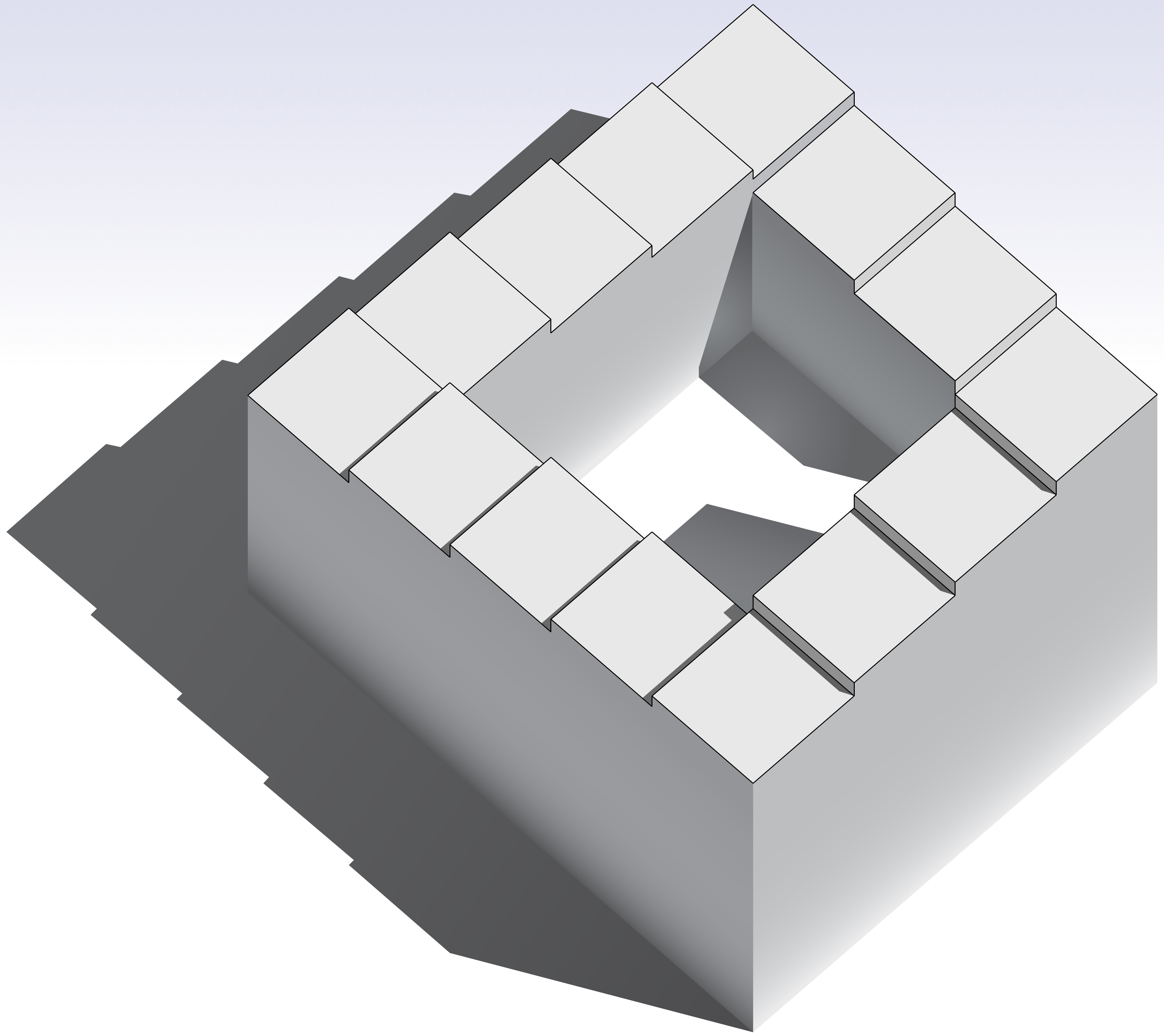
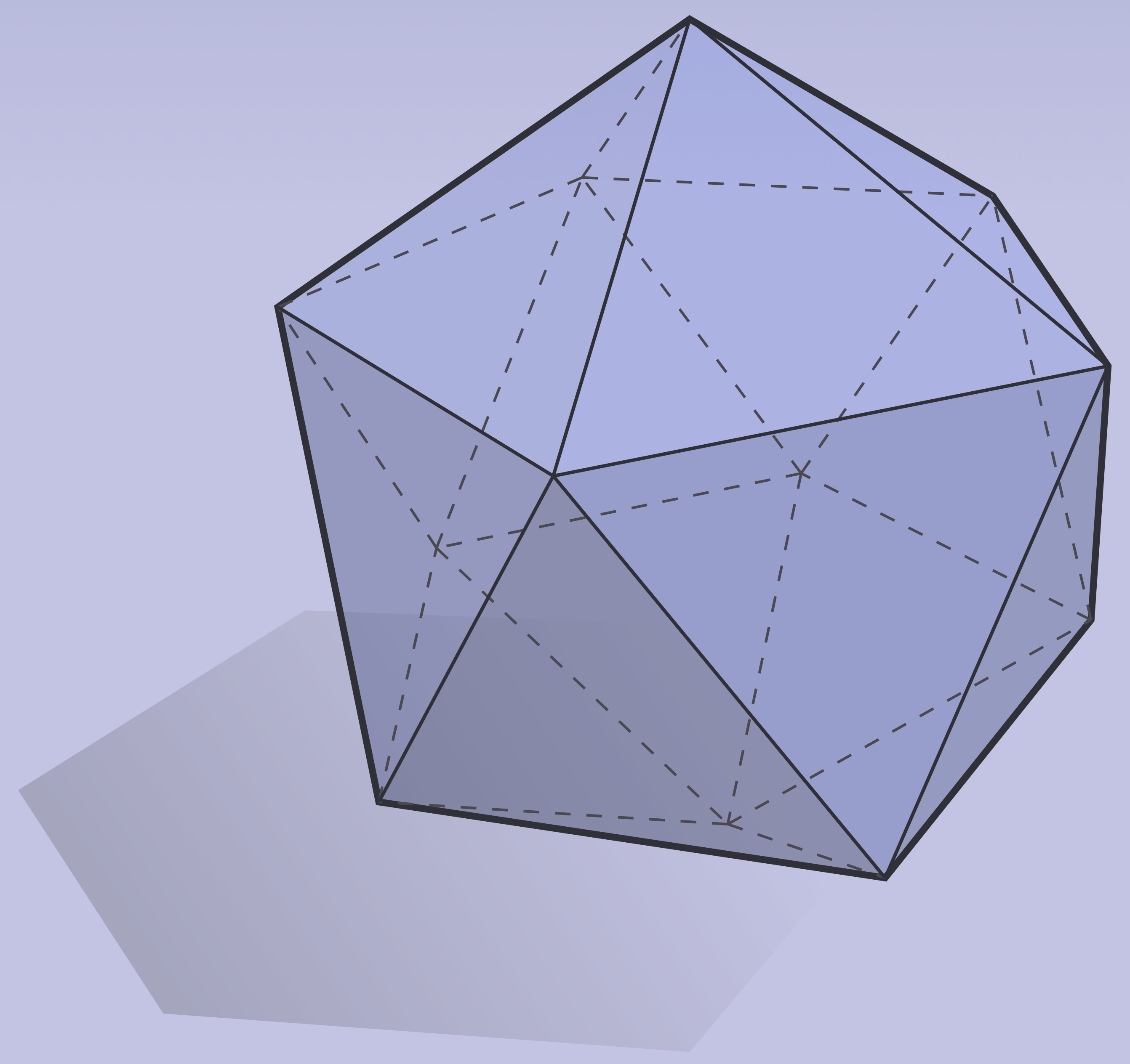


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 8:
Discrete Differential Forms



Discrete Differential Geometry: An Applied Introduction

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Review—Exterior Calculus

- Last lecture we saw *exterior calculus* (differentiation & integration of k-forms)
- As a review, let's try *solving an equation* involving differential forms in $\mathbb{R}^2$.
- Recall that *any* 1-form can be expressed as $\alpha = u dx + v dy$ for some pair of functions u,v
- Also recall that $dx \wedge dy = -dy \wedge dx$.
- Hence, know what u and v must look like
- In other words: $\alpha = \frac{1}{2}((x+a)dy - (y+b)dx)$
- …is this what you expected?

Given. Constant 2-form $\omega = dx \wedge dy$.

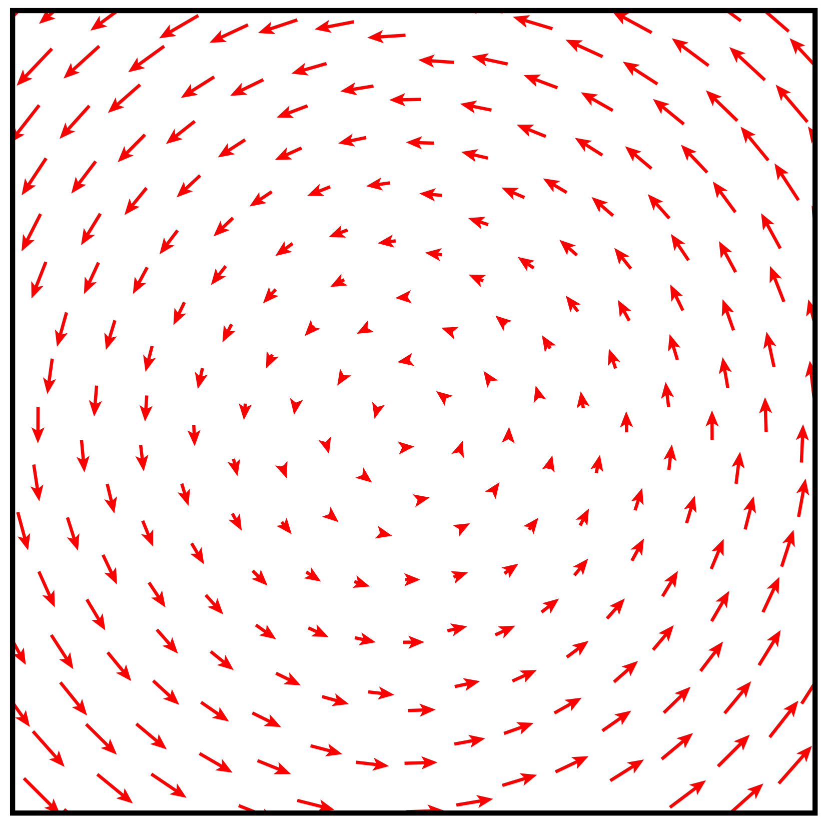
Find. A 1-form α such that $d\alpha = \omega$.

$$d\alpha = du \wedge dx + dv \wedge dy$$

$$\omega = -\tfrac{1}{2}dy \wedge dx + \tfrac{1}{2}dx \wedge dy$$

$$u(x,y) = -\tfrac{1}{2}y + a, \quad a \in \mathbb{R}$$

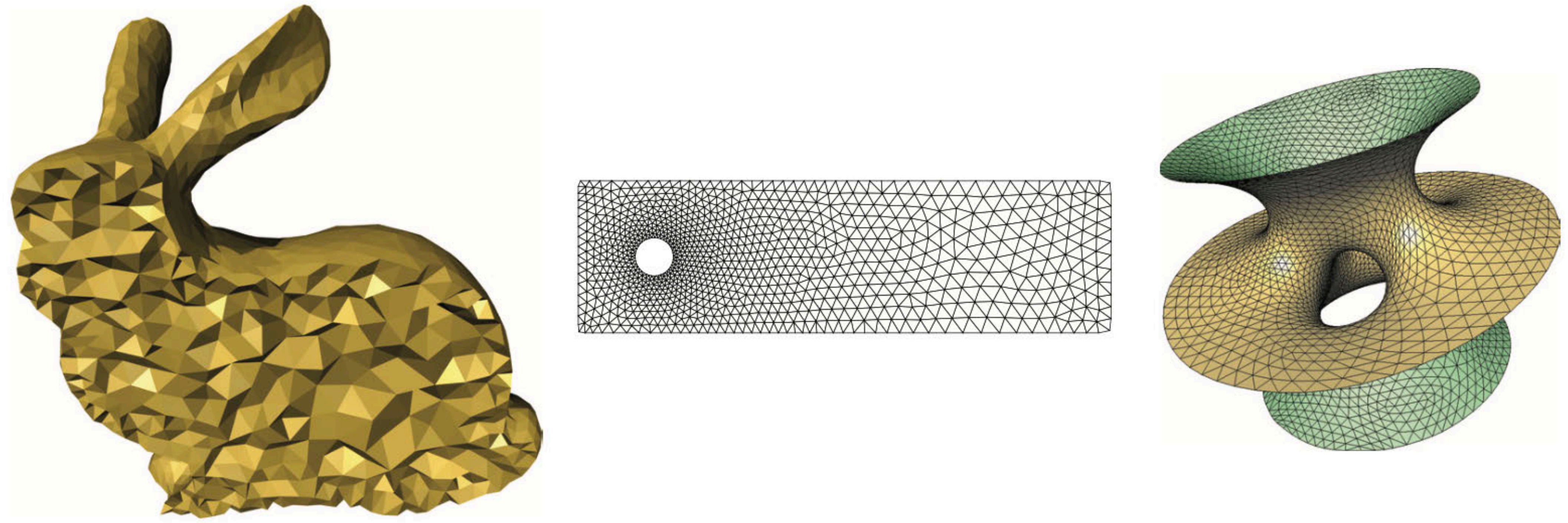
$$v(x,y) = \tfrac{1}{2}x + b, \quad b \in \mathbb{R}$$

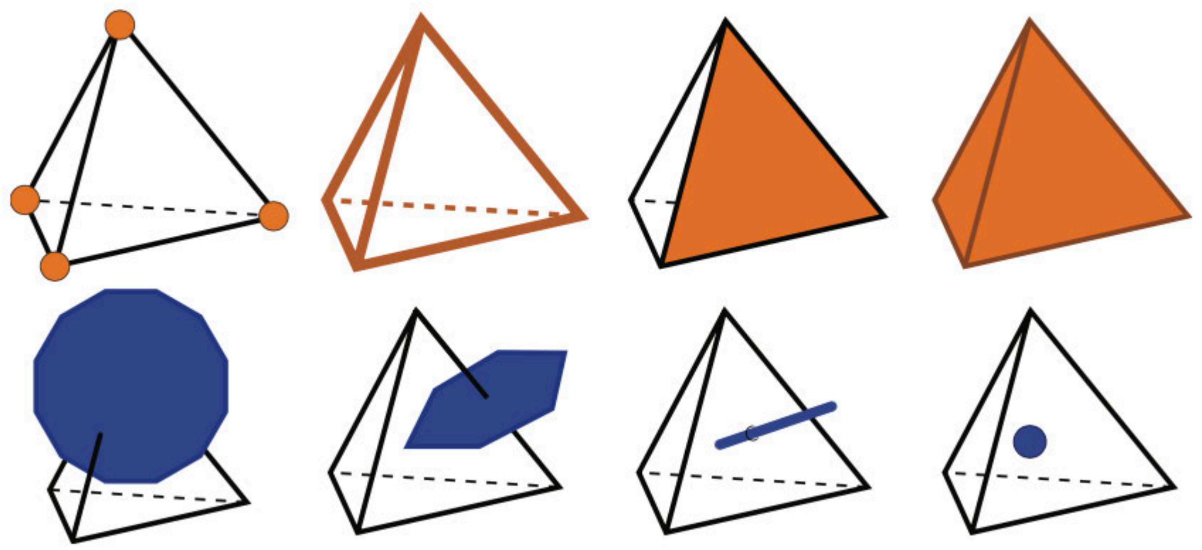


$\frac{1}{2}(xdy - ydx)$

Discrete Exterior Calculus—Motivation

- Solving even very easy differential equations by hand can be hard! (Imagine harder equations...)
- If equations involve measured data (e.g., domain geometry), forget about solving them by hand!
- Instead, use <u>computation</u> to approximate solutions
- **<u>Basic idea:</u>**
  - replace domain with mesh
    – *oriented simplicial complex*
  - replace differential forms with values on mesh
    – *differential k-form becomes values on k-simplices*
  - replace differential operators with matrices
    – *e.g., signed incidence matrices give exterior derivative*





$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

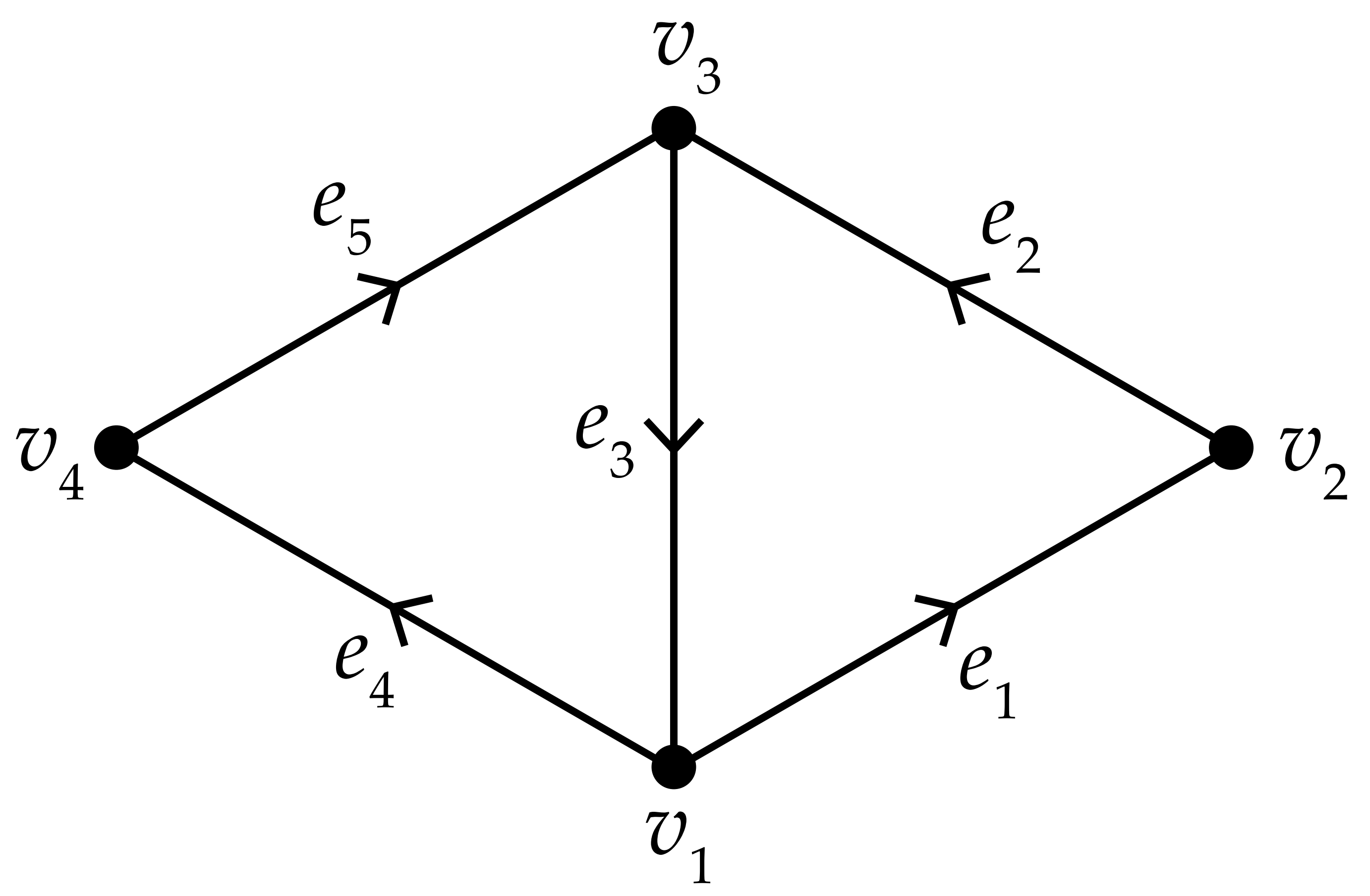
$$E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(pictures: Elcott et al, *"Stable, Circulation-Preserving, Simplicial Fluids"*)

Discrete Exterior Calculus — Basic Operations

- In smooth exterior calculus, we saw many operations (wedge product, Hodge star, exterior derivative, sharp, flat, …)
- In the discrete setting, the most commonly used operations are the **discrete exterior derivative** (d_k) and the **discrete Hodge star** ($\star_k$)
- Ultimately encoded as sparse matrices, applied to values stored on k-simplices of an oriented simplicial complex ("simplicial cochains").

$$\underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{d_0} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & w_4 & 0 \\ 0 & 0 & 0 & 0 & w_5 \end{bmatrix}}_{\star_1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}$$



Composition of Operators

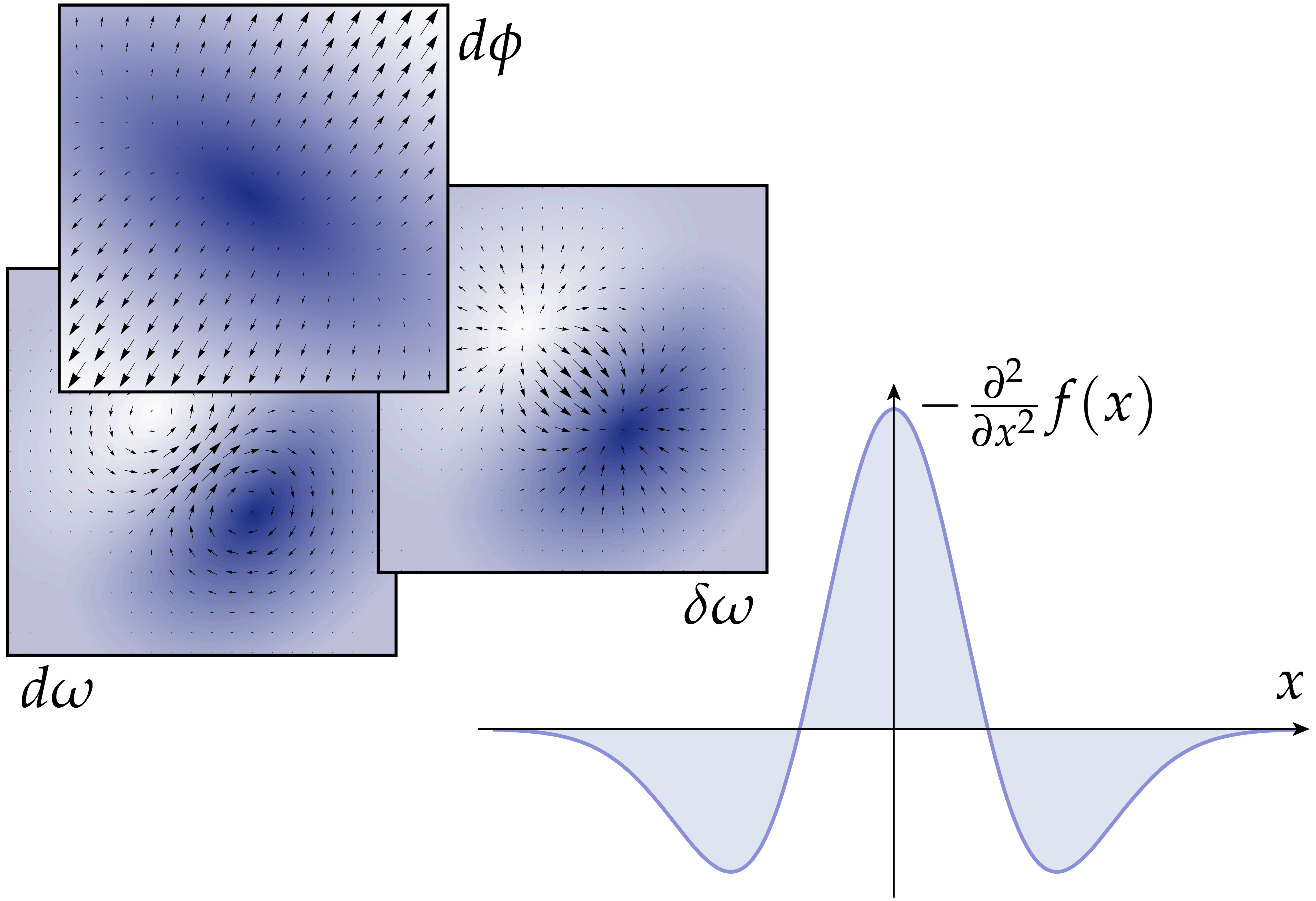
- By composing matrices, we can easily solve equations involving operators like those from vector calculus (grad, curl, div, Laplacian…) but in much greater generality (*e.g.*, curved surfaces, k-forms…) and on complicated domains (meshes)

$$\text{grad} \longrightarrow d_0 \qquad\qquad \text{curl} \longrightarrow \star_2 d_1$$

$$\text{div} \longrightarrow \star_0^{-1} d_0^T \star_1$$

$$\Delta \longrightarrow \star_0^{-1} d_0^T \star_1 d_0$$

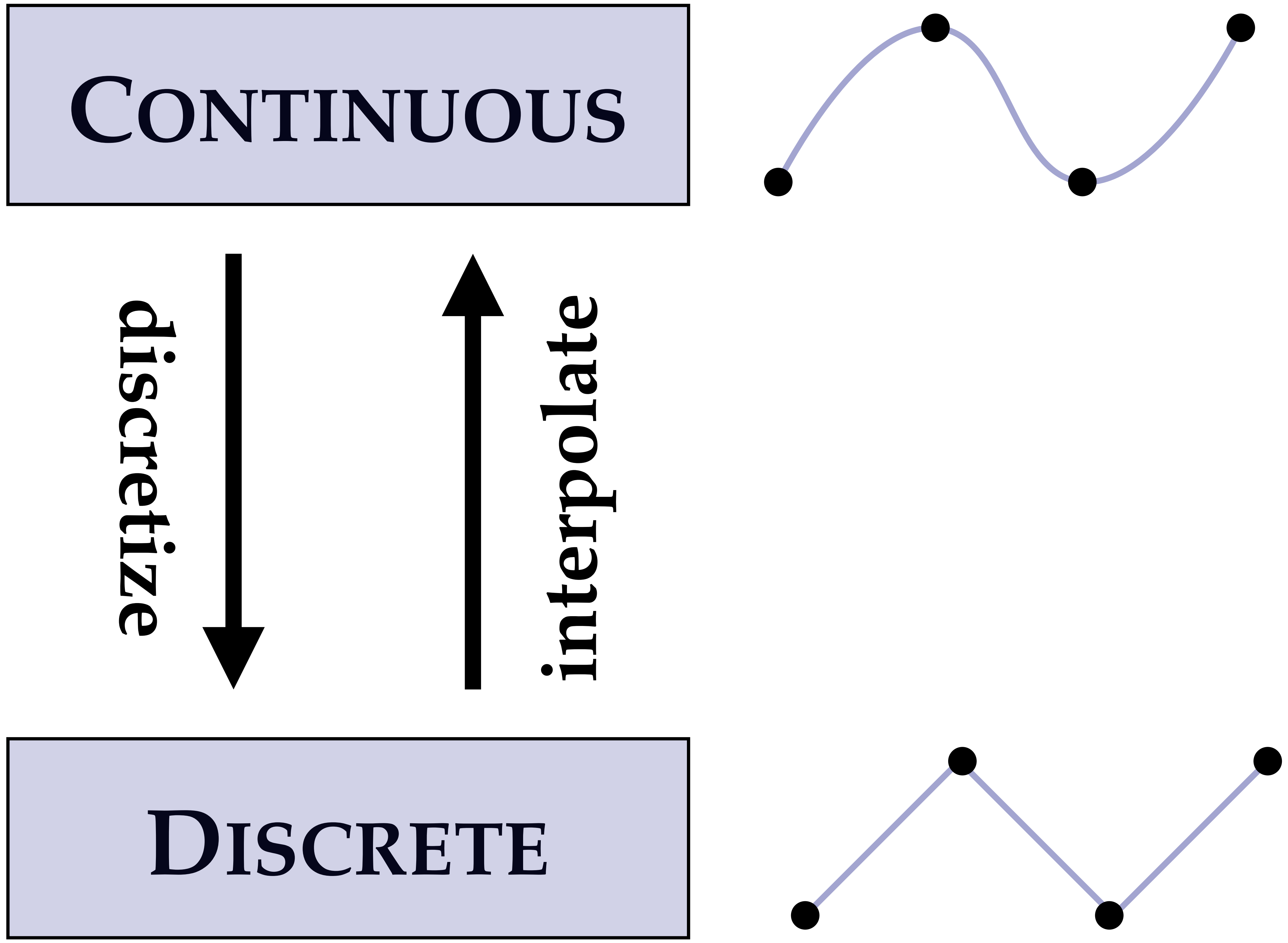
$$\Delta_k \longrightarrow d_{k-1} \star_{k-1}^{-1} d_{k-1}^T \star_k + \star_k^{-1} d_k^T \star_{k+1} d_k$$



Basic recipe: load a mesh, build a few basic matrices, solve a linear system.

