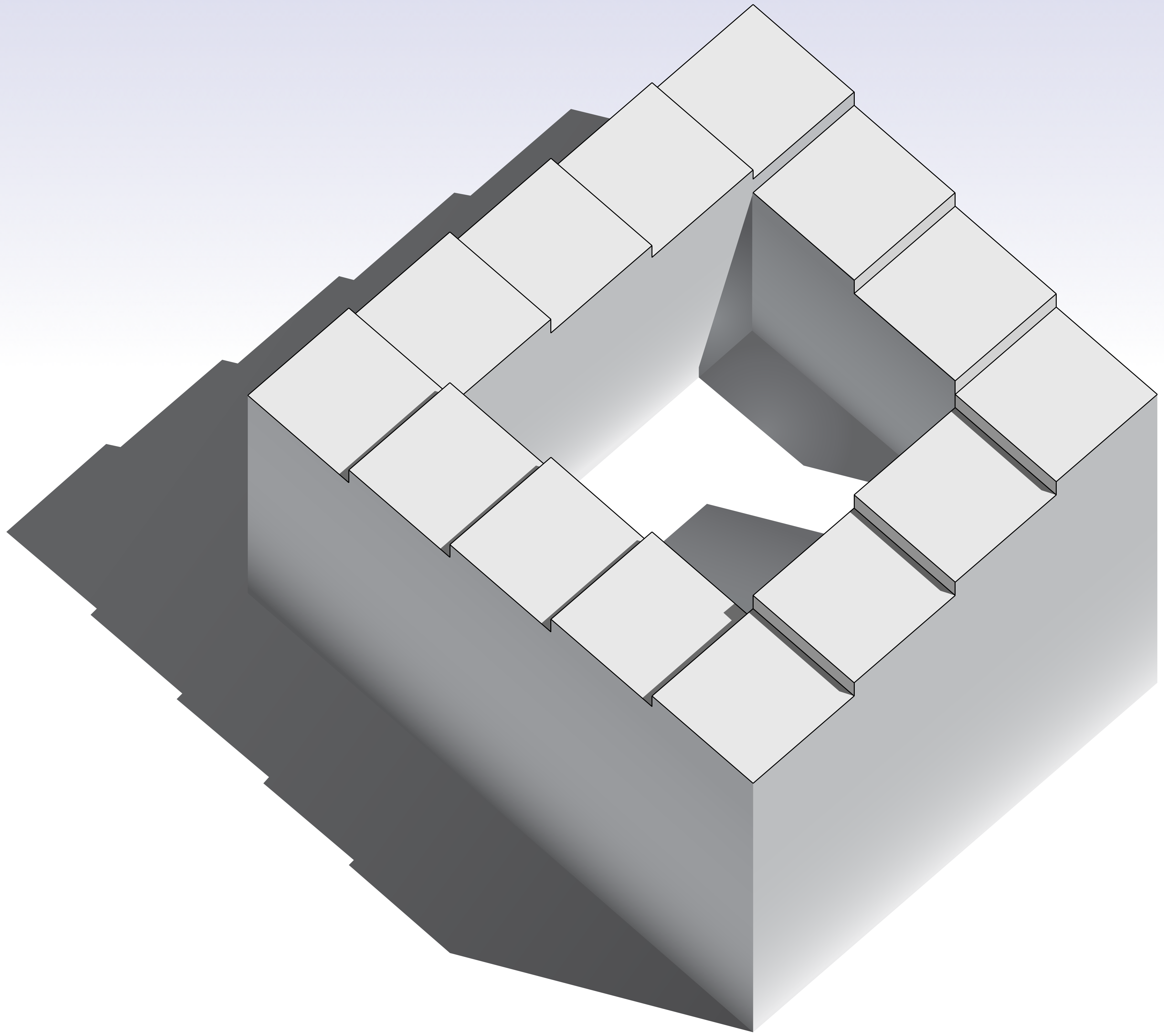
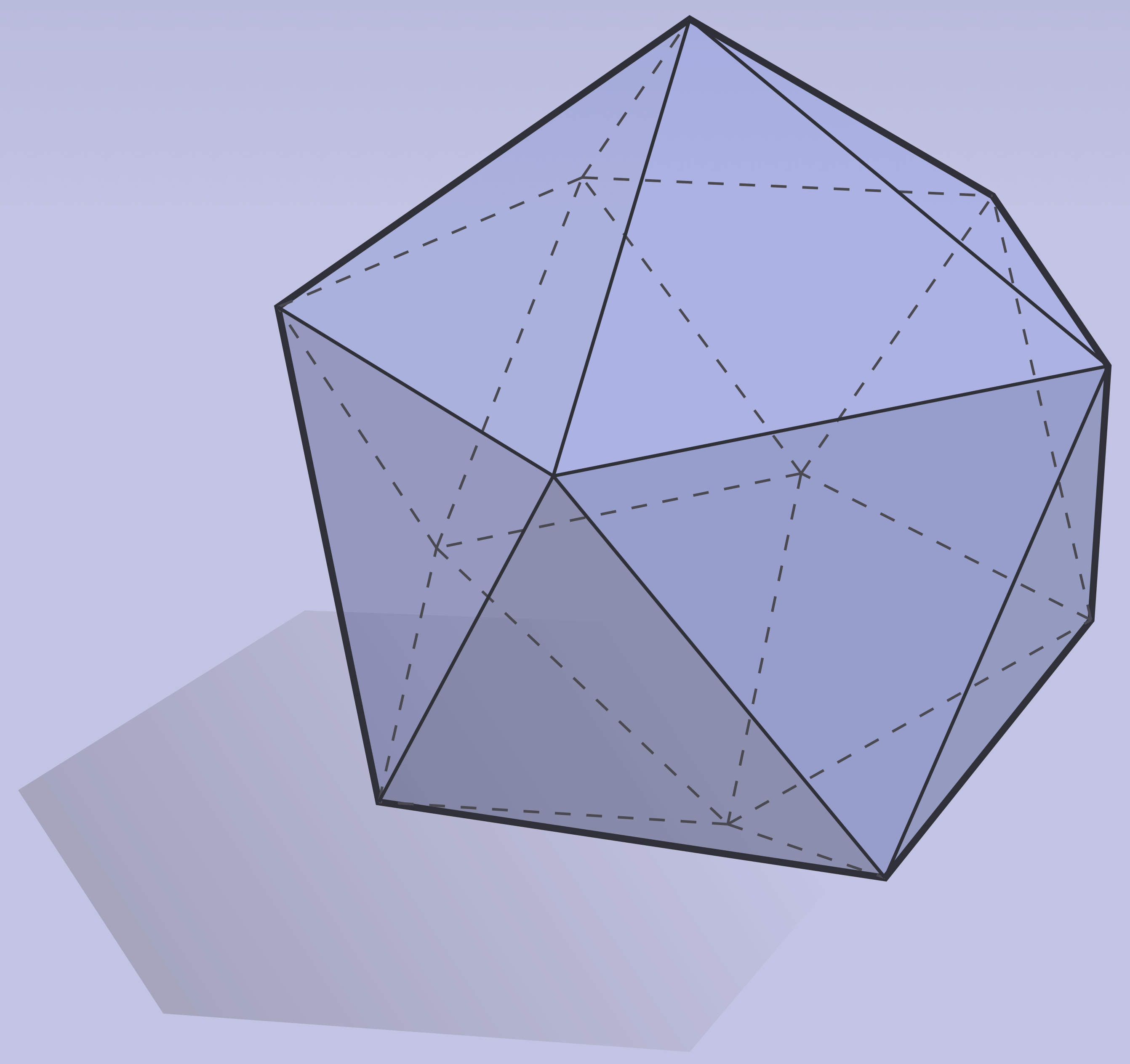


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 7:
Integration



Discrete Differential Geometry:
An Applied Introduction

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Integration and Differentiation

- Two big ideas in calculus:
  - **differentiation**
  - ***integration***
  - linked by *fundamental theorem of calculus*

$$\boxed{\int_a^b f'\,dx = f(b) - f(a)}$$

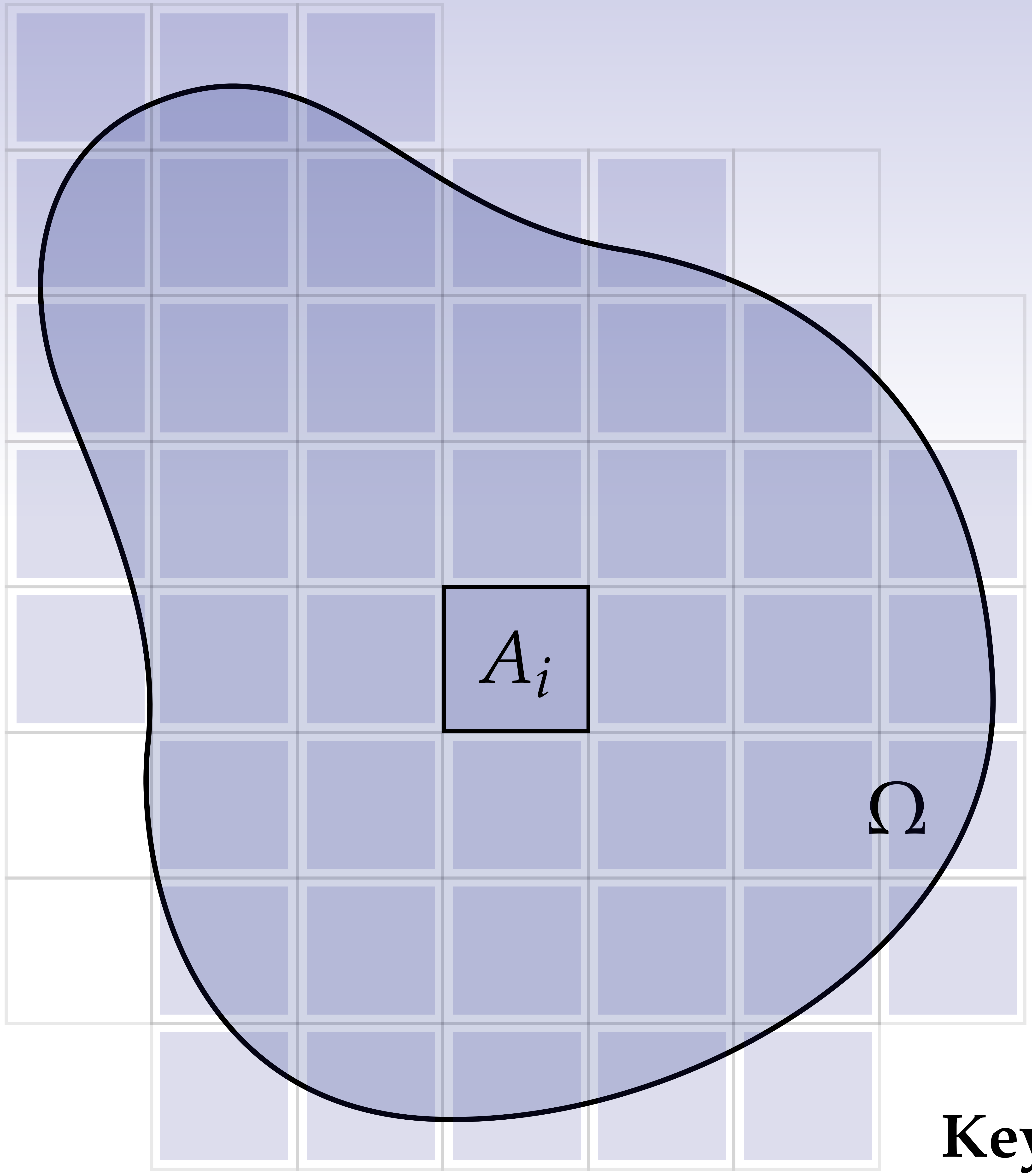
- Exterior calculus generalizes these ideas
  - **differentiation of *k*-forms** (exterior derivative)
  - **integration of *k*-forms** (measure volume)
  - linked by *Stokes' theorem*

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

- **Goal:** integrate differential forms over meshes to get *discrete exterior calculus (DEC)*

Integration of Differential k-Forms

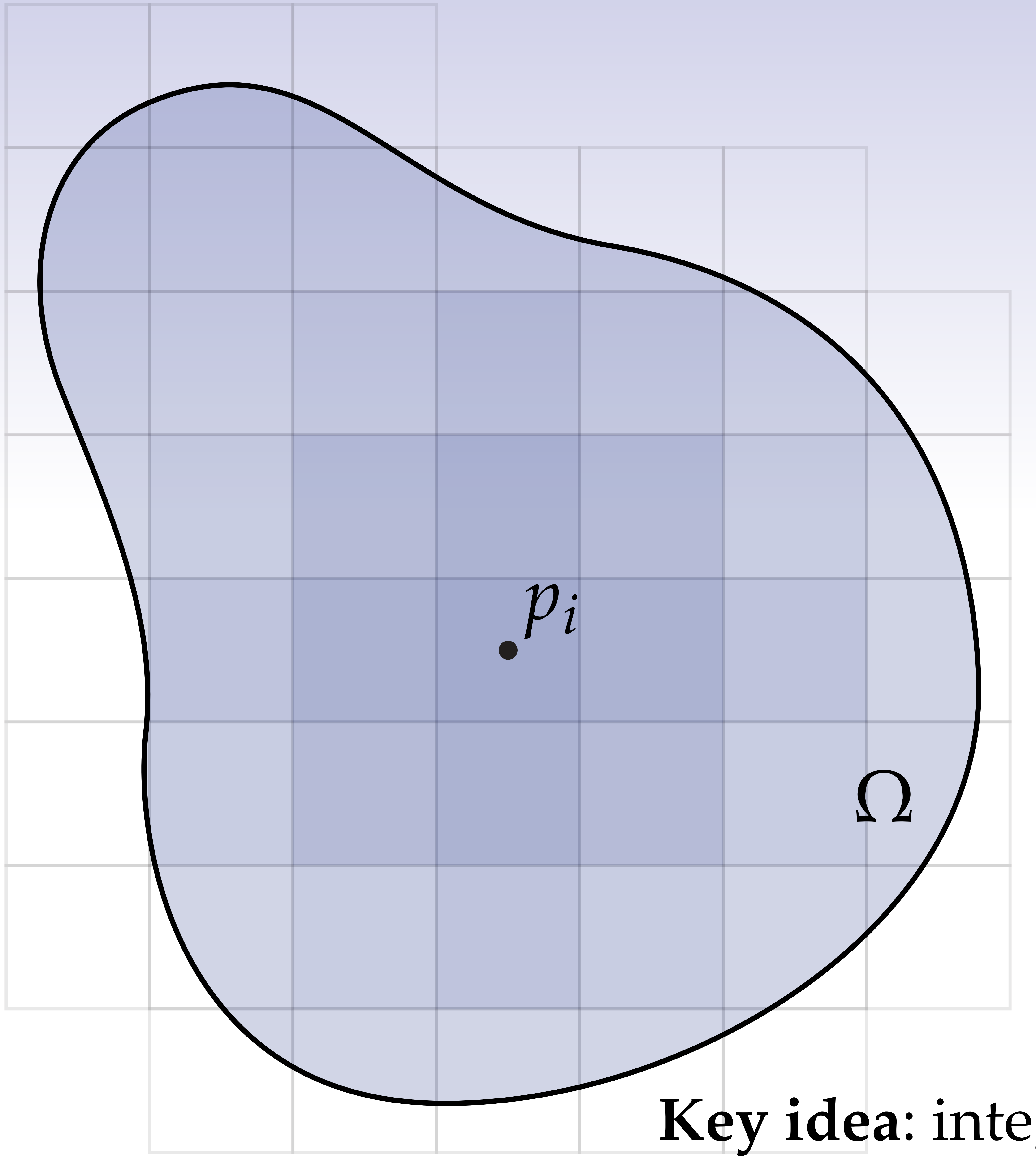
Review—Integration of Area



$$\sum_i A_i \implies \int_\Omega dA$$

Key idea: sums converge to integrals as we refine.

Review—Integration of Scalar Functions

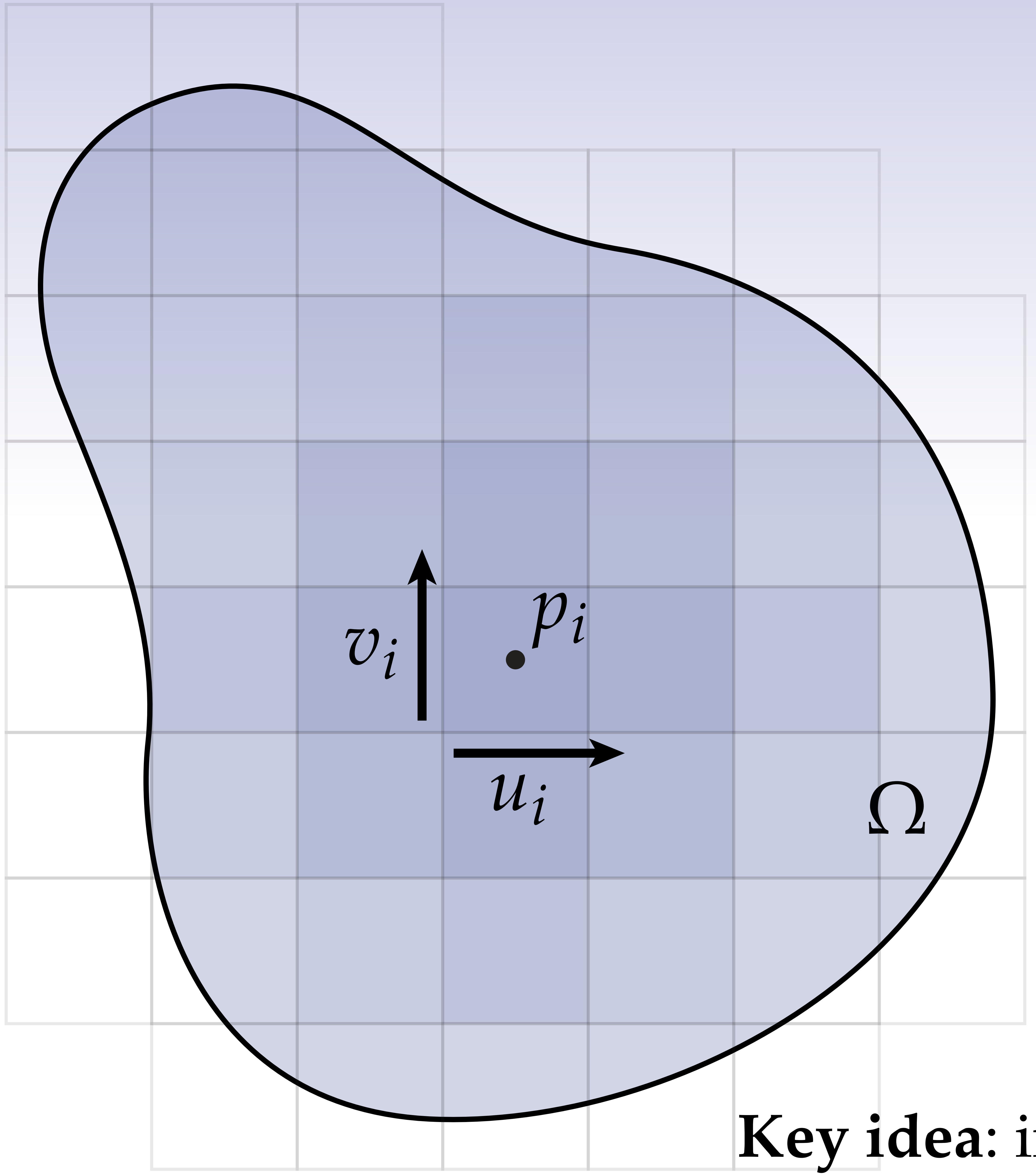


$$\phi : \Omega \to \mathbb{R}$$

$$\sum_i A_i \phi(p_i) \implies \int_\Omega \phi \, dA$$

Key idea: integrals of functions are weighted sums of area.

Integration of a 2-Form



ω — differential 2-form on Ω

$$\sum_i \omega_{p_i}(u_i, v_i) \quad \Longrightarrow \quad \int_\Omega \omega$$

Key idea: integration *always* involves differential forms!

Integration of Differential 2-forms — Example

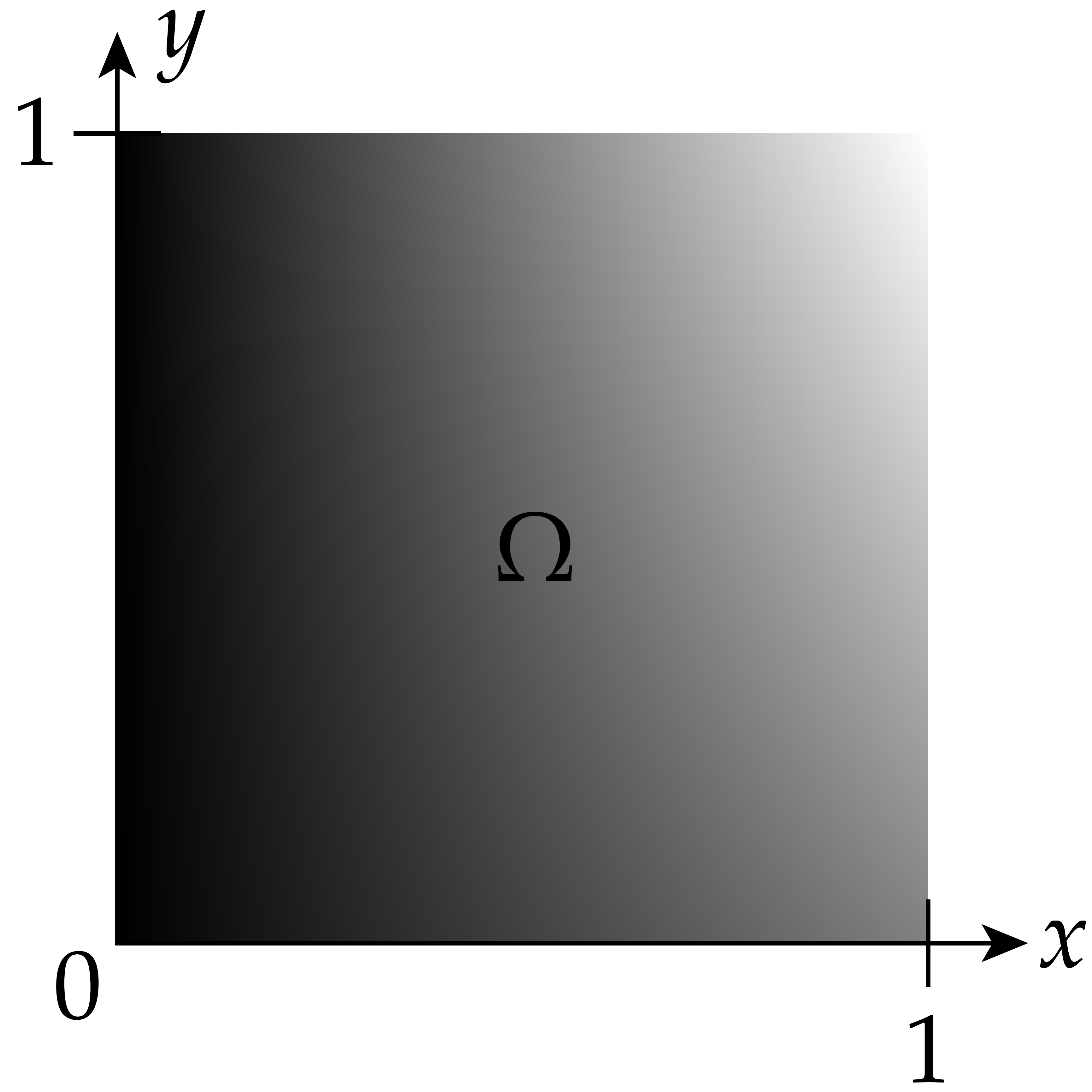
- Consider a differential 2-form on the unit square in the plane:

$$\omega := (x + xy)dx \wedge dy$$

$$\int_\Omega \omega = \int_\Omega (x + xy)dx \wedge dy$$

$$= \int_0^1 \int_0^1 (x + xy)dx \wedge dy$$

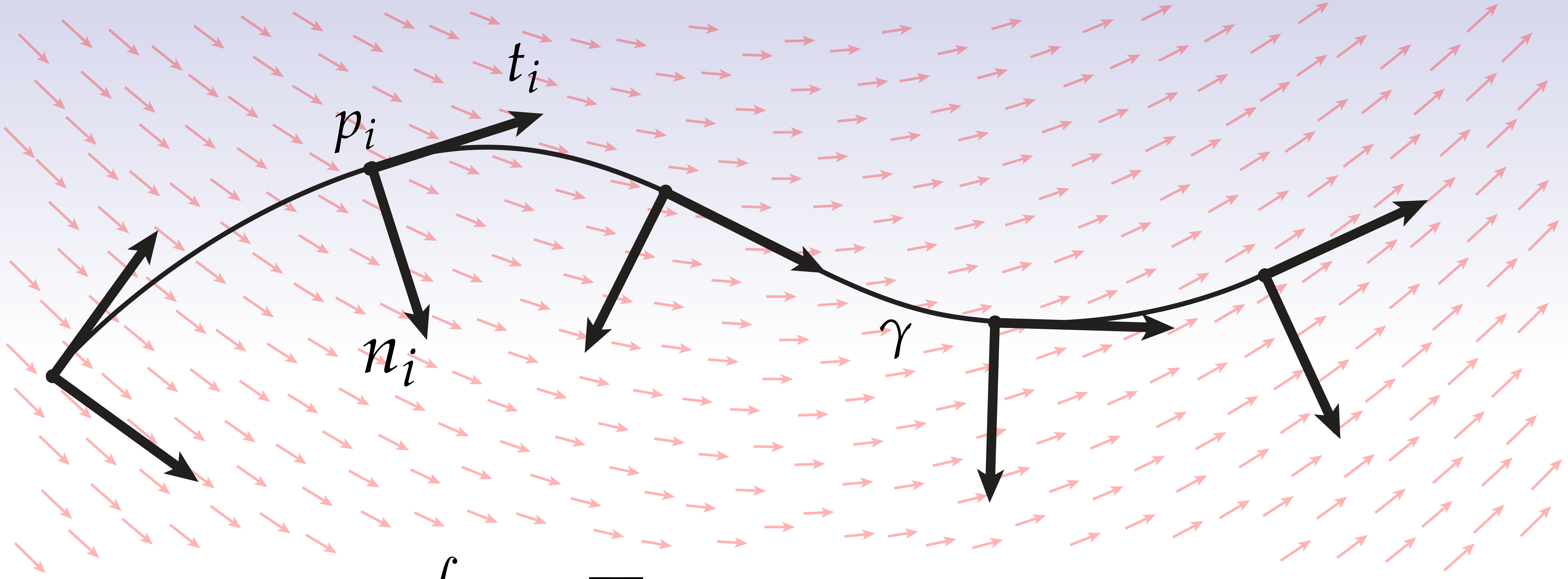
$$= \cdots = \frac{3}{4}$$



- In this case, no different from usual "double integration" of a scalar function.

Integration on Curves

α — differential 1-form on $\mathbb{R}^2$



$$\int_\gamma \alpha \approx \sum_i \alpha_{p_i}(t_i)$$

$$\int_\gamma \star\alpha \approx \sum_i \star\alpha_{p_i}(t_i) = \sum_i \alpha_{p_i}(n_i)$$

Integration on Curves—Example

- Consider for instance integrating a constant 1-form over the unit circle:

$$\int_{S^1} \alpha = \int_0^{2\pi} \alpha_{\gamma(s)}(T(s))\, ds =$$

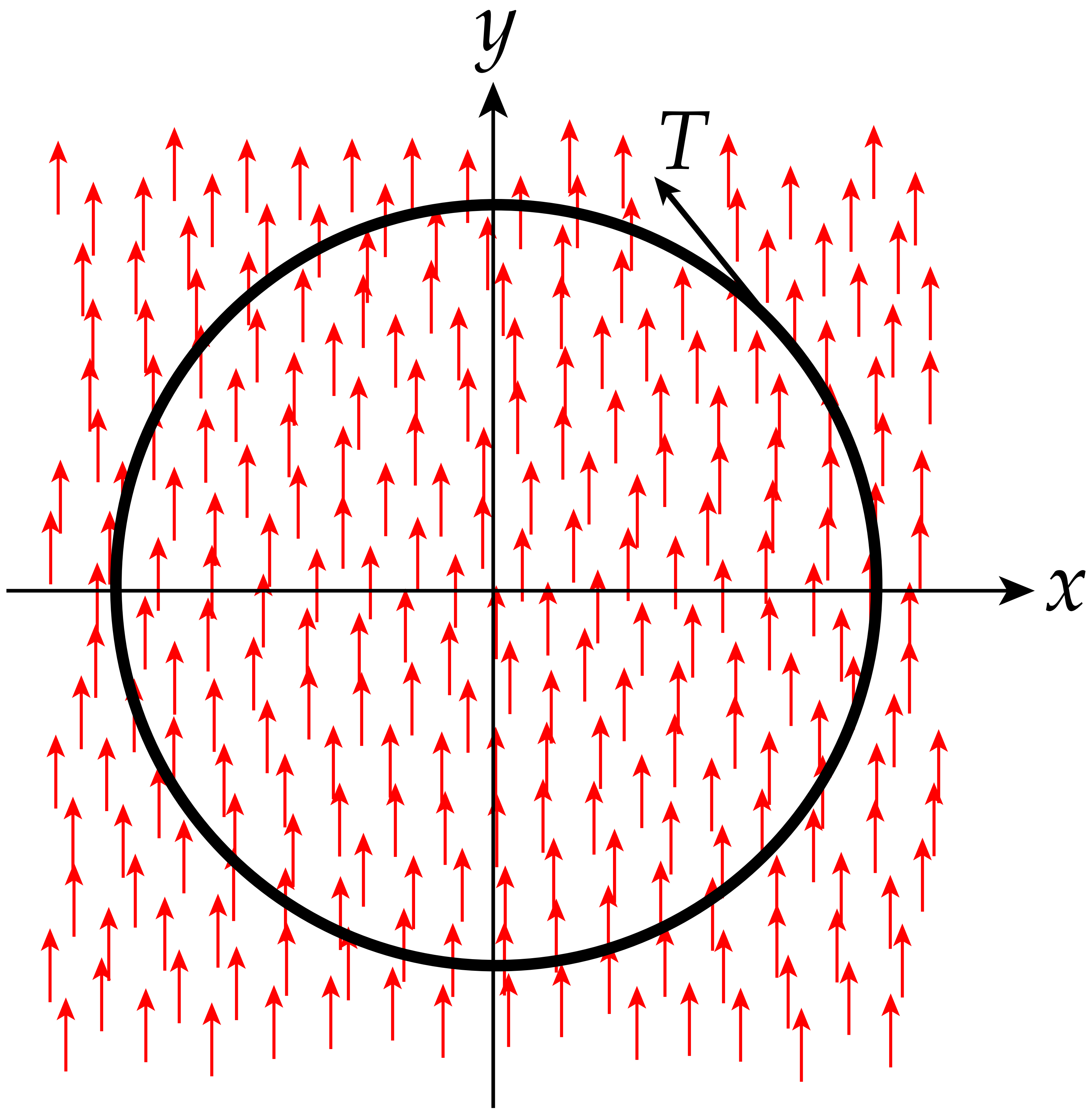
$$\int_0^{2\pi} \alpha_\gamma(s)(-\sin(s)\tfrac{\partial}{\partial x} + \cos(s)\tfrac{\partial}{\partial y})ds =$$

$$\int_0^{2\pi} dy(-\sin(s)\tfrac{\partial}{\partial x} + \cos(s)\tfrac{\partial}{\partial y})ds =$$

$$\int_0^{2\pi} \cos(s)\, ds = 0$$

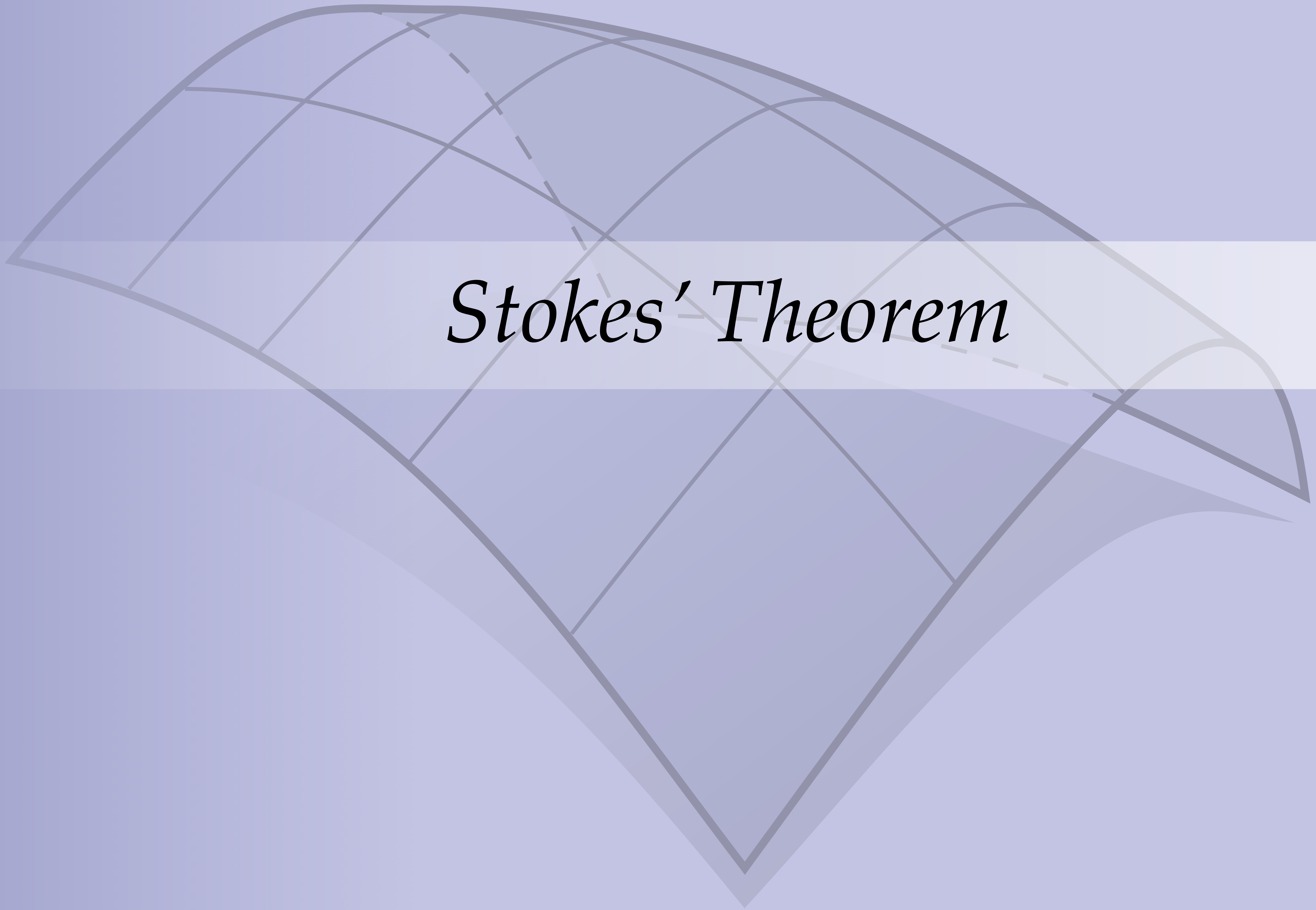
(Why does this result make sense geometrically?)

$\alpha = dy$

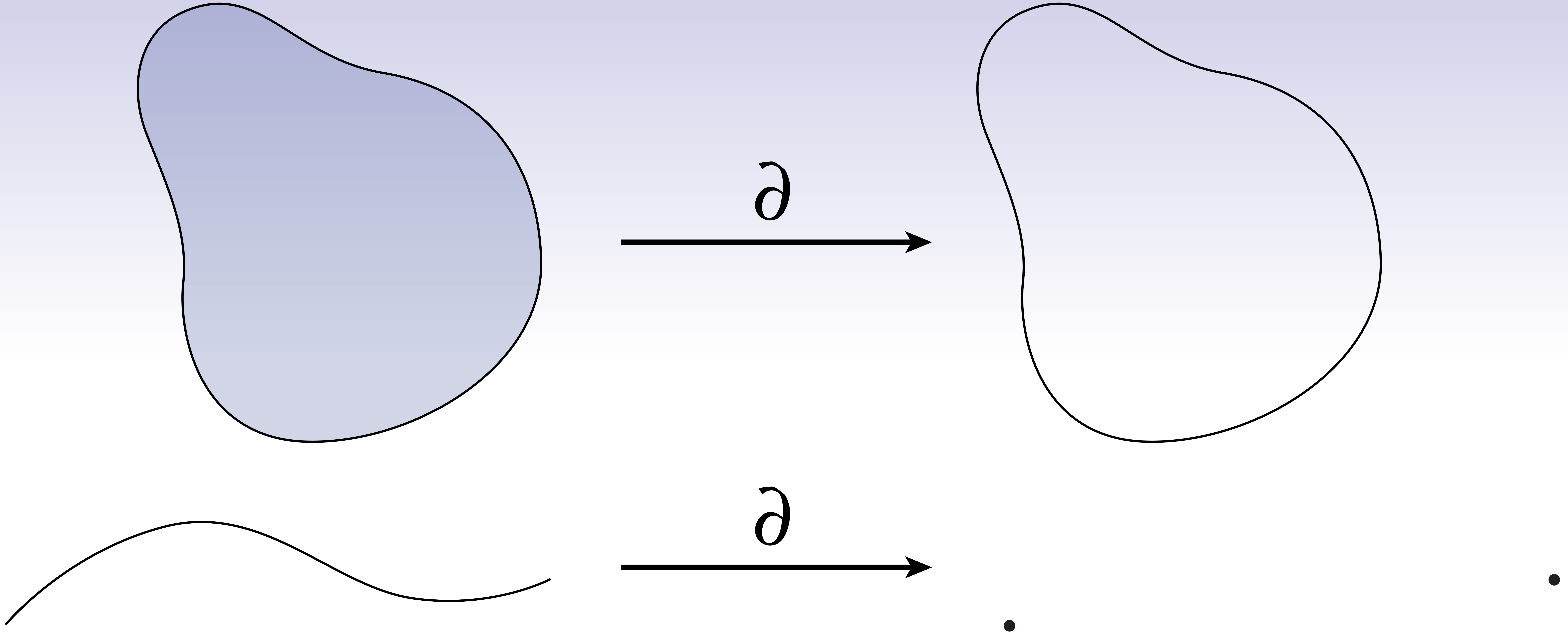


$\gamma : [0, 2\pi) \to \mathbb{R}^2; s \mapsto (\cos(s), \sin(s))$

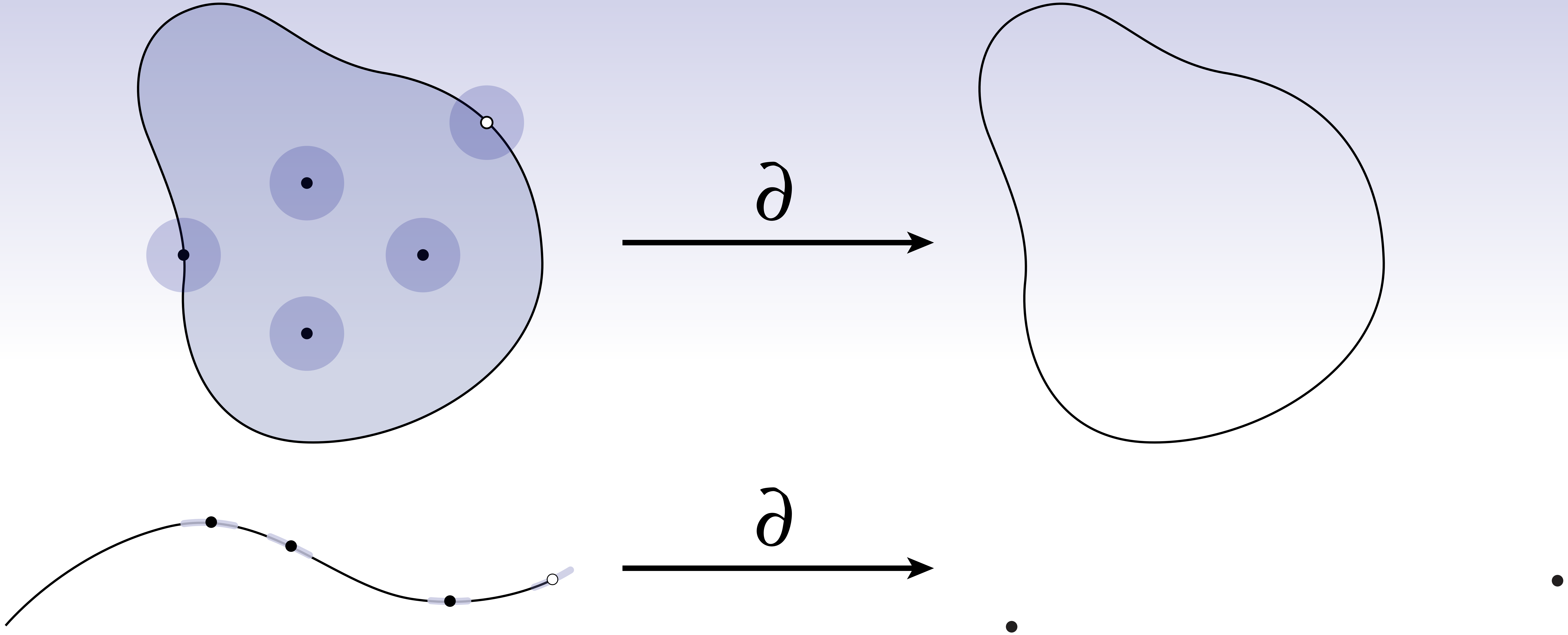
Stokes' Theorem



Boundary



Boundary

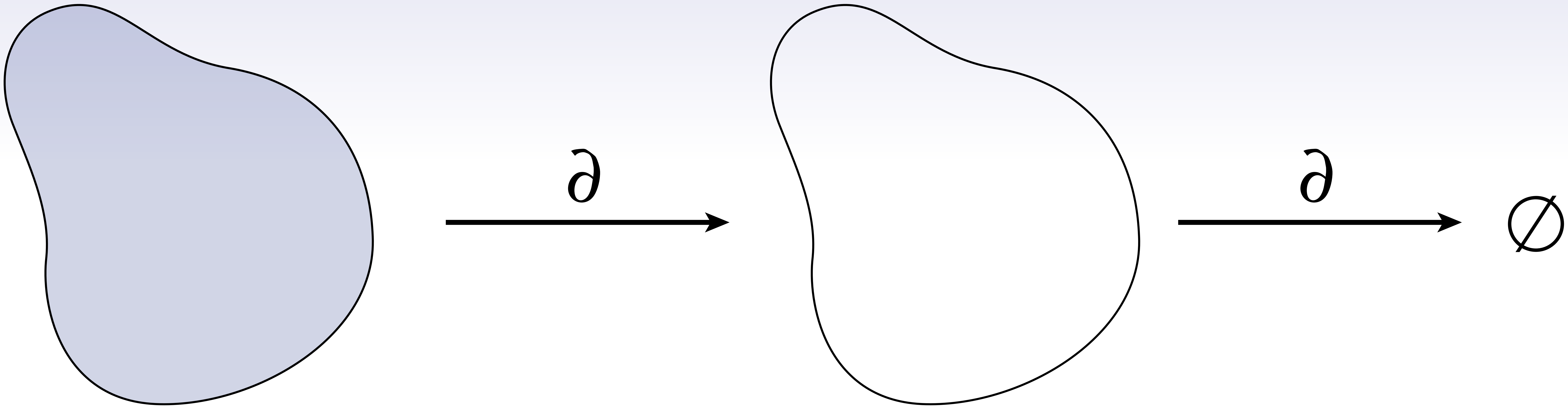


Basic idea: at an interior point p of a k-dimensional set the intersection of an open ball around p with the set looks like* an open k-ball; at a boundary point it doesn't.

*…is homeomorphic to, in the subspace topology.

Boundary of a Boundary

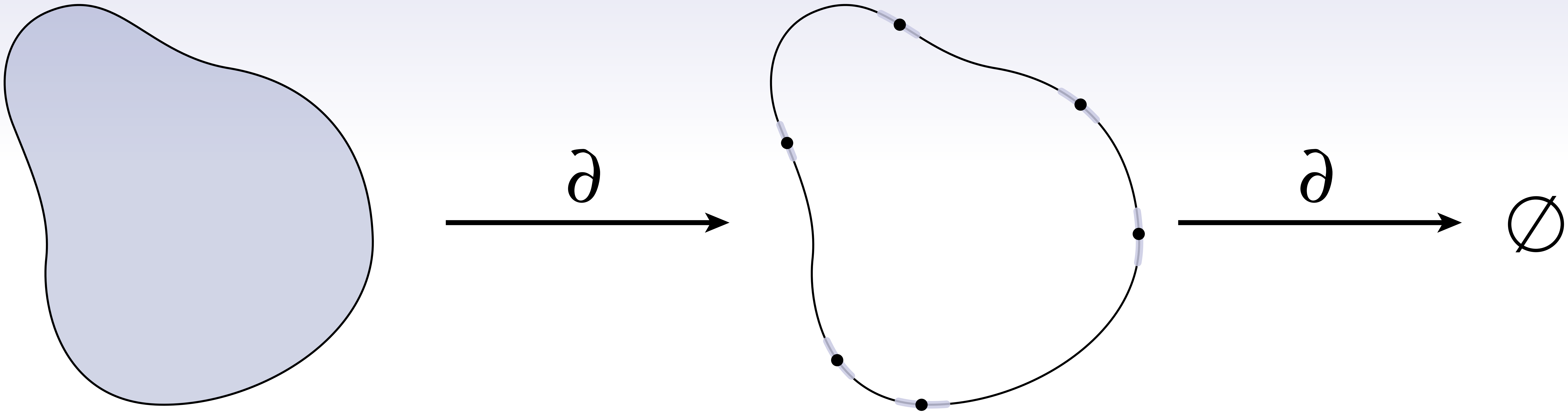
Q: Which points are in the boundary of the boundary?



A: No points! Boundary of a boundary is always *empty*.

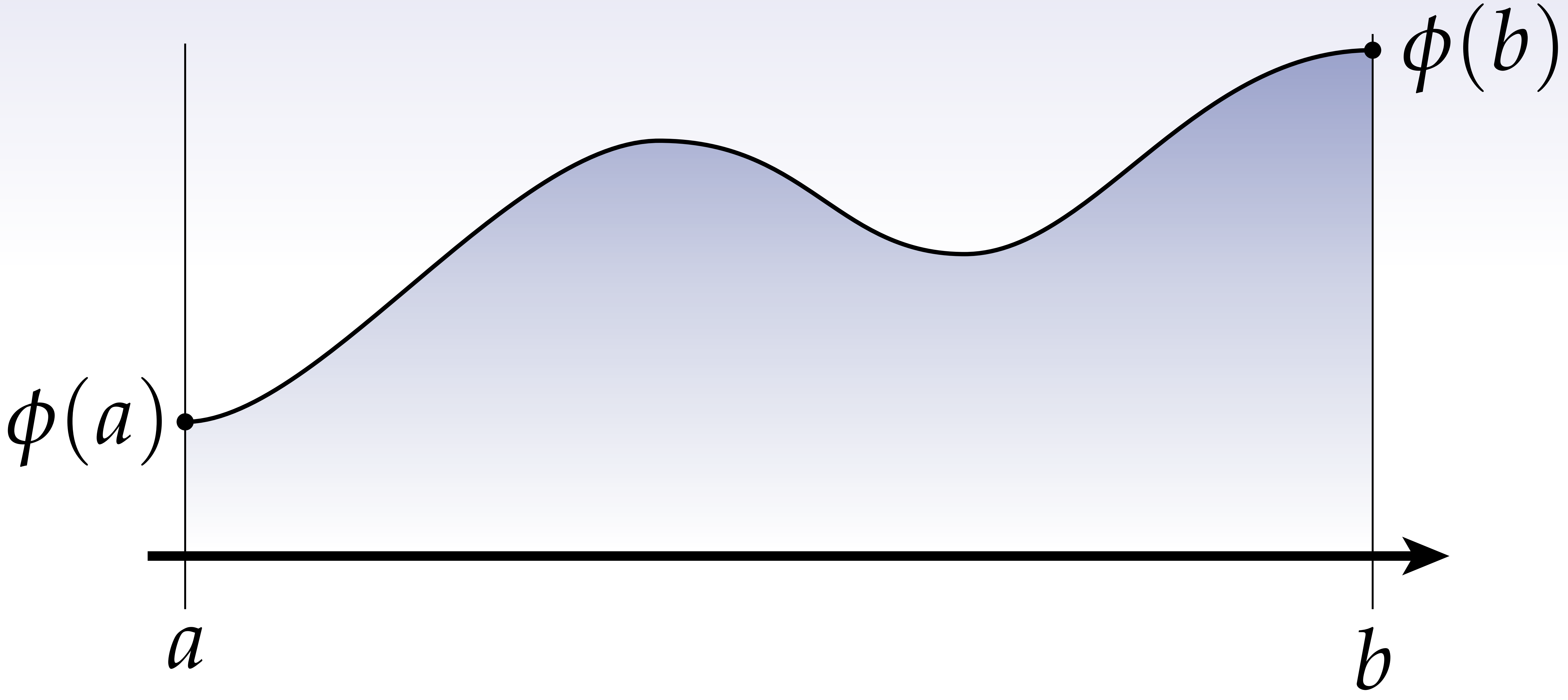
Boundary of a Boundary

Q: Which points are in the boundary of the boundary?



A: No points! Boundary of a boundary is always *empty*.

Review: Fundamental Theorem of Calculus



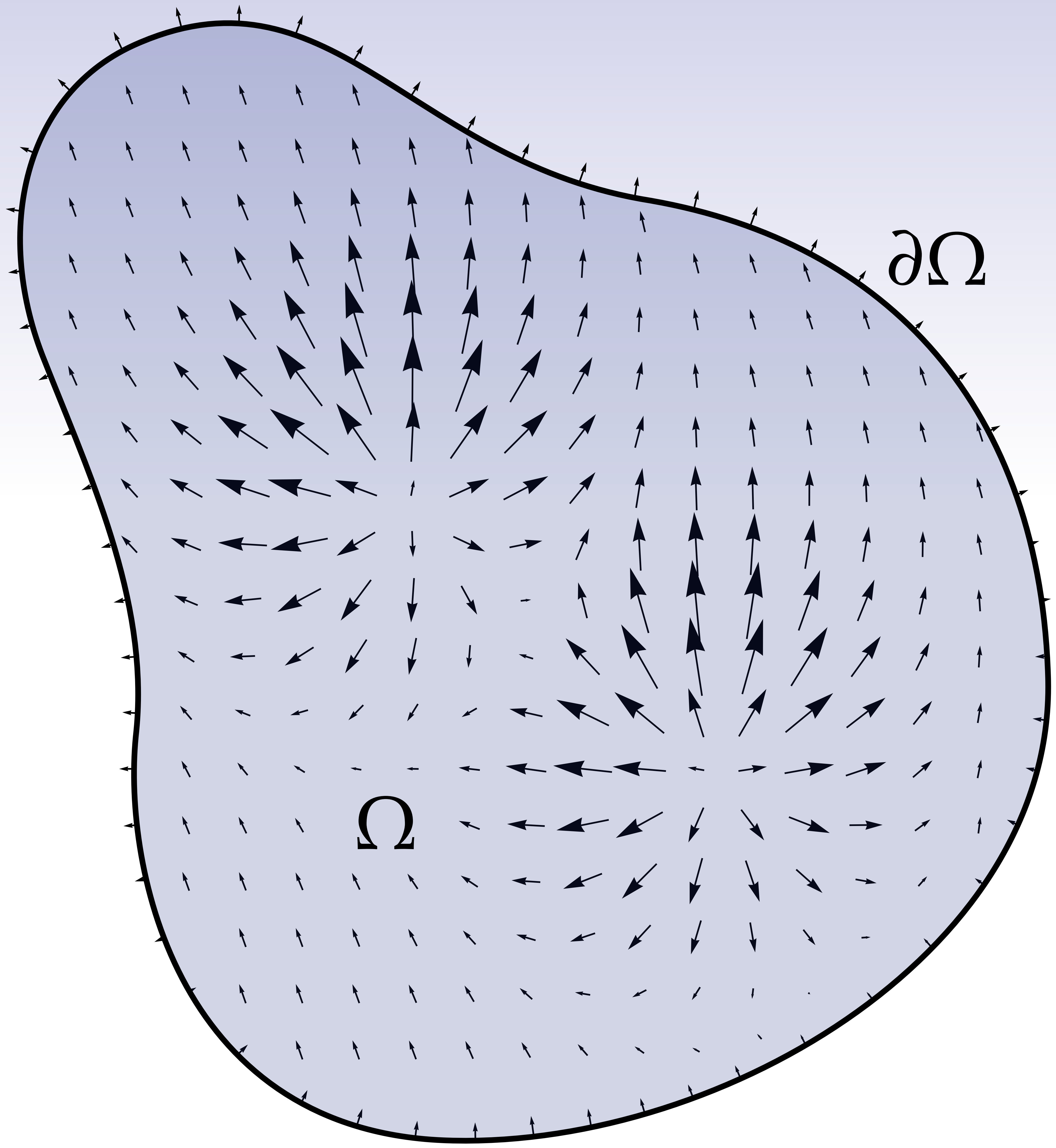
$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a)$$

Stokes' Theorem

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

Analogy: fundamental theorem of calculus

Example: Divergence Theorem



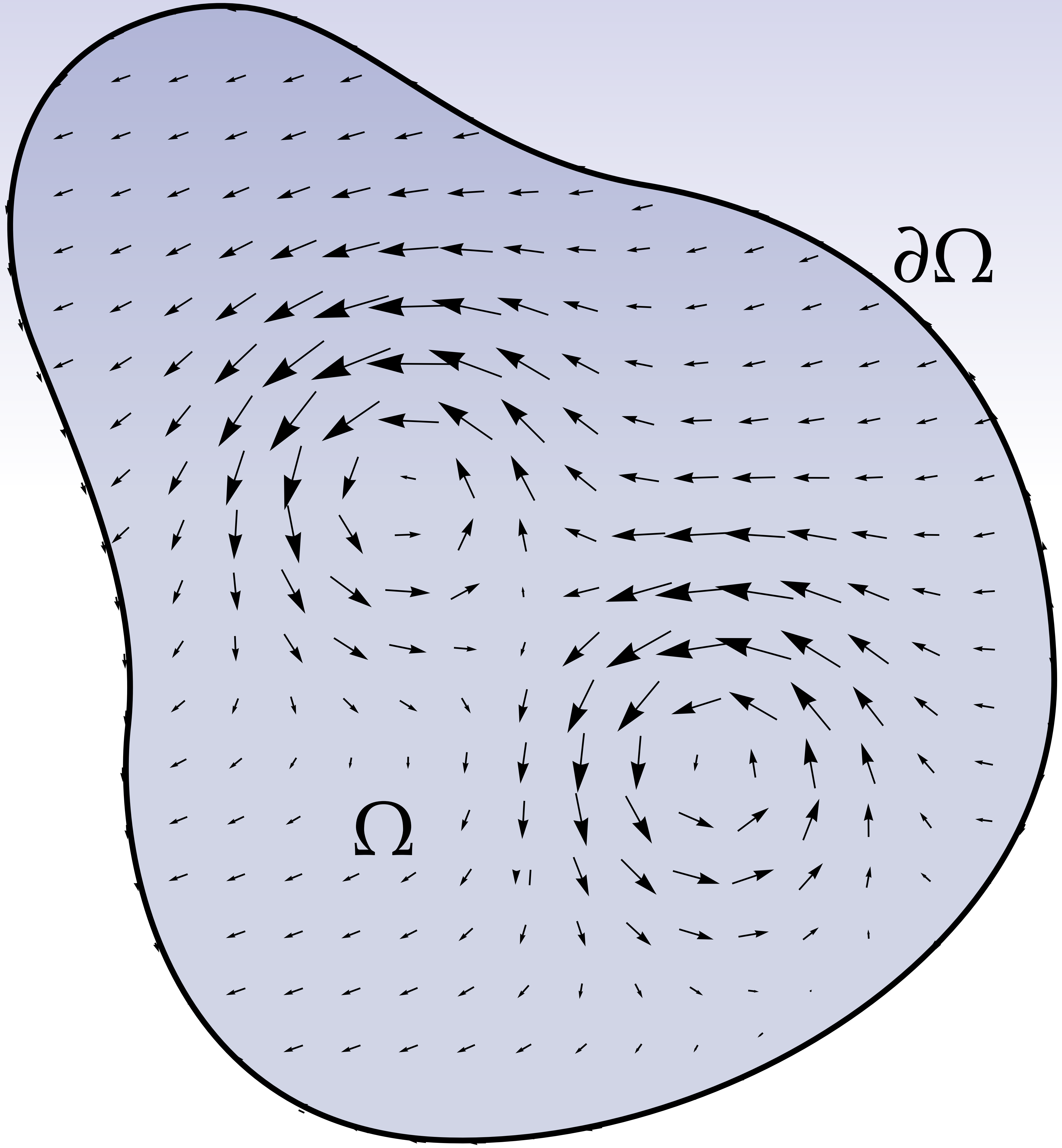
$$\int_\Omega \nabla \cdot X \, dA = \int_{\partial\Omega} n \cdot X \, d\ell$$

$$\iff$$

$$\int_\Omega d \star \alpha = \int_{\partial\Omega} \star\alpha$$

What goes in, must come out!

Example: Green's Theorem



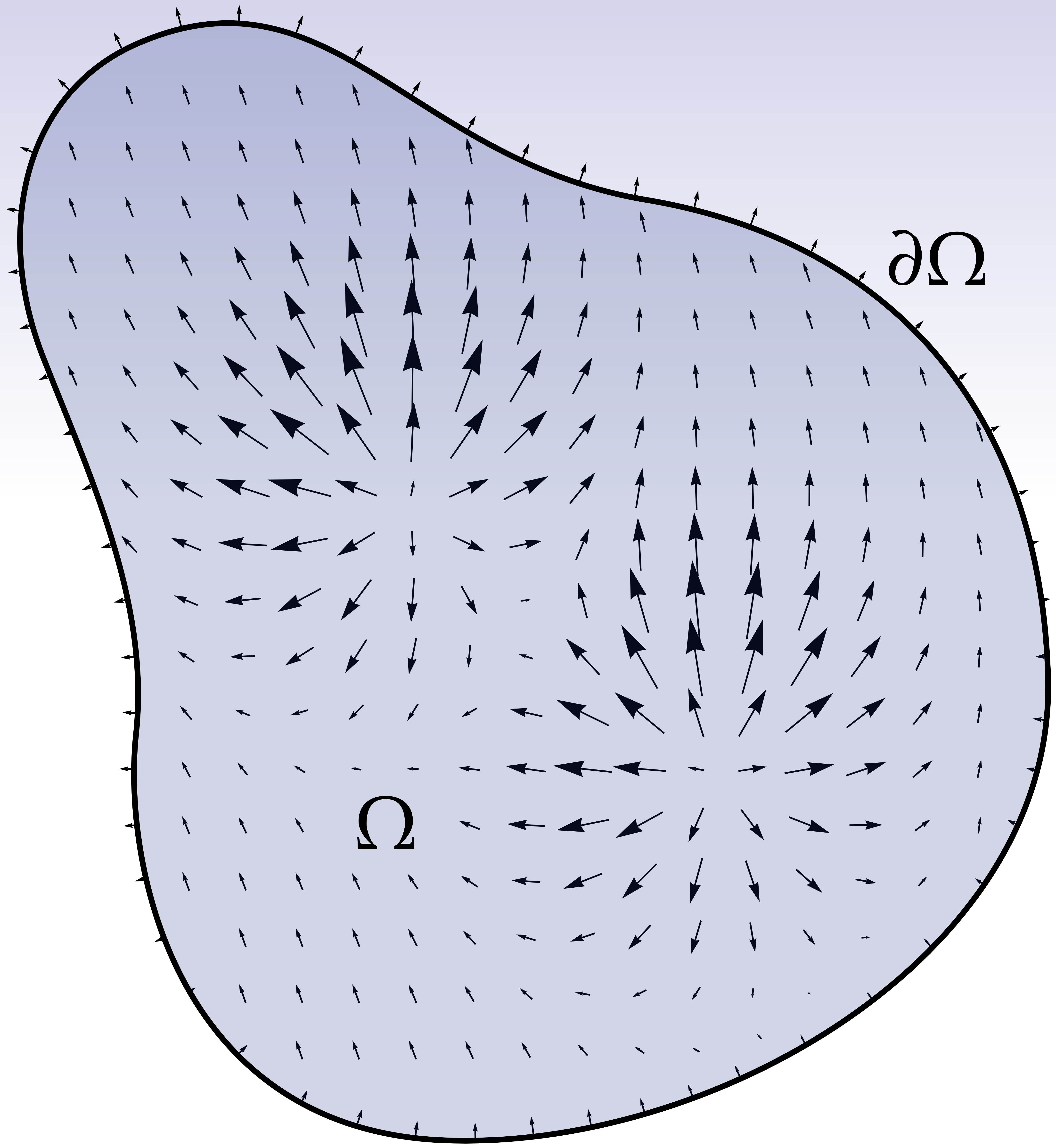
$$\int_\Omega \nabla \times X \, dA = \int_{\partial\Omega} t \cdot X \, d\ell$$

$$\Longleftrightarrow$$

$$\int_\Omega d\alpha = \int_{\partial\Omega} \alpha$$

What goes around comes around!

Stokes' Theorem

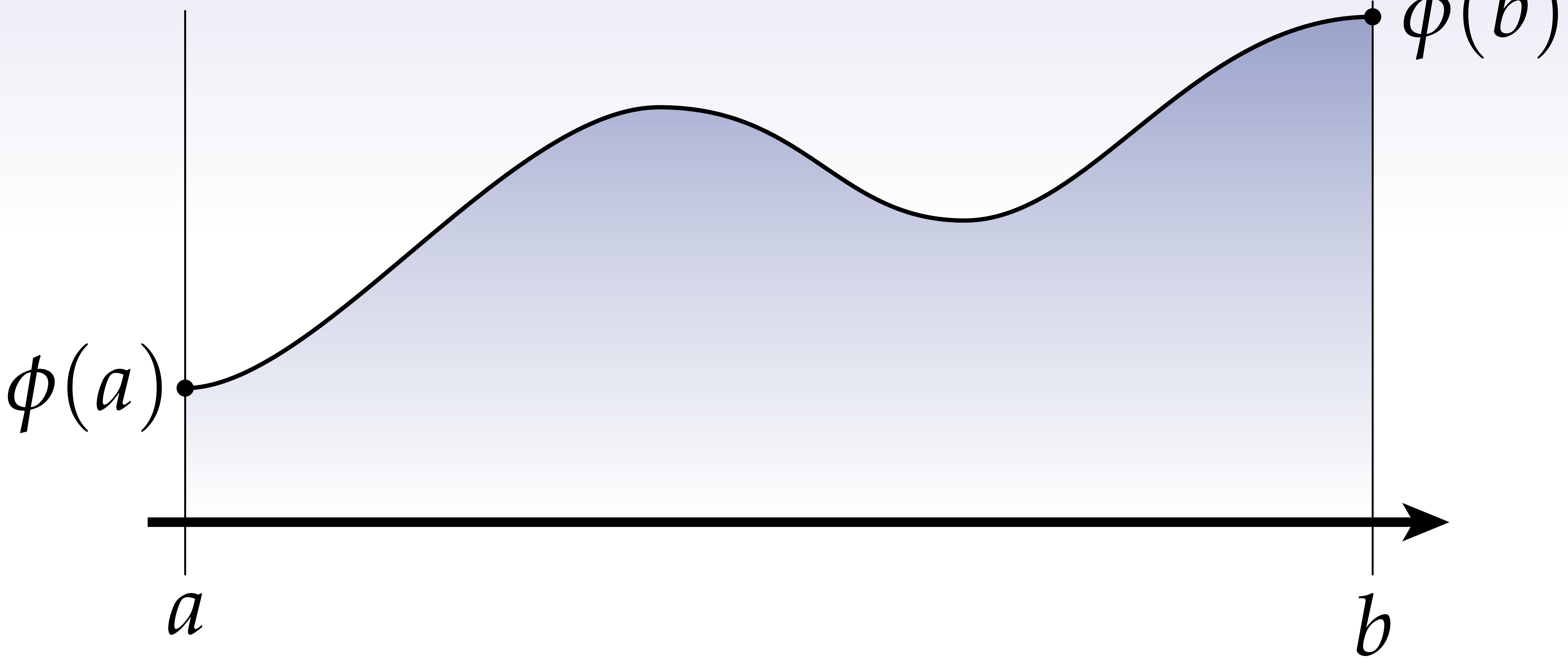


$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

"The change we see on the outside is purely a function of the change within."

—Zen koan

Fundamental Theorem of Calculus & Stokes'

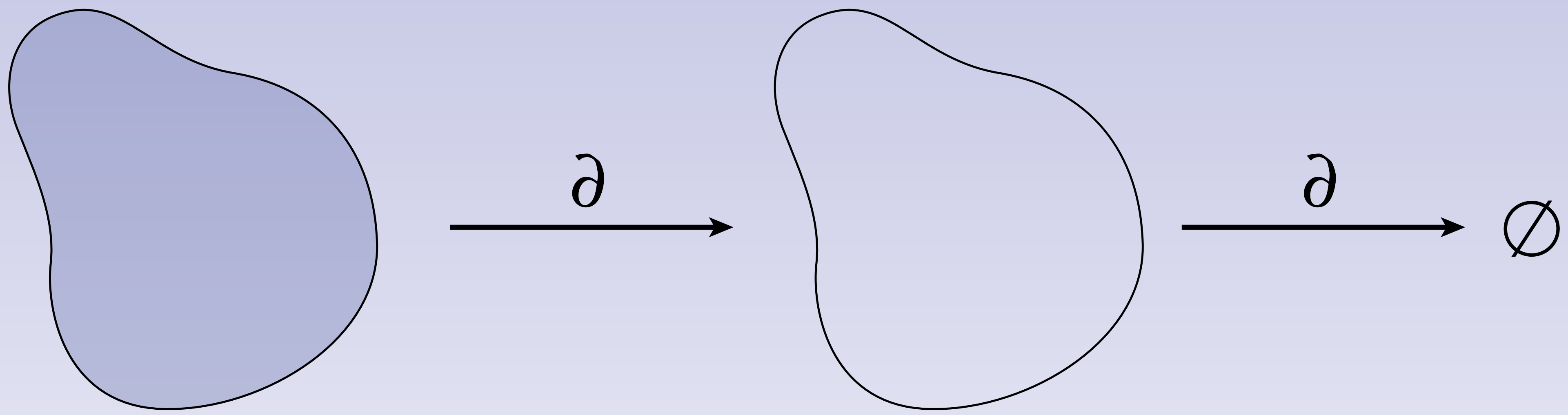


$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a)$$

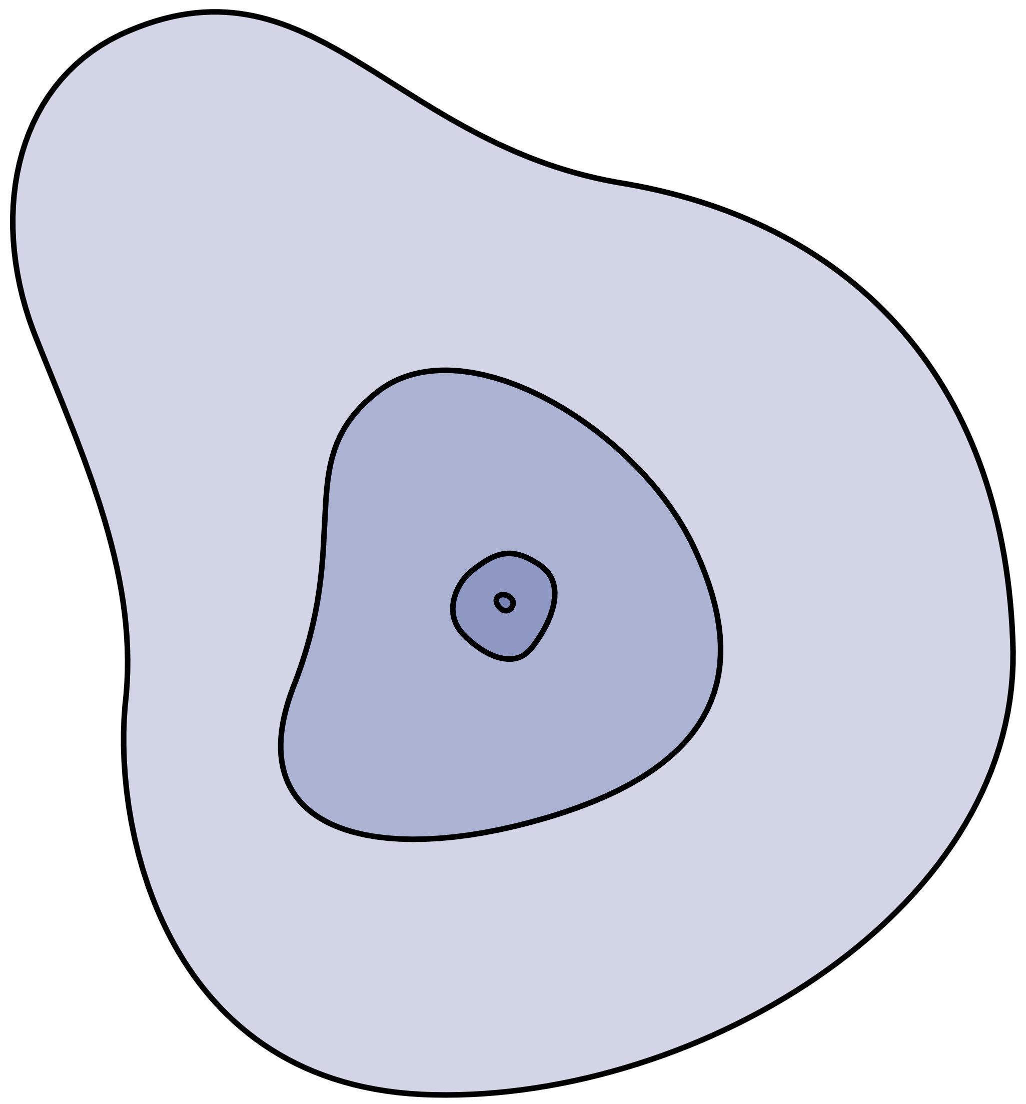
$$\Longleftrightarrow$$

$$\int_{[a,b]} d\phi = \int_{\partial [a,b]} \phi$$

Why is $d \circ d = 0$?



$$\int_\Omega dd\phi = \int_{\partial\Omega} d\phi = \int_{\underbrace{\partial\partial\Omega}_{\emptyset}} \phi = 0$$



...for *any* Ω (no matter how small!)

Why is $d \circ d = 0$?

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

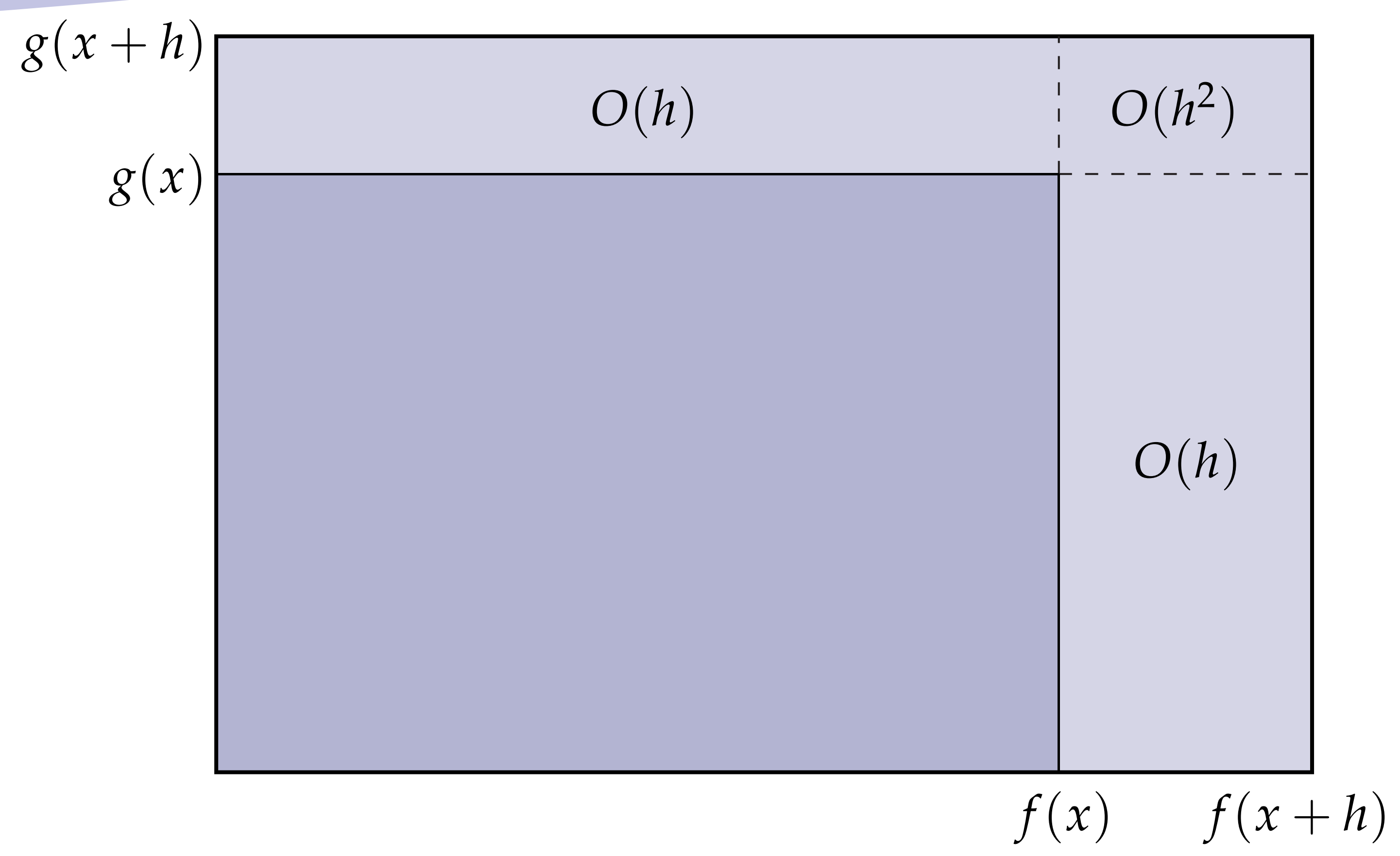
"behaves like gradient for functions"

differential $$d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \cdots + \frac{\partial \phi}{\partial x^n} dx^n$$

product rule $$d(\alpha \wedge \beta) = d\alpha \wedge \beta(-1)^k \alpha \wedge d\beta$$

~~**exactness**~~ $$\cancel{d \circ d = 0}$$

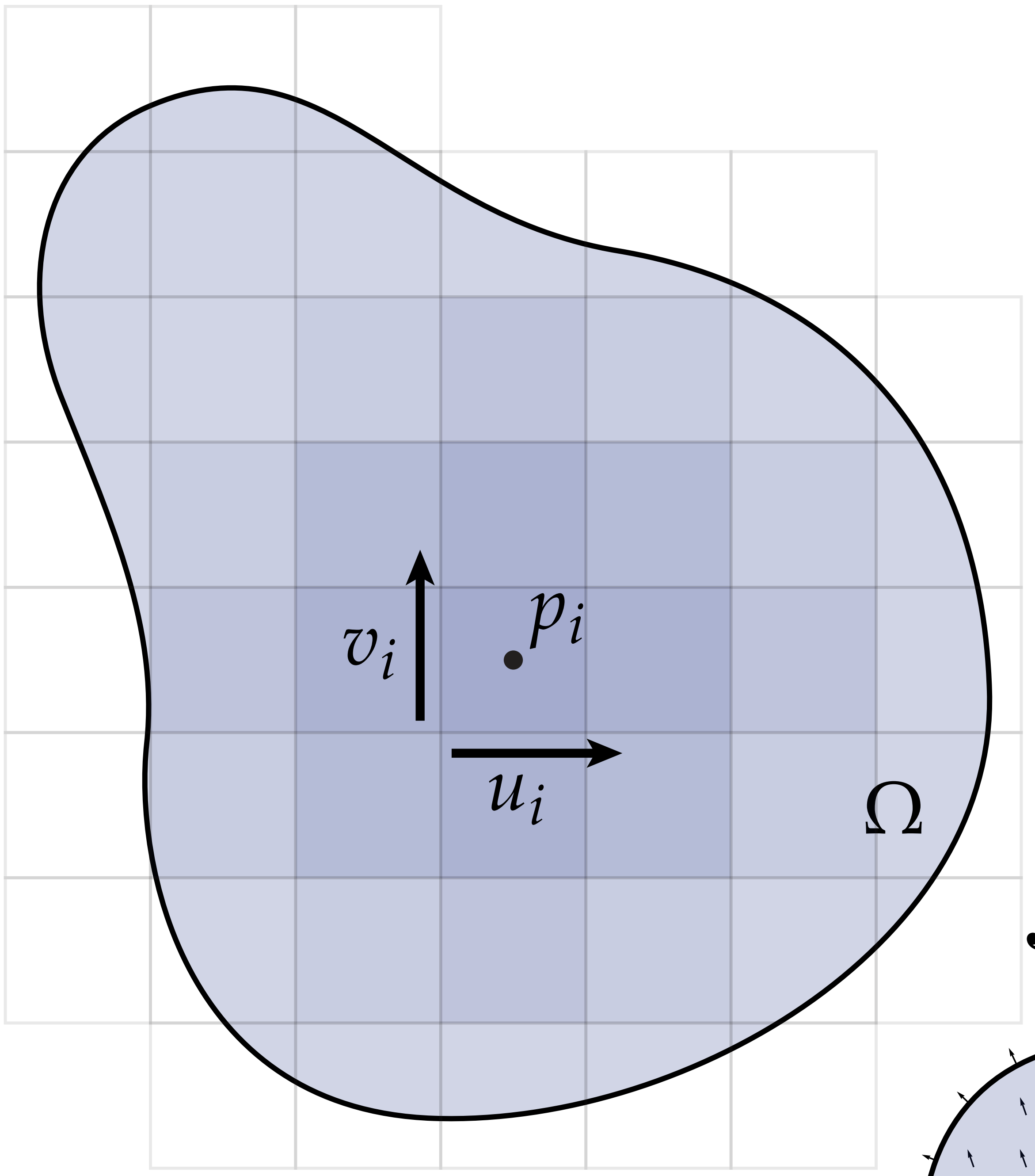
Stokes' theorem $$\int_\Omega d\alpha = \int_{\partial\Omega} \alpha$$



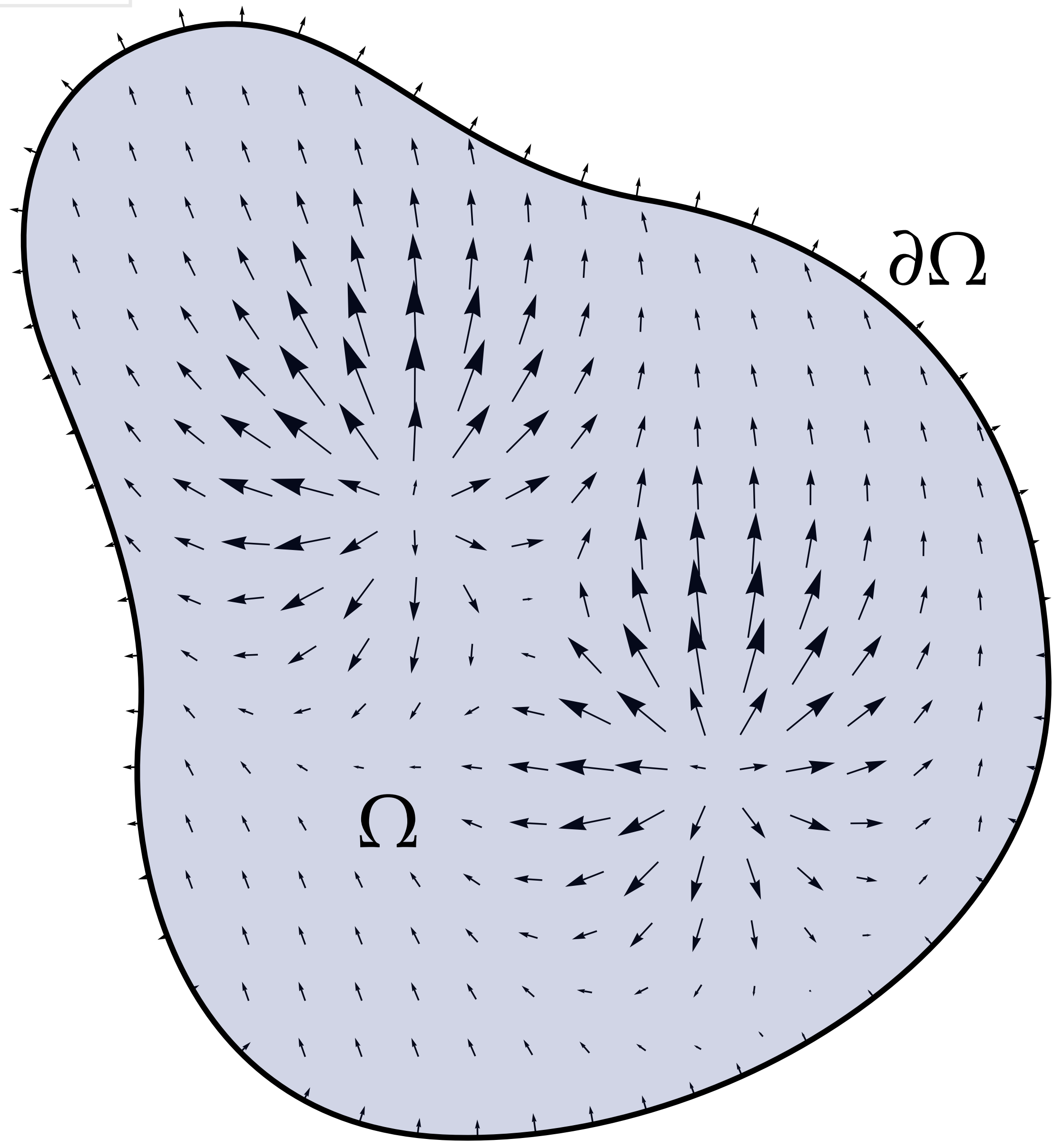
what goes in, must come out!

Integration & Stokes' Theorem - Summary

- Integration
  - break domain into small pieces
  - measure each piece with k-form
- Stokes' theorem
  - convert region integral to boundary integral
  - super useful—lets us "skip" a derivative
  - special cases: divergence theorem, Green's theorem, fundamental theorem of calculus, Cauchy's integral theorem… and *many more!*
  - Gets used *over and over* again in geometric computing
    - finite element methods, boundary element methods, …
    - **<u>discrete exterior calculus</u>**



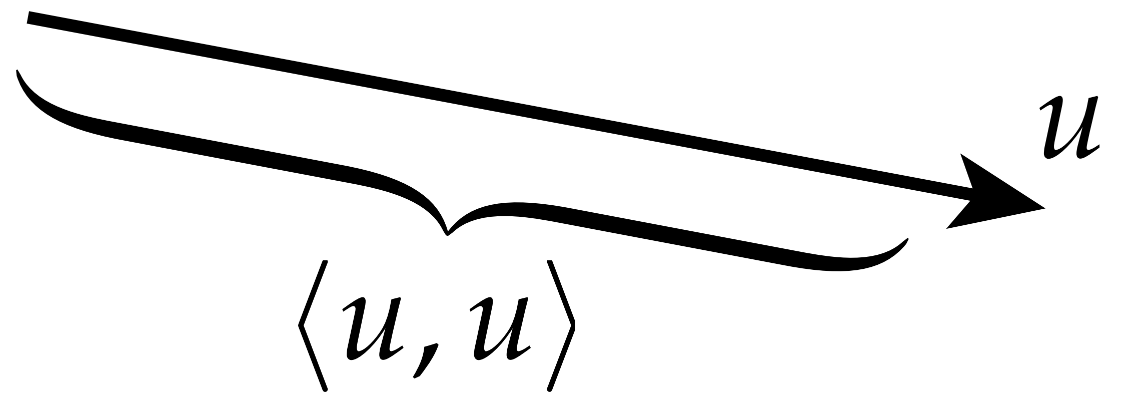
$$\int_M d\alpha = \int_{\partial M} \alpha$$



Inner Product on Differential k-Forms

Inner Product—Review

- Recall that a *vector space V* is any collection of “arrows” that can be added, scaled, …
- **Q:** What’s an *inner product* on a vector space?
- **A:** Loosely speaking, a way to talk about lengths, angles, etc., in a vector space
- More formally, a symmetric positive-definite bilinear map:



$$\langle \cdot\,, \cdot \rangle : V \times V \to \mathbb{R}$$

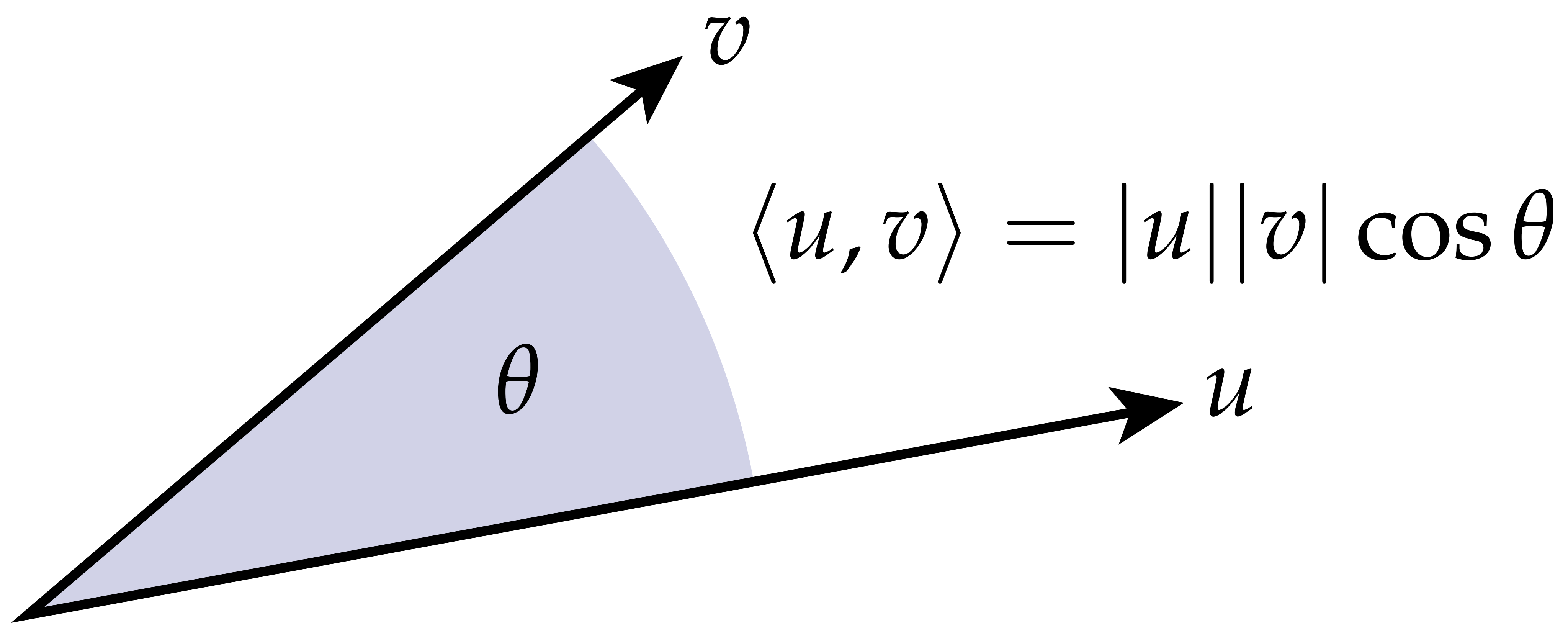
$$\langle u, v \rangle = \langle v, u \rangle$$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle au, v \rangle = a\langle u, v \rangle$$

$$\langle u, u \rangle \geq 0;\ \langle u, u \rangle = 0 \iff u = 0$$

for all vectors *u*,*v*,*w* in *V* and scalars *a*.



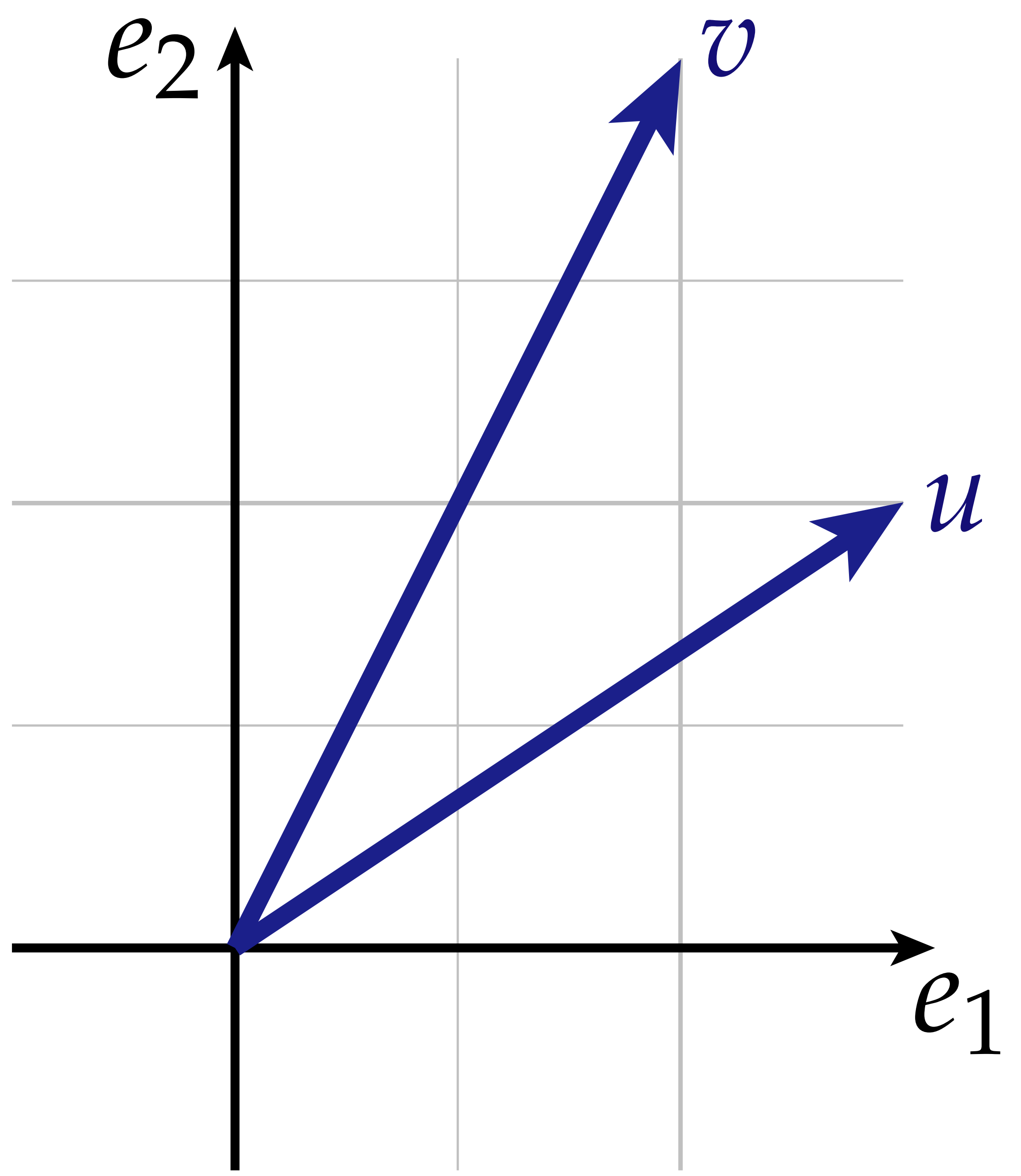
(Geometric interpretation of these rules?)

Euclidean Inner Product—Review

- Most basic inner product: inner product of two vectors in Euclidean $\mathbb{R}^n$
- Just sum up the product of components:

$$u = u^1 e_1 + \cdots + u^n e_n$$
$$v = v^1 e_1 + \cdots + v^n e_n$$

$$\langle u, v \rangle := \sum_{i=1}^{n} u^i v^i$$



<u>Example.</u>

$$u = 3e_1 + 2e_2$$
$$v = 2e_1 + 4e_2$$
$$\langle u, v \rangle = 3 \cdot 2 + 2 \cdot 4 = 14$$

(Does this operation satisfy all the requirements of an inner product?)

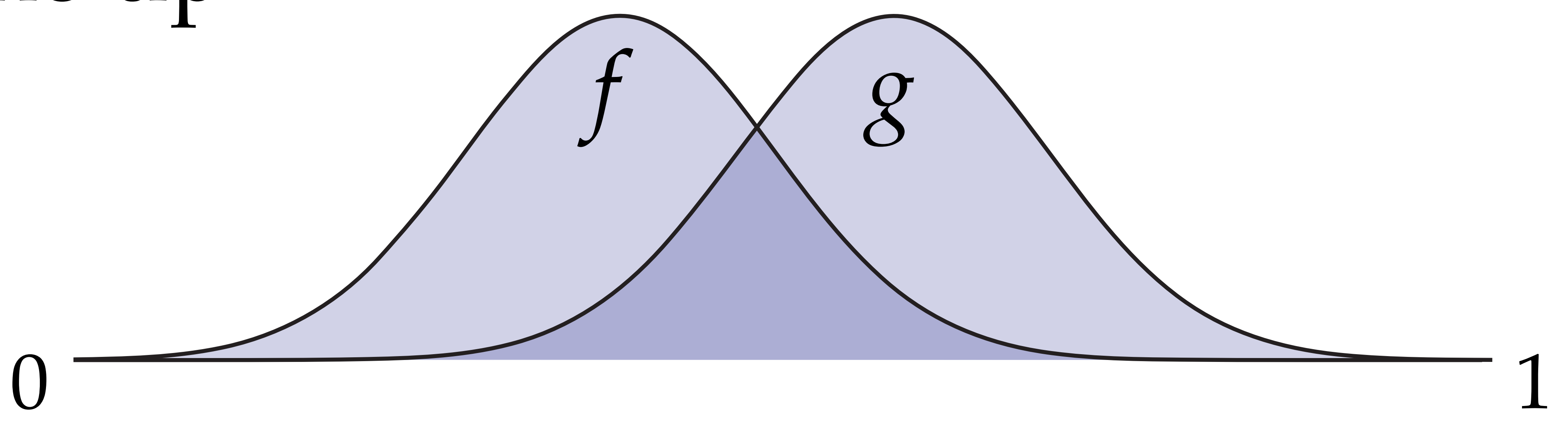
L^2 *Inner Product of Functions / 0-forms*

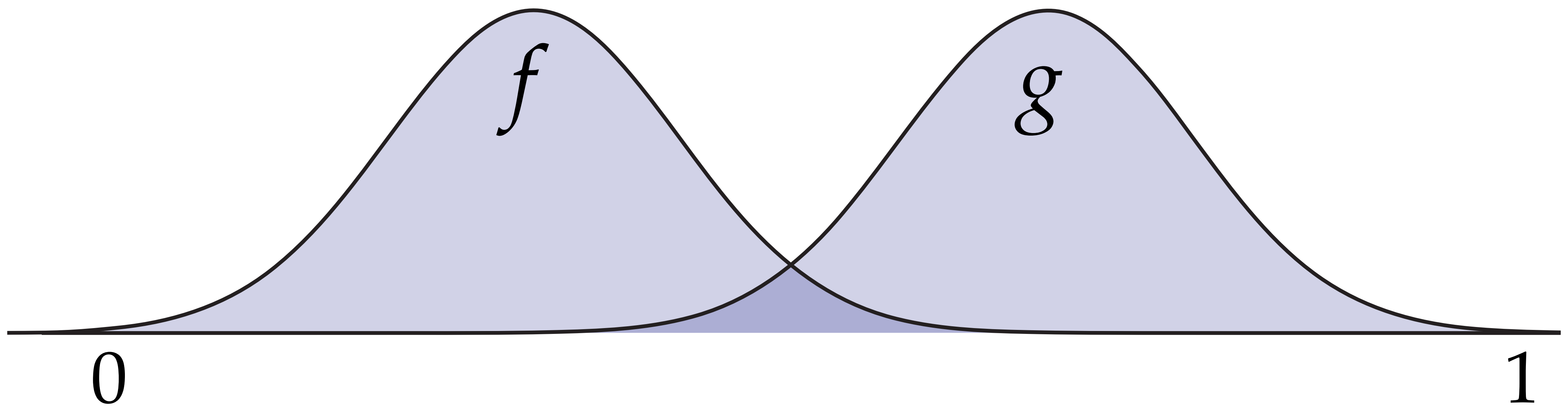
- Collections of *functions* are also vector spaces (*e.g.*, real integrable functions on [0,1])
- What does it mean to measure the inner product between functions?
- Want some notion of how well two functions "line up"
- One idea: just mimic the Euclidean dot product

$$f : [0,1] \to \mathbb{R}$$

$$g : [0,1] \to \mathbb{R}$$

$$\langle\langle f, g \rangle\rangle := \int_0^1 f(x)g(x)\,dx$$





- Called the L^2 inner product. (**Note:** defined on space of *square-integrable* functions)
- Does this capture notion of "lining up"? Does it obey rules of inner product?

Inner Product on k-Forms

Definition. Let $\alpha, \beta \in \Omega^k$ be any two differential k-forms. Their (L^2) inner product is defined as*

$$\langle\langle \alpha, \beta \rangle\rangle := \int_\Omega \star\alpha \wedge \beta$$

Q: What happens when k=0?

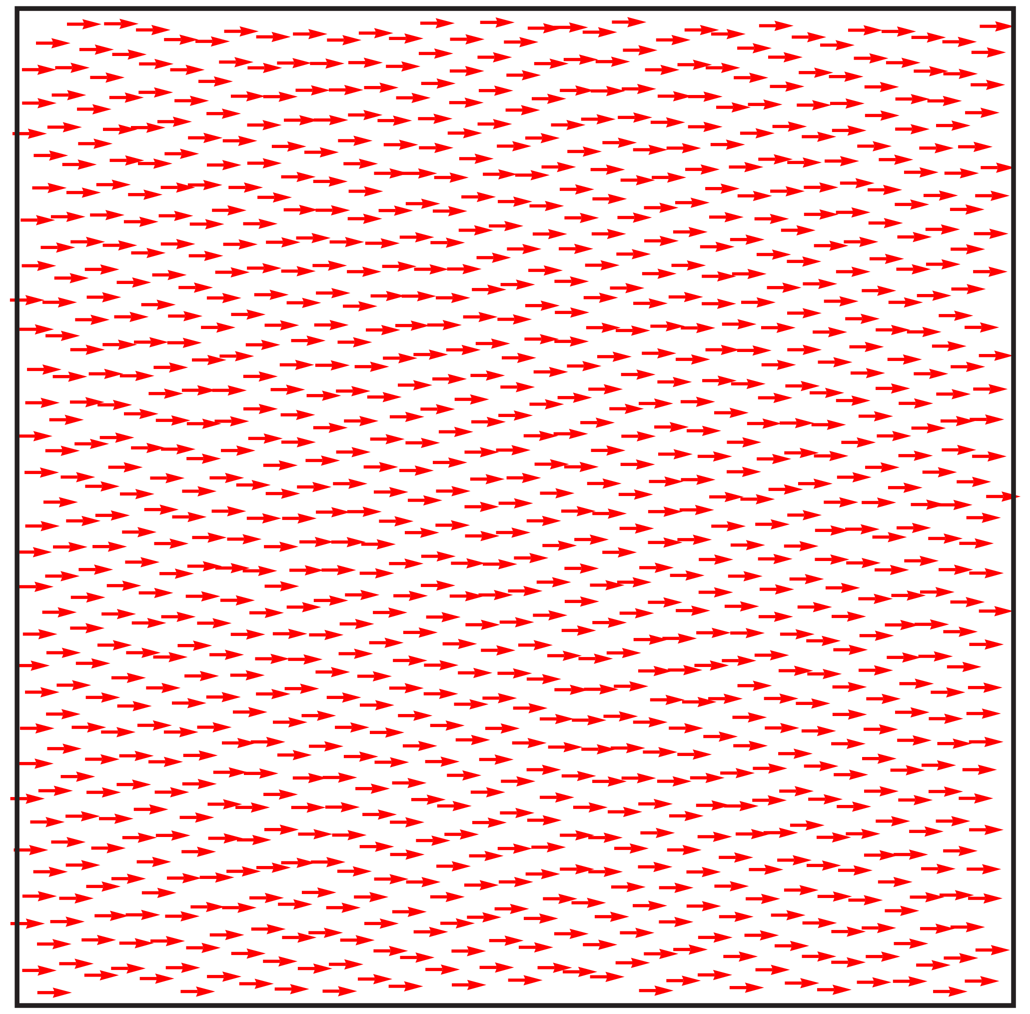
A: We just get the usual L^2 inner product on functions.

Q: What's the degree (k) of the integrand? Why is that important?

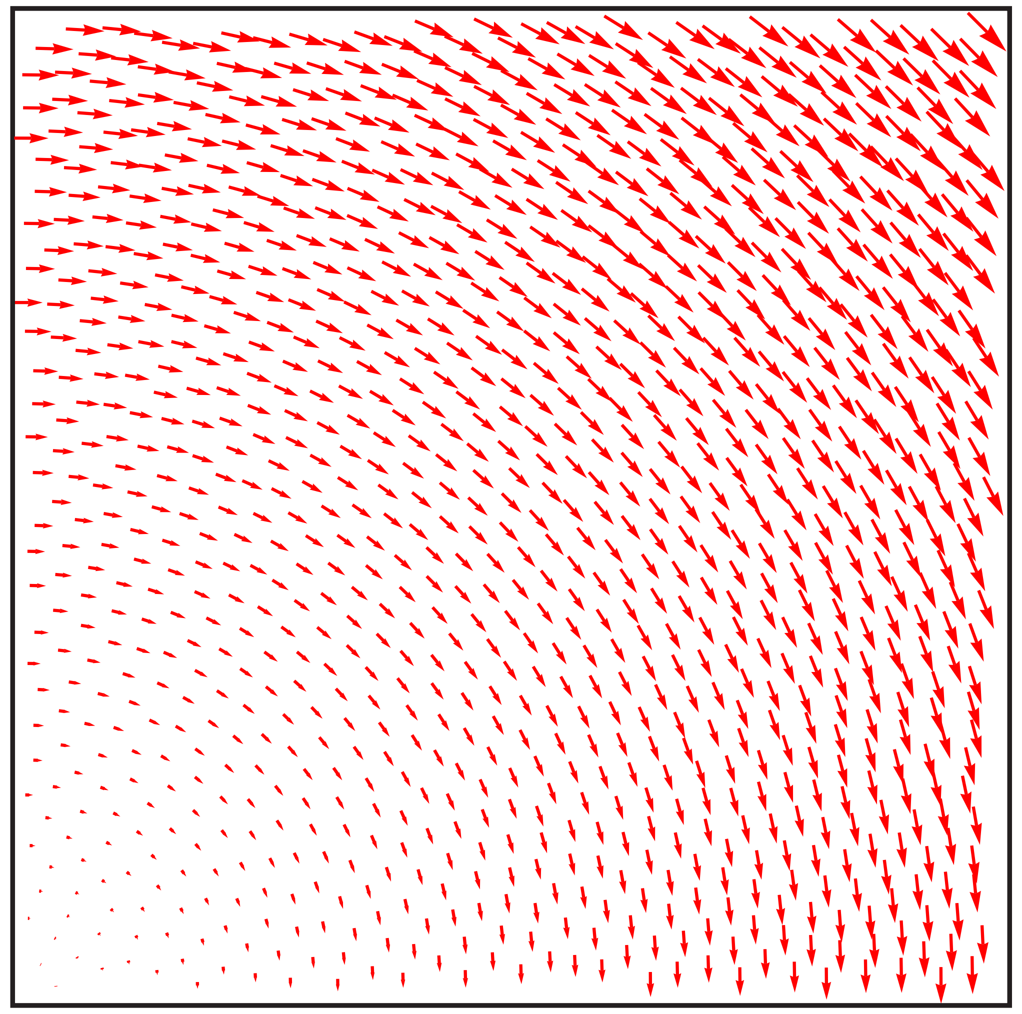
A: Integrand is always an n-form—which is the only thing we can integrate in n-D!

*Some authors define the integrand as $\alpha \wedge \star\beta$; our convention is consistent with the convention that in 2D the 1-form Hodge star is a *counter*-clockwise rotation.

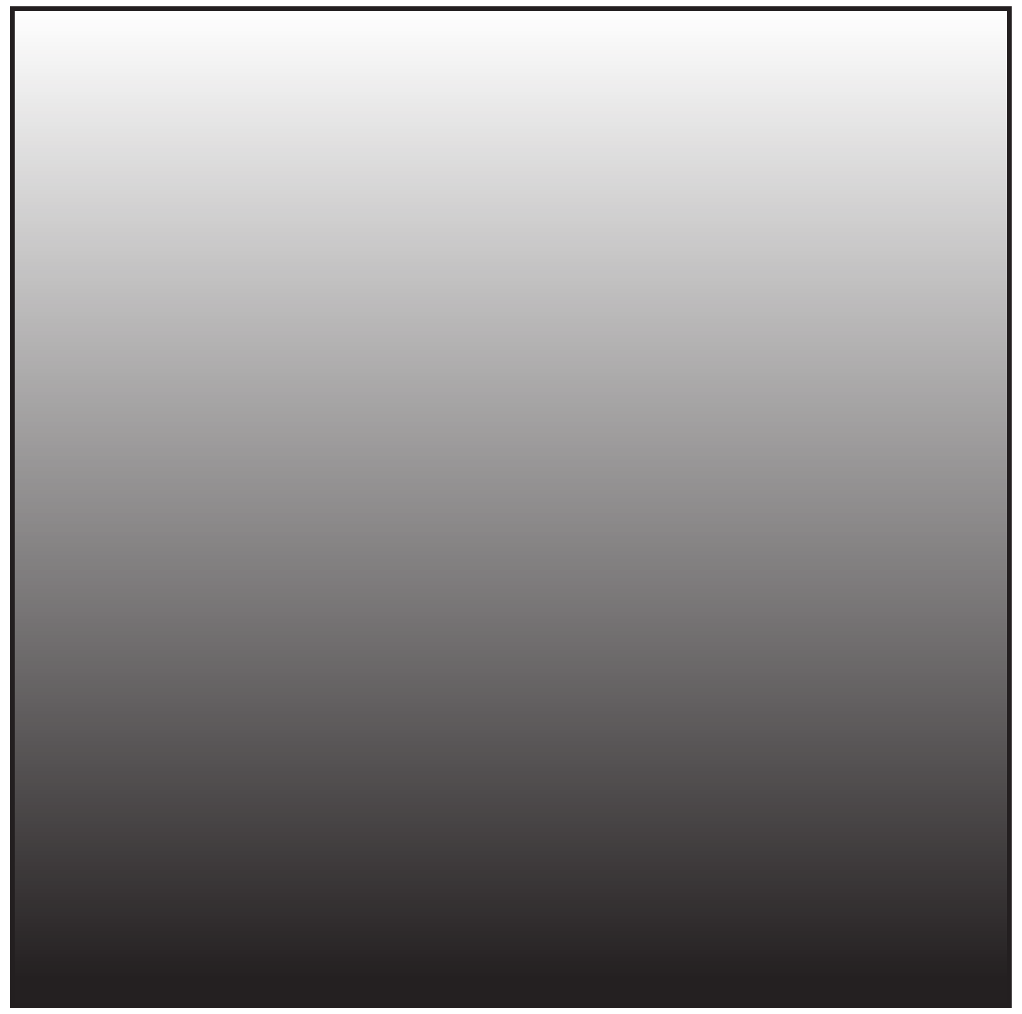
Inner Product of 1-Forms—Example



α



β



$\star\alpha \wedge \beta$

Example. Consider two 1-forms on the unit square $[0,1] \times [0,1]$ given by

$$\begin{aligned} \alpha &:= du, \\ \beta &:= v\,du - u\,dv. \end{aligned}$$

Their inner product is

$$\langle\langle \alpha, \beta \rangle\rangle = \int_0^1 \int_0^1 (\star\alpha) \wedge \beta =$$

$$\int_0^1 \int_0^1 dv \wedge (v\,du - u\,dv) =$$

$$-\int_0^1 \int_0^1 v\,du \wedge dv = -\frac{1}{2}$$

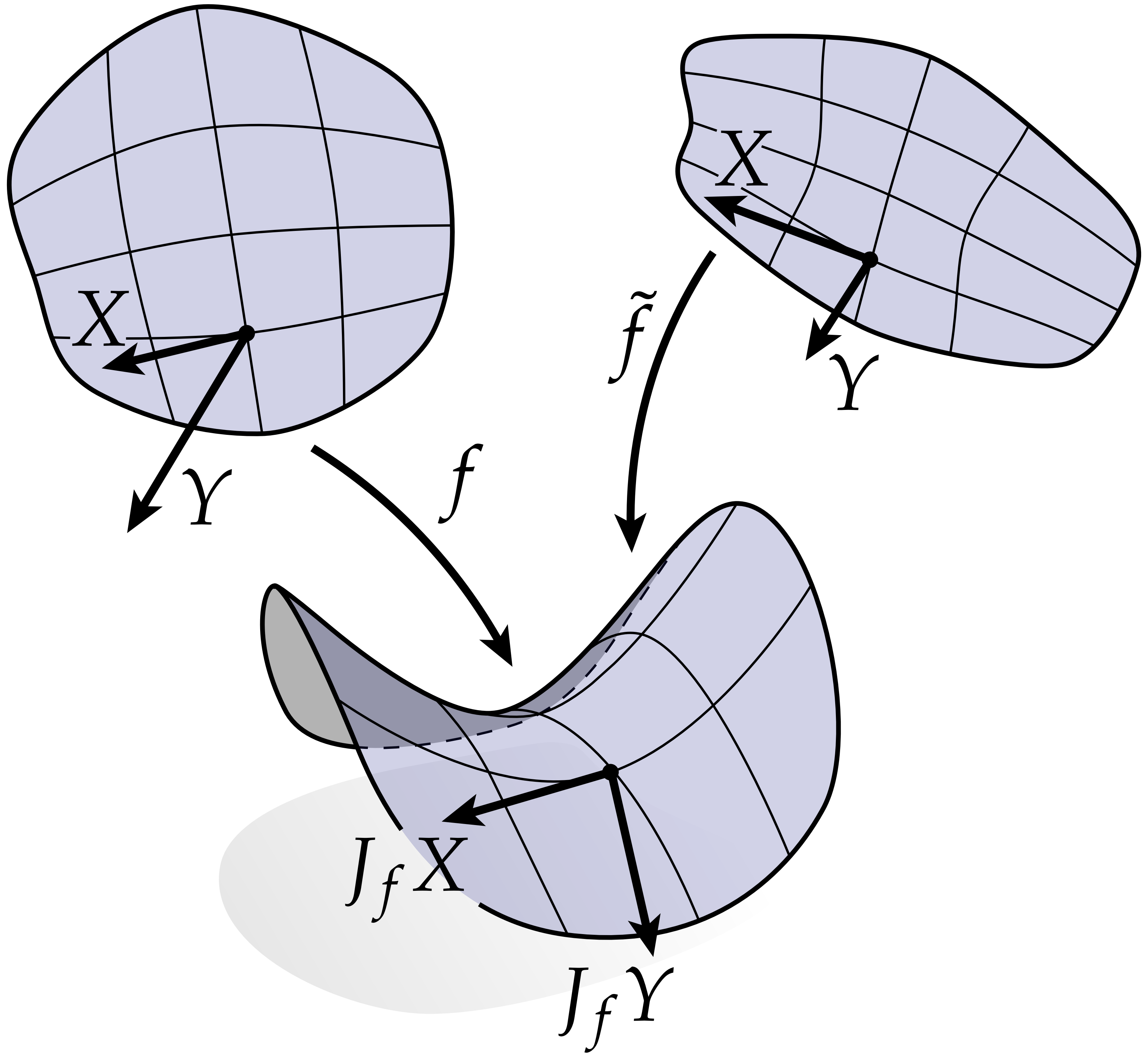
Summary

Exterior Calculus: Flat vs. Curved

- For simplicity, we introduced exterior calculus in **flat spaces** (R^n)
- Took care to make distinction between **vectors** and **covectors**, even though they often looked the same!
- But on **curved spaces** things will get more interesting, because the <u>inner product</u> is no longer just the ordinary "dot product"
- For instance, suppose we have two different parameterizations of a surface:
- 2D Euclidean dot product is the *wrong* way to measure angle between vectors!
- Will return to this perspective when we study smooth surfaces…

$$\alpha(X) \longleftrightarrow \langle \alpha^\sharp, X \rangle \qquad \langle X, Y \rangle \longleftrightarrow X^\flat(Y)$$

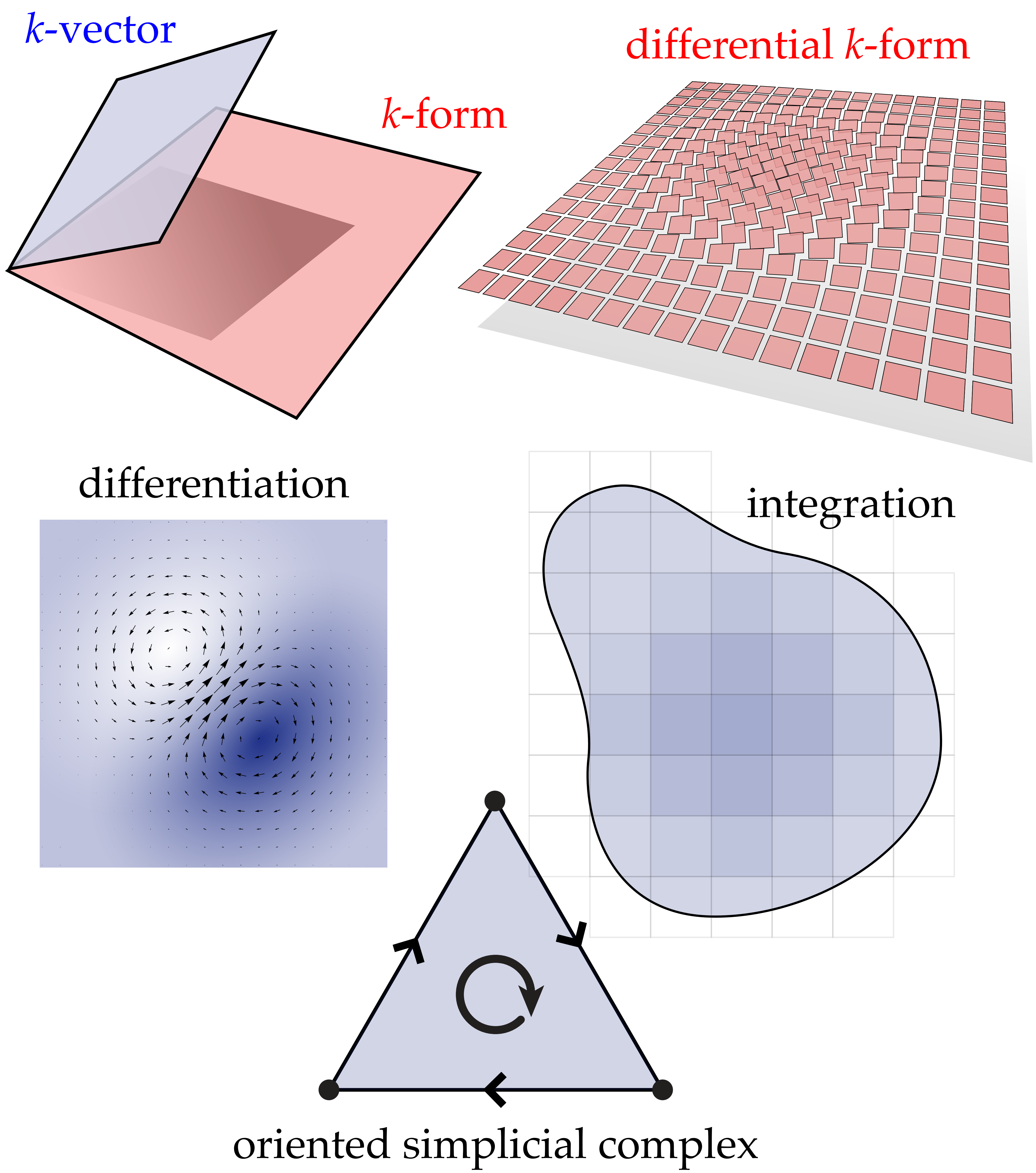
$$d\phi(X) \longleftrightarrow \langle \nabla\phi, X \rangle$$



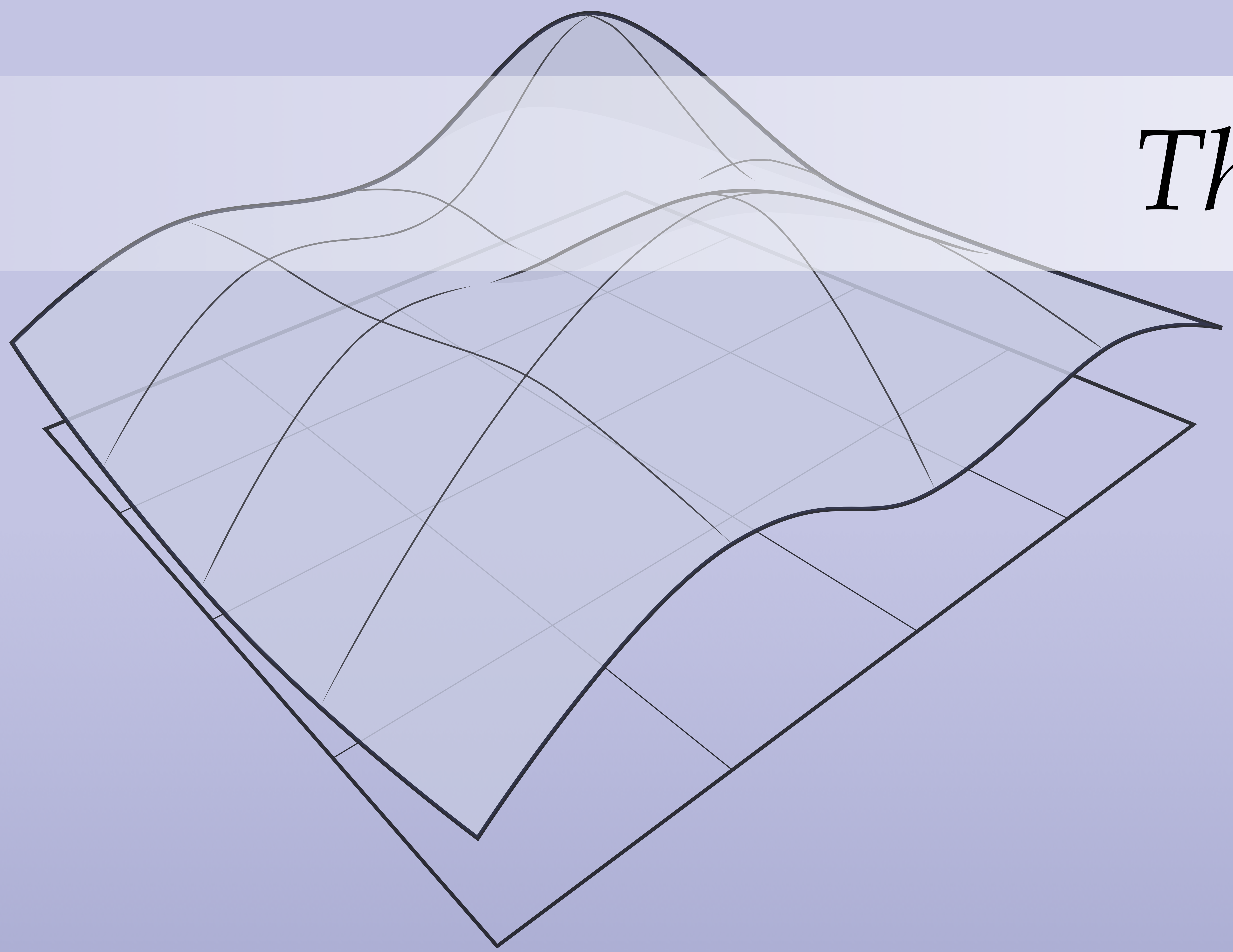
$$\langle X, Y \rangle := (J_f X)^T (J_f Y)$$

Exterior Calculus — Summary

- **What we've seen so far:**
- *Exterior algebra*: language of volumes (*k*-vectors)
- *k-form:* measures a k-dimensional volume
- *Differential forms:* *k*-form at each point of space
- *Exterior calculus*: differentiate / integrate forms
- *Simplicial complex*: mesh of *k*-simplices
- **Next up:**
  - Put all this machinery together
  - *Integrate* to get discrete exterior calculus (DEC)



Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION