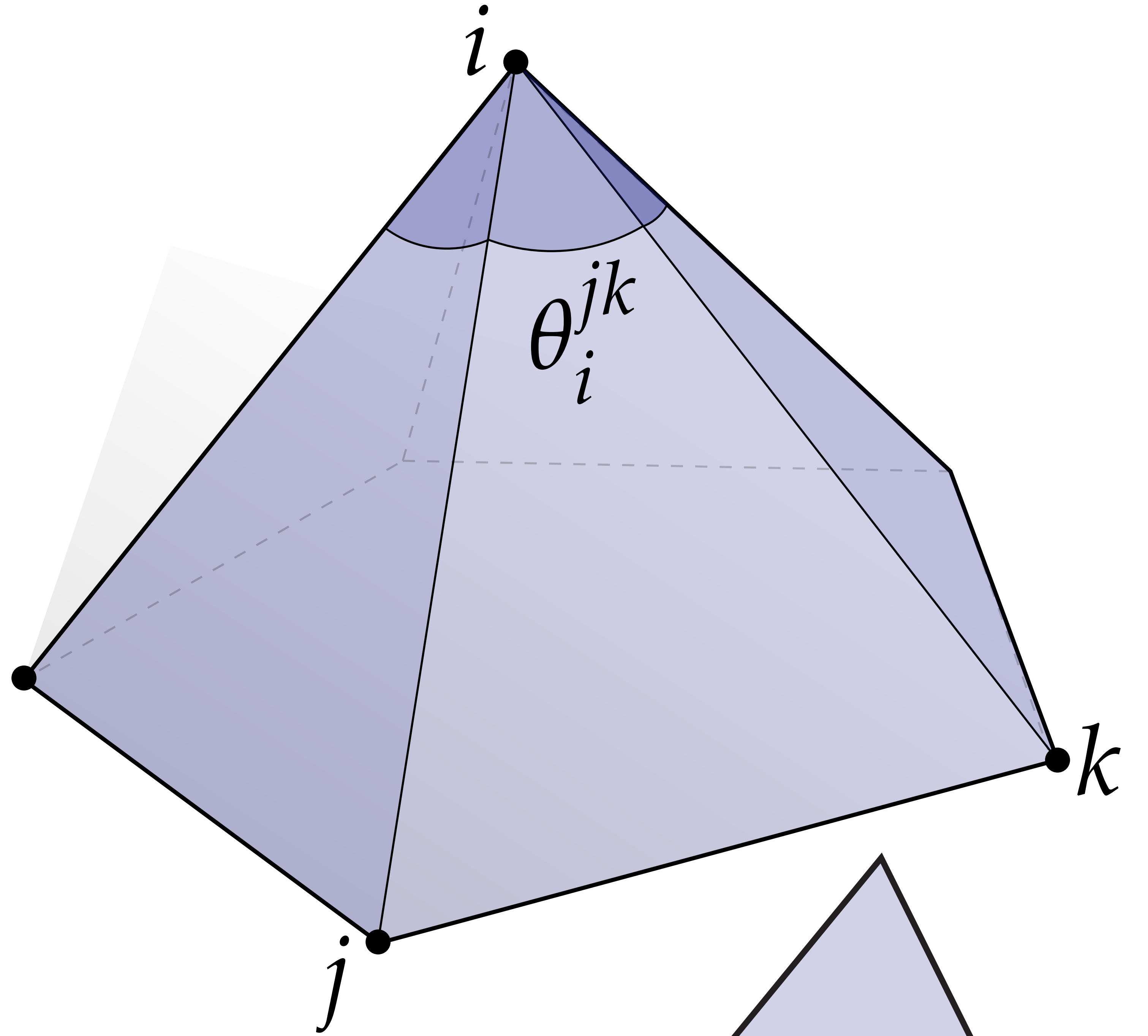
- Agrees with third expression in our sequence:

$$\delta\mathrm{mean}(f) = KN$$

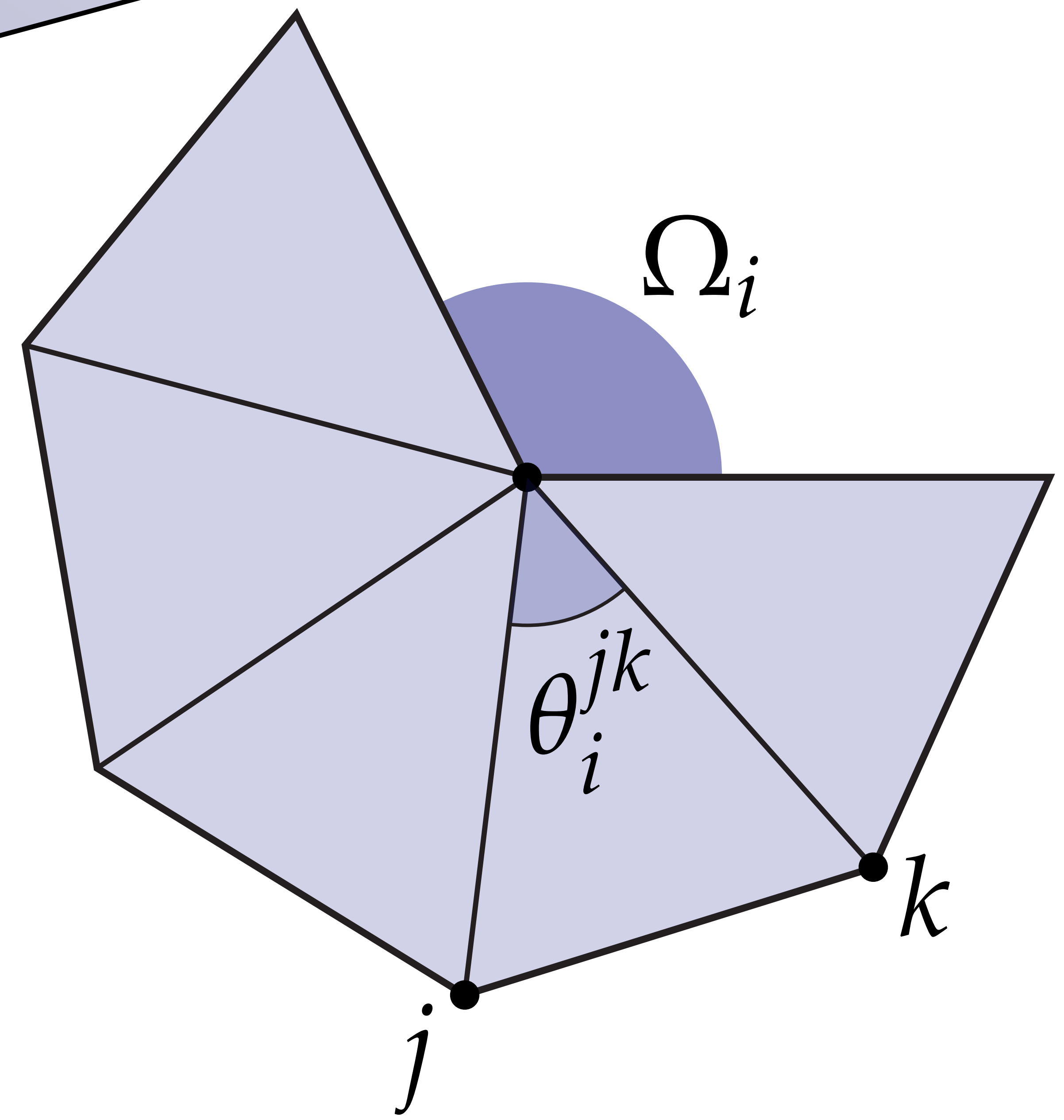
Total Gauss Curvature

- Total Gauss curvature of a discrete surface is the sum of angle defects

$$\boxed{\mathrm{Gauss}(f) = \sum_{i \in V} \left(2\pi - \sum_{ijk} \theta_i^{jk} \right)}$$



- From (discrete) Gauss-Bonnet theorem, we know this sum is always equal to just $2\pi\chi = 2\pi(V\text{-}E\text{+}F)$
- Gradient with respect to motion of any vertex is therefore *zero*—sequence ends here!



Summary

Summary—Scalar vs. Vector Curvature

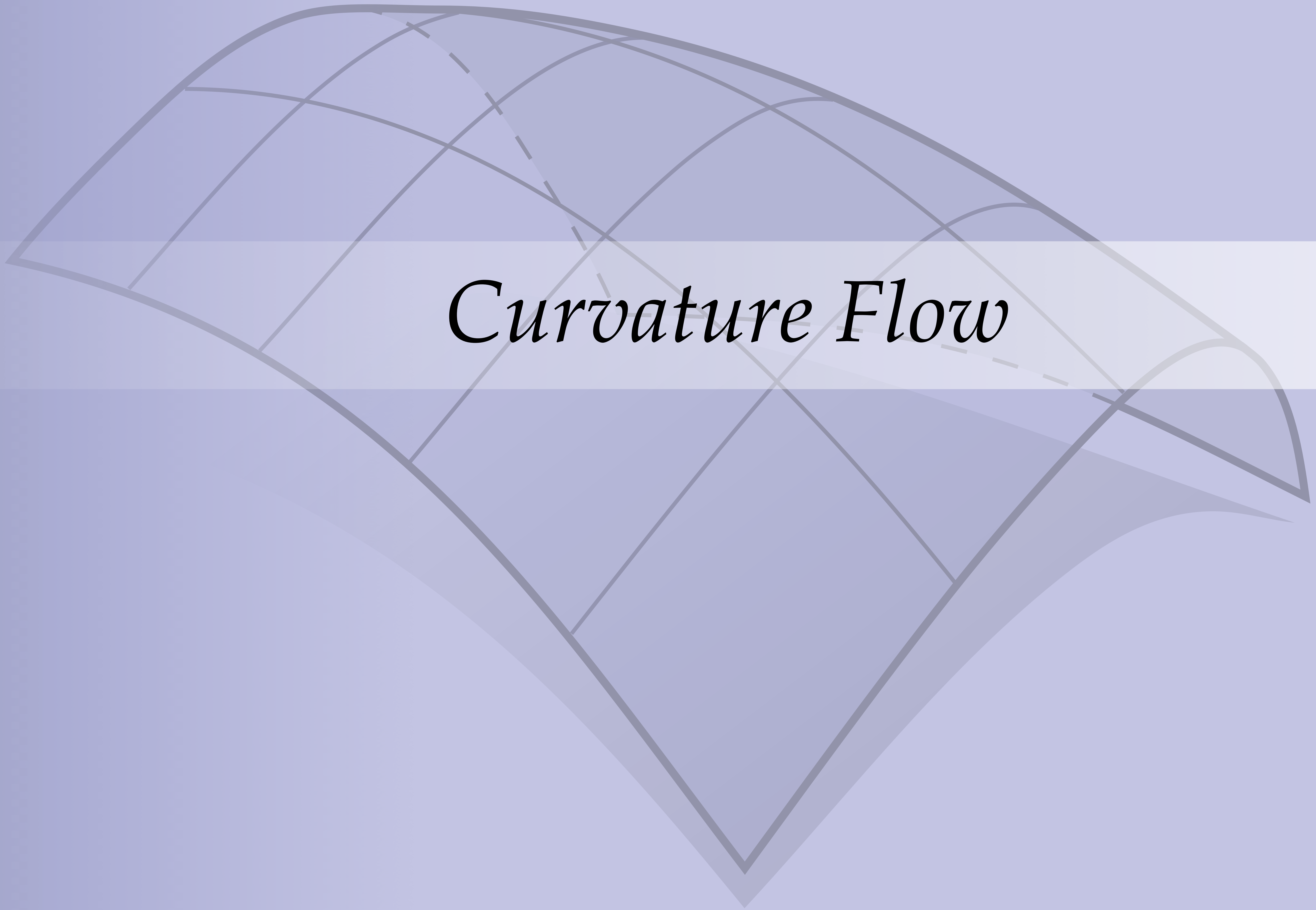
$$\text{volume} \xrightarrow{\delta f} \text{area} \xrightarrow{\delta f} \text{mean} \xrightarrow{\delta f} \text{Gauss} \xrightarrow{\delta f} 0$$

Discrete Curvature—Panoramic View

- In the end, all these pieces fit together nicely:
- **Scalar curvatures**
  - smooth out polyhedron and integrate (Steiner)
- **Curvature vectors**
  - integrate $df \wedge df$, $df \wedge dN$, $dN \wedge dN$ over dual cells
- Gradient of scalar curvatures also gives curvature vectors (making use of Gauss-Bonnet & Schläfli)
- Likewise, differentiating Steiner polynomial for volume gives scalar curvatures
- This “weaker” perspective generalizes to n-dimensions, piecewise-smooth surfaces, non-planar regions, … much further than “classic” definitions!
- *Easily implementable via simple formulas*

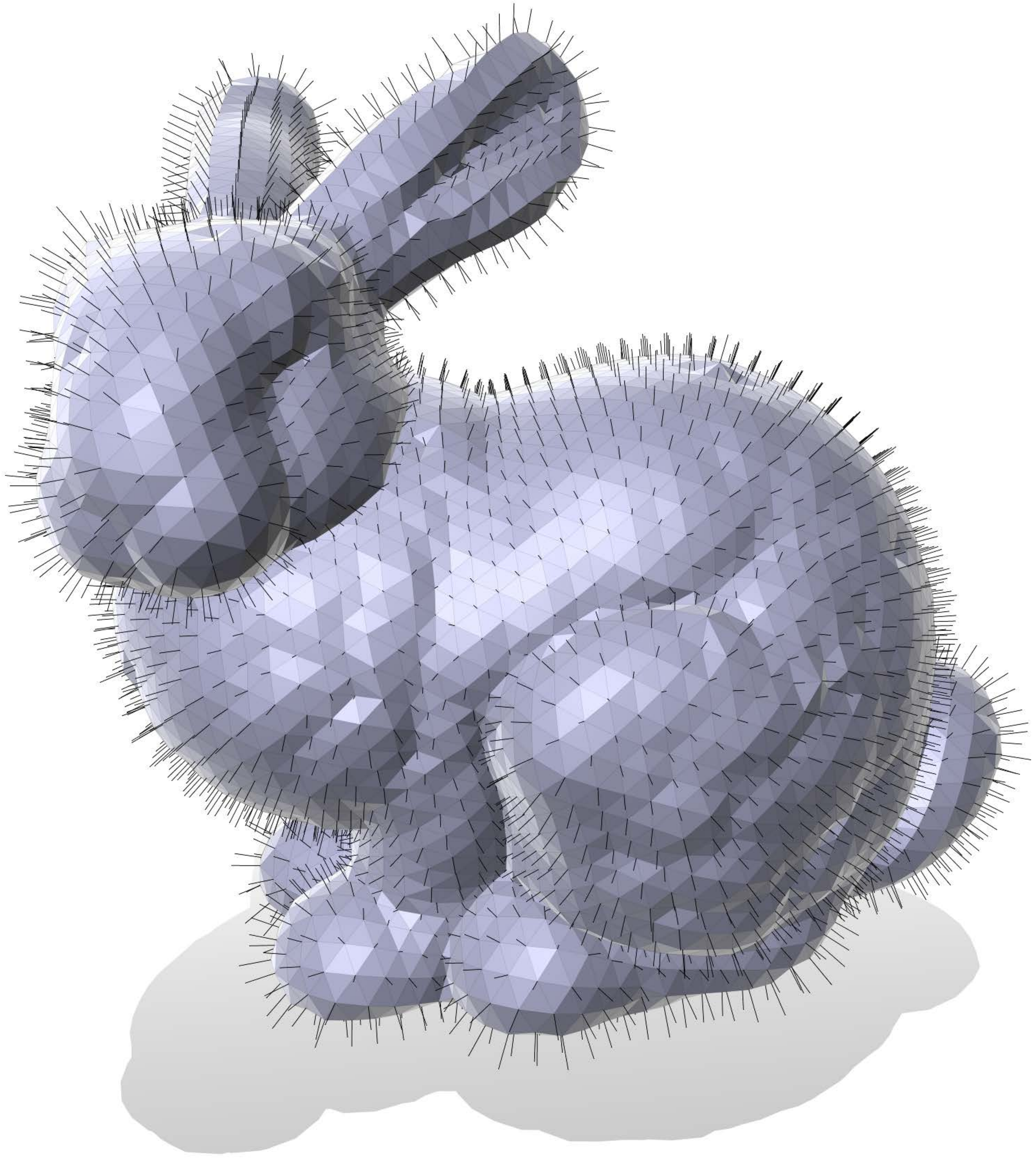
$$
\begin{tikzcd}
\underline{\text{Smooth}} & \underline{\text{Discrete}} & \underline{\text{Algebraic}} \\
\mathrm{vol}(M) \arrow[r] \arrow[d, "\delta f"'] & \sum_{ijk\in F} V_{0ijk} \arrow[d, "\nabla_f"'] & P(r) \arrow[l, "r\to 0"', "\text{(Steiner)}"] \arrow[dd, "\frac{d}{dr}"] \\
N\,dA \arrow[r, "\int_{V_i}"] \arrow[d, "\int_M"'] & \sum_{j\in\mathcal{N}(i)} A_j N_j & \\
\int_M dA \arrow[r] \arrow[d, "\delta f"'] & \sum_{ijk\in F} A_{ijk} \arrow[d, "\nabla_f"'] & P'(r) \arrow[l, "r\to 0"'] \arrow[dd, "\frac{d}{dr}"] \\
HN\,dA \arrow[r, "\int_{V_i}"] \arrow[d, "\int_M"'] & \frac{1}{2}\sum_{j\in\mathcal{N}(i)}(\cot\alpha_{ij}+\cot\beta_{ij})(f_j-f_i) & \\
\int_M H\,dA \arrow[r] \arrow[d, "\delta f"'] & \sum_{ij\in E} \theta_{ij}\ell_{ij} \arrow[d, "\nabla_f"'] & P''(r) \arrow[l, "r\to 0"'] \arrow[dd, "\frac{d}{dr}"] \\
KN\,dA \arrow[r, "\int_{V_i}", "\text{(Schläfli)}"'] \arrow[d, "\int_M"'] & \frac{1}{2}\sum_{j\in\mathcal{N}(i)} \frac{\theta_{ij}}{\ell_{ij}}(f_j-f_i) & \\
\int_M K\,dA \arrow[r] \arrow[dr, "\delta f"'] & \sum_{i\in V}\left(2\pi - \sum_{ijk\in F}\varphi_i^{jk}\right) \arrow[d, "\text{(Gauss-Bonnet)}\ \nabla_f"'] & P'''(r) \arrow[l, "r\to 0"'] \arrow[dl, "\frac{d}{dr}"] \\
 & 0 &
\end{tikzcd}
$$

Curvature Flow

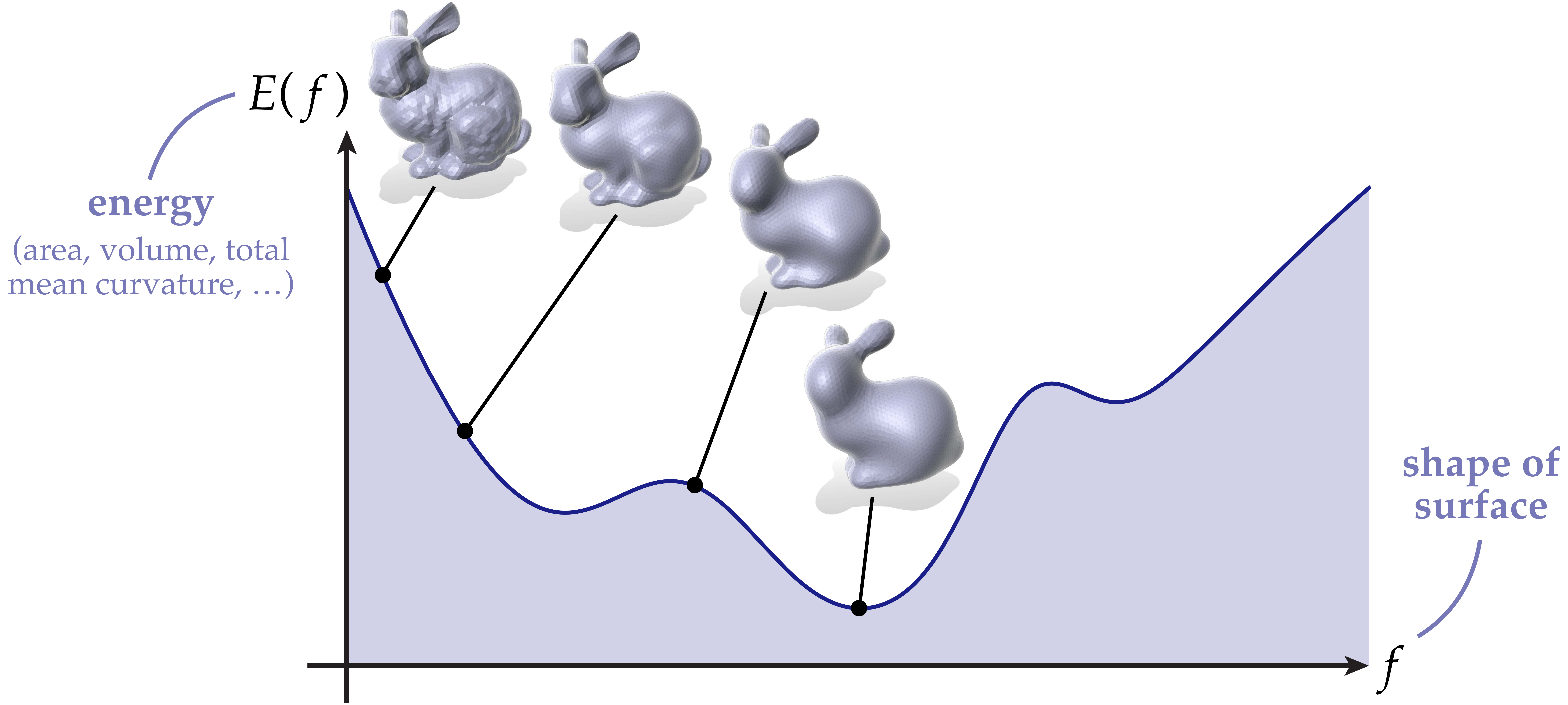


Curvature Flow

- Can use *curvature flow* to process surfaces
- Common task: smooth out surface / remove noise
- Basic strategy:
  - compute some function of curvature at each vertex
  - move in normal direction w/ speed proportional to curvature
  - *repeat*



Curvature Flow—Variational Perspective



Key idea: many curvature flows can be viewed as minimization of some energy

Curvature Flow—Numerical Integration

- Consider an energy E that assigns a "score" to any immersed surface f

$$\min_f E(f)$$

- Can reduce energy via gradient descent: "wiggle" surface in a way that decreases energy as quickly as possible

- **Smooth picture:** time derivative of the immersion f is equal to (minus) the first-order variation of energy with respect to f

$$\tfrac{d}{dt} f(t) = -\delta E(f(t))$$

- **Discrete picture:** replace time derivative with *difference* in time (time step τ)

$$\frac{f_i^{k+1} - f_i^k}{\tau} = -\nabla_{f_i} E(f)$$

  - evaluating energy gradient at current time step k gives "forward Euler" update

$$f_i^{k+1} = f_i^k - \tau \nabla_{f_i^k} E(f^k)$$

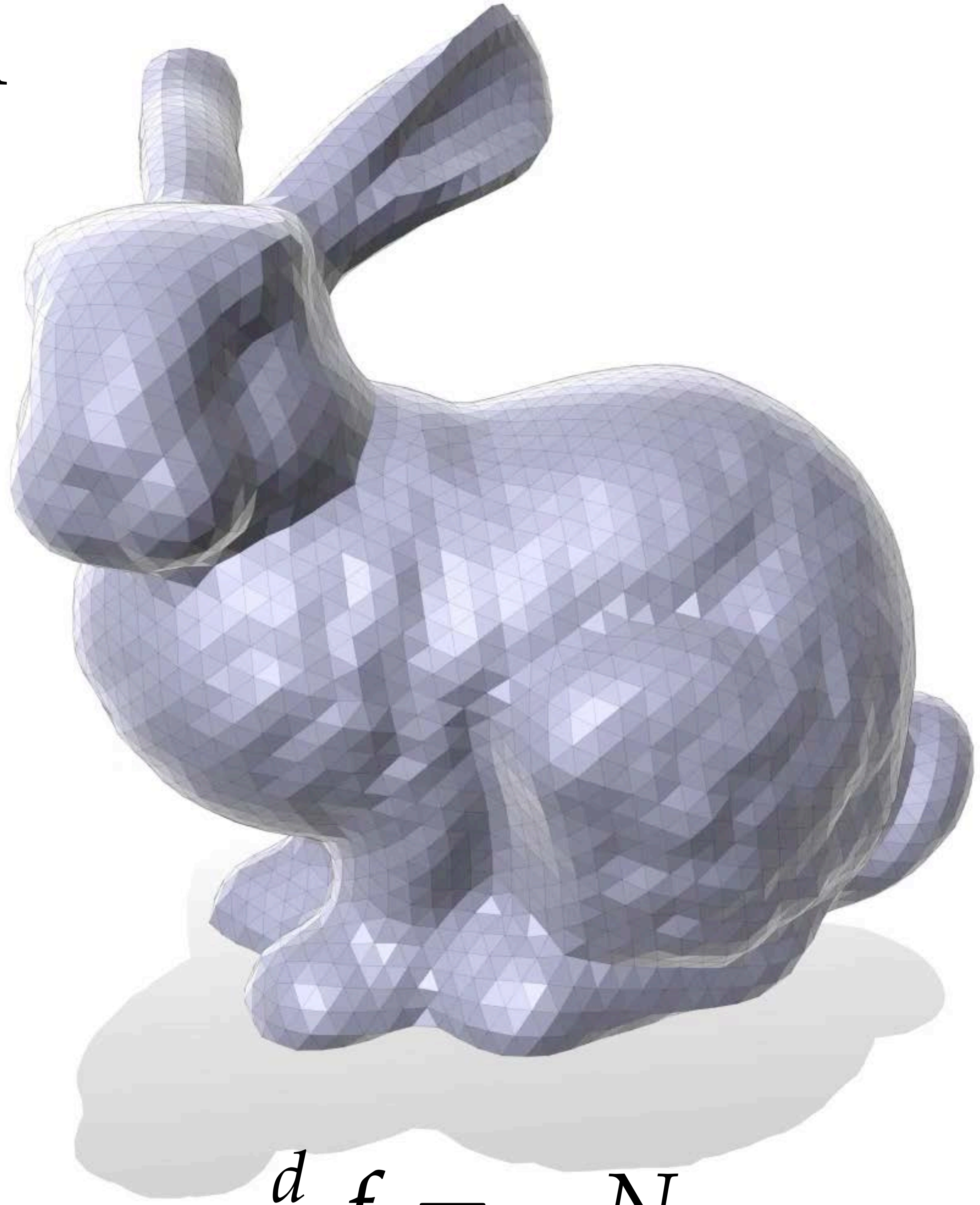
Normal Flow

$\boxed{\text{volume} \xrightarrow{\delta f} \text{area}} \xrightarrow{\delta f} \text{mean} \xrightarrow{\delta f} \text{Gauss} \xrightarrow{\delta f} 0$

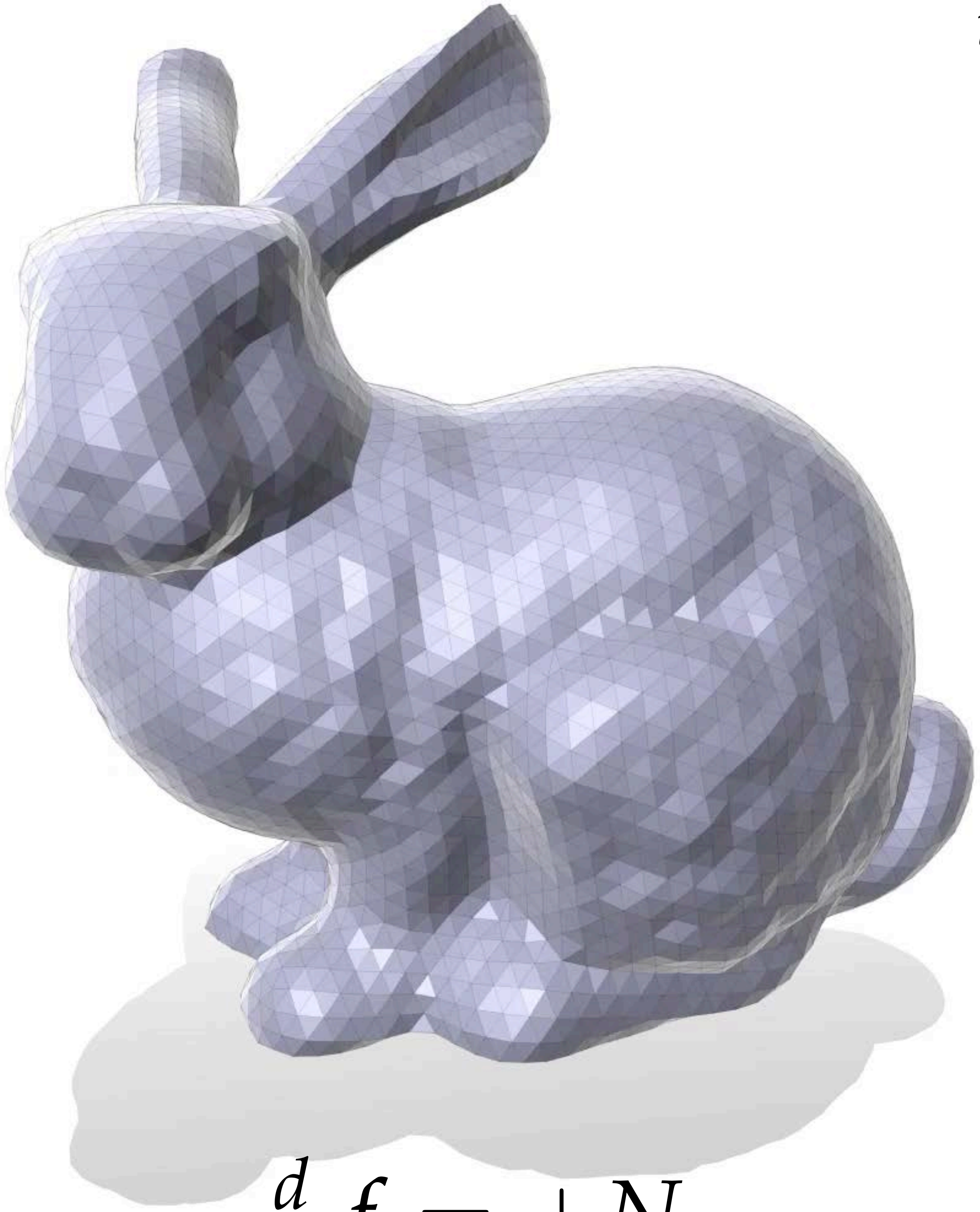
$E(f) = \text{volume}(f)$

$\delta E = N dA$

$$f_i^{k+1} = f_i^k - \frac{\tau}{6} \sum_{ijk \in \text{St}(i)} f_j \times f_k$$



$\frac{d}{dt} f = -N$



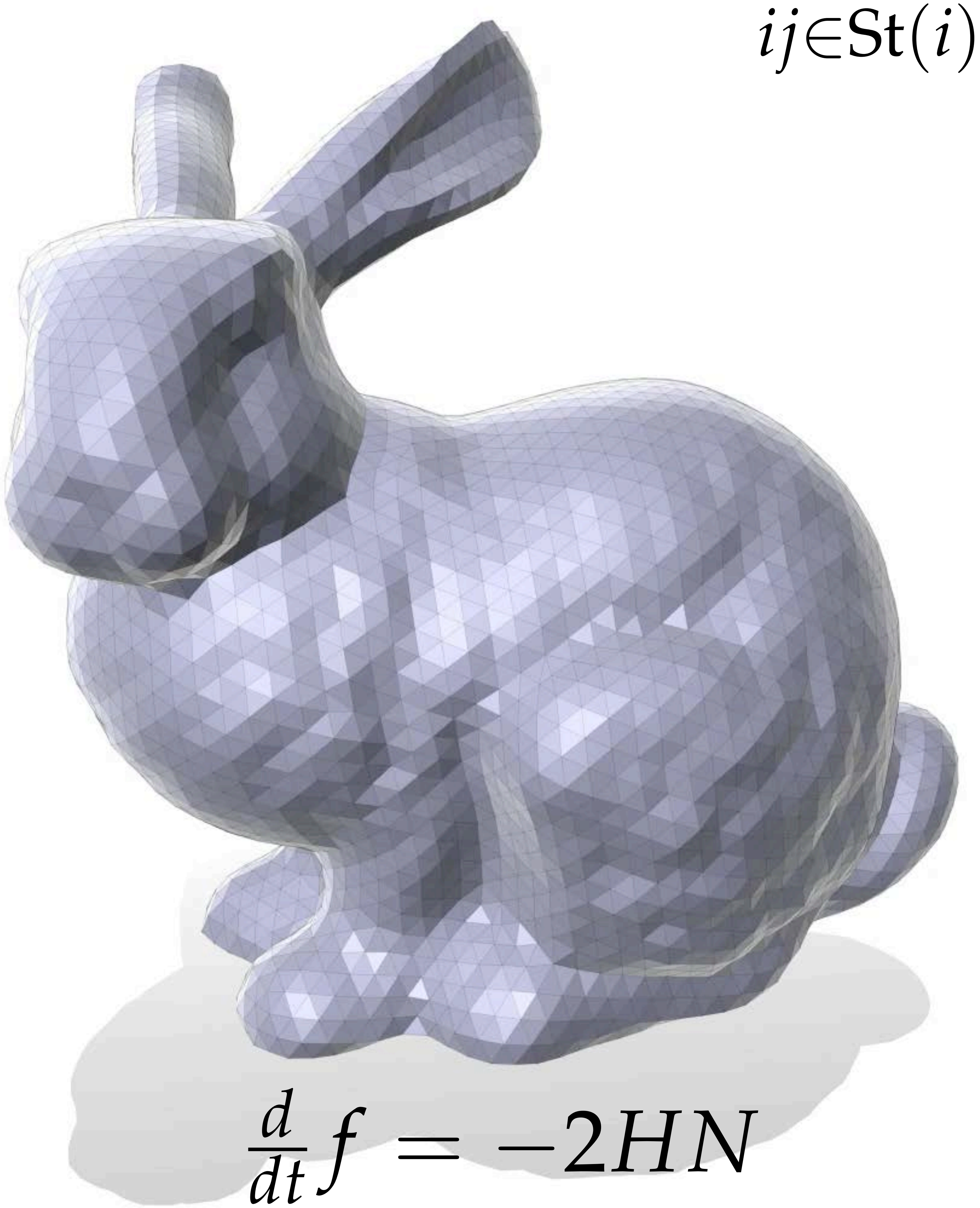
$\frac{d}{dt} f = +N$

Mean Curvature Flow volume $\xrightarrow{\delta f}$ $\boxed{\text{area} \xrightarrow{\delta f} \text{mean}}$ $\xrightarrow{\delta f}$ Gauss $\xrightarrow{\delta f}$ 0

$$E(f) = \int_M dA$$

$$\delta E = 2HNdA$$

$$f_i^{k+1} = f_i^k - \frac{\tau}{2} \sum_{ij \in \mathrm{St}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j^k - f_i^k)$$

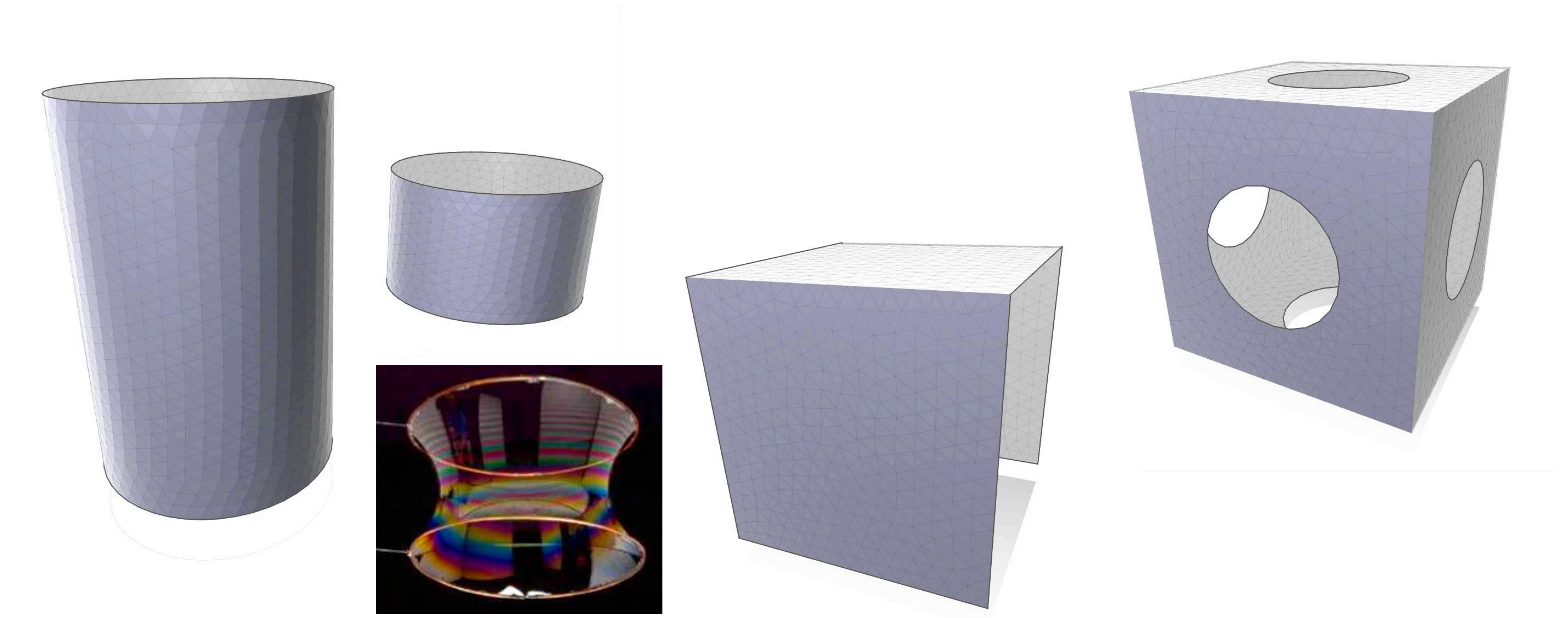


$$\frac{d}{dt} f = -2HN$$

Mean Curvature Flow volume $\xrightarrow{\delta f}$ $\boxed{\text{area} \xrightarrow{\delta f} \text{mean}}$ $\xrightarrow{\delta f}$ Gauss $\xrightarrow{\delta f}$ 0

$E(f) = \int dA$

$f^{k+1} = f^k - \tau \sum (\cot \alpha_{ij} + \cot \beta_{ij})(f^k_j - f^k_i)$



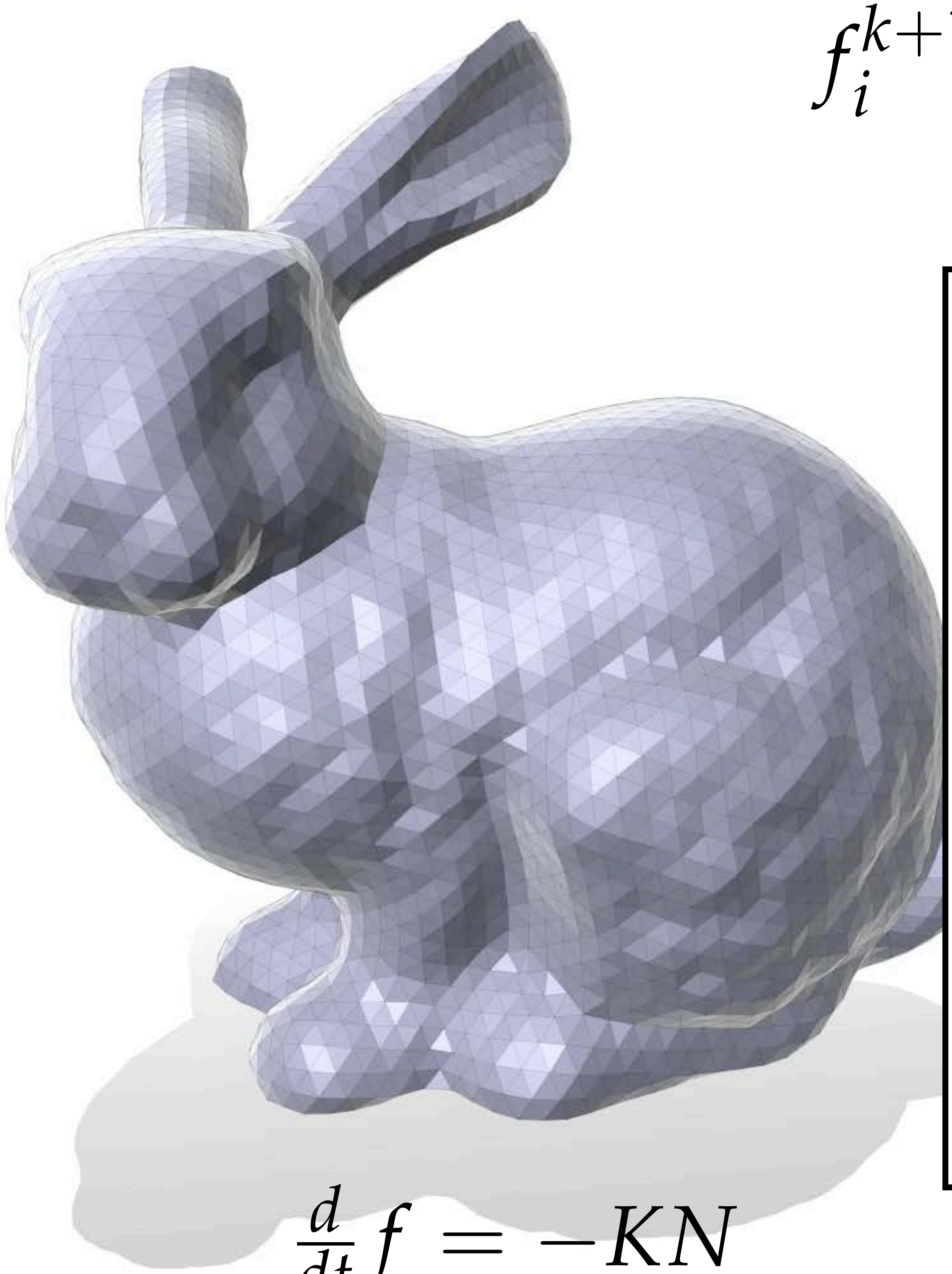
Plateau problem: find surface of smallest area with given boundary ("minimal surface")

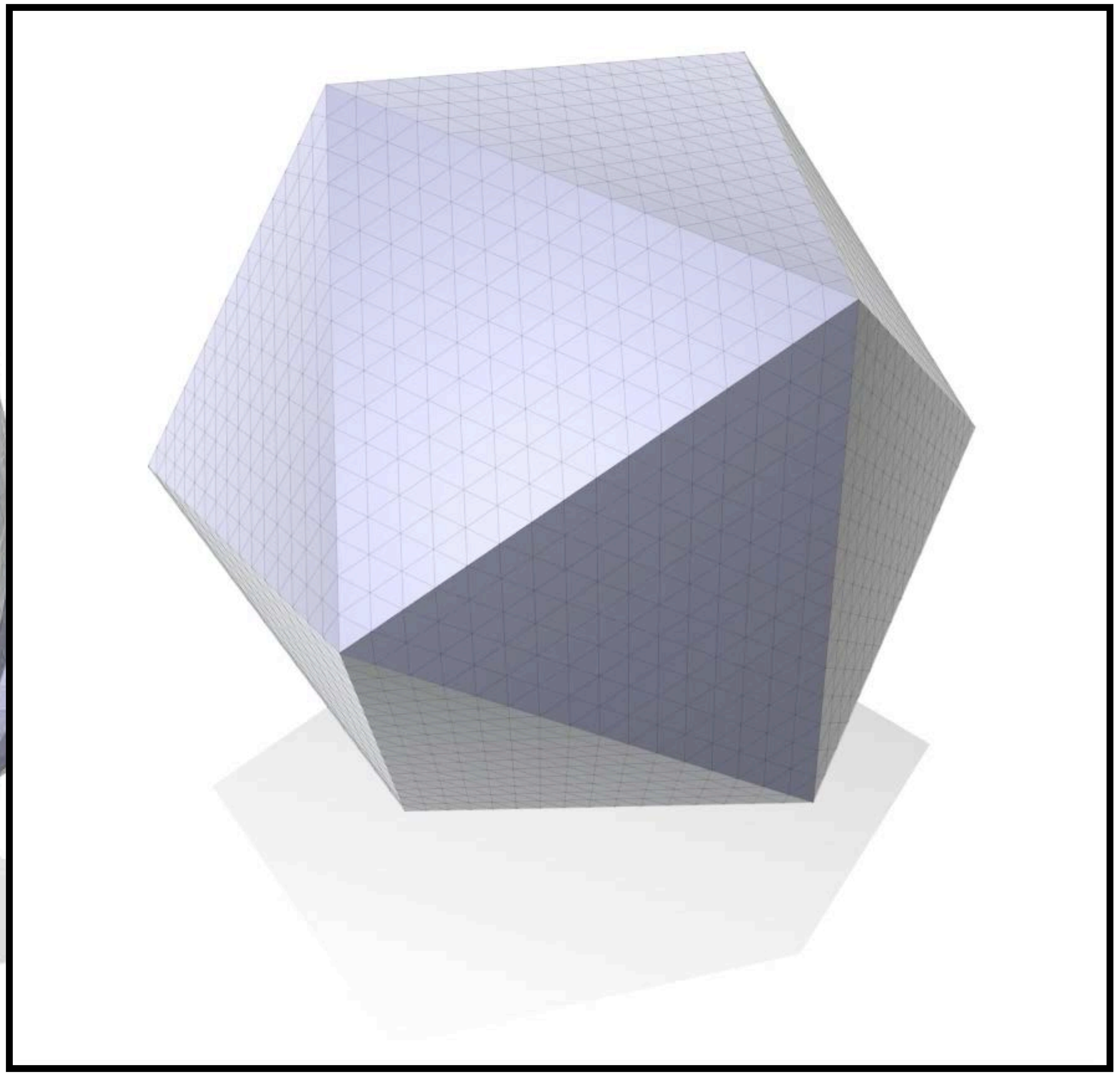
Gauss Curvature Flow volume $\xrightarrow{\delta f}$ area $\xrightarrow{\delta f}$ $\boxed{\text{mean} \xrightarrow{\delta f} \text{Gauss}}$ $\xrightarrow{\delta f}$ 0

$$E(f) = \int_M H\, dA$$

$$\delta E = KN\, dA$$

$$f_i^{k+1} = f_i^k - \frac{\tau}{2} \sum_{ij \in \mathrm{St}(i)} \frac{\varphi_{ij}}{\ell_{ij}} (f_j^k - f_i^k)$$





$$\frac{d}{dt} f = -KN$$

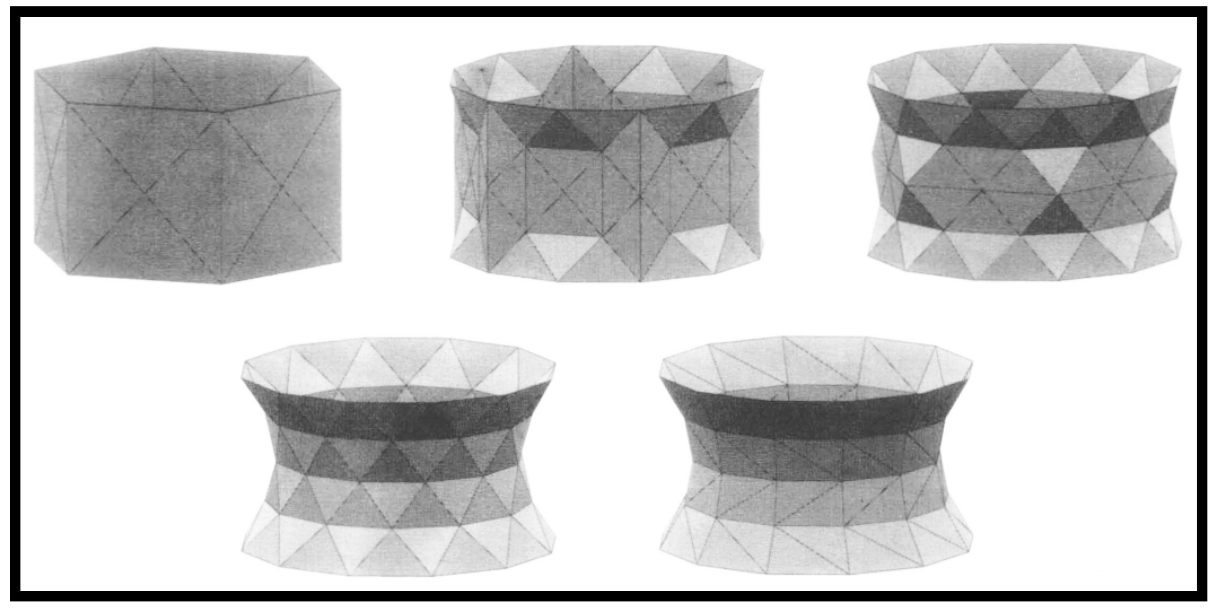
Willmore Flow

$$E = \int_M H^2 \, dA$$

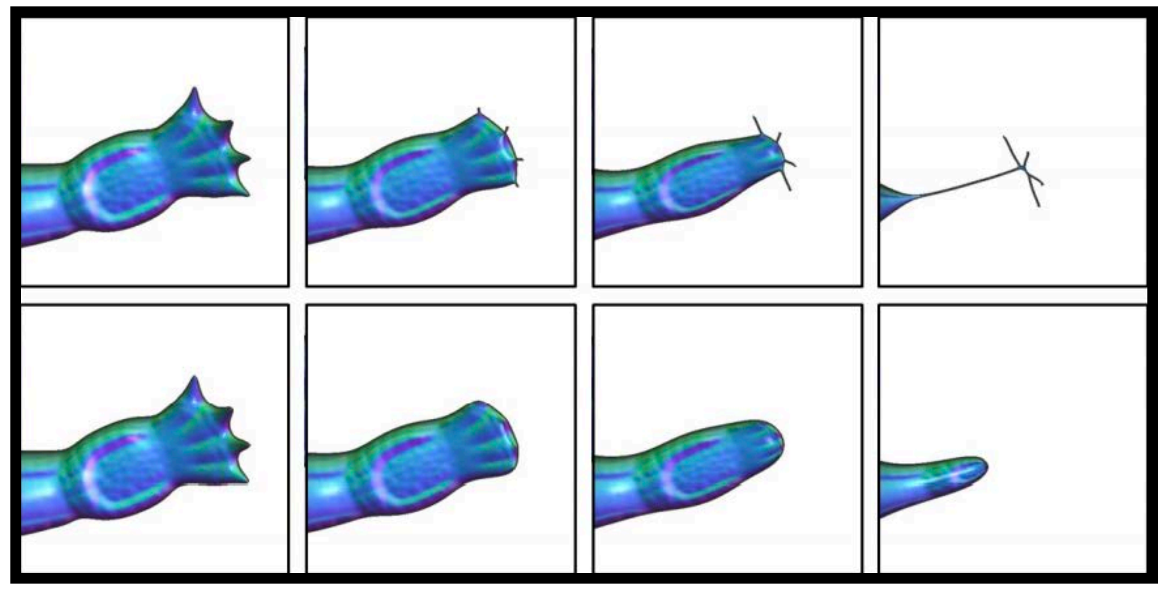
$$E_{\text{discrete}} = \sum_{i \in V} (HN)_i^2 / A_i$$

$$\frac{d}{dt} = -\nabla_f E$$

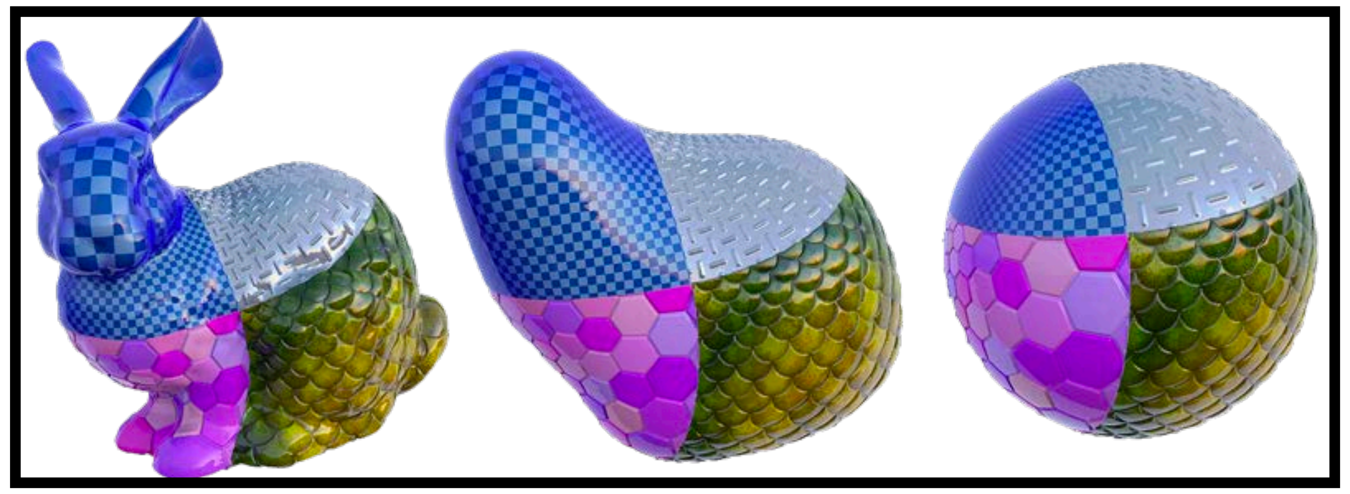
Curvature Flow Algorithms—Further Reading



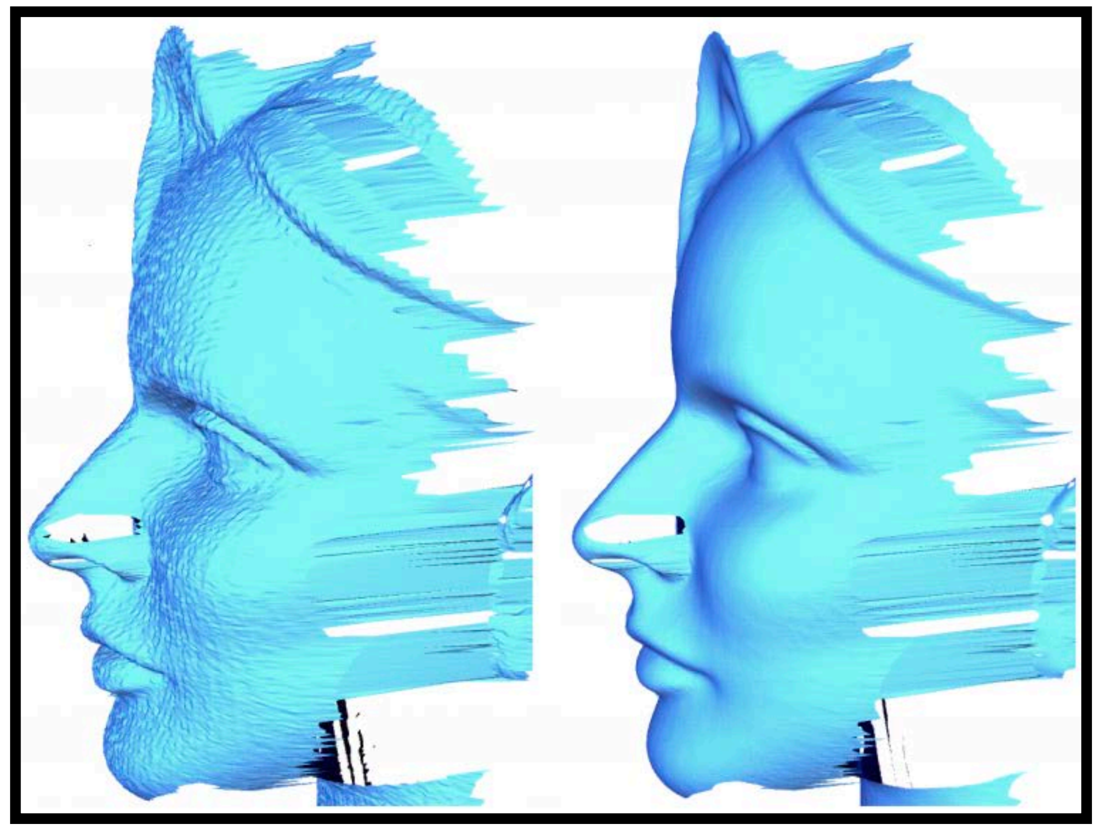
Brakke, *"The Surface Evolver"* (1992)



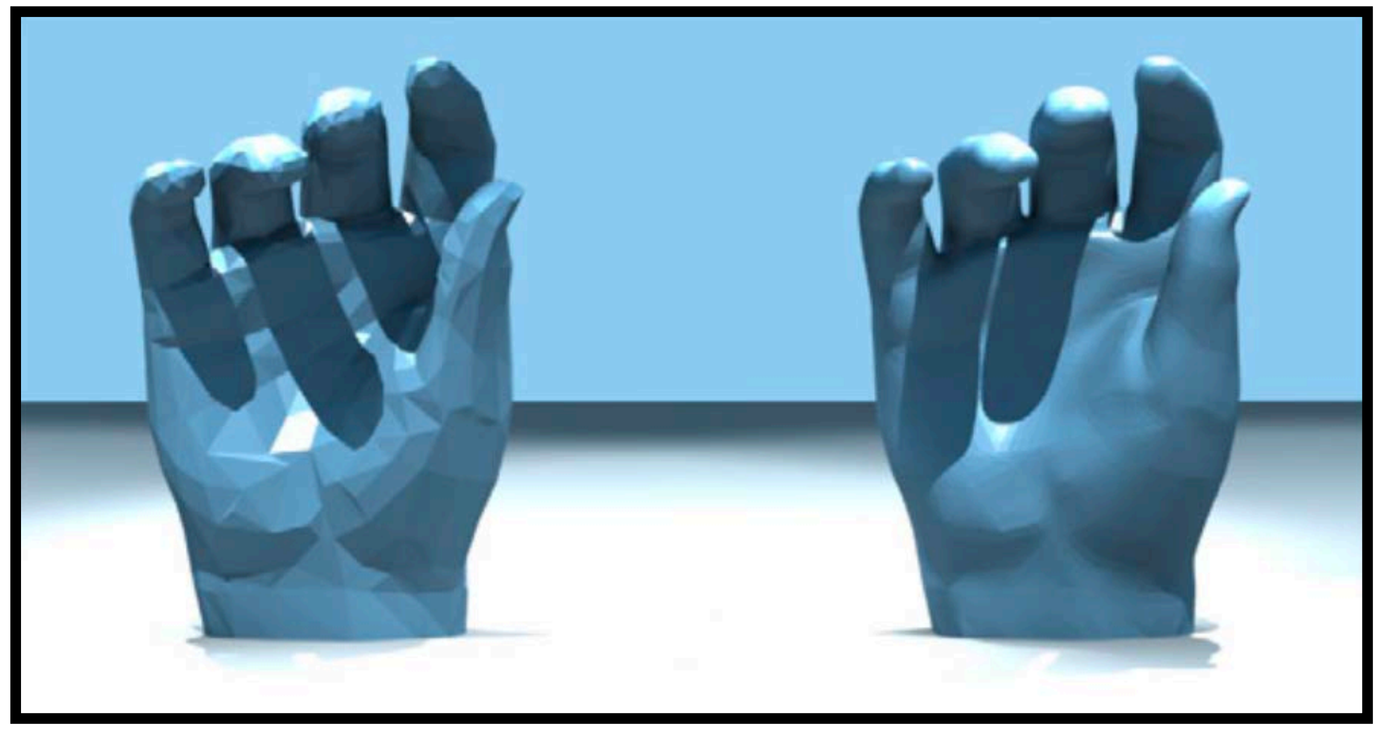
Kazhdan et al, *"Can Mean-Curvature Flow be Modified to be Non-singular?"* (2012)



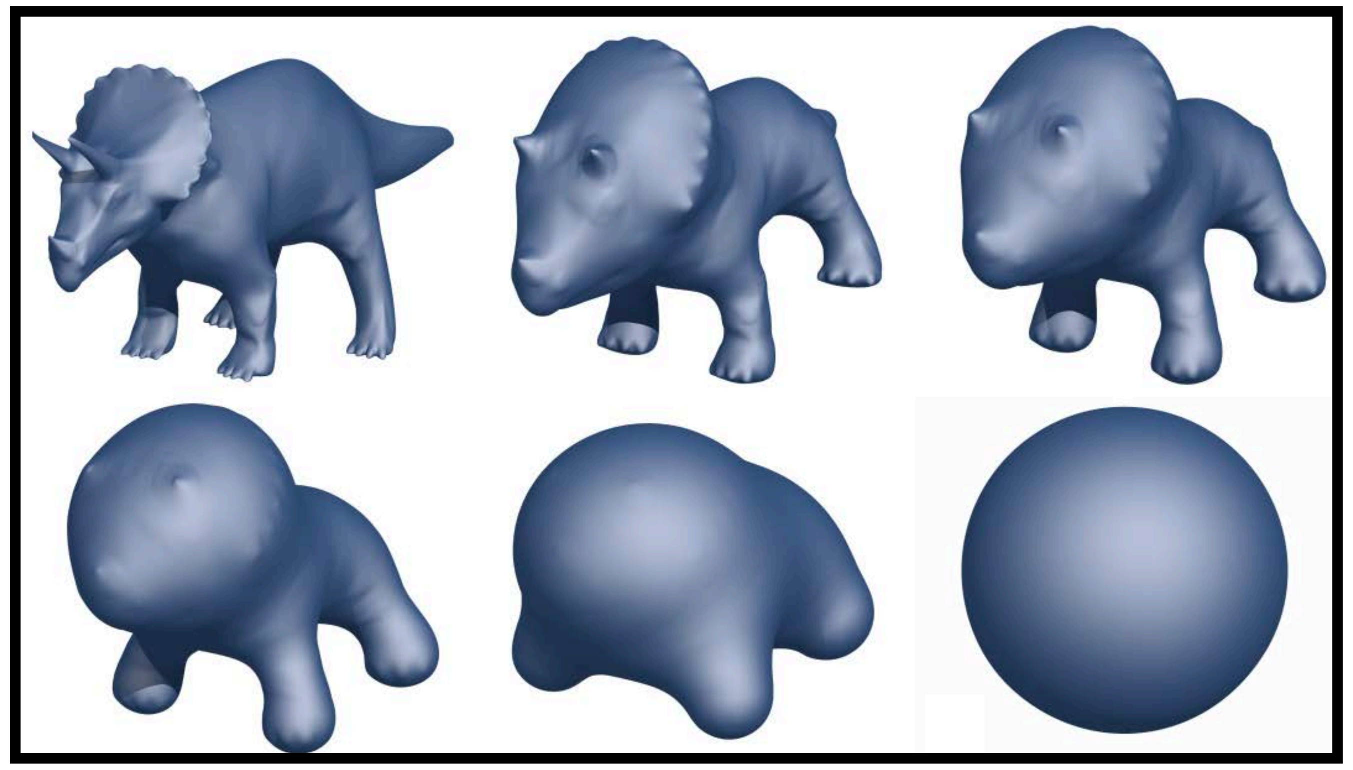
Crane et al, *"Robust Fairing via Conformal Curvature Flow"* (2013)



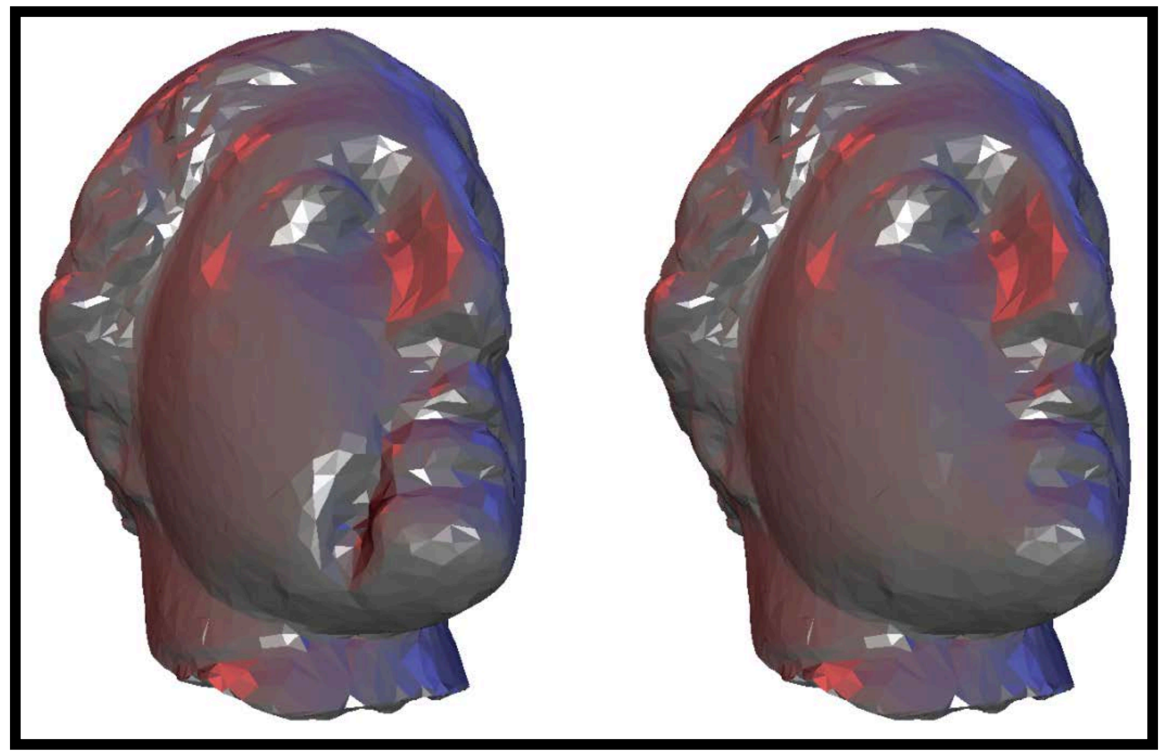
Desbrun et al, *"Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow"* (1999)



Wardetzky et al, *"Discrete Quadratic Curvature Energies"* (2007)

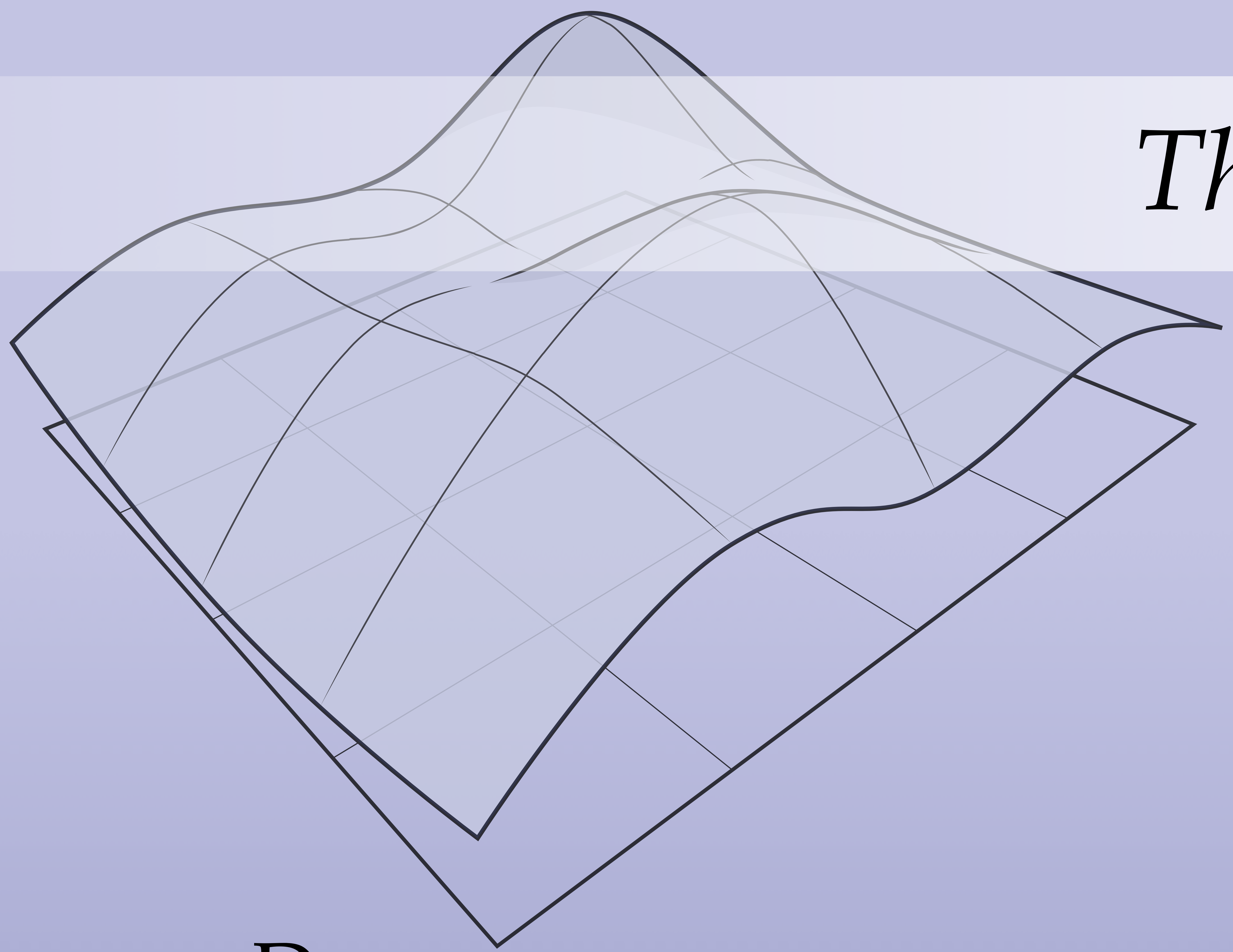


Schumacher, *"On H2 Gradient Flows for the Willmore Energy"* (2017)



Bobenko & Schröder, *"Discrete Willmore Flow"* (2005)

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858