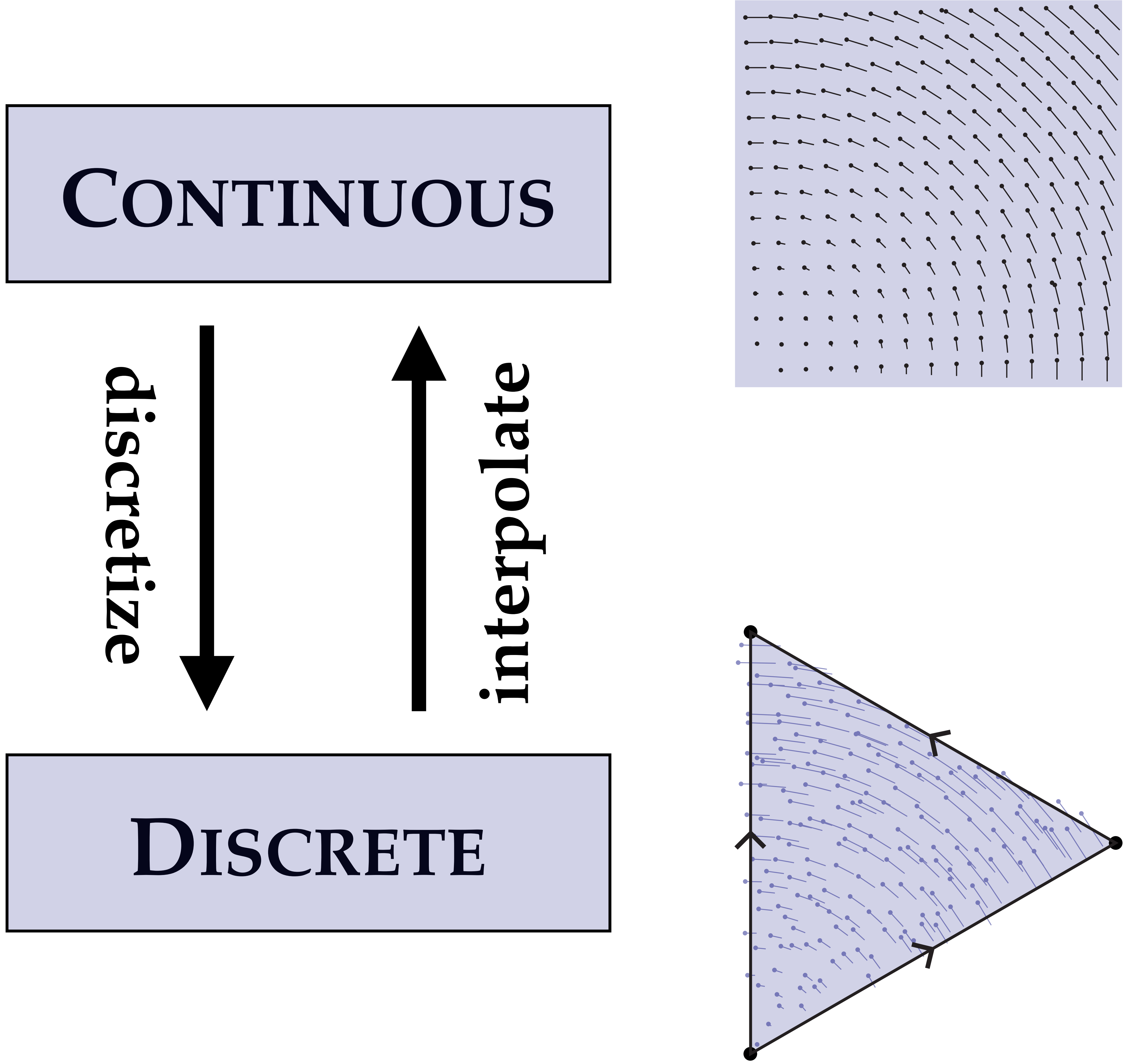
Discretization & Interpolation

- Two basic operations needed to translate between smooth & discrete quantities:
  - **Discretization** — given a continuous object, how do I turn it into a finite (or *discrete*) collection of measurements?
  - **Interpolation** — given a discrete object (representing a finite collection of measurements), how do I come up with a continuous object that agrees with (or *interpolates*) it?

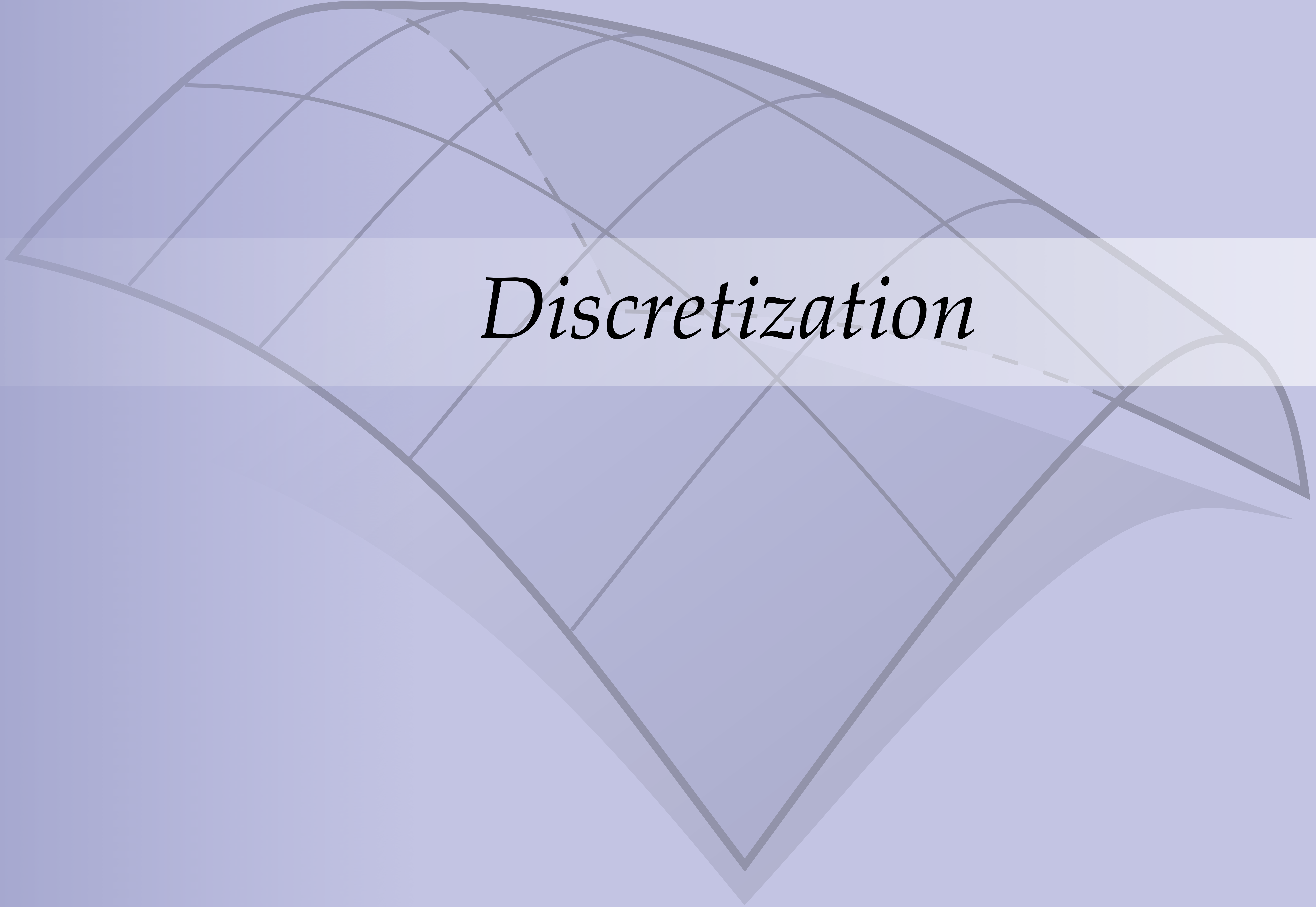


Discretization & Interpolation — Differential Forms

- In the case of differential k-forms:
  - **Discretization** happens via *integration* over oriented k-simplices (known as the *de Rham map*)
  - **Interpolation** is performed by taking linear combinations of continuous functions associated with k-simplices (known as *Whitney interpolation*)
- With these operations, becomes easy to translate some pretty sophisticated equations into algorithms!

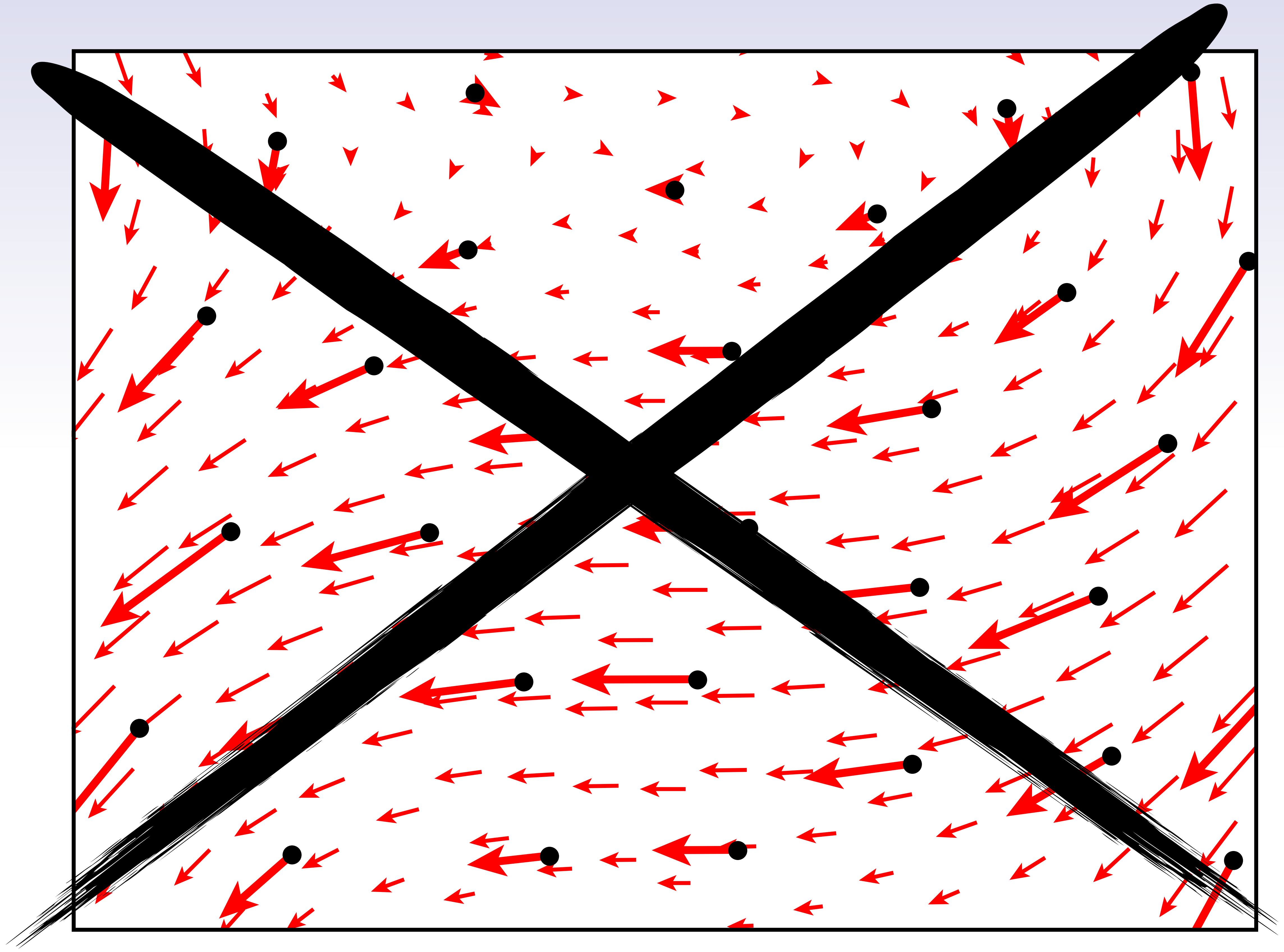


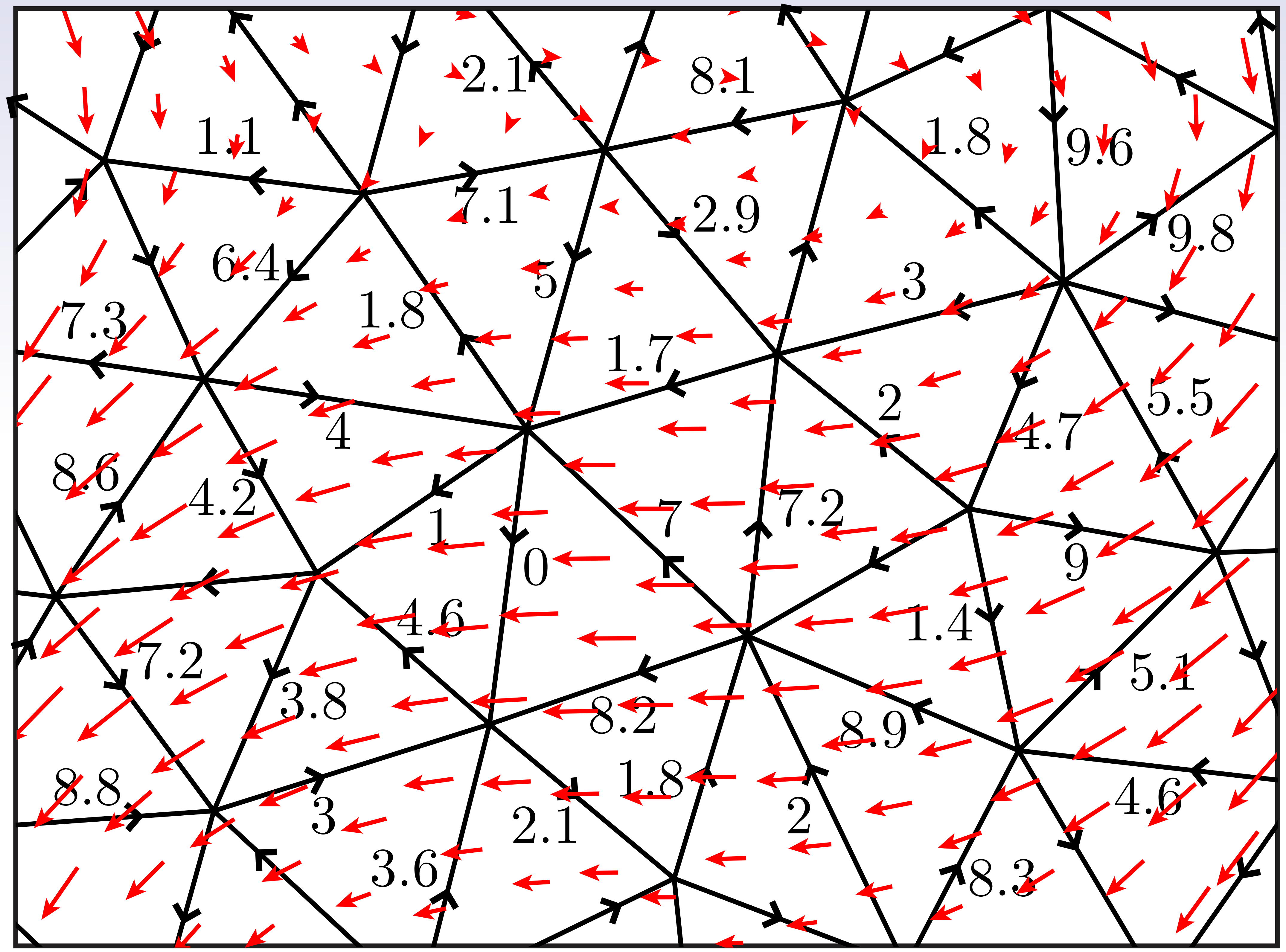
Discretization



Discretization — Basic Idea

How can we approximate a differential form with a finite amount of information?





Basic idea: integrate k-forms over k-simplices.

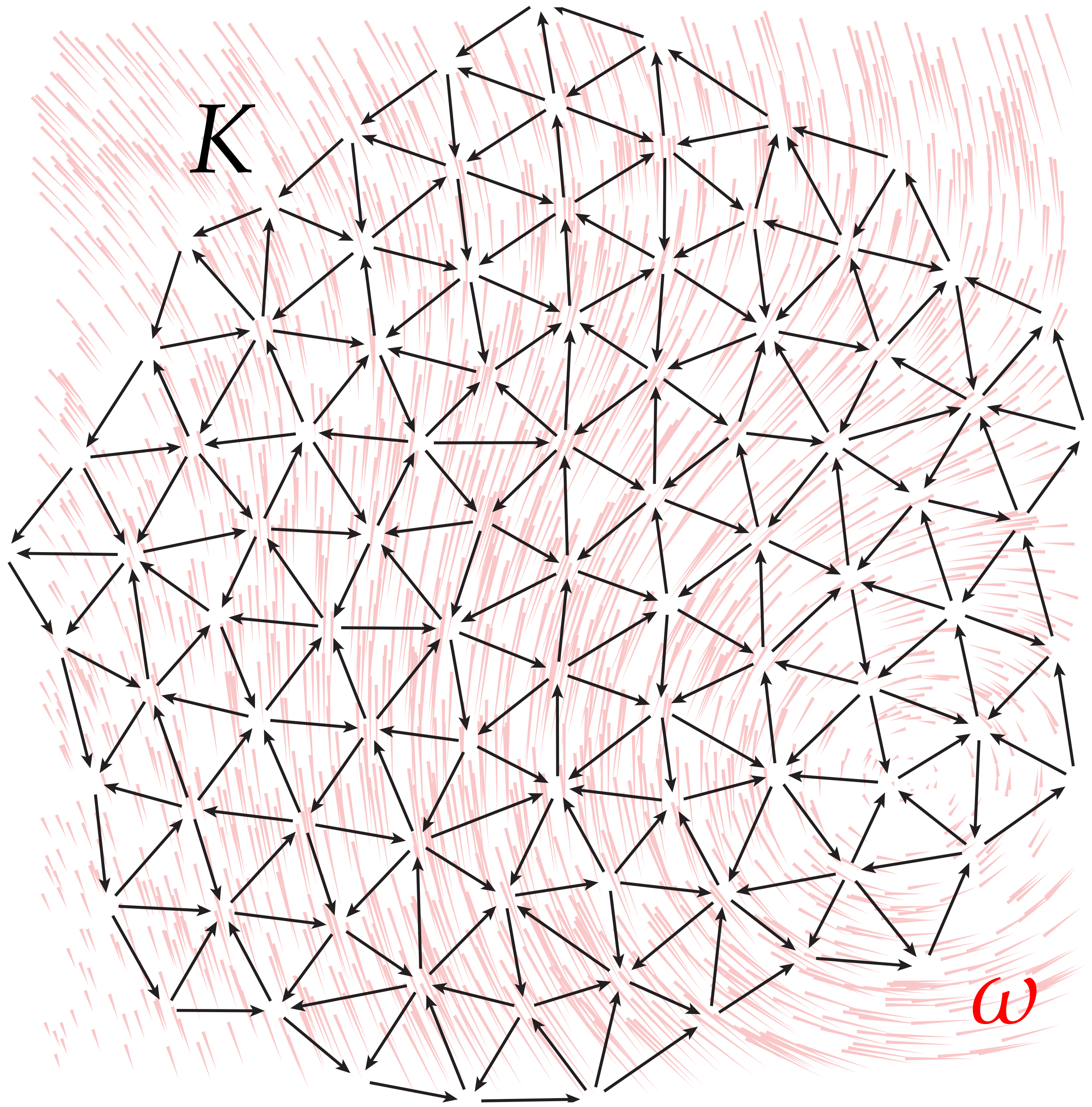
Doesn't tell us *everything* about the form… but enough to solve equations!

Discretization of Forms (de Rham Map)

Let ω be a differential k-form on $\mathbb{R}^n$, and let K be an oriented simplicial complex. For each k-simplex σ in K, the corresponding value of the discrete k-form is

$$\boxed{\hat{\omega}_\sigma := \int_\sigma \omega}$$

The map from continuous forms to discrete forms is called the *discretization map*, or sometimes the *de Rham map*.



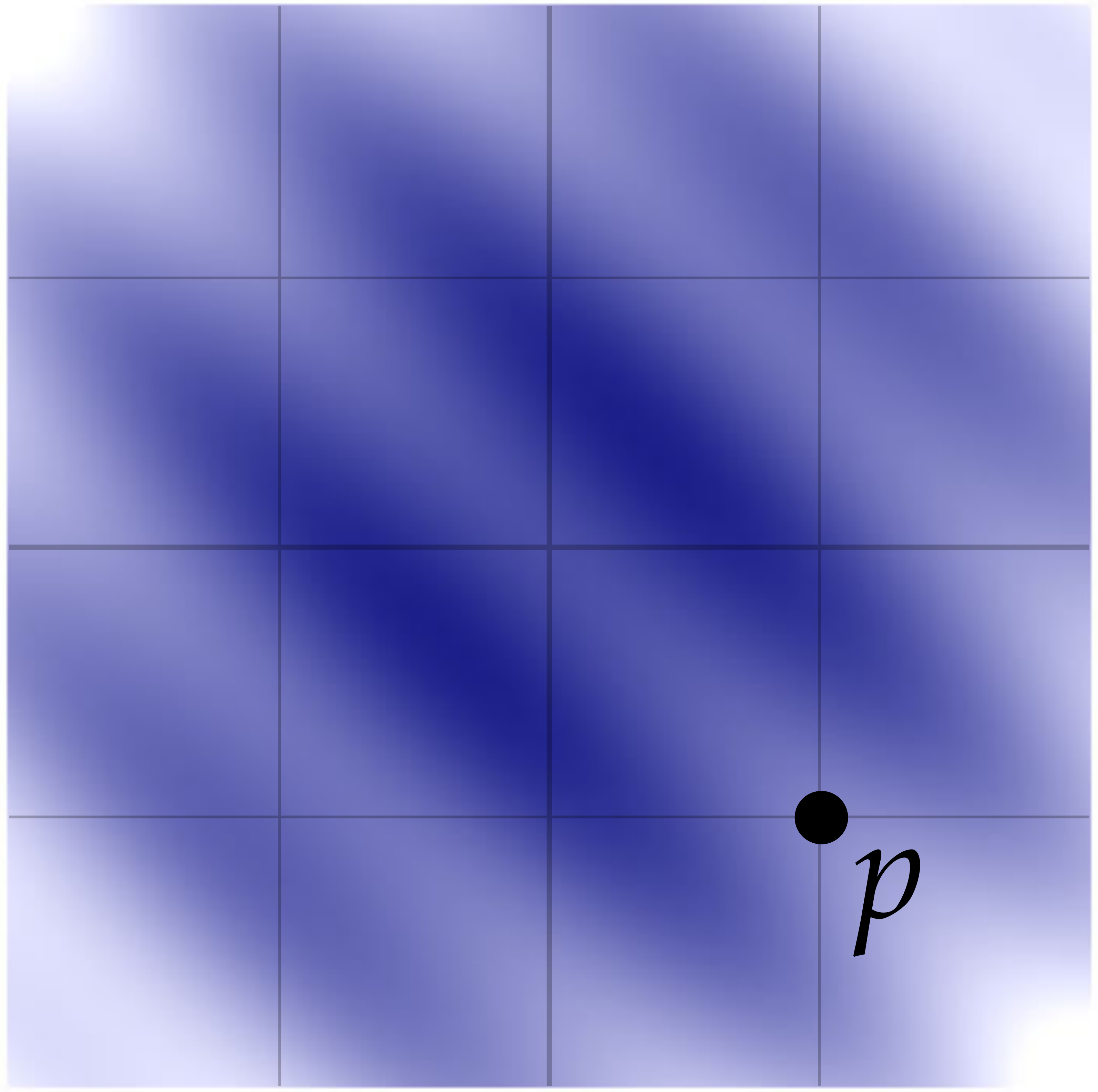
$$\mapsto \begin{bmatrix} 1.27 \\ 8.34 \\ -4.92 \\ 0.01 \\ \vdots \\ 2.96 \end{bmatrix}$$

$\hat{\omega}$

Key idea: *discretization* just means "integrate a k-form over k-simplices." Result is just a list of values.

Integrating a 0-form over Vertices

- Suppose we have a 0-form ϕ
- What does it mean to integrate it over a vertex v?
- Easy: just take the value of the function at the location p of the vertex!



Example:

$$\phi(x,y) := x^2 + y^2 + \cos(4(x+y))$$

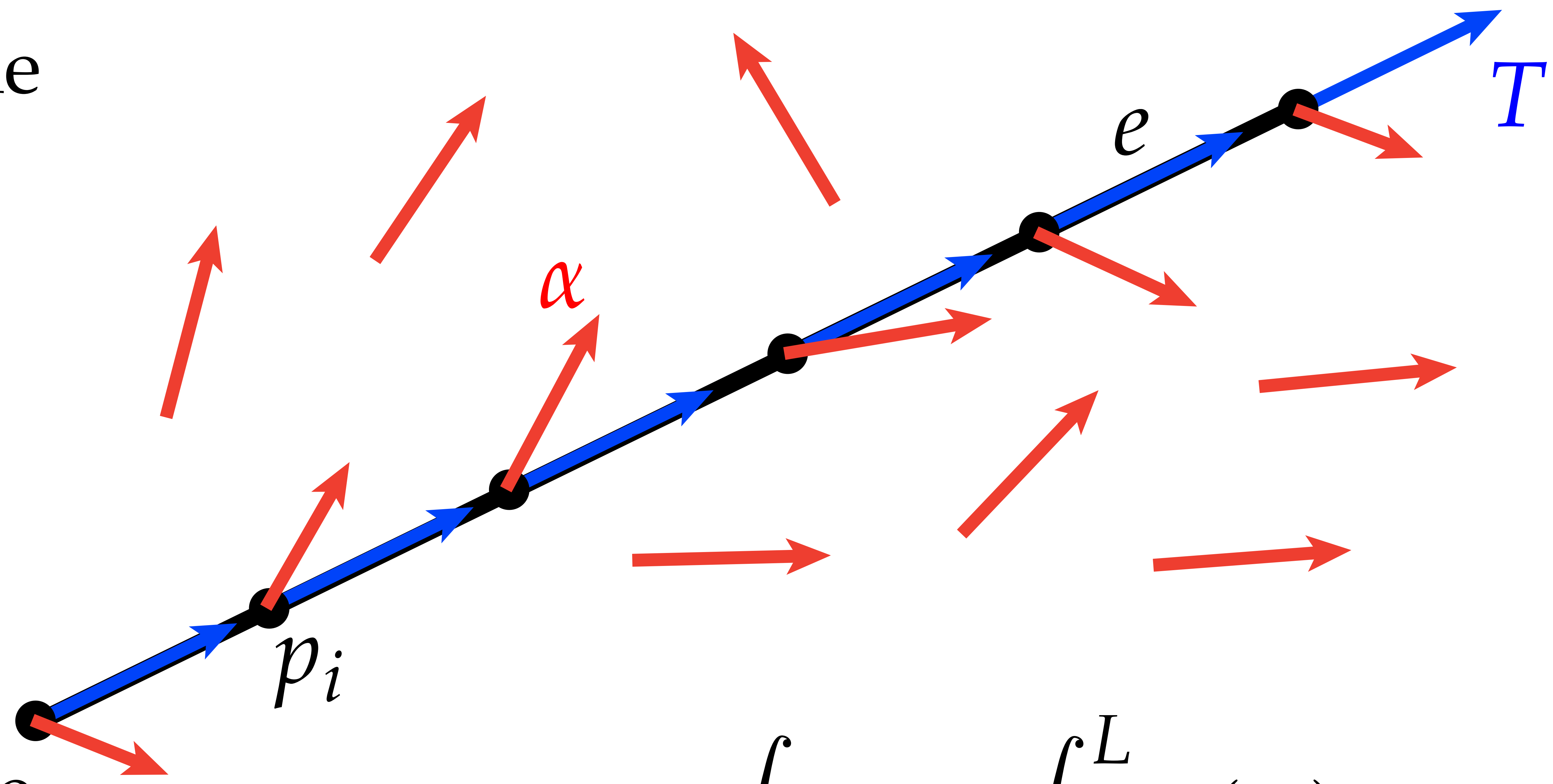
$$p = (1,-1)$$

$$\int_v \phi = \phi(p) = 1 + 1 + \cos(0) = 3$$

Key idea: integrating a 0-form at vertices of a mesh just "samples" the function

Integrating a 1-form over an Edge

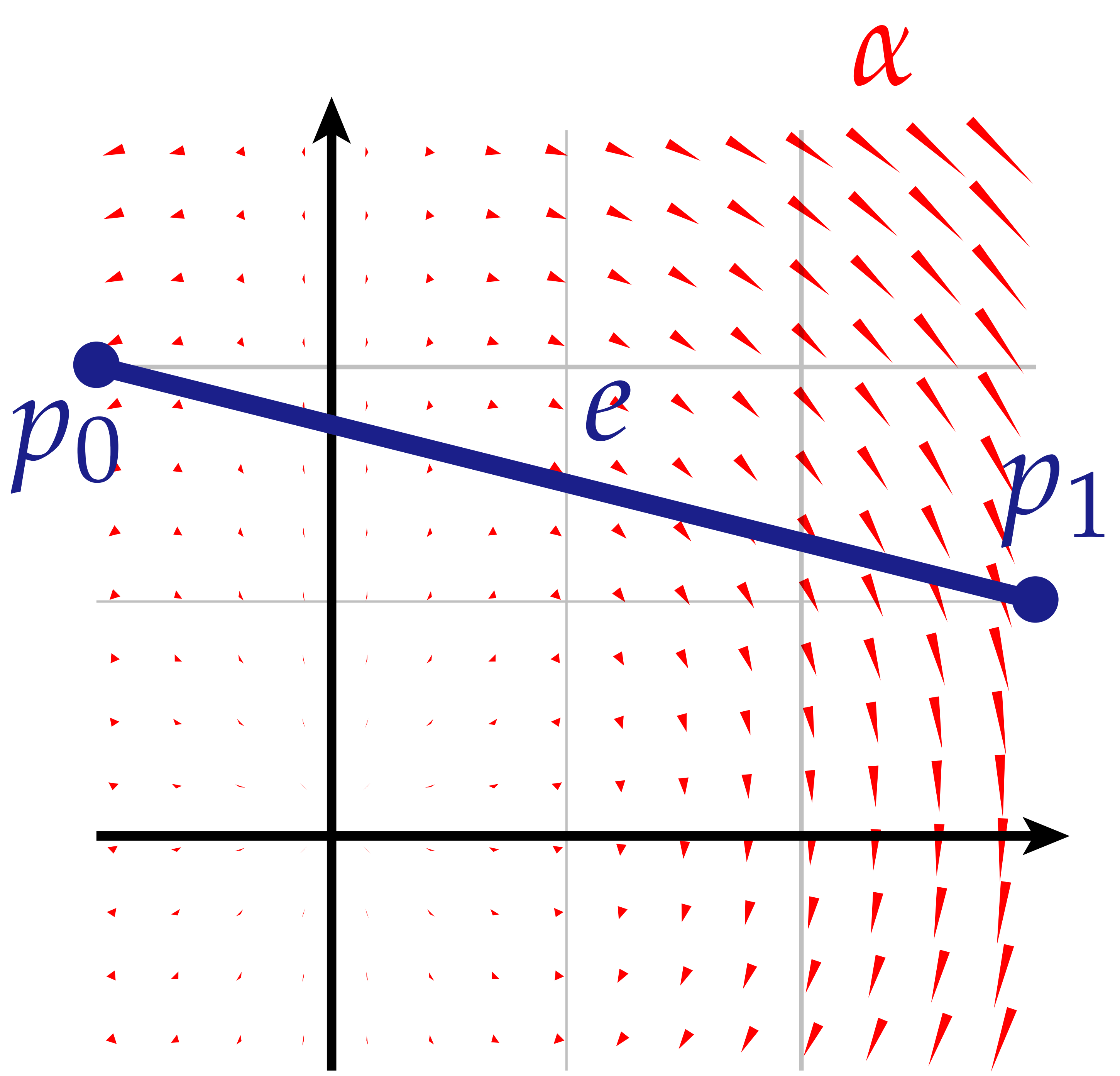
- Suppose we have a 1-form α in the plane
- How do we integrate it over an edge e?
- **Basic recipe:**
  - Compute unit tangent T
  - Apply α to T, yielding function $\alpha(T)$
  - Integrate this scalar function over edge
- Result gives "total circulation"
- Can use *numerical quadrature* for tough integrals
  - In practice, rare to actually integrate!
  - More often, discrete 1-form values come from, *e.g.*, operations on discrete 0-form



$$\hat{\alpha}_e := \int_e \alpha = \int_0^L \alpha(T)\, ds$$

$$\int_e \alpha \approx \text{length}(e) \left(\frac{1}{N} \sum_{i=1}^{N} \alpha_{p_i}(T) \right)$$

Integrating a 1-Form over an Edge—Example



In $\mathbb{R}^2$, consider a 1-form $\alpha := xydx - x^2dy$
and an edge e with endpoints
$$\begin{aligned} p_0 &:= (-1,2) \\ p_1 &:= (3,1) \end{aligned}$$

Q: What is $\int_e \alpha$?

A: Let's first compute the edge length L and unit tangent T:

$$L := |p_1 - p_0| = \sqrt{17} \qquad T := (p_1 - p_0)/L = (4,-1)/\sqrt{17}$$

Hence, $\alpha(T) = (4xy + x^2)/\sqrt{17}$.

An arc-length parameterization of the edge is given by

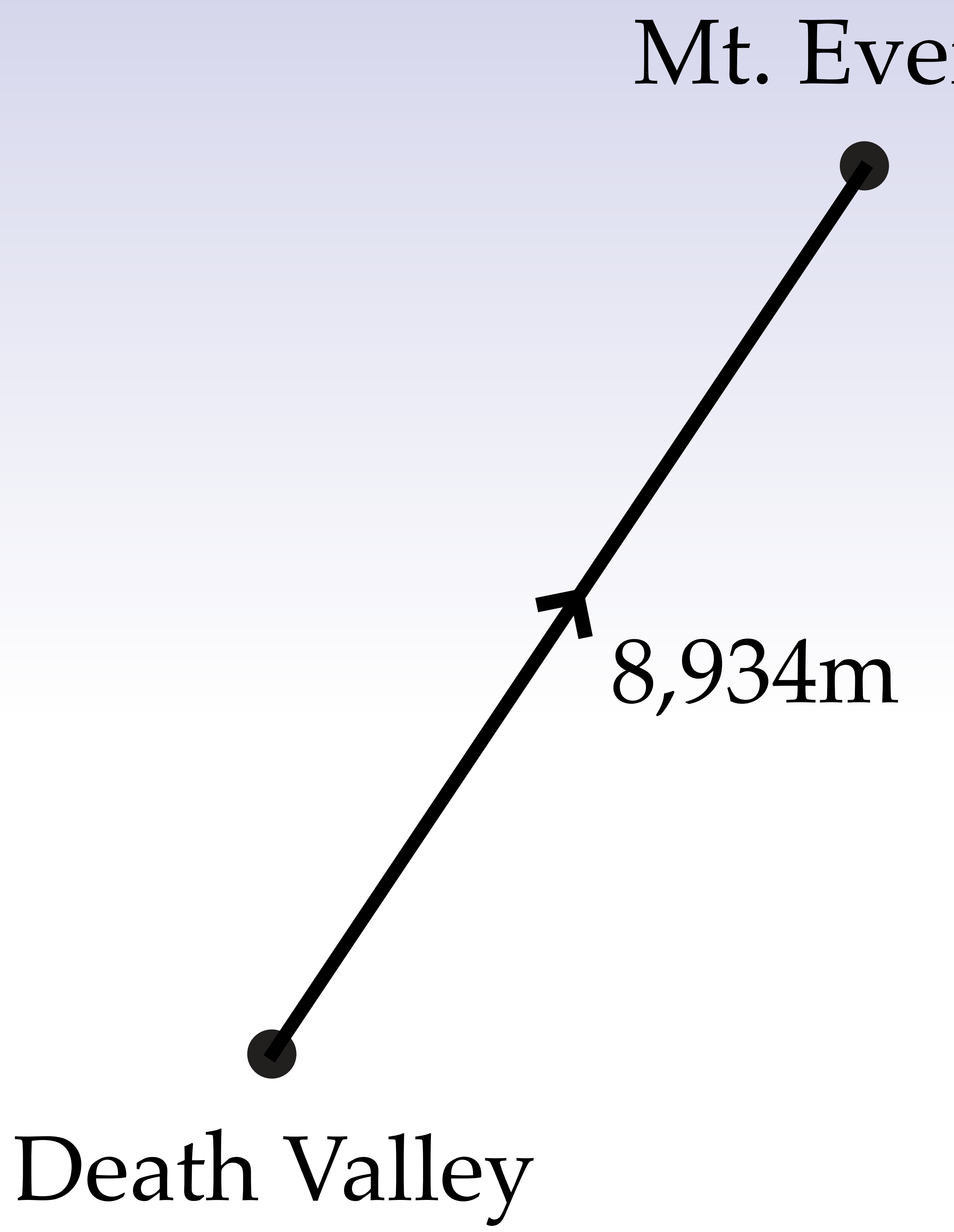
$$p(s) := p_0 + \tfrac{s}{L}(p_1 - p_0), \quad s \in [0,L]$$

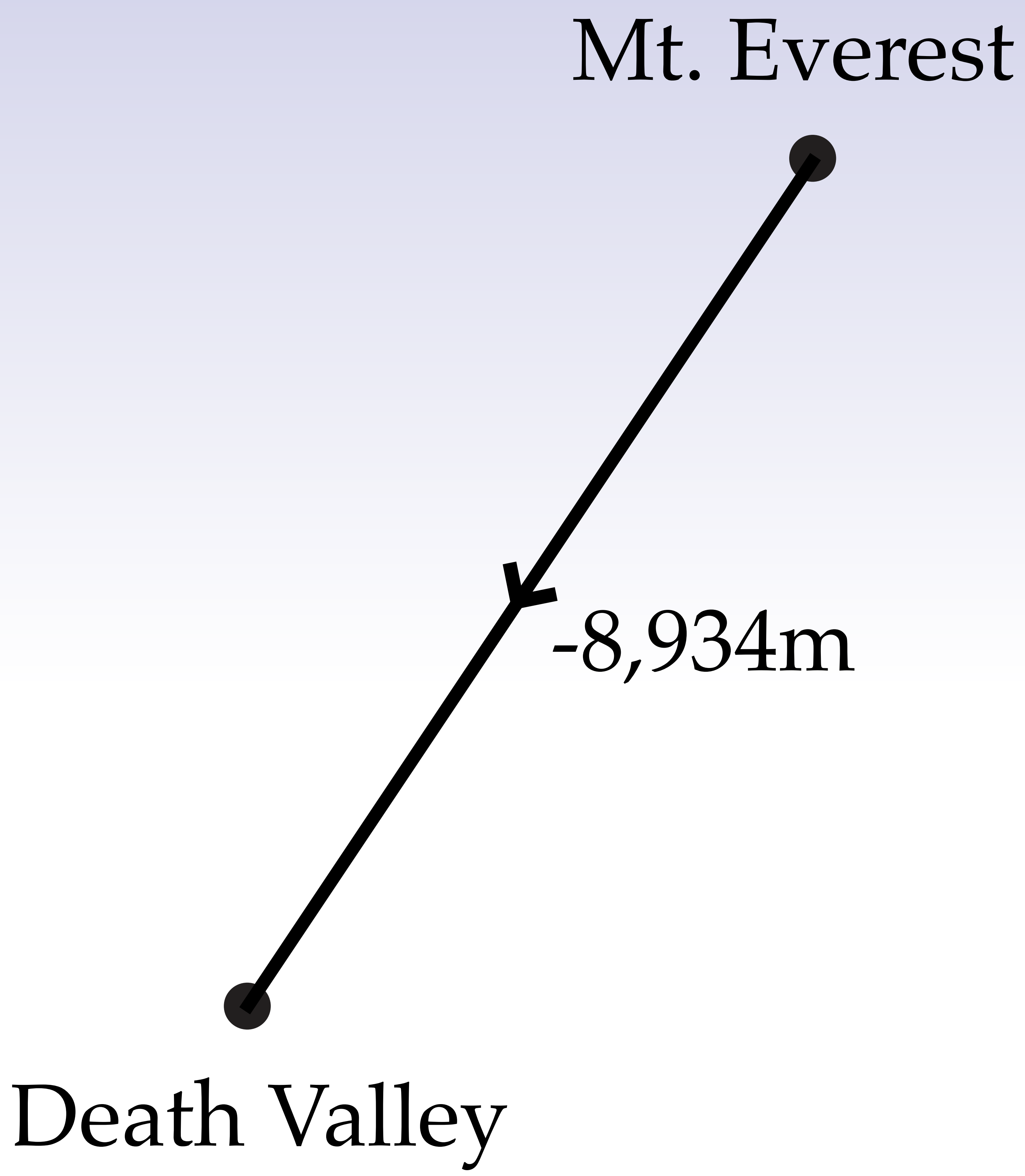
By plugging in all these expressions/values, our integral simplifies to

$$\int_0^L \alpha(T)_{p(s)}\, ds = \tfrac{7}{17}\int_0^L 4s - L\, ds = 7$$

…why not let $T := (p_0\text{-}p_1)/L$?

Orientation & Integration

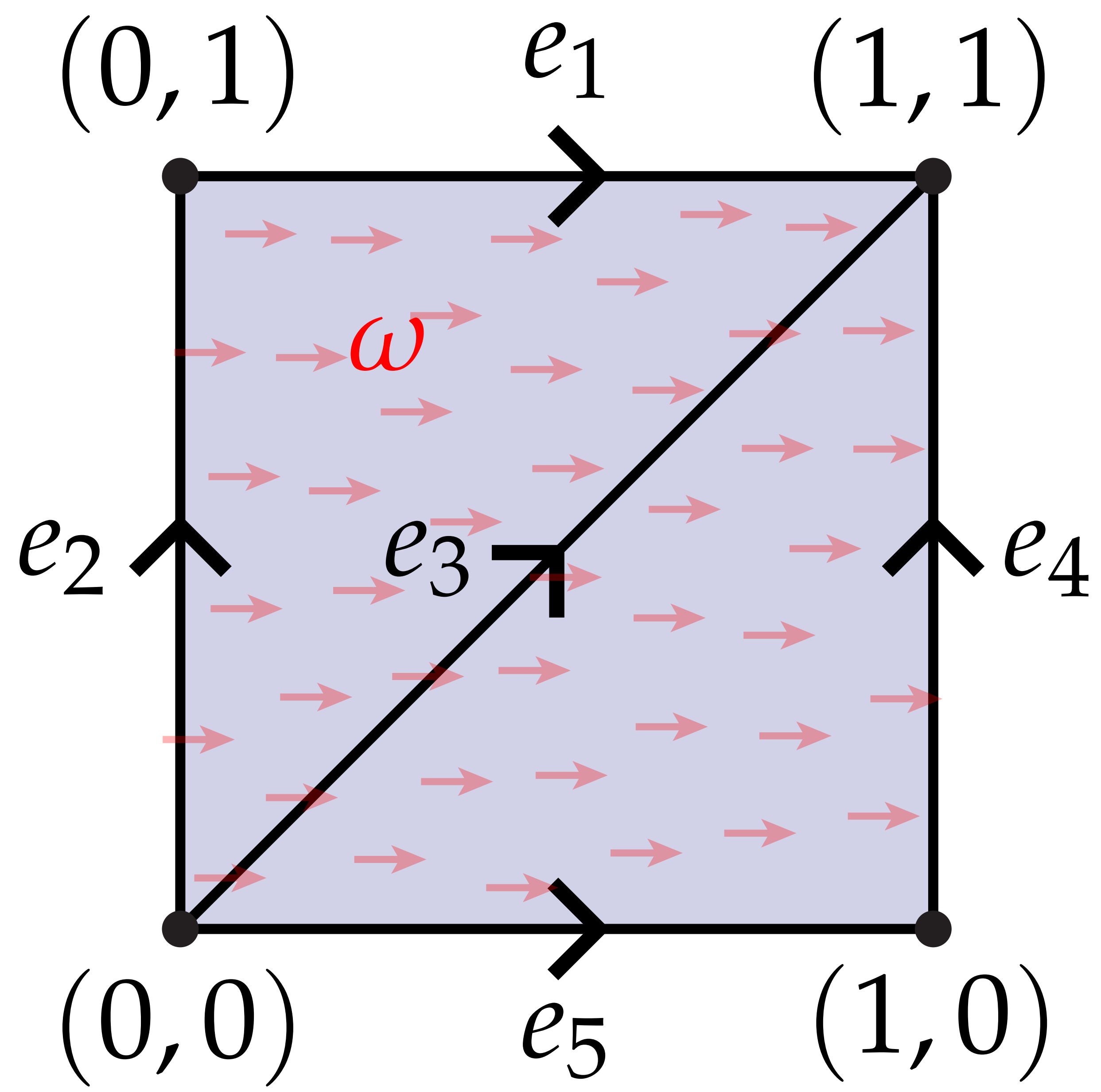




$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a) = -(\phi(a) - \phi(b)) = -\int_b^a \frac{\partial \phi}{\partial x} dx$$

Discretizing a 1-form — Example

Example. Consider the unit square $[0,1]^2$ with coordinates (x,y). Let K be the oriented simplicial complex shown on the right, and consider the differential 1-form $\omega := 2dx$. We can discretize ω by integrating it over each edge of K:



$$\widehat{\omega}_1 = \int_{e_1} \omega = \int_0^1 \omega\left(\tfrac{\partial}{\partial x}\right) d\ell = \int_0^1 2\, d\ell = 2.$$

$$\widehat{\omega}_2 = \int_{e_2} \omega = \int_0^1 \omega\left(\tfrac{\partial}{\partial y}\right) d\ell = \int_0^1 0\, d\ell = 0.$$

$$\widehat{\omega}_3 = \int_{e_3} \omega = \int_0^{\sqrt{2}} \omega\left(\tfrac{1}{\sqrt{2}}\left(\tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial y}\right)\right) d\ell = \int_0^{\sqrt{2}} \tfrac{2}{\sqrt{2}}\, d\ell = 2.$$

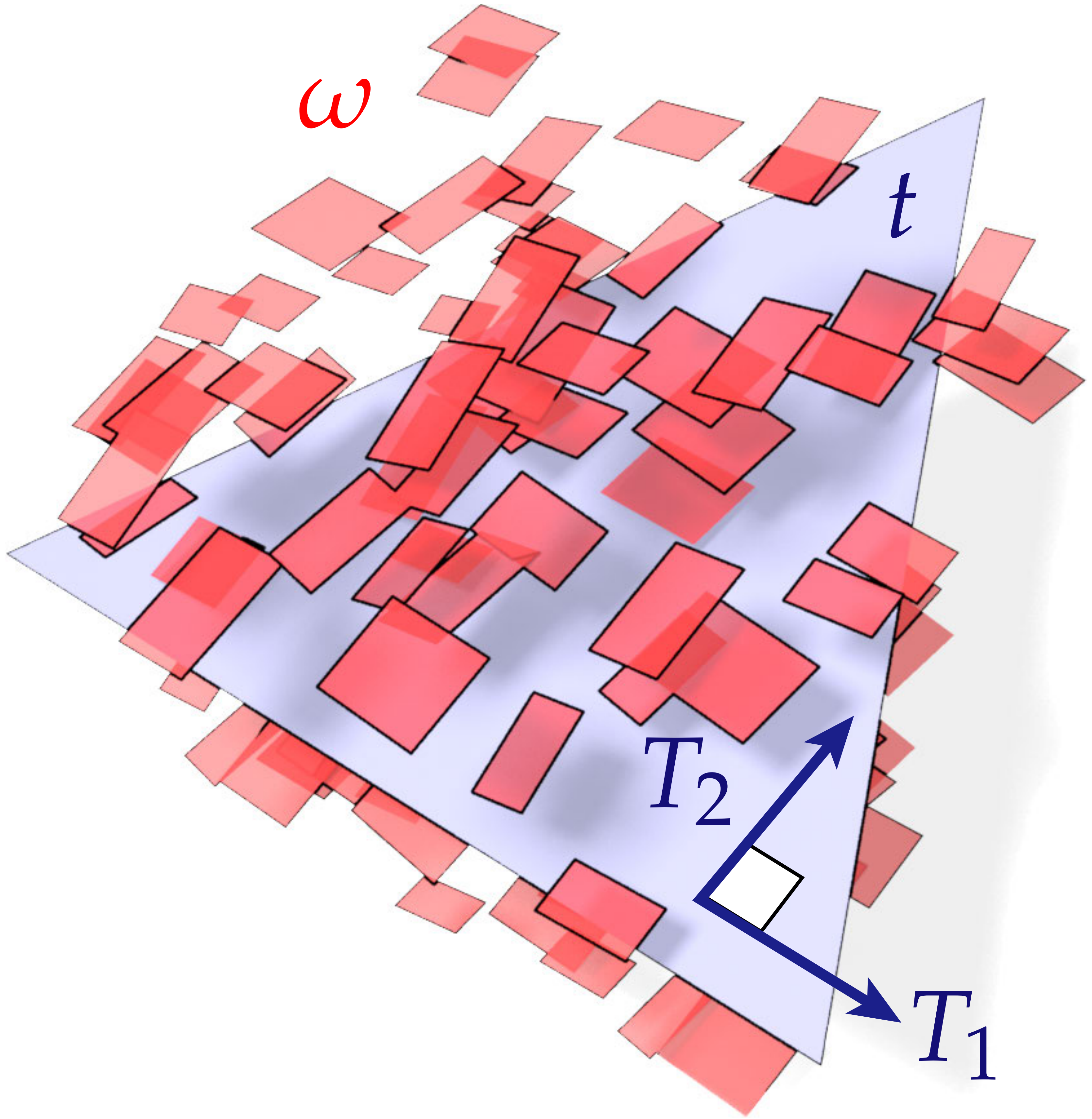
$\ldots$

Question: Why does $\widehat{\omega}_1 = \widehat{\omega}_3$?

Integrating a 2-form Over a Triangle

- Suppose we have a 2-form ω in R^3
- How do we integrate it over a triangle t?
- Similar recipe to 1-form:
  - Compute orthonormal basis T_1,T_2 for triangle
  - Apply ω to T_1,T_2, yielding a function $\omega(T_1,T_2)$
  - Integrate this scalar function over triangle
- Value encodes how well triangle is "lined up" with 2-form on average, times area of triangle
- Again, rare to actually integrate explicitly!

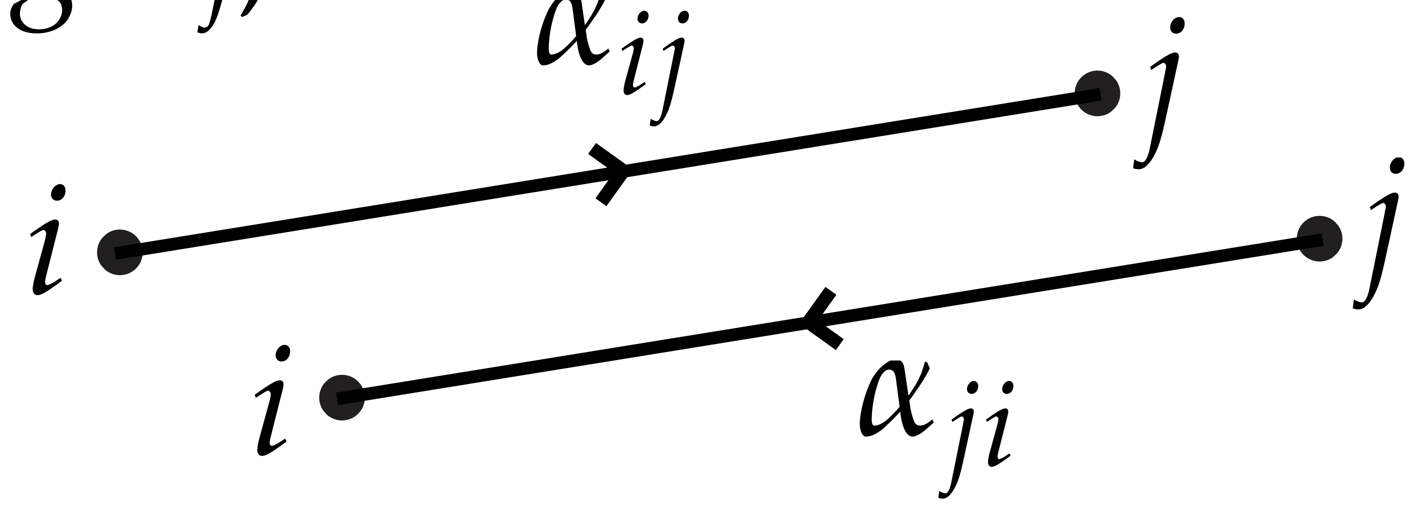
Q: Here, what determines the *orientation* of t?



$$\int_t \omega = \int_t \omega(T_1, T_2)\, dA$$
$$\approx \text{area}(t) \left(\frac{1}{N} \sum_{i=1}^{N} \omega_{p_i}(T_1, T_2) \right)$$

Orientation and Integration

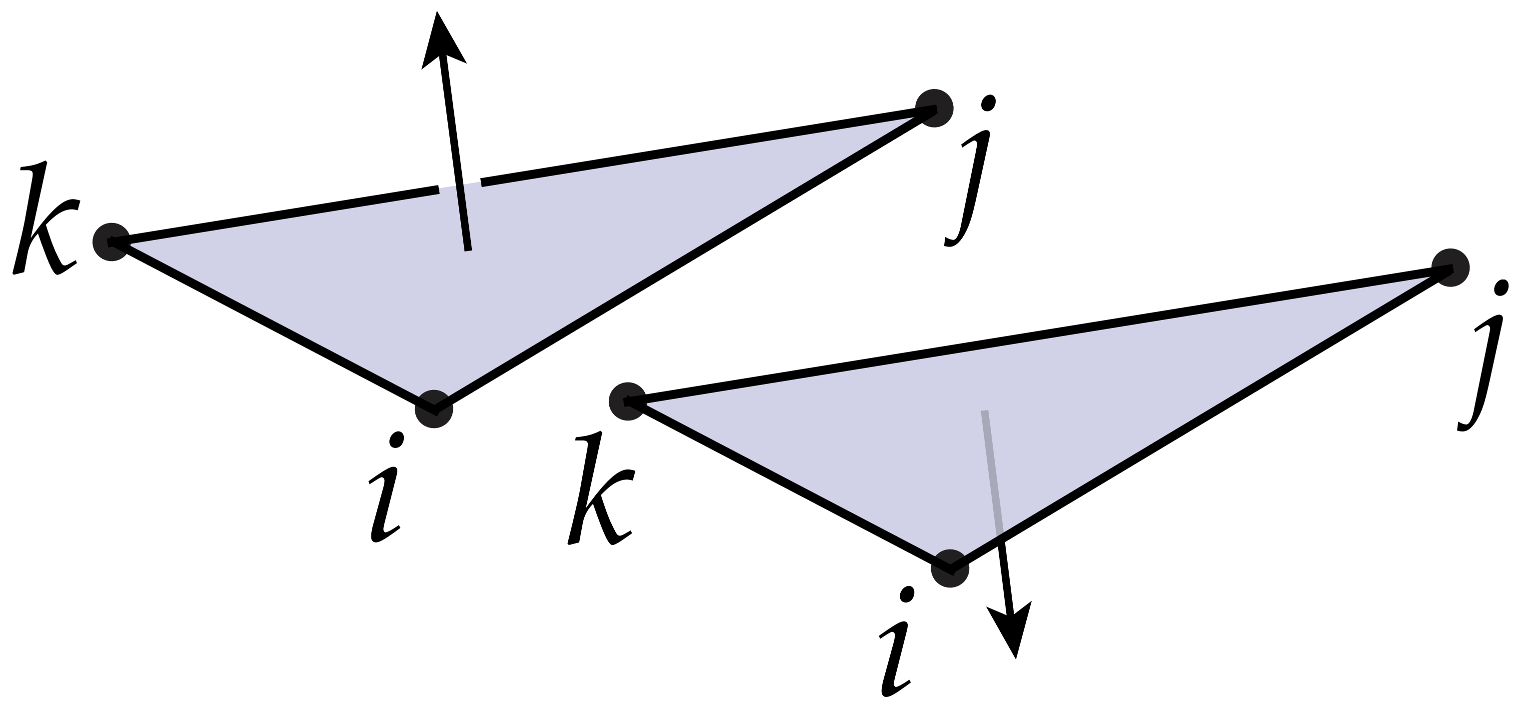
- In general, reversing the **orientation** of a simplex will reverse the **sign** of the integral.
- *E.g.*, suppose we have a discrete 1-form α. Then for each edge ij,



$$\alpha_{ij} = -\alpha_{ji}$$

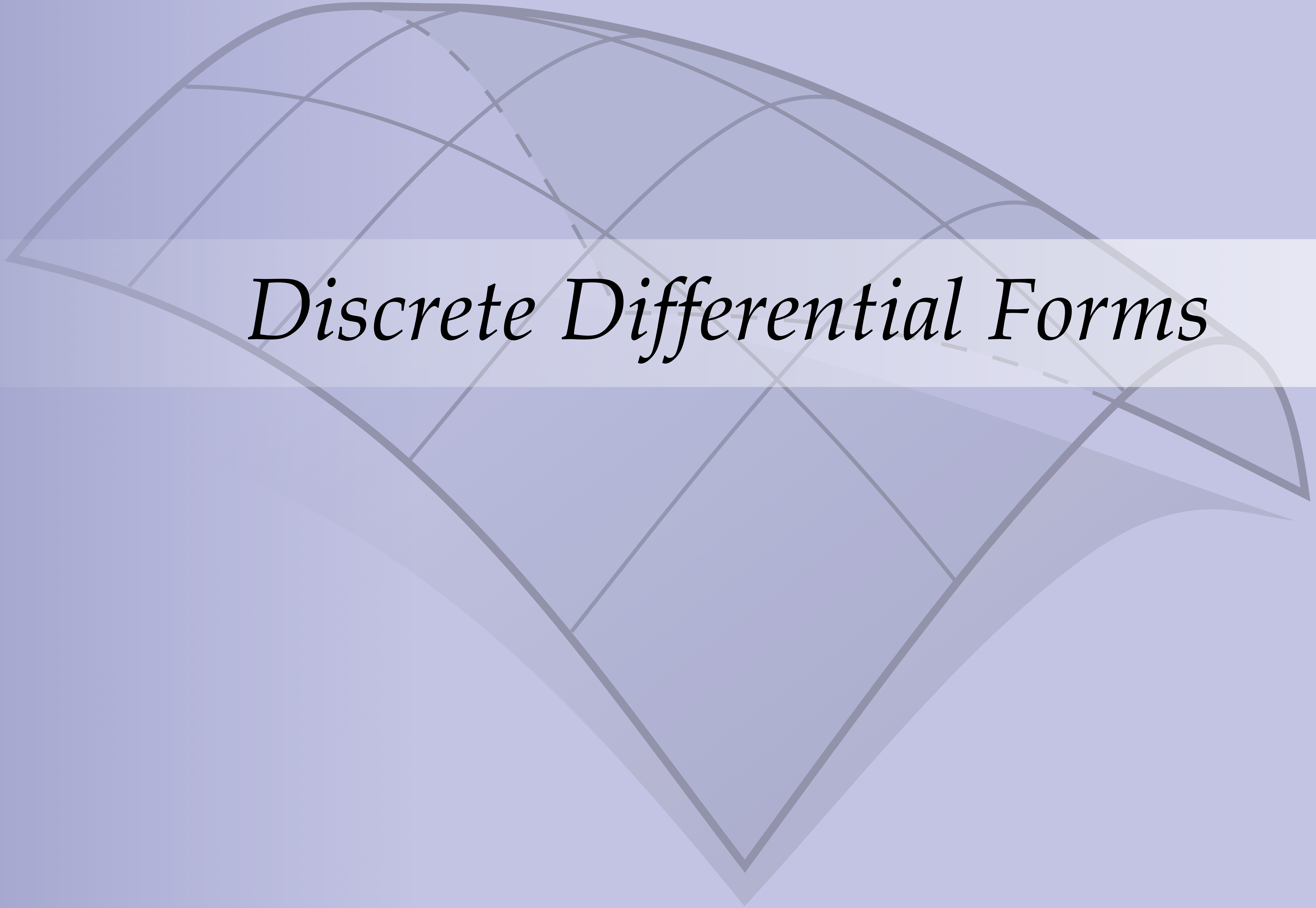
- **Q:** Suppose we have a 2-form β. What do you think the relationship is between…

$$\beta_{ijk} \;=\; \beta_{jki} \qquad\qquad \beta_{jik} = -\beta_{kij}$$



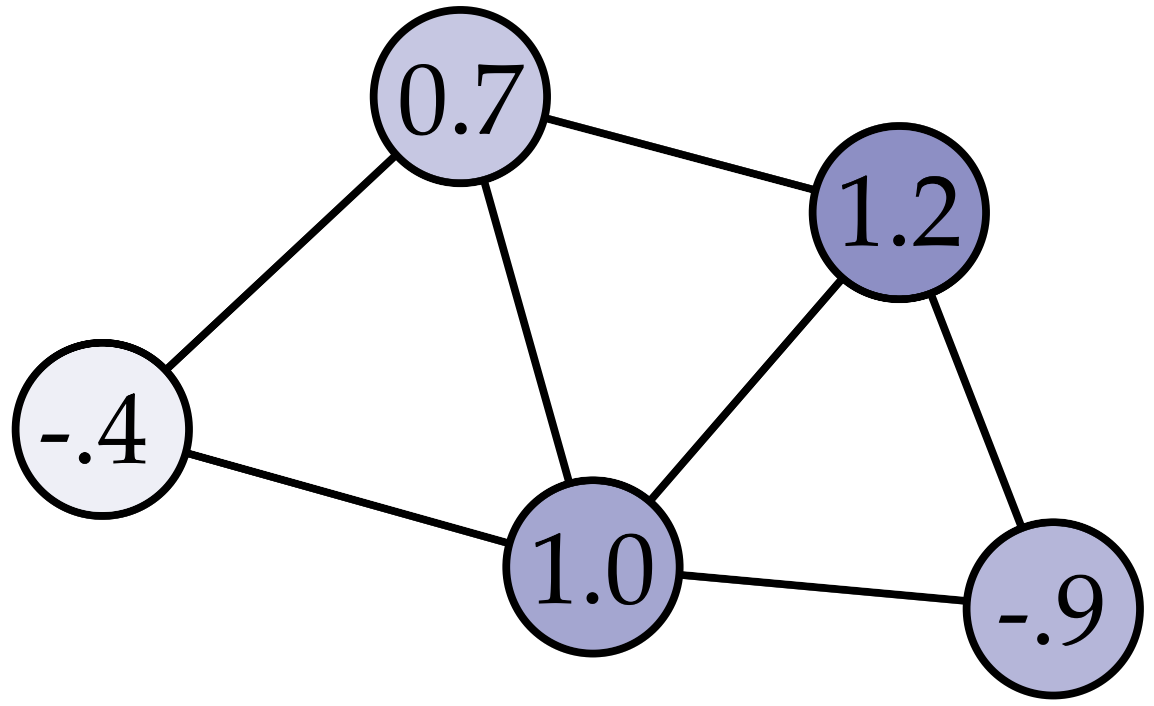
- **Q:** What's the rule in general?
- **A:** Discrete k-form values change sign under odd permutation. (Sound familiar?)

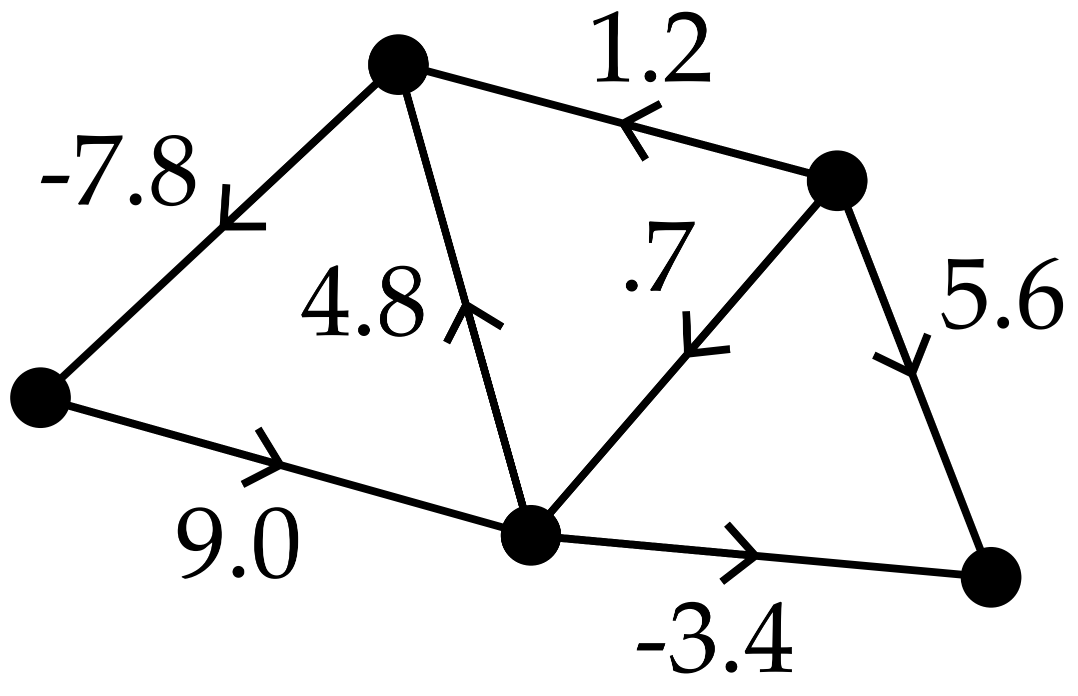
Discrete Differential Forms

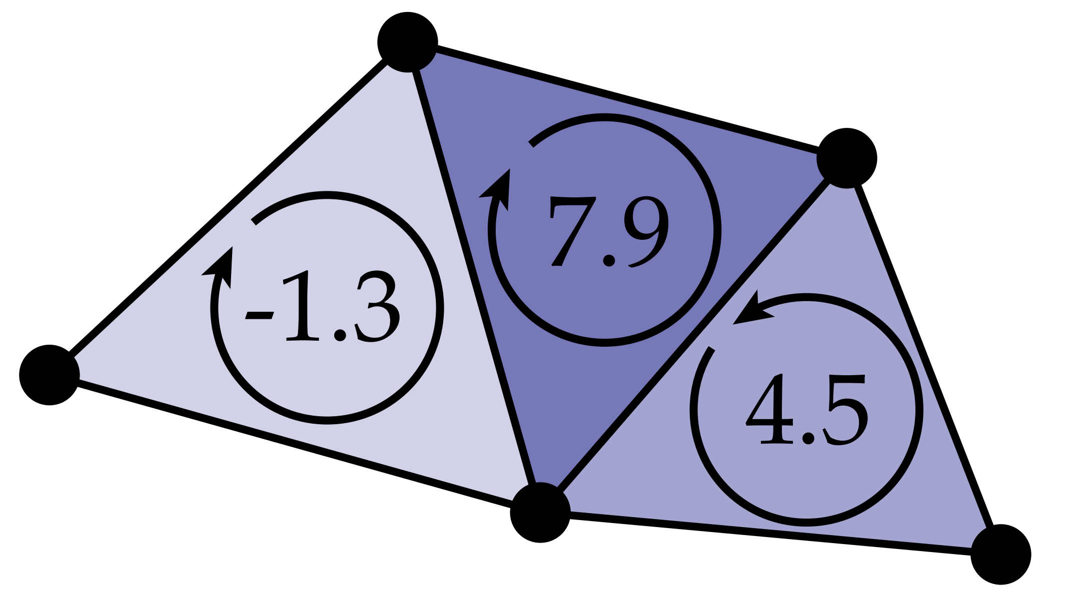


Discrete Differential k-Form

- Abstractly, a *discrete differential k-form* is just any assignment of a value to each oriented *k*-simplex.
- For instance, in 2D:
  - values at **vertices** encode a discrete **0-form**
  - values at **edges** encode a discrete **1-form**
  - values at **faces** encode a discrete **2-form**
- *Conceptually*, values represent integrated k-forms
- *In practice*, almost never comes from direct integration!
- Typically, values start at vertices (samples of some function); 1-forms, 2-forms, *etc.*, arise from applying operators like the *discrete exterior derivative* (next lecture)



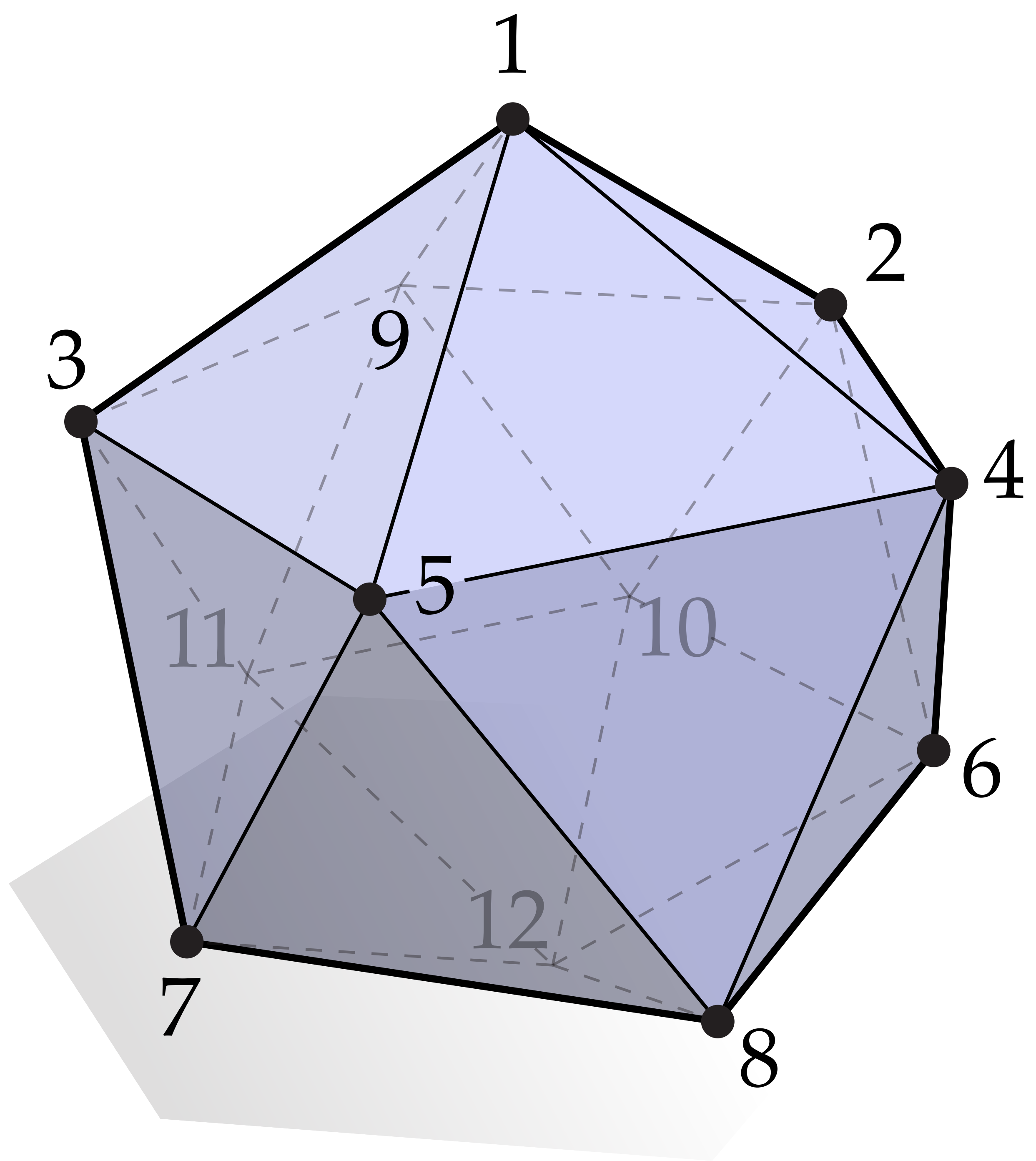




Matrix Encoding of Discrete Differential k-Forms

- We can encode a discrete k-form as a column vector with one entry for every k-simplex.
- Simplest example: a discrete 0-form can be encoded as a vector with $|V|$ entries
- To do so, we need to first assign a unique *index* to each k-simplex
  - The order of these indices can be completely arbitrary
  - Just need some way to put elements of the mesh into correspondence with entries of the vector

$$\phi : V \to \mathbb{R}$$

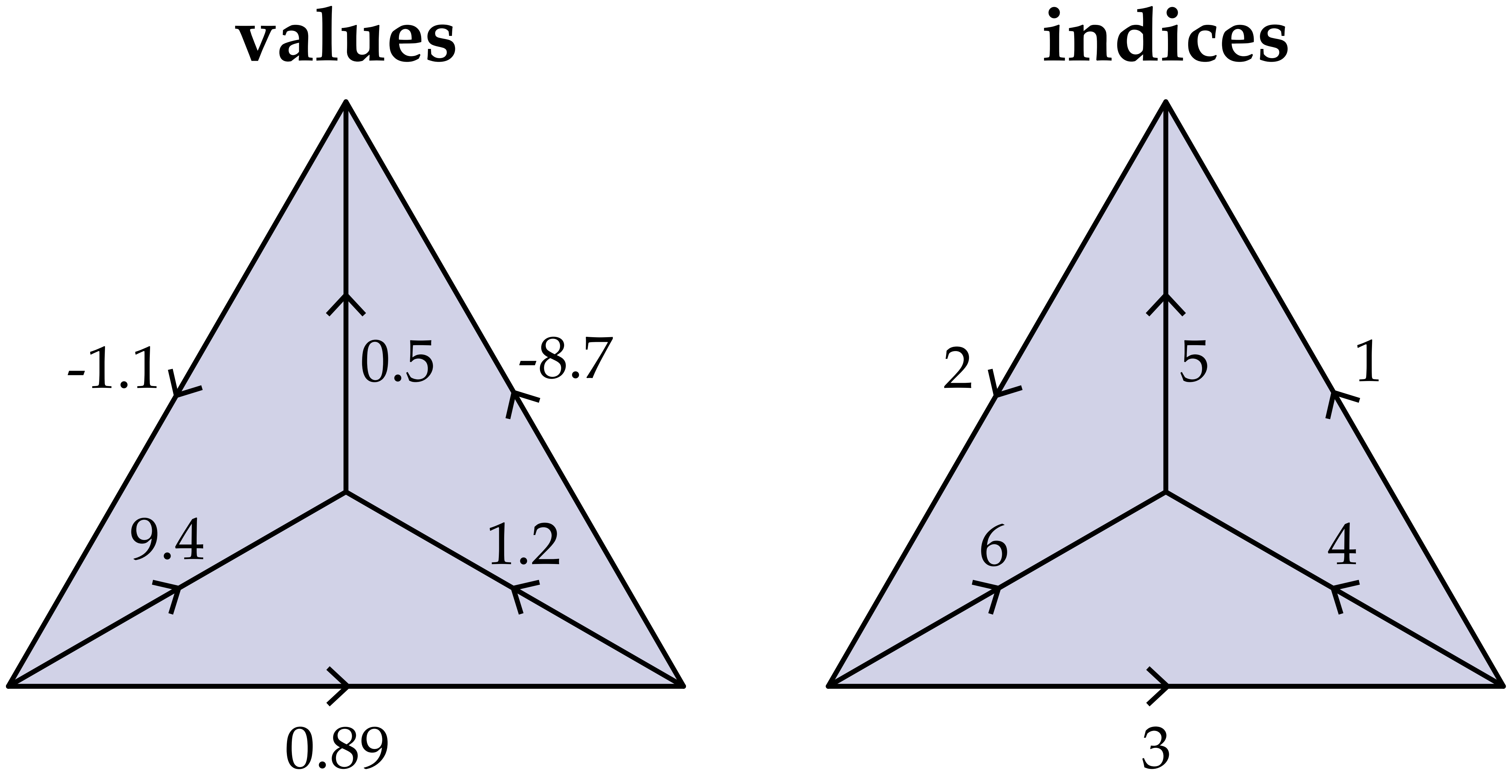


$$\phi = \left[\begin{array}{ccc} \phi_1 & \cdots & \phi_{|V|} \end{array}\right]$$

Careful: In code, indices often start from 0 rather than 1!

Matrix Encoding of Discrete Differential 1-Form

- A discrete differential 1-form is a value per edge of an oriented simplicial complex.
- To encode these values as a column vector, we must first assign a unique index to each edge of our complex.
- We can then assign values to the entries of a vector $\hat{\alpha} \in \mathbb{R}^{|E|}$ encoding the discrete 1-form.

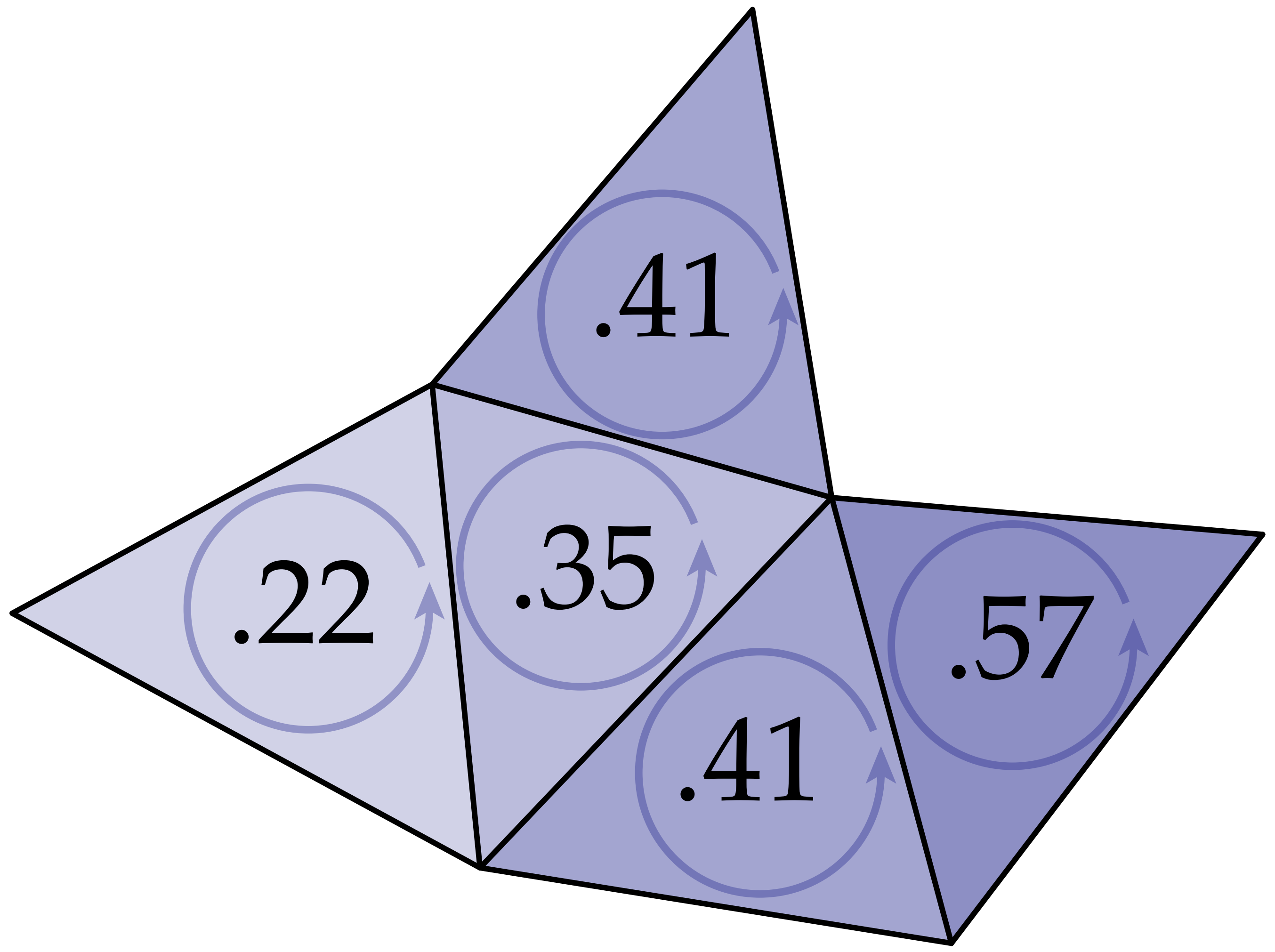


$$\hat{\alpha} = \left[\begin{array}{cccccc} -8.7 & -1.1 & 0.89 & 1.2 & 0.5 & 9.4 \end{array} \right]$$

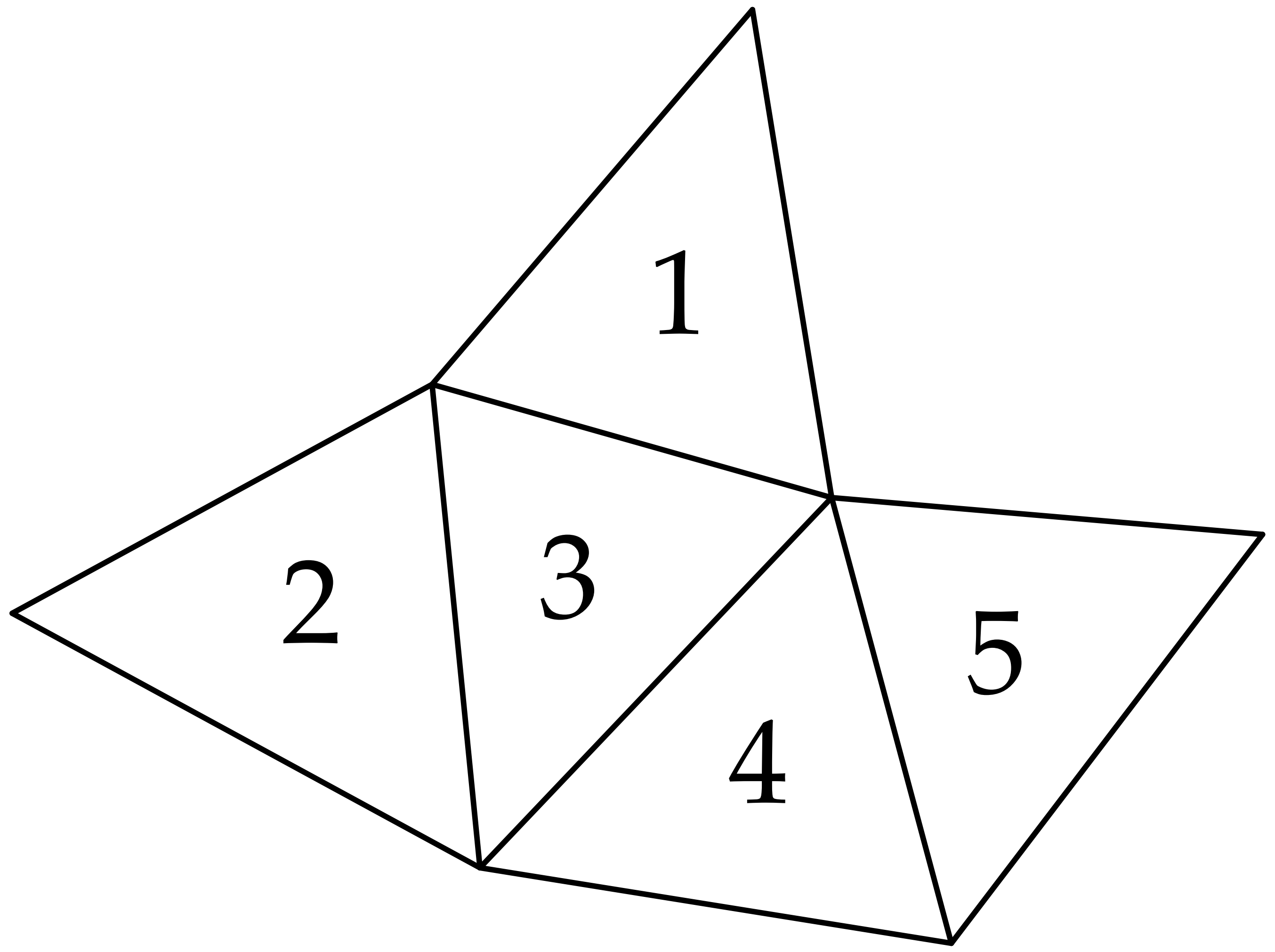
Careful: if we ever change the orientation of an edge, we must also negate the value in our vector!

Matrix Encoding of Discrete Differential 2-Form

- Same idea for encoding a discrete differential 2-form as a vector $\hat{\omega} \in \mathbb{R}^{|F|}$
- Assign indices to each 2-simplex; now we know which values go in which entries



values



indices

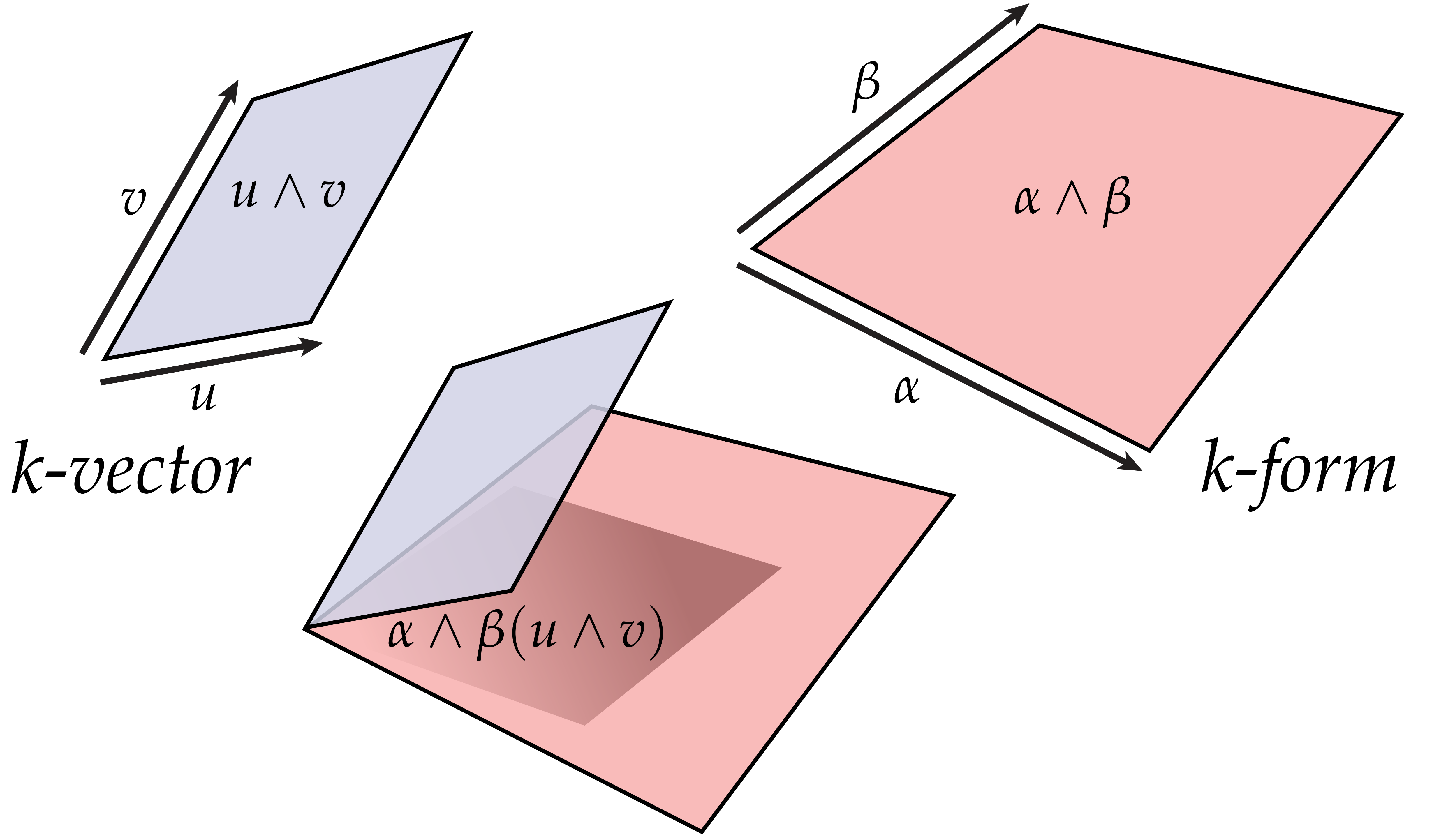
$$\omega = \left[\ .41 \quad .22 \quad .35 \quad .41 \quad .57\ \right]$$

As always, changing the orientation of a triangle *ijk* will reverse the sign of the corresponding entry.

Chains & Cochains

In the discrete setting, duality between *"things that get measured"* (k-vectors) and *"things that measure"* (k-forms) is captured by notion of *chains* and *cochains*.

SMOOTH



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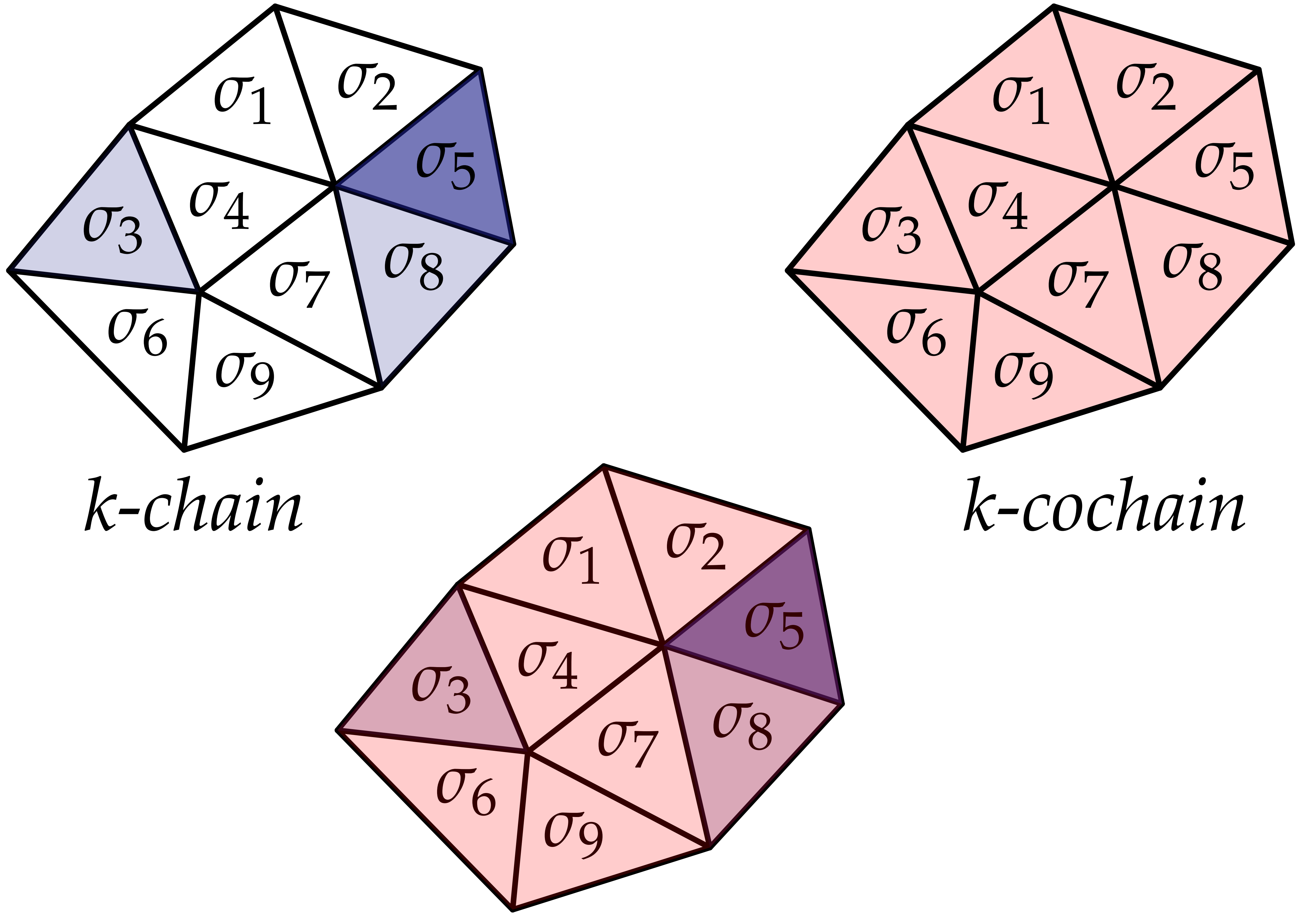
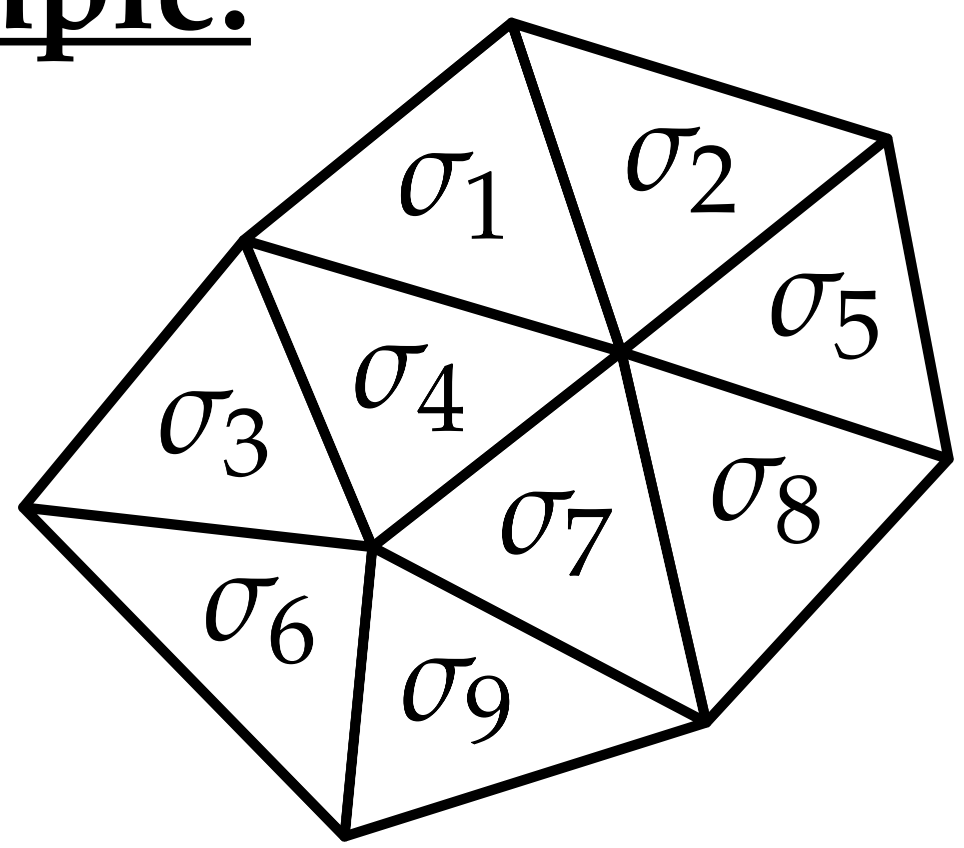


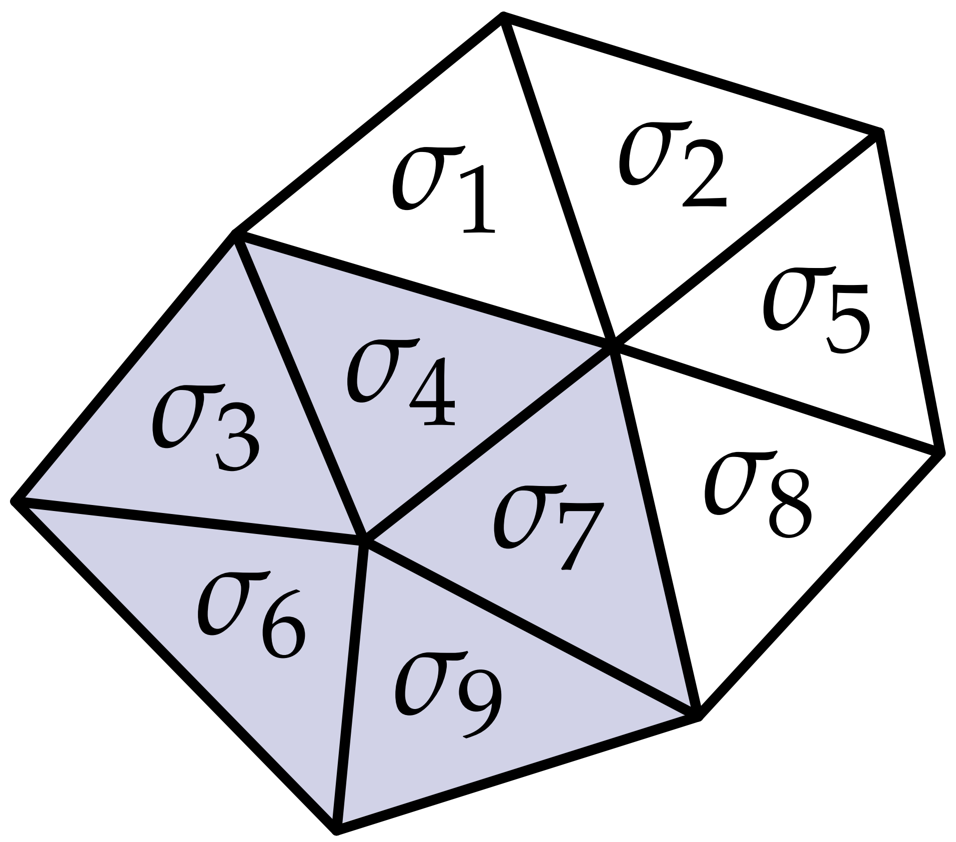
Simplicial Chain

- Suppose we associate <u>every</u> *k*-simplex with its own basis vector
- Can specify some region of a mesh via a linear combination of simplices

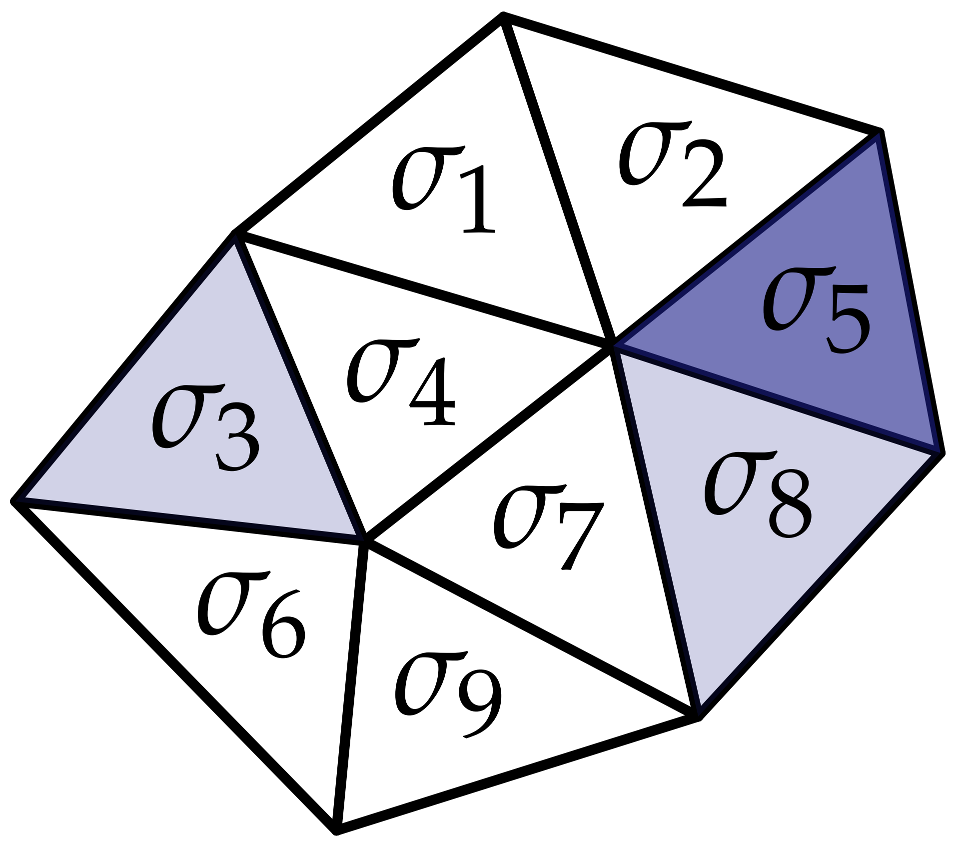
<u>Example.</u>



0



$\sigma_3+\sigma_4+\sigma_6+\sigma_7+\sigma_9$



$\sigma_3 + 3\sigma_5 + \sigma_8$

Q: What does it means when we have a coefficient other than 0 or 1? (Or *negative*?)

A: Roughly speaking, "*n* copies" of that simplex. (Or opposite *orientation*.)

(Formally: *chain group* C_k is the free abelian group generated by the *k*-simplices.)

Arithmetic on Simplicial Chains

$$c_1 = e_3 - e_{12} + e_{18} - e_{15} + e_6 - e_1$$

$$c_2 = e_{15} + e_{19} - e_{17} - e_8 - e_2 - e_6$$

$$c_1 + c_2 = e_3 - e_{12} + e_{18} - \cancel{e_{15}} + \cancel{e_6} - e_1 + \cancel{e_{15}} + e_{19} - e_{17} - e_8 - e_2 - \cancel{e_6}$$

$$= e_3 - e_{12} + e_{18} - e_1 + e_{19} - e_{17} - e_8 - e_2 =: c_3$$

Boundary Operator on Simplices

Definition. Let $\sigma := (v_{i_0}, \ldots, v_{i_k})$ be an oriented k-simplex.
Its *boundary* is the oriented $k-1$-chain

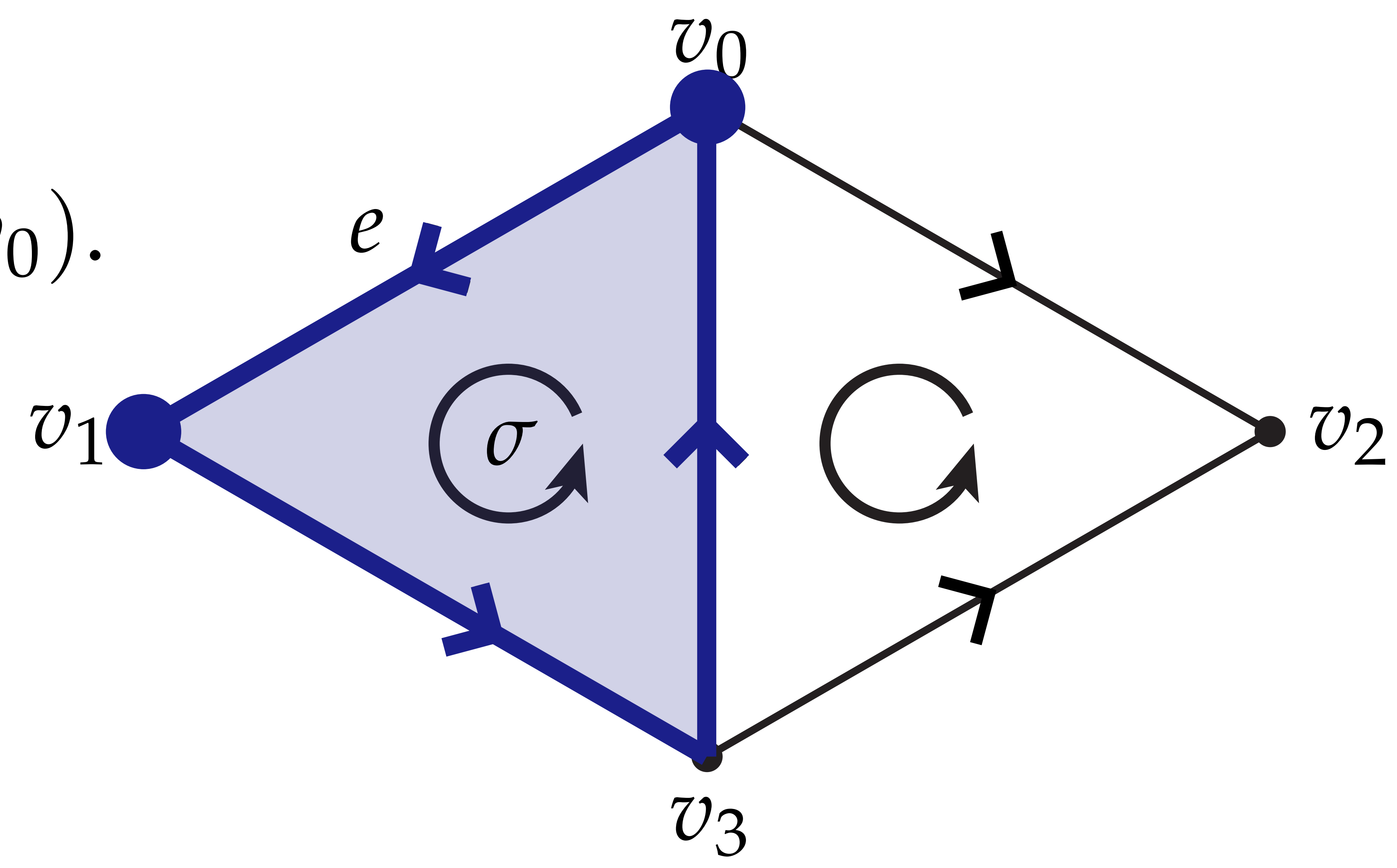
$$\partial\sigma := \sum_{p=0}^{k} (-1)^p (v_{i_0}, \ldots, \cancel{v_{i_p}}, \ldots, v_{i_k}),$$

where $\cancel{v_{i_p}}$ indicates that the pth vertex has been omitted.

Example. Consider the 2-simplex $\sigma := (v_0, v_1, v_3)$.
Its boundary is the 1-chain $(v_0, v_1) + (v_1, v_3) + (v_3, v_0)$.

Example. Consider the 1-simplex $e := (v_0, v_1)$.
Its boundary is the 0-chain $\partial e = v_1 - v_0$.

Example. Consider the 0-simplex (v_1).
Its boundary is the empty set.

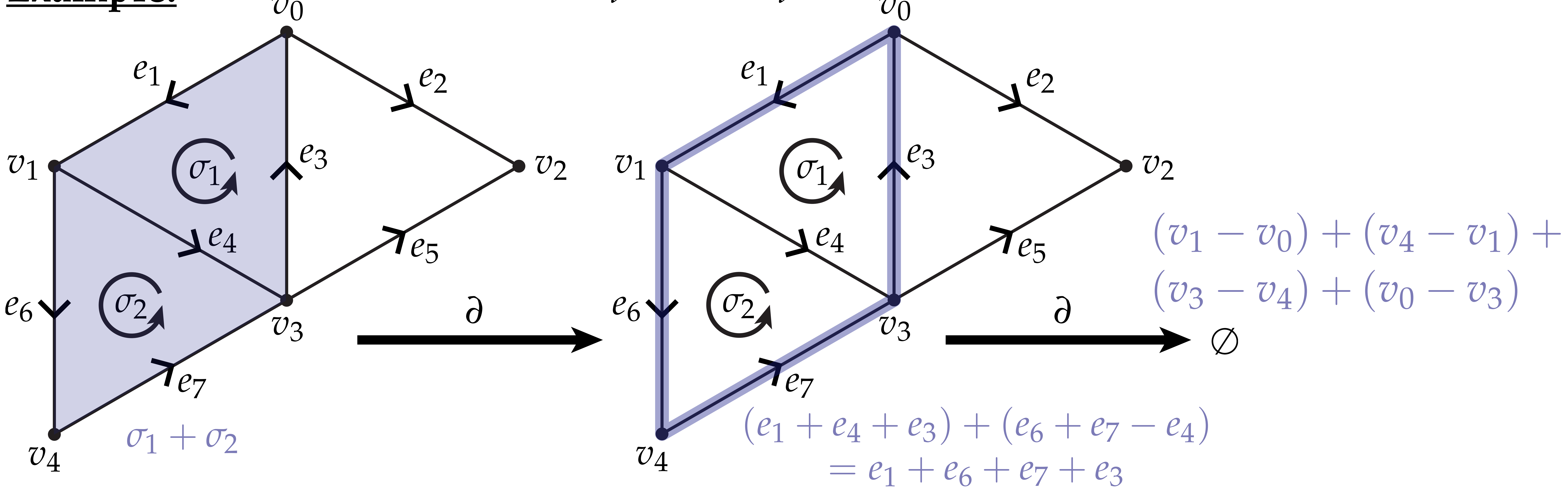


Boundary Operator on Simplicial Chains

The boundary operator can be extended to any chain by linearity, *i.e.*,

$$\partial \sum_i c_i \sigma_i = \sum_i c_i \partial_i \sigma_i.$$

Example.

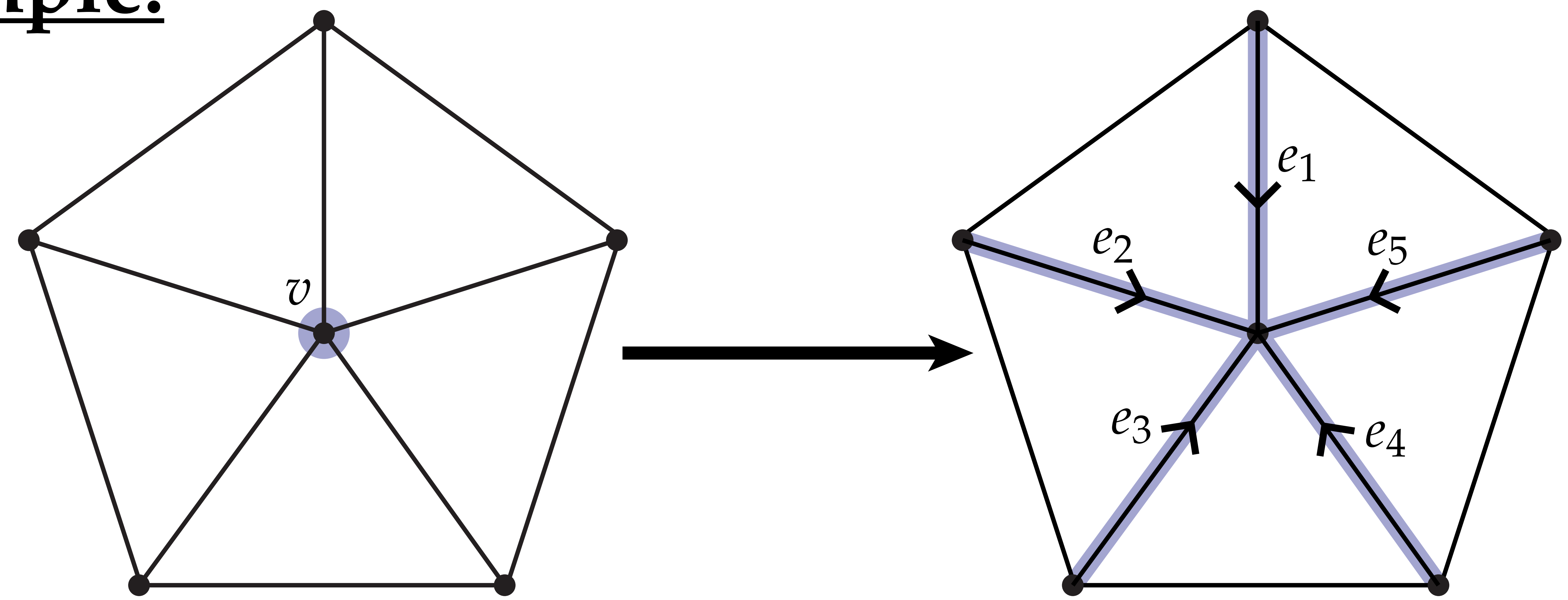


Notice: boundary of boundary is *always* empty!

Coboundary Operator on Simplices

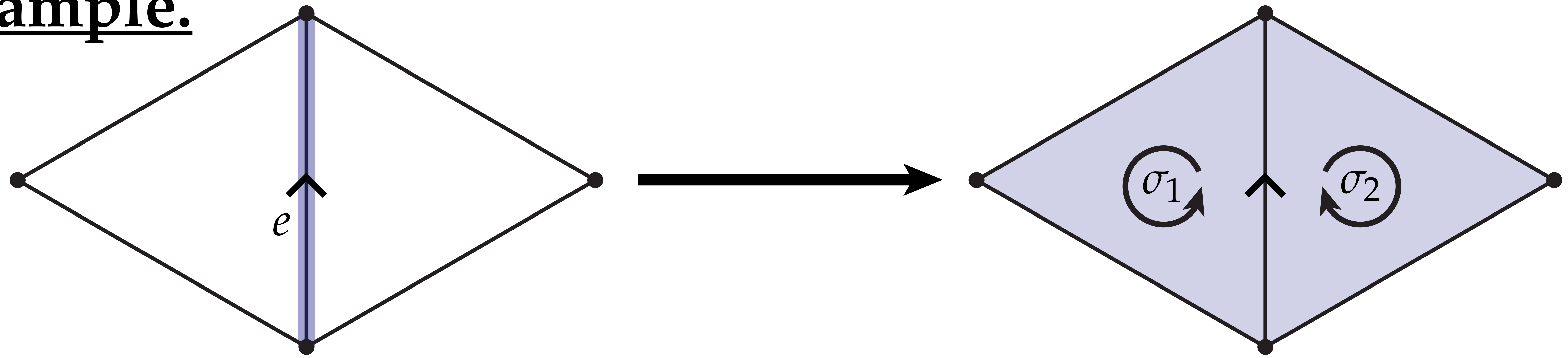
The *coboundary* of an oriented k-simplex σ is the collection of all oriented $(k+1)$-simplices that contain σ, and which have the same relative orientation.

Example.



Q: Why do the arrows point in?

Example.



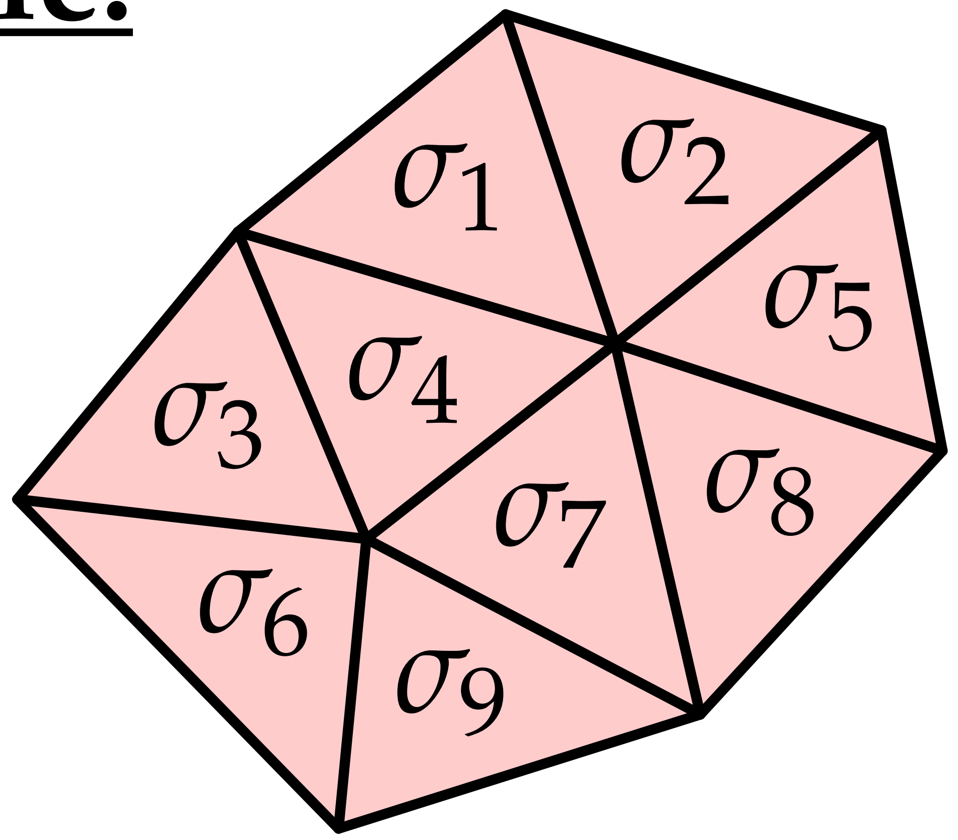
(**Analogy:** simplicial star)

Simplicial Cochain

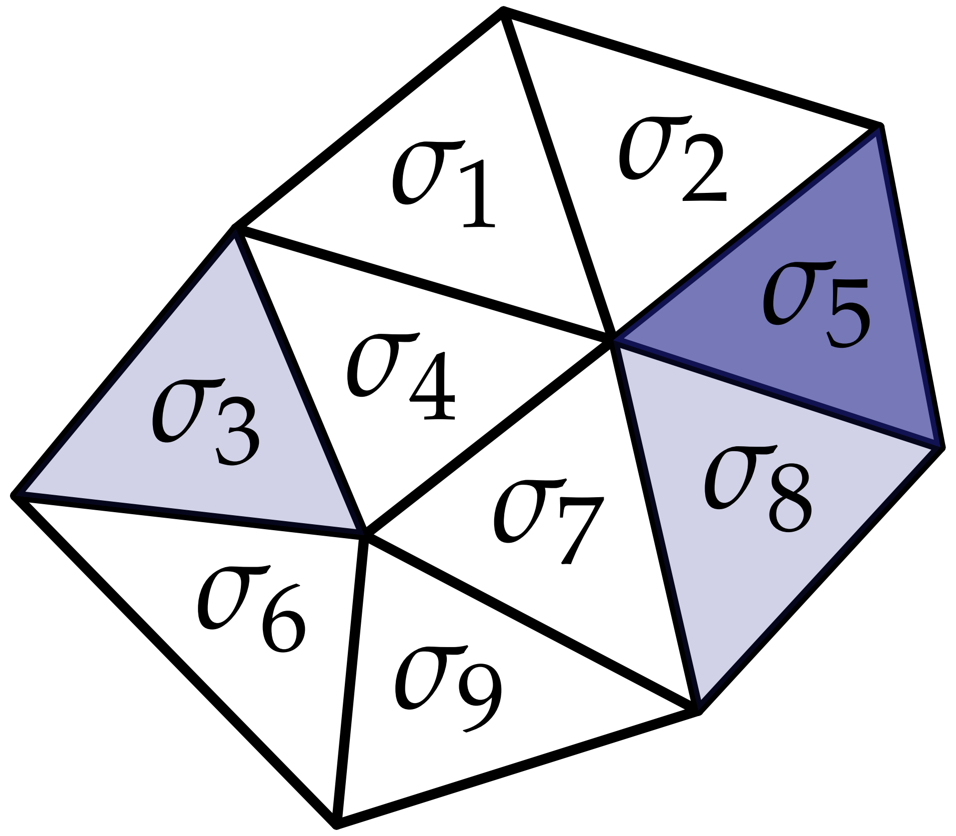
A *simplicial k-cochain* α is a **linear** map taking a simplicial k-chain to a number:

$$\alpha(c_1\sigma_1 + \cdots + c_n\sigma_n) = \sum_{i=1}^{n} \alpha_i c_i$$

Example.



$\forall i, \alpha(\sigma_i) = 1$



$\sigma_3 + 3\sigma_5 + \sigma_8$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 1 + 3 + 1 = 5$$

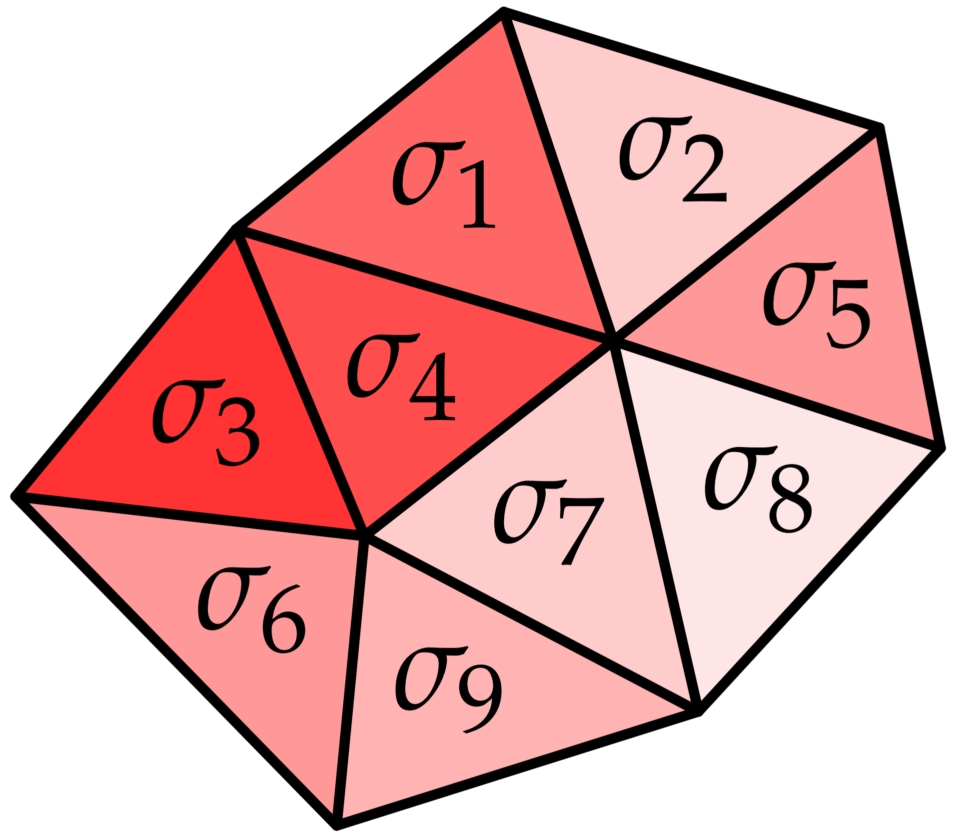
(Formally: *cochain group* C^k is group of homomorphisms from k-chains to the reals.)

Simplicial Cochains & Discrete Differential Forms

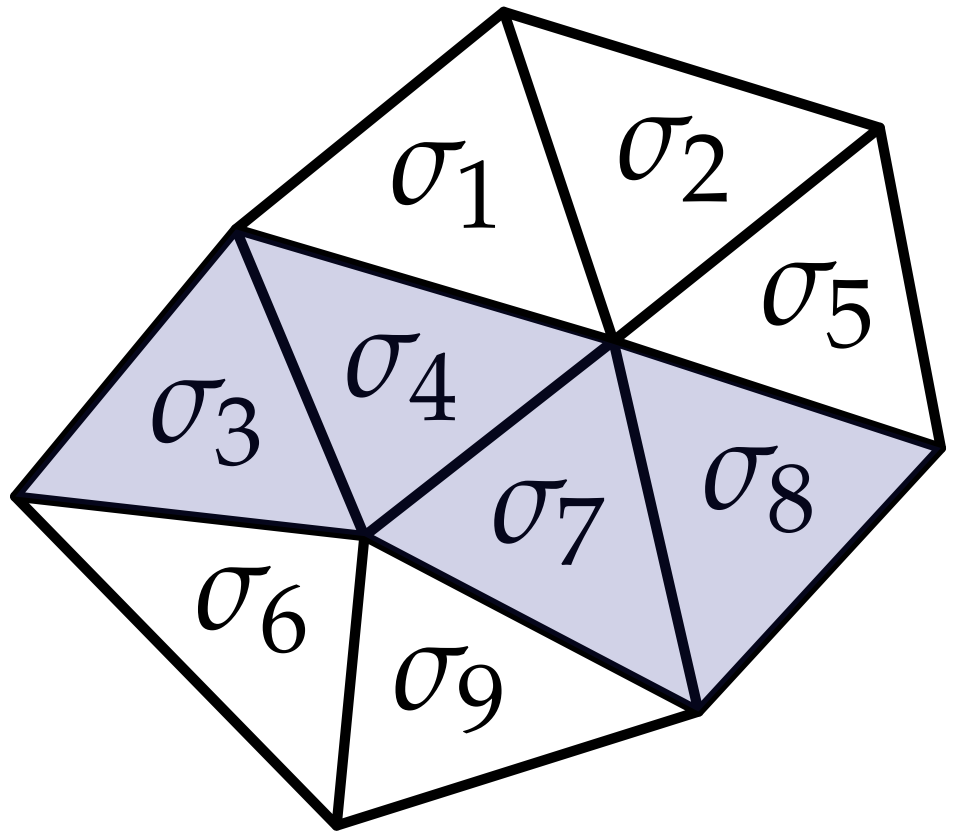
Suppose a simplicial k-cochain is given by the integrated values from a discrete k-form

Q: What does it mean (geometrically) when we apply it to a simplicial k-chain?

A: Our discrete k-form values come from integrating a smooth k-form over each k-simplex. So, we just get the integral over the region specified by the chain:



$$\hat{\alpha}_i := \int_\sigma \alpha$$



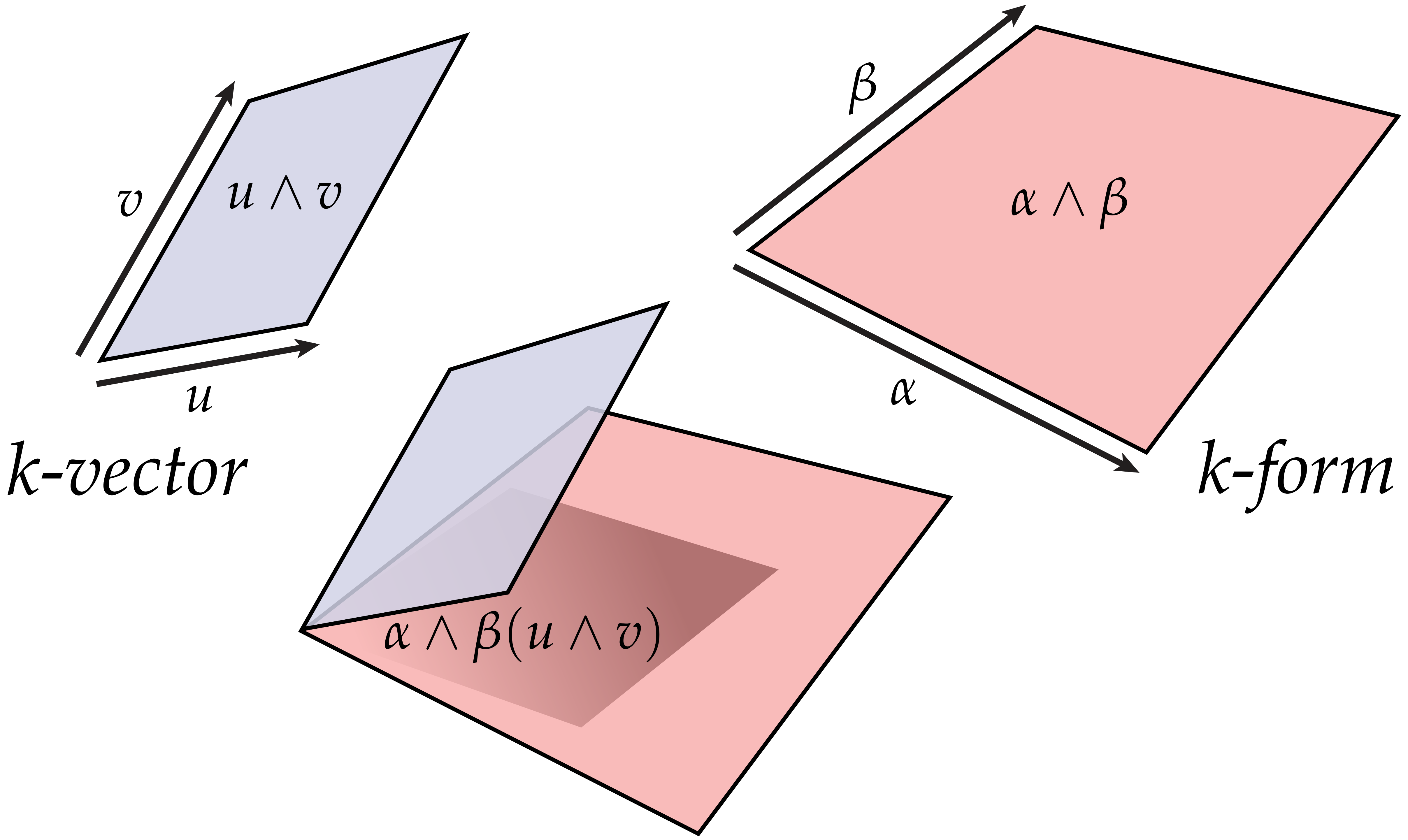
$$c = \sigma_3 + \sigma_4 + \sigma_7 + \sigma_8$$

$$\begin{aligned} \hat{\alpha}(c) &= \hat{\alpha}_3 + \hat{\alpha}_4 + \hat{\alpha}_7 + \hat{\alpha}_8 \\ &= \int_{\sigma_3 \cup \sigma_4 \cup \sigma_7 \cup \sigma_8} \alpha \end{aligned}$$

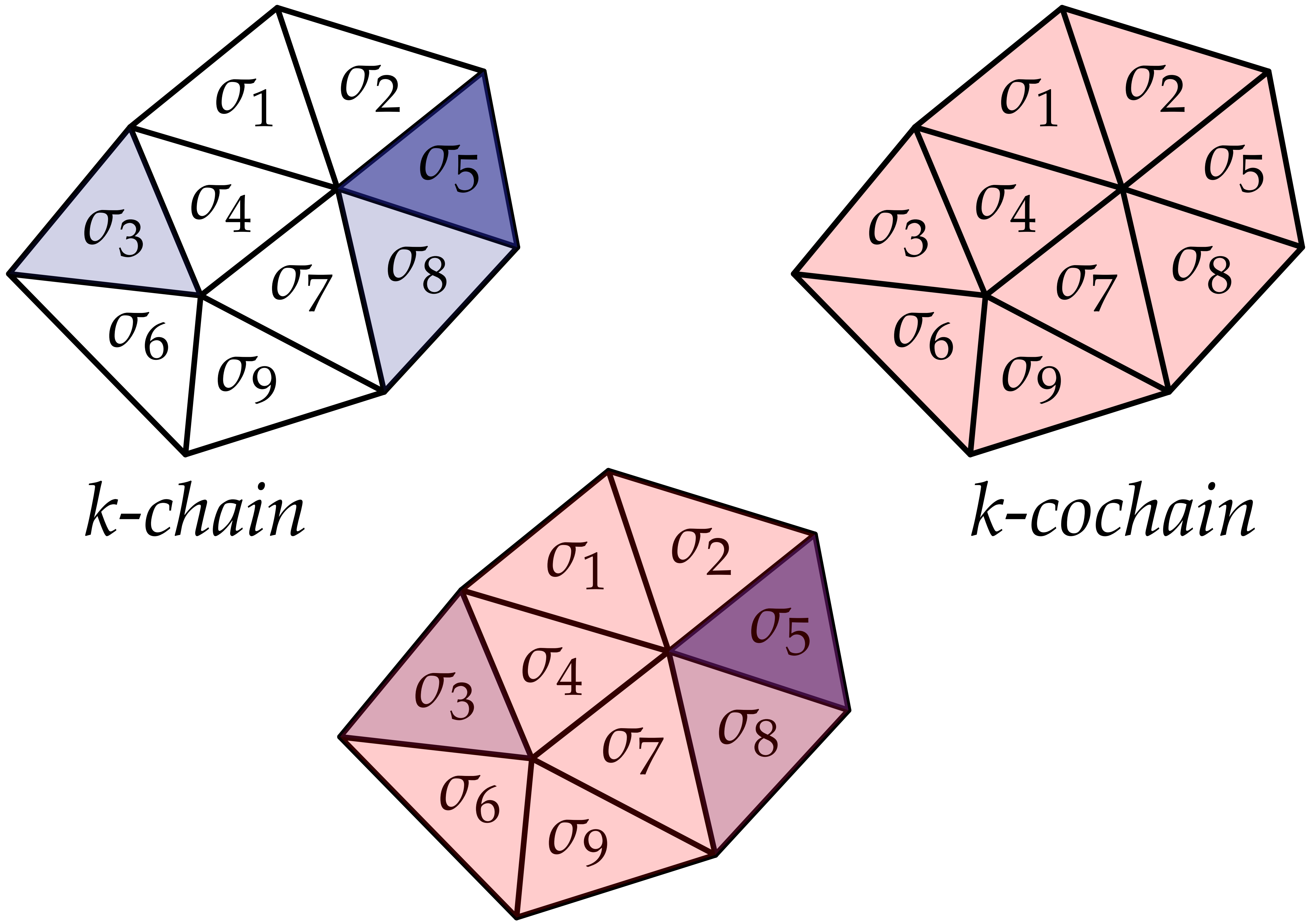
Discrete Differential Form — Abstract Definition

Definition. A *discrete differential k-form* is an assignment of a number to each k-simplex of an oriented simplicial complex. $\widehat{\Omega}_k$ denotes the space of discrete k-forms (k-cochains).

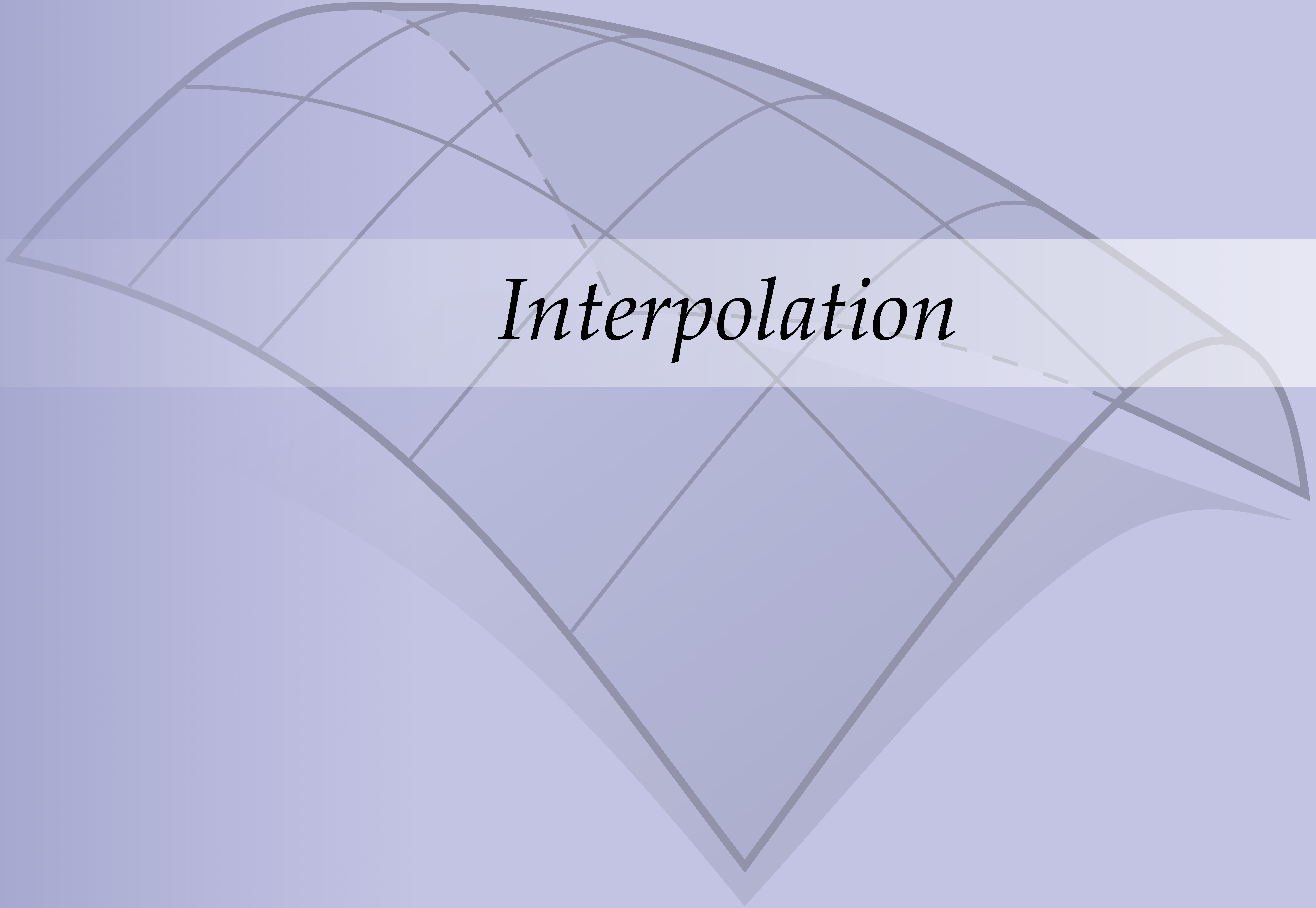
SMOOTH



DISCRETE



Interpolation



Interpolation — 0-Forms

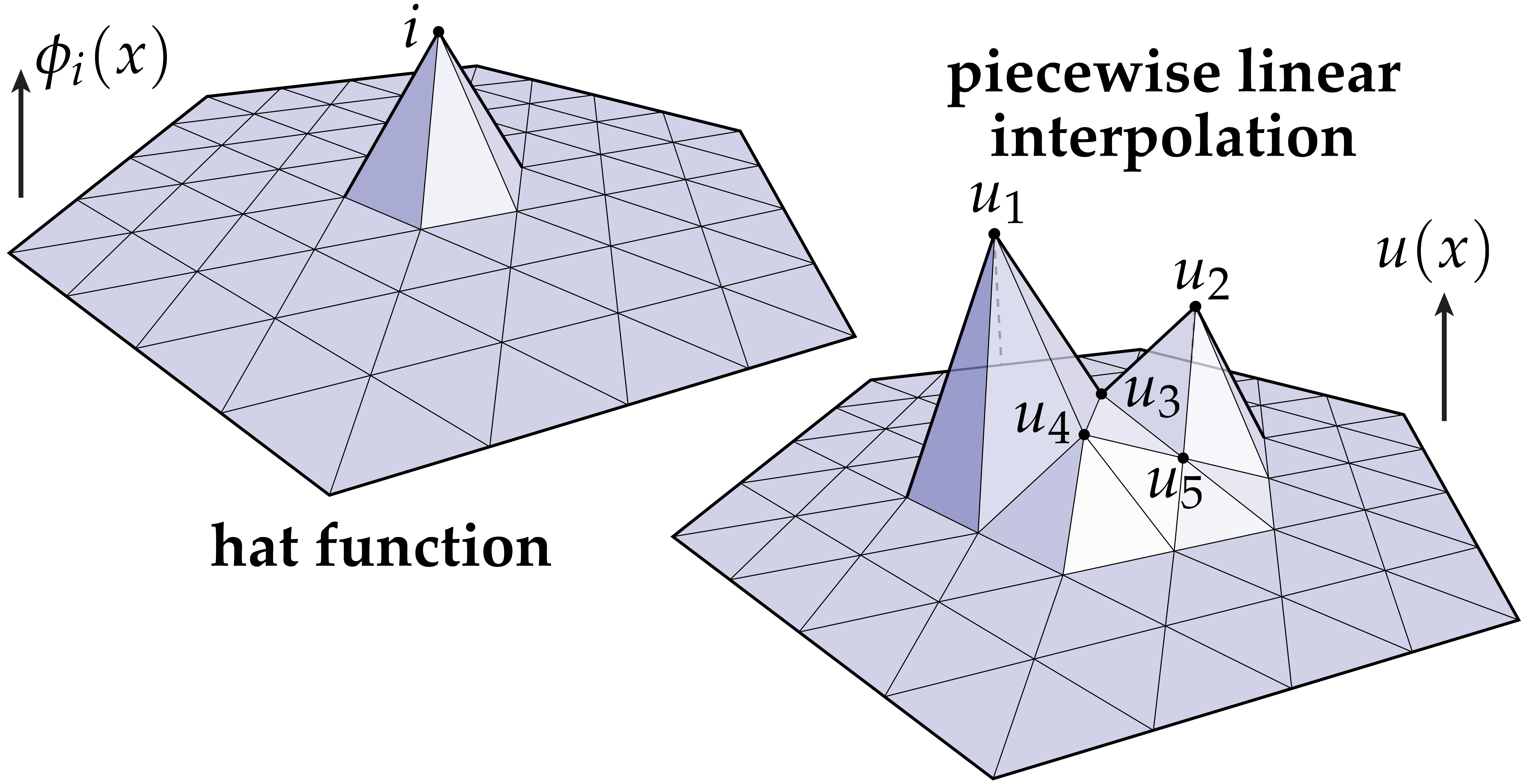
On any simplicial complex K, the *hat function* a.k.a. *Lagrange basis* ϕ_i is a real-valued function that is linear over each simplex and satisfies

$$\phi_i(v_j) = \delta_{ij},$$

for each vertex v_j, *i.e.*, it equals 1 at vertex i and 0 at vertex j. Given a discrete 0-form $u : V \to \mathbb{R}$, we can construct an *interpolating* 0-form via

$$u(x) = \sum_i u_i \phi_i(x)$$

i.e., we simply weight the hat functions by values at vertices.

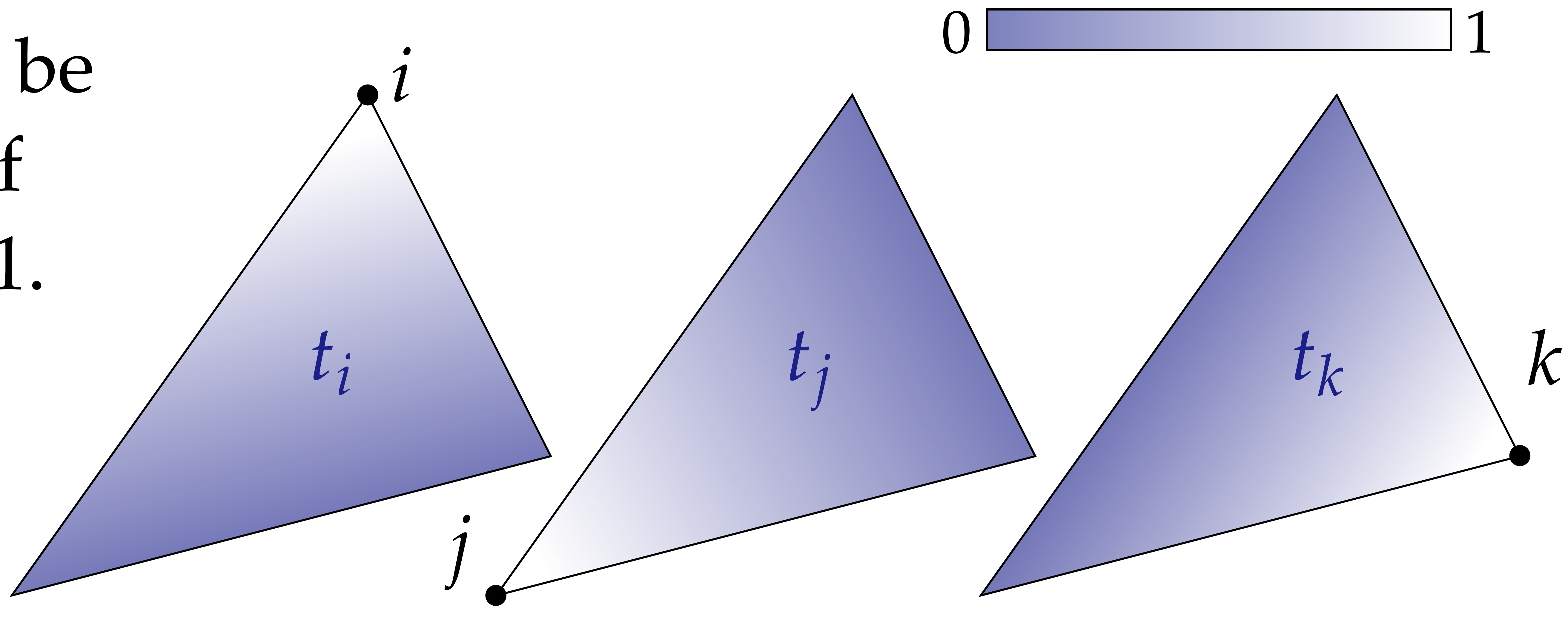


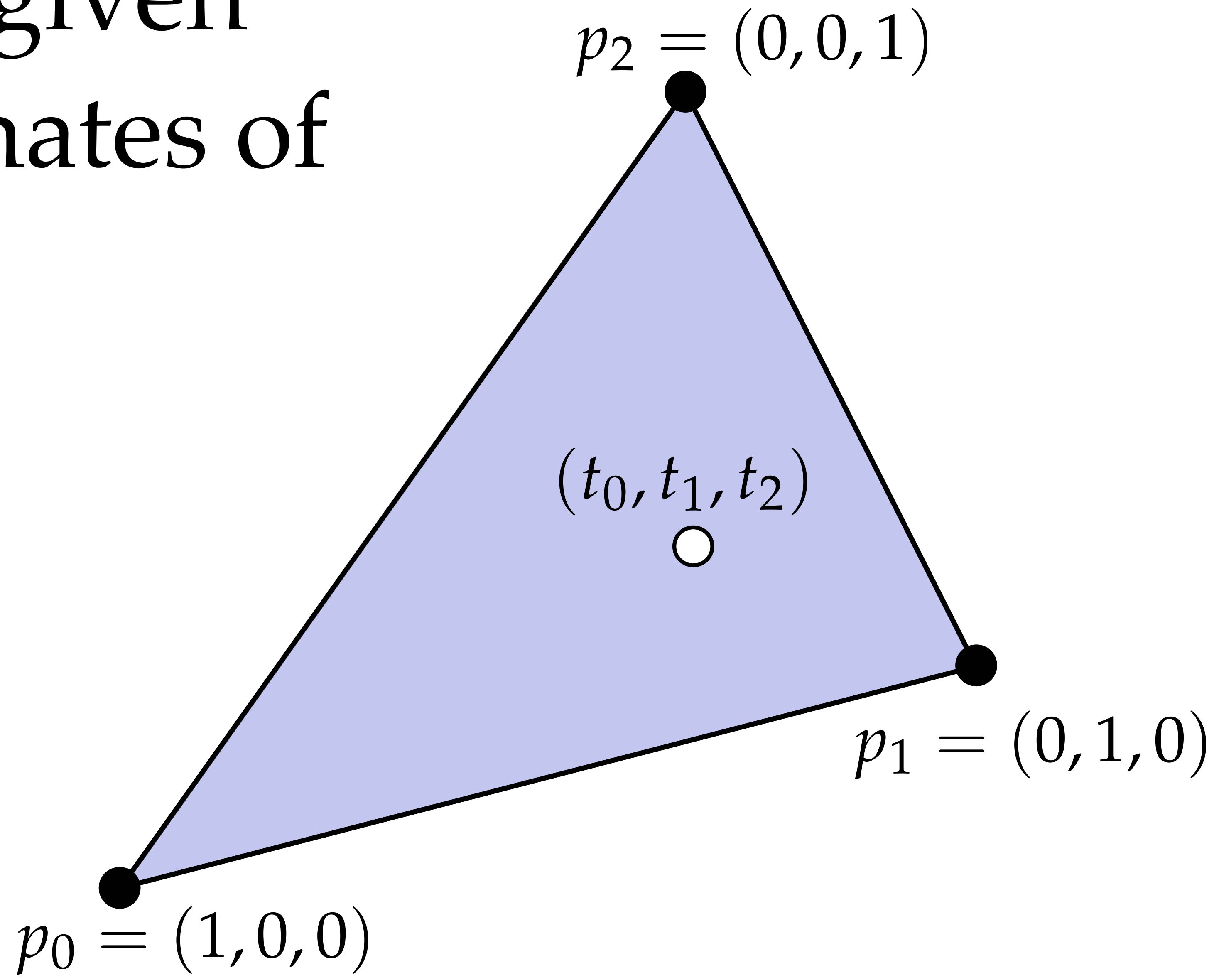
Note: result is a *continuous* 0-form.

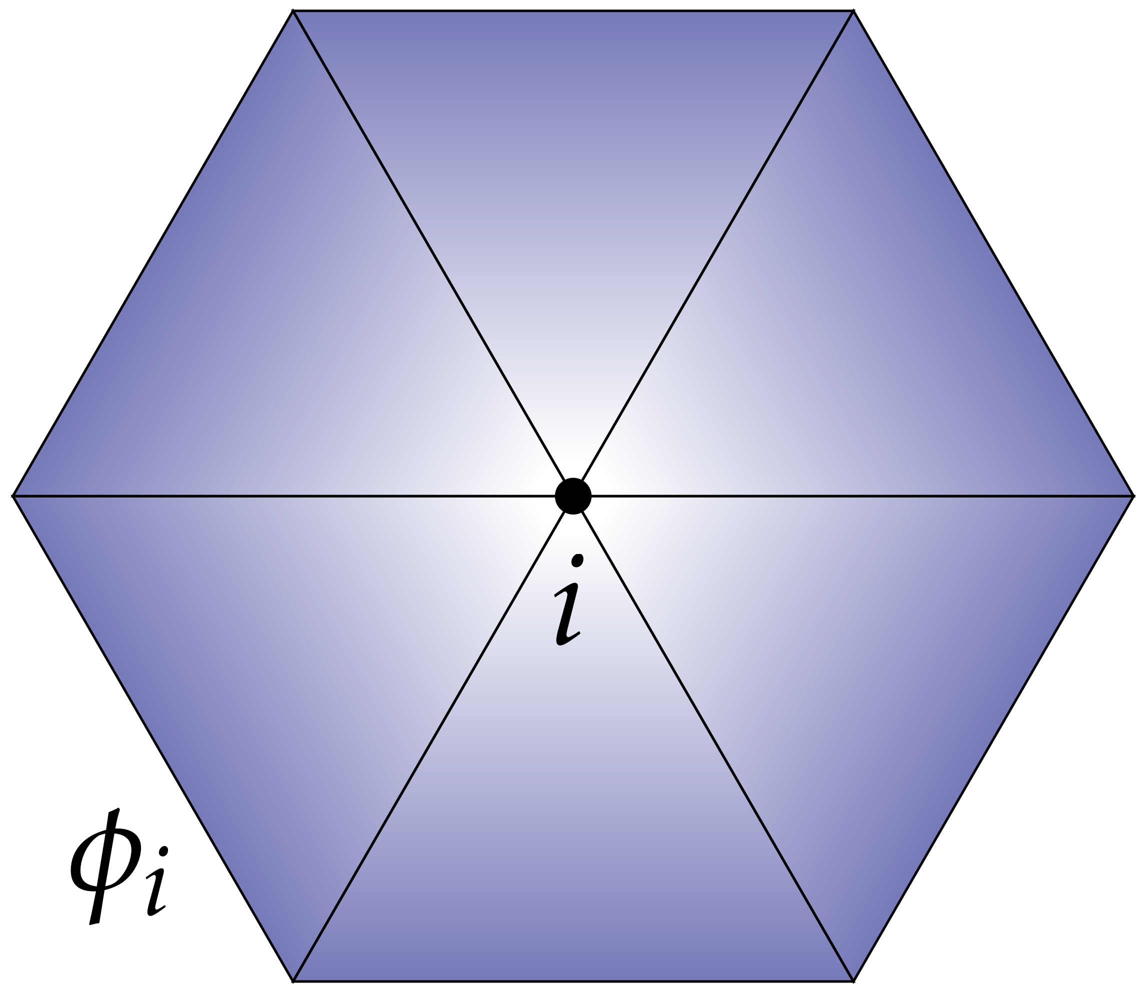
Barycentric Coordinates — Revisited

- Recall that any point in a k-simplex can be expressed as a weighted combination of the vertices, where the weights sum to 1.
- The weights t_i are called the *barycentric coordinates.*
- The Lagrange basis for a vertex i is given explicitly by the barycentric coordinates of i in each triangle containing i.

$$\sigma = \left\{ \sum_{i=0}^{k} t_i p_i \;\middle|\; \sum_{i=0}^{k} t_i = 1,\ t_i \geq 0\ \forall i \right\}$$







Interpolation—k-Forms (Whitney Map)

Definition. Let ϕ_i be the hat functions on a simplicial complex. The *Whitney 1-forms* are differential 1-forms associated with each oriented edge ij, given by

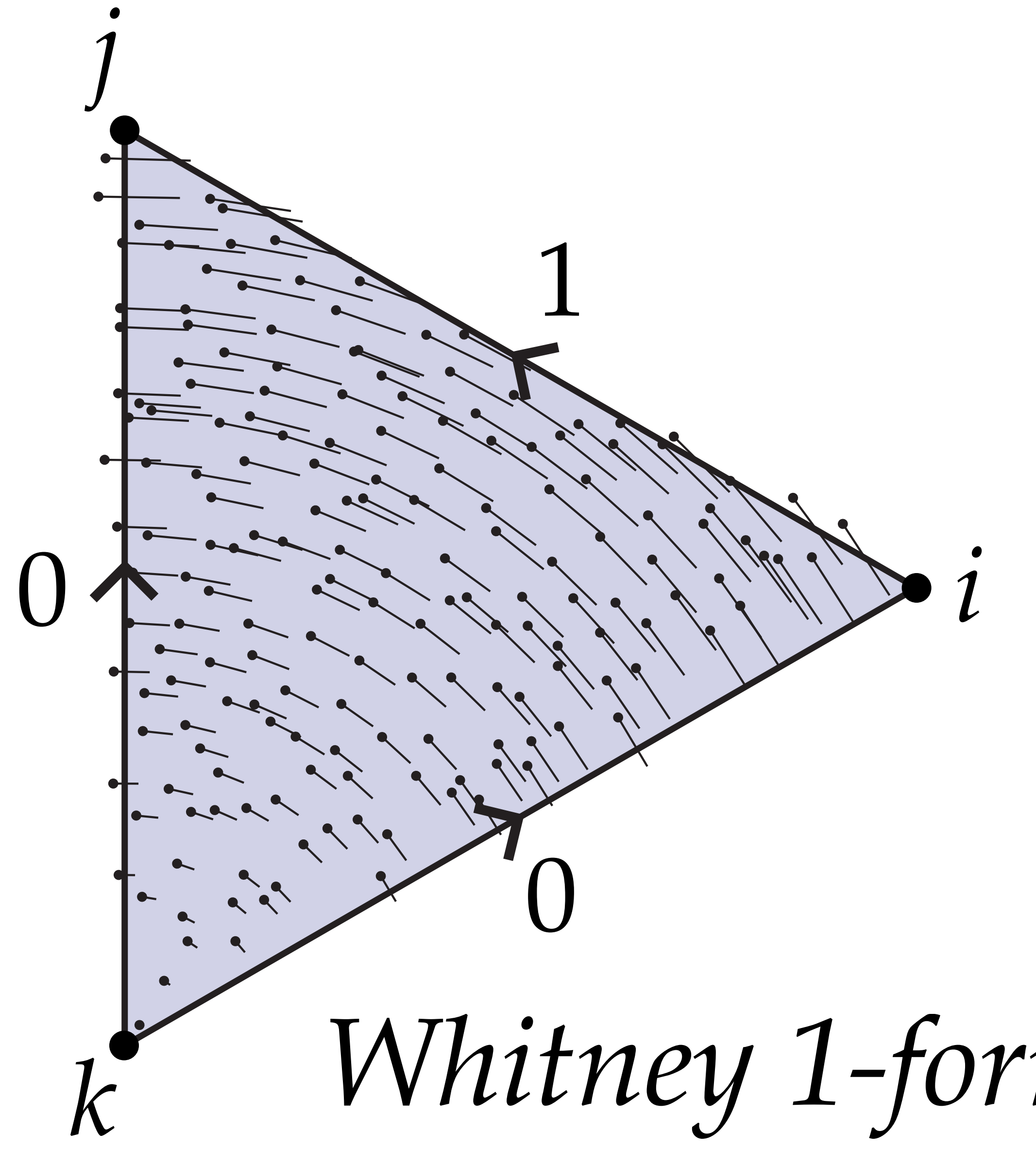
$$\phi_{ij} := \phi_i \, d\phi_j - \phi_j \, d\phi_i$$

(Note that $\phi_{ij} = -\phi_{ji}$). The Whitney 1-forms can be used to interpolate a discrete 1-form $\widehat{\omega}$ (value per edge) via

$$\sum_{ij} \widehat{\omega}_{ij} \phi_{ij}.$$

More generally, the *Whitney k-form* associated with an oriented k-simplex $(i_0, \ldots, i_k)$ is given by

$$\sum_{p=0}^{k} (-1)^p \phi_{i_p} d\phi_{i_0} \wedge \cdots \wedge \cancel{d\phi_{i_p}} \wedge \cdots \wedge d\phi_{i_k}$$



Whitney 1-form ϕ_{ij}

Discretization & Interpolation

- **<u>Fact:</u>** Suppose we have a discrete differential k-form. If we interpolate by Whitney bases, then discretize via the de Rham map (*i.e.*, by integration), then we recover the exact same discrete k-form.

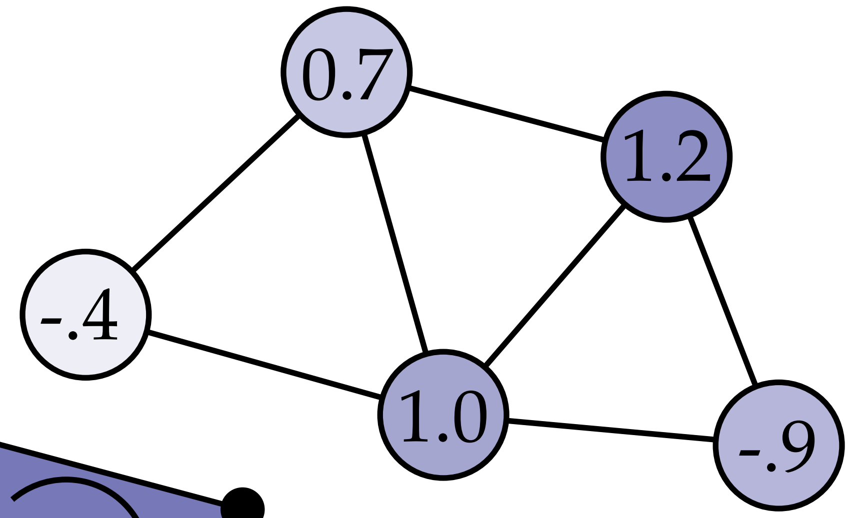
$$
\begin{tikzcd}
\Omega_k \ \textit{(smooth differential k-forms)} \arrow[d, shift right=1ex, "\textit{(discretize)}\ \int"'] \\
\widehat{\Omega}_k \ \textit{(discrete differential k-forms)} \arrow[u, shift right=1ex, "\phi\ \ \textit{(interpolate)}"']
\end{tikzcd}
$$

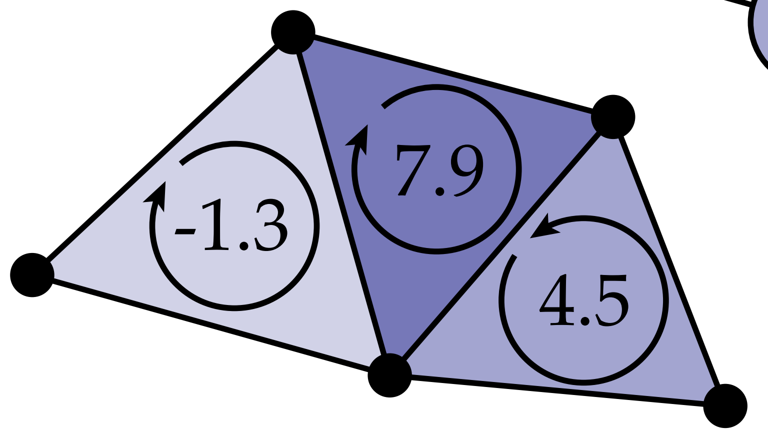
Q: What about the other direction? If we discretize a continuous k-form then interpolate, will we always recover the same continuous k-form?

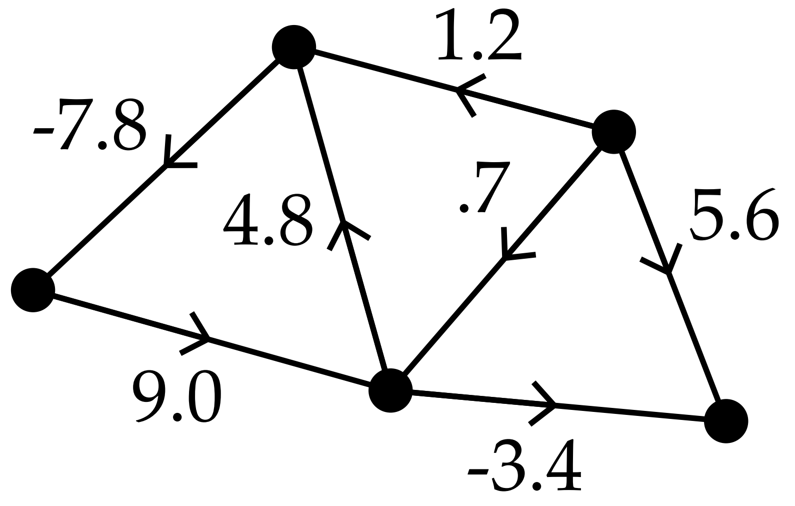
Summary

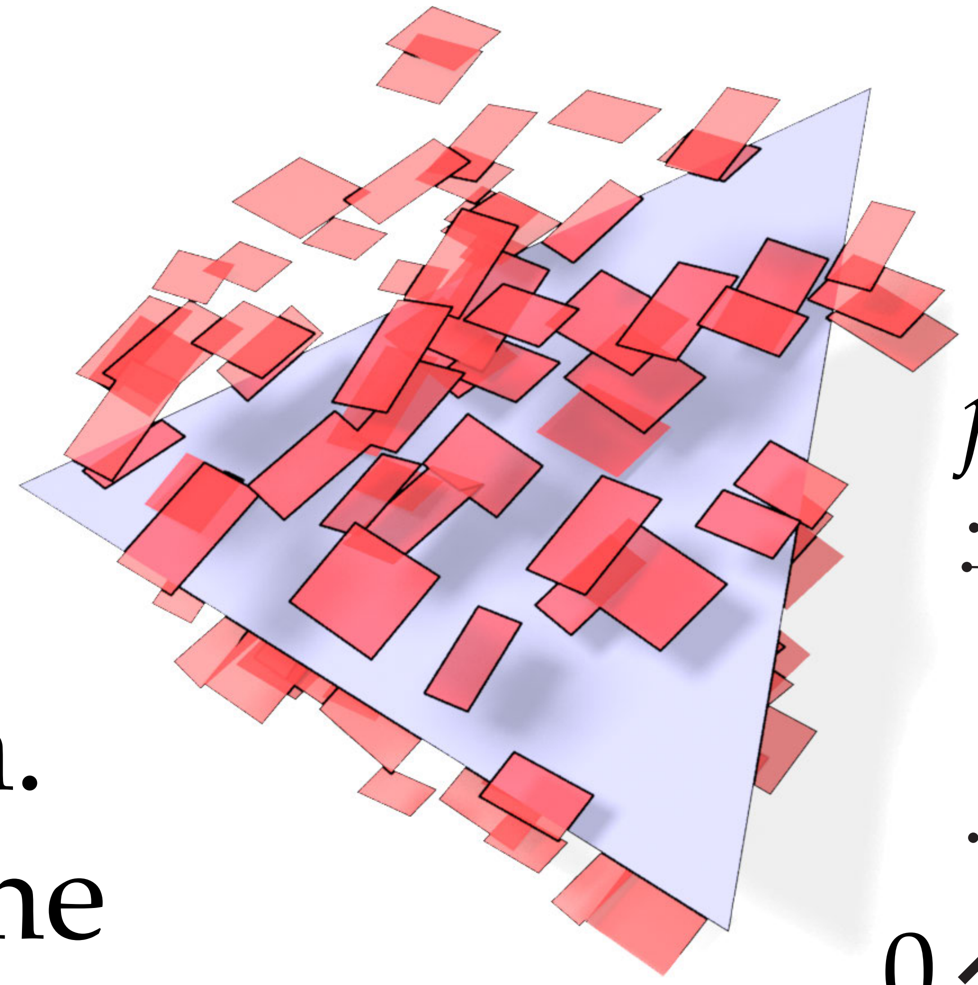
Discrete Differential Forms—Summary

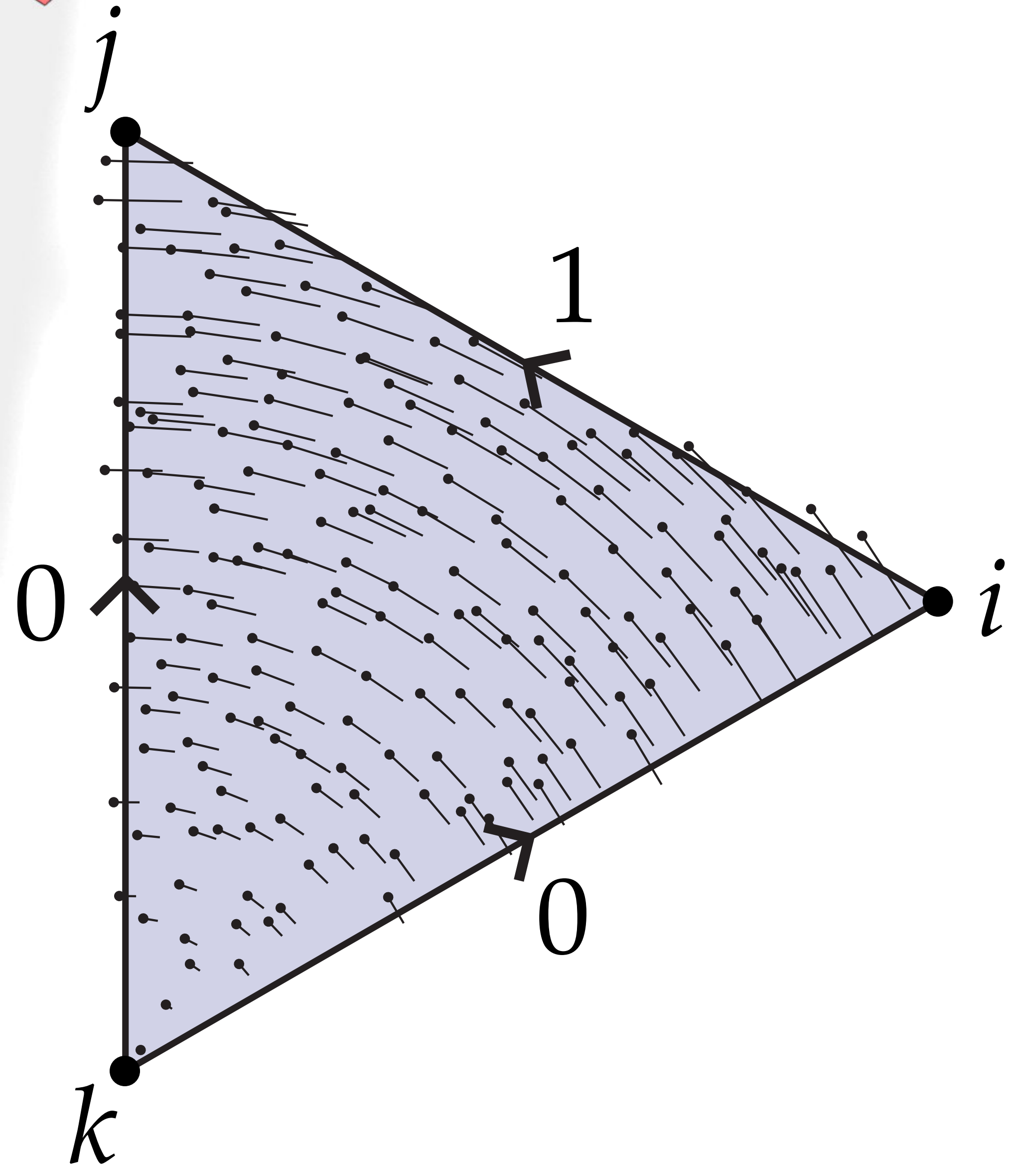
- A *discrete differential k-form* amounts to a value stored on each oriented k-simplex
- **Discretization:** given a smooth differential k-form, can approximate by a discrete differential k-form by integrating over each k-simplex
- **Interpolation:** given a discrete differential k-form, construct a continuous one by taking a weighted sum of basis k-forms
- *In practice,* almost never comes from direct integration. More typically, values start at vertices (samples of some function); 1-forms, 2-forms, *etc.*, arise from applying operators like the (discrete) exterior derivative.
- Next lecture: develop these operators!



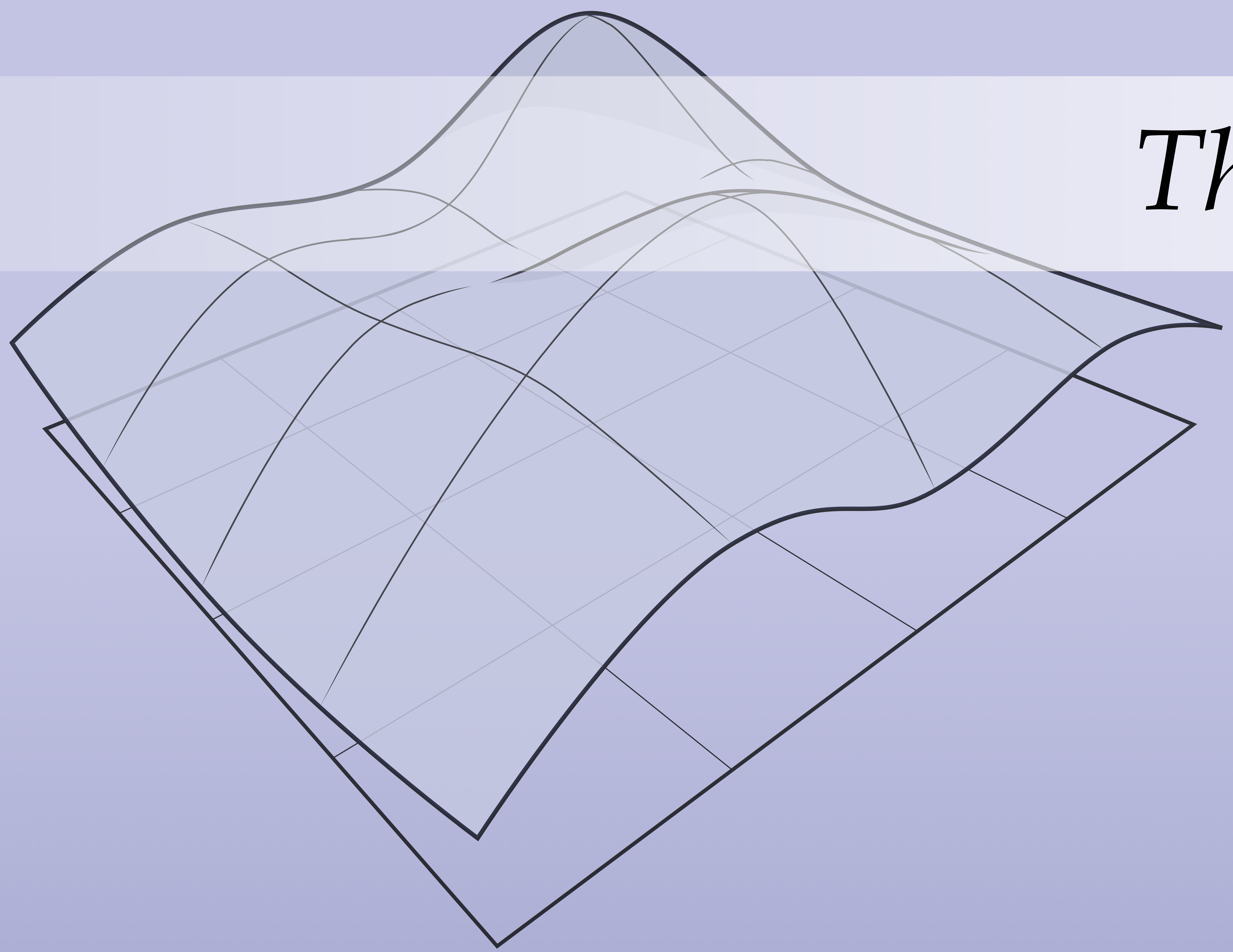








Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION