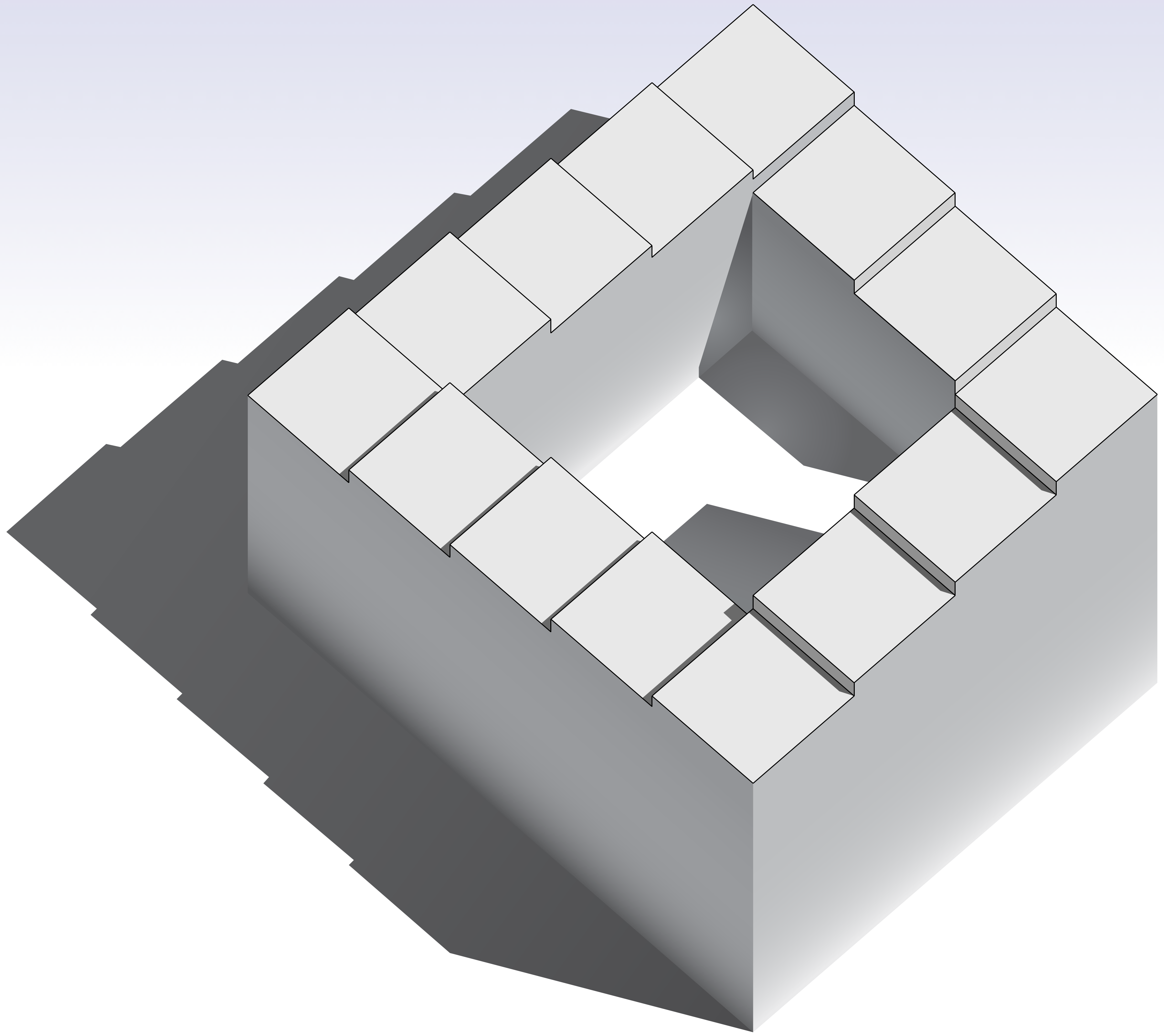
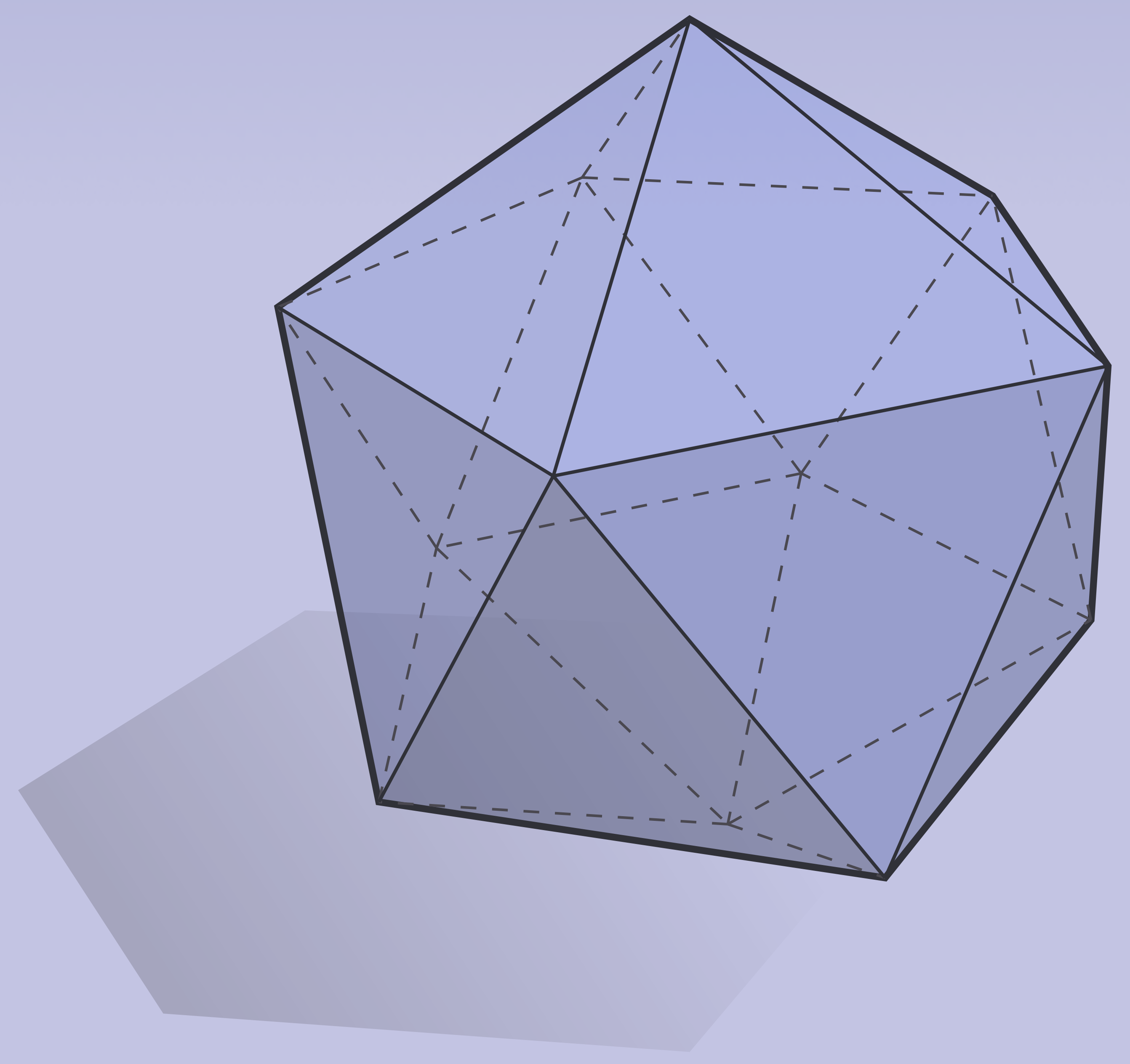


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 4:
k-FORMS

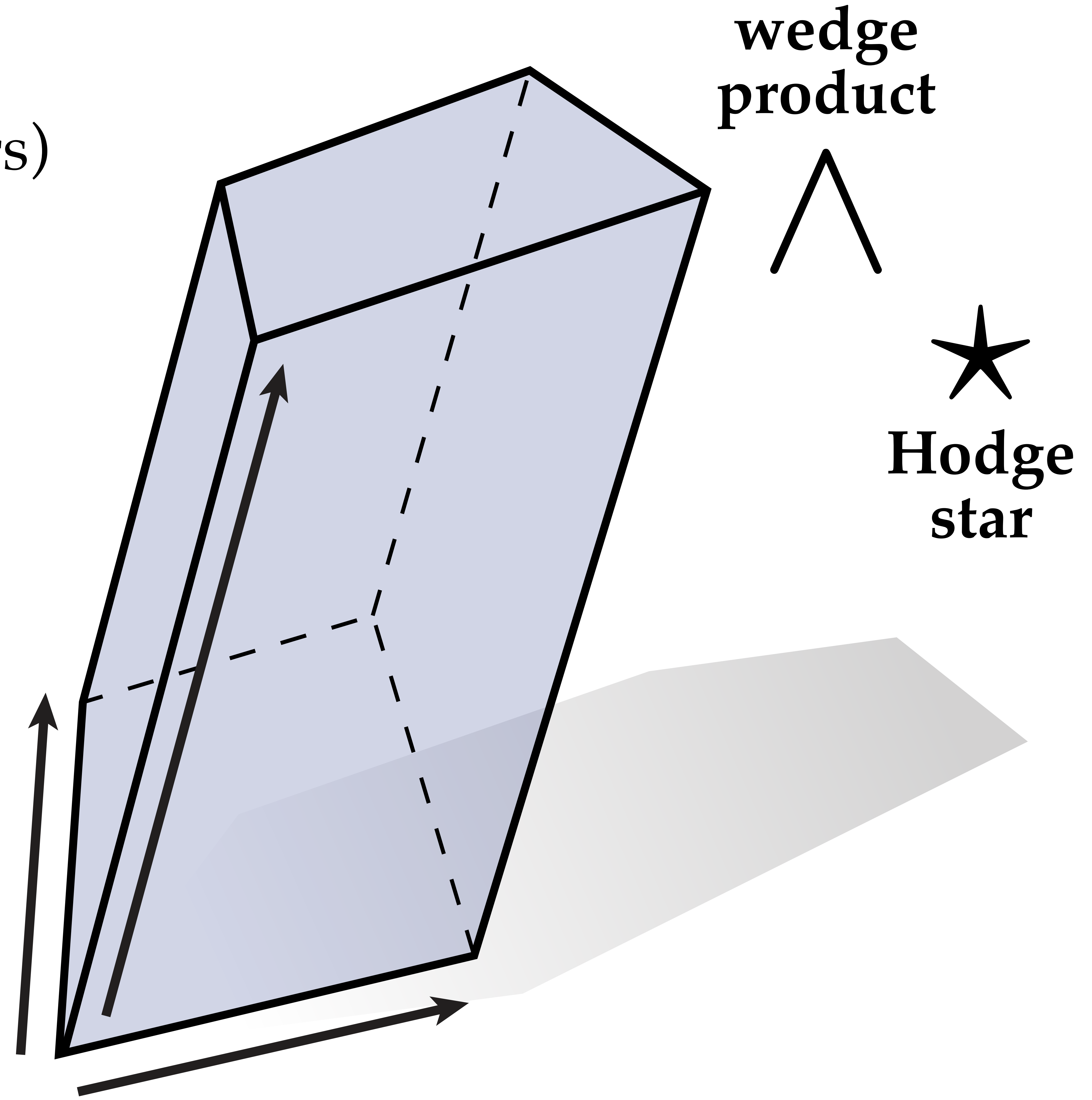


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858

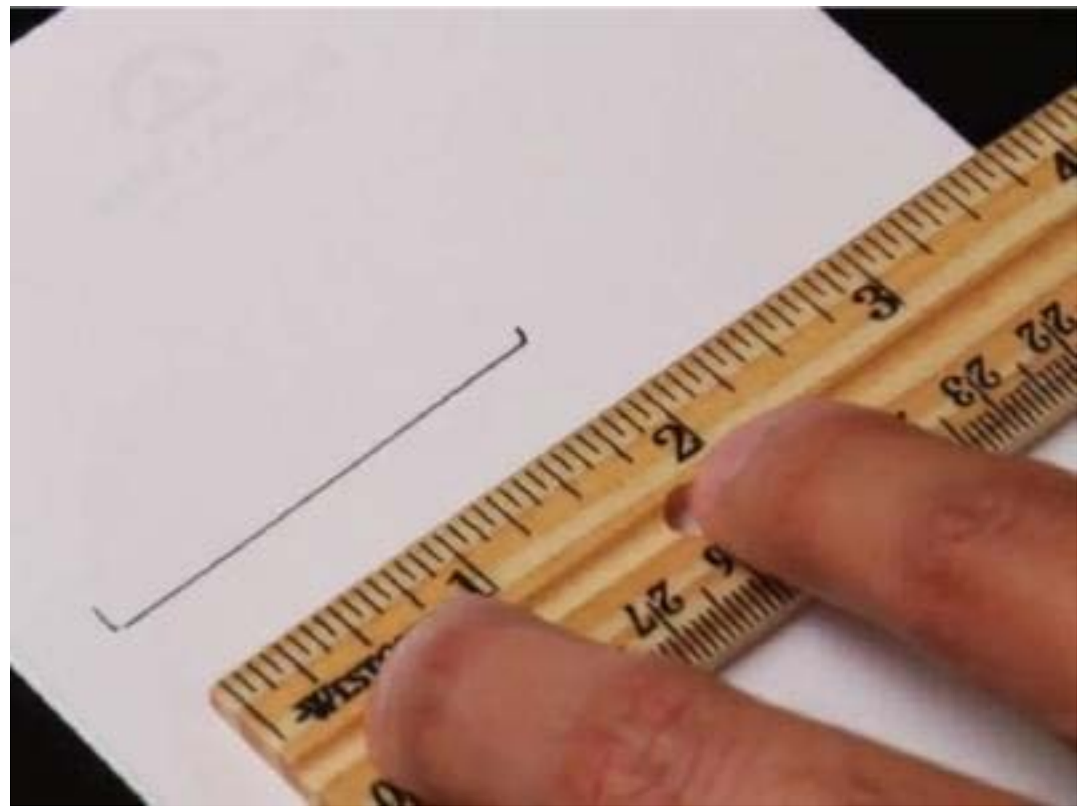
k-Vectors and k-Forms — Overview

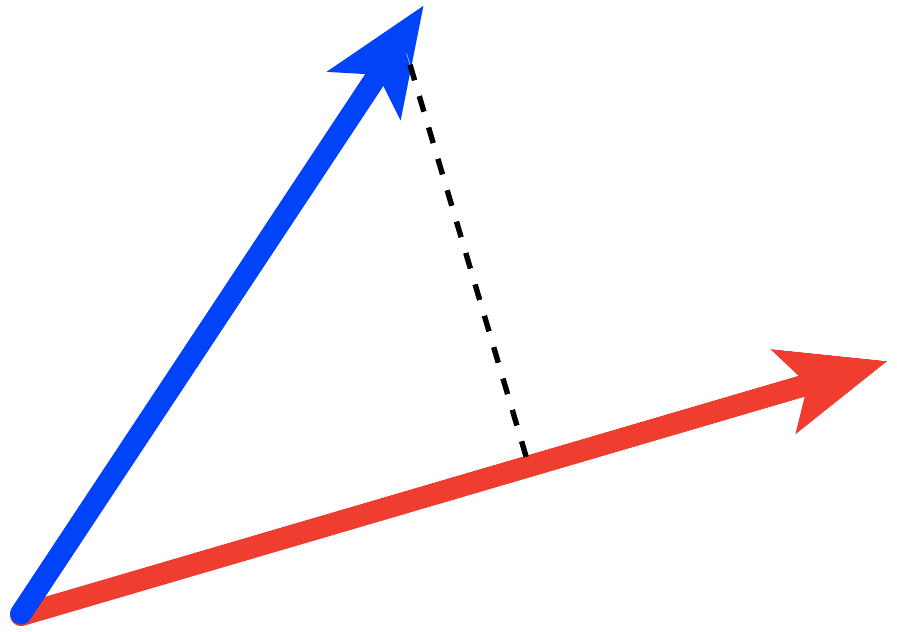
- *Last time:*
  - **Exterior algebra**—"little volumes" (k-vectors)
- *Where we're headed:*
  - **Exterior calculus**—how do lengths, areas, volumes change over curved surfaces?
  - Essential language for geometry & physics
- *Today:*
  - Focus on how to *measure* little volumes
  - Key idea: volumes are measured by other volumes!
  - Will call such volumes "*k*-forms"

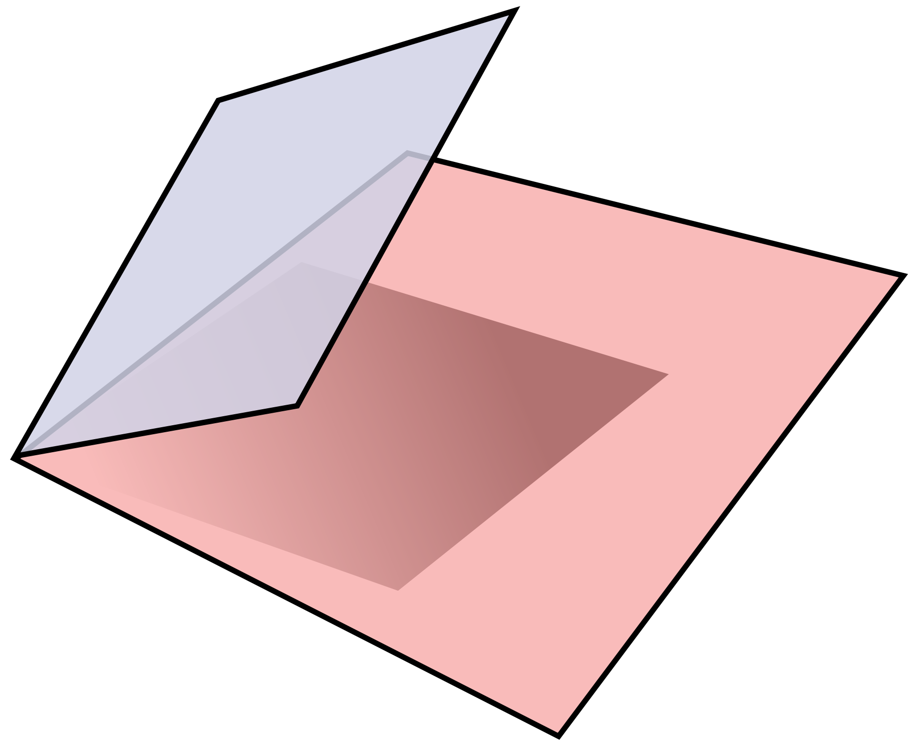


Measurement and Duality

- **Interesting observation:** measurement devices have the *same dimension* as the thing they're measuring:
  - to measure length, use something one-dimensional (ruler, string, *etc.*)
  - to measure volume, use something three-dimensional (*e.g.*, liquid measure)
  - *etc.*
- Same idea shows up in linear algebra:
  - a vector can be "paired" with another vector to get a measurement (inner product)
- Exterior calculus will generalize this idea:
  - a *k*-dimensional volume gets "paired" with a *dual* *k*-dimensional volume to get a measurement

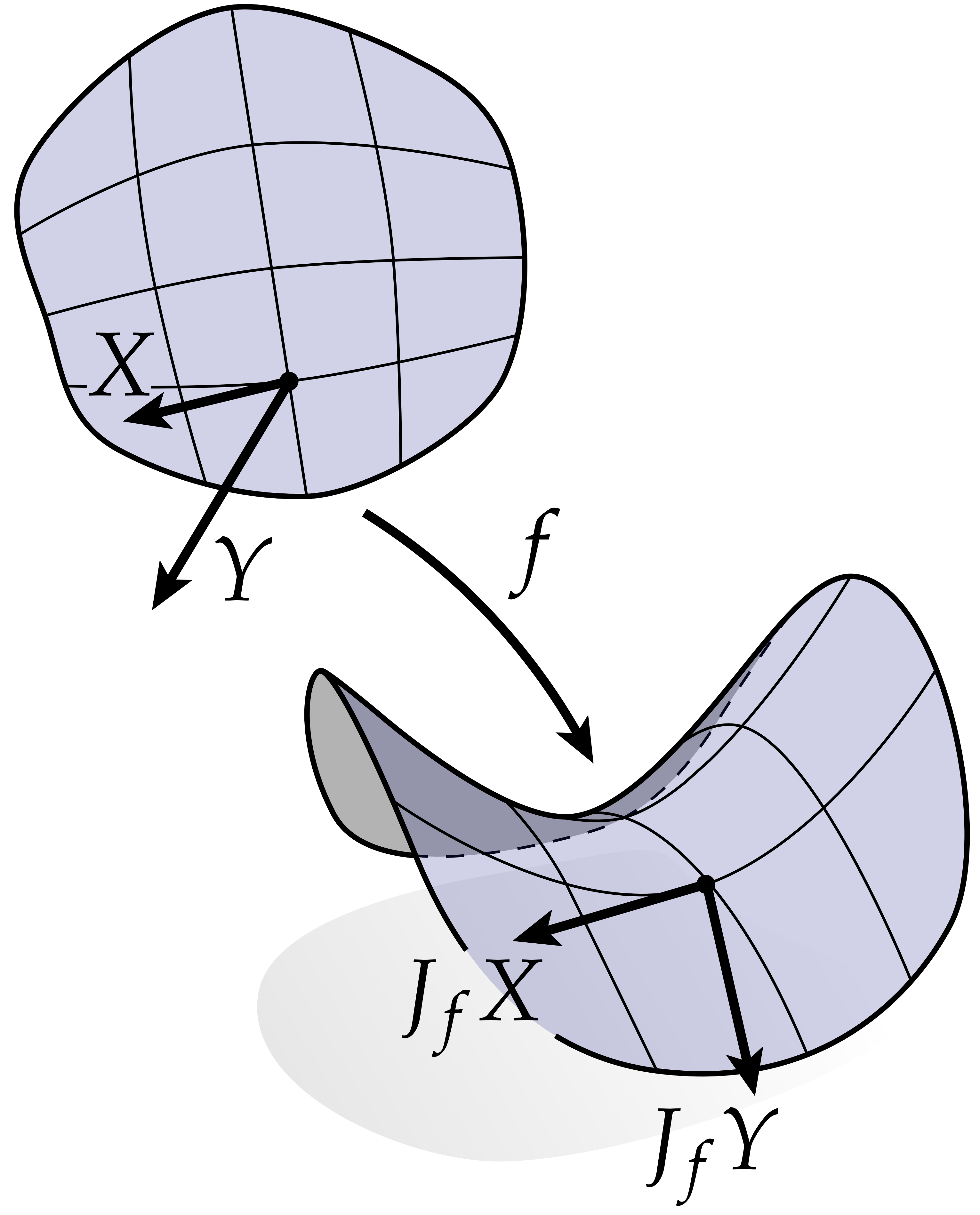






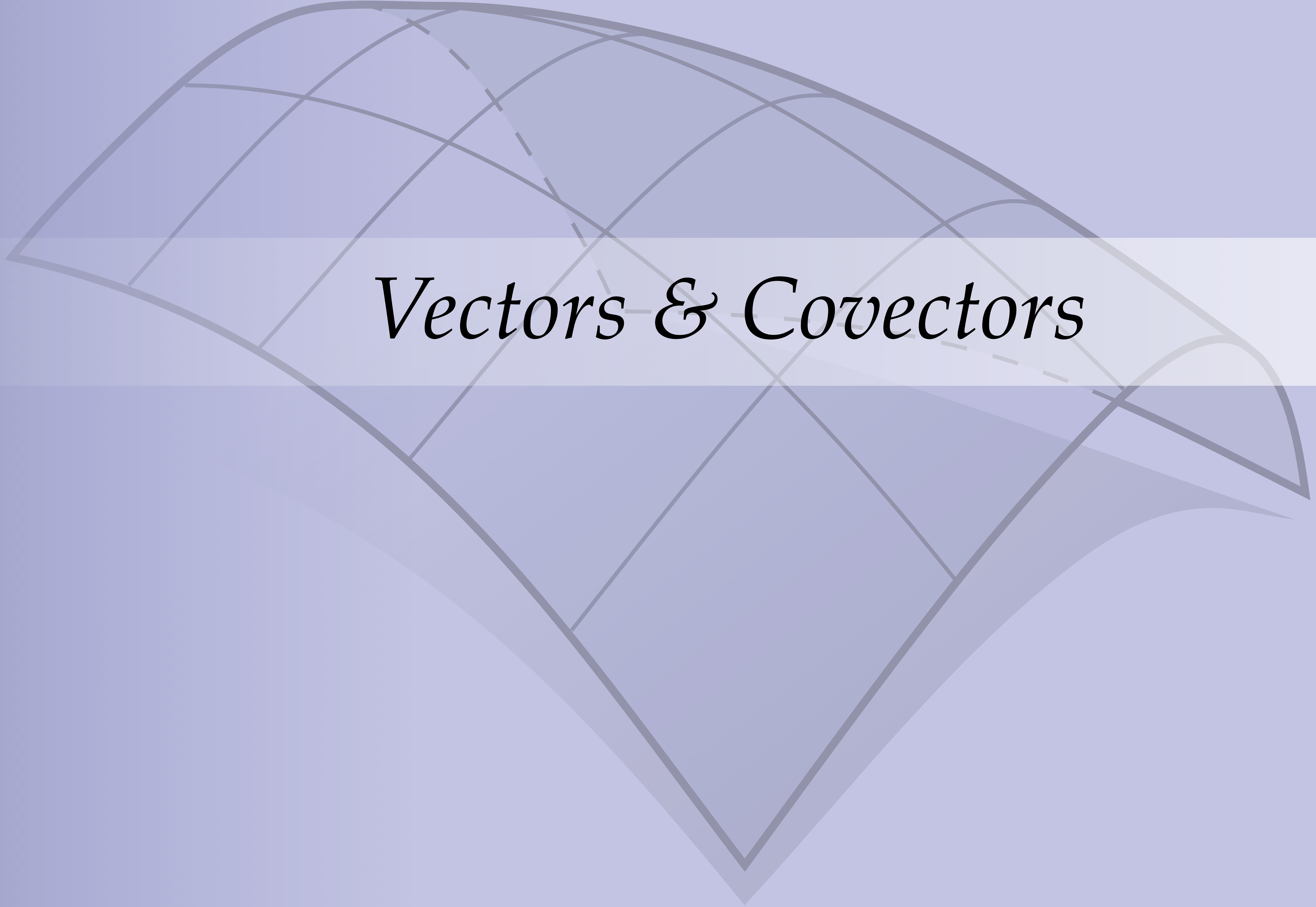
Motivation: Measurement in Curved Spaces

- For simplicity, we will first study exterior calculus in **flat spaces** ($\mathbb{R}^n$)
- May seem like much ado about nothing: e.g., *pairing* vectors and dual vectors will look no different from inner product
- On **curved spaces** things get more interesting—*e.g.*, "bending" the plane gives a different inner product at each point *(Riemmanian metric)*
- Exterior calculus will help us incorporate the Riemannian metric into our calculations in a systematic way



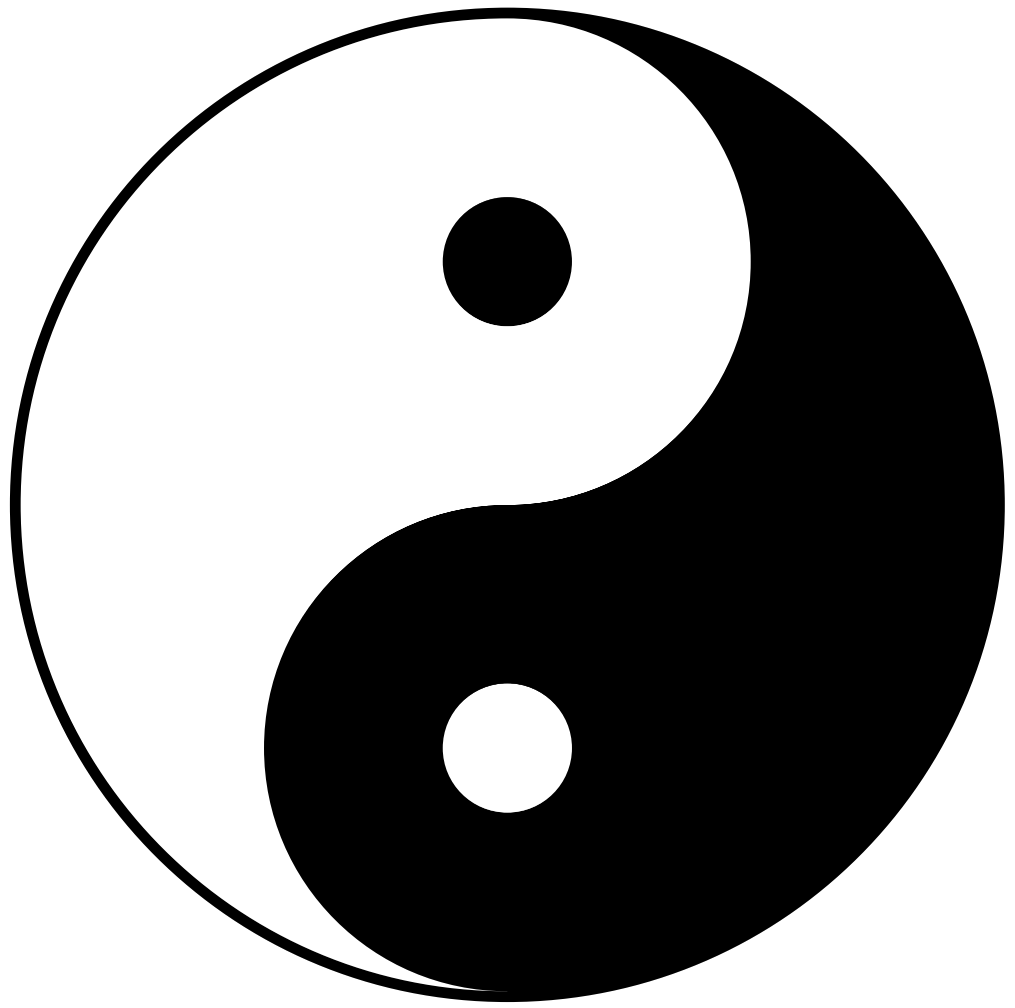
$$\langle X, Y \rangle := (J_f X)^T (J_f Y)$$

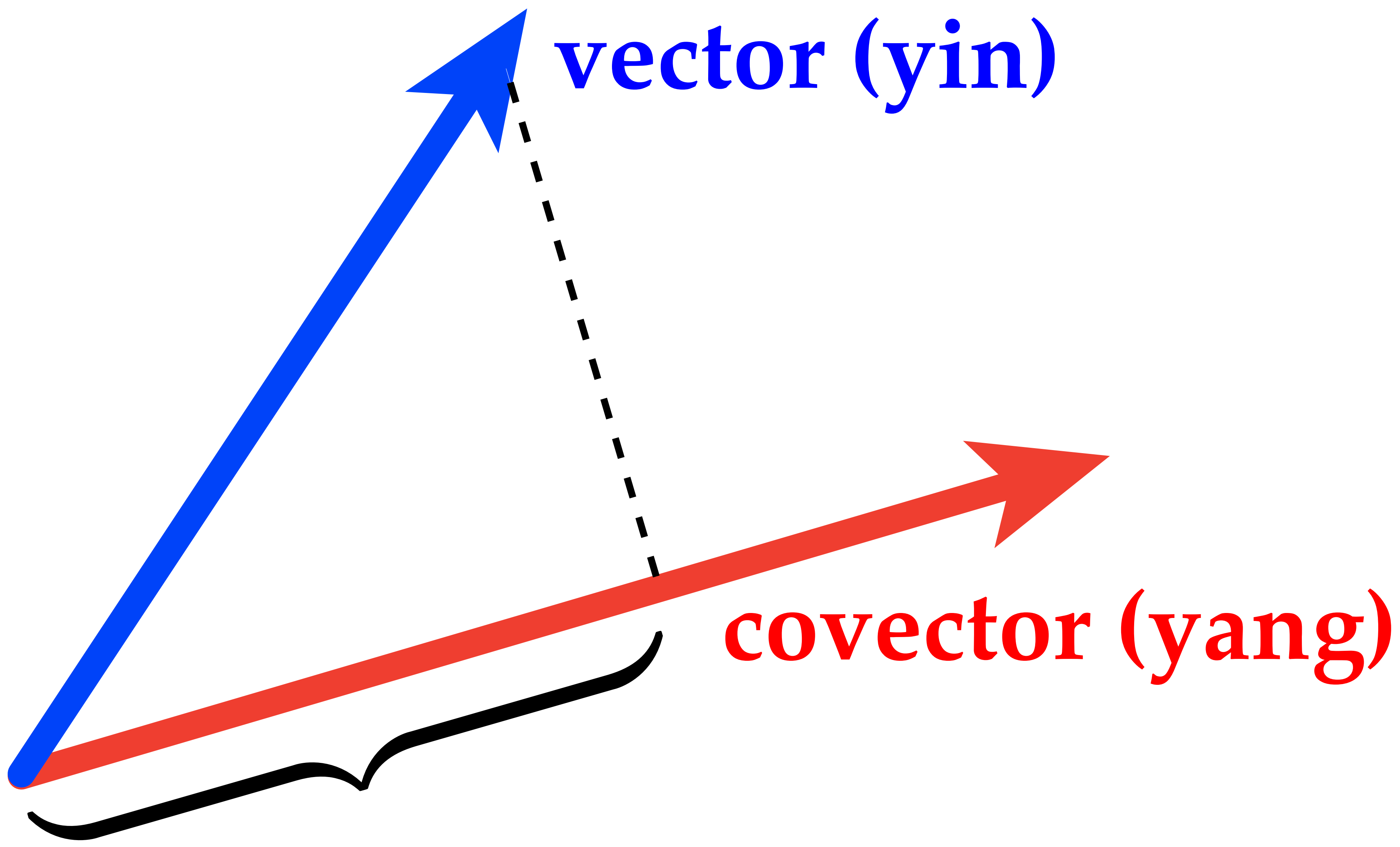
Vectors & Covectors



Vector-Covector Duality

- *Duality* is a pervasive idea in mathematics—two sets of objects that are in one-to-one correspondence, but play complementary roles.
- Important duality in differential geometry and exterior calculus: *vectors* vs. *covectors*.
- Loosely speaking:
  - **covectors** are objects that *"measure"*
  - **vectors** are objects that *"get measured"*
- Just as wedging together vectors yields k-vectors, wedging together covectors will yield *k-forms*, which are dual to k-vectors.





Analogy: Row & Column Vectors

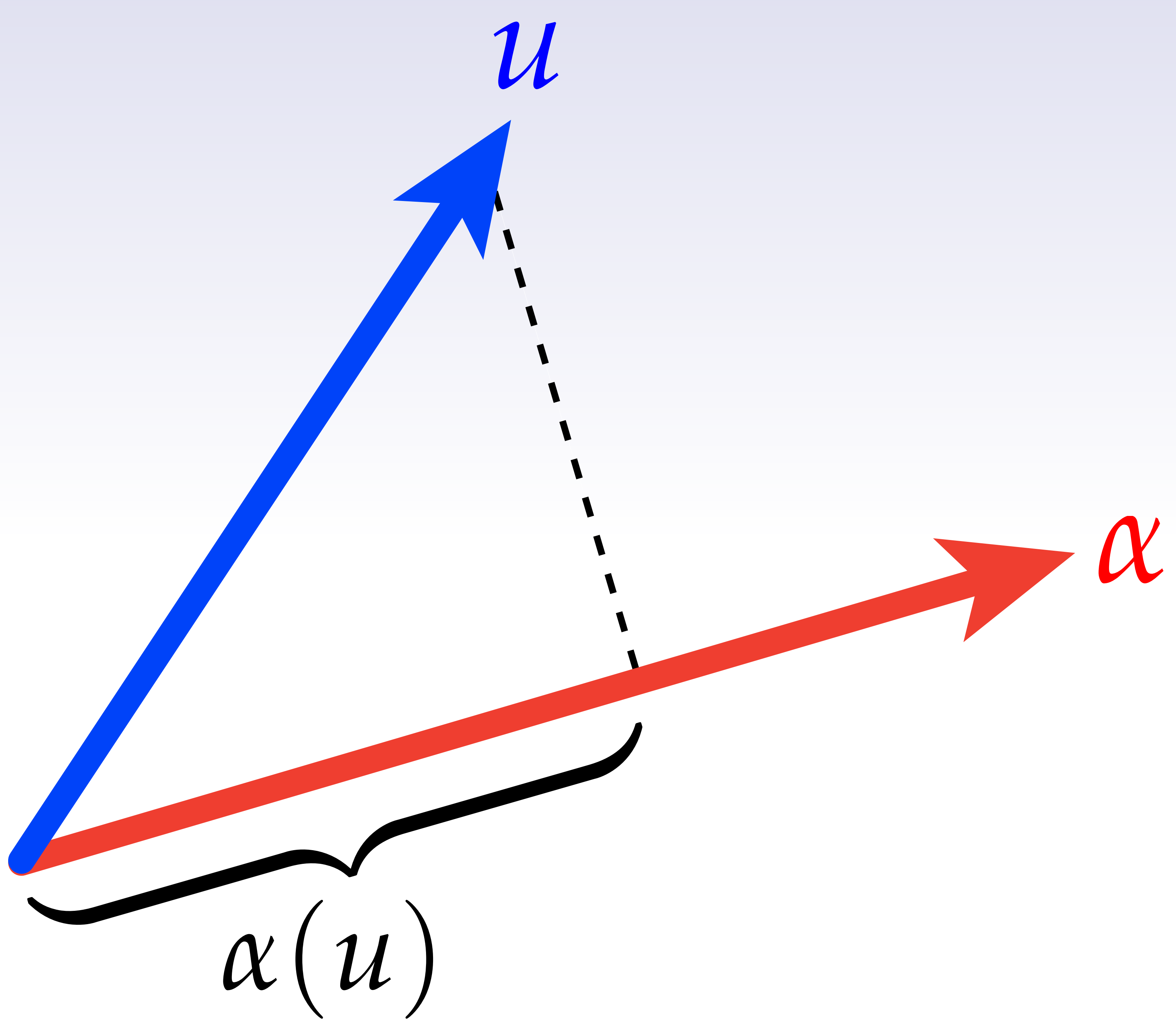
In matrix algebra, we make a distinction between *row vectors* and *column vectors*:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Q: Why do we make the distinction? What does it mean geometrically? What does it mean as a linear map? *Is this distinction useful?*

Vectors and Covectors

α — covector with unit magnitude
u — vector of any magnitude



Key idea: a covector *measures* length of vector along a particular direction

Dual Space & Covectors

Definition. Let V be any real vector space. Its *dual space* V^* is the collection of all linear functions $\alpha : V \to \mathbb{R}$ together with the operations of *addition*

$$(\alpha + \beta)(u) := \alpha(u) + \beta(u)$$

and *scalar multiplication*

$$(c\alpha)(u) := c(\alpha(u))$$

for all $\alpha, \beta \in V^*$, $u \in V$, and $c \in \mathbb{R}$.

Definition. An element of a dual vector space is called a *dual vector* or a *covector*.

(Note: unrelated to *Hodge dual*!)

Covectors — Example ($\mathbb{R}^3$)

- As a concrete example, let's consider the vector space $V = \mathbb{R}^3$
- Recall that a map *f* is *linear* if for all vectors **u**, **v** and scalars *a*, we have

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \qquad \textbf{and} \qquad f(a\mathbf{u}) = af(\mathbf{u})$$

- **Q:** What's an example of a *linear* map from $\mathbb{R}^3$ to $\mathbb{R}$?
  - Suppose we express our vectors in coordinates $\mathbf{u} = (x,y,z)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

vector

- **A:** One of *many* possible examples: $f(x,y,z) = x + 2y + 3z$
- **Q:** What are *all* the possibilities?

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

covector

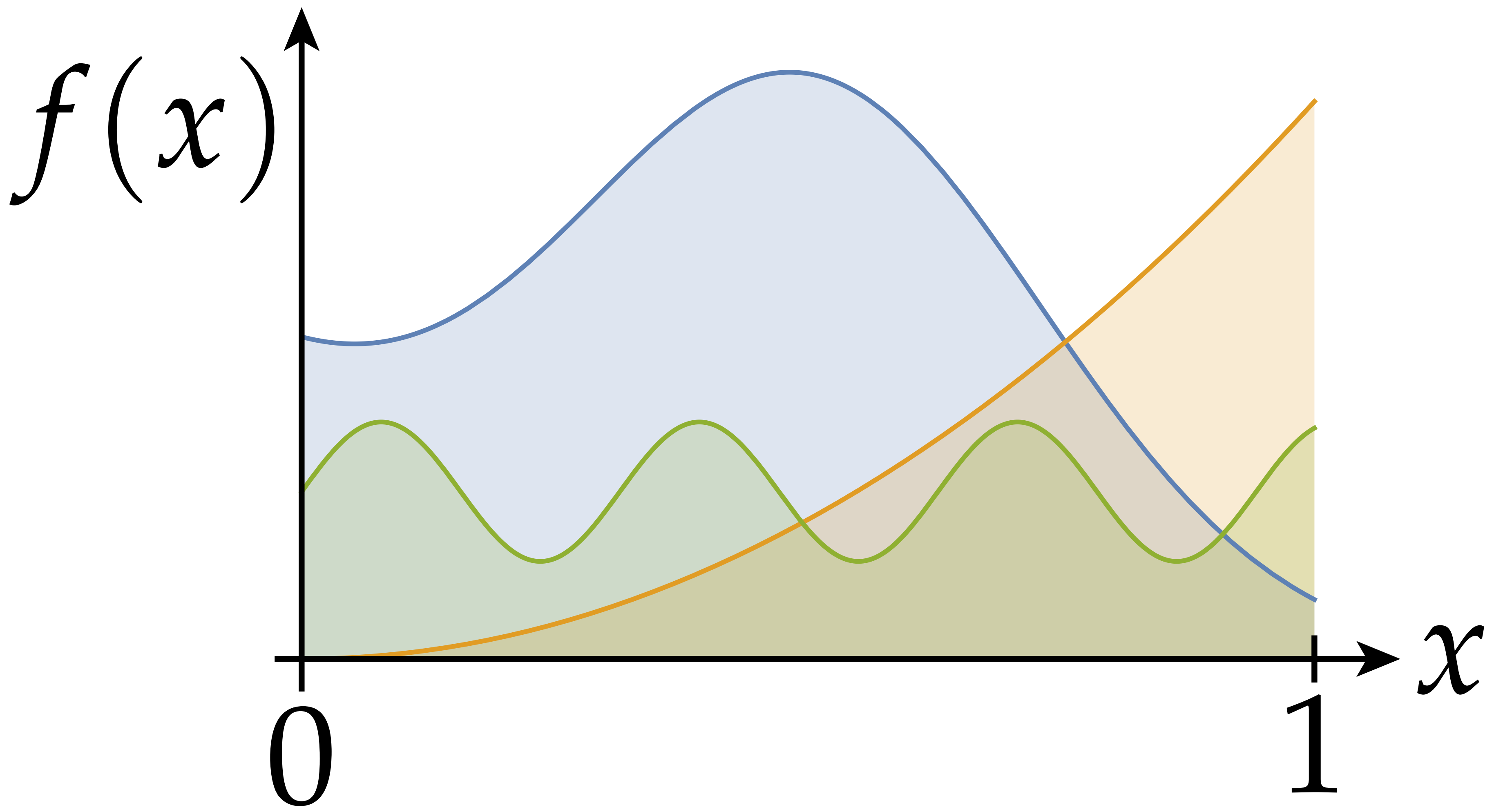
- **A:** They all just look like $f(x,y,z) = ax + by + cz$ for constants a,b,c
- In other words in Euclidean R^3, a *covector* looks like just another 3-vector!

Covectors—Example (Functions)

- If covectors are just the same as vectors, why even bother?
- Here's a more interesting example:

Example. Let V be the set of integrable functions $f : [0,1] \to \mathbb{R}$, and consider maps

- $\phi : V \to \mathbb{R}; f \mapsto \int_0^1 f(x)\, dx$
- $\delta : V \to \mathbb{R}; f \mapsto f(0)$



Is V a vector space? Are ϕ and δ covectors?

Key idea: the difference between primal & dual vectors is not merely superficial!

Sharp and Flat

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xrightarrow{\mathsf{T}} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u, v \xrightarrow{\flat} u^\flat(v)$$

$$\alpha, \beta \xrightarrow{\sharp} \alpha(\beta^\sharp)$$

Analogy: *transpose*

(Why use musical symbols? Will see a bit later…)

Sharp and Flat w/ Inner Product

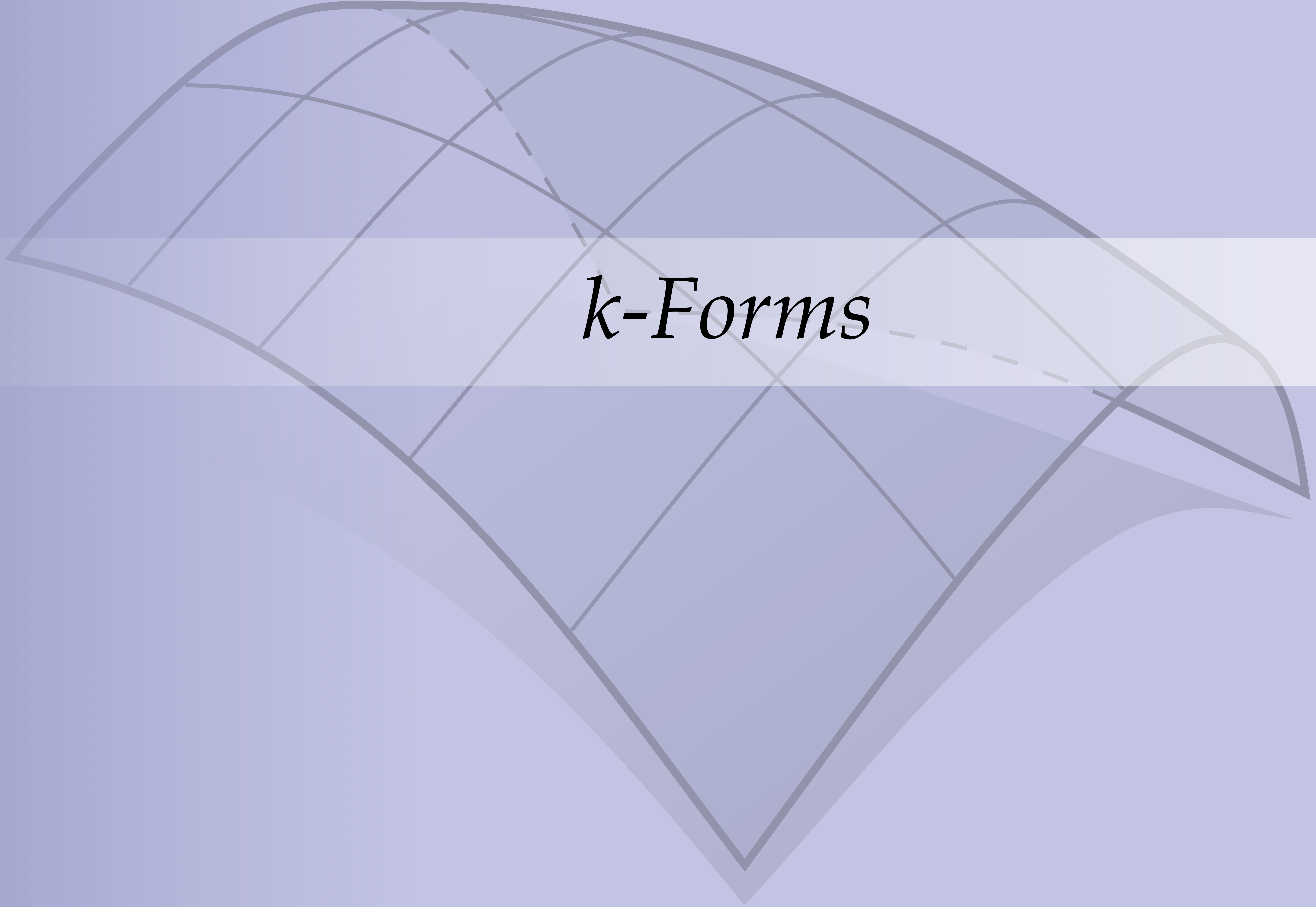
$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u^\flat(v) = u^\mathsf{T} M v \quad \Longleftrightarrow \quad u^\flat(\cdot) = \langle u, \cdot \rangle$$

$$\alpha(\beta^\sharp) = \alpha M^{-1} \beta^\mathsf{T} \quad \Longleftrightarrow \quad \langle \alpha^\sharp, \cdot \rangle = \alpha(\cdot)$$

Basic idea: applying the flat of a vector is the same as taking the inner product; taking the inner product w/ the sharp is same as applying the original covector.

k-Forms



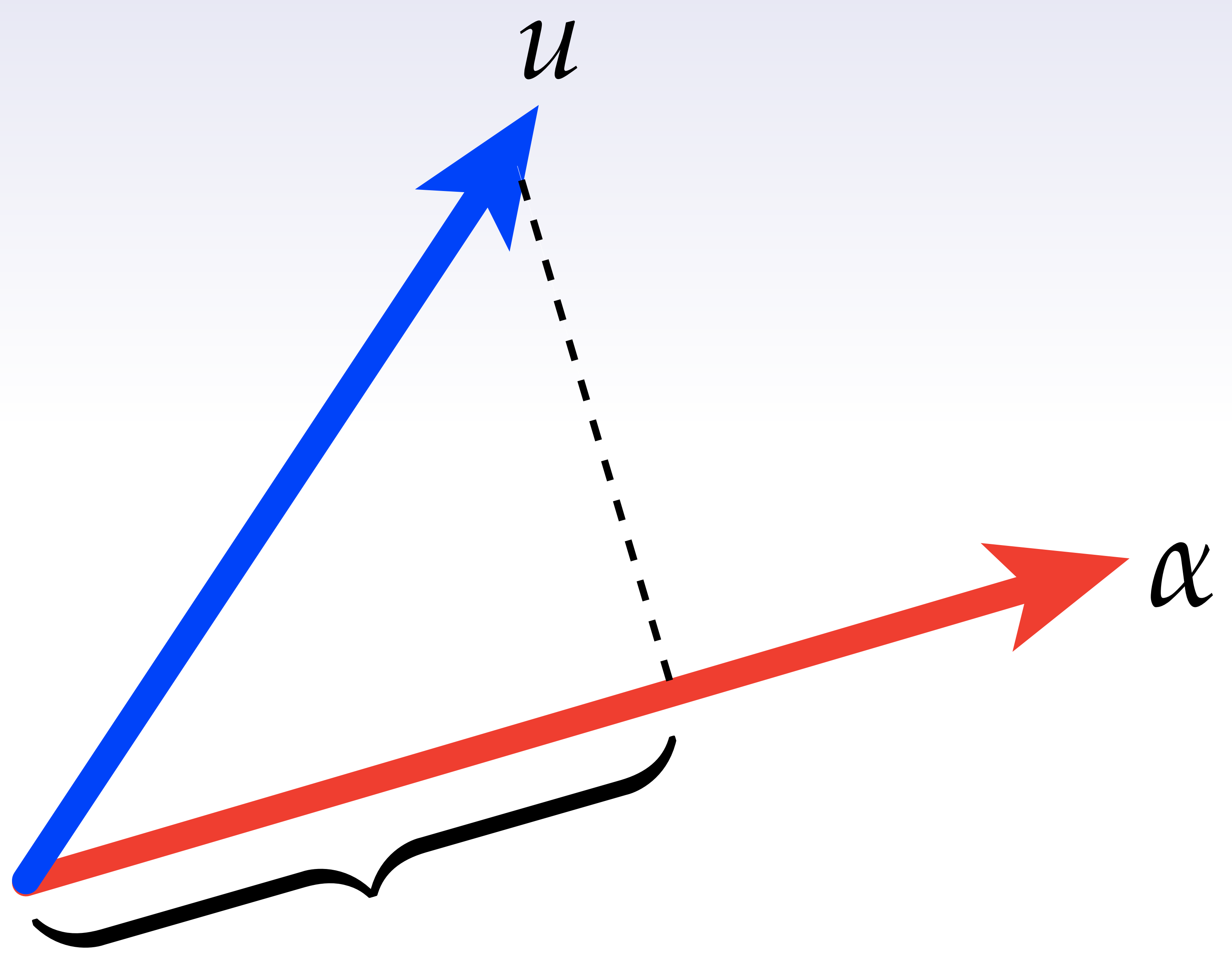
Covectors, Meet Exterior Algebra

- So far we've studied two distinct concepts
- Starting with an ordinary vector space…
  - **exterior algebra**—build up "volumes" from vectors
  - **covectors—**linear maps from vectors to scalars
- Combine to get an *exterior algebra of covectors*
  - Will call these objects *k-forms*
  - Just as a covector measures vectors, a *k*-form will measure *k*-vectors
  - In particular, measurements will be **multilinear**, *i.e.*, linear in each 1-vector

	primal	**dual**
linear algebra	vectors	covectors
exterior algebra	*k*-vectors	*k*-forms

Measurement of Vectors

Geometrically, what does it mean to take a **linear** measurement of a single vector?

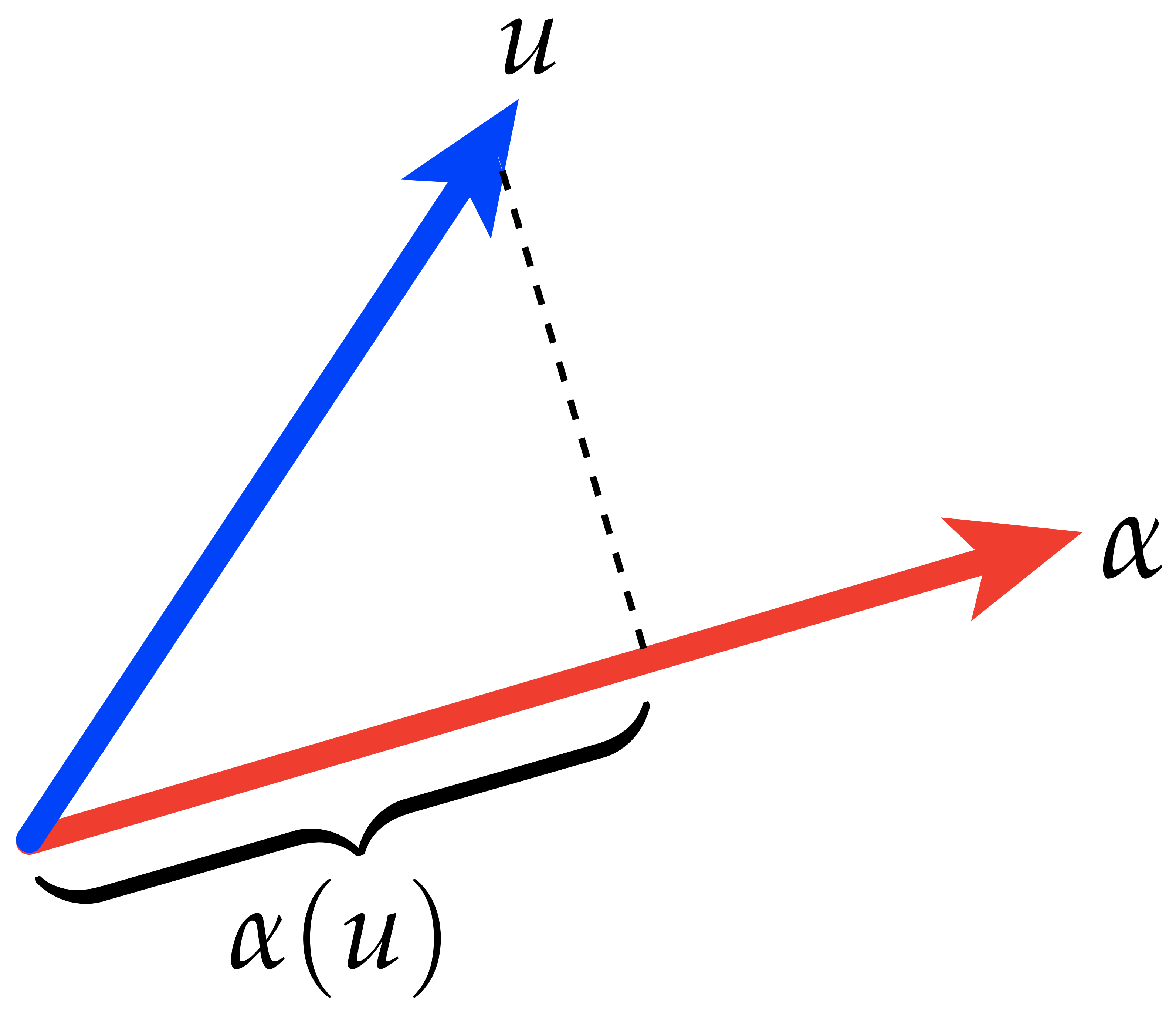


Observation: only thing we can measure is extent along some direction.

Computing the Projected Length

- Concretely, how do we compute projected length of one vector along another?
  - If α has unit norm, then we can just take the usual *dot product*
  - Since we think of u as the vector *"getting measured"* and α as the (co)vector *"doing the measurement"*, we'll write this as a function application $\alpha(u)$:

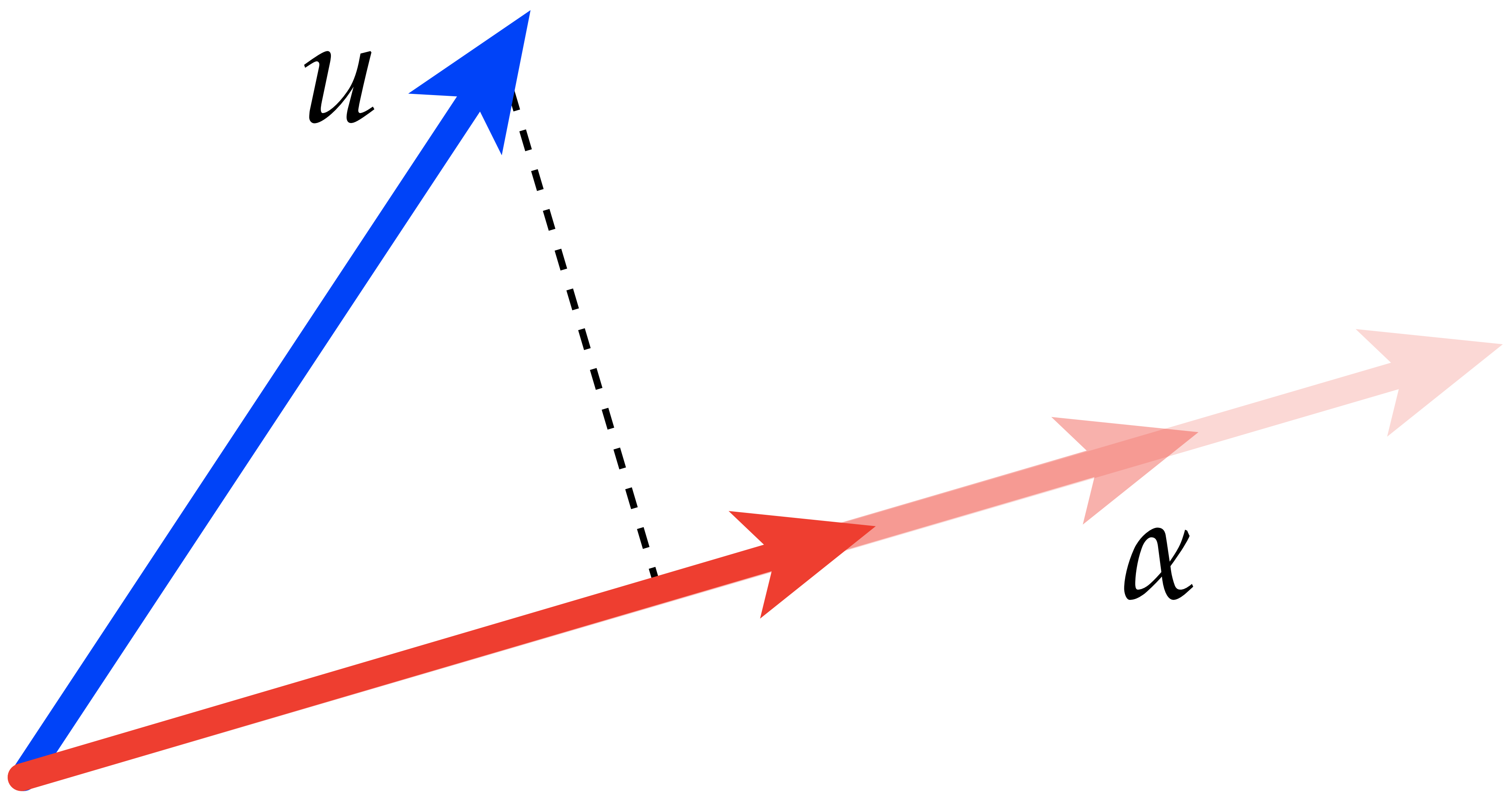
$$\alpha(u) = \sum_{i=1}^{n} \alpha_i u^i$$



1-form

We can of course apply this same function when α does <u>not</u> have unit length:

$$\alpha(u) := \sum_i \alpha_i u^i$$

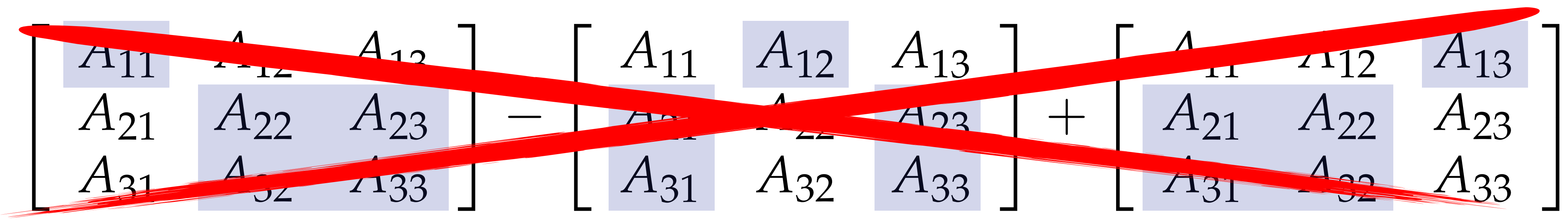


Interpretation?

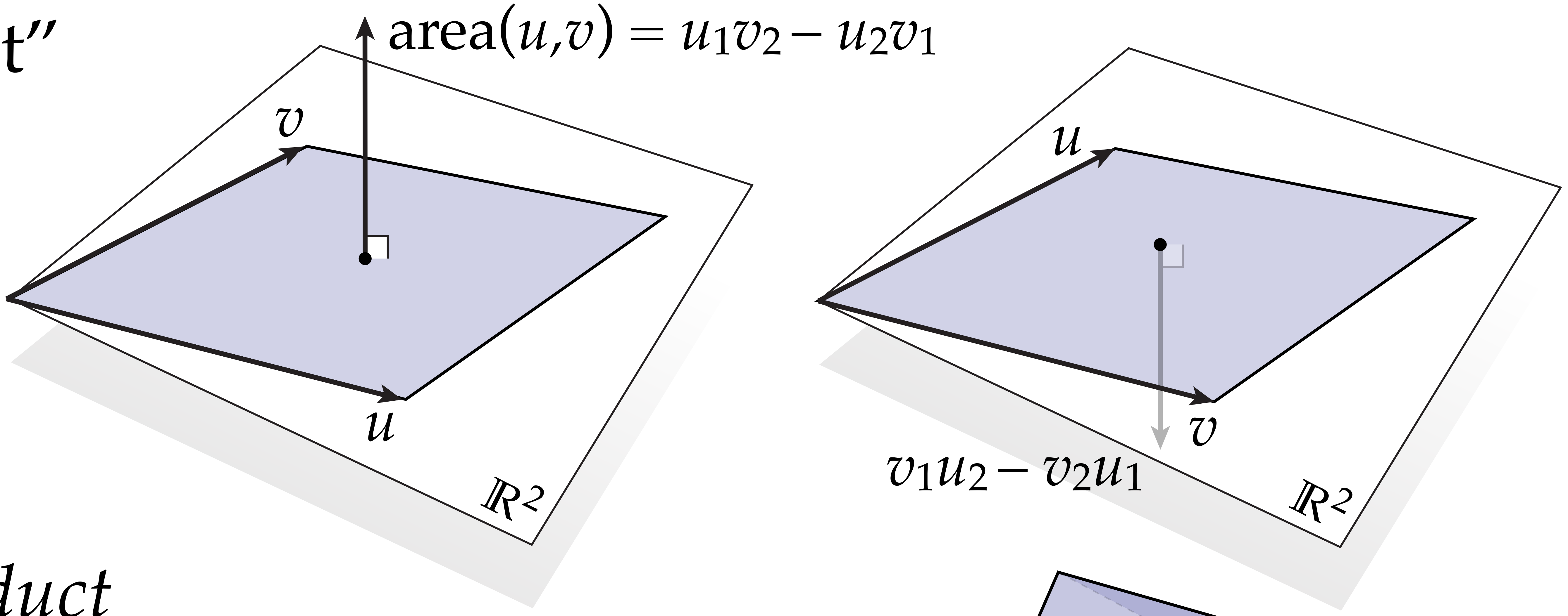
Projected length gets scaled by magnitude of α.

Review: Determinants & Signed Volume

- The determinant of a square matrix is often introduced via some formula or algorithm.



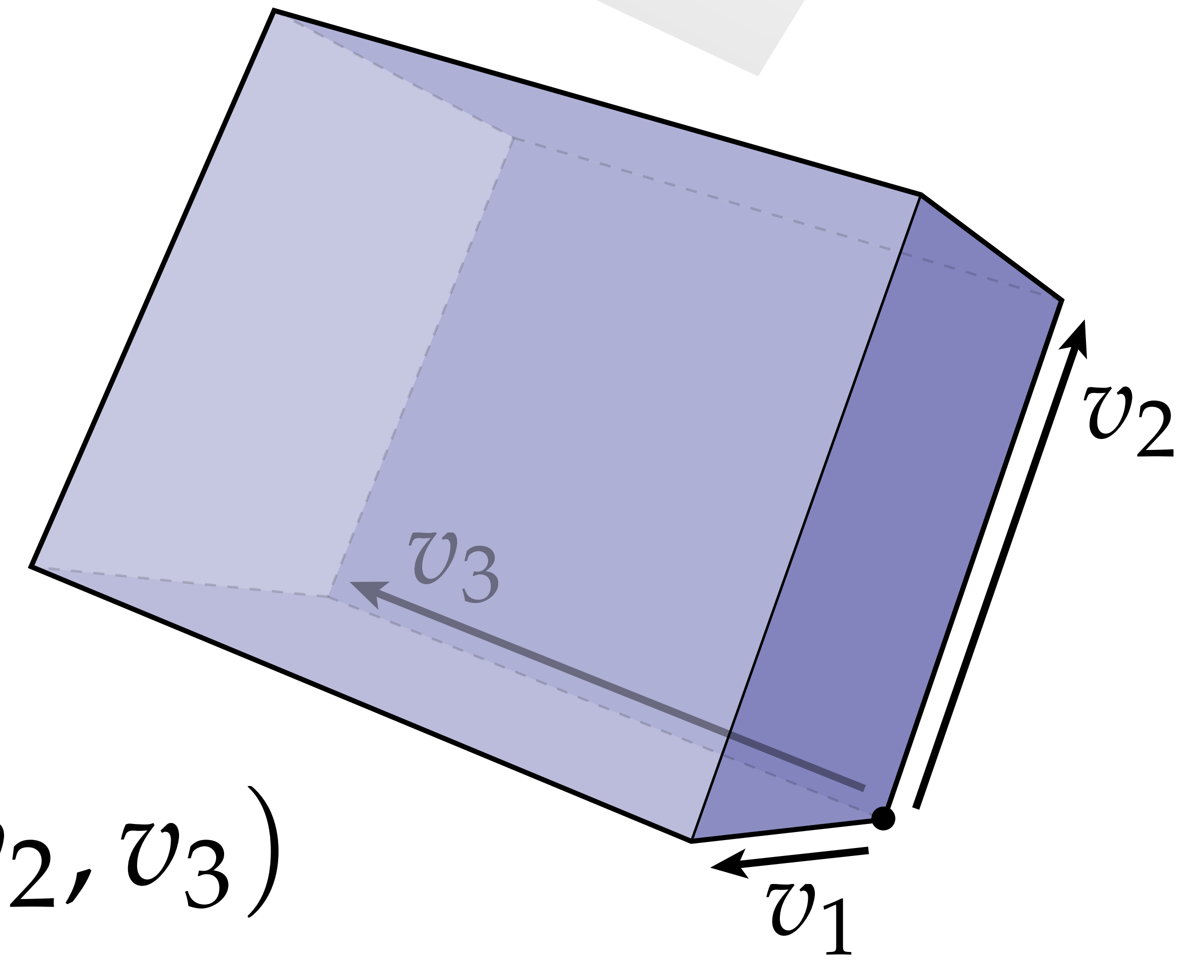
- When you hear the word "determinant" you should instead think "volume"
  - more precisely: *signed volume*
  - sign flips with orientation



- E.g., 2D signed area given by *cross product*
- More generally, the determinant of a collection of vectors $v_1, \ldots, v_n$ is the signed volume of the parallelepiped defined by these vectors

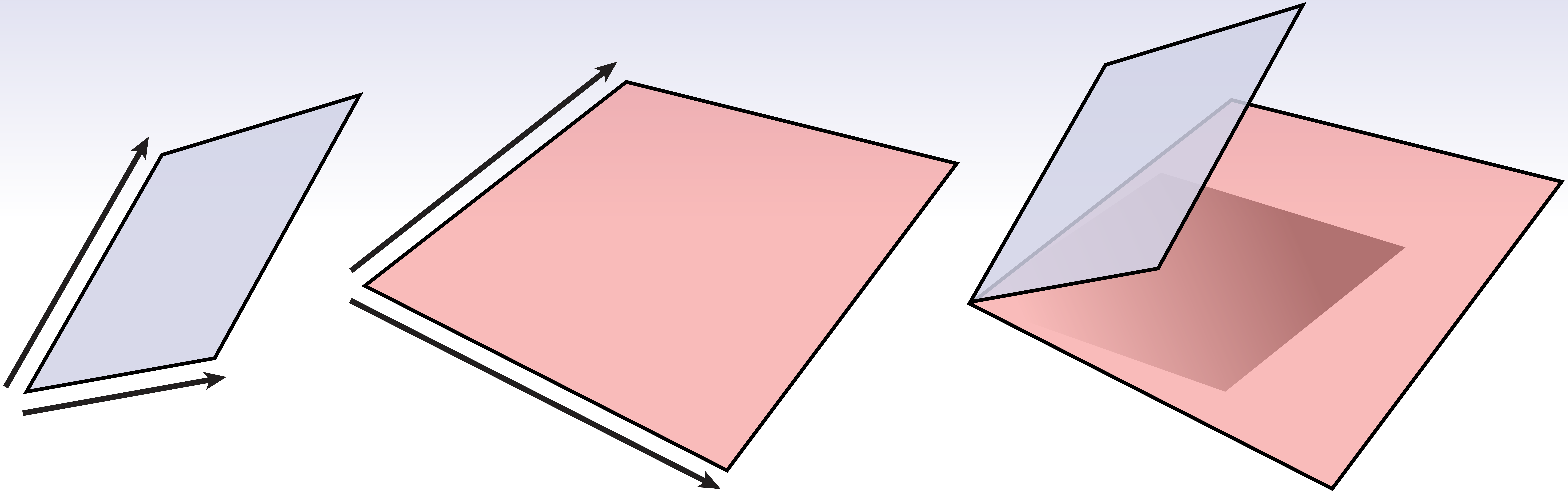
$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

$$\det(A) = \mathrm{vol}(v_1, v_2, v_3)$$



Measurement of 2-Vectors

Geometrically, what does it mean to take a **multilinear** measurement of a 2-vector?



Intuition: size of "shadow" of one parallelogram on another.

Computing the Projected Area

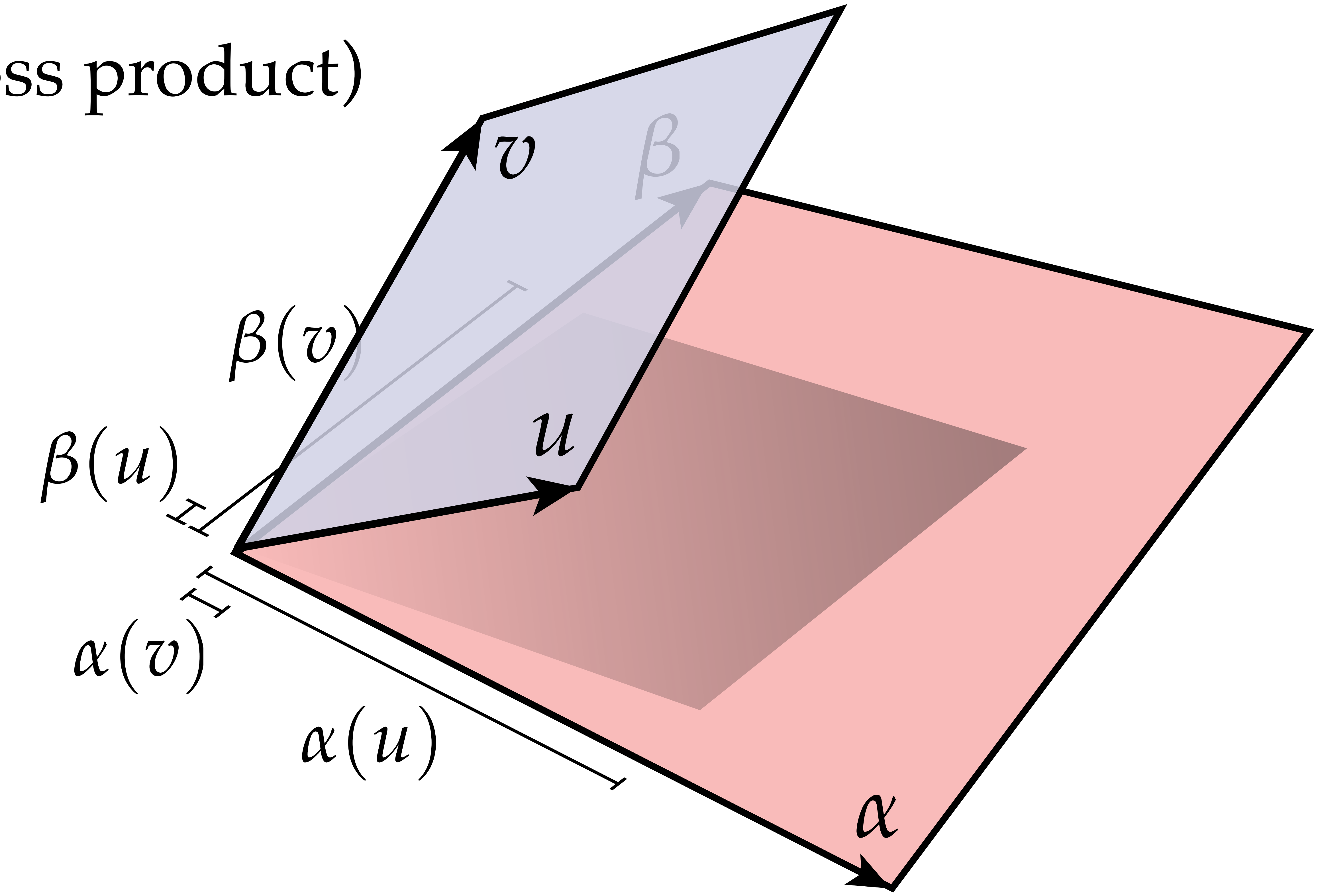
- How do we compute projected area of a parallelogram (u,v) onto a plane?
  - pick any orthonormal basis α,β for the plane
  - project vectors onto plane
  - then apply standard formula for area (cross product)

<u>**Projection**</u>

$$u \mapsto (\alpha(u), \beta(u))$$
$$v \mapsto (\alpha(v), \beta(v))$$

<u>**Area**</u>

$$\alpha(u)\beta(v) - \alpha(v)\beta(u)$$

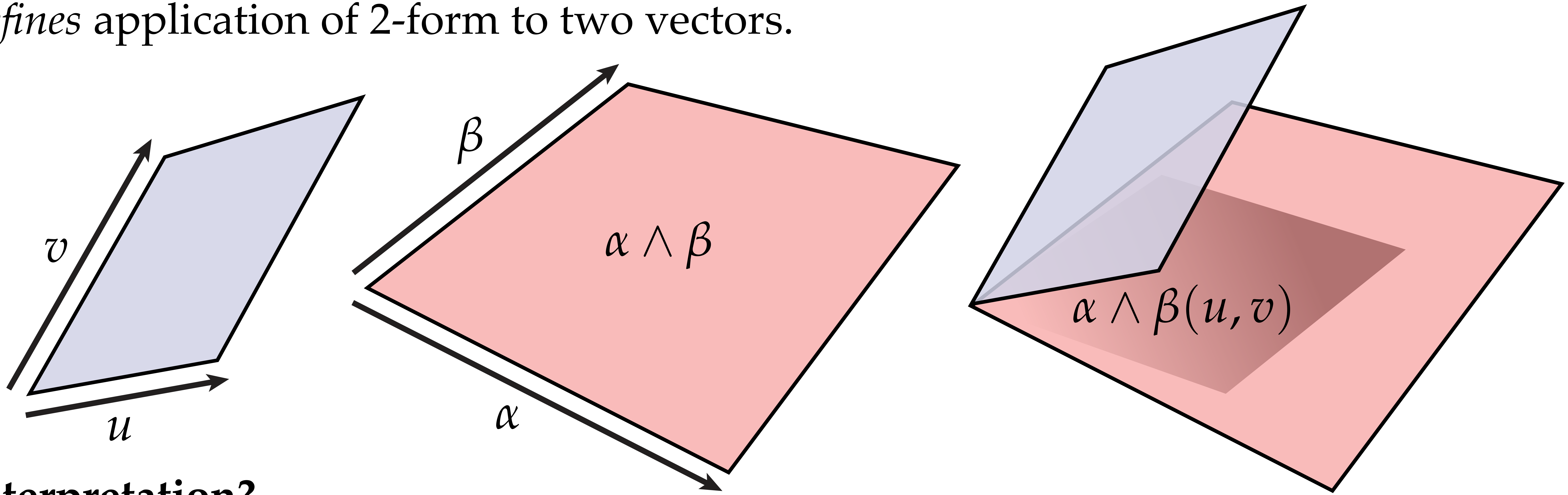


2-form

We can of course apply this same expression when α,β are not orthonormal:

$$(\alpha \wedge \beta)(u, v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

Defines application of 2-form to two vectors.



Interpretation?

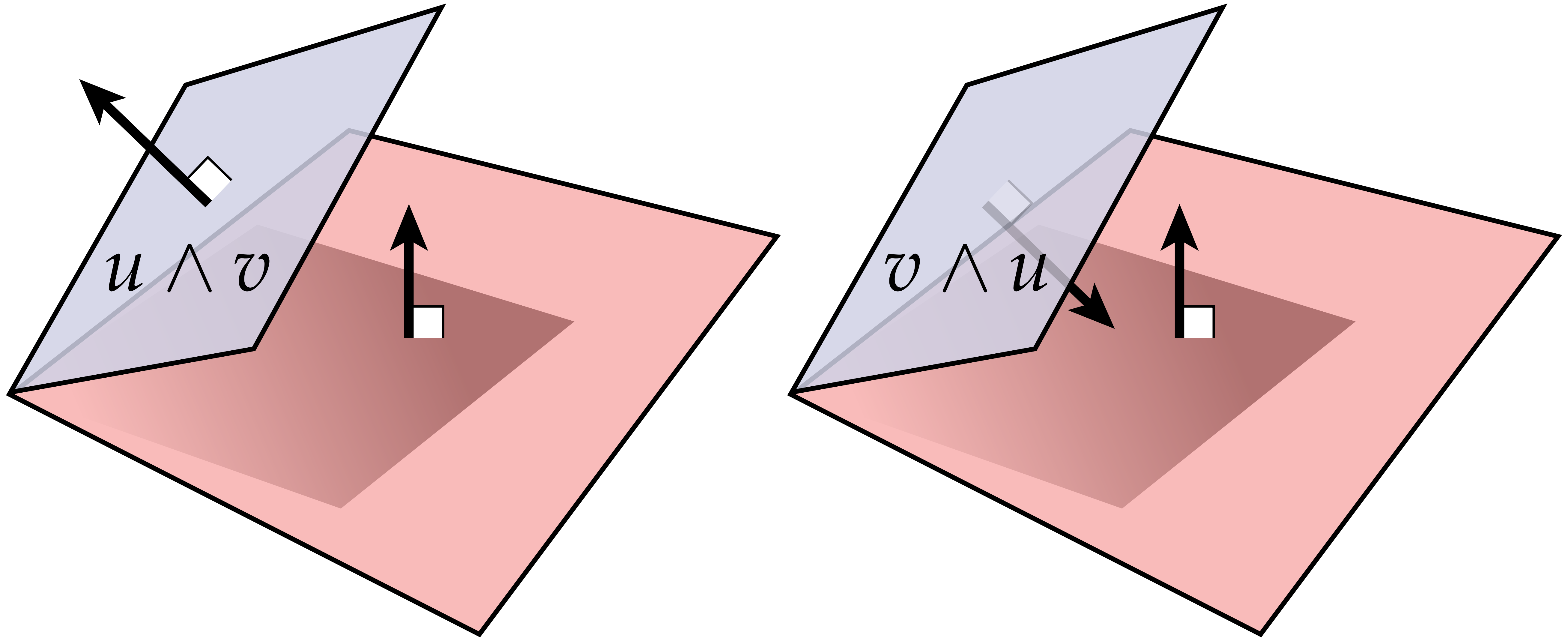
Projected area of u,v gets scaled by area of parallelogram with edges α,β.

Antisymmetry of 2-Forms

Notice that exchanging the arguments of a 2-form reverses sign:

$$\begin{aligned}(\alpha \wedge \beta)(v,u) &= \alpha(v)\beta(u) - \alpha(u)\beta(v) \\ &= -(\alpha(u)\beta(v) - \alpha(v)\beta(u)) \\ &= -(\alpha \wedge \beta)(u,v)\end{aligned}$$

Q: What does this *antisymmetry* mean geometrically?



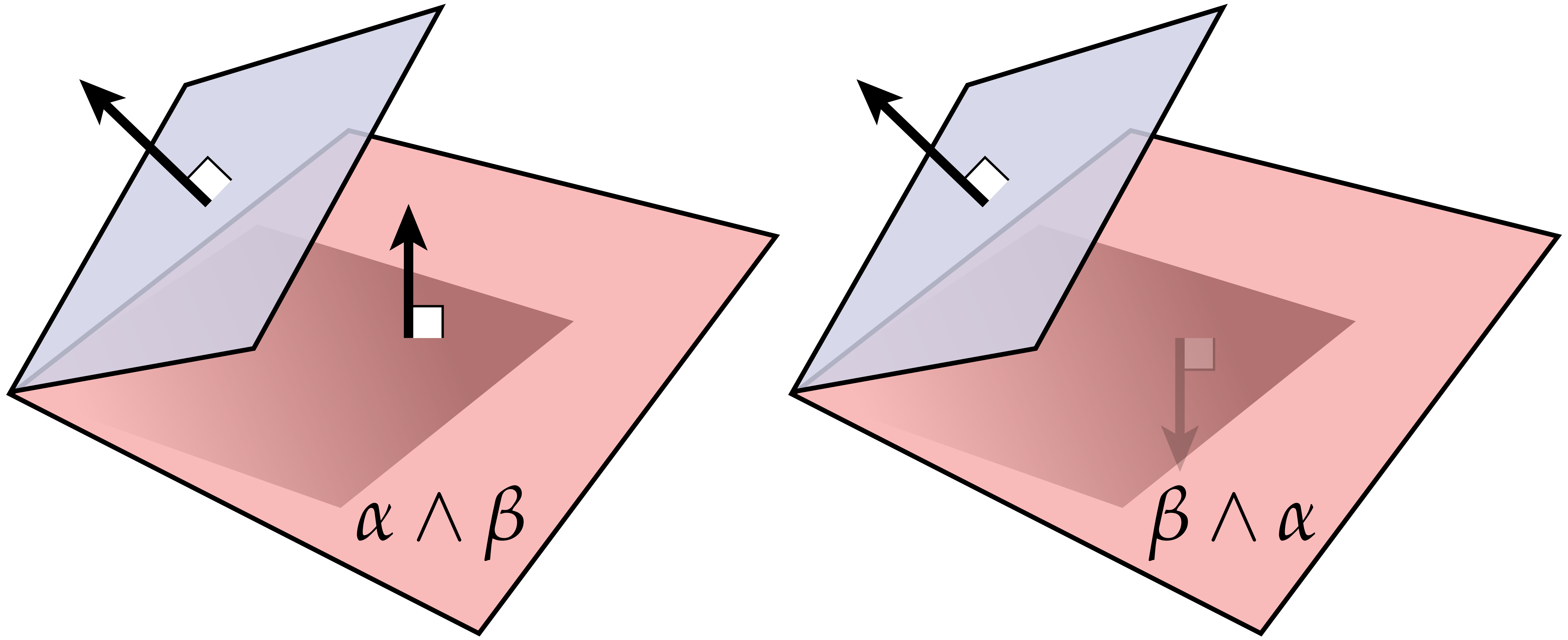
A: It means we care about the *relative orientation* of the two parallelograms.

Antisymmetry of 2-Forms

Recall that exchanging the arguments to a wedge product *also* reverses sign:

$$\begin{aligned}(\beta \wedge \alpha)(u,v) &= \beta(u)\alpha(v) - \beta(v)\alpha(u) \\ &= -(\alpha(u)\beta(v) - \alpha(v)\beta(u)) \\ &= -(\alpha \wedge \beta)(u,v)\end{aligned}$$

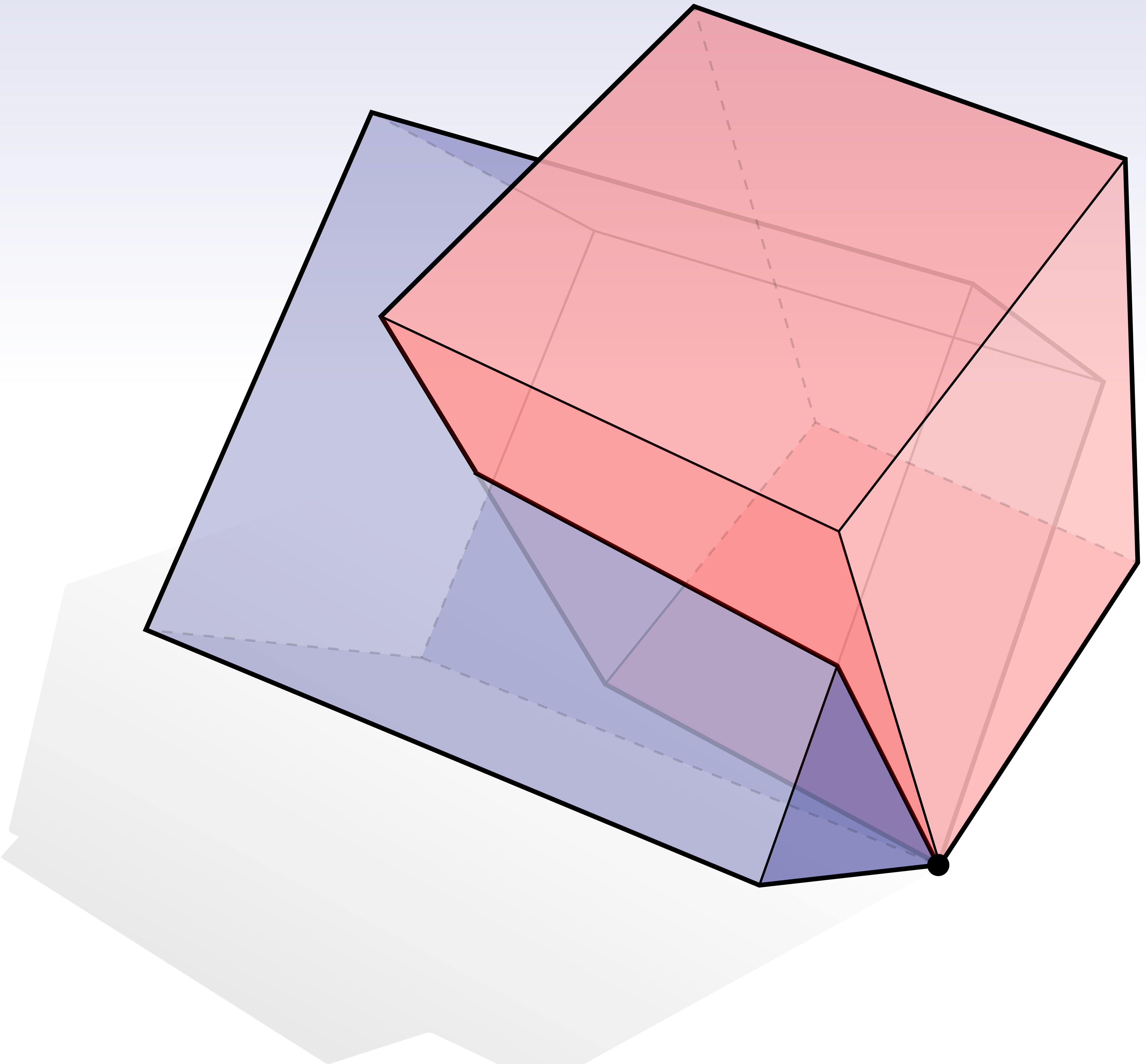
Q: What does this *other* antisymmetry mean geometrically?



A: It accounts for the orientation of the 2-vector (*"what do we want to measure?"*)

Measurement of 3-Vectors

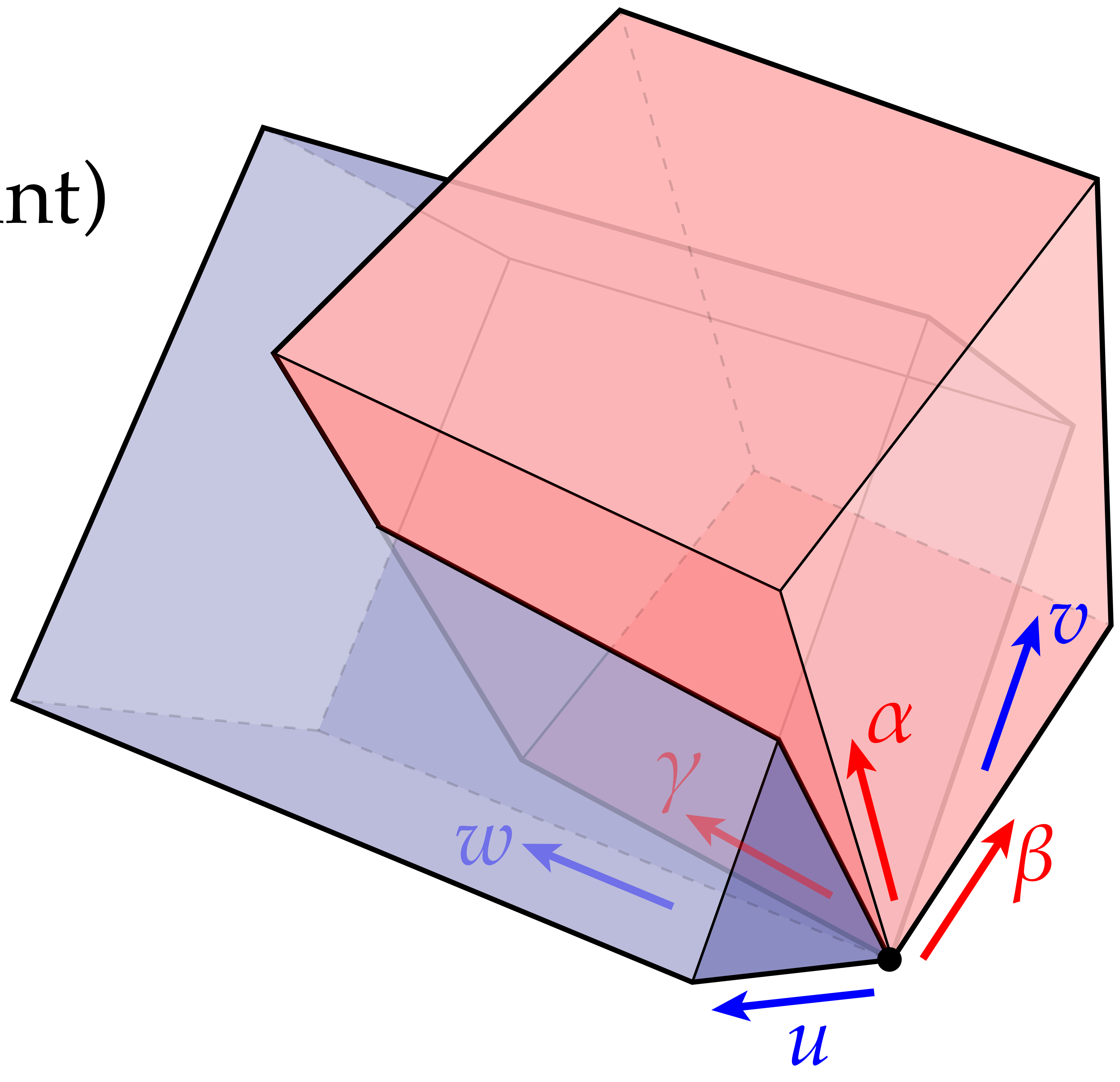
Geometrically, what does it mean to take a **multilinear** measurement of a 3-vector?



Observation: in $\mathbb{R}^3$, all 3-vectors have same *"direction."* Can only measure *magnitude*.

Computing the Projected Volume

- Concretely, how do we compute the volume of a parallelepiped w/ edges u,v,w?
  - Suppose (α,β,γ) ,is any orthonormal basis
  - Project vectors u,v,w onto this basis
  - Then apply standard formula for volume (determinant)



Projection

$$u \mapsto (\alpha(u), \beta(u), \gamma(u))$$
$$v \mapsto (\alpha(v), \beta(v), \gamma(v))$$
$$w \mapsto (\alpha(w), \beta(w), \gamma(w))$$

Volume

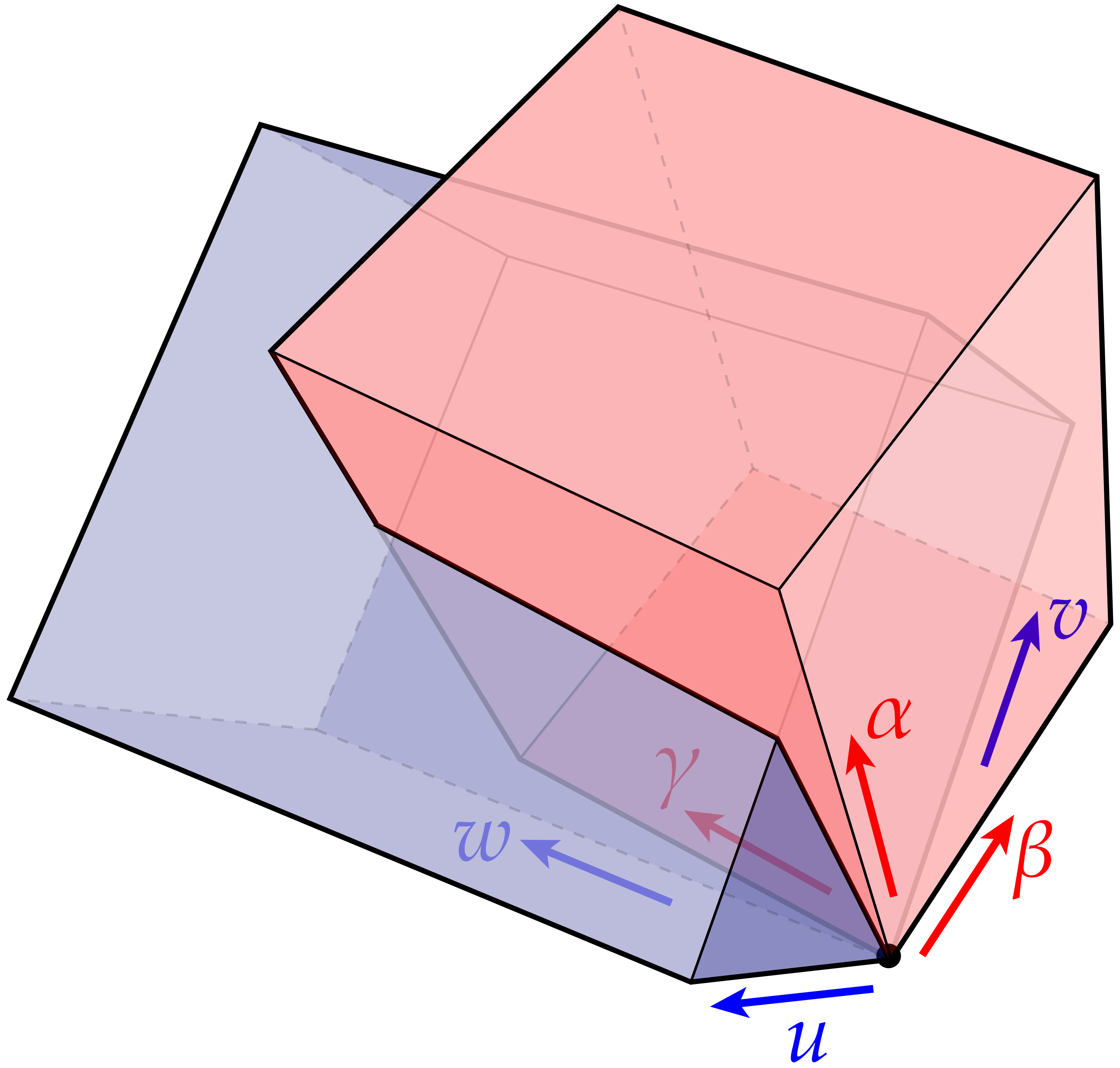
$$\begin{vmatrix} \alpha(u) & \alpha(v) & \alpha(w) \\ \beta(u) & \beta(v) & \beta(w) \\ \gamma(u) & \gamma(v) & \gamma(w) \end{vmatrix}$$

$$= \alpha(u)\beta(v)\gamma(w) + \alpha(v)\beta(w)\gamma(u) + \alpha(w)\beta(u)\gamma(v)$$
$$- \alpha(u)\beta(w)\gamma(v) - \alpha(w)\beta(v)\gamma(u) - \alpha(v)\beta(u)\gamma(w)$$

3-form

We can of course apply this same expression when α,β,γ are not orthonormal:

$$\begin{aligned}(\alpha \wedge \beta \wedge \gamma)(u, v, w) := \alpha(u)\beta(v)\gamma(w) + \alpha(v)\beta(w)\gamma(u) + \alpha(w)\beta(u)\gamma(v) \\ - \alpha(u)\beta(w)\gamma(v) - \alpha(w)\beta(v)\gamma(u) - \alpha(v)\beta(u)\gamma(w)\end{aligned}$$



Interpretation (in $\mathbb{R}^3$)?

Volume of u,v,w gets scaled by volume of α,β,γ.

k-Form

- More generally, k-form is a *fully antisymmetric, multilinear* measurement of a k-vector.
- Typically think of this as a map from k vectors to a scalar:

$$\alpha : \underbrace{V \times \cdots V}_{k \text{ times}} \to \mathbb{R}$$

- *Multilinear* means "linear in each argument." *E.g.,* for a 2-form:

$$\begin{aligned} \alpha(au + bv, w) &= a\alpha(u, w) + b\alpha(v, w) \\ \alpha(u, av + bw) &= a\alpha(u, v) + b\alpha(u, w) \end{aligned} \qquad \forall u, v, w \in V, a, b \in \mathbb{R}$$

- *Fully antisymmetric* means exchanging two arguments reverses sign. E.g., 3-form:

$$\alpha(u, v, w) = \alpha(v, w, u) = \alpha(w, u, v) =$$
$$-\alpha(u, w, v) = -\alpha(w, v, u) = -\alpha(v, u, w)$$

k-Forms and Determinants

- For 3-forms, saw that we could express application of a *k*-form via a *determinant*
- Captures the fact that *k*-forms are measurements of *volume*
- How does this work more generally?
- **Conceptually**: "project" onto *k*-dimensional space and measure volume there

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(u_1, \ldots, u_k) := \begin{vmatrix} \alpha_1(u_1) & \cdots & \alpha_1(u_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(u_1) & \cdots & \alpha_k(u_k) \end{vmatrix}$$

k=1:

$$\det\left(\begin{bmatrix} \alpha_1(u_1) \end{bmatrix}\right) = \alpha_1(u_1)$$

(Determinant of a 1x1 matrix is just the one entry of that matrix!)

k=2:

$$\det\left(\begin{bmatrix} \alpha_1(u_1) & \alpha_1(u_2) \\ \alpha_2(u_1) & \alpha_2(u_2) \end{bmatrix}\right)$$

$$= \alpha_1(u_1)\alpha_2(u_2) - \alpha_1(u_2)\alpha_2(u_1)$$

A Note on Notation

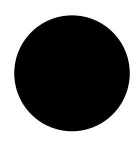
- A k-form effectively measures a k-vector
- For whatever reason, *nobody* writes the argument k-vector using a wedge
- Instead, the convention is to write a list of vectors:

$$(\alpha \wedge \beta)(u \wedge v) \qquad (\alpha \wedge \beta)(u, v)$$

(At least type can be inferred from notation: if there's a wedge, it's a k-form!)

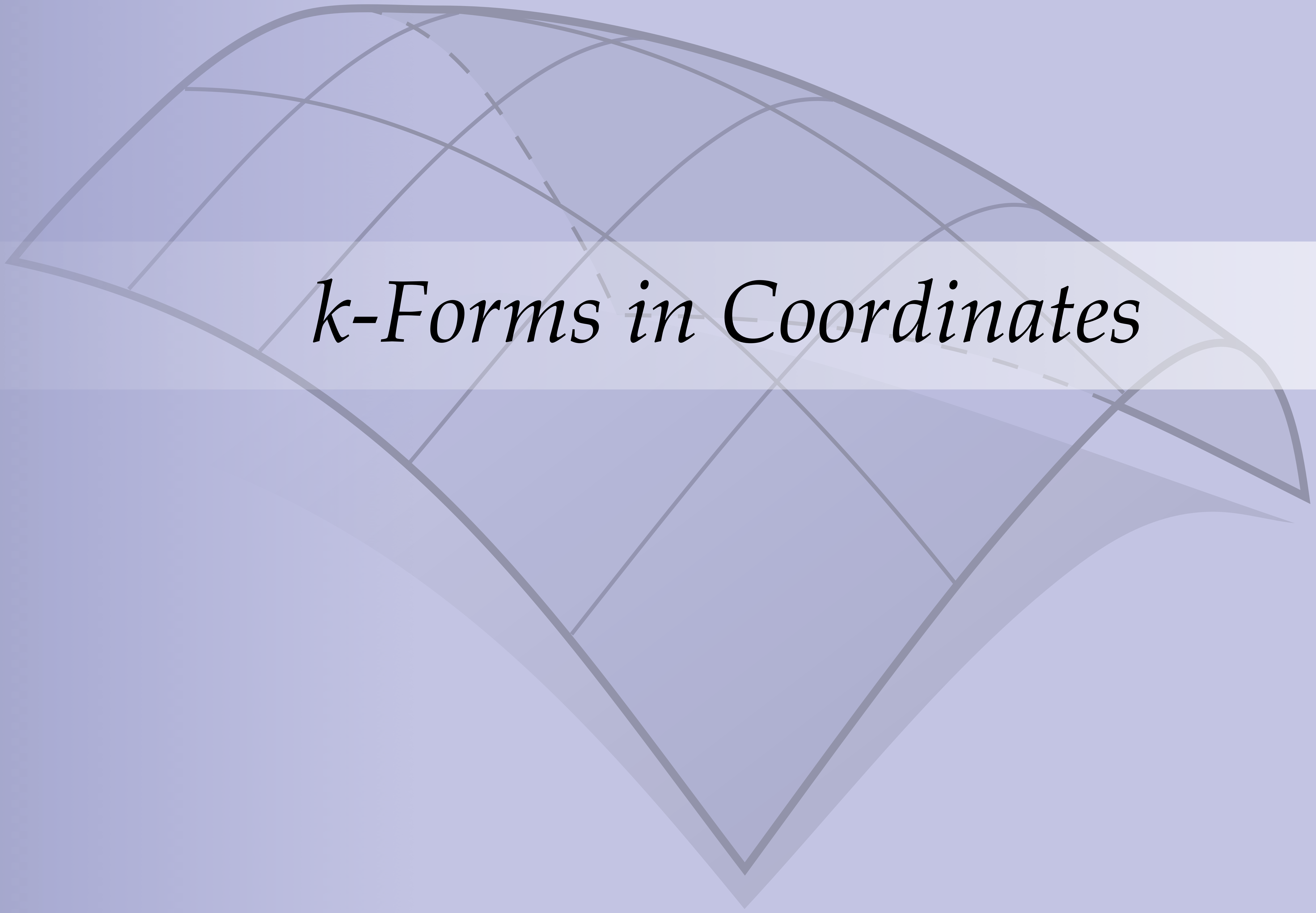
0-Forms

- What's a 0-form?
  - In general, a *k*-form takes *k* vectors and produces a scalar
  - So a 0-form must take 0 vectors and produce a scalar
  - I.e., ***a 0-form is a scalar!***
- Basically looks like this:



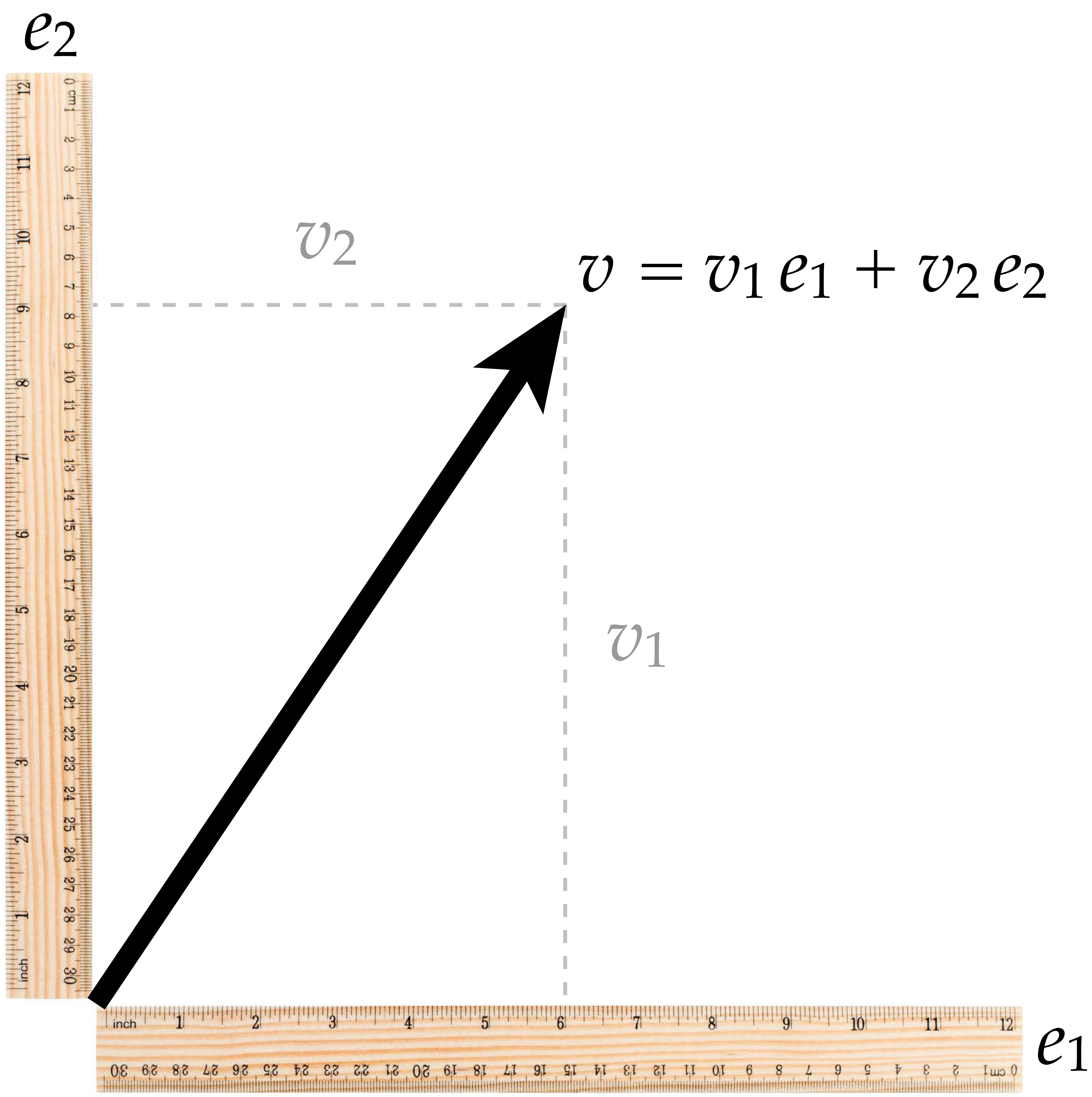
Note: still has *magnitude*, even though it has only one possible "direction."

k-Forms in Coordinates



Measurement in Coordinates

- Idea of measurement becomes very concrete once you have a coordinate system
- *E.g.*, for a vector:
  - just measure along each coordinate axis
  - use these measurements to take a weighted linear combination of bases



Let's see how this works for *k*-forms…

Dual Basis

In an n-dimensional vector space V, can express vectors v in a basis $e_1, \ldots, e_n$:

$$v = v^1 e_1 + \cdots + v^n e_n$$

The scalar values v^i are the *coordinates* of v.

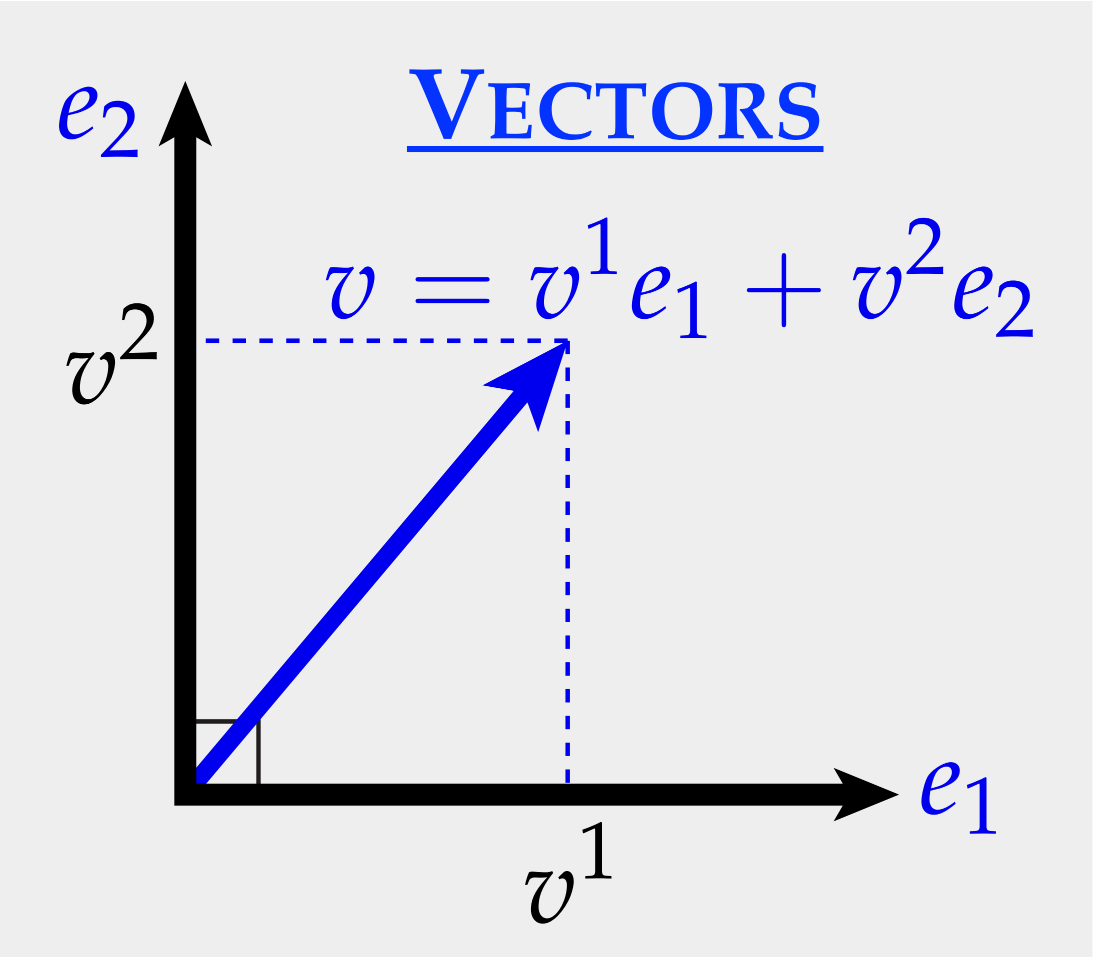
We can also write *covectors* α in a so-called *dual basis* $e^1, \ldots, e^n$:

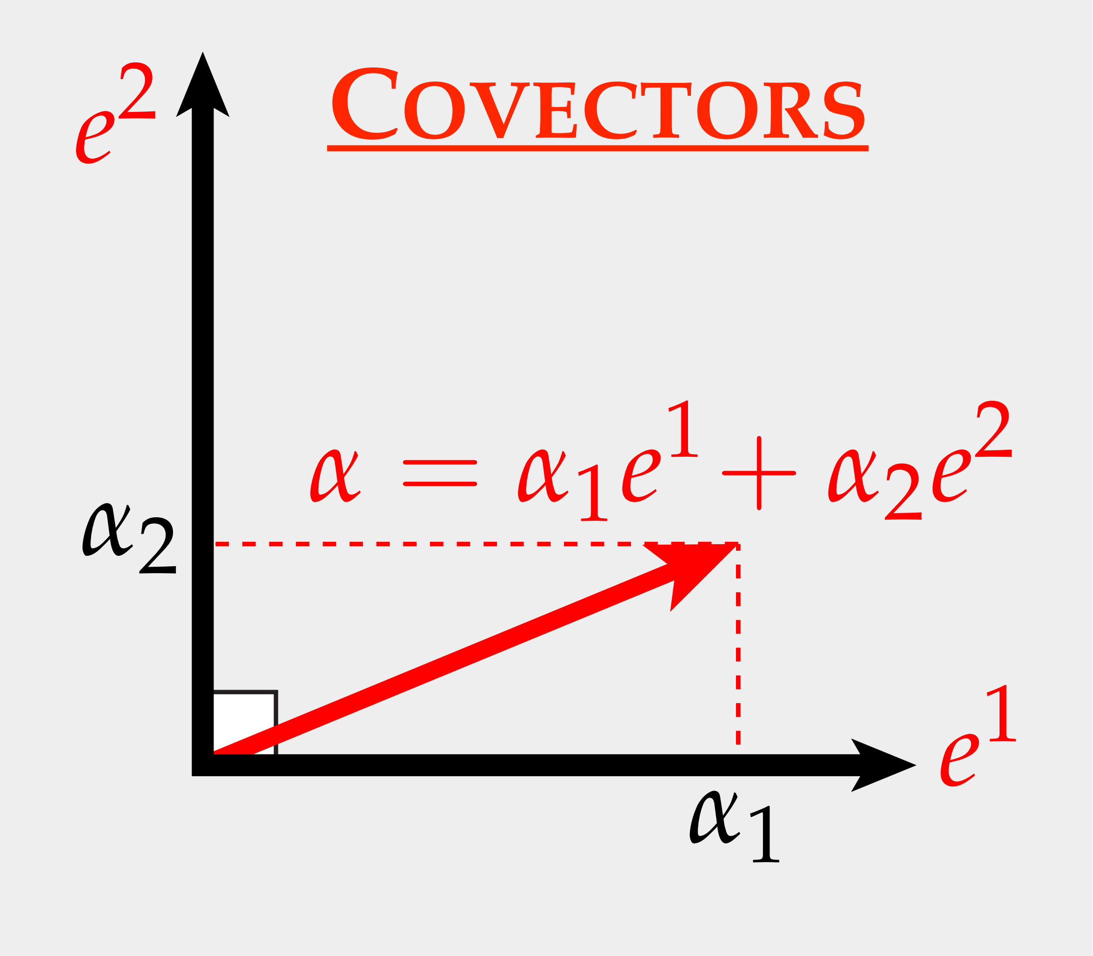
$$\alpha = \alpha_1 e^1 + \cdots + \alpha_n e^n$$

These bases have a special relationship, namely:

$$e^i(e_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

(**Q:** What does e^i mean, geometrically?)



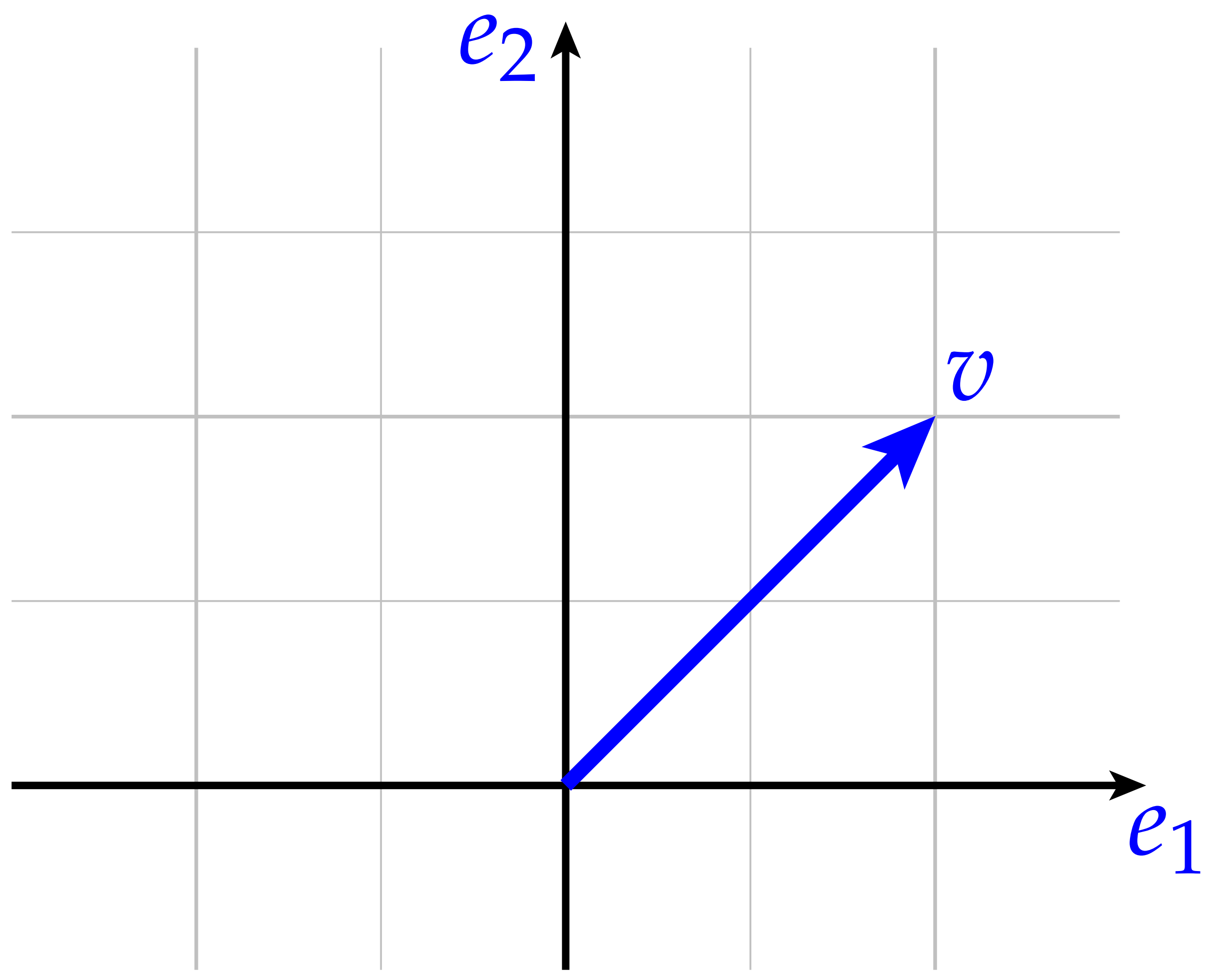


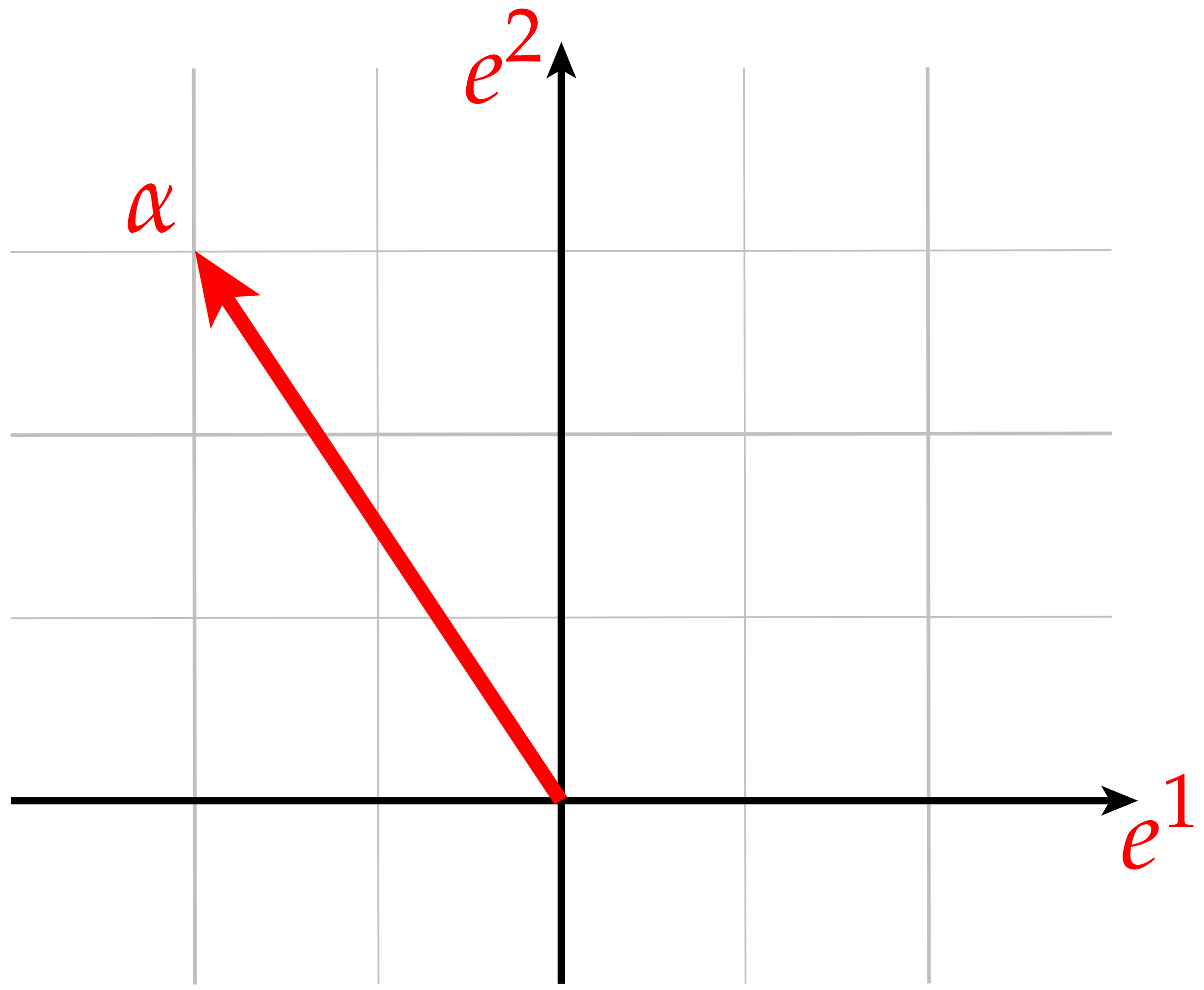
1-form — Example in Coordinates

- Some simple calculations in coordinates help to solidify understanding of k-forms.
- Let's start with a vector v and a 1-form α in the plane:

$$v = 2e_1 + 2e_2$$

$$\alpha = -2e^1 + 3e^2$$





$$\begin{aligned}
\alpha(v) &= (-2e^1 + 3e^2)(2e_1 + 2e_2) \\
&= -2e^1(2e_1 + 2e_2) + 3e^2(2e_1 + 2e_2) \\
&= -4\cancelto{1}{e^1(e_1)} - 4\cancelto{0}{e^1(e_2)} + 6\cancelto{0}{e^2(e_1)} + 6\cancelto{1}{e^2(e_2)} \\
&= -4 + 6 \quad \text{(Just like a \textit{dot product!})} \\
&= 2.
\end{aligned}$$

2-form—Example in Coordinates

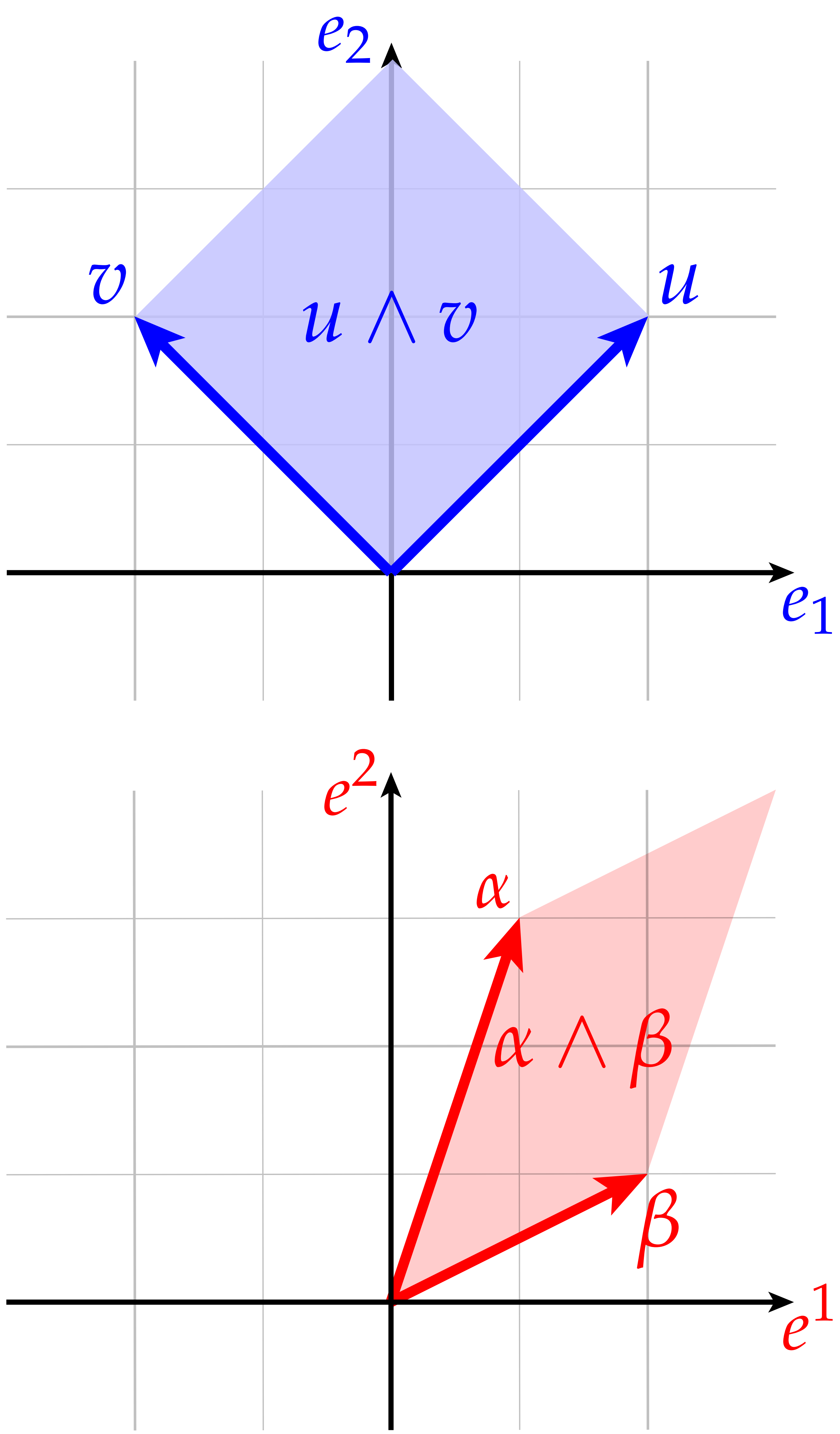
Consider the following vectors and covectors:

$$u = 2e_1 + 2e_2 \qquad \alpha = e^1 + 3e^2$$
$$v = -2e_1 + 2e_2 \qquad \beta = 2e^1 + e^2$$

We then have:

$$(\alpha \wedge \beta)(u,v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$
$$\alpha(u) = 1 \cdot 2 + 3 \cdot 2 = 8$$
$$\beta(v) = \cdots = -2$$
$$\alpha(v) = \cdots = 4$$
$$\beta(u) = \cdots = 6$$
$$\Rightarrow (\alpha \wedge \beta)(u,v) = 8 \cdot (-2) - 4 \cdot 6 = -40.$$

Q: What does this value mean, geometrically? Why is it *negative*?

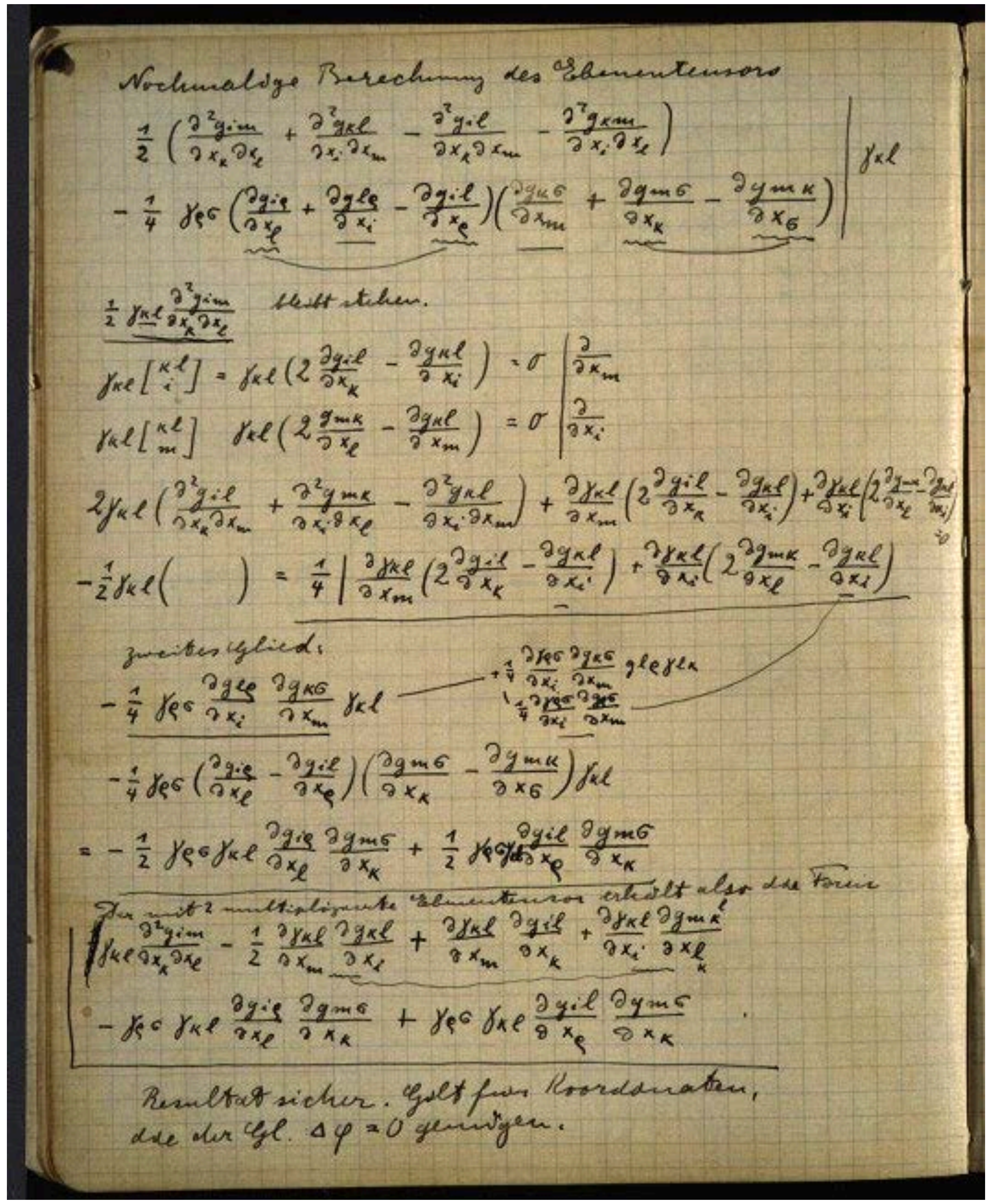


Einstein Summation Notation

Why are some indices *“up”* and others *“down”*?

Bemerkung zur Vereinfachung der Schreibweise der Ausdrücke.

Ein Blick auf die Gleichungen dieses Paragraphen zeigt, daß über Indizes, die zweimal unter einem Summenzeichen auftreten [z. B. der Index ν in (5)], stets summiert wird, und zwar *nur* über zweimal auftretende Indizes. Es ist deshalb möglich, ohne die Klarheit zu beeinträchtigen, die Summenzeichen wegzulassen. Dafür führen wir die Vorschrift ein: Tritt ein Index in einem Term eines Ausdruckes zweimal auf, so ist über ihn stets zu summieren, wenn nicht ausdrücklich das Gegenteil bemerkt ist.

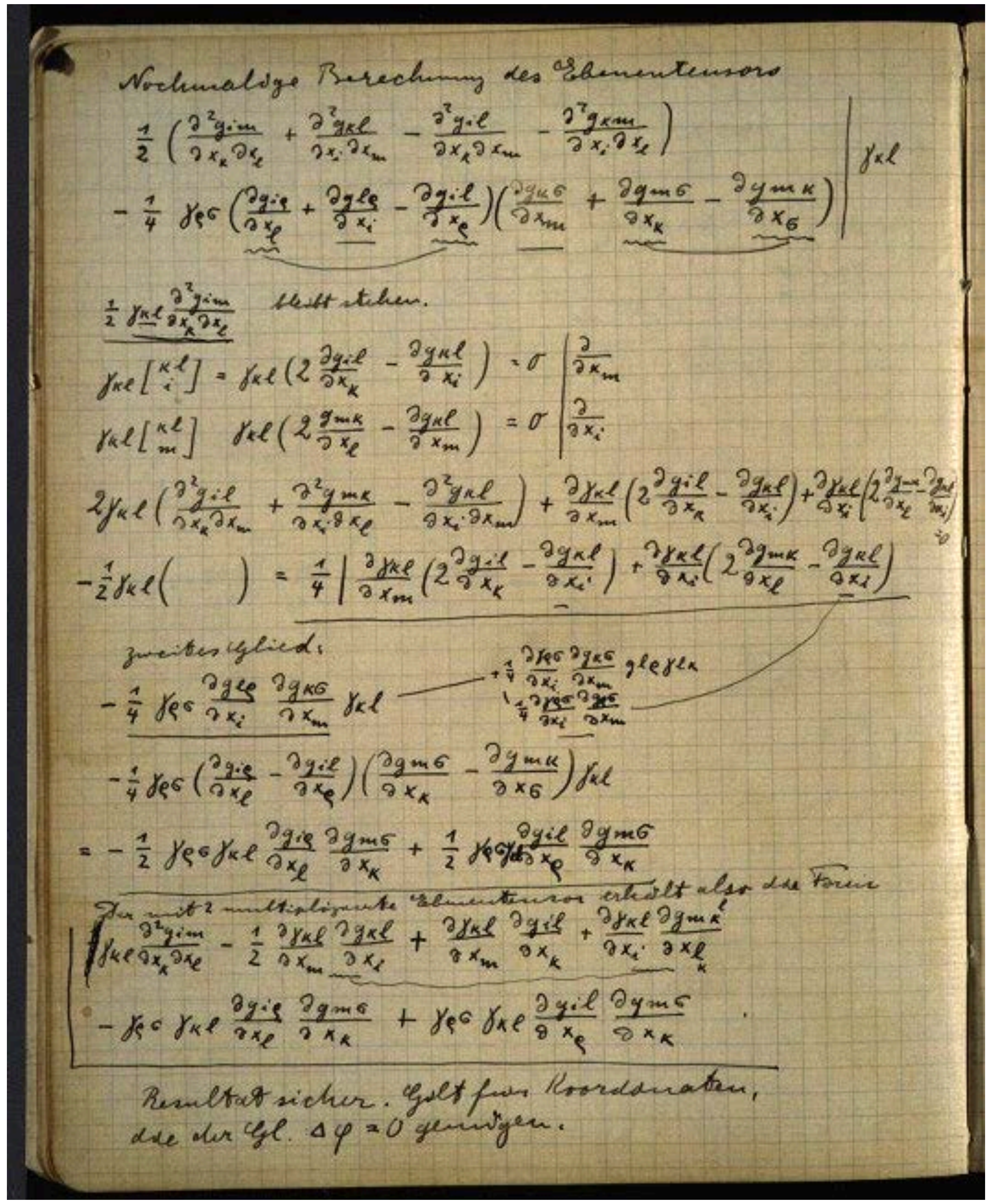


— Einstein, “Die Grundlage der allgemeinen Relativitätstheorie” (1916)

Einstein Summation Notation

Key idea: sum over repeated indices.

$$x^i y_i := \sum_{i=1}^{n} x^i y_i$$



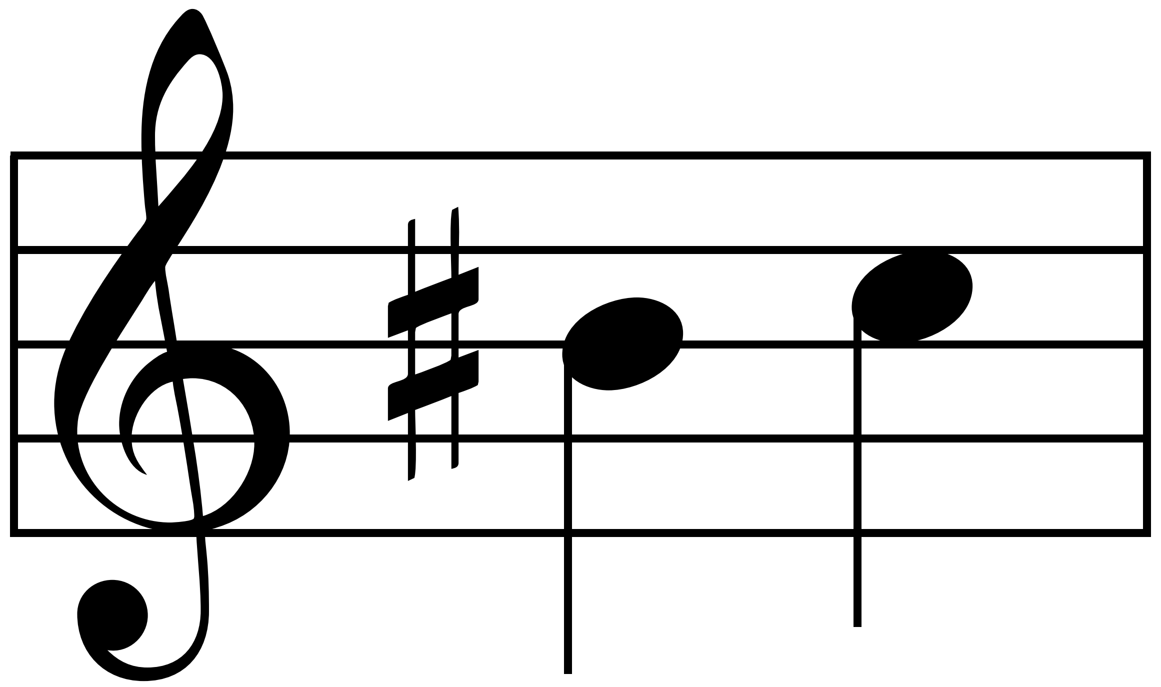
NOTE ON A SIMPLIFIED WAY OF WRITING EXPRESSIONS

A look at the equations of this paragraph show that there is always a summation over indices which occur twice, and only for twice-repeated indices. It is therefore possible, without detracting from clarity, to omit the sum sign. For this we introduce a rule: if an index in an expression appears twice, then a sum is implicitly taken over this index, unless specifically noted to the contrary.

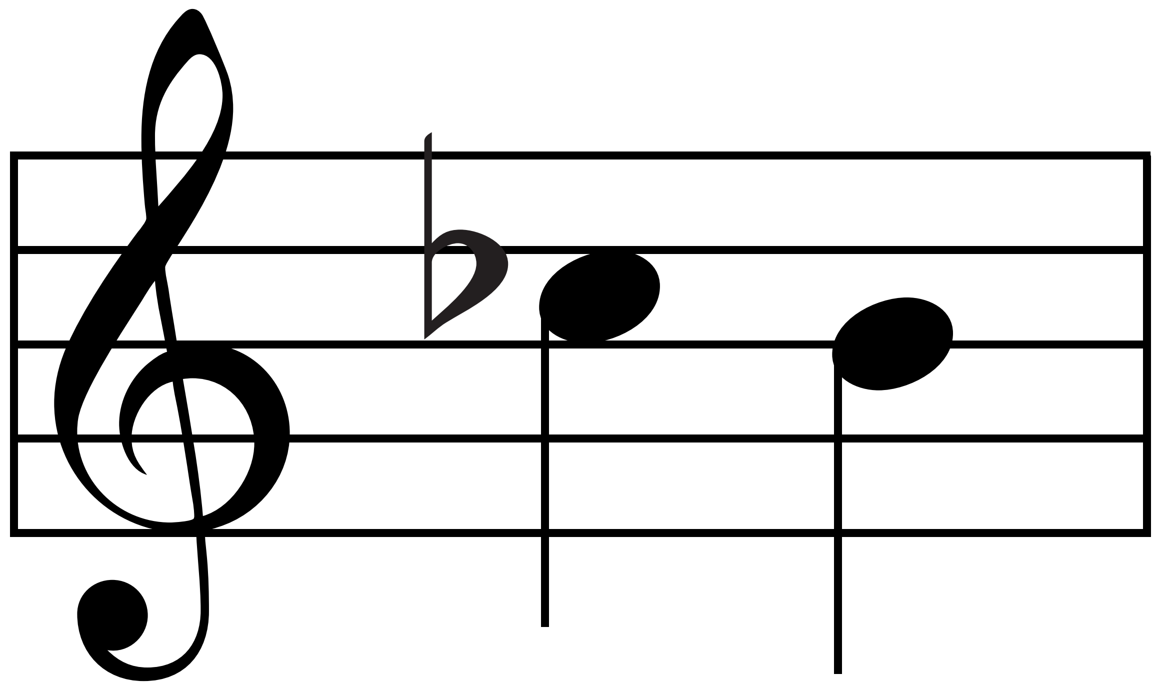
— *Einstein, "The Basis of General Relativity" (1916)*

Sharp and Flat in Coordinates

Q: What do sharp and flat do on a musical staff?



(raise pitch)



(lower pitch)

Likewise, sharp and flat *raise* and *lower* indices of coefficients for 1-forms/vectors.

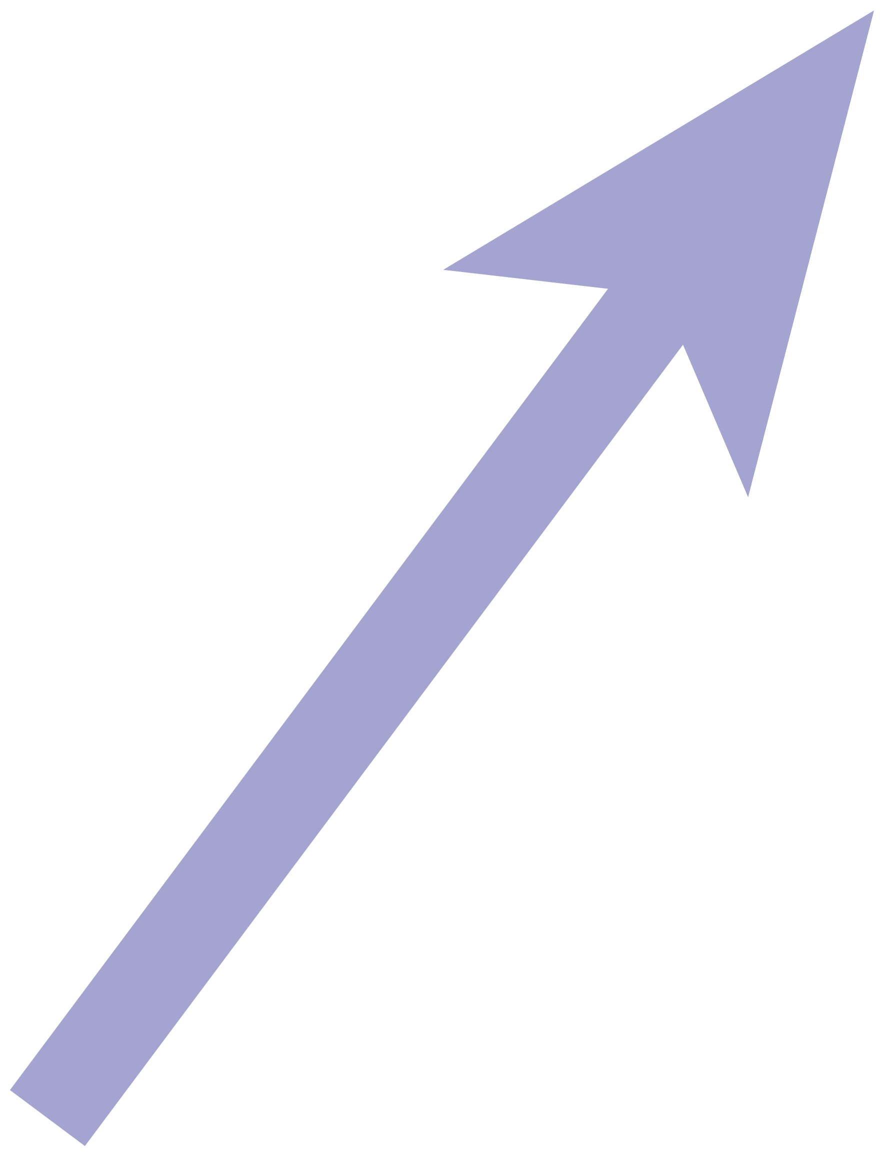
Suppose for instance that $u^\sharp = \alpha$ and $\alpha^\flat = u$. Then

$$\alpha = \alpha_1 e^1 + \cdots + \alpha_n e^n \quad \underset{\overset{\flat}{\Longleftarrow}}{\overset{\sharp}{\Longrightarrow}} \quad u = u^1 e_1 + \cdots + u^n e_n$$

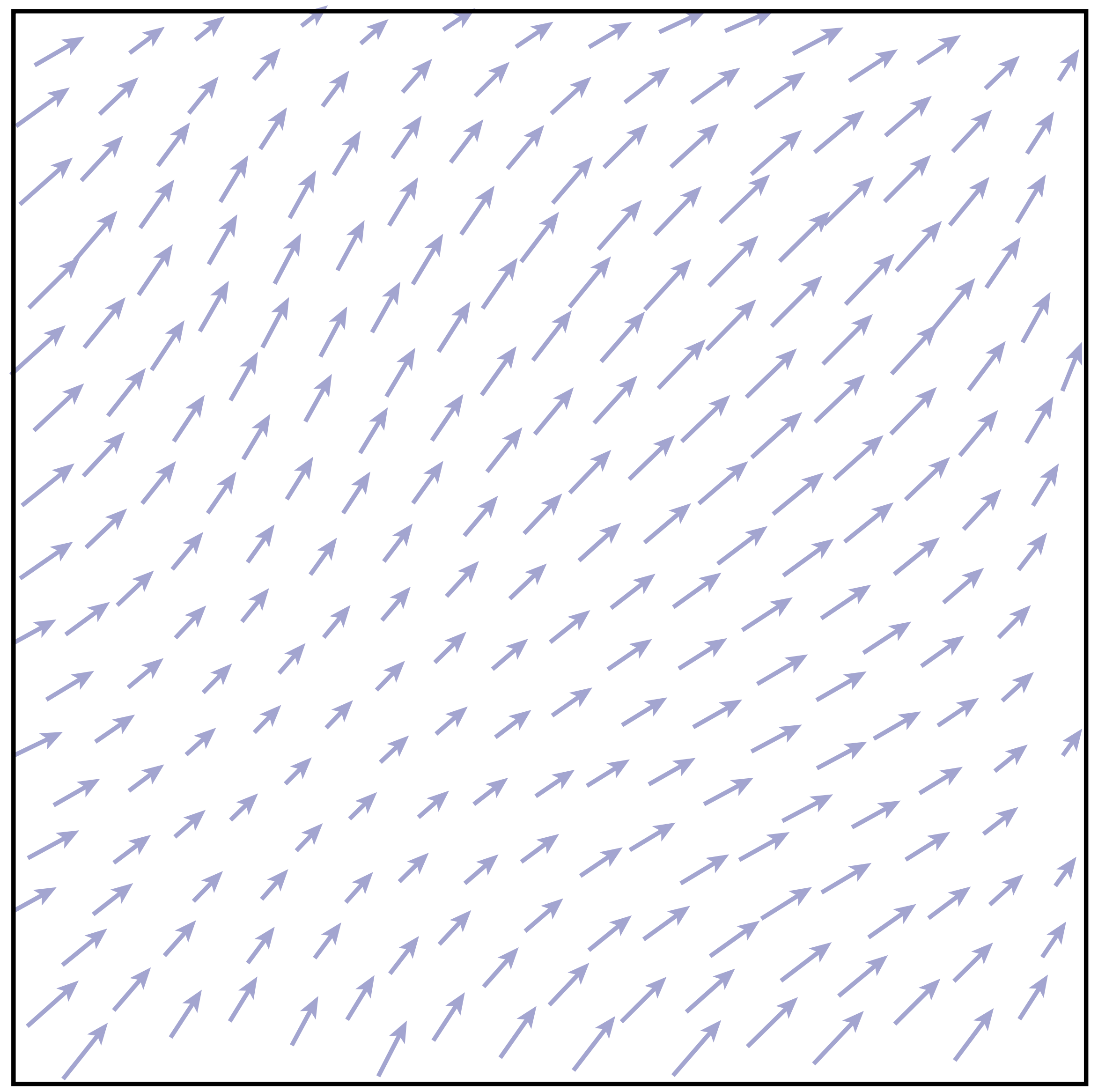
(Sometimes called the *musical isomorphisms.)*

Coming Up: Differential Forms

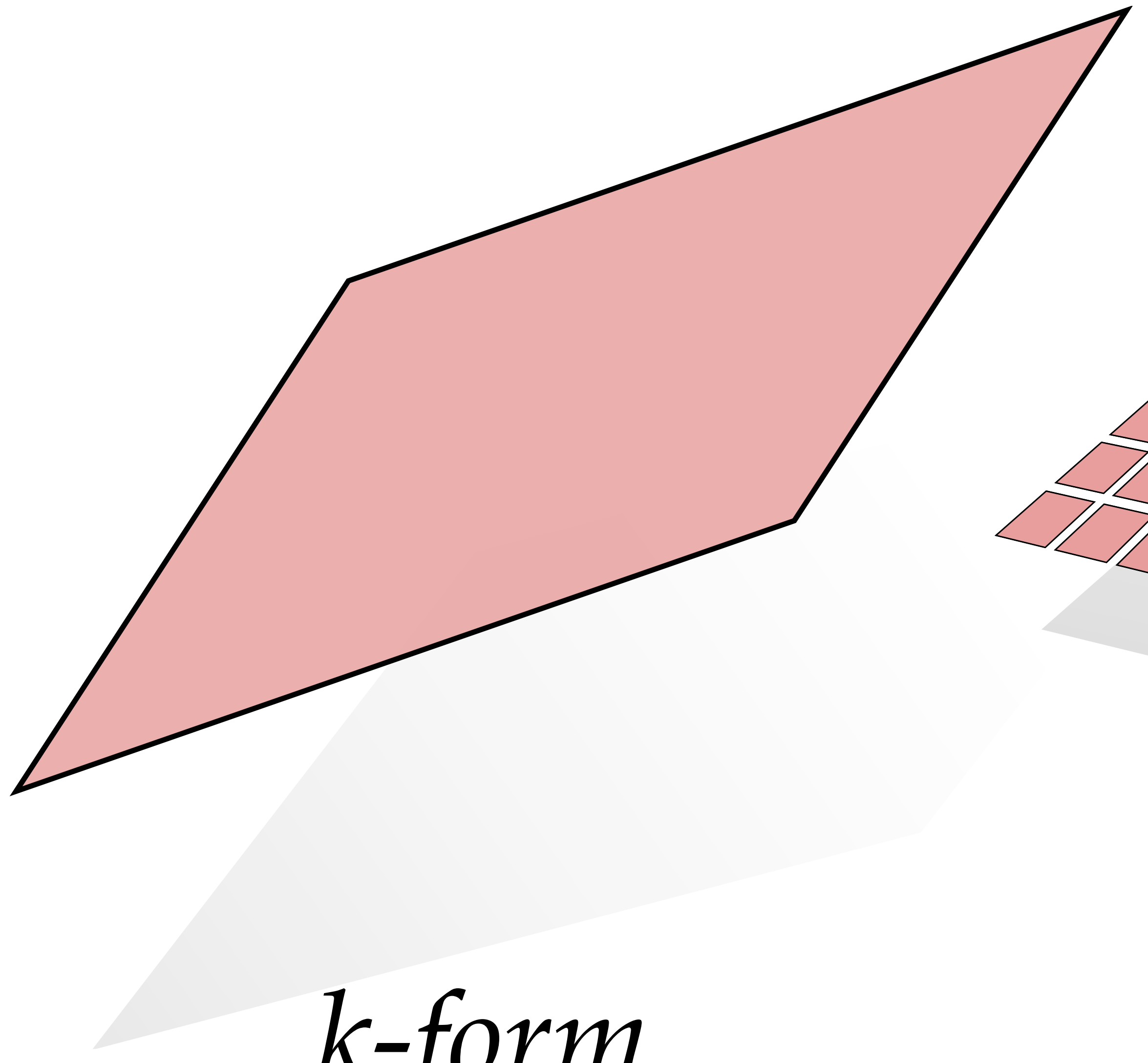
- Often useful to attach a vector to each point to obtain a *vector field* (fluid flow, gradient, …)
- **Next time** we will likewise attach a *k*-form to each point to obtain a *differential k-form*



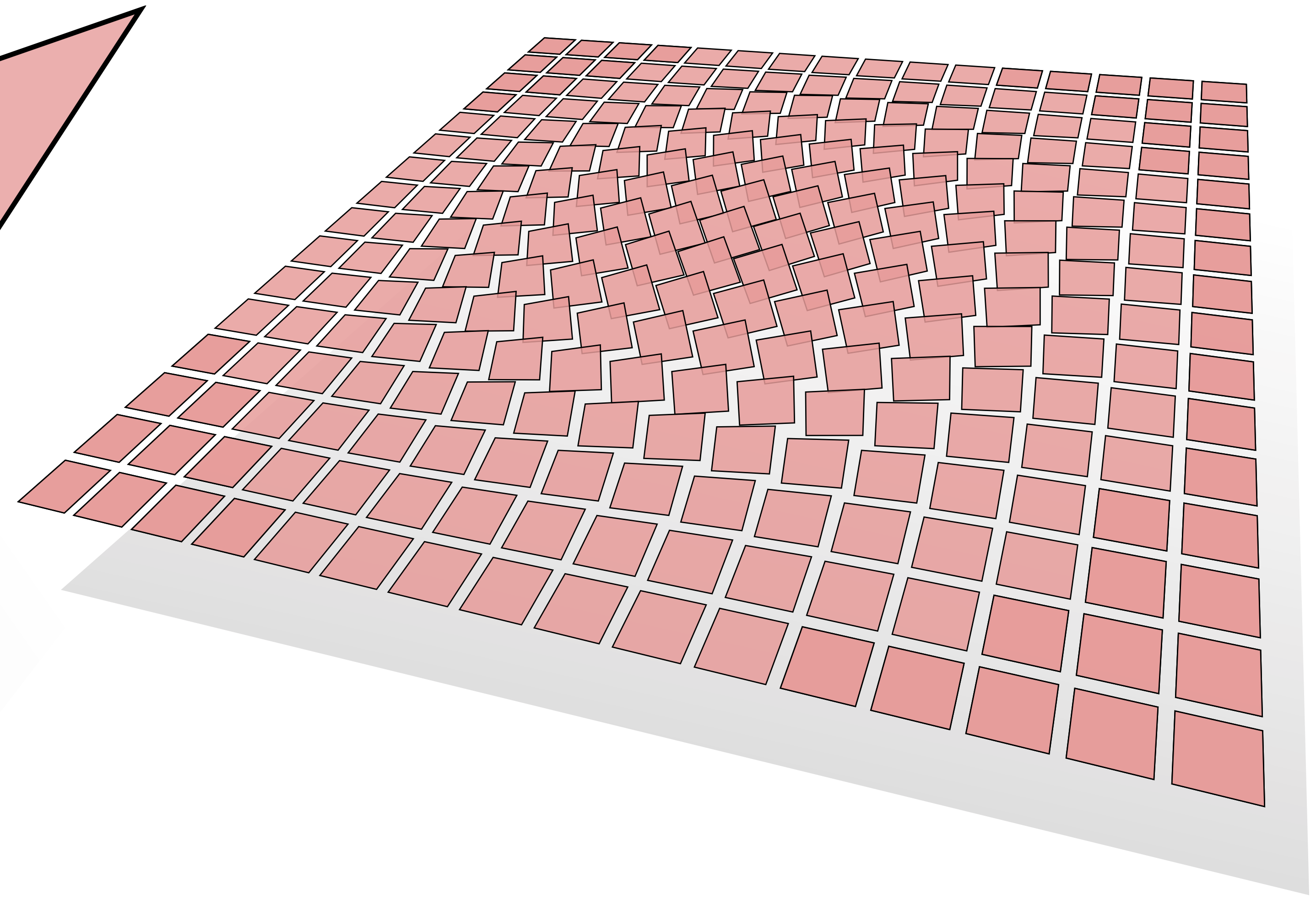
vector



vector field

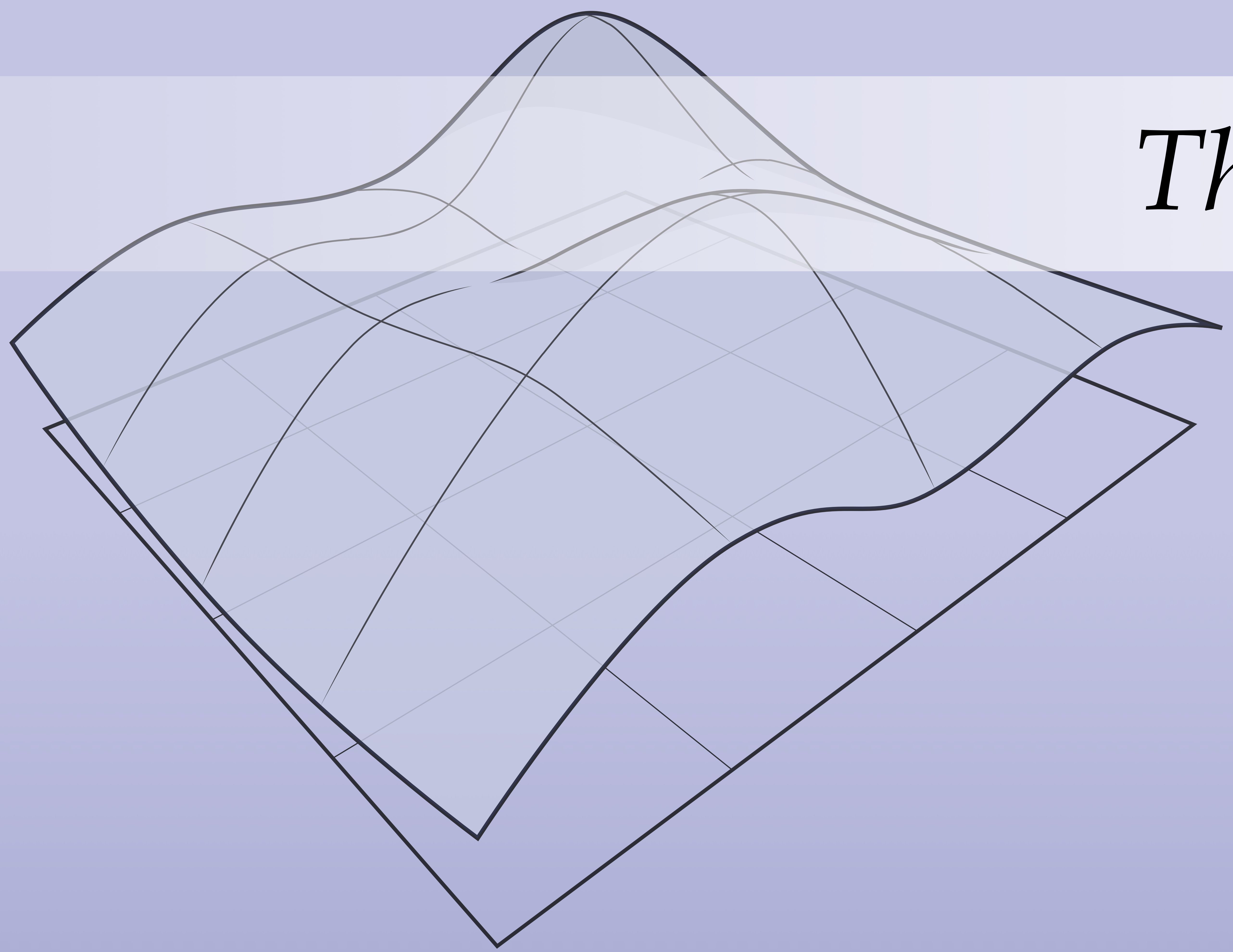


k-form



differential k-form

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION