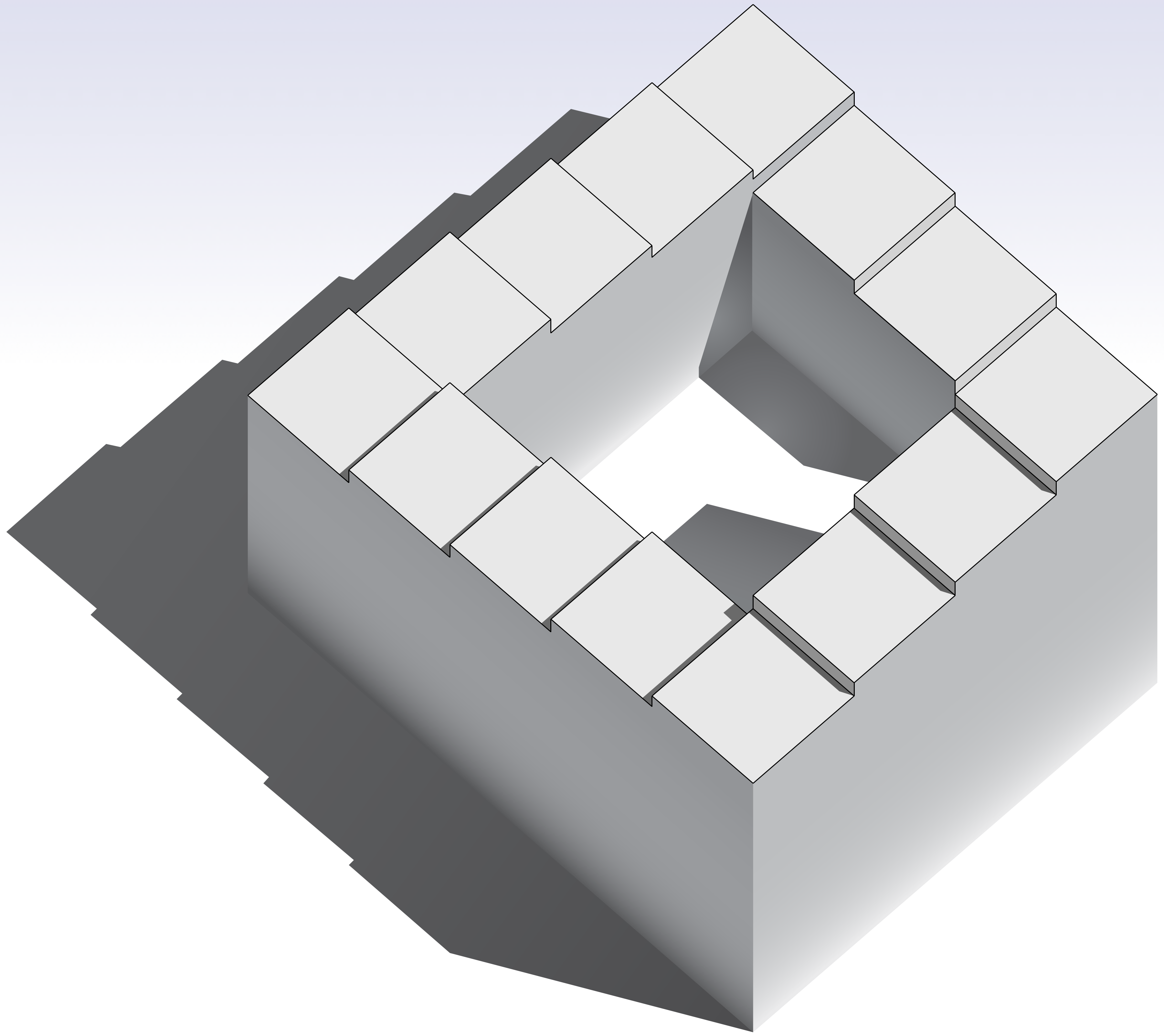
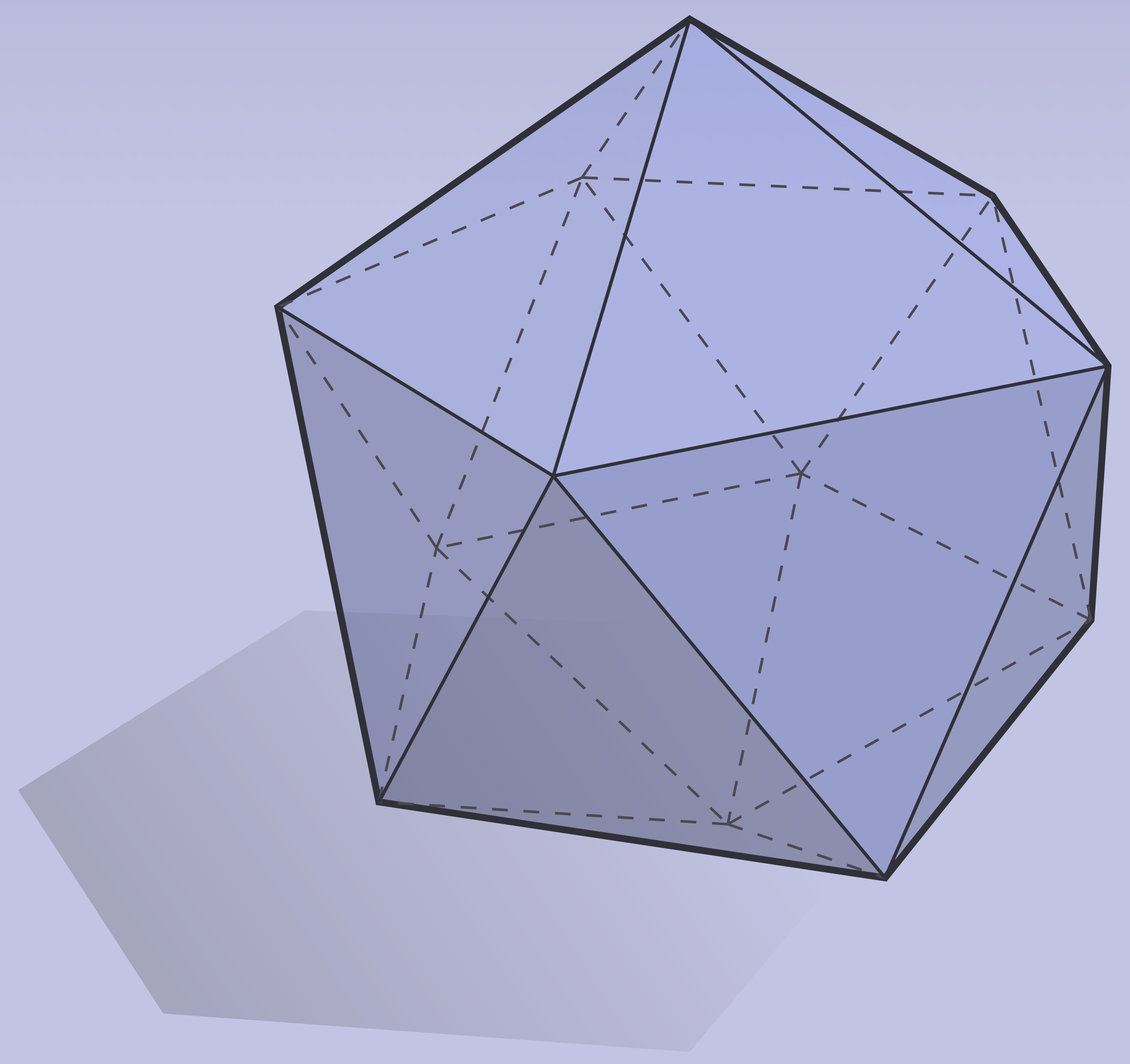


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 14:
DISCRETE SURFACES



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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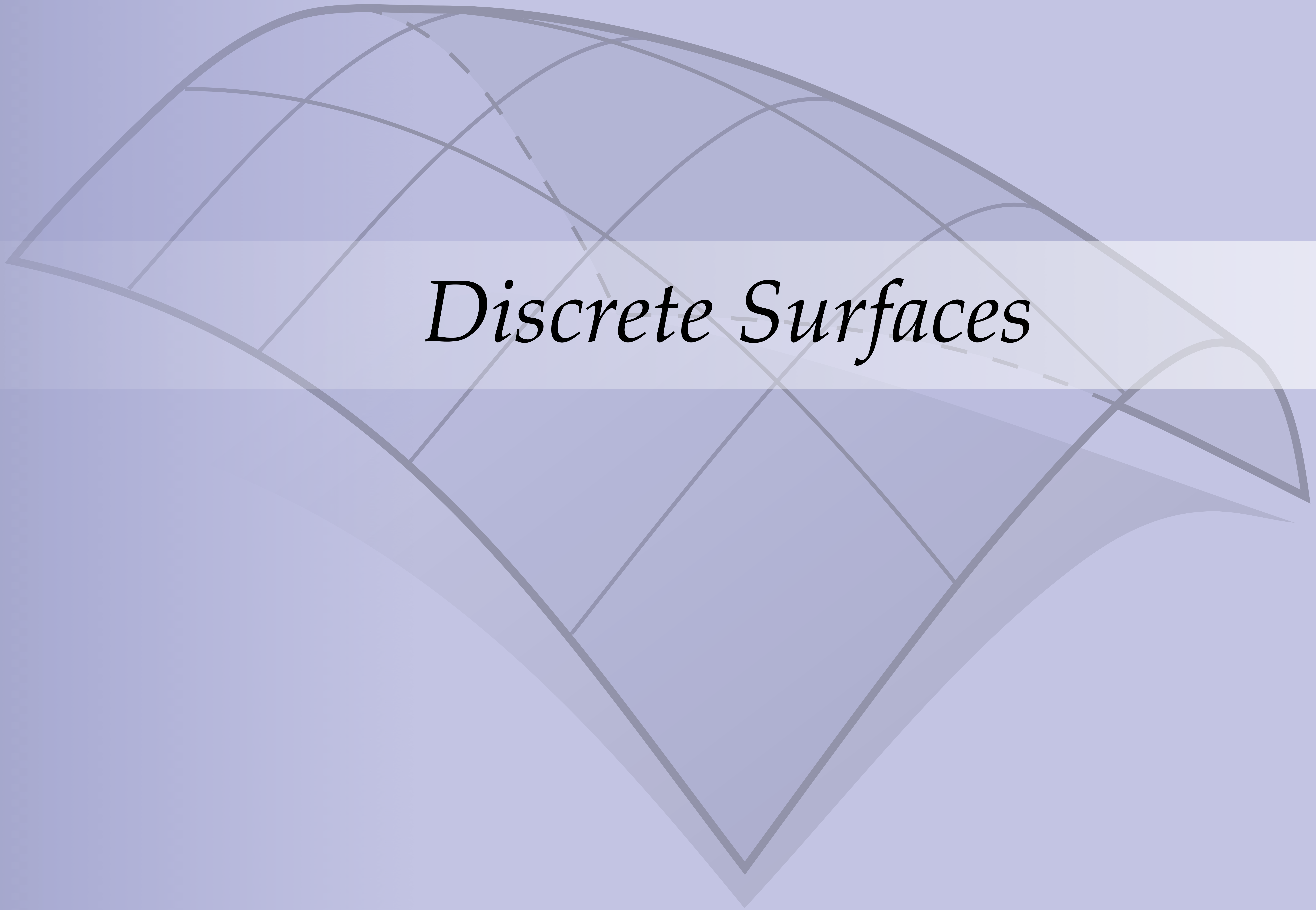
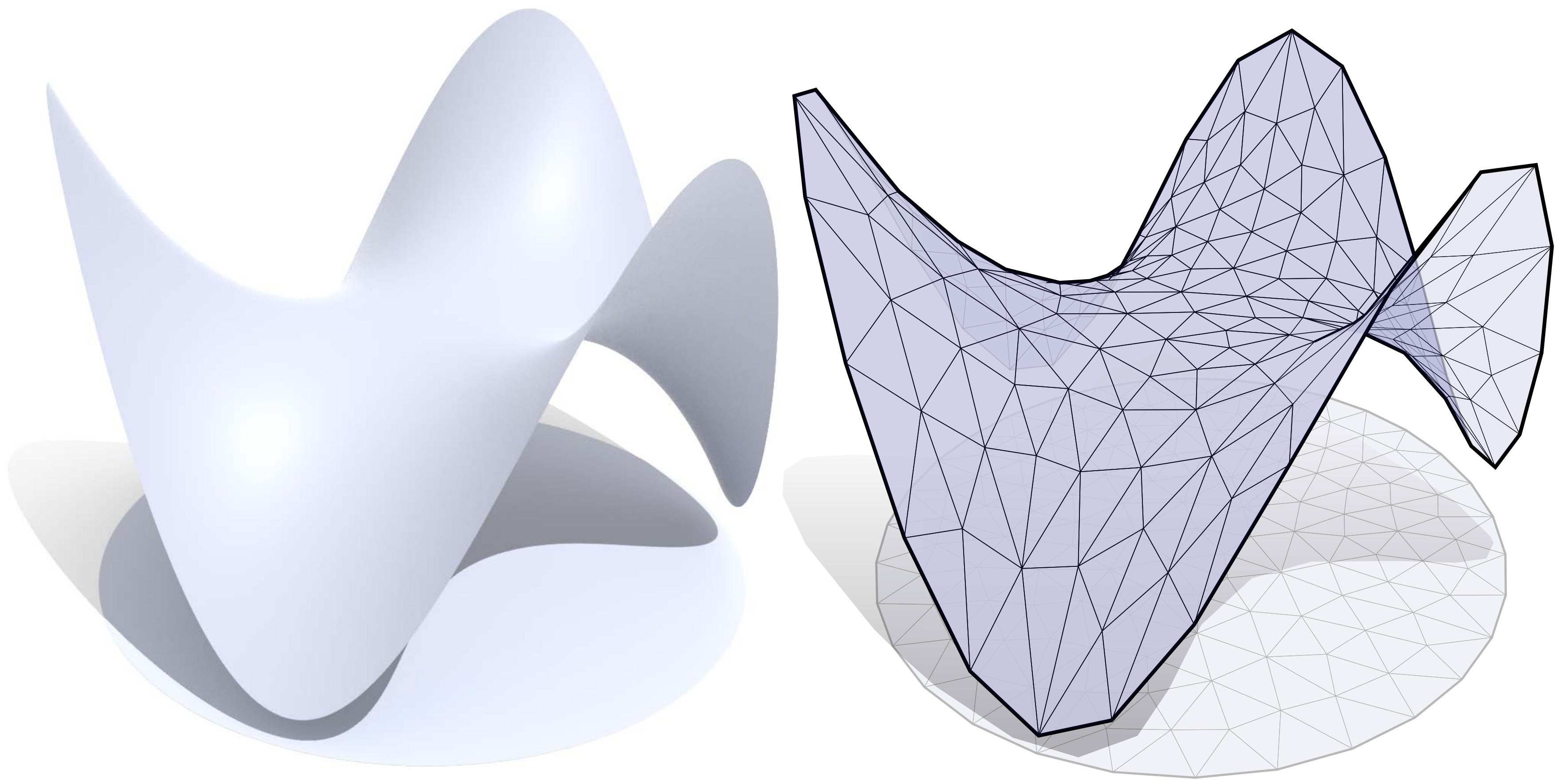
Discrete Surfaces

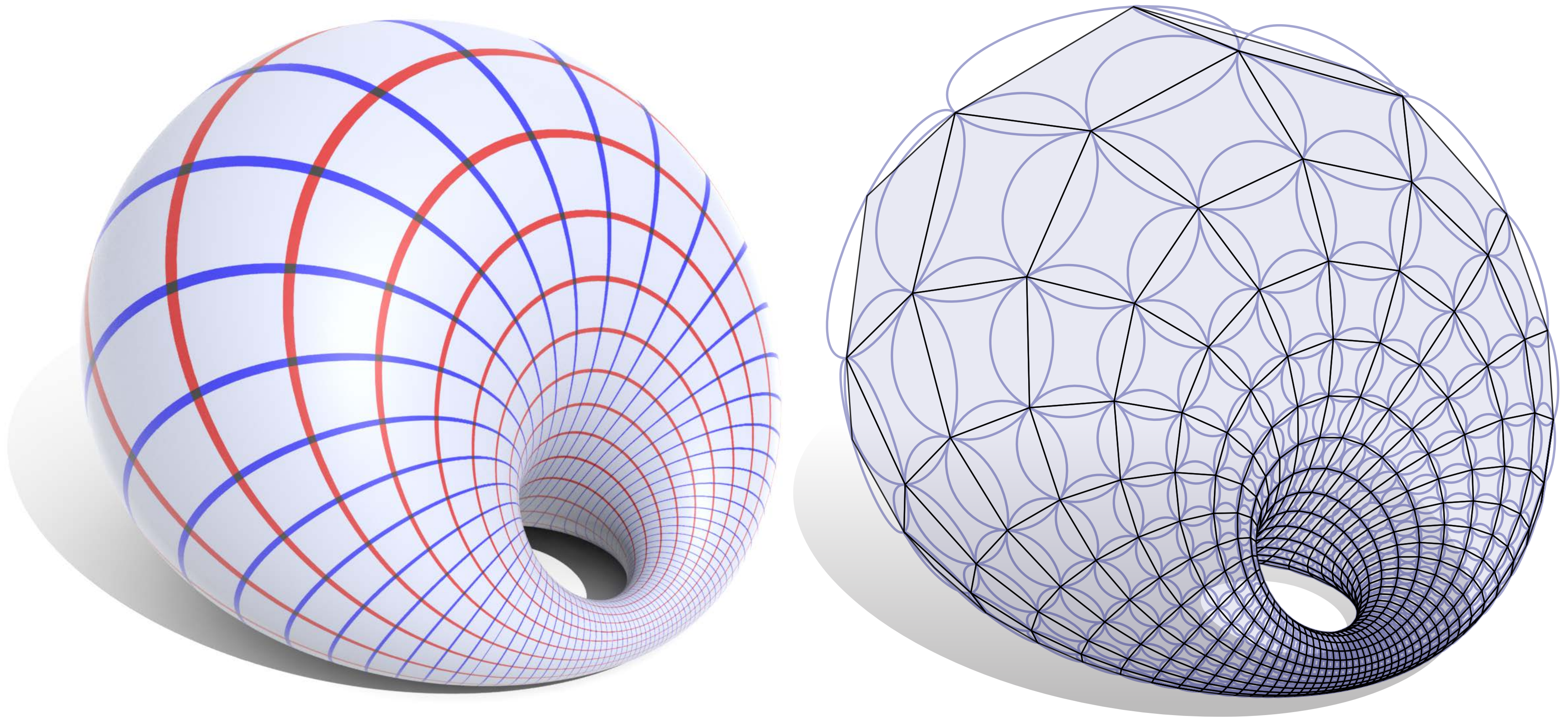


Discrete Models of Surfaces

- Two primary models of surfaces in discrete differential geometry:
- **Simplicial**
  - – surfaces are simplicial 2-manifolds
  - – natural fit with discrete exterior calculus



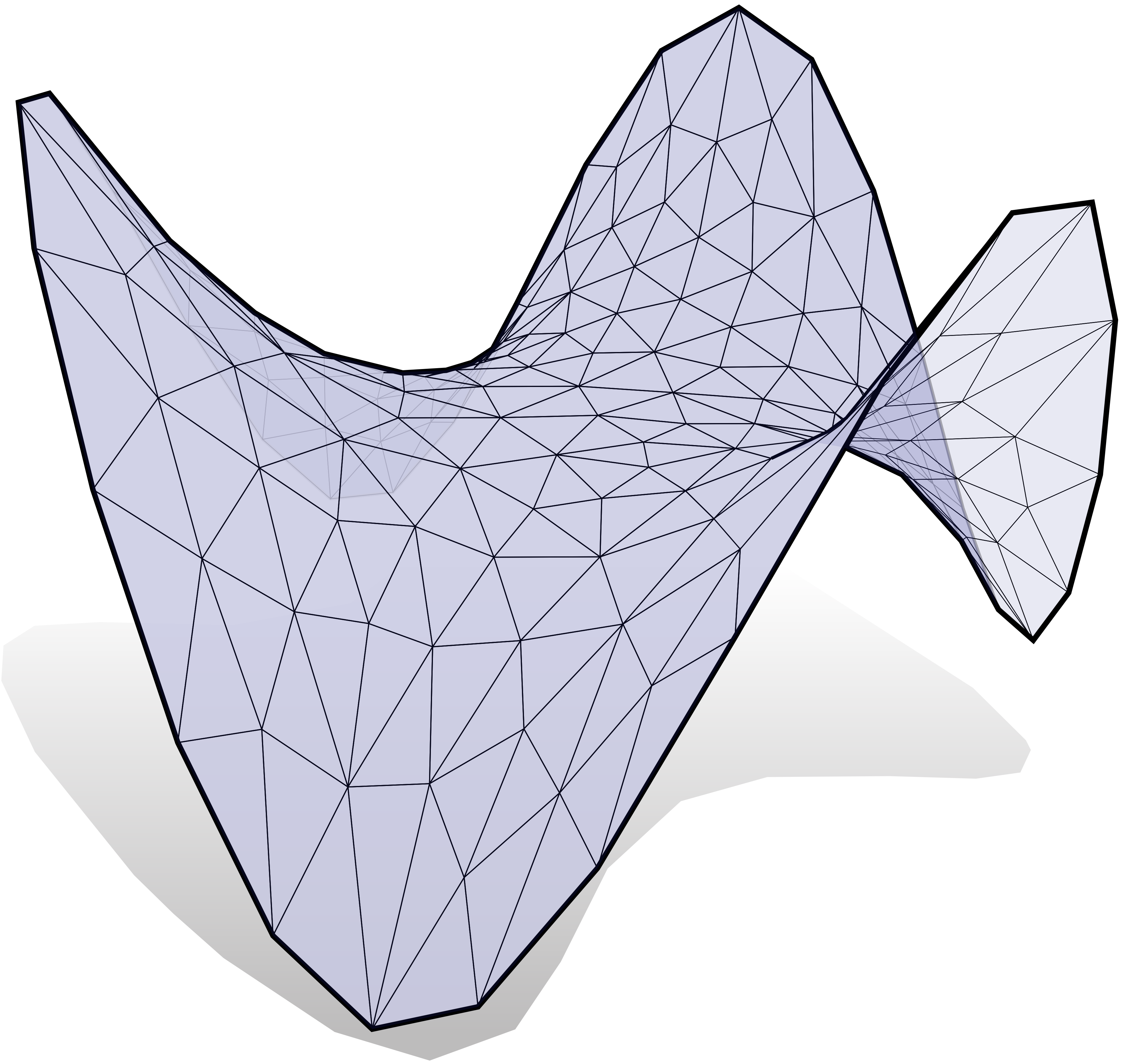
- **Nets**
  - – surfaces are piecewise integer lattices
  - – natural fit with *discrete integrable systems*



- Simplicial surfaces more common in applications; focus of our course

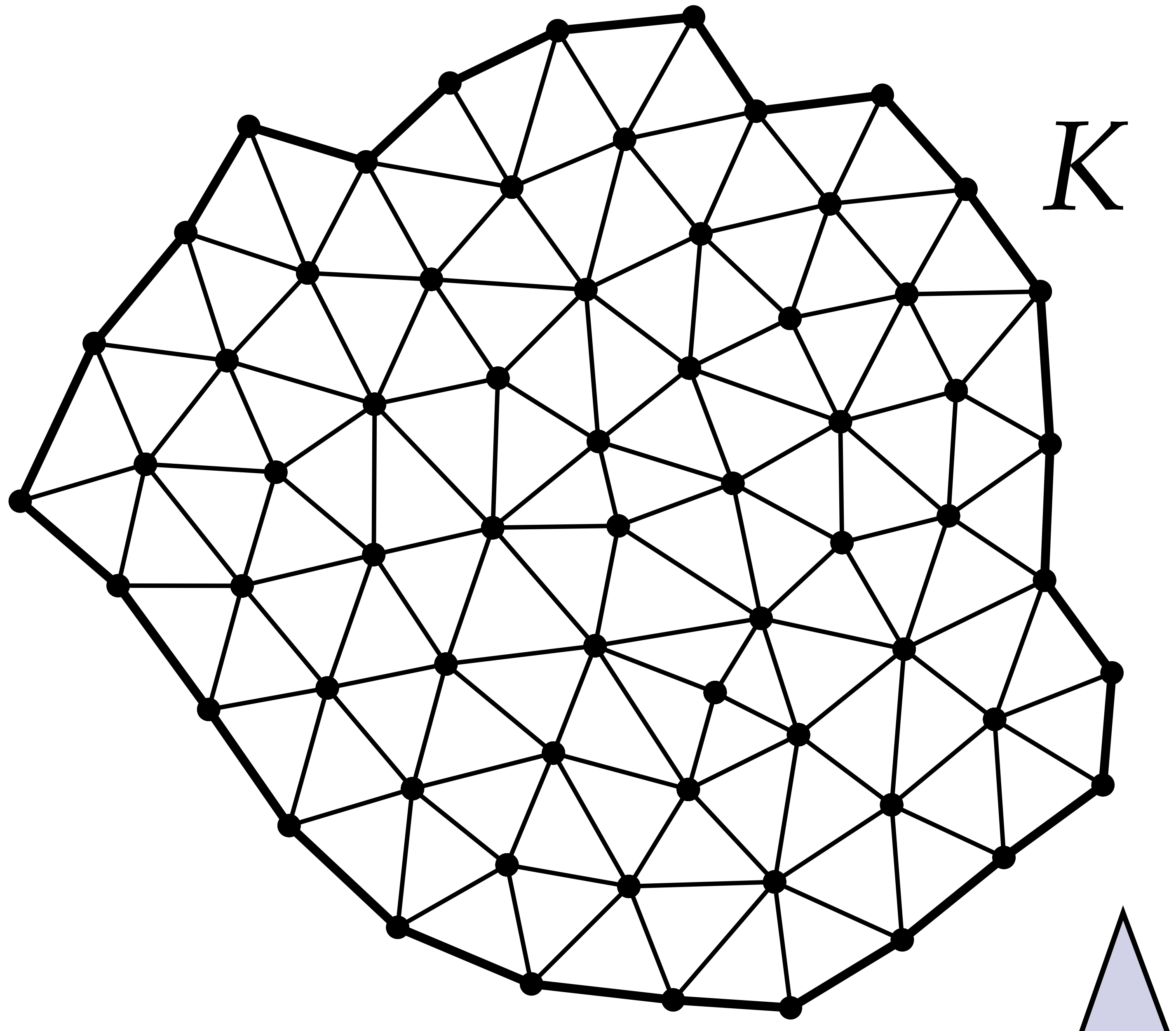
Simplicial Surface—Short Story

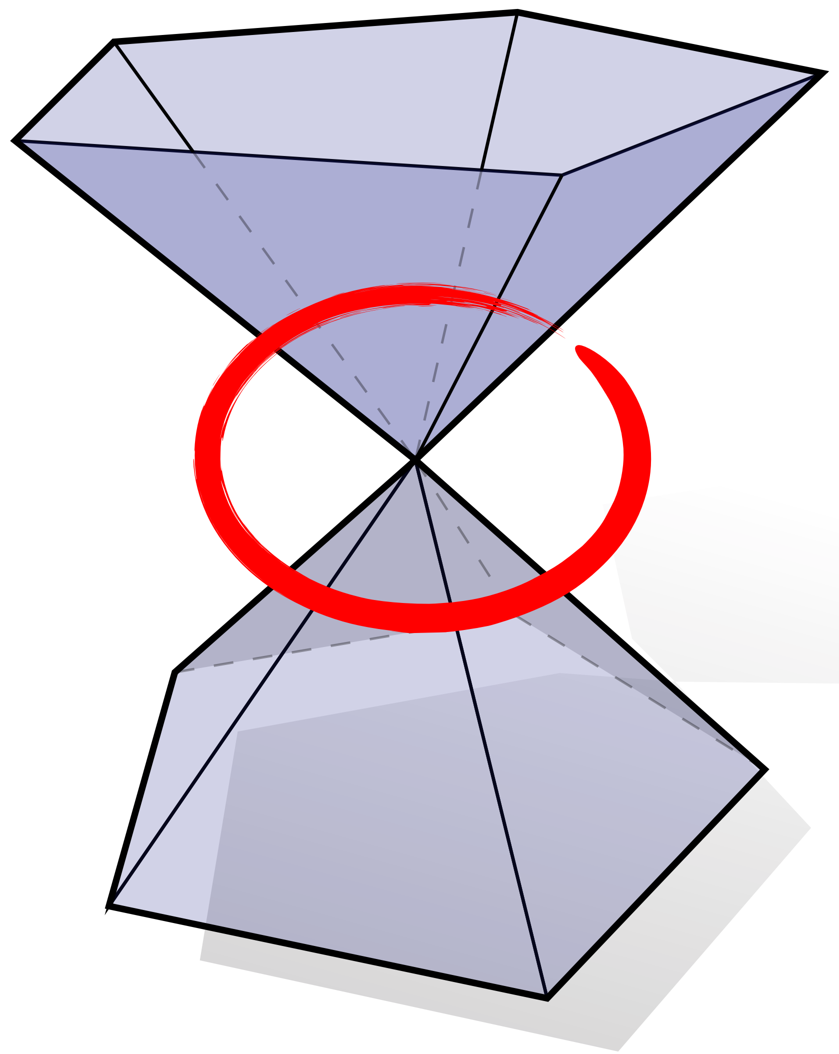
- Loosely speaking, a **simplicial surface** is just a *"triangle mesh"*
- But, being more careful about this definition enables us to connect "triangle meshes" to differential geometry
- As with smooth surfaces, will have regularity conditions that make life easier:
  - **topology:** connectivity is *manifold*
  - **geometry:** vertex coordinates describe a *simplicial immersion*

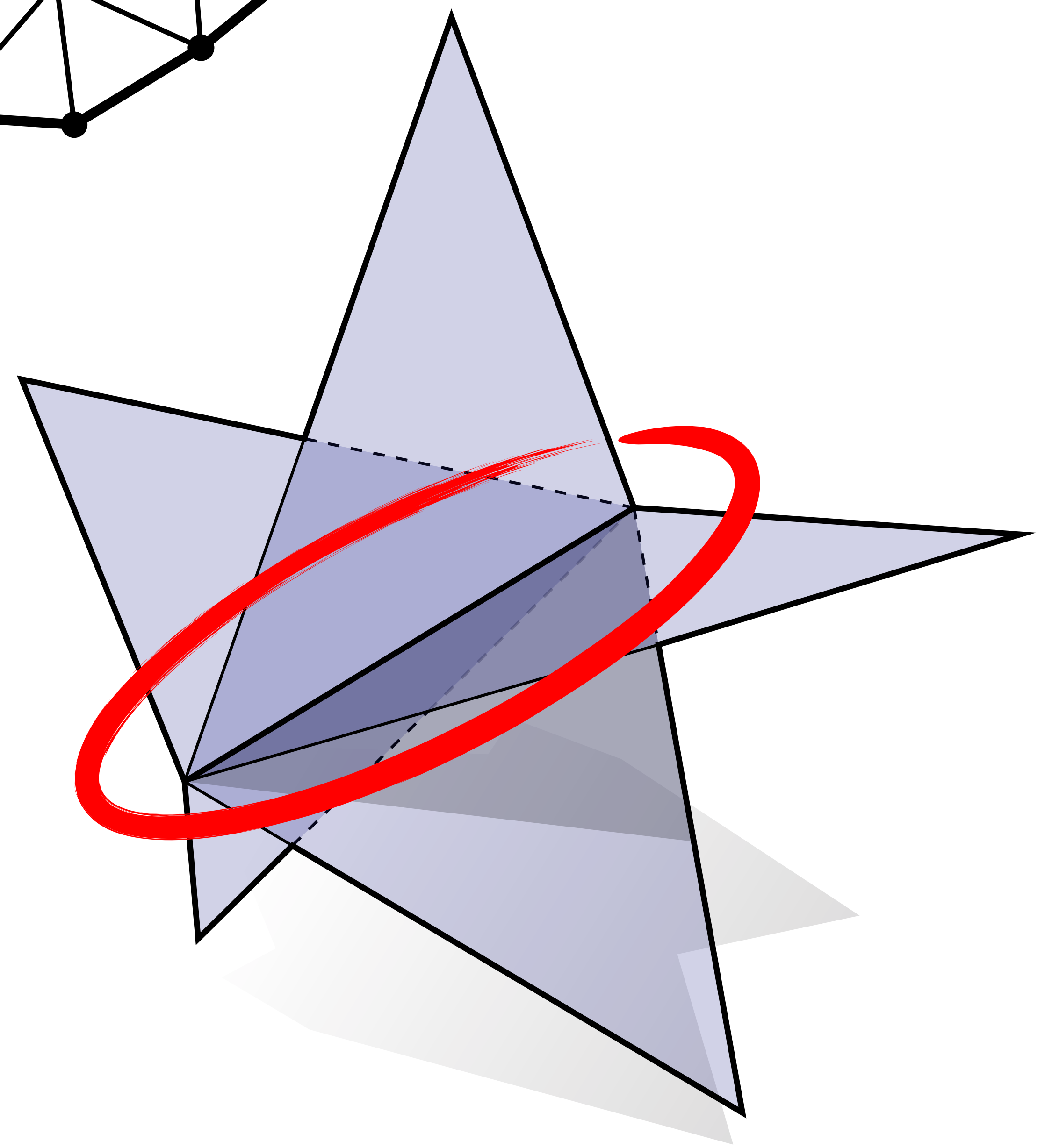


Abstract Simplicial Surface

- An **(abstract) simplicial <u>surface</u>** is a <u>manifold</u> simplicial 2-complex
  – highest-degree simplices are triangles
  – every edge contained in two triangles (or one, along boundary)
  – every vertex contained in a single edge-connected cycle of triangles (or path, along boundary)
- Will typically denote by $K=(V,E,F)$



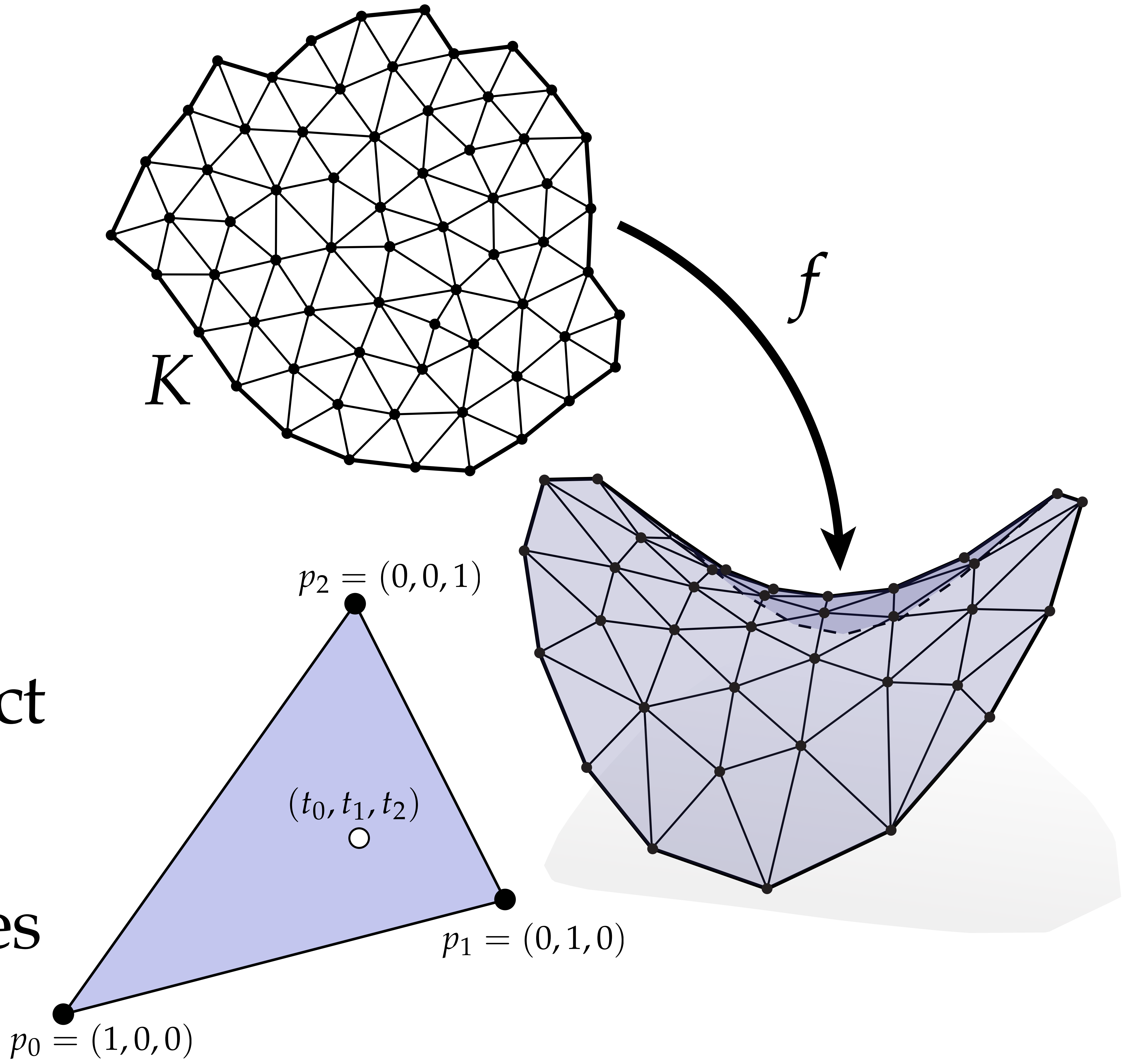




Key idea: no "shape"—just connectivity

Simplicial Map

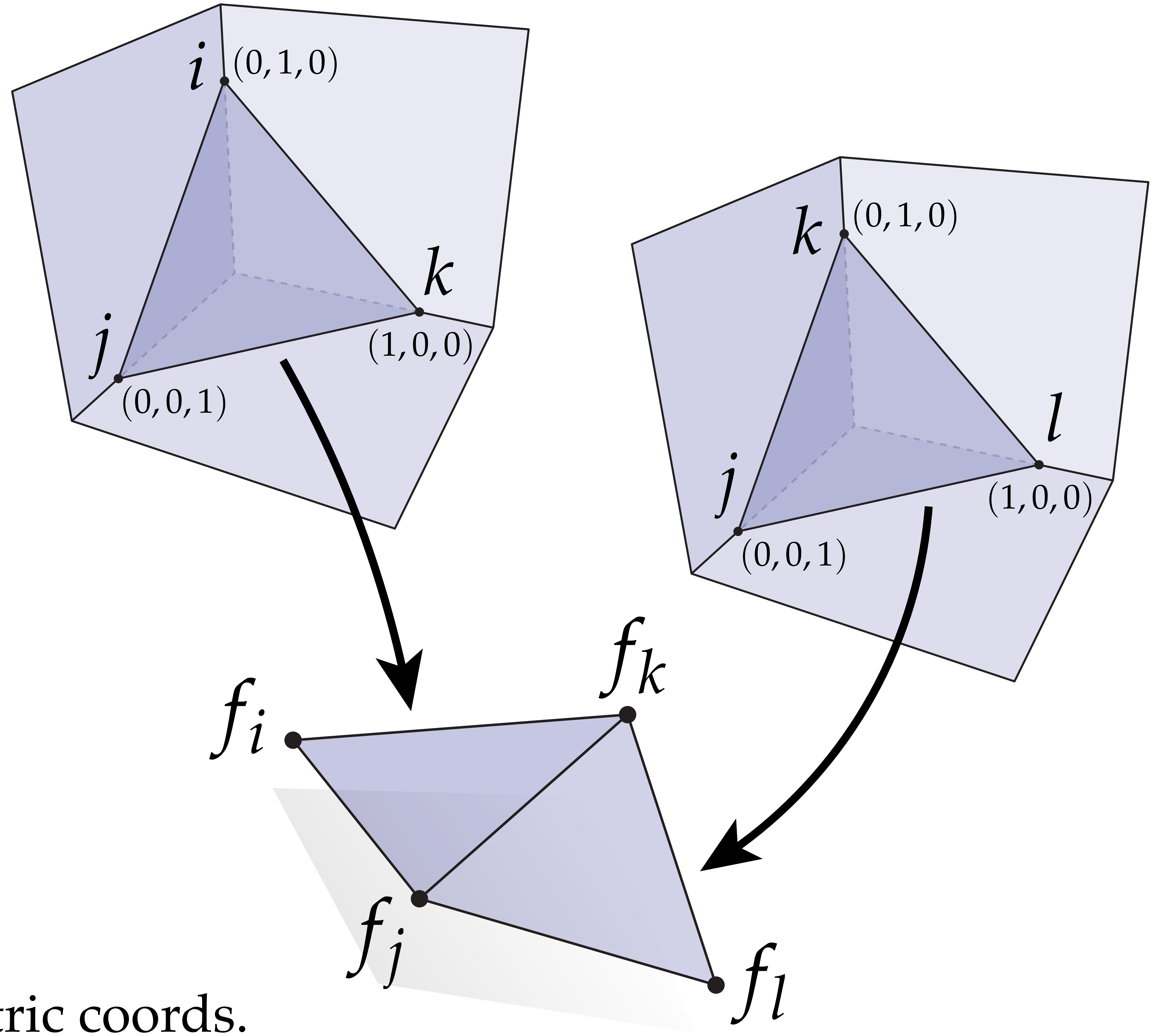
- How do we give a "shape" to an abstract simplicial surface?
- Assign coordinates f_i to each vertex (discrete R^n-valued 0-form)
- Linearly interpolate over edges, triangles via *barycentric coordinates*
- Image of each simplex in our abstract surface is now a simplex in R^n
- Any map from simplices to simplices is called a **simplicial map**



Simplicial Map, continued

- What's really going on here? I.e., what's the domain of our map f?
- Abstract simplicial complex is just a set of subsets… How do we talk about points "inside" a simplex?
- Barycentric coordinates effectively associate each abstract simplex with a a copy of the *standard simplex*
- Domain of f is then the (disjoint) union of all these simplices, "glued" together along shared edges*

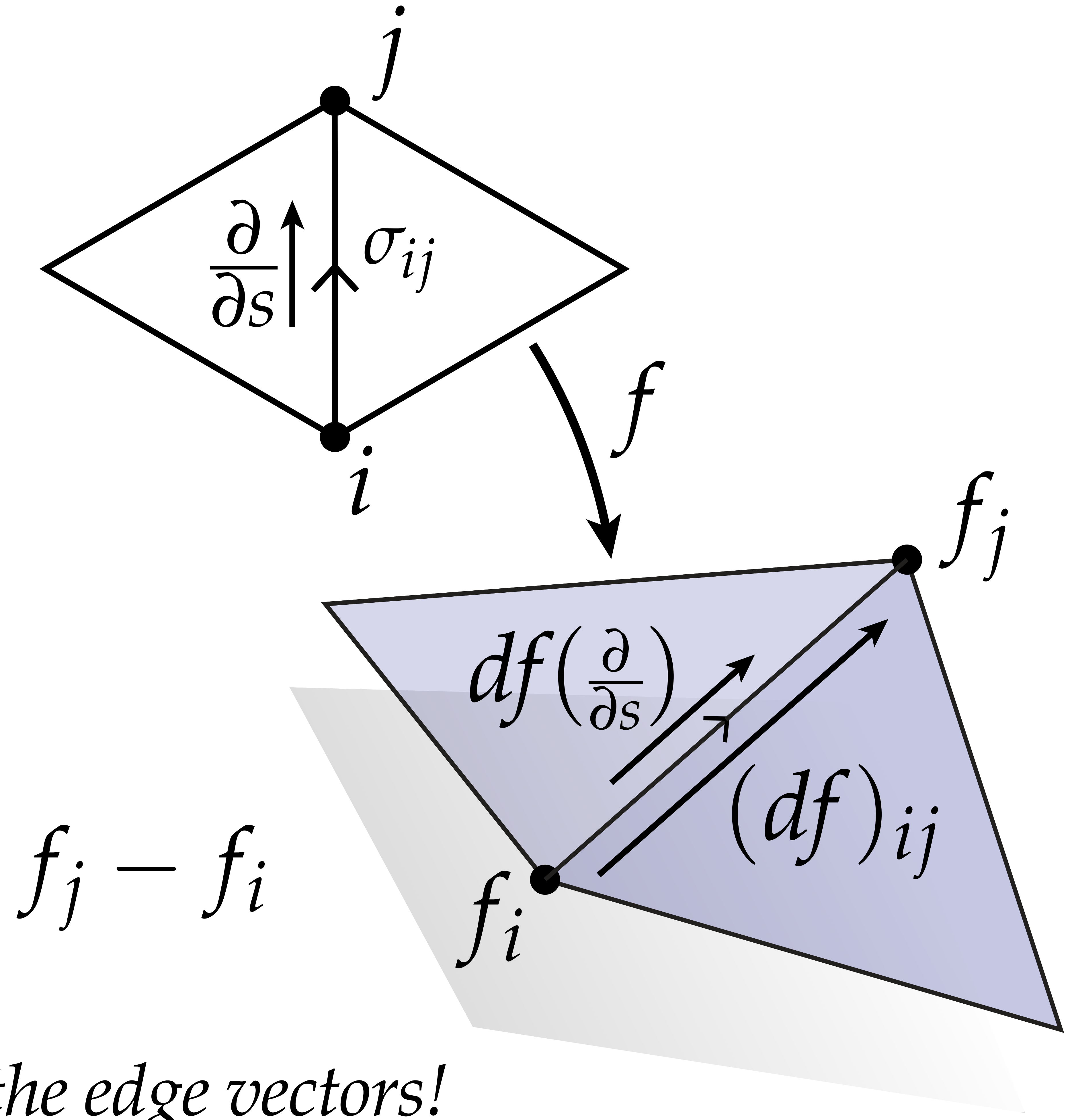
$K = \{\ \{i,j,k\}, \{j,k,l\}, \{i,j\}, \{j,k\}, \{k,i\}, \ldots\}$



*Formally: quotient w.r.t. an equivalence relation on barycentric coords.

Discrete Differential

- Map f is a discrete, R^n-valued 0-form
- **Discrete differential** df is just discrete exterior derivative of f — one value per oriented edge ij
- What do these values mean geometrically?
- Recall that a discrete 1-form represents the integral of a smooth 1-form over a 1-simplex σ_{ij}:

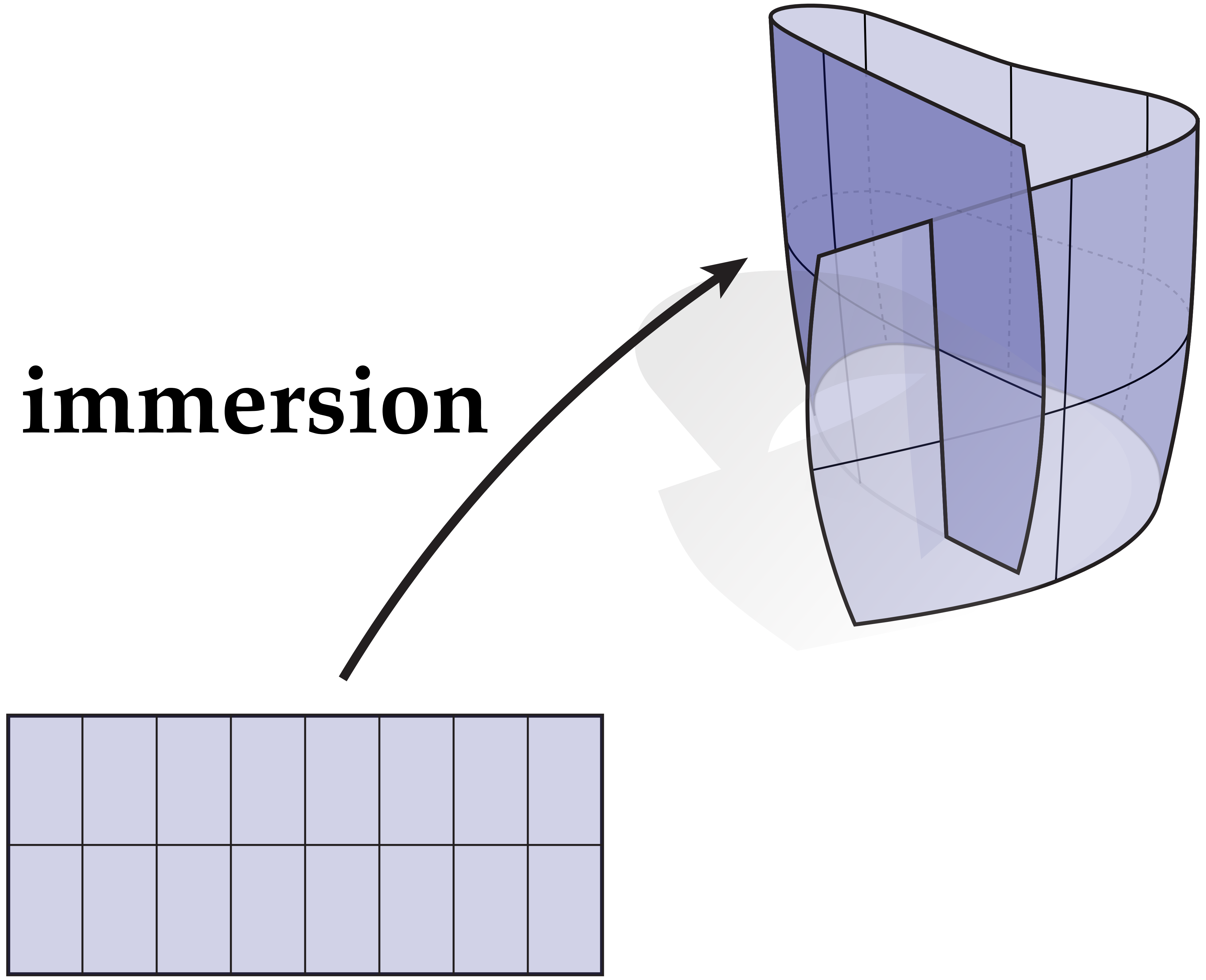


$$(df)_{ij} := \int_{\sigma_{ij}} df\left(\tfrac{d}{ds}\right) ds = \int_{\sigma_{ij}} df = \int_{\partial\sigma_{ij}} f = f_j - f_i$$

- In other words, *discrete differential is nothing more than the edge vectors!*
- Like any other 1-form, antisymmetric w.r.t. orientation: $(df)_{ji} = -(df)_{ij}$

Review: Immersion

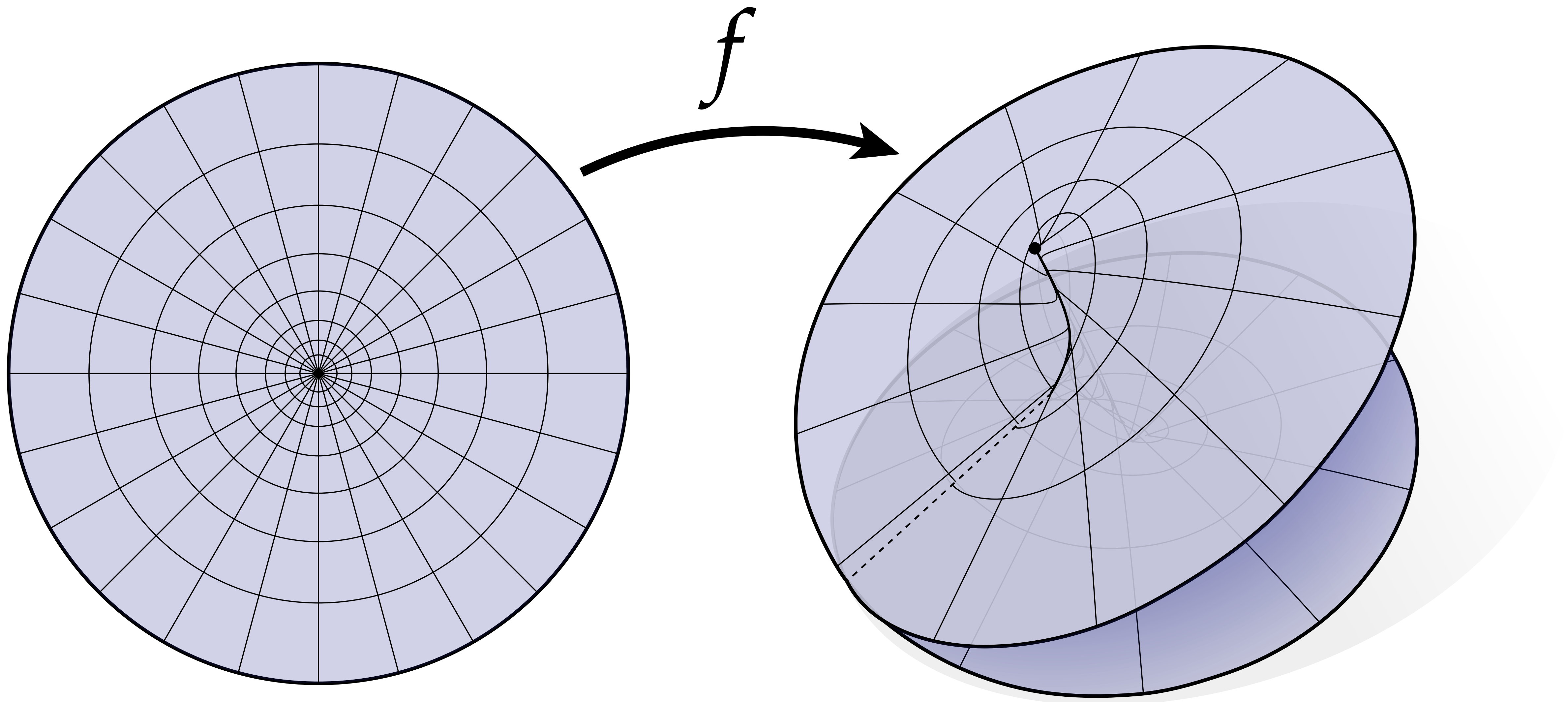
A parameterized surface f is an *immersion* if its differential is nondegenerate, *i.e.*, if $df(X) = 0$ if and only if $X = 0$.

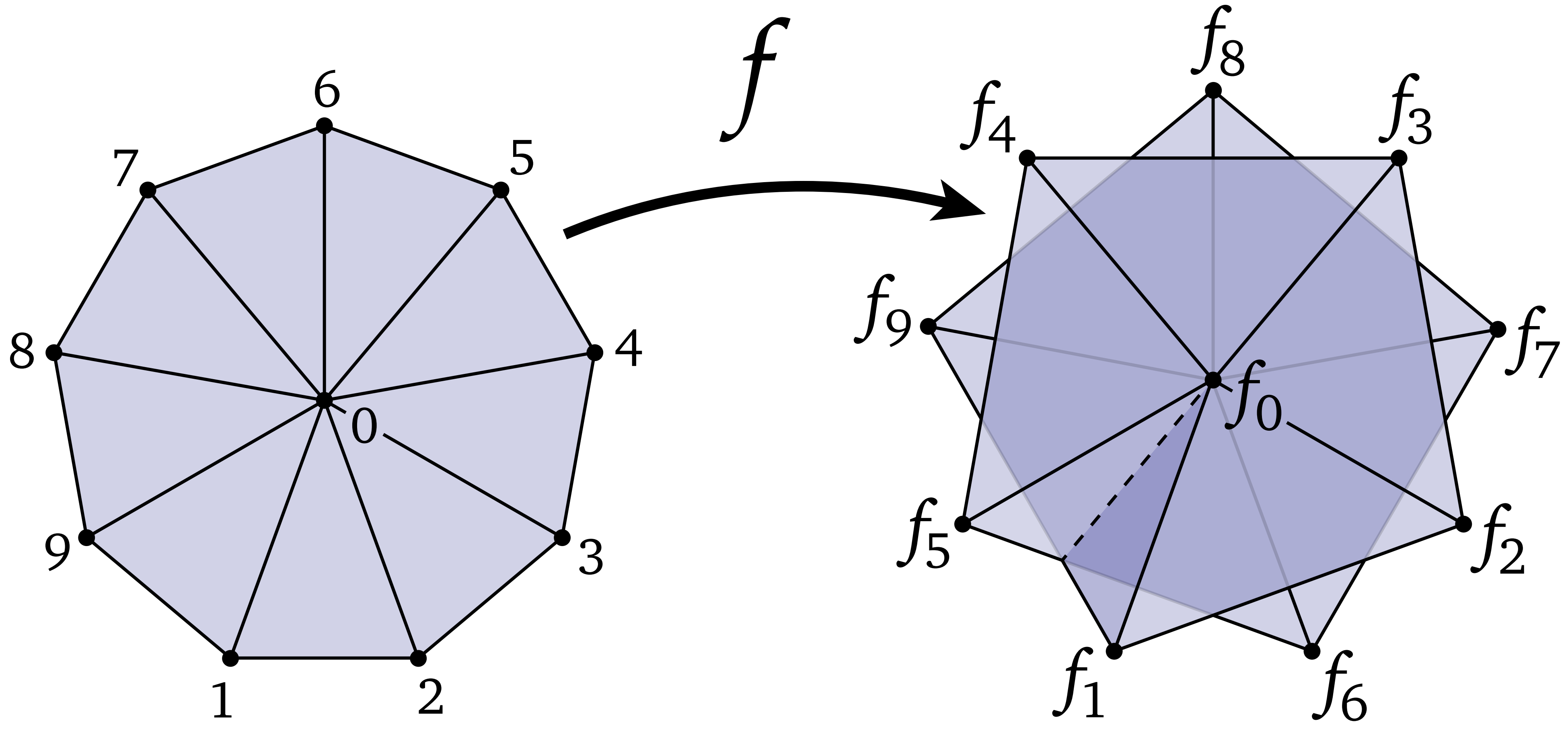


Motivation: map is "nice enough" to define other differential quantities

Discrete Immersion

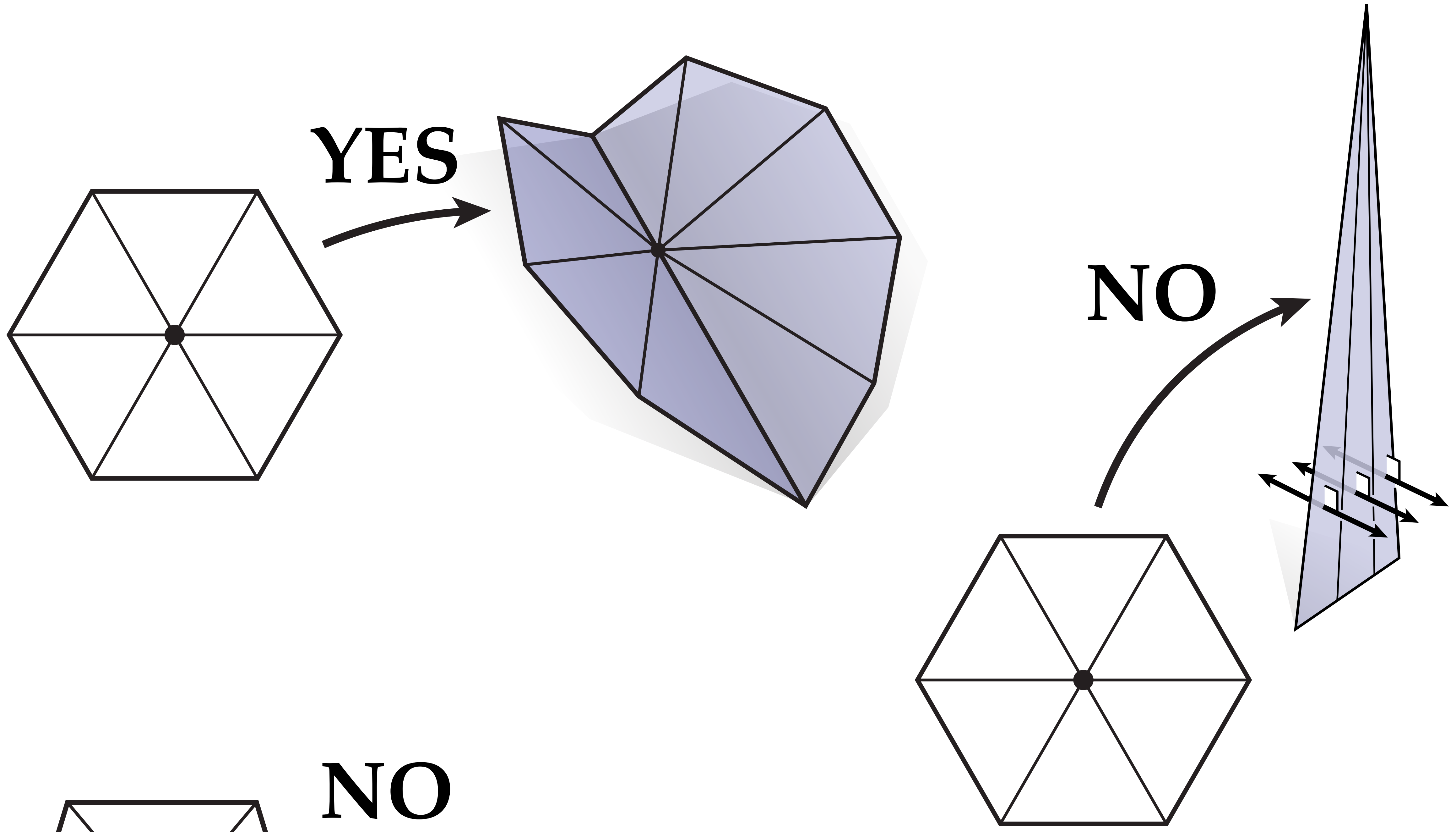
- How do we faithfully translate this "nondegenerate" condition into the discrete setting?
- Naïvely, a nondegenerate *discrete* differential just means there are no zero edge lengths…
- Doesn't faithfully capture important features of smooth immersions!
  - *E.g.*, no *branch points*

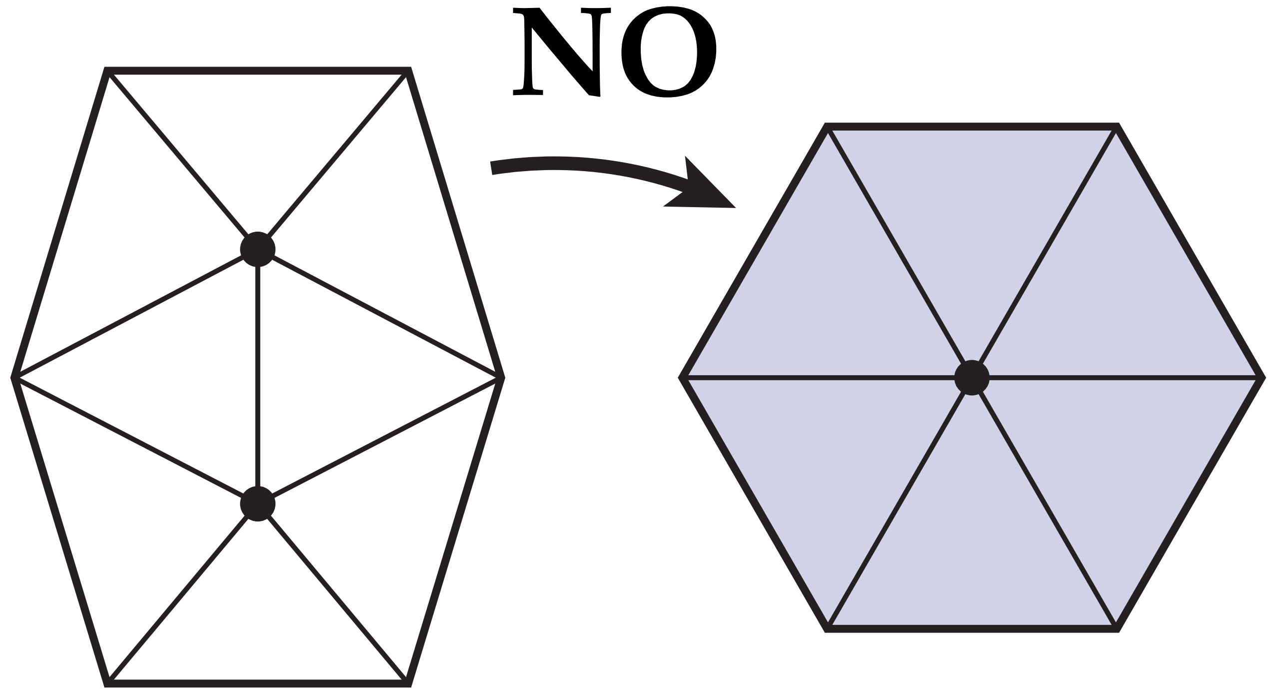


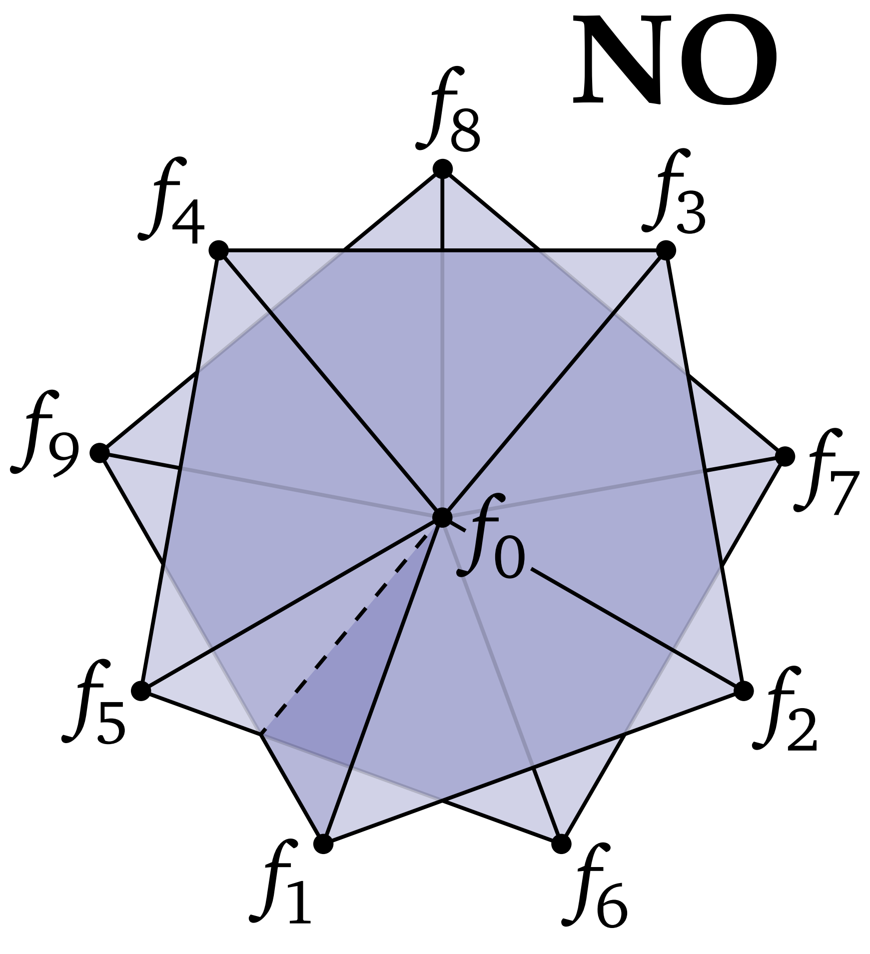


Simplicial Immersion

- Instead, capture more basic property of smooth immersions: local injectivity
- **Definition.** A *discrete immersion* is a locally injective simplicial map
- Basic notion of <u>regularity</u> for discrete surfaces
- **Fact.** A simplicial map is locally injective if and only if every vertex star is embedded

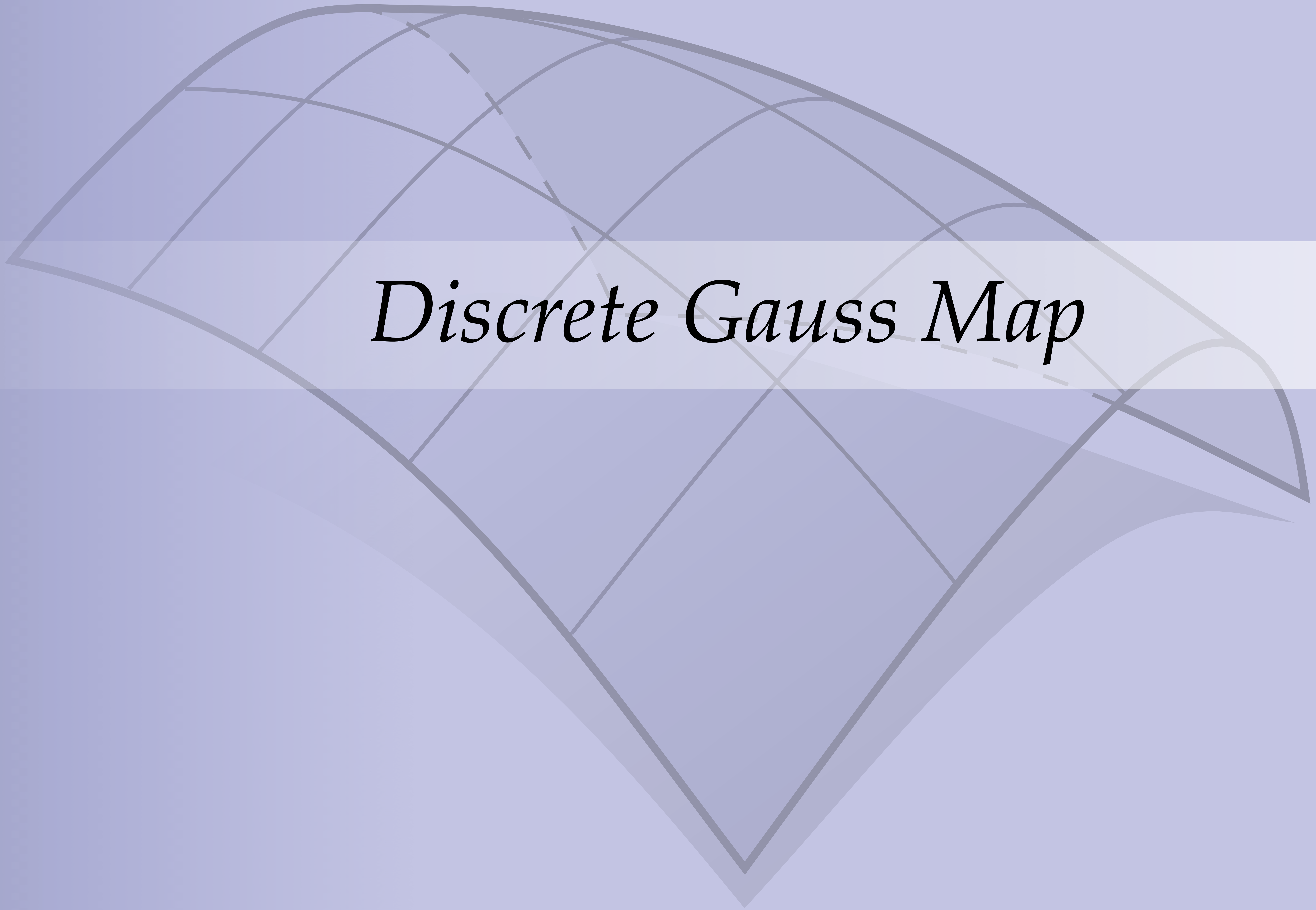






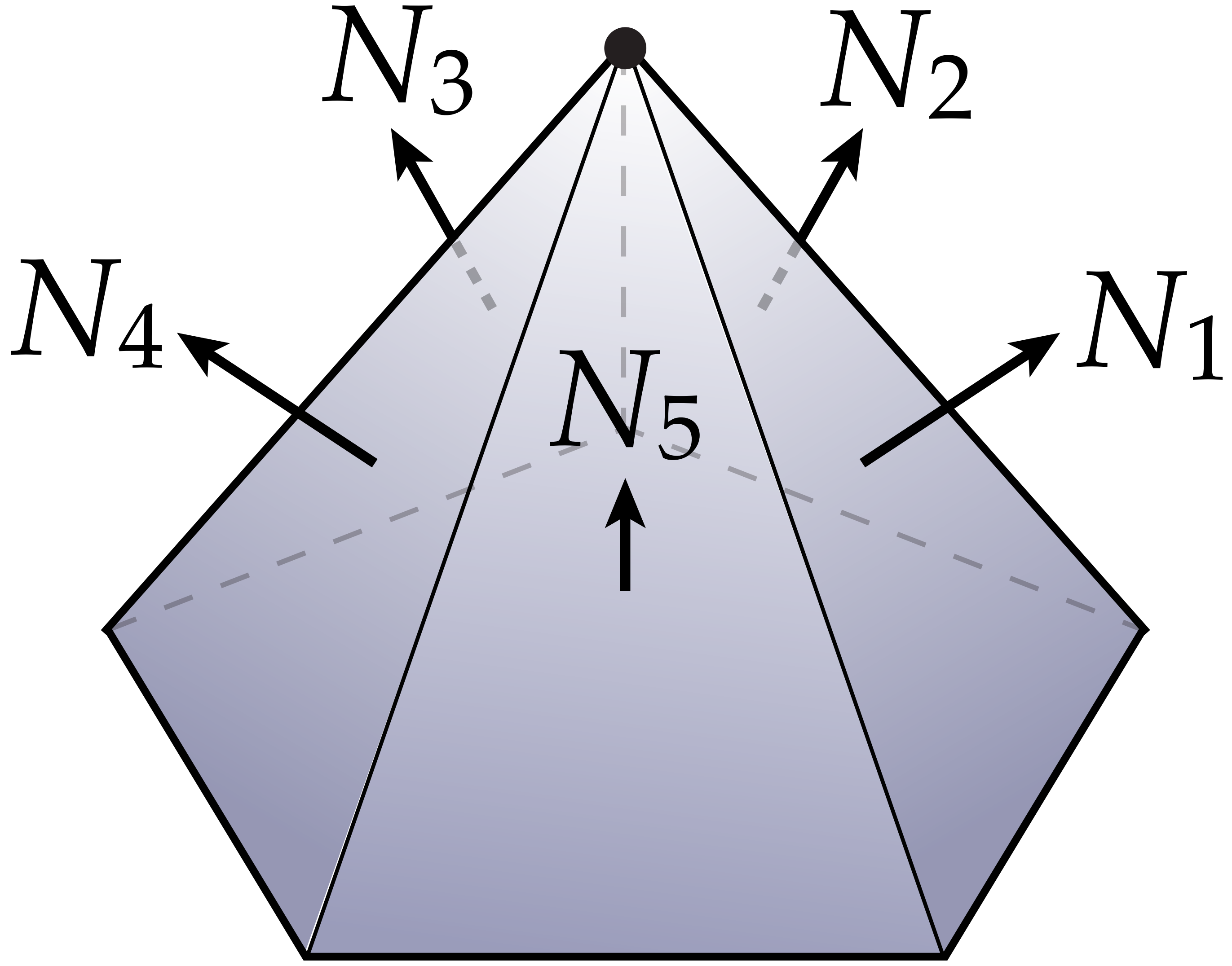
Key idea: *"nonzero areas / lengths / angles"* is necessary, but not sufficient!

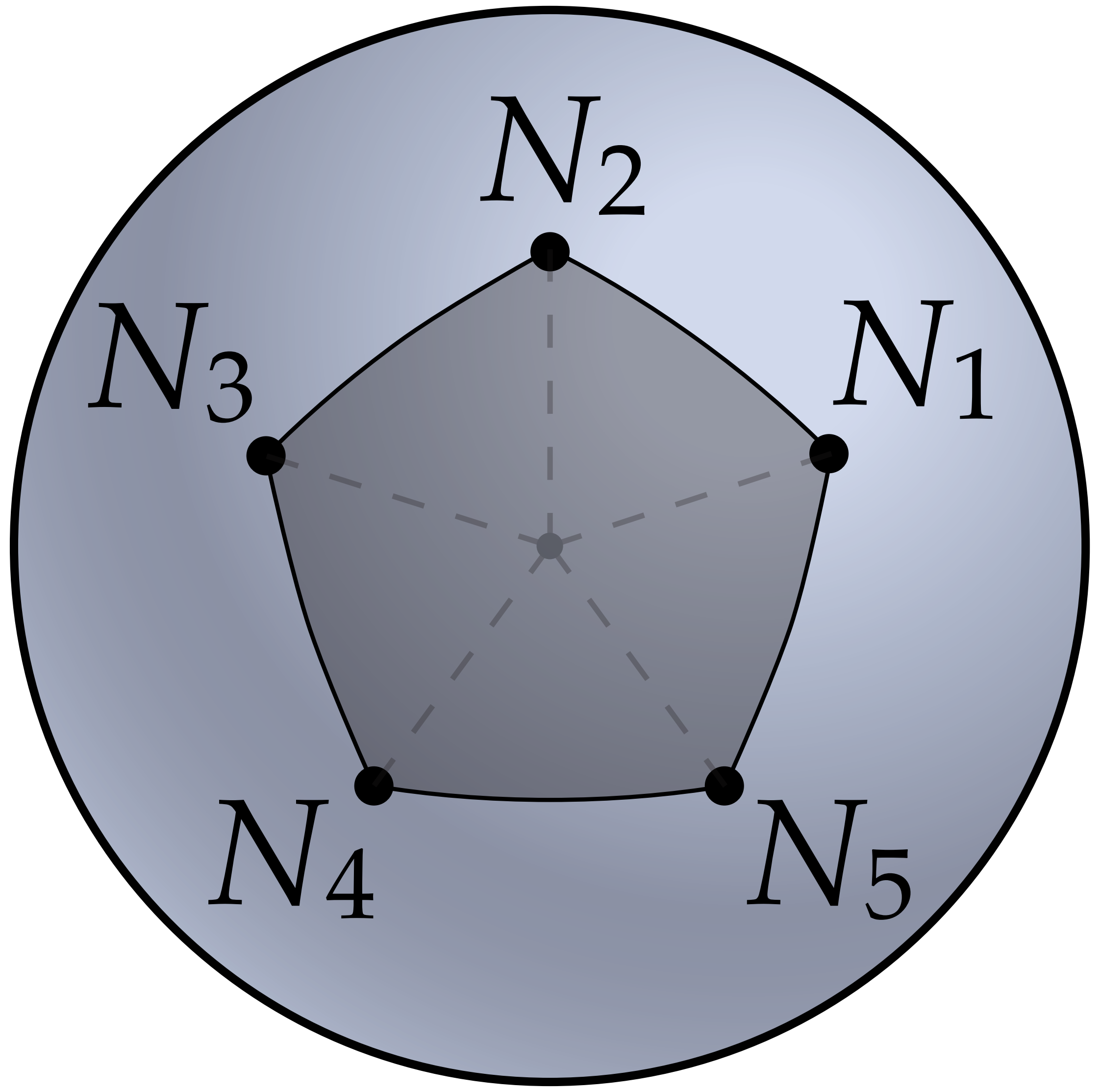
Discrete Gauss Map



Discrete Gauss Map

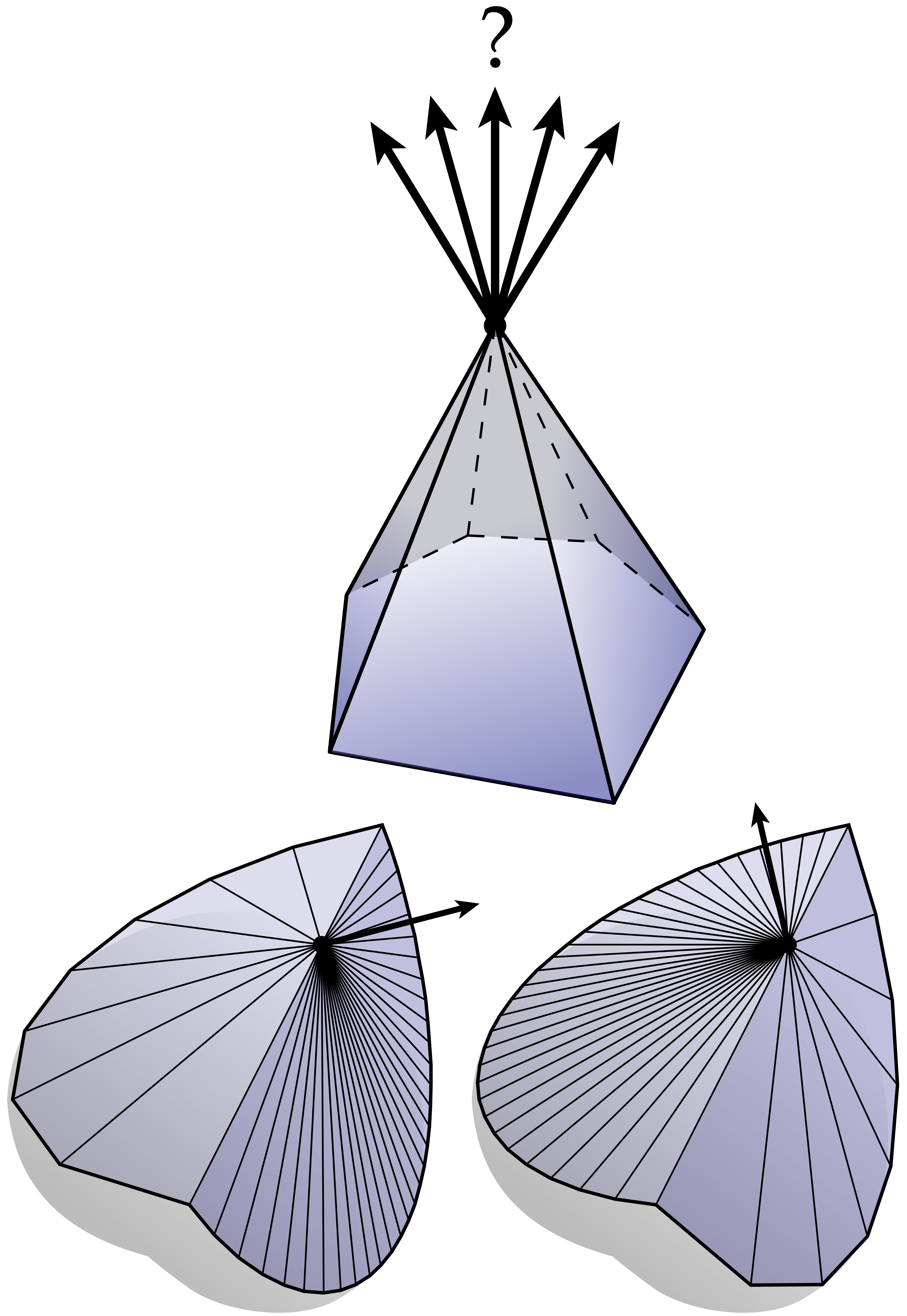
- For a discrete immersion, the Gauss map is simply the triangle normals
- **Discrete exterior calculus:** dual discrete R^3-valued 0-form (vector per triangle)
- Can visualize as points on the unit sphere
- Connecting adjacent normals by arcs corresponds to family of normals orthogonal to edge





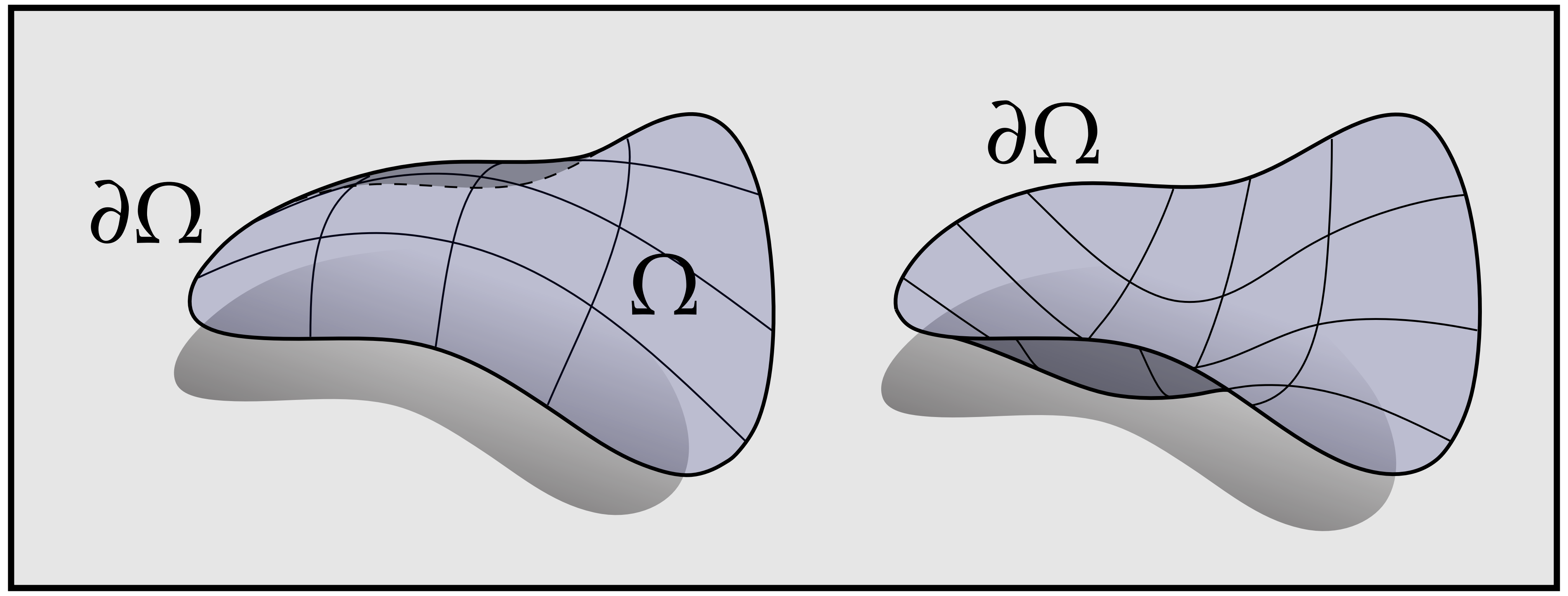
Discrete Vertex Normal?

- Discrete Gauss map still doesn't define normals at vertices (or edges)
- Many possible *ad-hoc* definitions for vertex normal, but may behave poorly…
- *E.g.*, uniformly averaging face normals yields results that <u>depend on tessellation</u> rather than geometry
- Better approach: start in the smooth setting & apply principled discretization



Discrete Vector Area

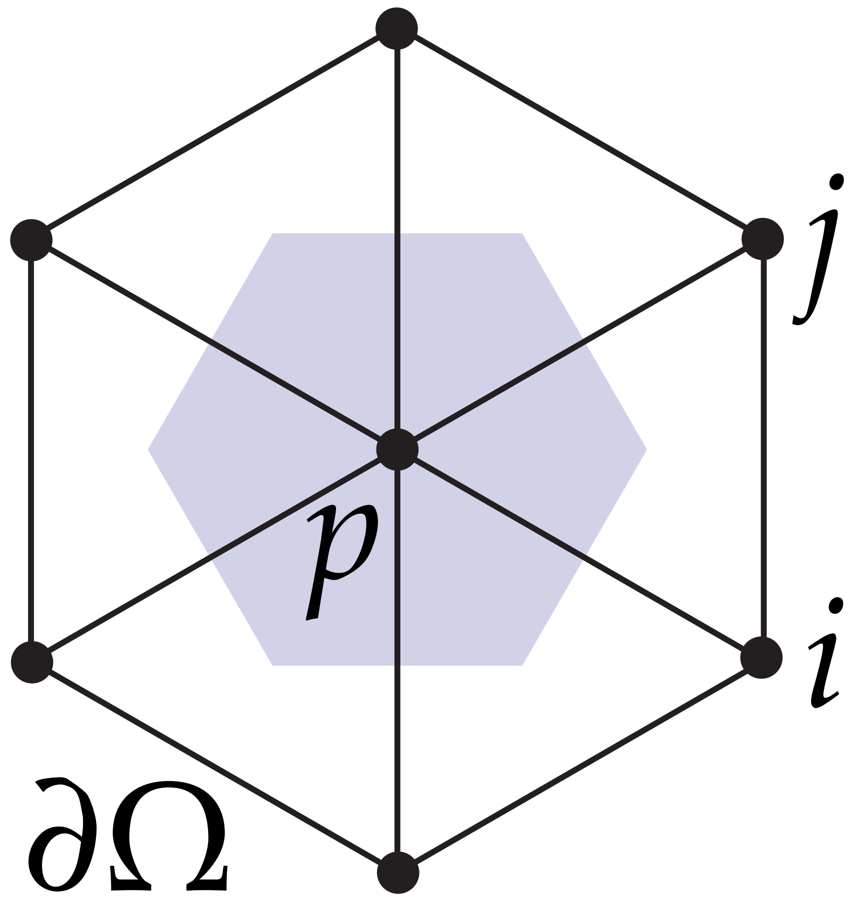
- Recall smooth vector area: $\int_\Omega N\,dA = \frac{1}{2}\int_\Omega df \wedge df = \frac{1}{2}\int_{\partial\Omega} f \times df$
- **Idea:** Integrate NdA over dual cell to get normal at vertex p



$$\frac{1}{3}\int_\Omega N\,dA = \frac{1}{6}\int_{\partial\Omega} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\int_{e_{ij}} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\frac{f_i+f_j}{2} \times (f_j - f_i) = \boxed{\frac{1}{6}\sum_{ij\in\partial\Omega} f_i \times f_j}$$



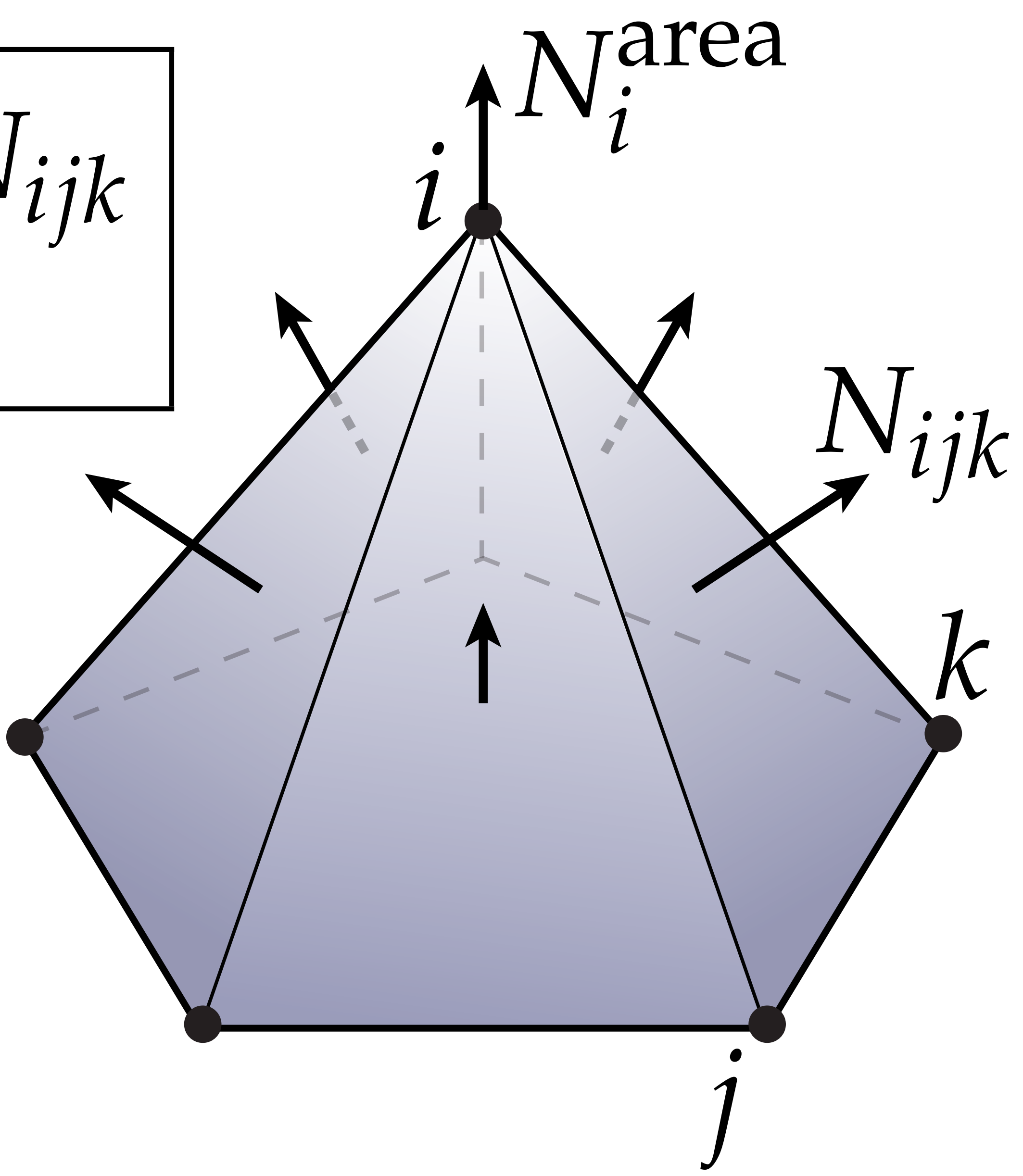
Note: Doesn't depend on the location of p!

Other Natural Definitions

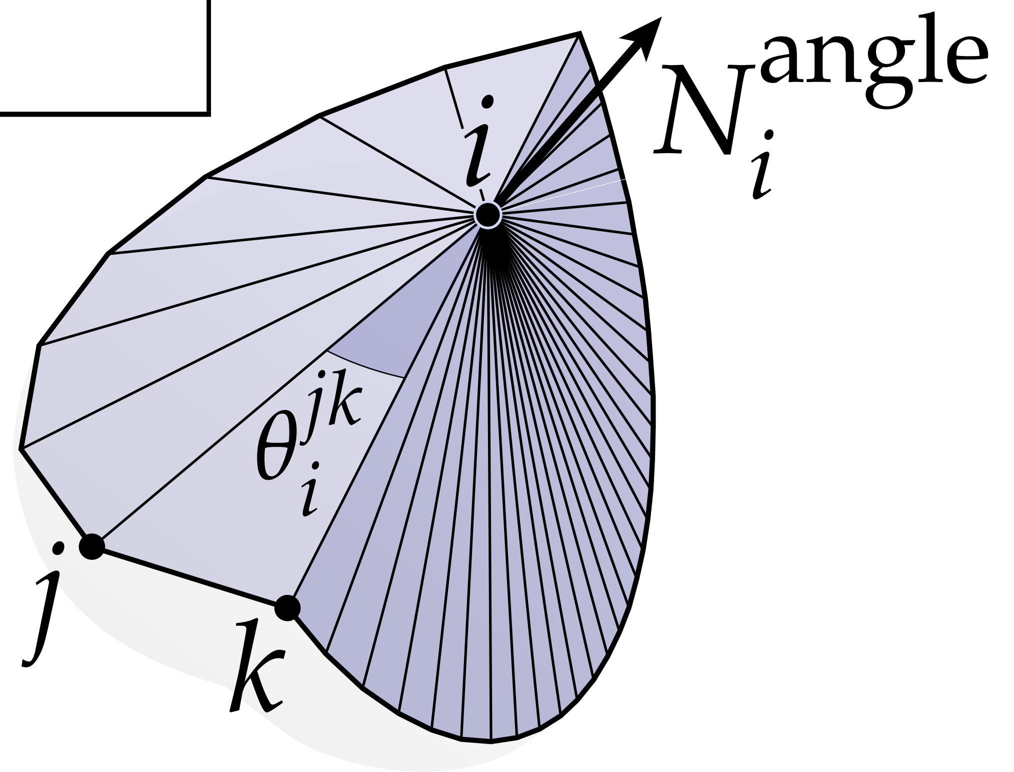
- area-weighted vertex normal
  - sum of triangle normals times triangle areas
  - corresponds to exact *volume variation*
- angle weighted vertex normal
  - sum of triangle normals times interior angles
  - gives same result, independent of triangulation

Please: just *anything* but uniformly weighted!

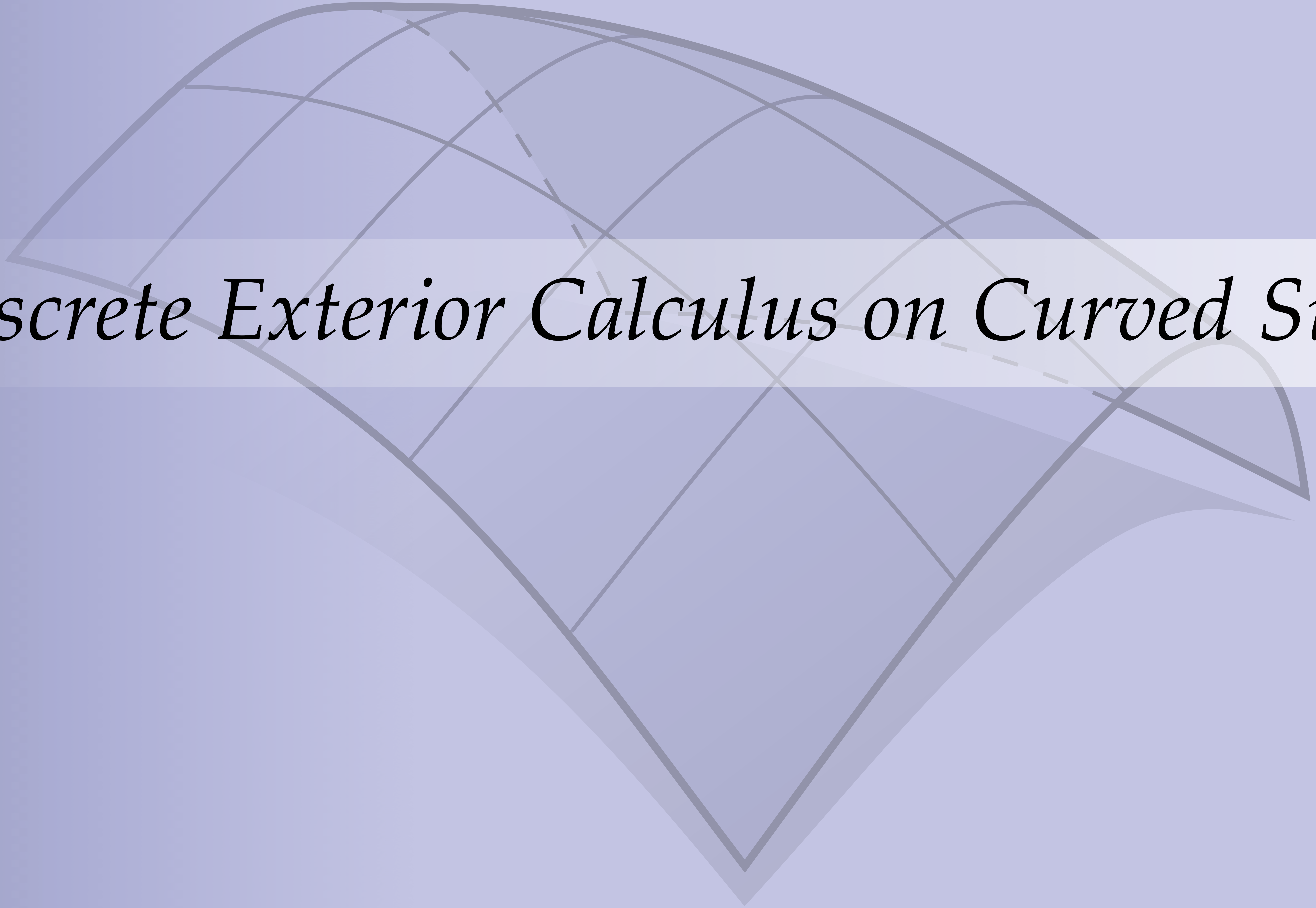
$$\sum_{ijk} A_{ijk} N_{ijk}$$



$$\sum_{ijk} \theta_i^{jk} N_{ijk}$$

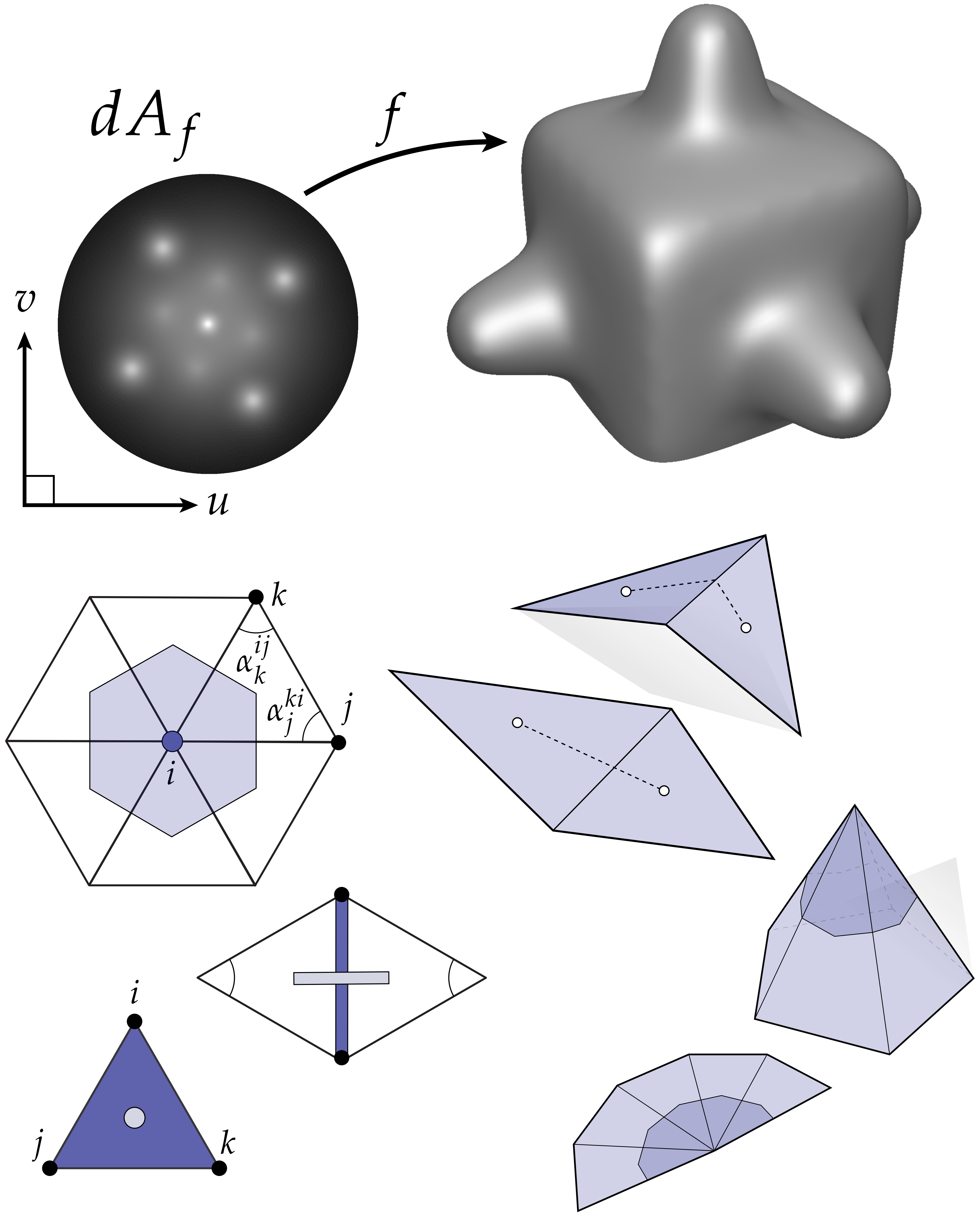


Discrete Exterior Calculus on Curved Surfaces



Discrete Exterior Calculus on Curved Surfaces

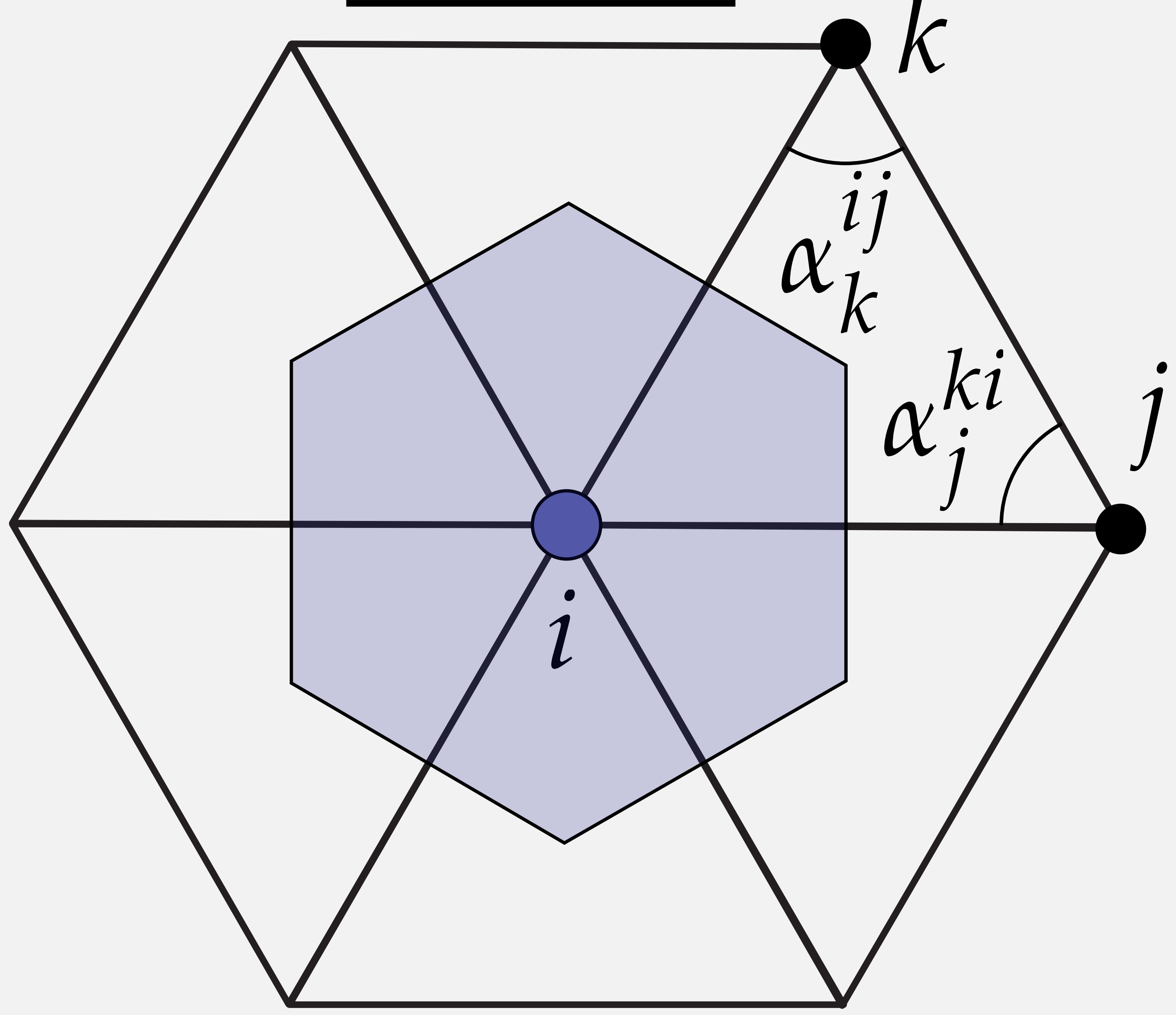
- In the smooth setting, we first defined exterior calculus in R^n, then saw how to augment it to work on curved surfaces
- **Key observation:** just need to change the Hodge star, which encodes all geometric information (length, angle, area, …)
- For simplicial surfaces in R^3, life is even easier: each simplex is already flat!
- Will have to make essentially no change to our discrete Hodge star from R^n…



Diagonal Hodge Star on a Surface

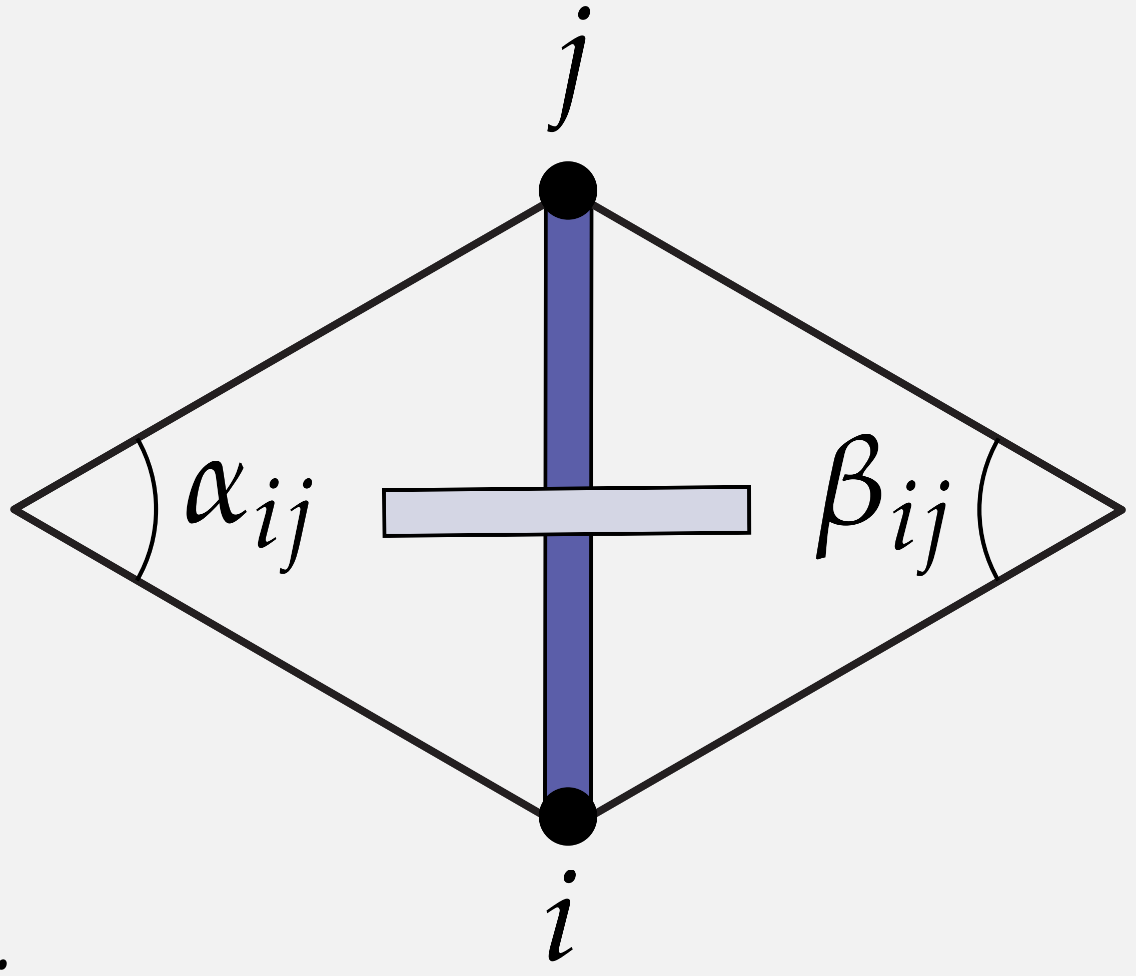
Recall that on a simplicial surface, we discretized the Hodge star via diagonal matrices storing *primal-dual volume ratios:*

0-forms



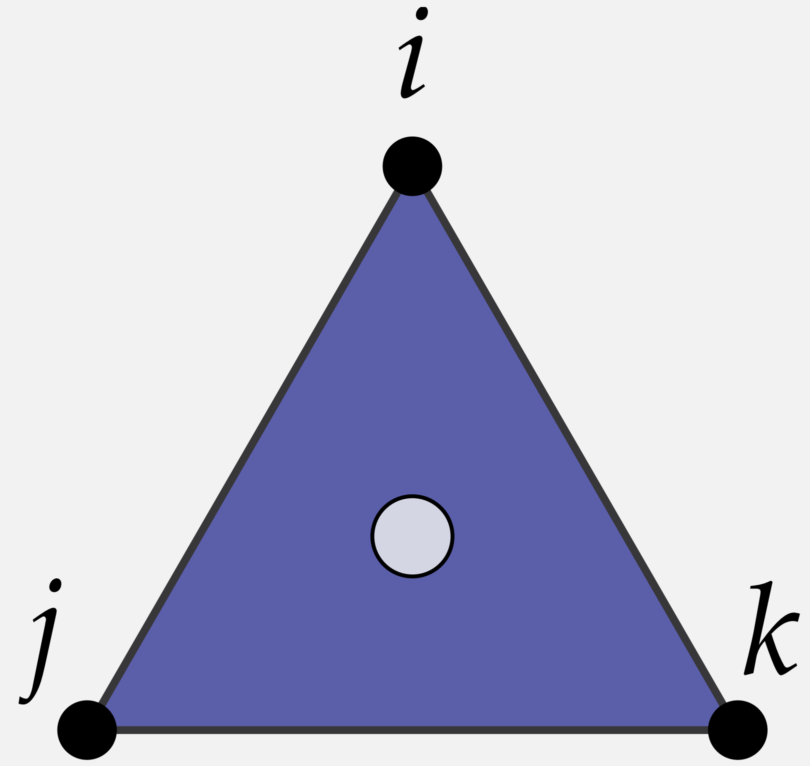
$$A_i^* / 1 = \frac{1}{8} \sum_{ijk \in F} (\ell_{ij}^2 \cot \alpha_k^{jk} + \ell_{ik}^2 \cot \alpha_j^{ki})$$

1-forms



$$\frac{\ell_{ij}^*}{\ell_{ij}} = \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})$$

2-forms



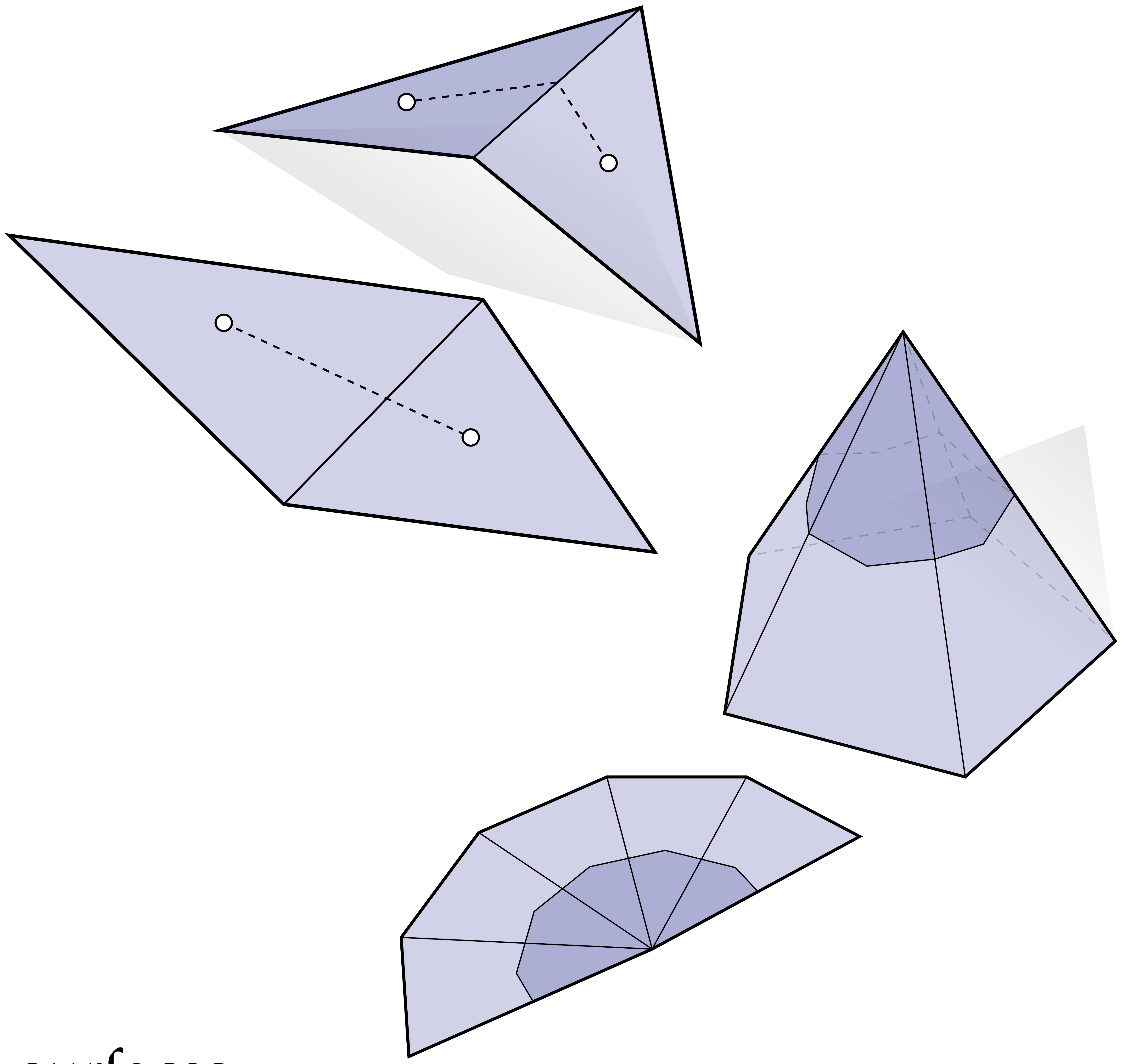
$$\frac{1}{A_{ijk}} = \frac{1}{\sqrt{s(s-\ell_{ij})(s-\ell_{jk})(s-\ell_{ki})}}$$

$$s = \frac{1}{2}(\ell_{ij} + \ell_{jk} + \ell_{ki})$$

Q: What happens if our mesh is no longer flat?

Diagonal Hodge Star on a Curved Surface

- **A:** Nothing changes! We can still apply the same formulas—which depend only on *primal lengths* and *interior angles*
- *E.g.*, for the 1-form Hodge star, we are effectively taking a length ratio involving the dual distance "along" the surface
- For 0-/2-form Hodge star, just summing up little areas from pieces of triangles
- This makes sense: Hodge star operators are purely "**intrinsic**": they do not depend at all on how a surface sits in space.



Key idea: 2D formulas also work for simplicial surfaces

Discrete Laplace-Beltrami Operator

- From here, we can immediately build discrete differential operators for *curved* surfaces by just composing our existing discrete exterior derivative and discrete Hodge star operators

- For instance, the ordinary 2D Laplacian now becomes the *Laplace-Beltrami operator*

$$\Delta\phi = *d * d\phi$$

- Using our expressions for the discrete Hodge star, can write the discrete Laplace-Beltrami operator via the famous *cotan formula:*

$$(\Delta u)_i = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

Recovery of Discrete Surfaces

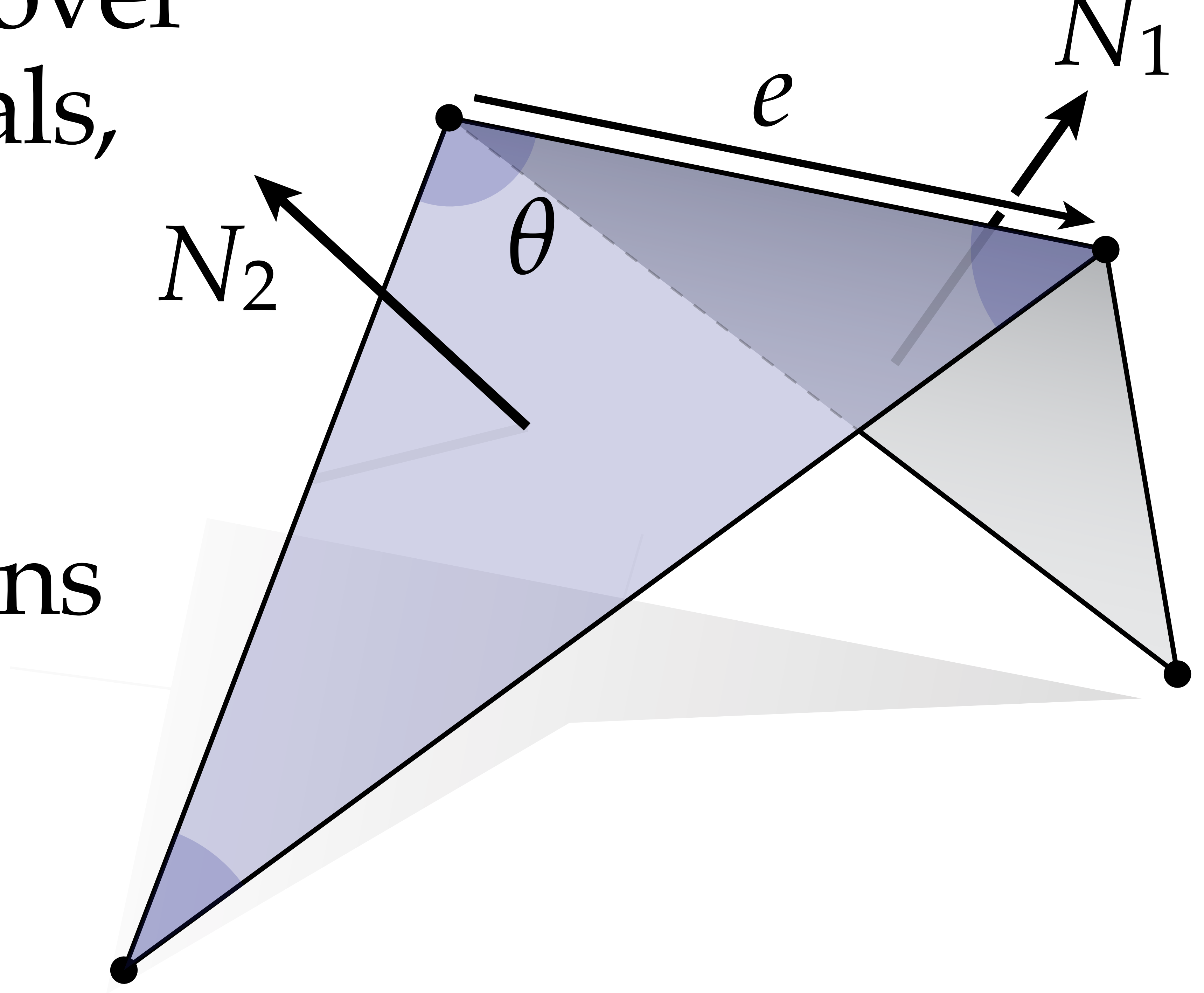
Recovery of Discrete Surfaces

- In a variety of situations, geometry can be recovered from differential quantities:
  - (**Ordinary functions** can be recovered from their derivative)
  - **Plane curves** can be recovered from their curvature
  - **Space curves** can be recovered from their curvature and torsion
  - **Smooth surfaces** can be recovered from 1st & 2nd fundamental form
  - **Convex surfaces** can be recovered from their Riemannian metric…

Q: What data is sufficient to describe a *discrete* surface?

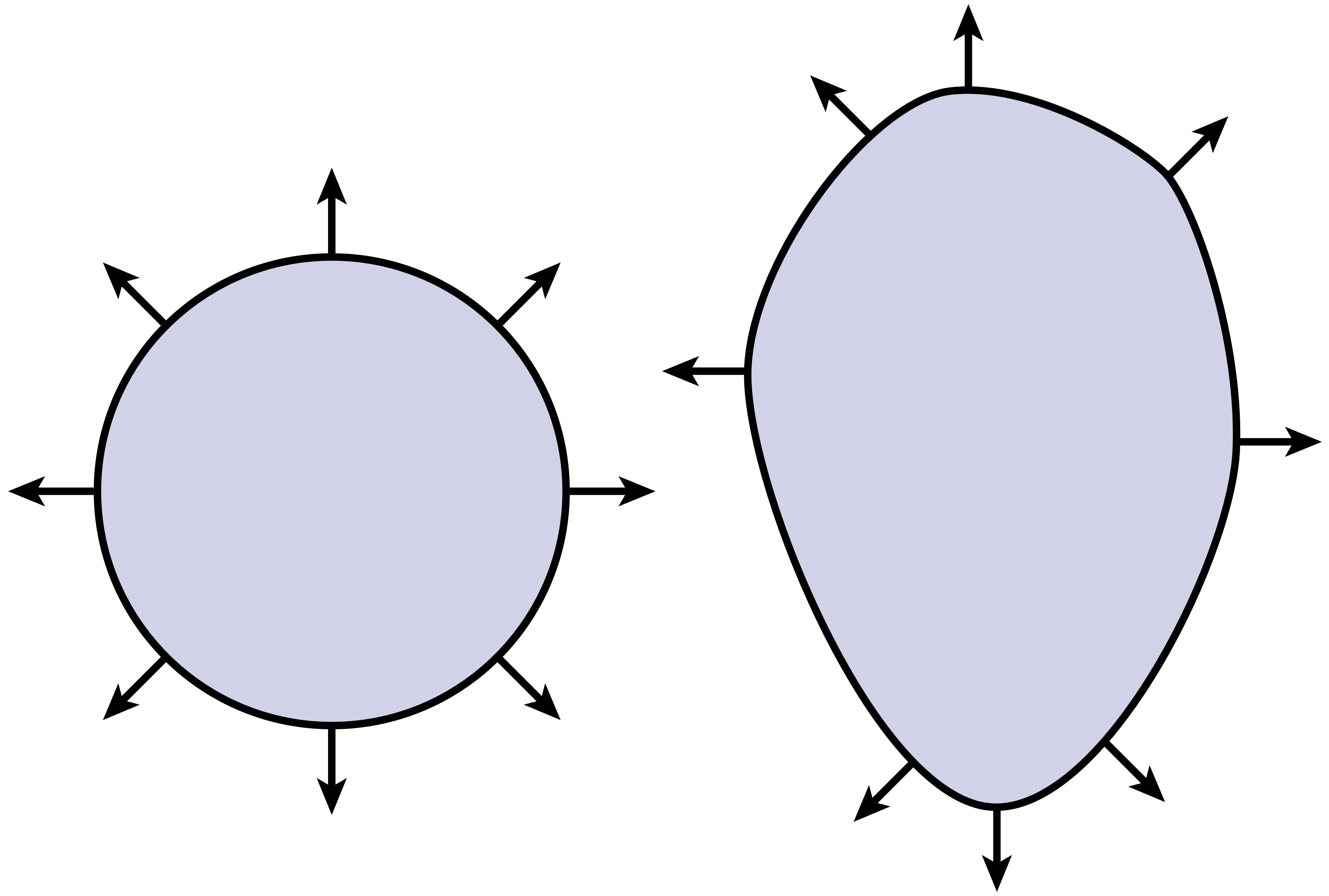
Shape from Normals—Simplicial

- **Q:** Given only discrete Gauss map, can we recover the immersion? (*I.e.*, given only triangle normals, can we get vertex positions?)
- **A:** Yes! Basic recipe:
  - Cross product of normals gives edge directions
  - Dot product of edges gives interior angles
  - Three angles determine triangle up to scale; normal determines plane of each triangle
  - Build triangles one-by-one and "glue" together
- **Q:** Does this recipe *always* work?

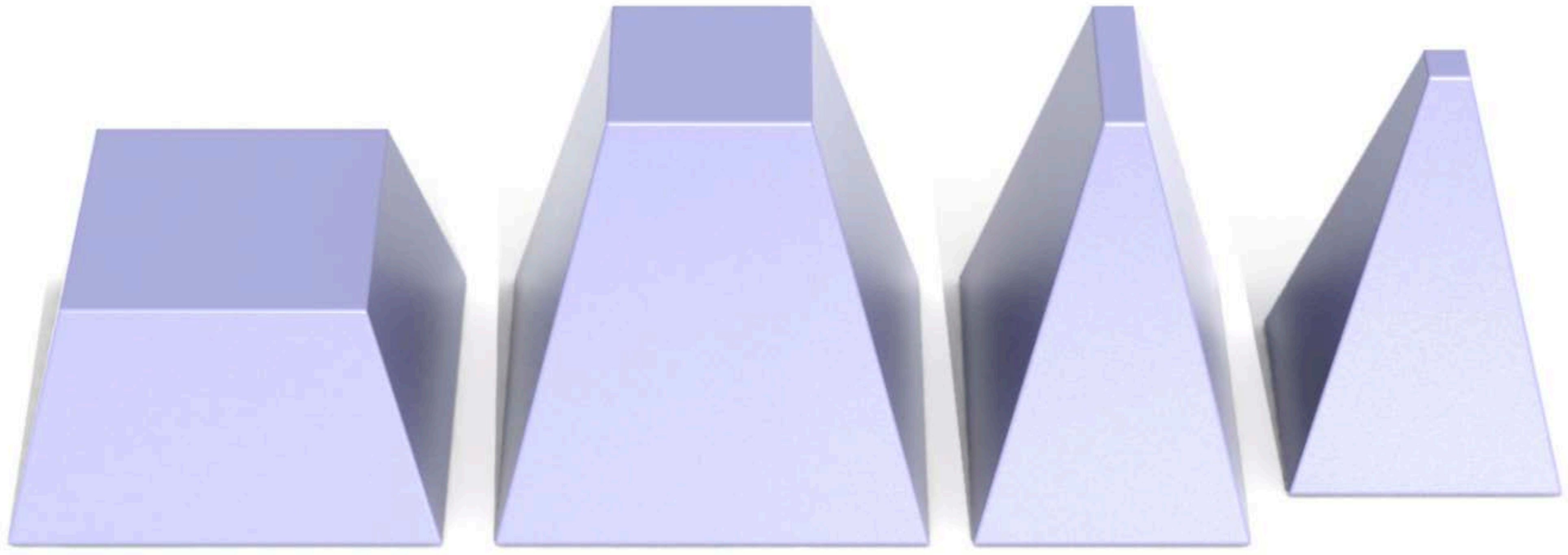


Shape from Normals—Smooth

- **Q:** Is it *strange* that we can recover a discrete surface from Gauss map? Can we do something similar in the smooth setting?
- Consider a simpler case: Gauss map on a *curve*
- $N(s) := (\cos(s), \sin(s))$
- **Problem:** unless we know curve is arc-length parameterized, N is the Gauss map of *any* convex curve! Need additional data (parameterization)
- Same story for any convex discrete curves, or any convex smooth surfaces: normals are not enough!

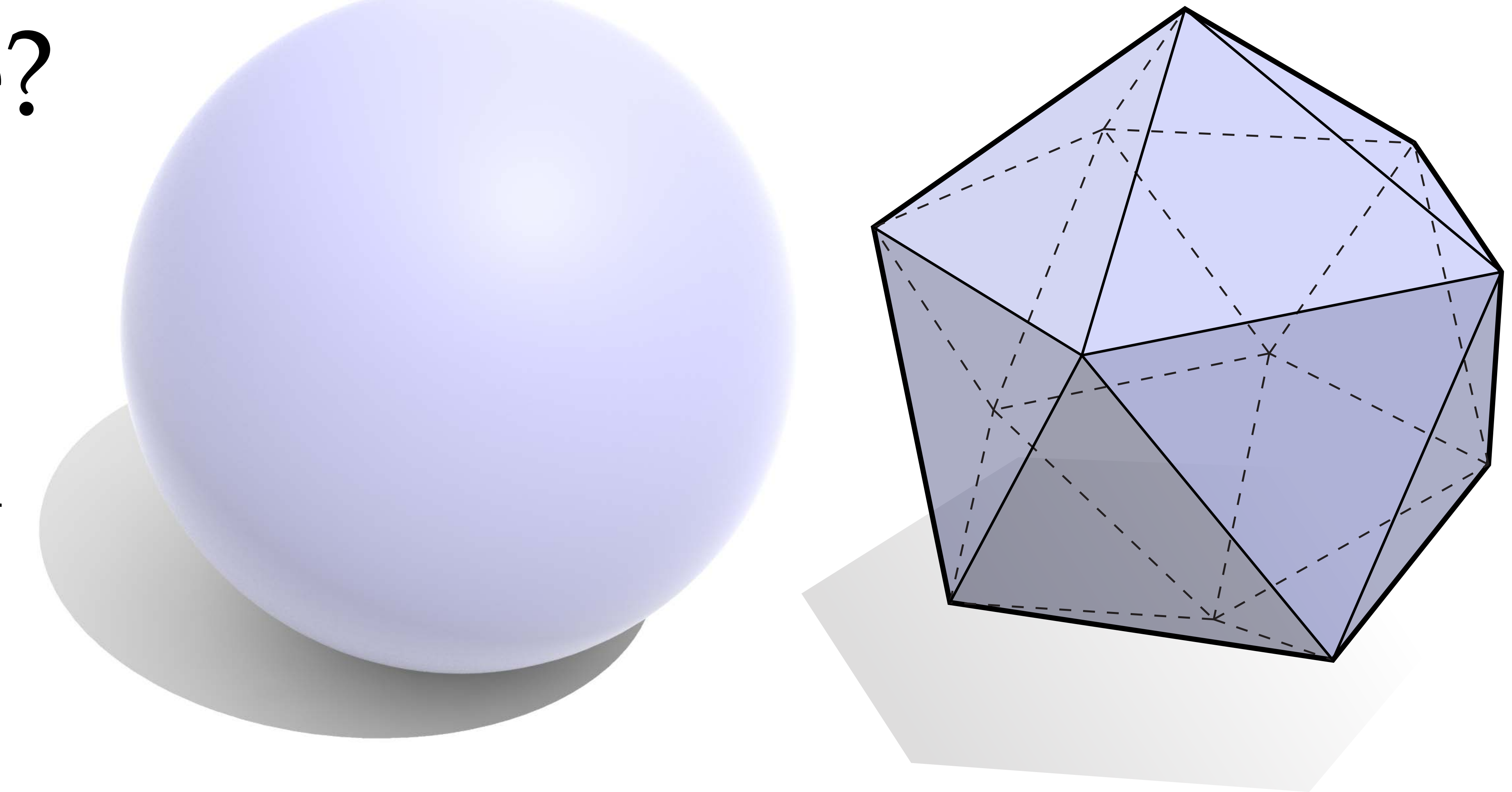


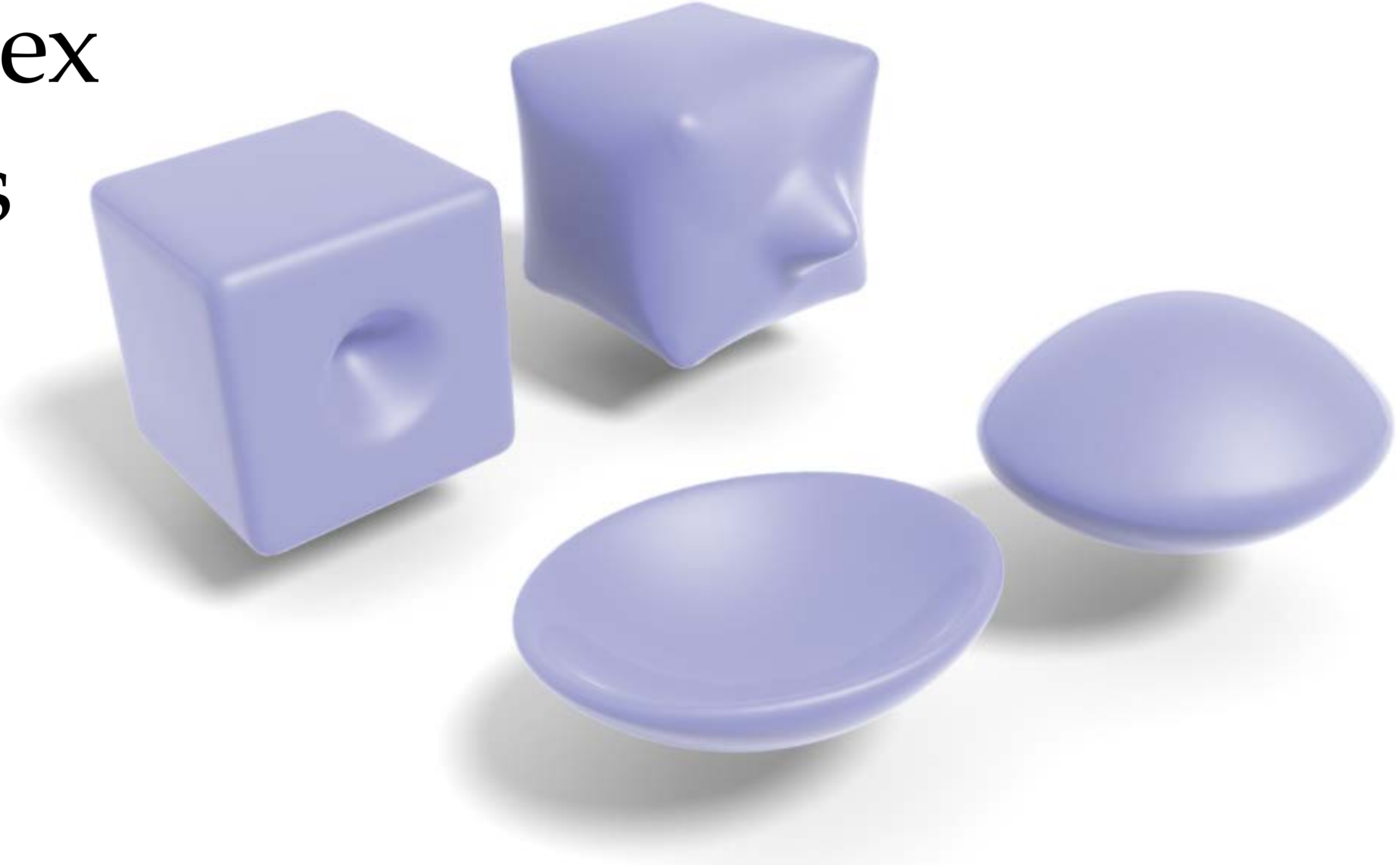
Mystery: Why don't we need additional data to recover *simplicial* surfaces? (Even convex ones…)



Recovery from Metric

- What data *is* sufficient to describe a surface?
- **Theorem.** (Cohn-Vossen) Smooth convex surface is uniquely determined (up to rigid motions) by its *Riemannian metric*.
- **Theorem.** (Alexandrov-Connelly) A convex polyhedron is uniquely determined by its *edge lengths*.



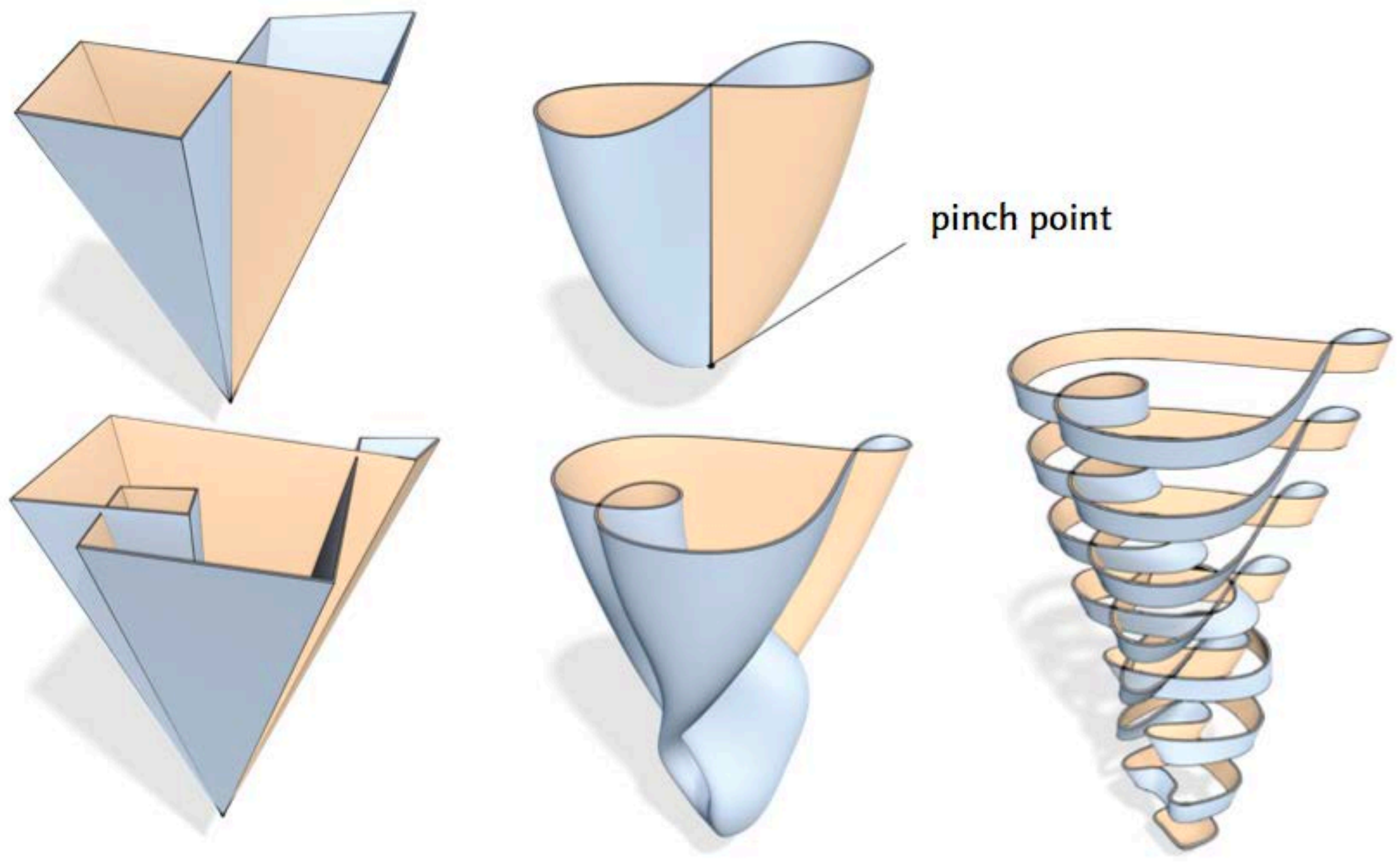


Note: not always true in nonconvex case!

Recovery of Nonconvex Shapes from Metric?

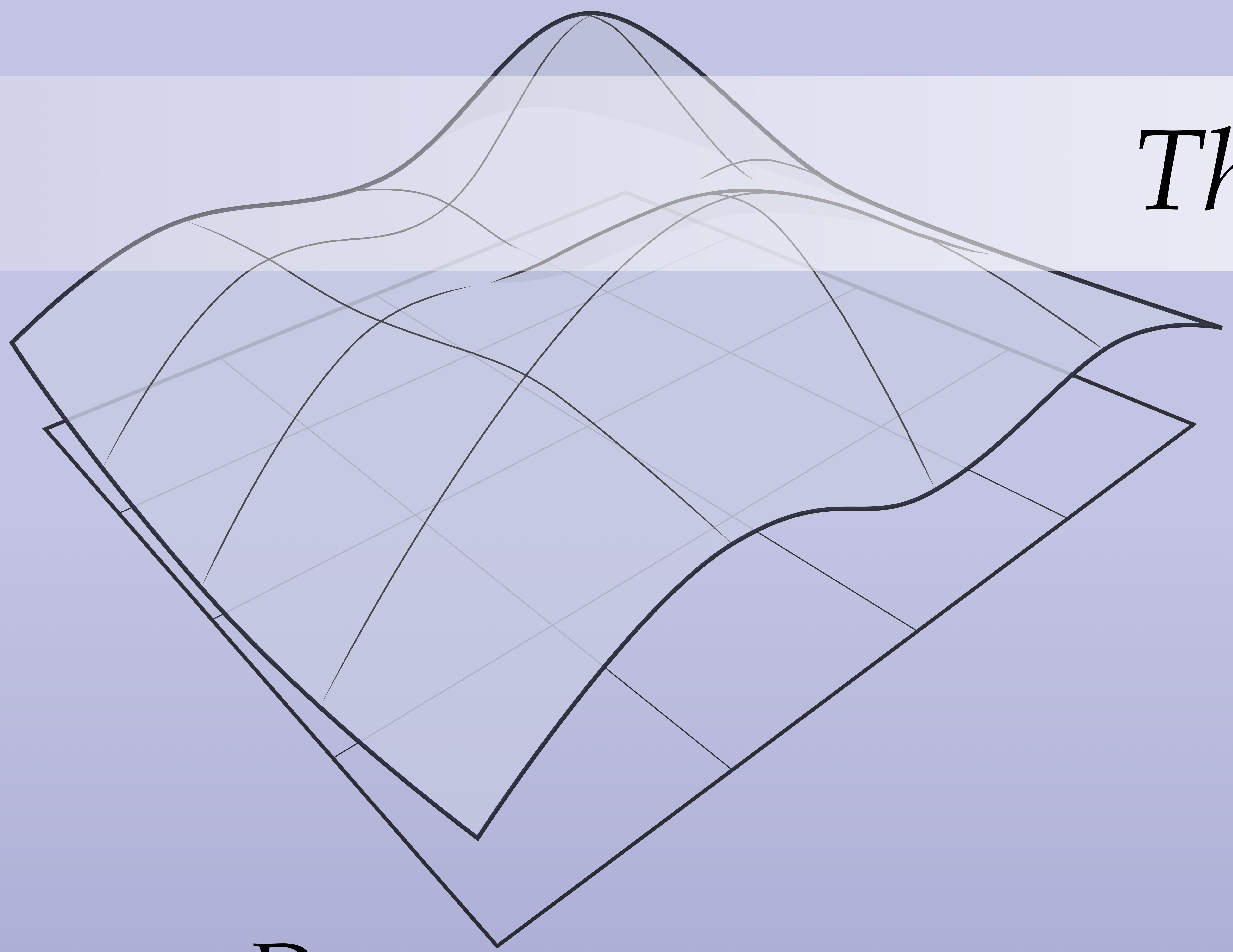
Algorithm: Shape from Metric

- Recent algorithm (*approximately! usually!*) recovers mesh from lengths
  - Chern et al, "Shape from Metric" (2018)
  - Nice read if you want to get deeper into discrete surfaces: discrete immersion, discrete spin structure…



http://page.math.tu-berlin.de/~chern/projects/ShapeFromMetric/

Thanks!



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