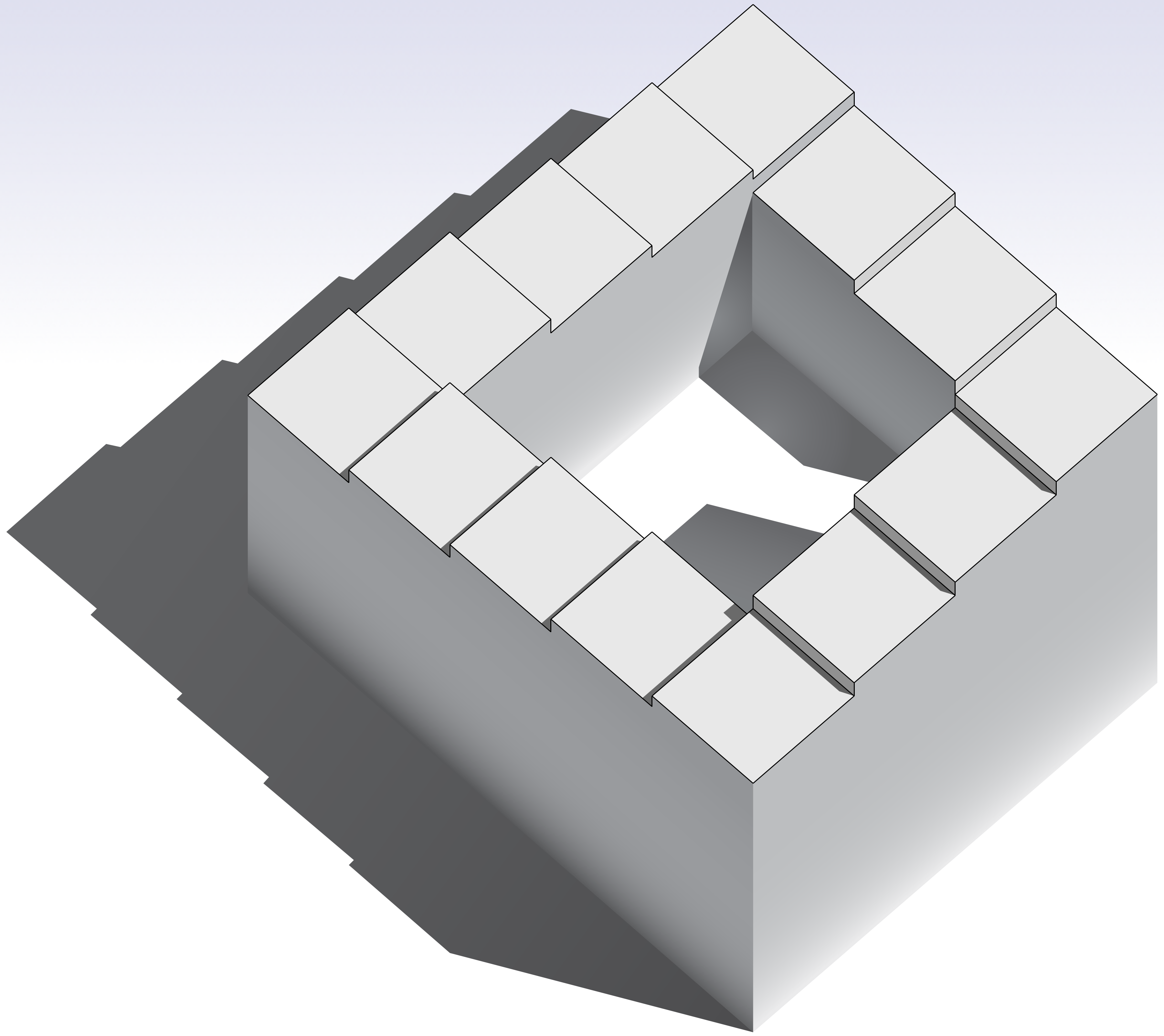
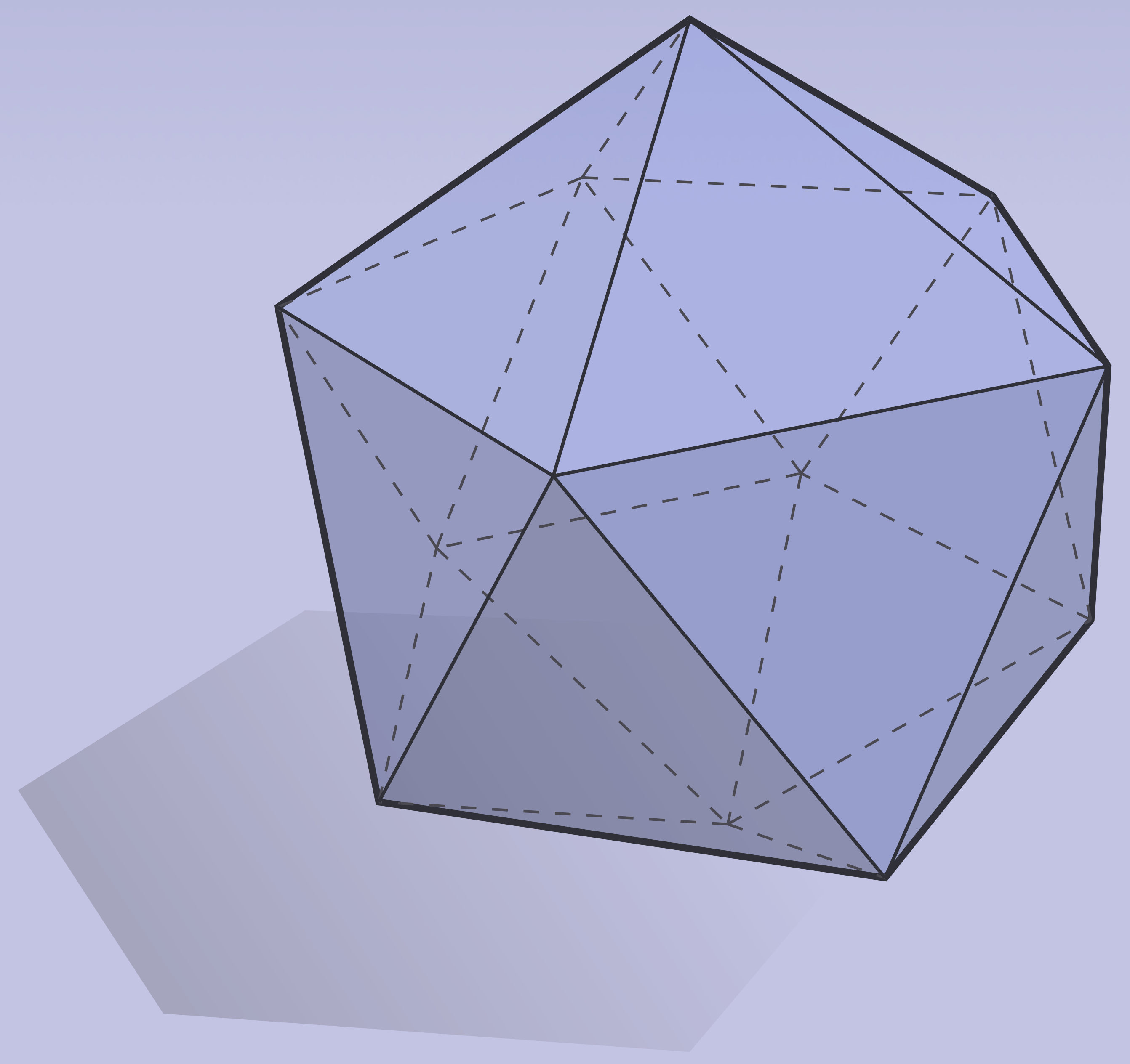


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 20:
GEODESICS



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

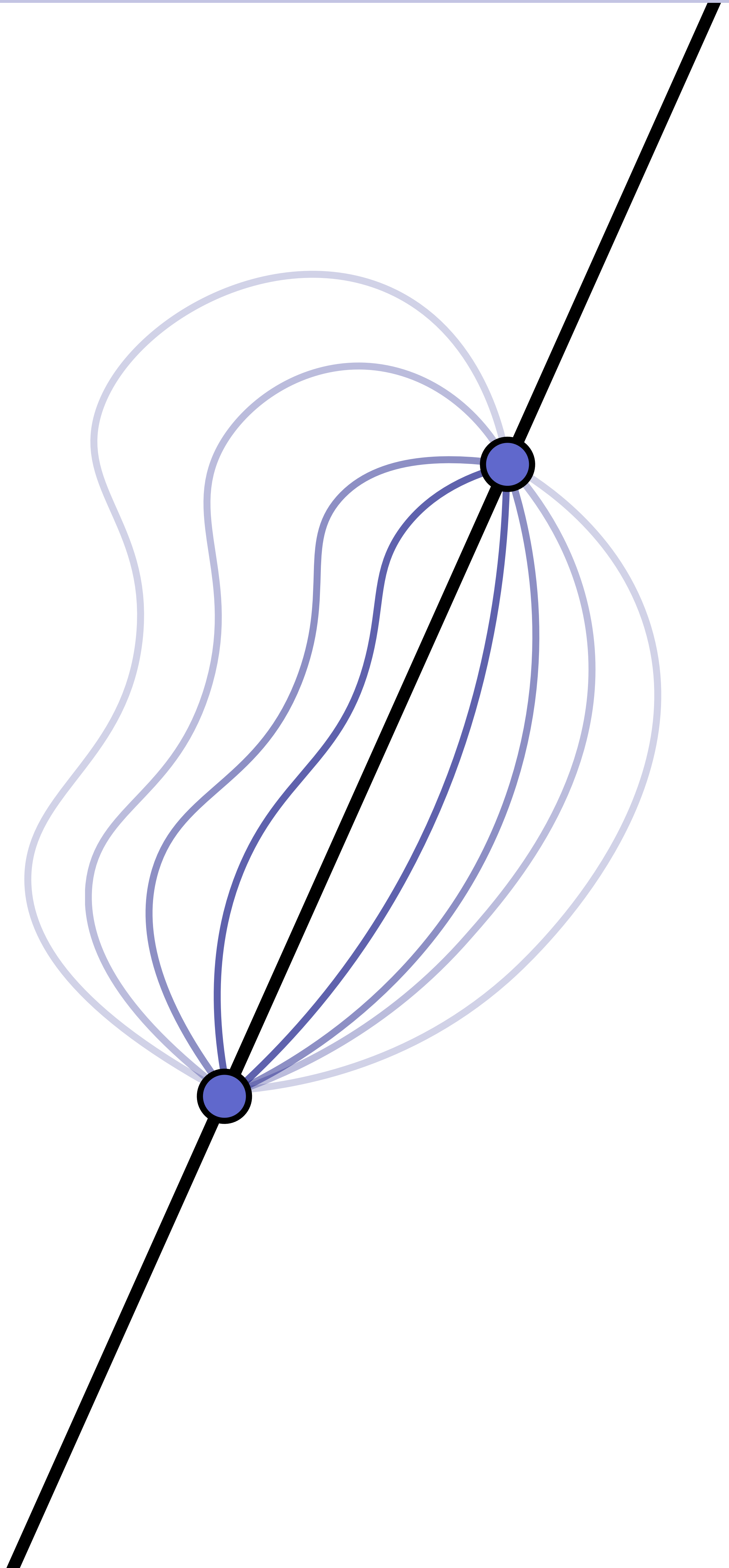
CMU 15-458/858 • Keenan Crane

Geodesics — Overview

- Generalize the notion of *lines* to curved spaces
- Ordinary lines have two basic features:

1. **straightest** — no curvature / acceleration
2. **shortest** — (locally) minimize length

- Geodesics share these same *local* properties, but may exhibit different behavior *globally*
- Part of the "origin story" of both classical and differential geometry…



Key idea: geodesic is straightest, (locally) shortest curve

Euclidean Geometry

Euclid (c. 300BC) used five basic “postulates” as a starting point for geometry:

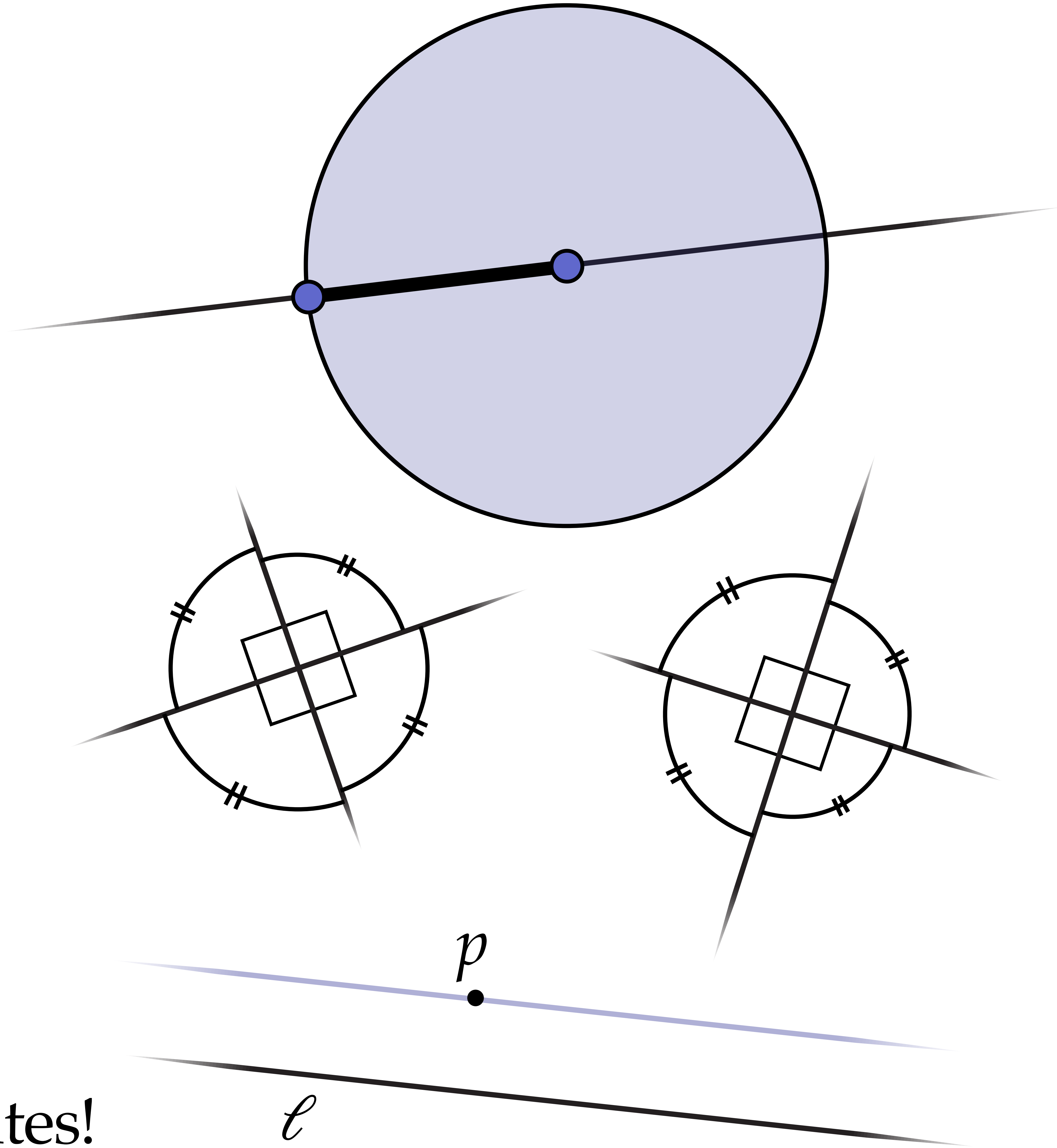
I. Any two points can be connected by a straight line segment

II. Any line segment can be extended into a line

III. For any segment, there’s a circle centered at one endpoint, with the segment as a radius

IV. All right angles are congruent

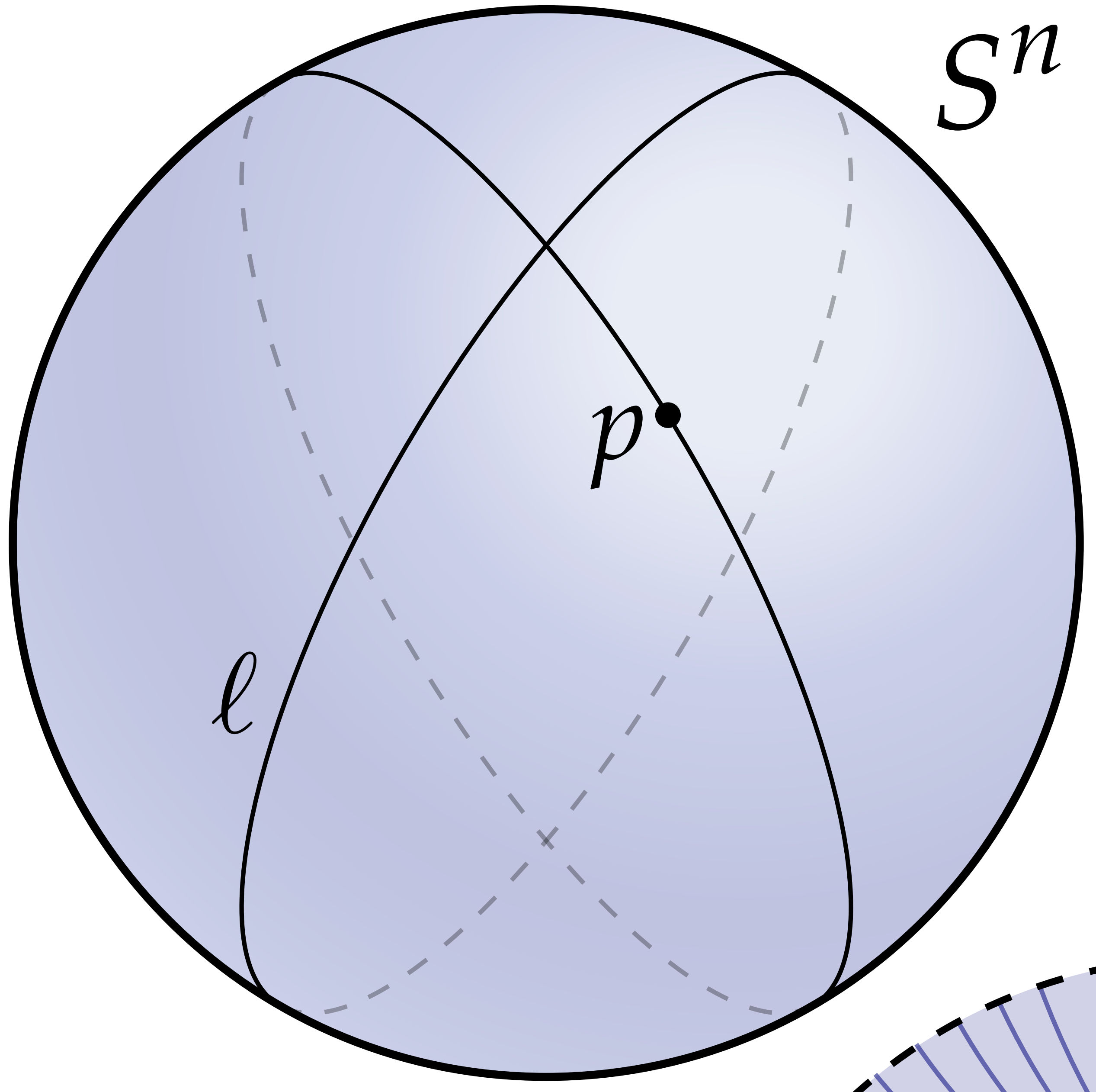
V. For any line ℓ and point p not on ℓ, there’s a unique line parallel to ℓ passing through p

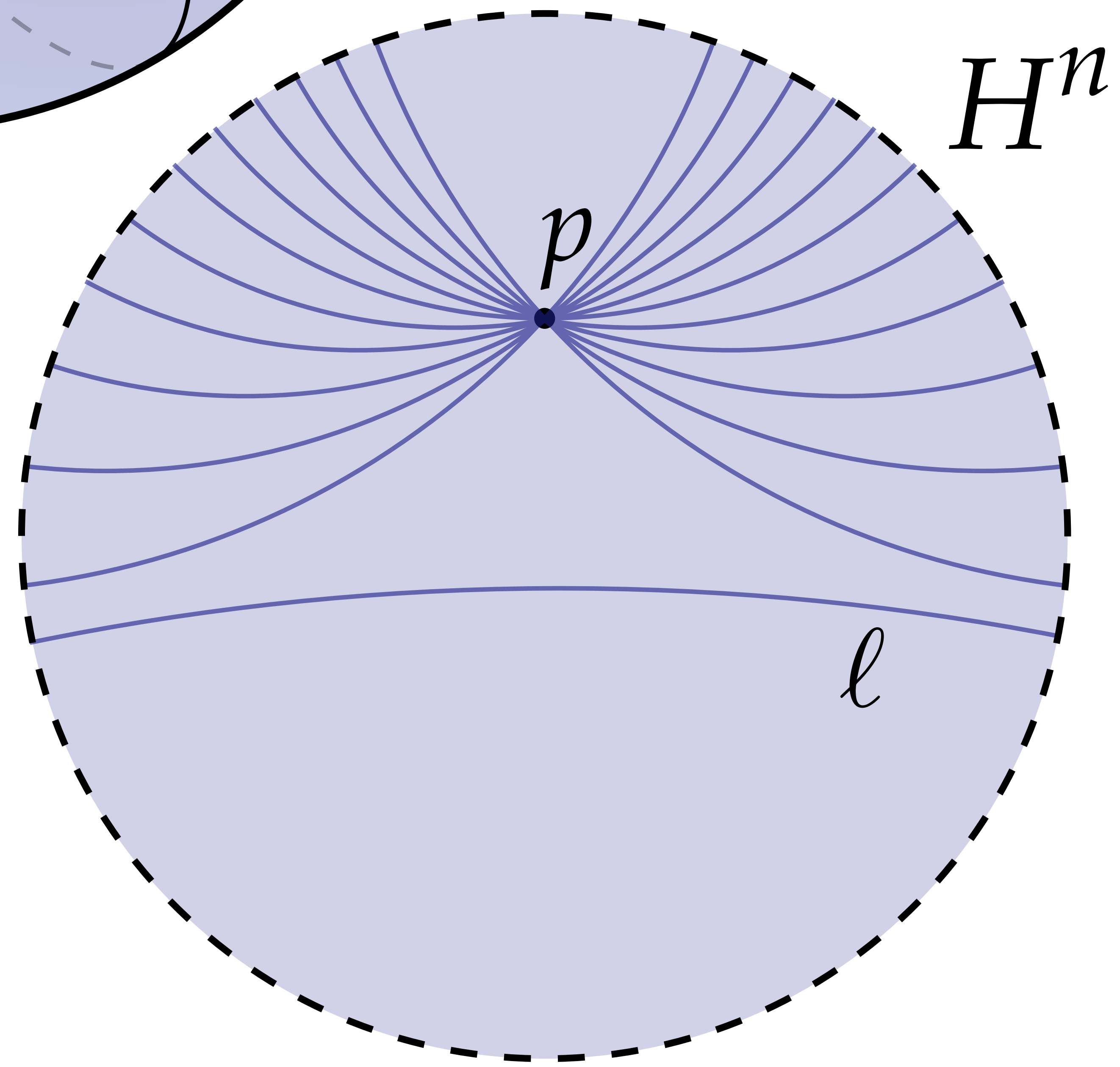


Idea: *everything* else can be proved from these postulates!

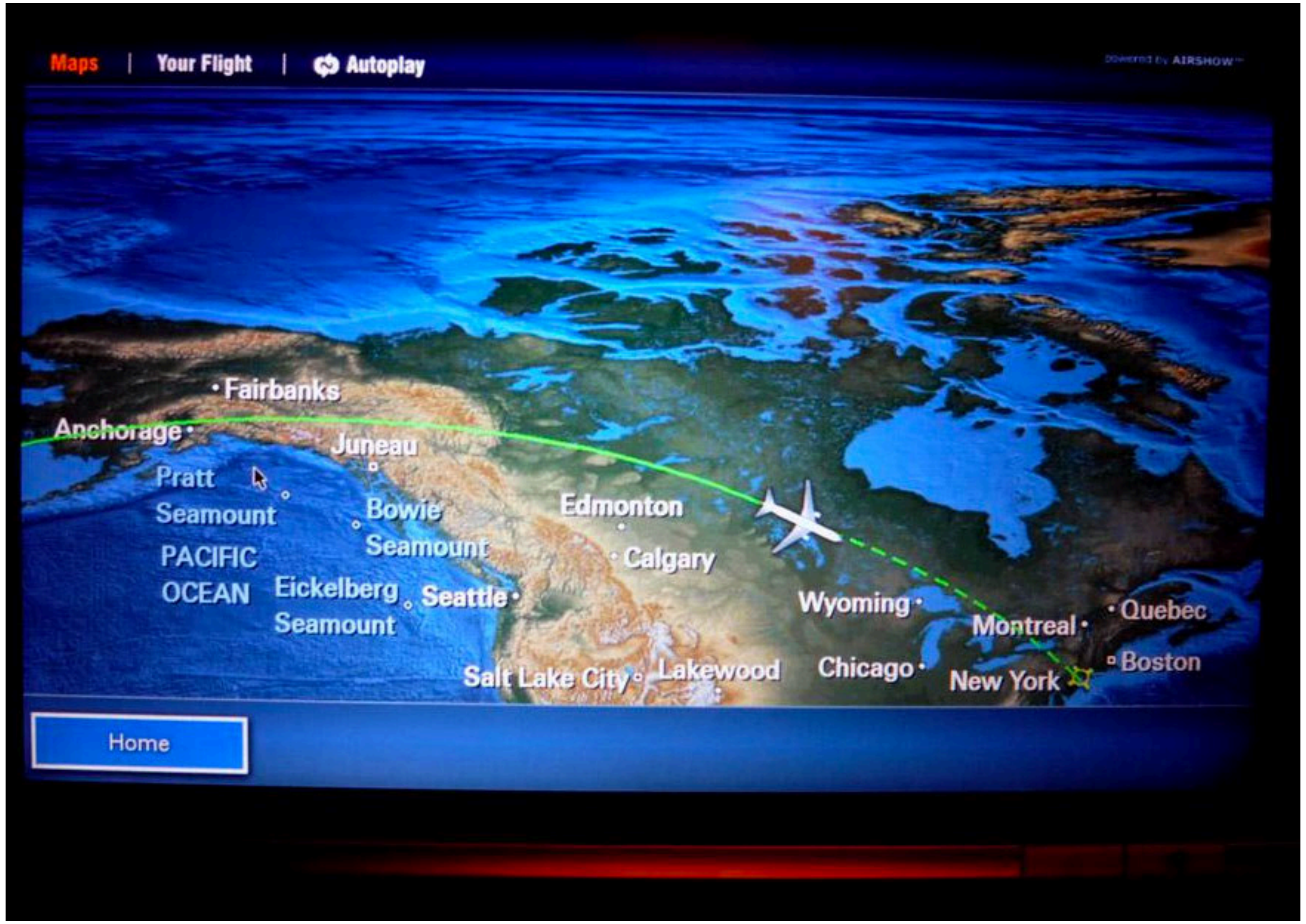
Non-Euclidean Geometry

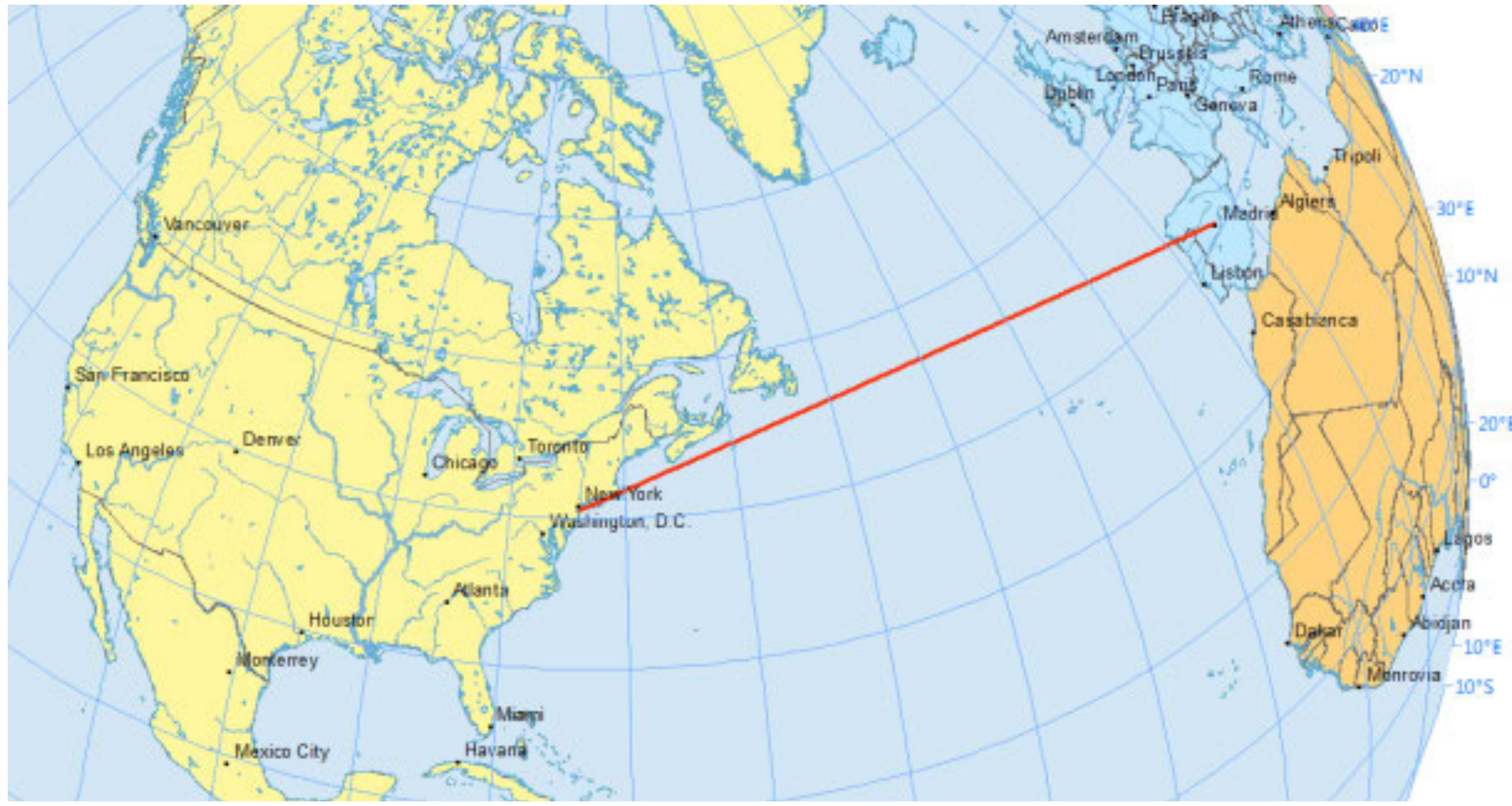
- *Many* attempts to prove parallel postulate from first four. After two thousand years…
- (Lobachevsky, Bolyai, Gauss, …) Not possible! There are other logically consistent geometries where parallel postulate doesn't hold:
  - **Elliptic:** no parallel lines through a point—all lines intersect
  - **Hyperbolic:** parallel line through point is not unique
- More generally: "lines" or *geodesics* on curved surfaces will behave differently than in the plane
  - Will try to understand this behavior today…



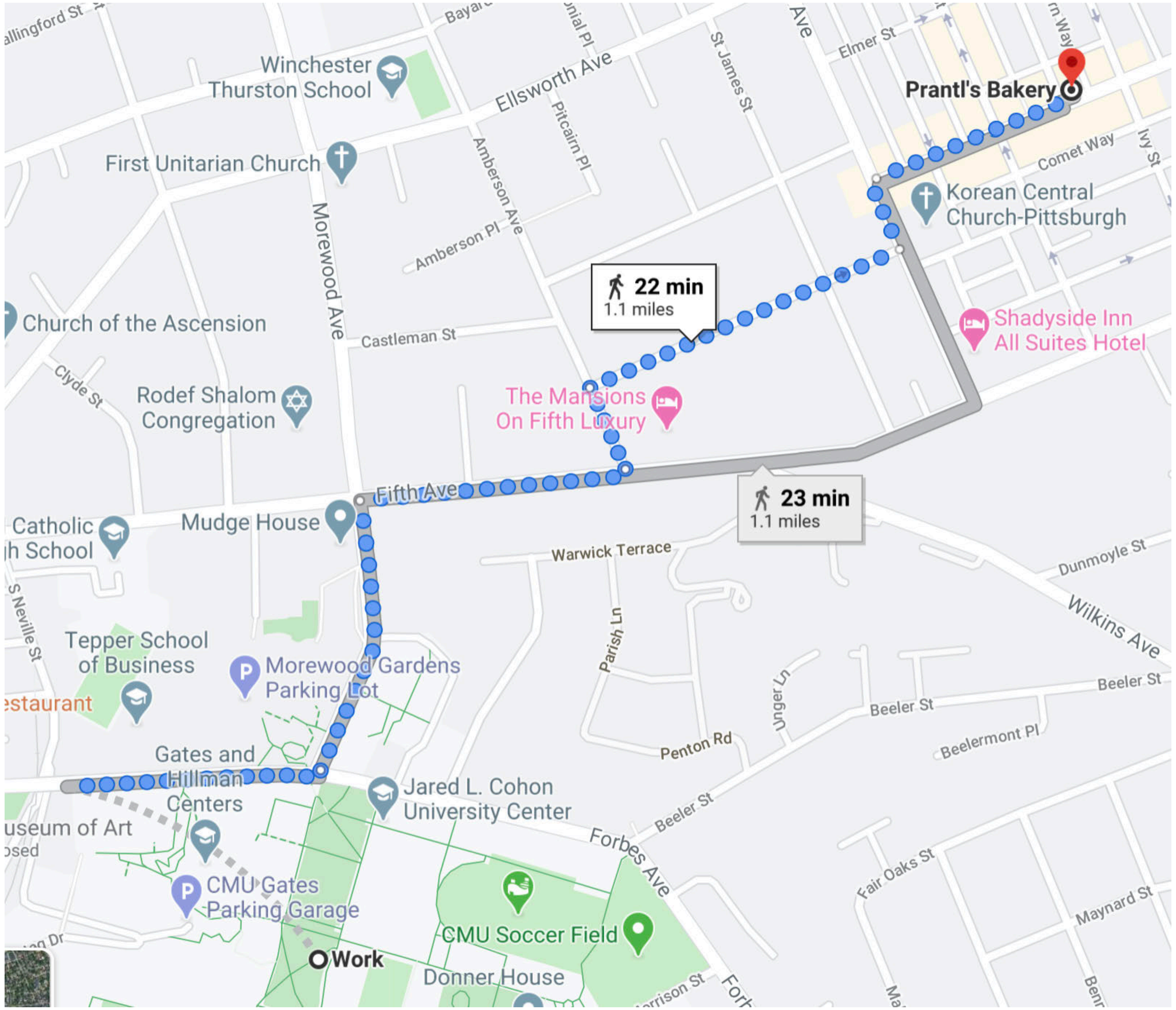


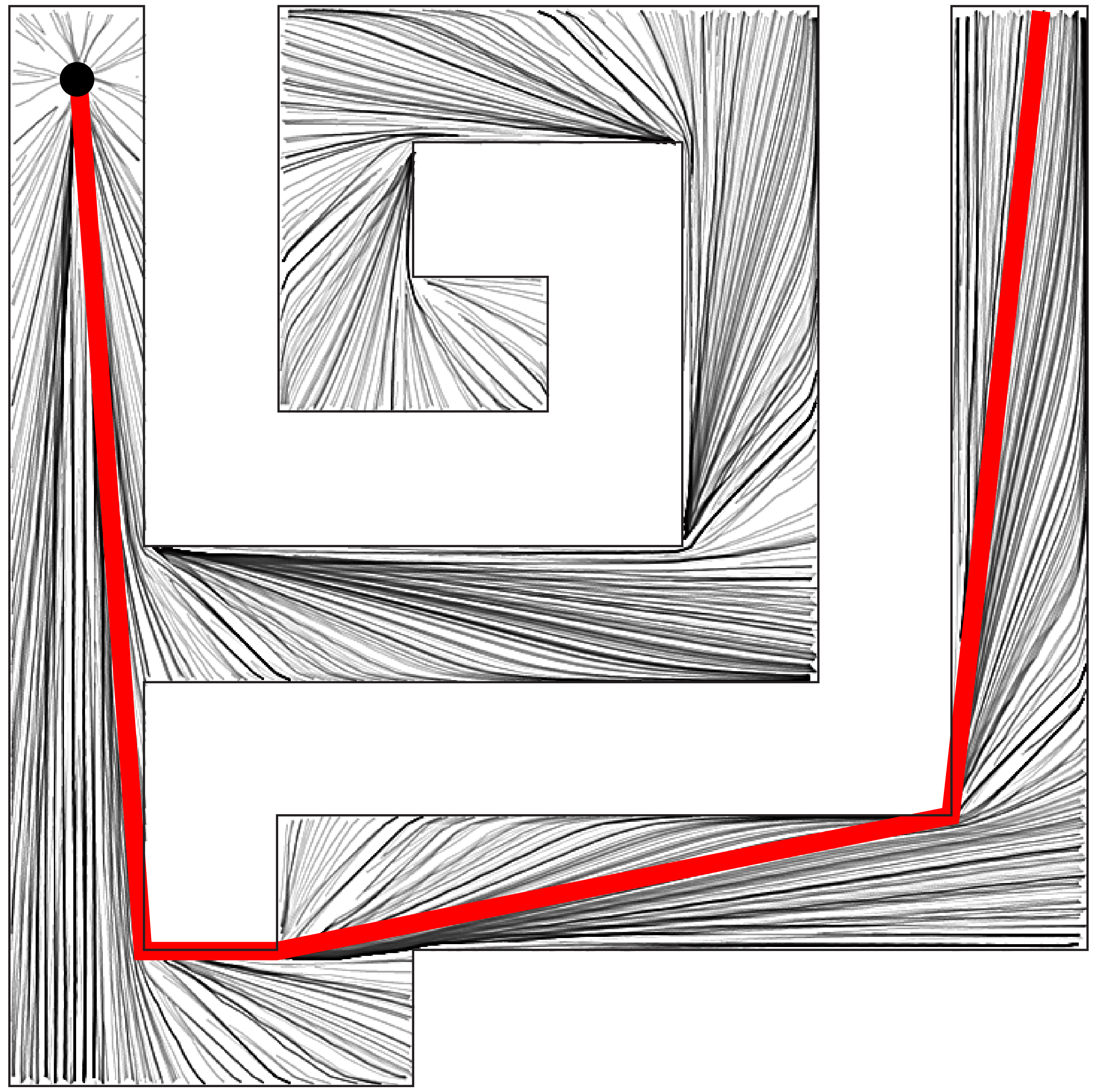
Examples of Geodesics — Great Arcs on the Sphere

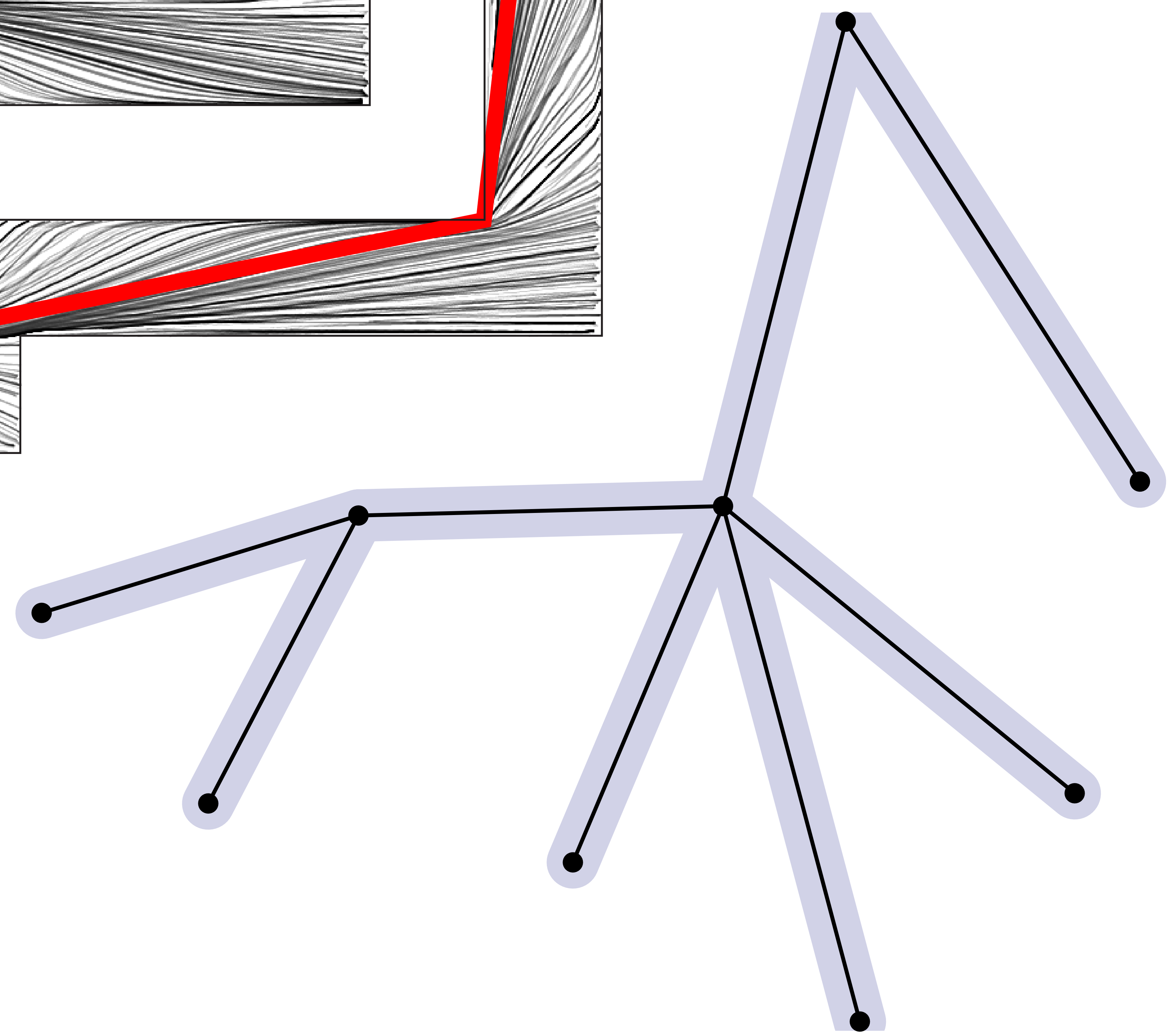




Examples of Geodesics — Shortest Paths in Graphs

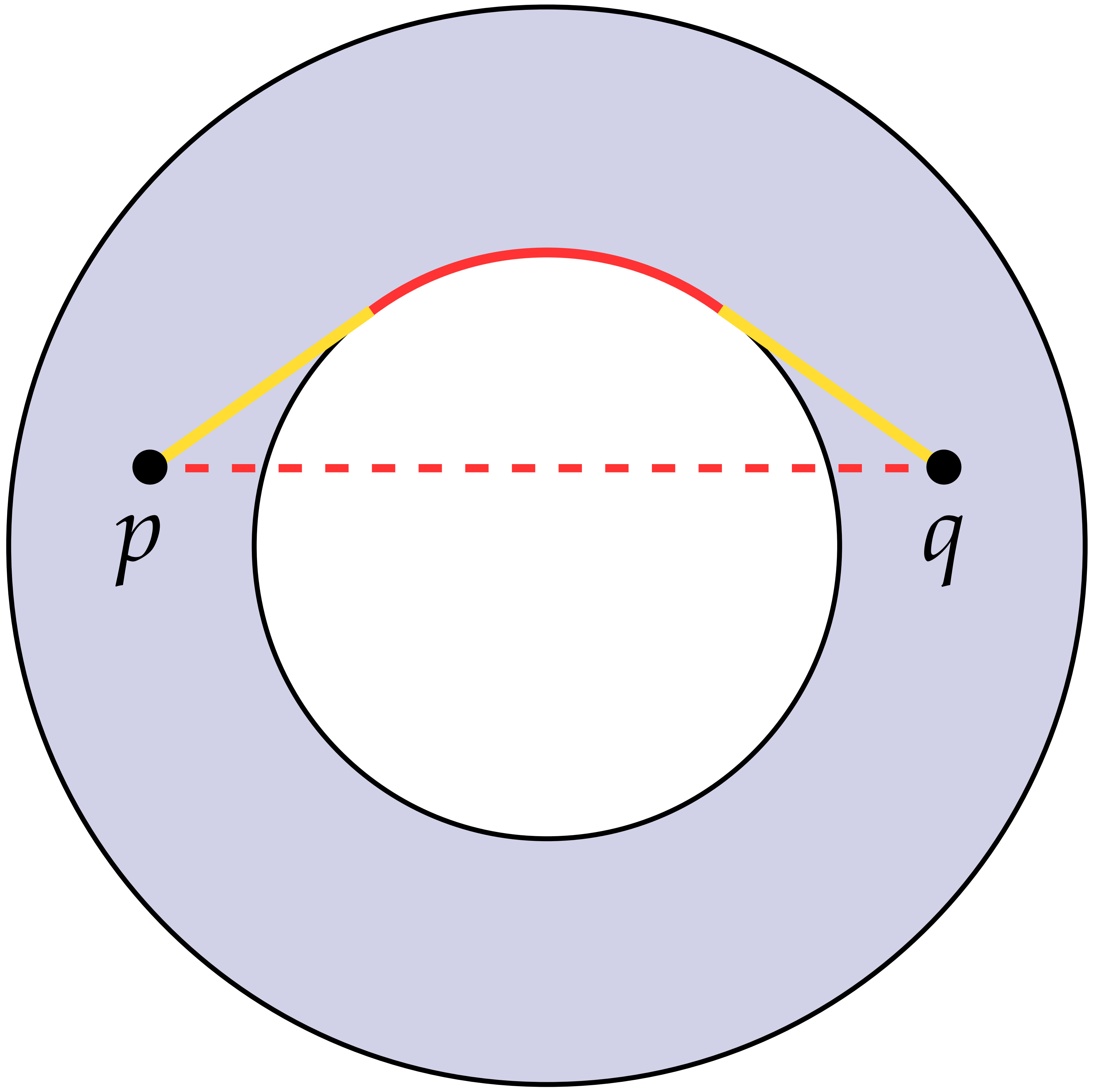




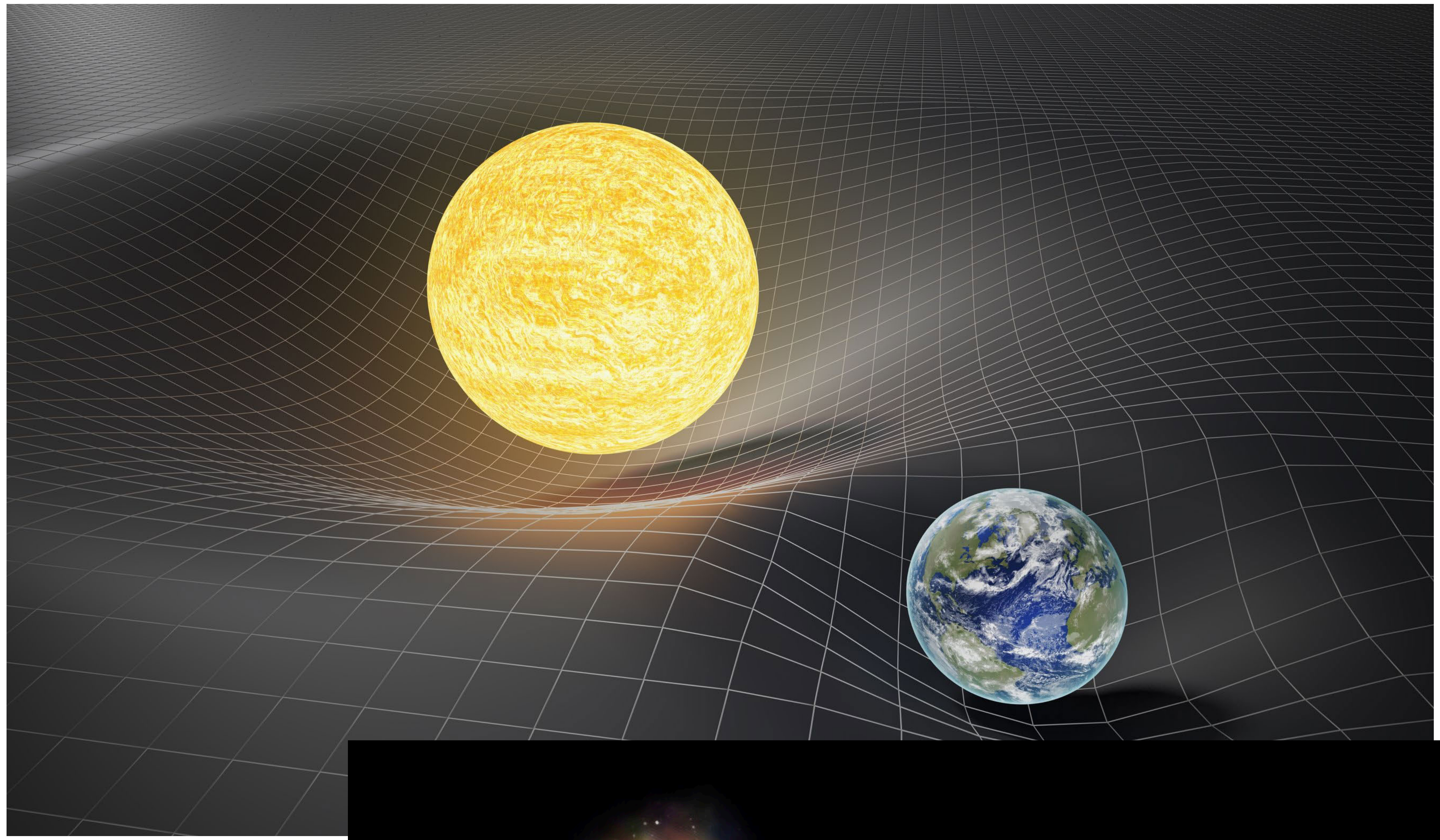


Aside: Geodesics on Domains with Boundary

- On domains with boundary, *shortest* path will not always be *straight*
  - can also "hug" pieces of the boundary (curvature will match boundary curvature, acceleration will match boundary normal)
  - on the interior, path will still be both shortest & straightest
- For simplicity, we will mainly consider domains without boundary



Examples of Geodesics — Paths of Light



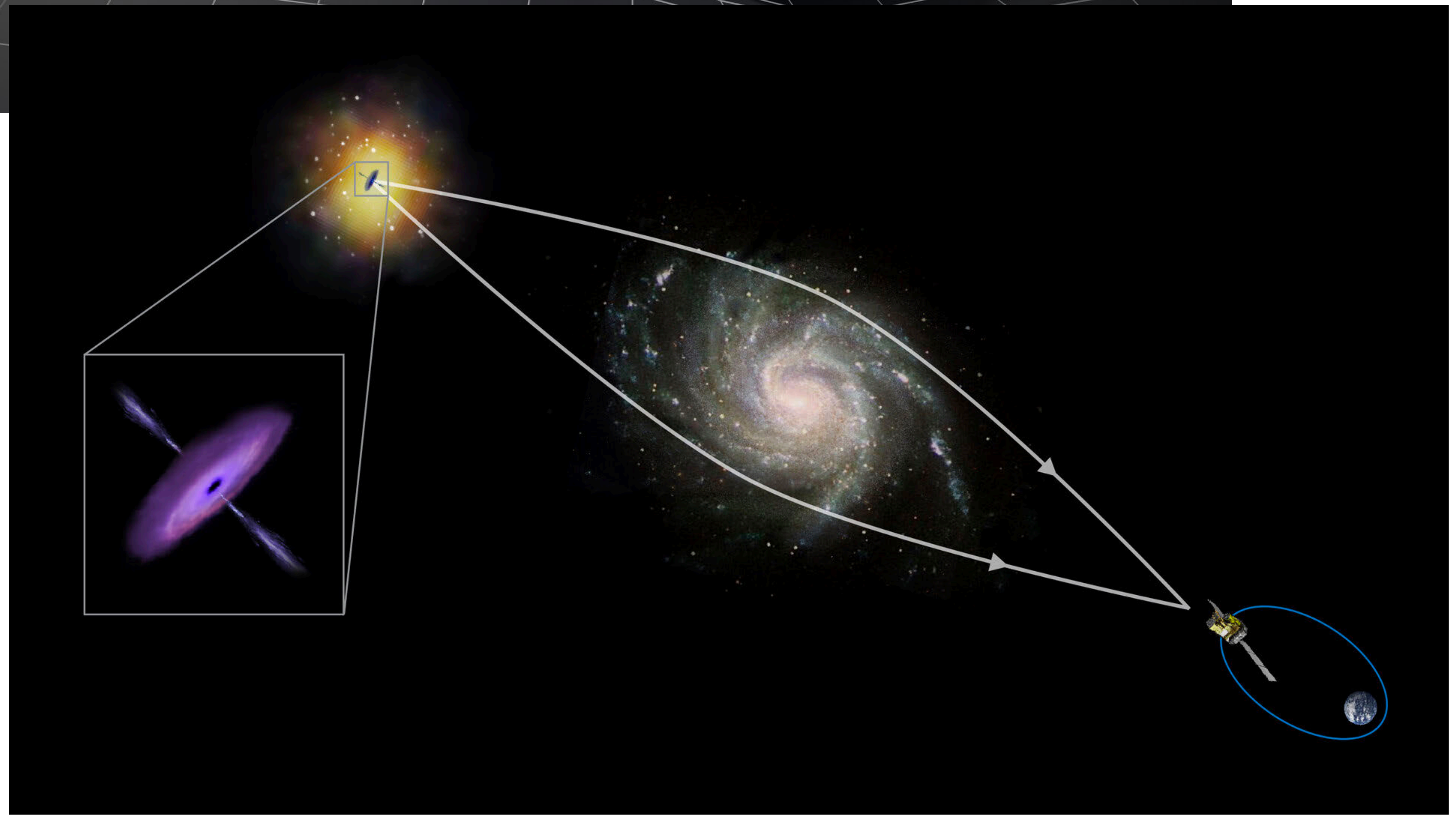


image credit: European Space Agency

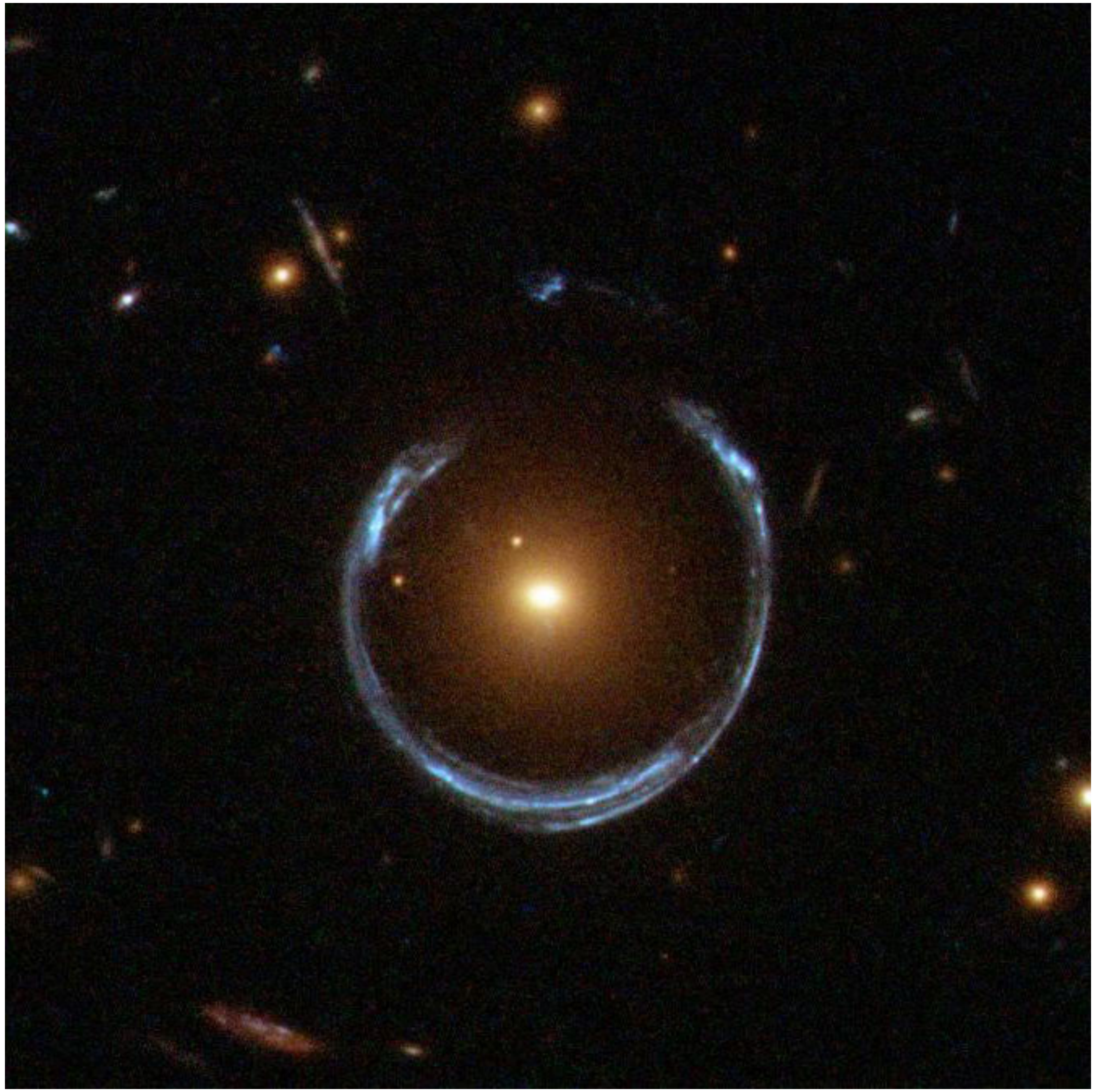
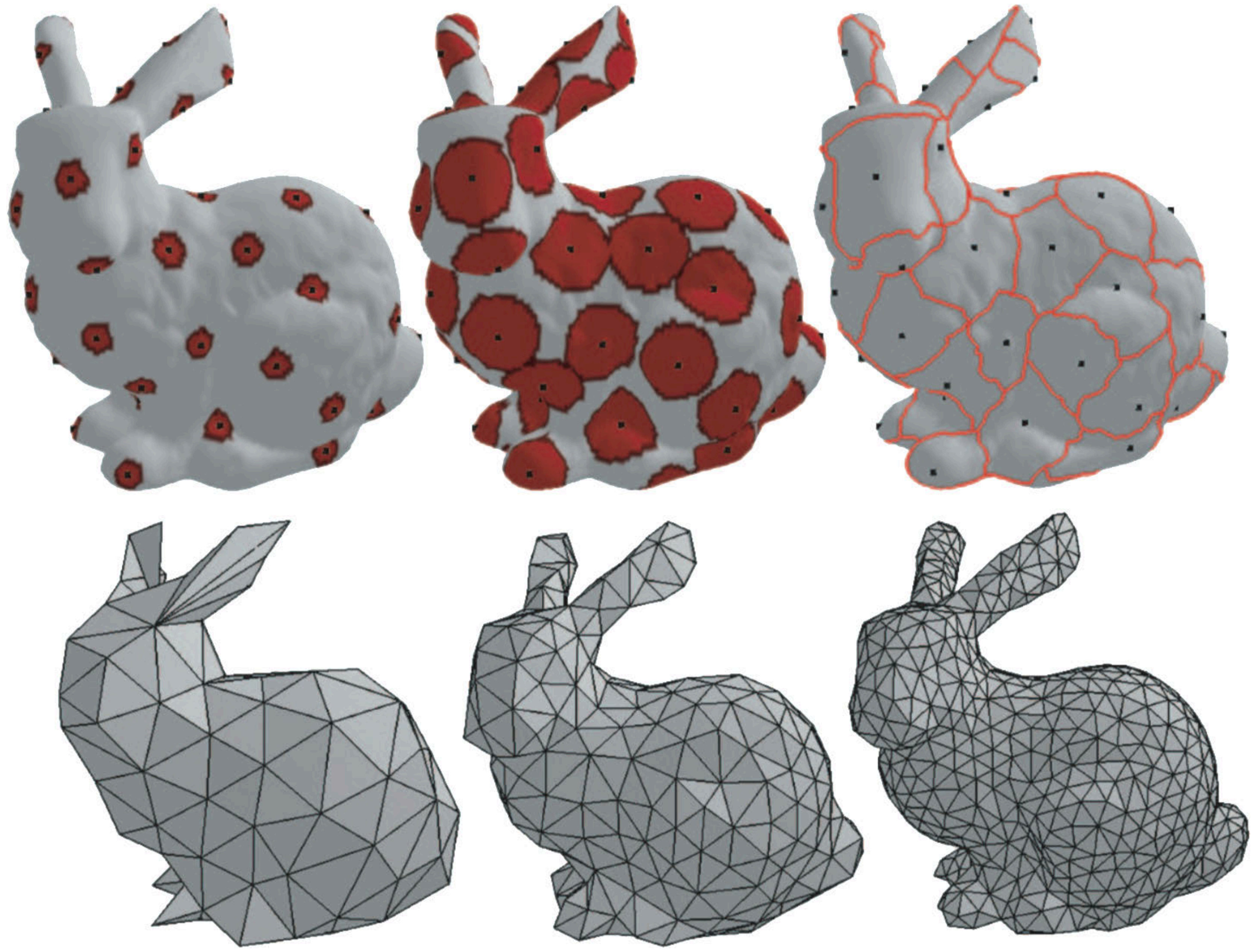


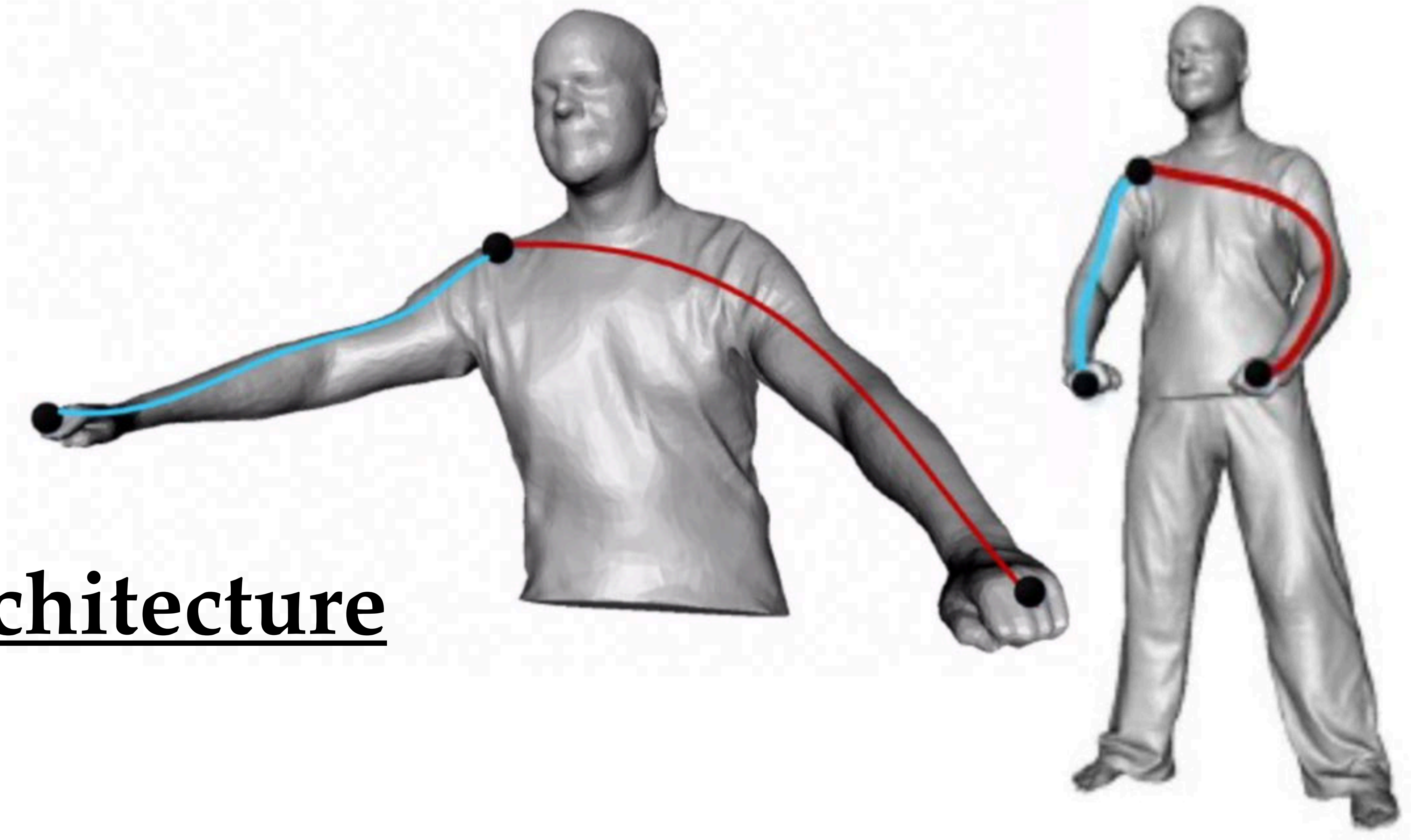
image credit: ESA/Hubble & NASA

Examples of Geodesics — Geometry Processing

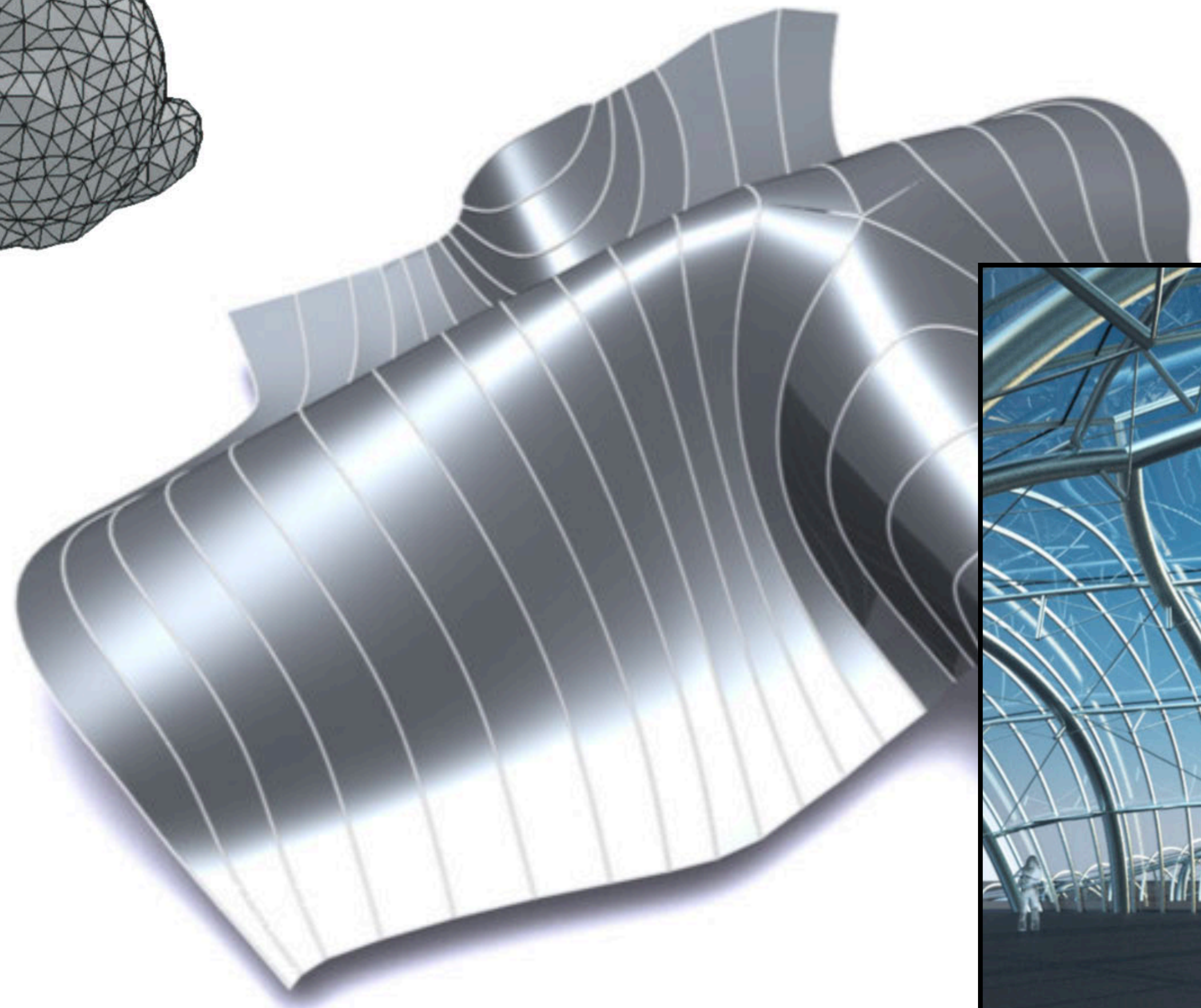
surface remeshing

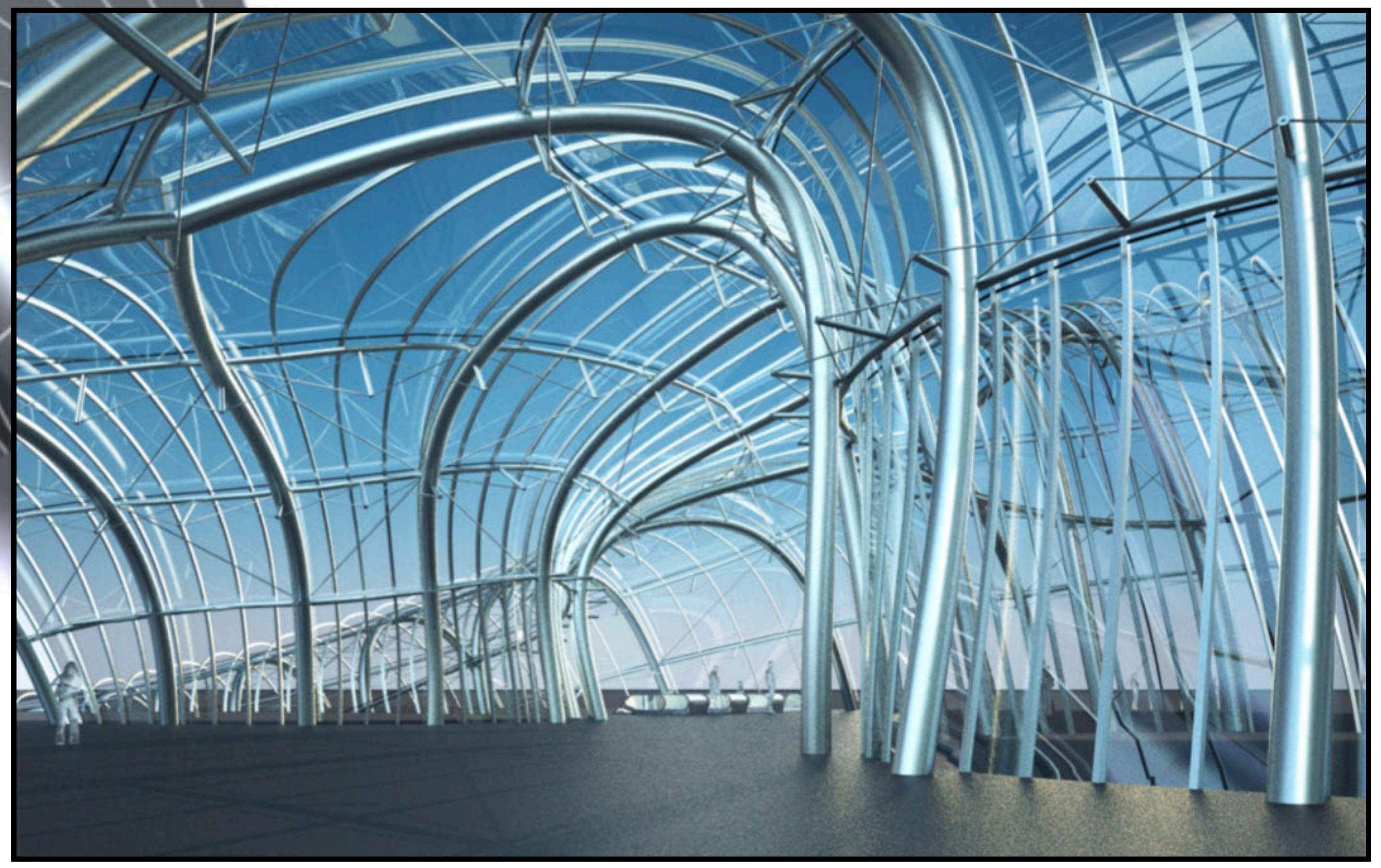


shape analysis / correspondence



computational architecture





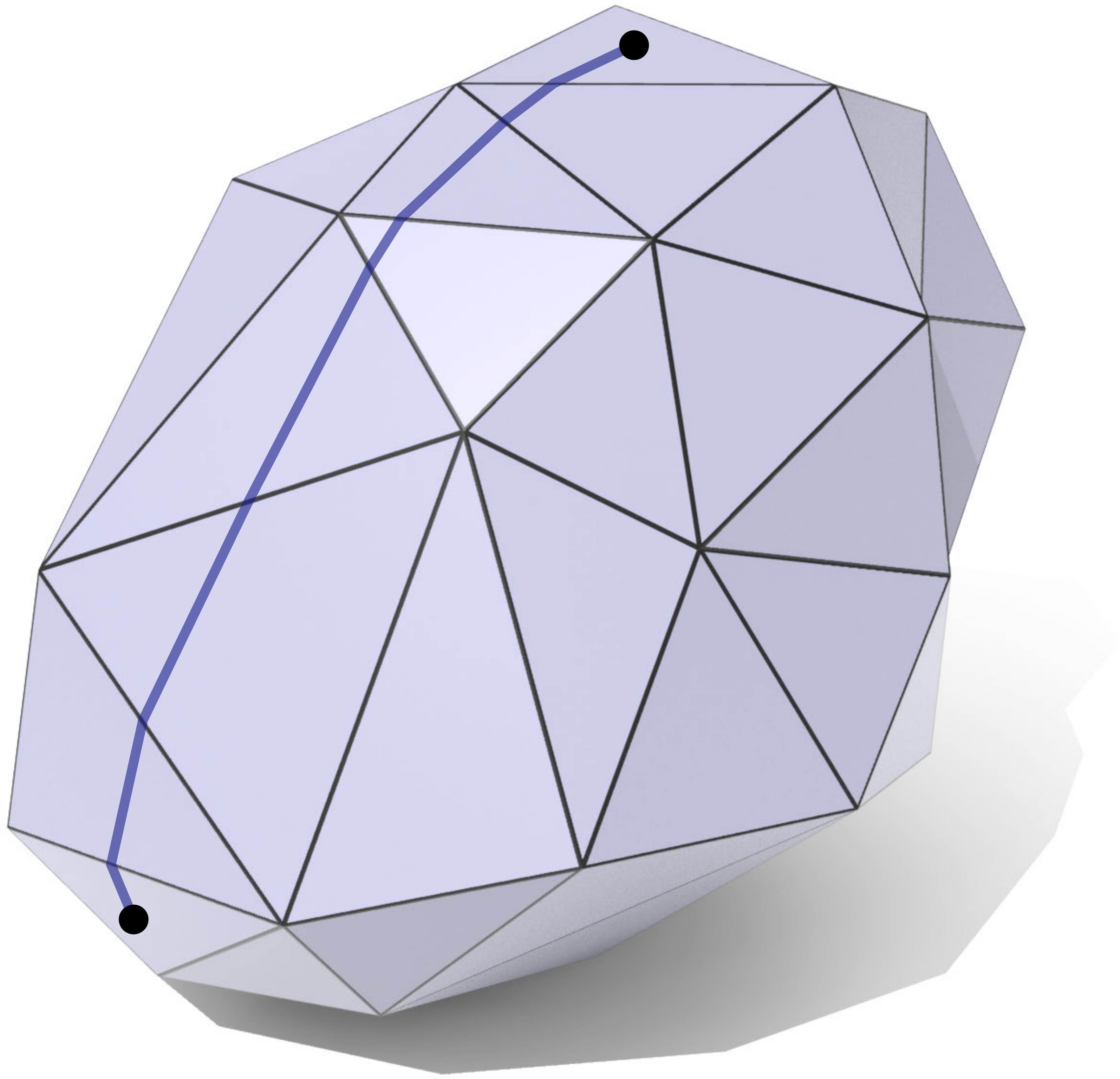
Isometry Invariance of Geodesics

- *Isometries* are special deformations that do not change the <u>intrinsic</u> geometry
  - Formally: preserves the *Riemannian metric* (which measures lengths & angles of tangent vectors)
- For instance, folding up a map doesn't change angle between north and south, or areas of land masses
- Likewise, the shortest path between two cities does not change if we roll up a map

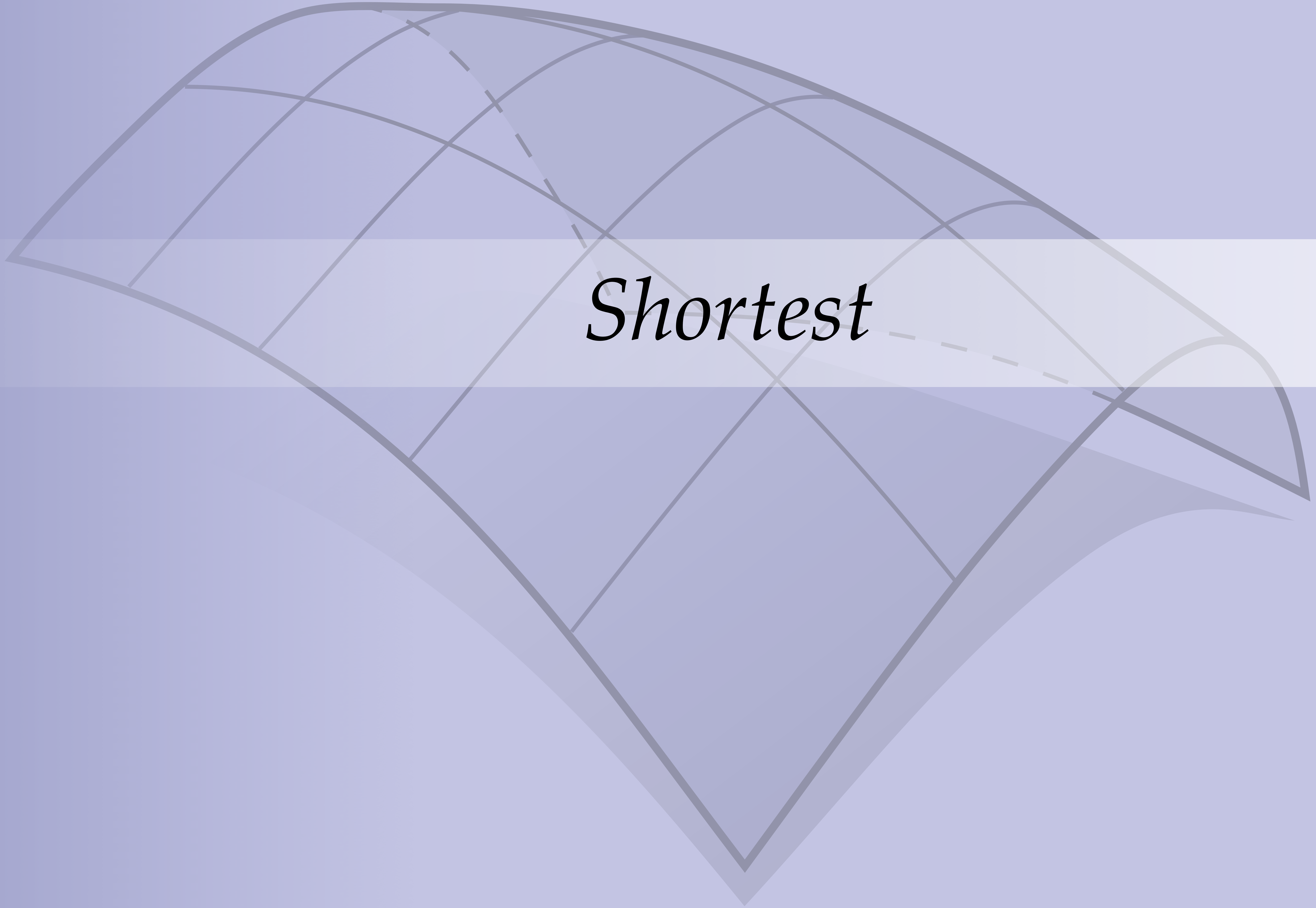
Key fact: geodesics are *isometry invariant.*

Discrete Geodesics

- How can we come up with a definition of *discrete* geodesics?
- Play "The Game" of DDG and consider different smooth starting points:
  - *straightest (zero acceleration)*
  - *locally shortest*
  - *no geodesic curvature*
  - *harmonic map from interval to manifold*
  - *gradient of distance function*
  - …
- Each starting point will have different consequences



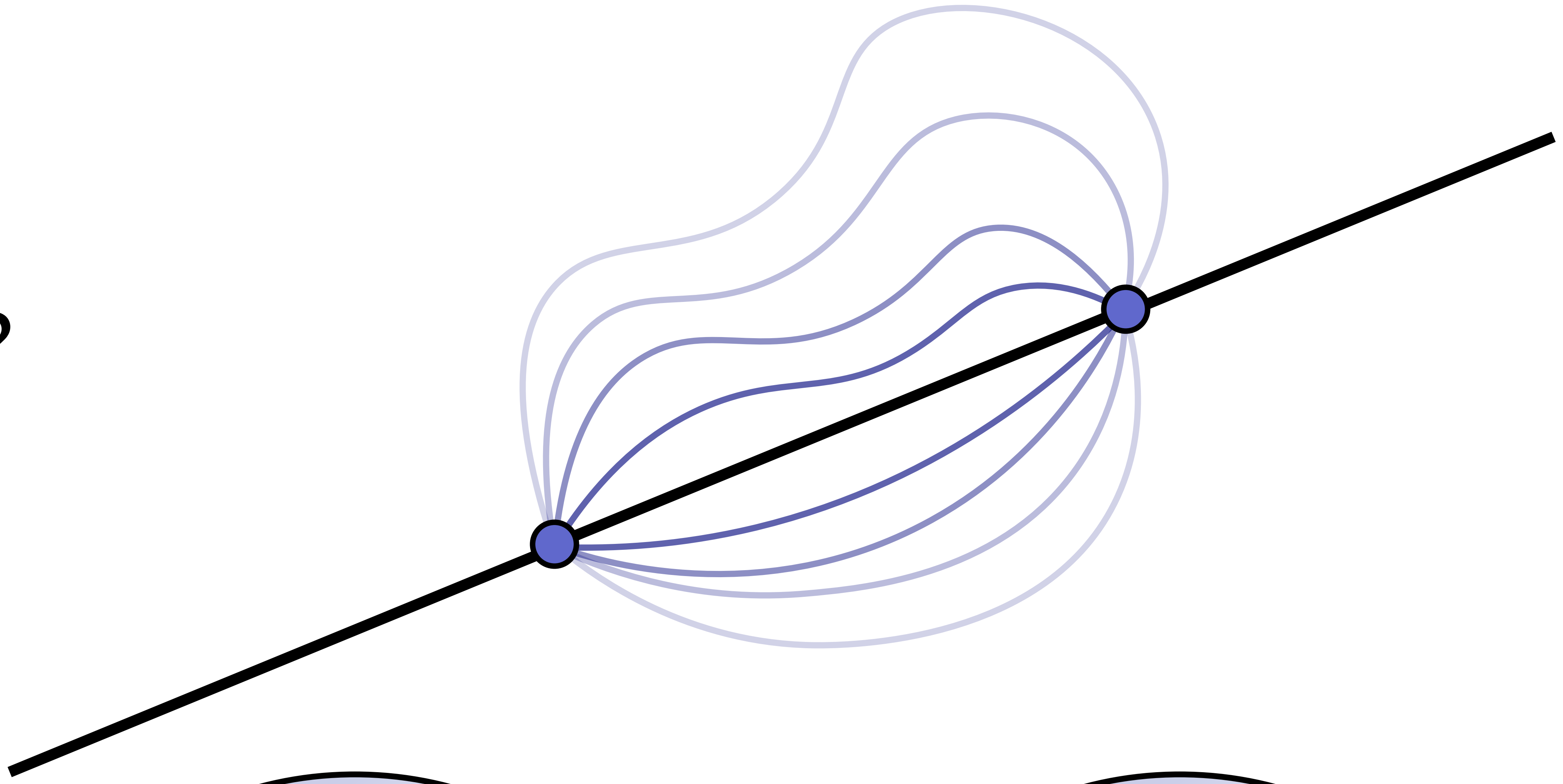
Observation: for simplicial surfaces will see that **<u>shortest</u>** and **<u>straightest</u>** disagree

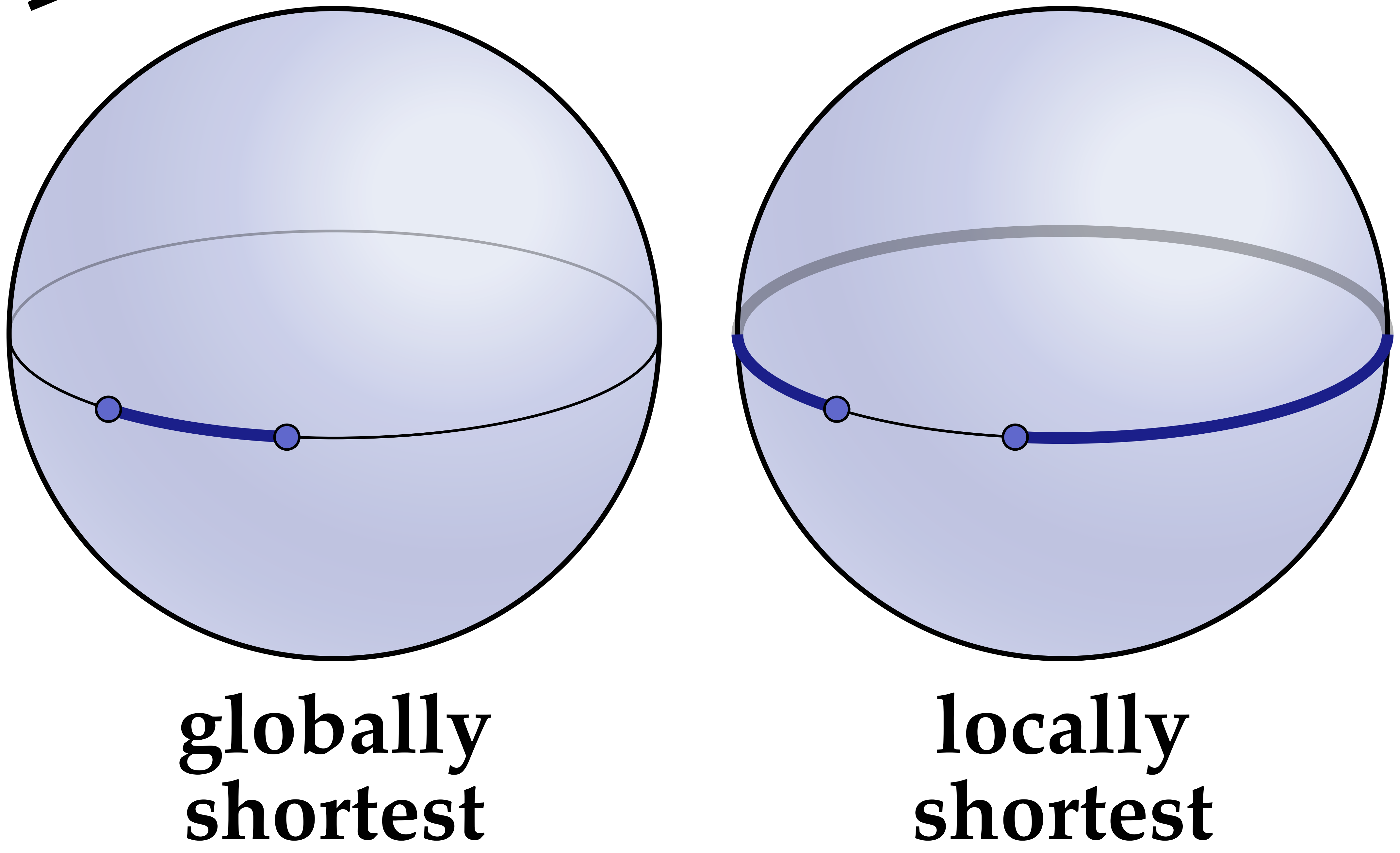


Shortest

Locally Shortest Paths

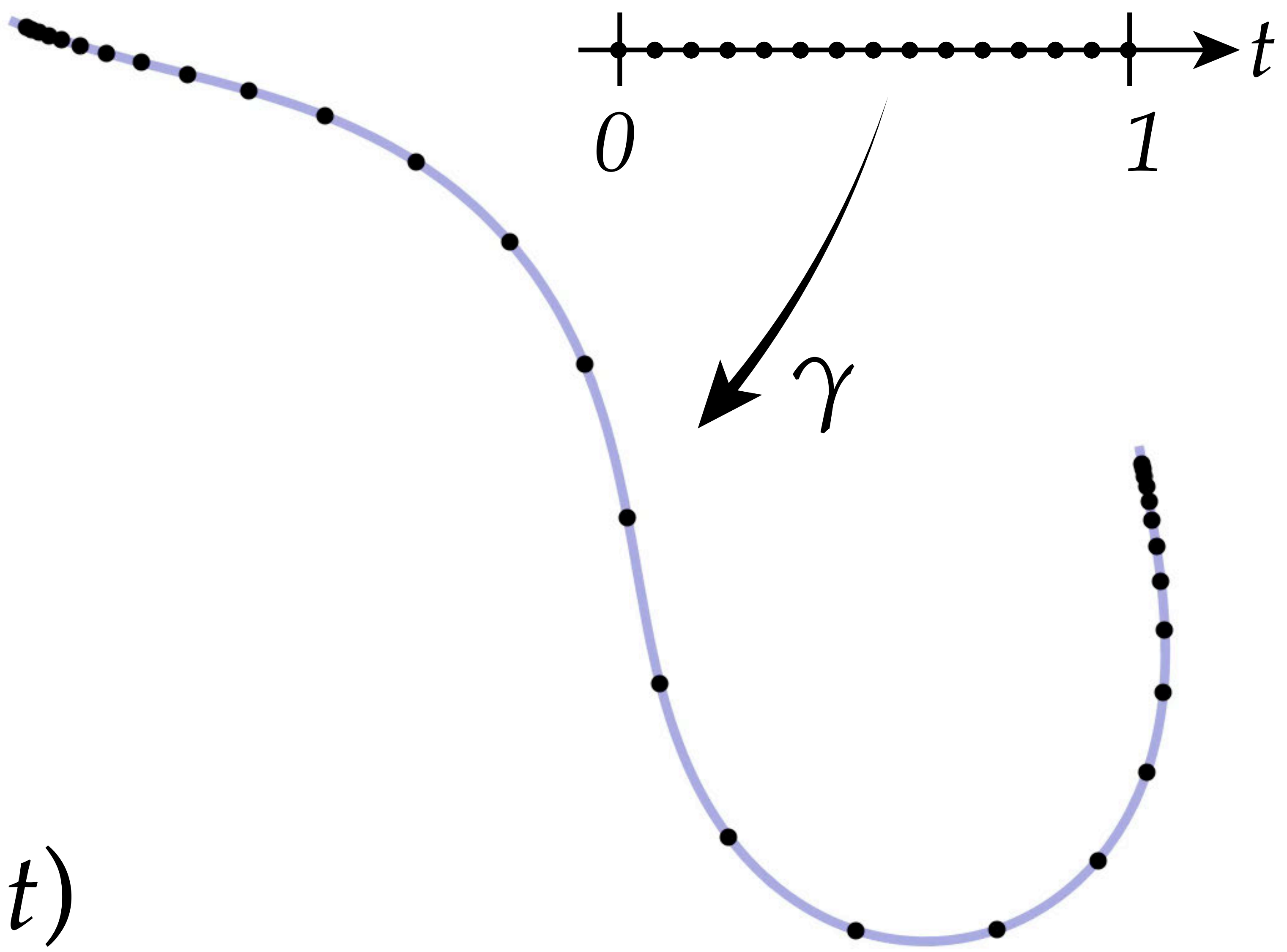
- A Euclidean line segment can be characterized as the shortest path between two distinct points
- How can we characterize a whole Euclidean line?
  - …where are the endpoints?
- Say that it's *locally shortest*: for any two "nearby" points on the path, can't find a shorter route
  - "nearby" means shortest path is **unique**
- This description directly gives us one possible definition for geodesics
- Note that *locally* shortest does not imply *globally* shortest!
  - <u>Both</u> are geodesic paths





Dirichlet Energy and Curve Length

Recall *Dirichlet energy*, which measures “smoothness”:



planar curve

$$\gamma : [0,1] \to \mathbb{R}^2$$

Dirichlet energy

$$E_D(\gamma) = \int_0^1 |\gamma'(t)|^2 \, dt$$

Can write γ as a reparameterization of a unit-speed curve:

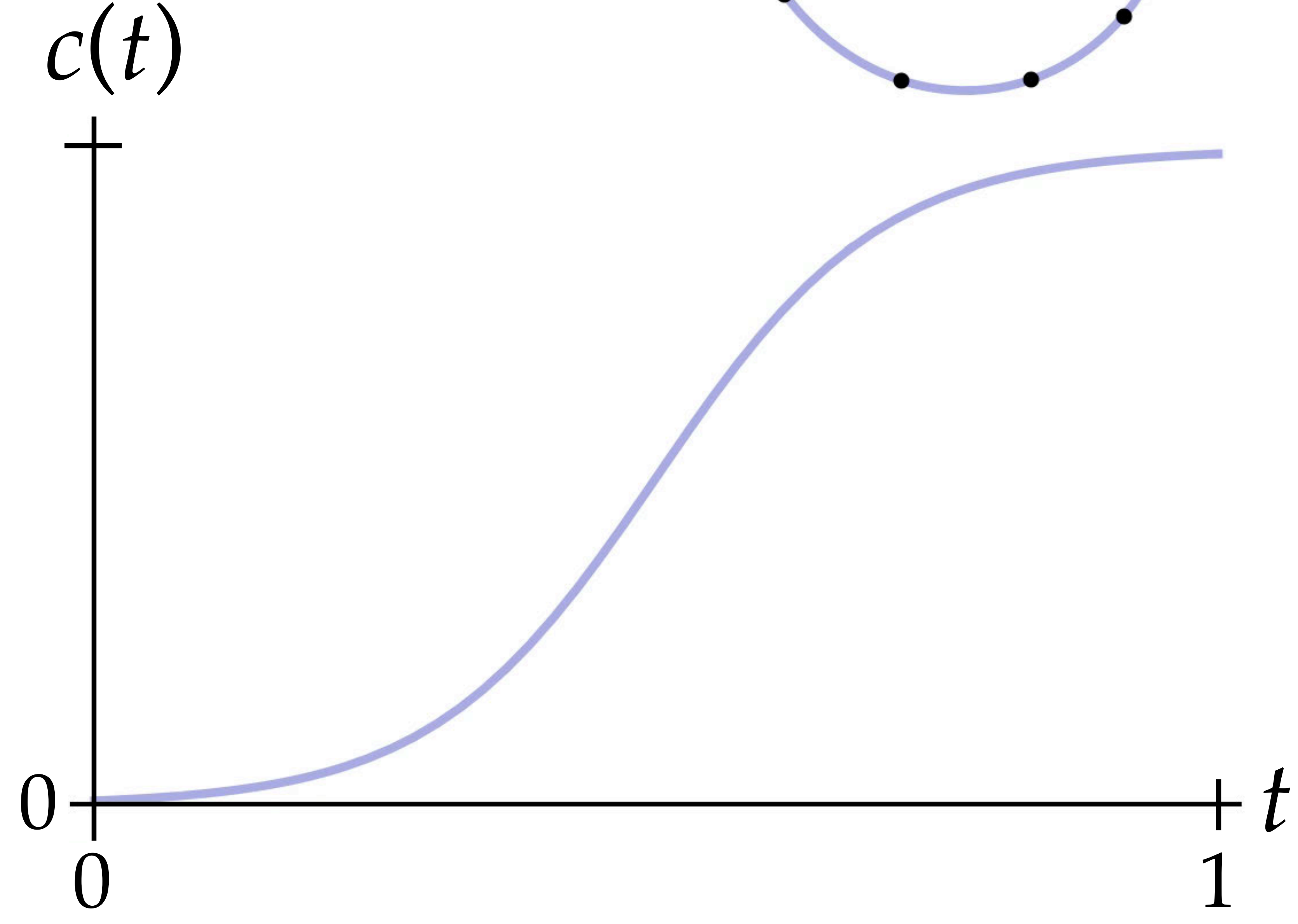
unit-speed curve

$$\widehat{\gamma} : [0, L] \to \mathbb{R}^2$$

speed function

$$c : [0,1] \to \mathbb{R}$$
$$c(0) = 0, \; c(1) = L$$

$$\boxed{\gamma(t) = \widehat{\gamma}(c(t))}$$
$$|\gamma'(t)| = |c'(t)|$$



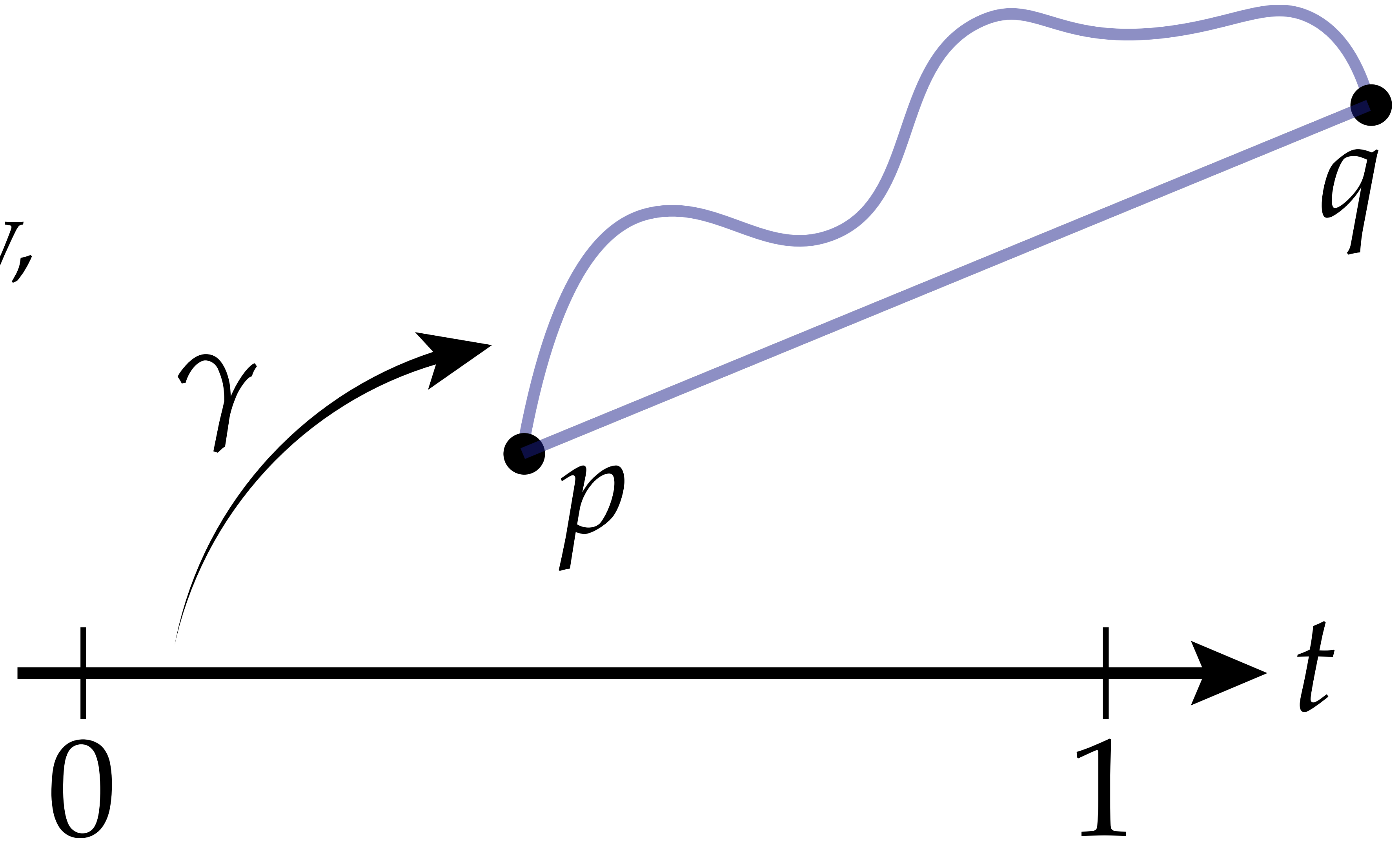
Now let’s try to find the *smoothest* curve…

$$\min_{\gamma} E_D(\gamma) = \min_{\widehat{\gamma}} \left(\min_{c} \int_0^1 \underbrace{(c'(t))^2}_{\to L} \, dt \right) = \min_{\widehat{\gamma}} L^2$$

Key idea: for a curve, minimizing Dirichlet energy will minimize length.

Shortest Planar Curve—Variational Perspective

- Consider again a curve $\gamma(t)$: $[0,1] \longrightarrow R^2$
- Can find shortest path by minimizing Dirichlet energy, subject to fixed endpoints $\gamma(0)=p$, $\gamma(1)=q$:



$$\min_{\gamma} \int_0^1 |\gamma'(t)|^2 \, dt$$

(integration by parts)

$$\iff \min_{\gamma} -\int_0^1 \langle \gamma(t), \gamma''(t) \rangle \, dt$$

- Taking gradient w.r.t. γ yields a 1D Poisson equation
- **Q:** Solution? **A:** Linear function!

$$\boxed{\begin{aligned} \frac{\partial^2}{\partial t^2}\gamma(t) &= 0 \\ \gamma(0) &= p \\ \gamma(1) &= q \end{aligned}}$$

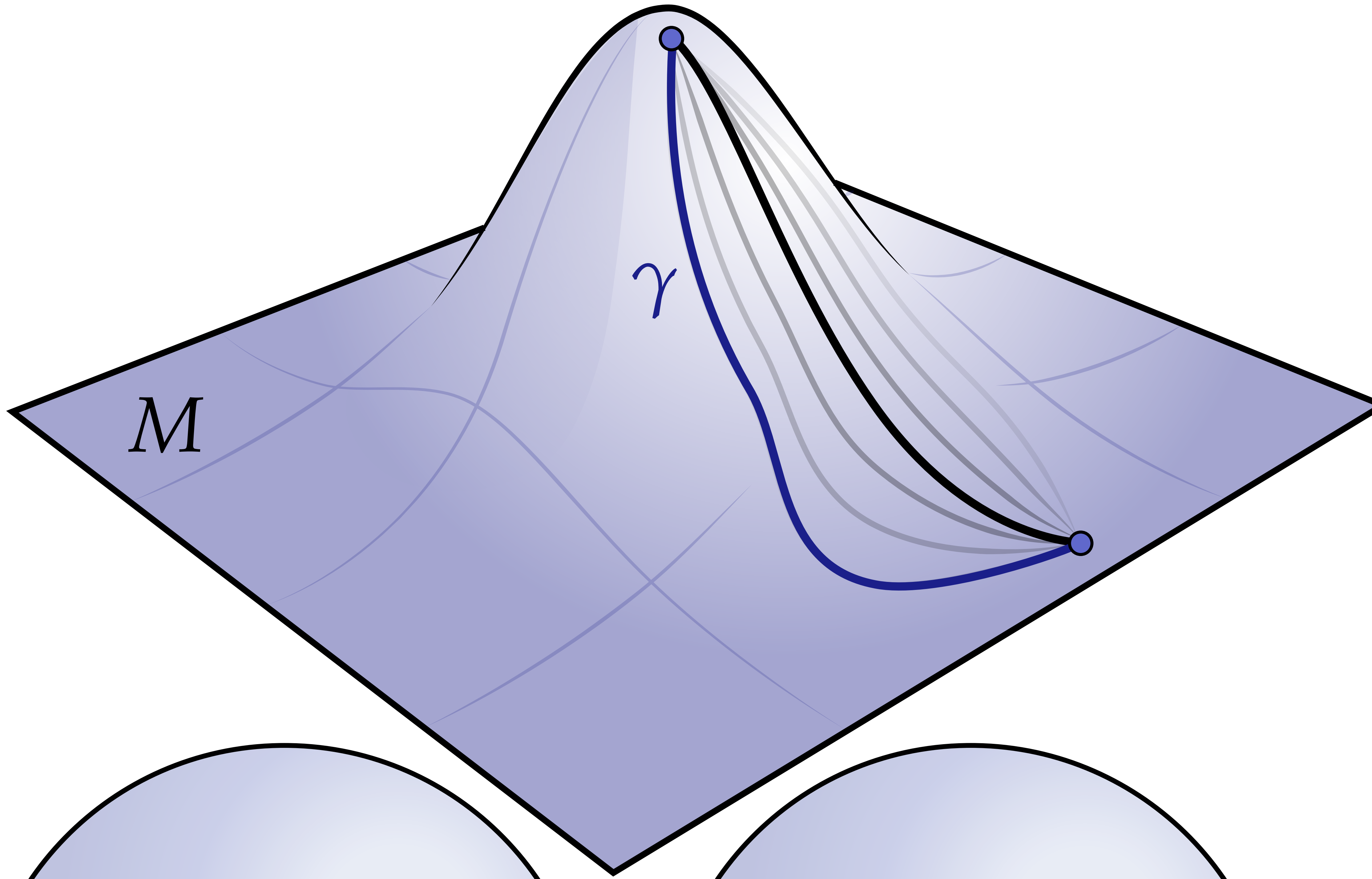
Key idea: geodesics are <u>harmonic functions</u>

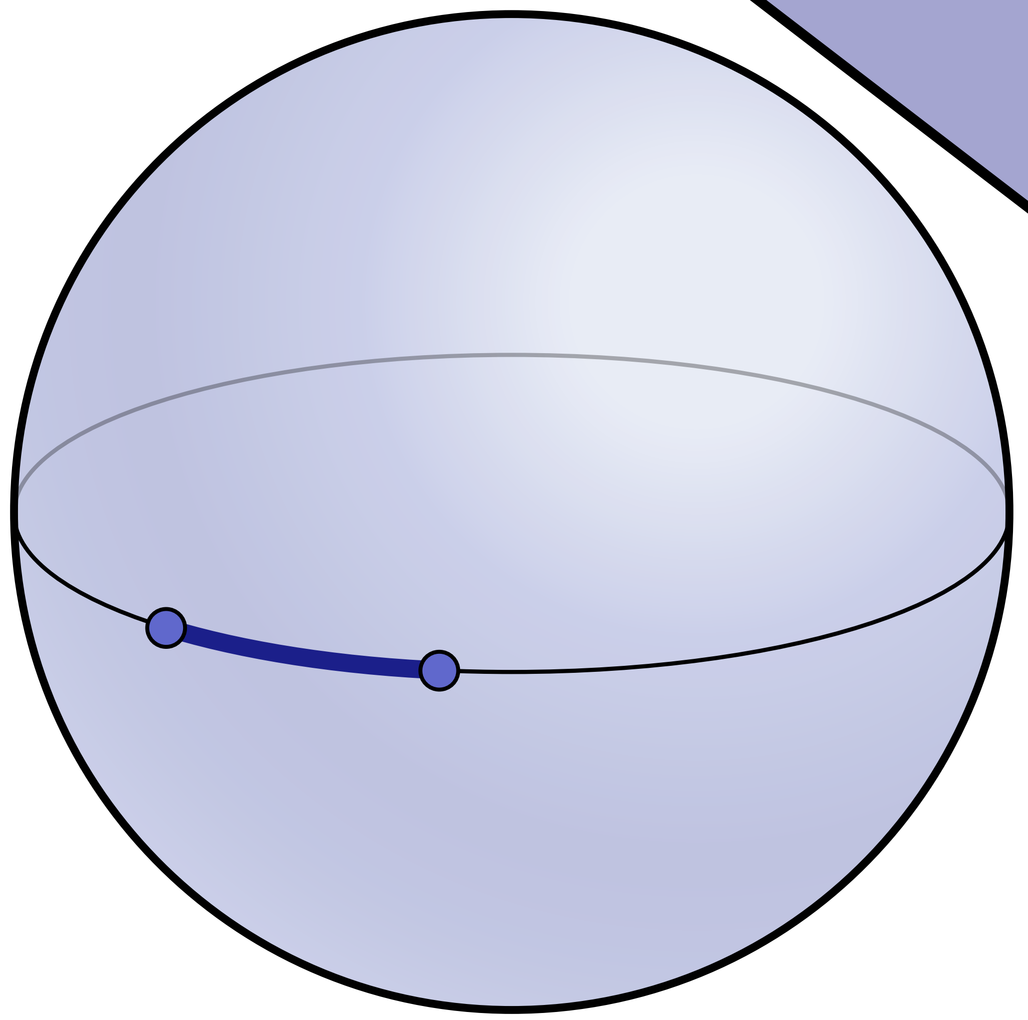
Shortest Geodesic—Variational Perspective

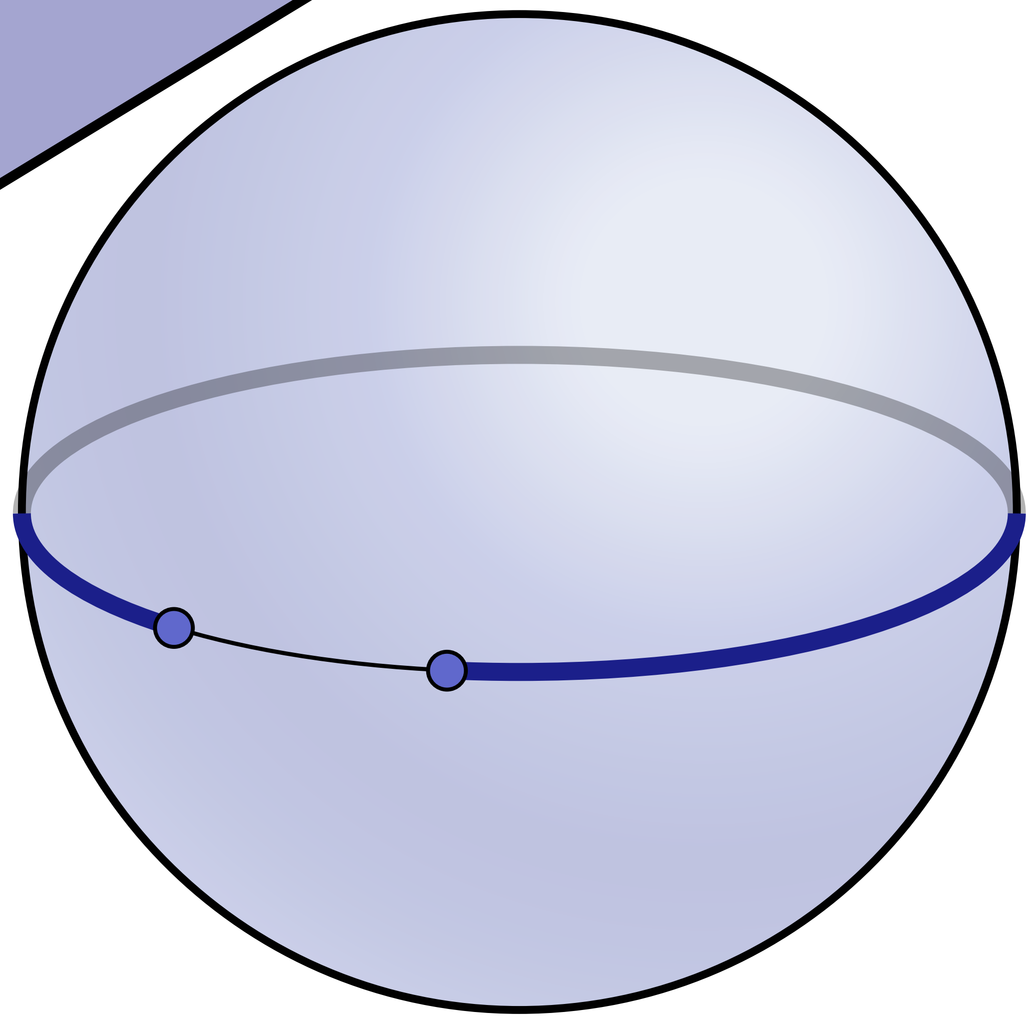
- Essentially same story on a curved surface (M,g)
- Consider a differentiable curve γ: $[0,1] \longrightarrow M$
- Dirichlet energy is then

$$E_D(\gamma) = \int_0^1 |\gamma'(t)|^2\, dt = \int_0^1 g\left(\gamma'(t), \gamma'(t)\right)\, dt$$

- Geodesics are still <u>critical points</u> (*harmonic*)
- May no longer be global minimizers




minimum


saddle point

Shortest Geodesic—Variational Perspective

- Essentially same story on a curved surface (M,g)
- Consider a differenti $[0,1] \longrightarrow M$
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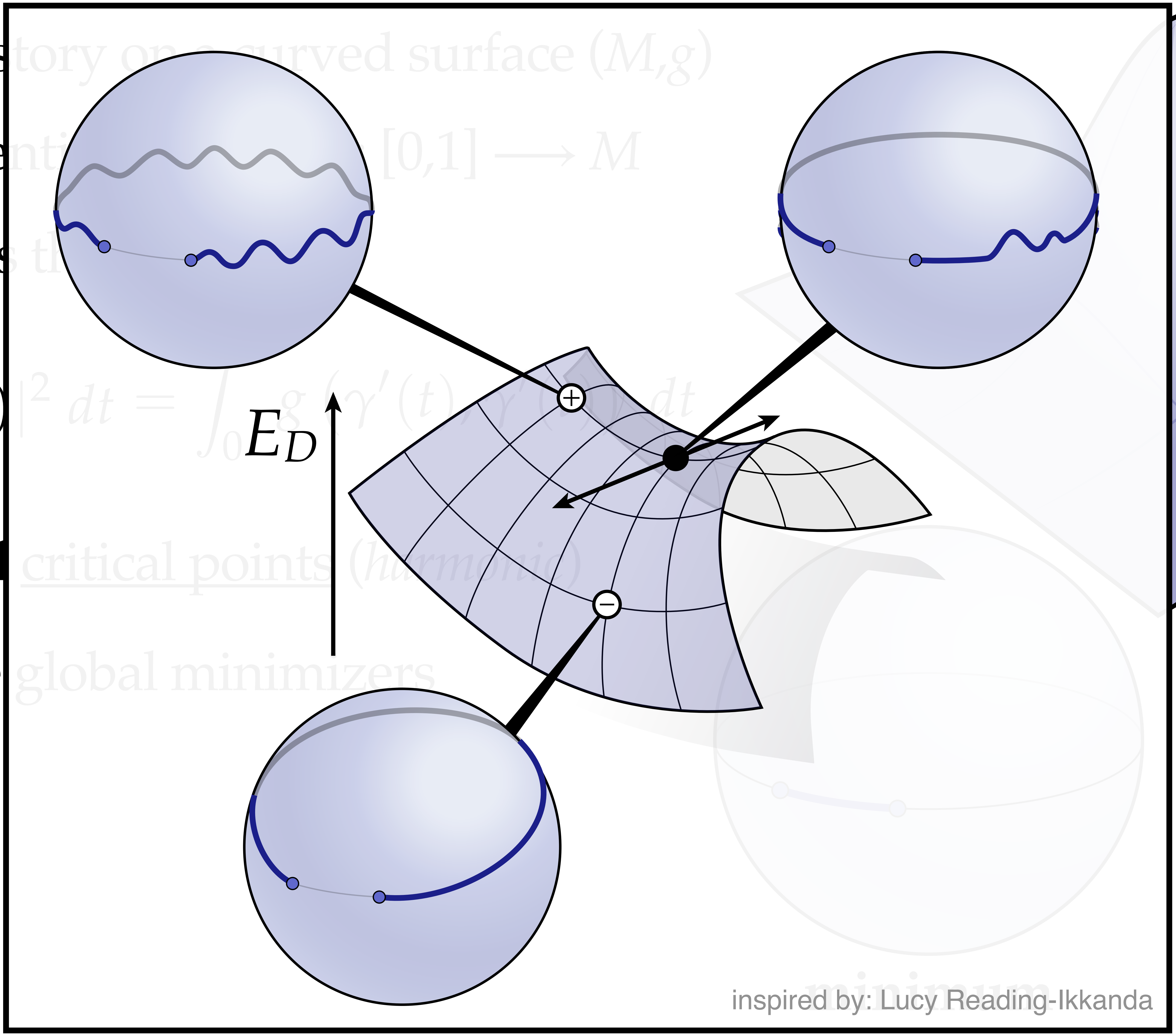

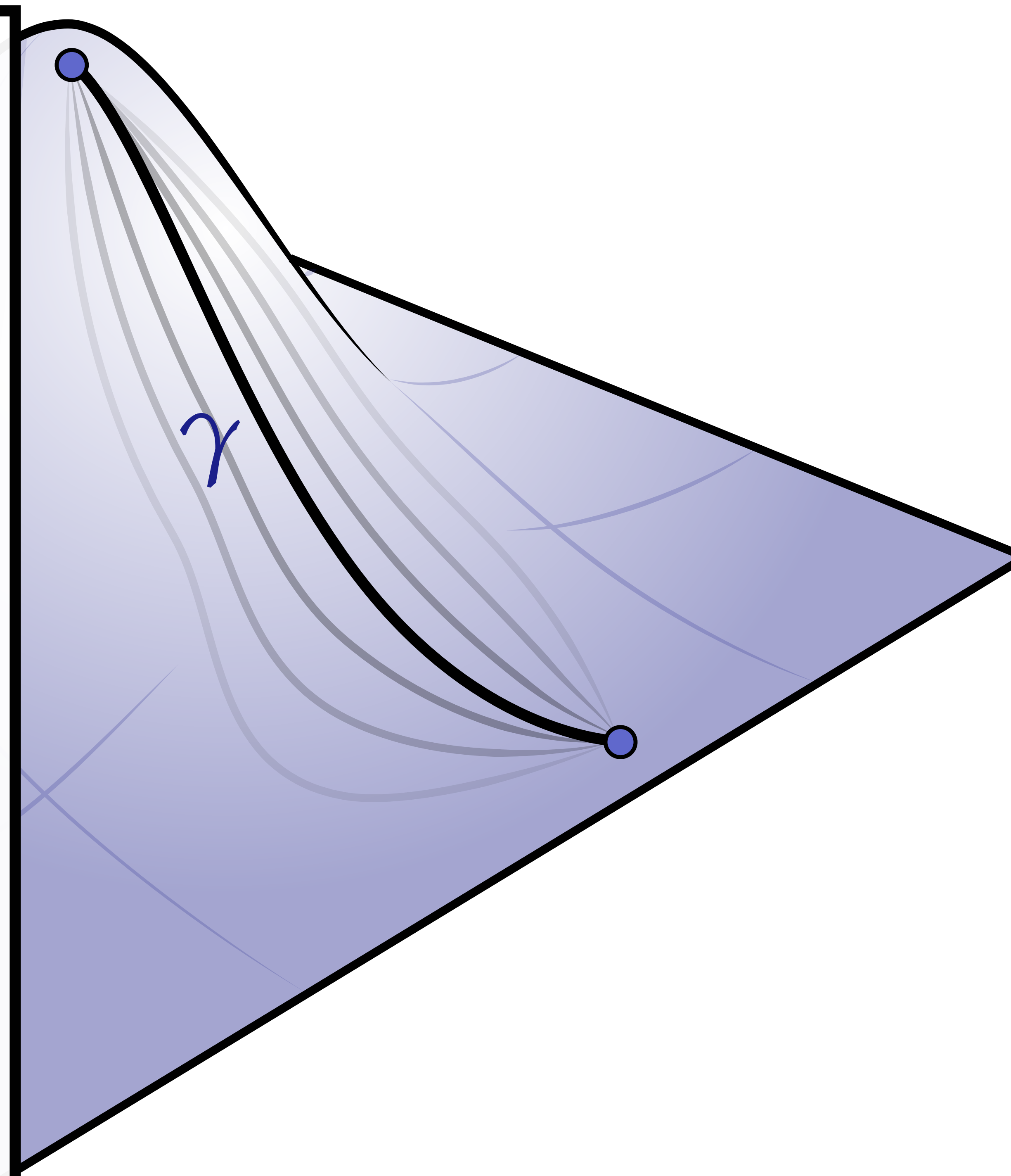

Shortest Geodesic—Variational Perspective

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- Consider a differentiable curve γ: $[0,1] \longrightarrow M$
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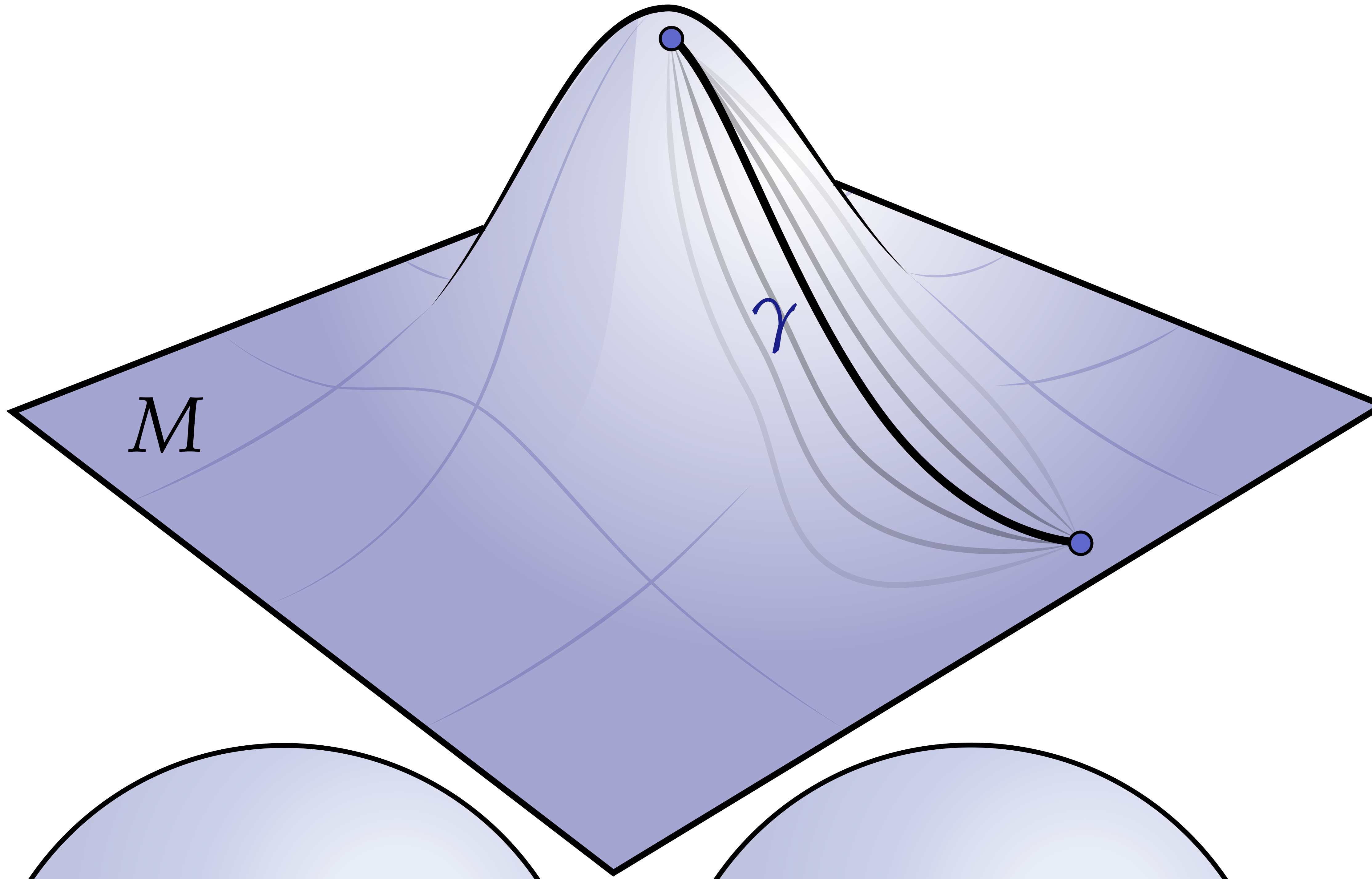
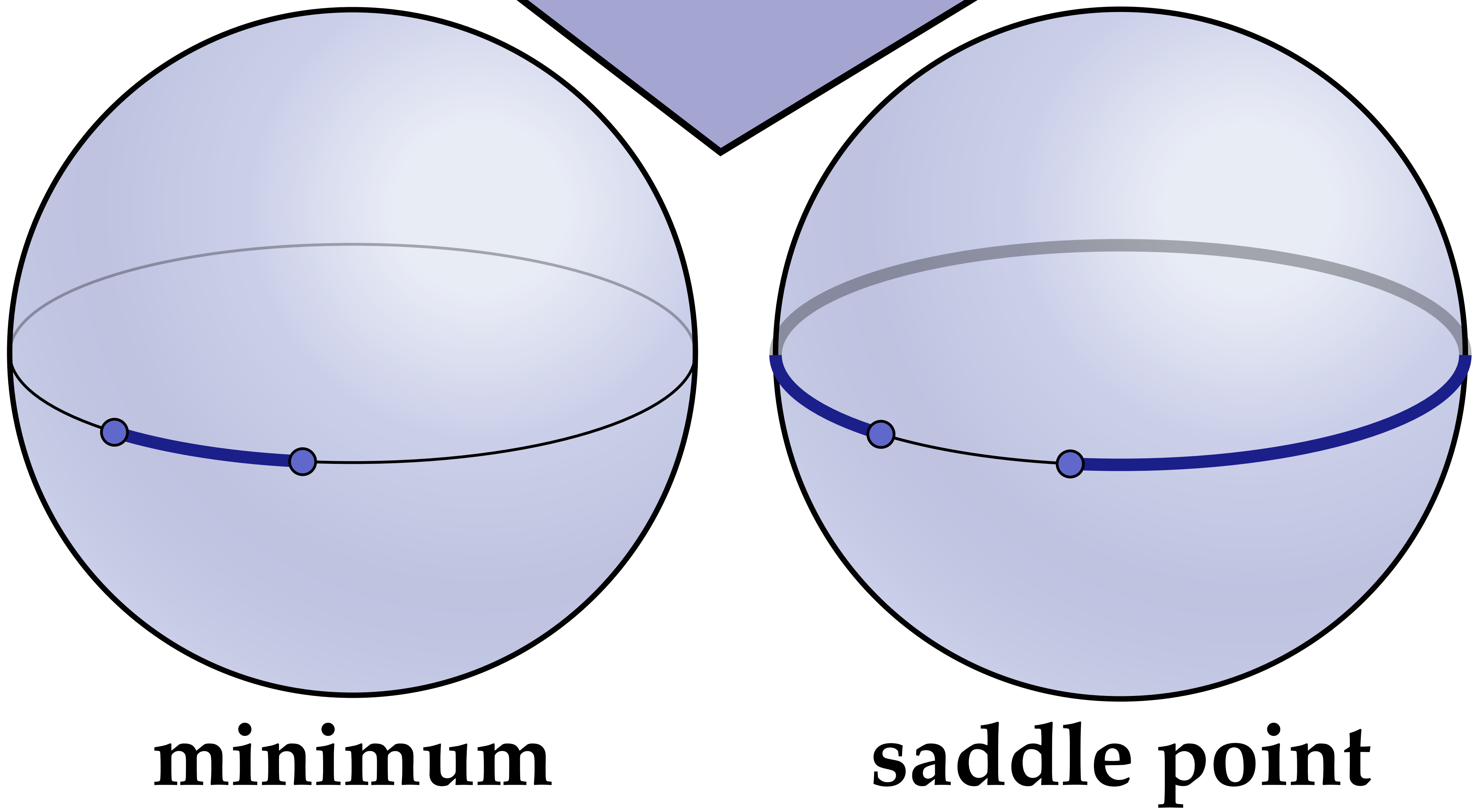
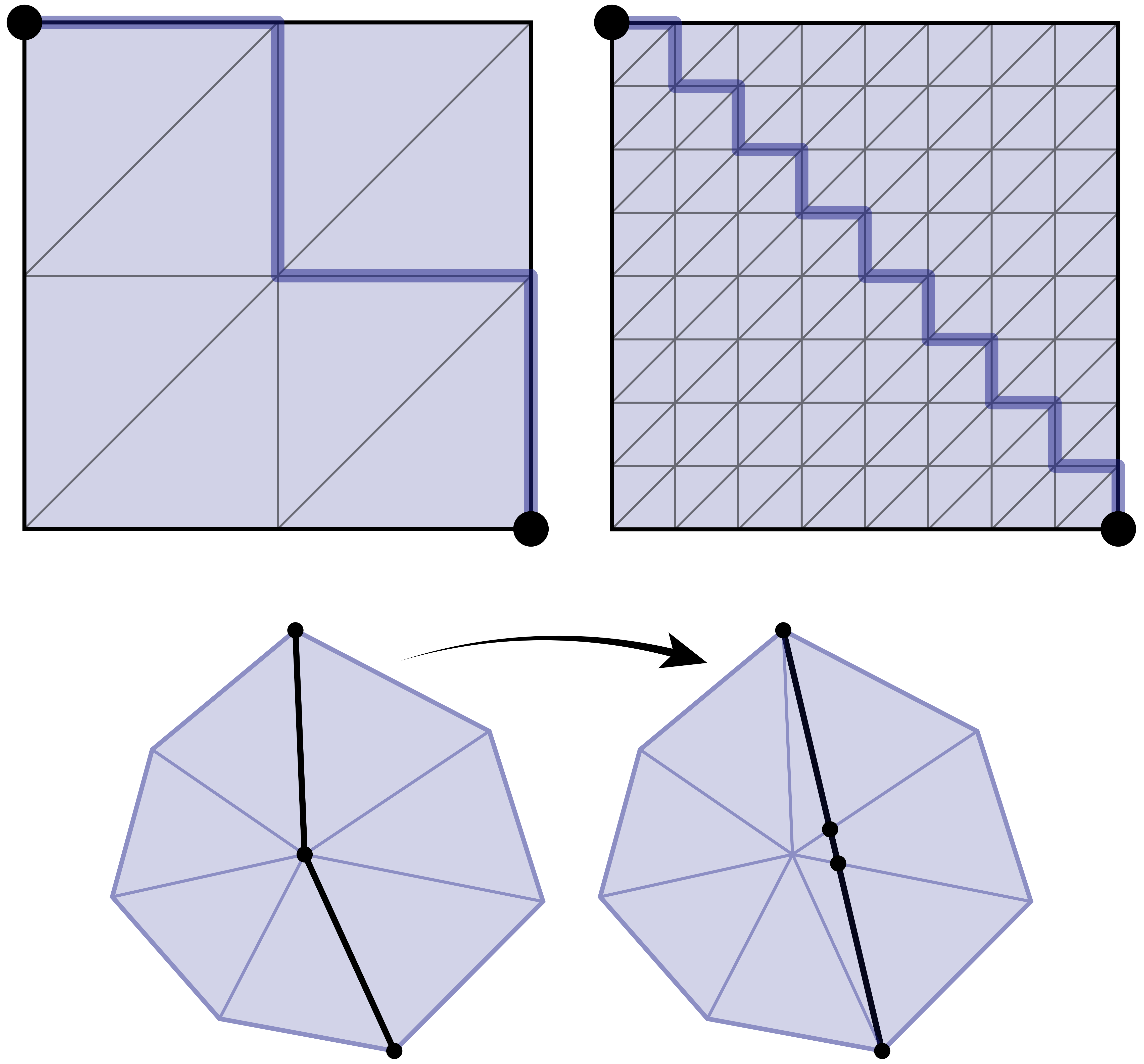
- Geodesics are still critical points (*harmonic*)
- May no longer be global minimizers
- Hence, geodesics no longer found by solving easy linear equation (Laplace)
  - Will *need numerical algorithms!*





Discrete Shortest Paths—Boundary Value Problem

- **Q:** How can we find a shortest path in the discrete case?
- Dijkstra's algorithm obviously comes to mind, but a shortest path in the edge graph is almost never geodesic
  - even if you refine the mesh!
- To get *locally* shortest path, could iteratively straighten Dijkstra path by until no more progress can be made
- What if we want to compute the *globally* shortest path?



Martínez et al, "Computing Geodesics on Triangular Meshes" (2005)

Discrete Shortest Paths — Vertices

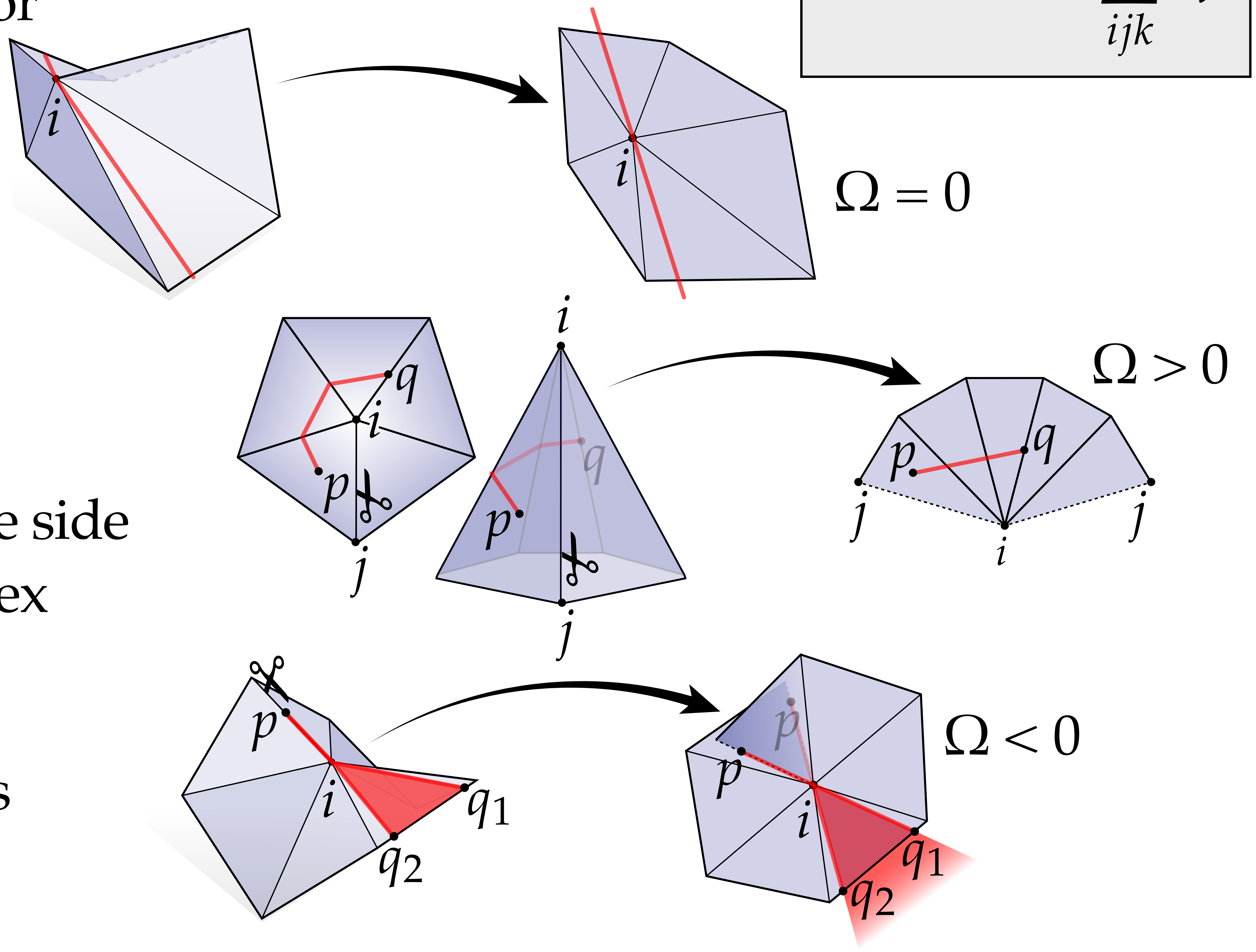
$$\Omega_i = 2\pi - \sum_{ijk} \theta_i^{jk}$$

- Even *locally* shortest paths near vertices require some care—different behavior depending on angle defect Ω
- **Flat** ($\Omega = 0$)

  Shortest path simply goes straight through the vertex
- **Cone** ($\Omega > 0$)

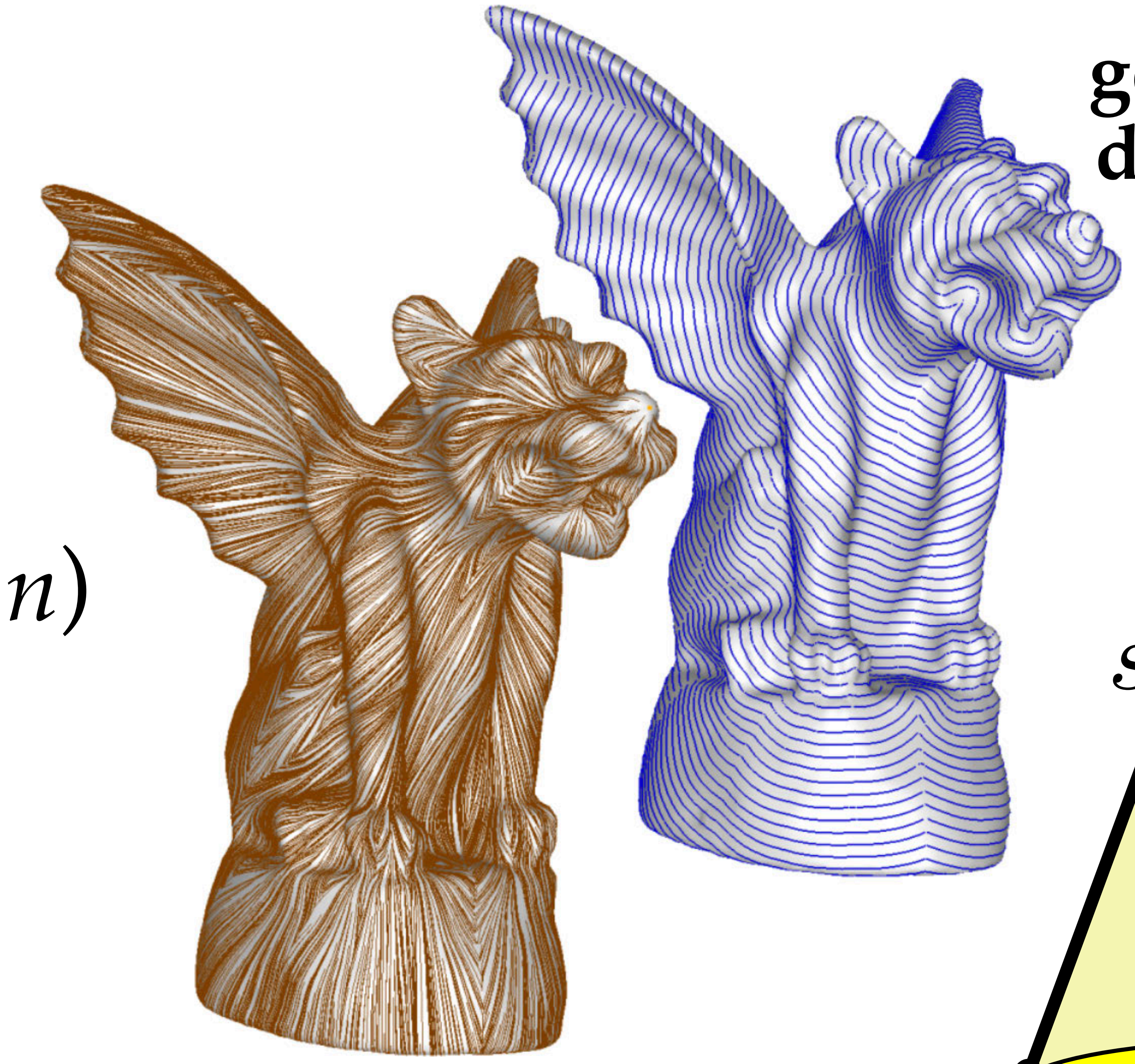
  Can always faster to go around one side or the other; never *through* the vertex
- **Saddle** ($\Omega < 0$)

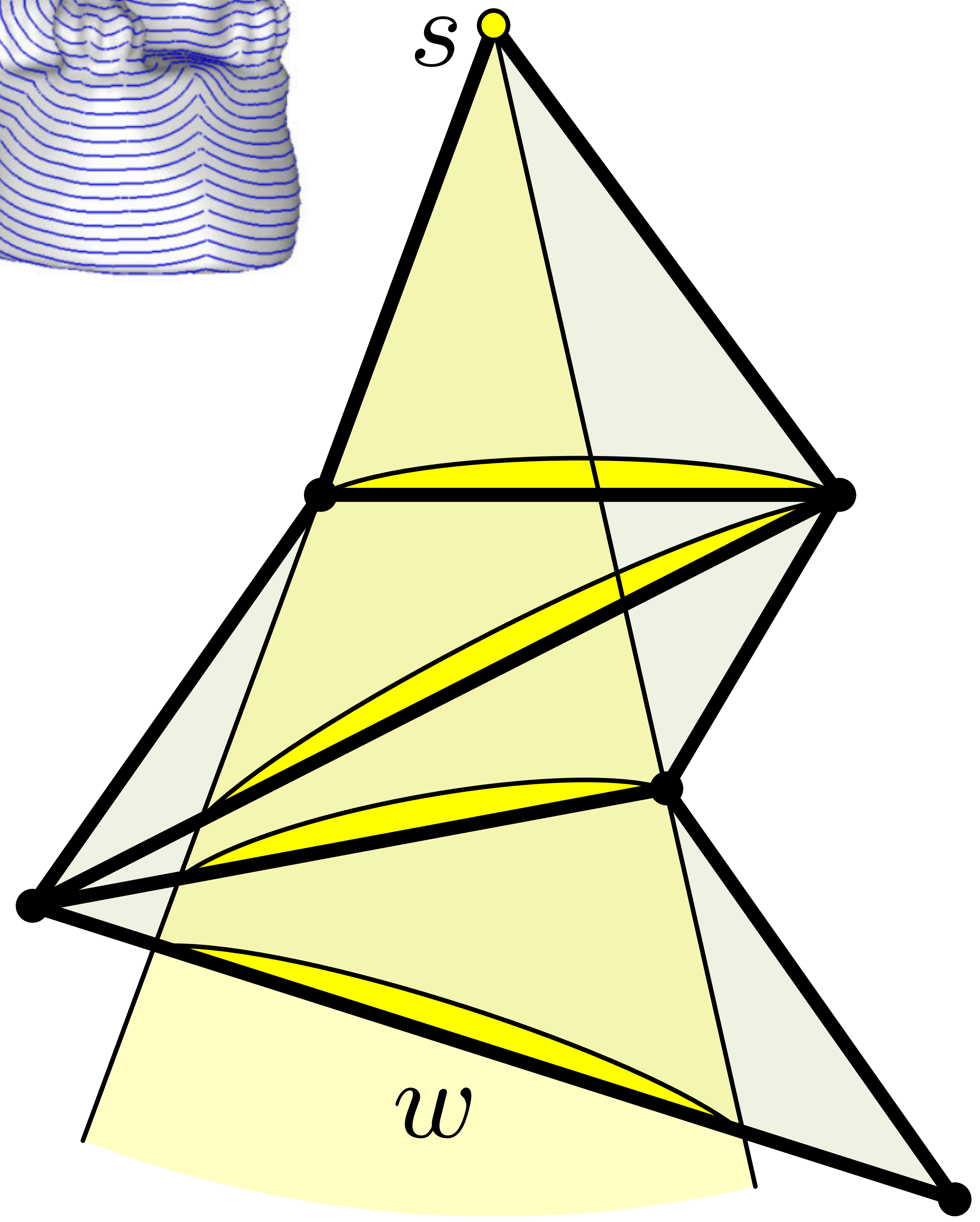
  Always *many* locally shortest paths passing through a saddle vertex.



Algorithms for Shortest Polyhedral Geodesics

- Algorithms for *shortest* polyhedral geodesics generalize Dijkstra's algorithm to include paths <u>through</u> triangles
- Mitchell, Mount, Papadimitrou (MMP) *"The Discrete Geodesic Problem"* (1986) — $O(n^2 \log n)$
- **Basic idea:** track intervals or "windows" of common geodesic paths
- Many subsequent improvements by pruning windows, approximation, … though still quite expensive (same asymptotic complexity)

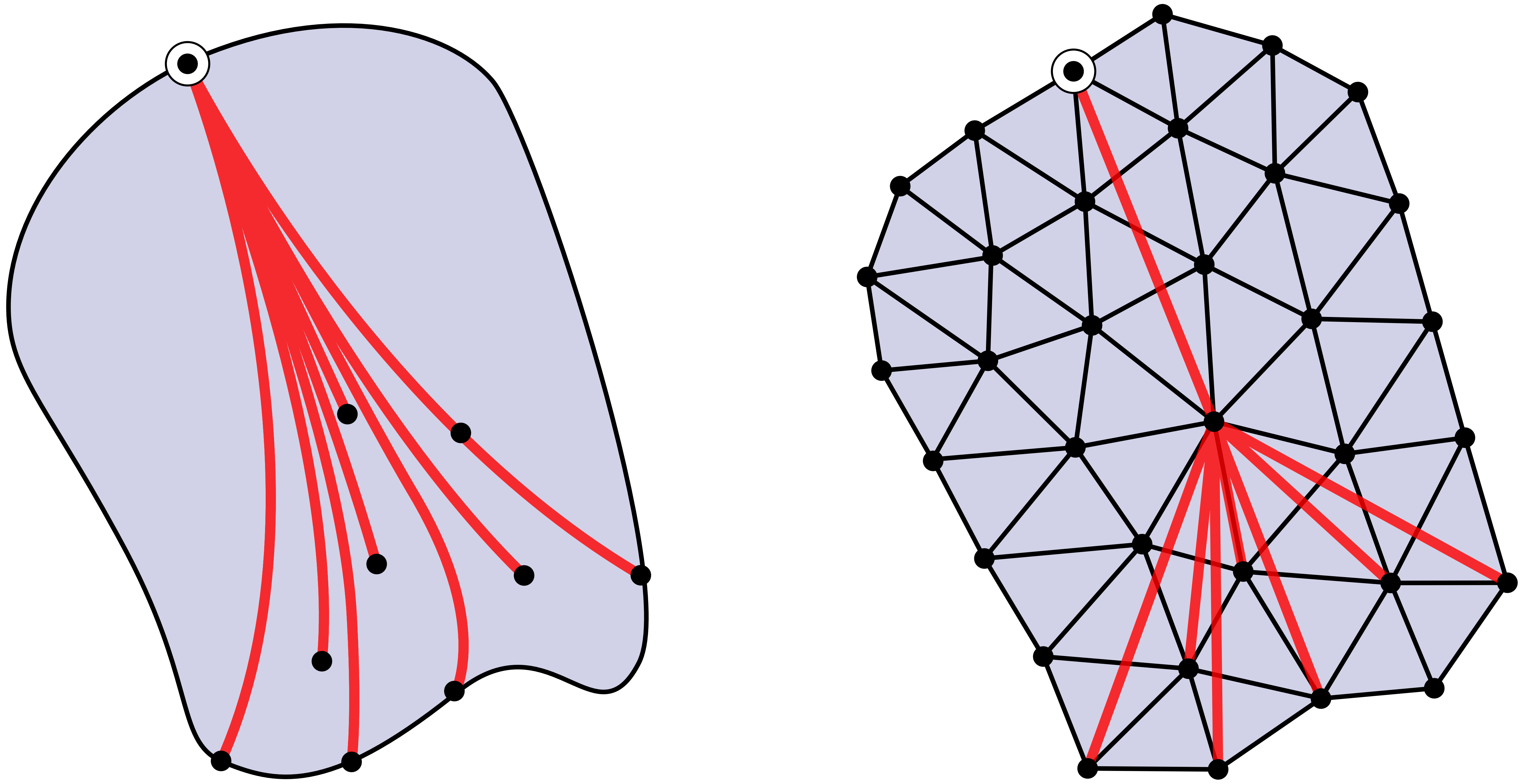




See: Surazhsky et al. *"Fast Exact and Approximate Geodesics on Meshes"* (2005)

Shortest Geodesics—Smooth vs. Discrete

- **Smooth:** two minimal geodesics γ_1, γ_2 from a source p to distinct points p_1, p_2 (resp.) intersect only if $\gamma_1 \subseteq \gamma_2$ or $\gamma_2 \subseteq \gamma_1$
- **Discrete:** many geodesics can coincide at saddle vertex (*"pseudo-source"*)



Note: *Shortest polyhedral geodesics may not faithfully capture behavior of smooth ones!*

Closed Geodesics

- **Theorem.** (Birkhoff 1917) Every smooth convex surface contains a simple closed geodesic, *i.e.*, a geodesic loop that does not cross itself (*"Birkhoff equator"*)
- **Theorem.** (Luysternik & Shnirel'man 1929) Actually, there are at least three—and this result is sharp: *only* three on some smooth surfaces.

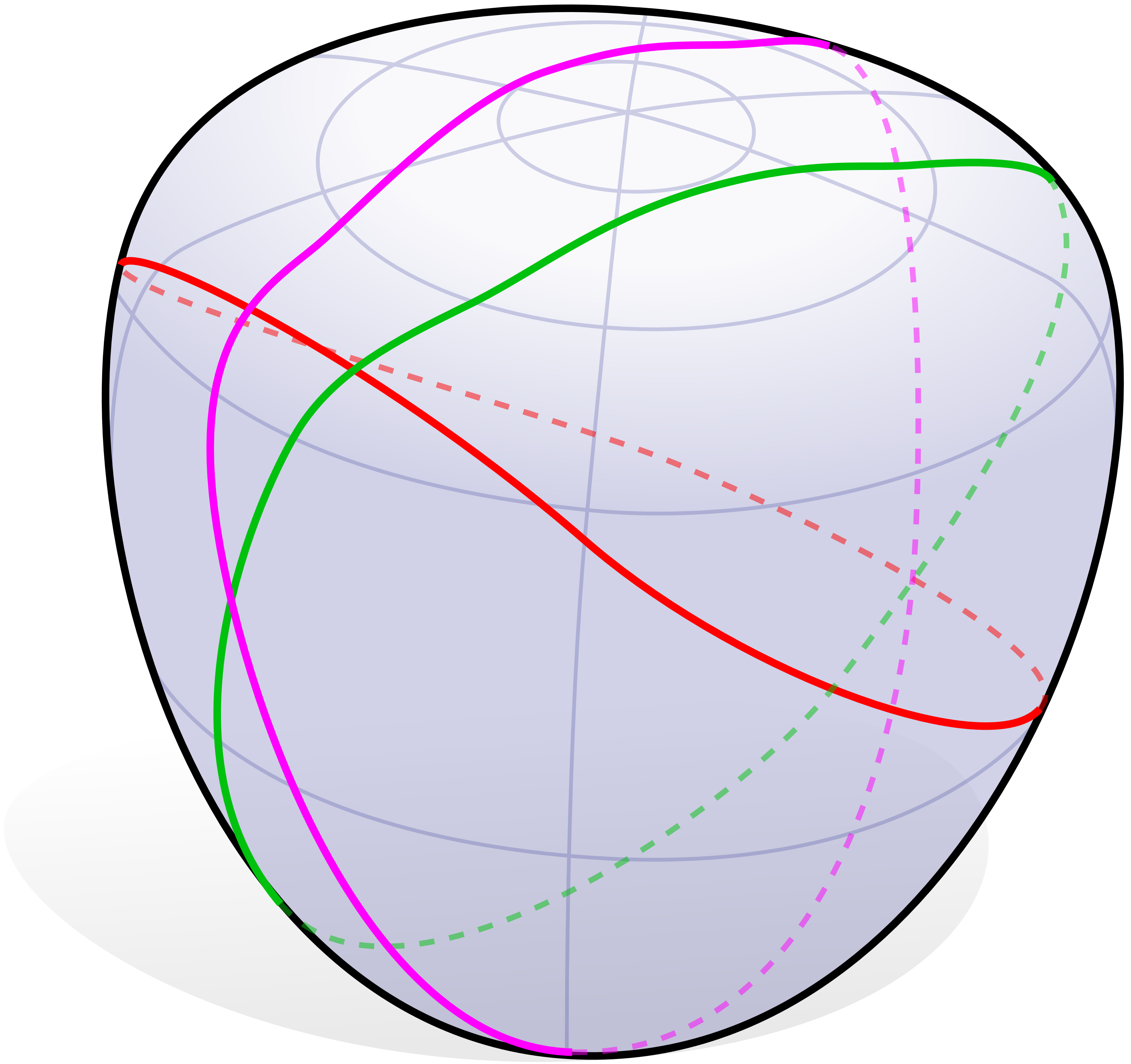
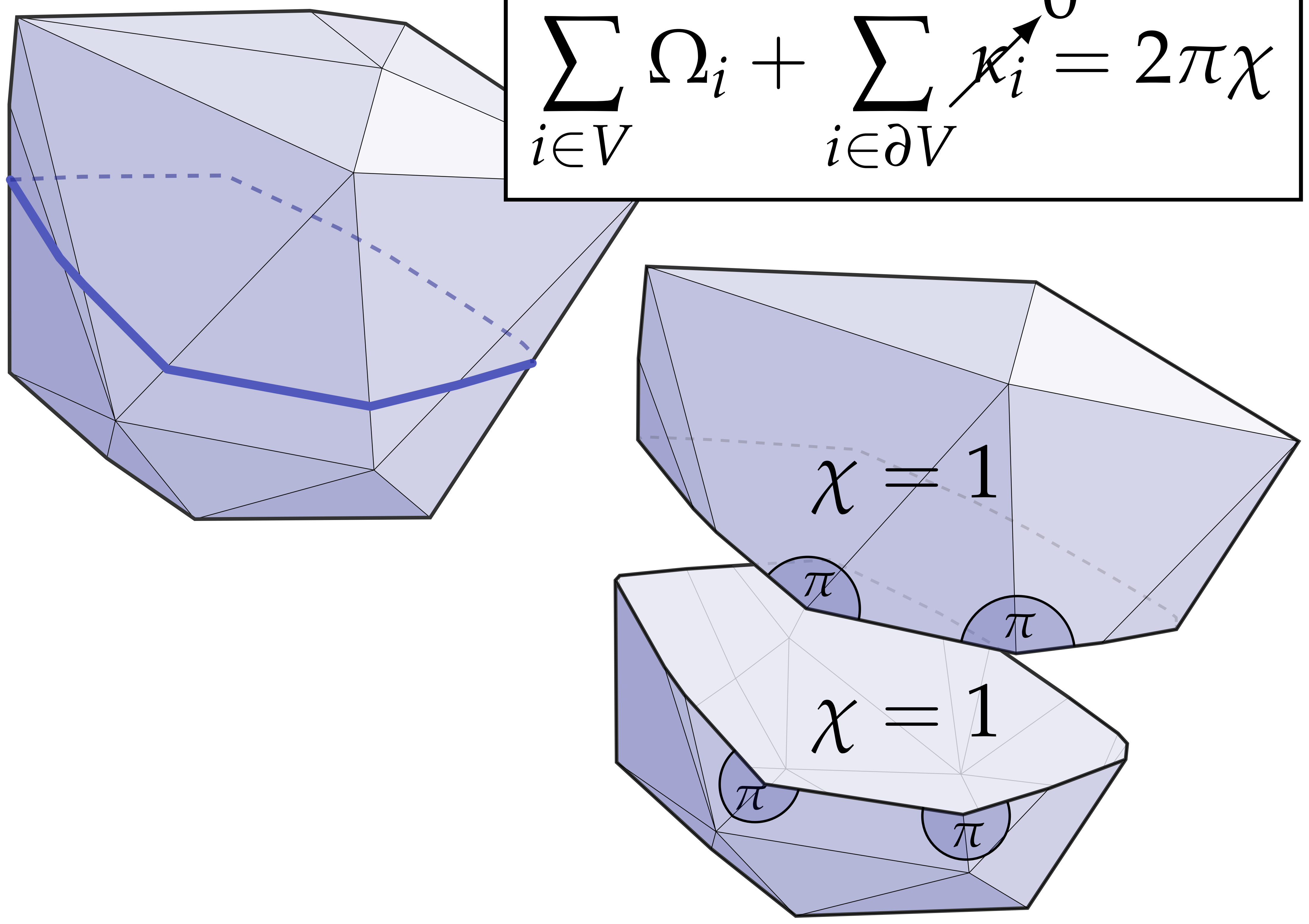


image adapted from Radeschi, "Closed Geodesics on Surfaces and Riemannian Manifolds"

Closed Geodesics

- **Theorem.** (Birkhoff 1917) Every smooth convex surface contains a simple closed geodesic, *i.e.*, a geodesic loop that does not cross itself (*"Birkhoff equator"*)
- **Theorem.** (Luysternik & Shnirel'man 1929) Actually, there are at least three—and this result is sharp: *only* three on some smooth surfaces.
- **Theorem.** (Galperin 2002) *Most* convex polyhedra <u>do not</u> have simple closed geodesics (in the sense of discrete *shortest* geodesics).
- *Shortest* characterization of discrete geodesics again fails to capture properties from smooth setting.

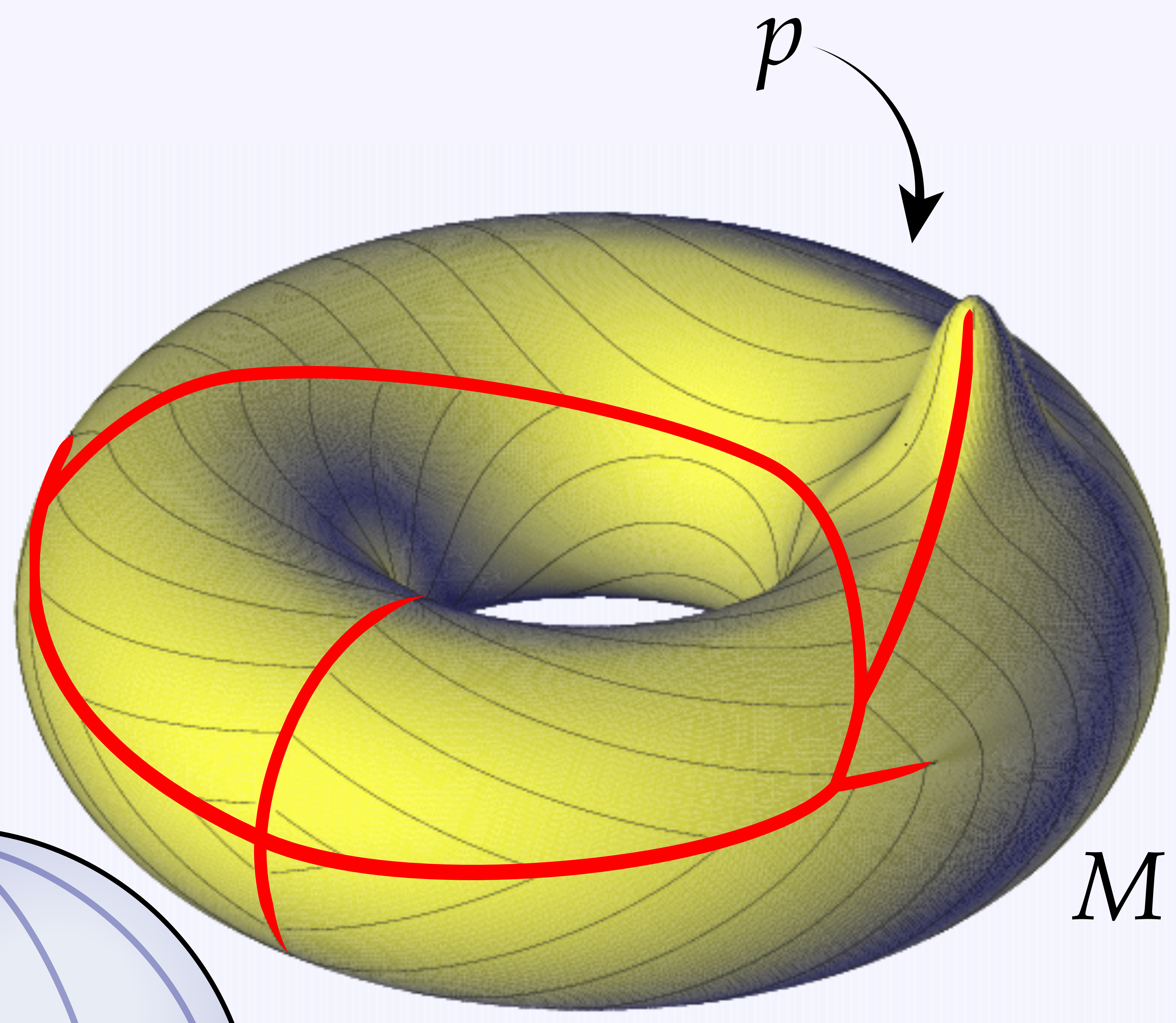
$$\sum_{i\in V}\Omega_i + \sum_{i\in\partial V}\cancel{\kappa_i}^{\,0} = 2\pi\chi$$

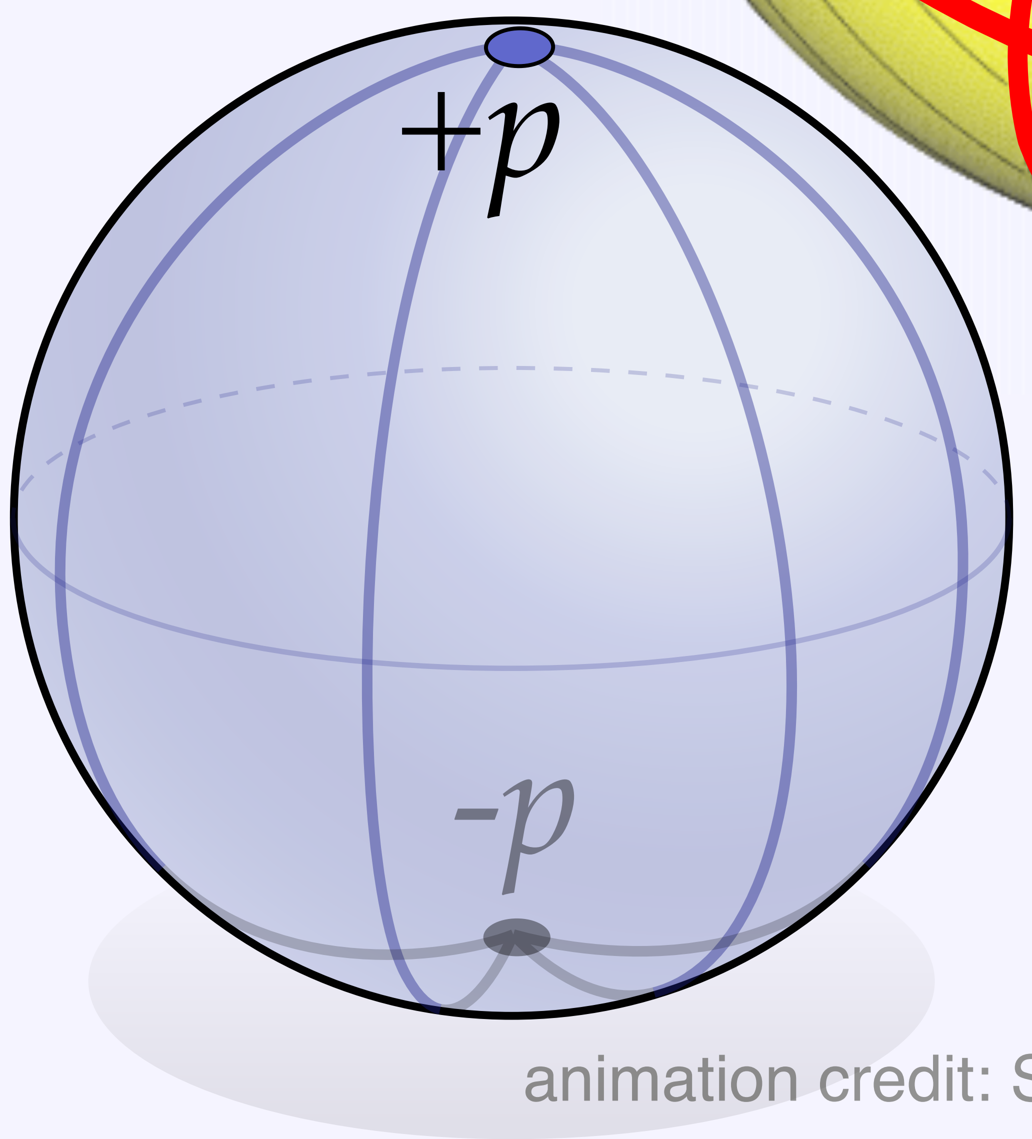


A *shortest* geodesic can't pass through convex vertices. So, by Gauss-Bonnet, a closed geodesic would have to partition vertices into two sets that each have total angle defect of **<u>exactly</u>** 2π.

Cut Locus & Injectivity Radius

- For a source point p on a smooth surface M, the *cut locus* is the set of all points q such that there is not a unique (globally) shortest geodesic between p and q.
  - *injectivity radius* is the distance to the closest point on the cut locus
- *E.g.*, on a sphere cut locus of any point $+p$ is the antipodal point $-p$
  - injectivity radius covers whole sphere
- In general can be *much* more complicated (and smaller injectivity radius…)

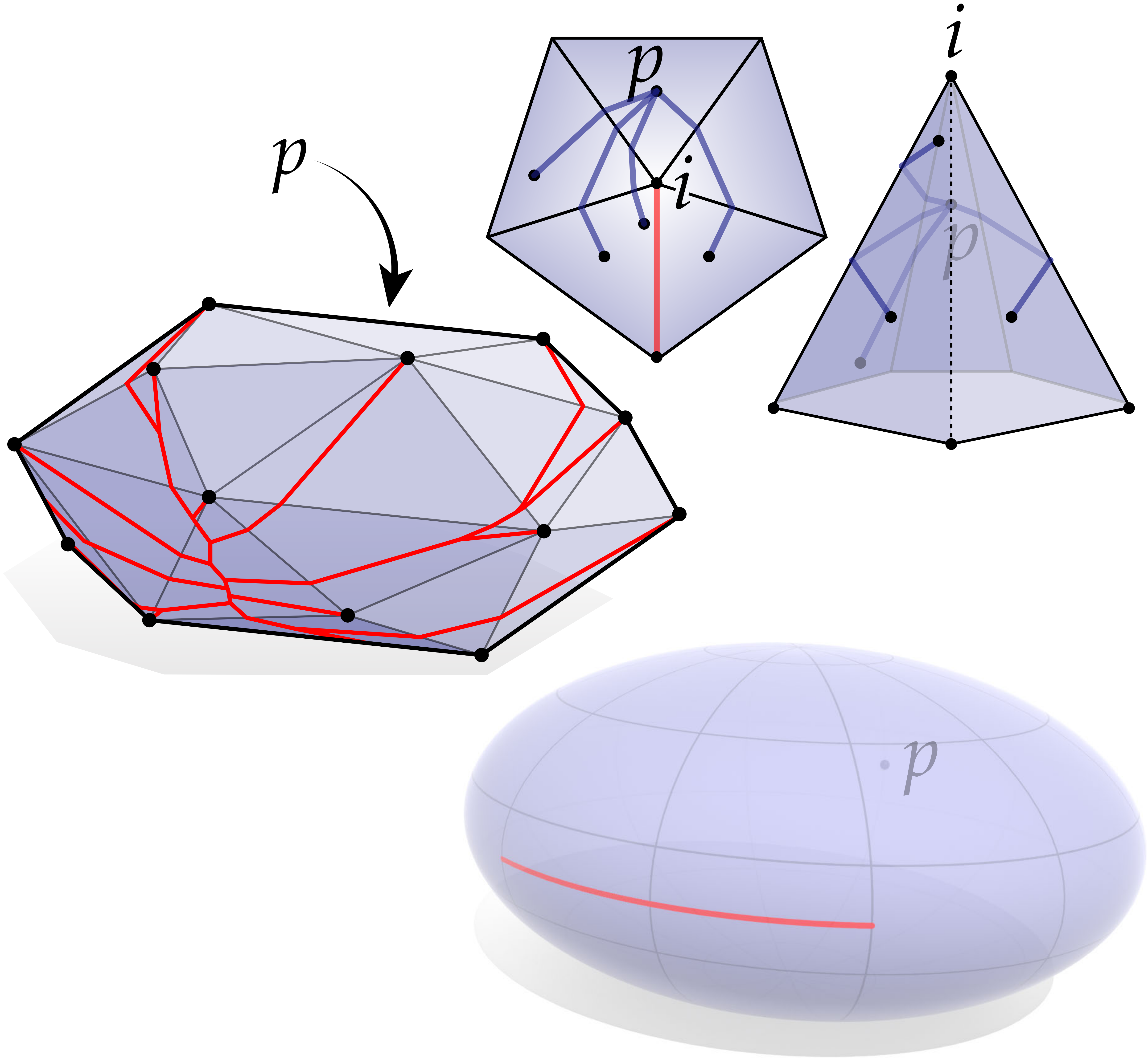




animation credit: S. Markvorsen and P.G. Hjorth (The Cut Locus Project)

Discrete Cut Locus

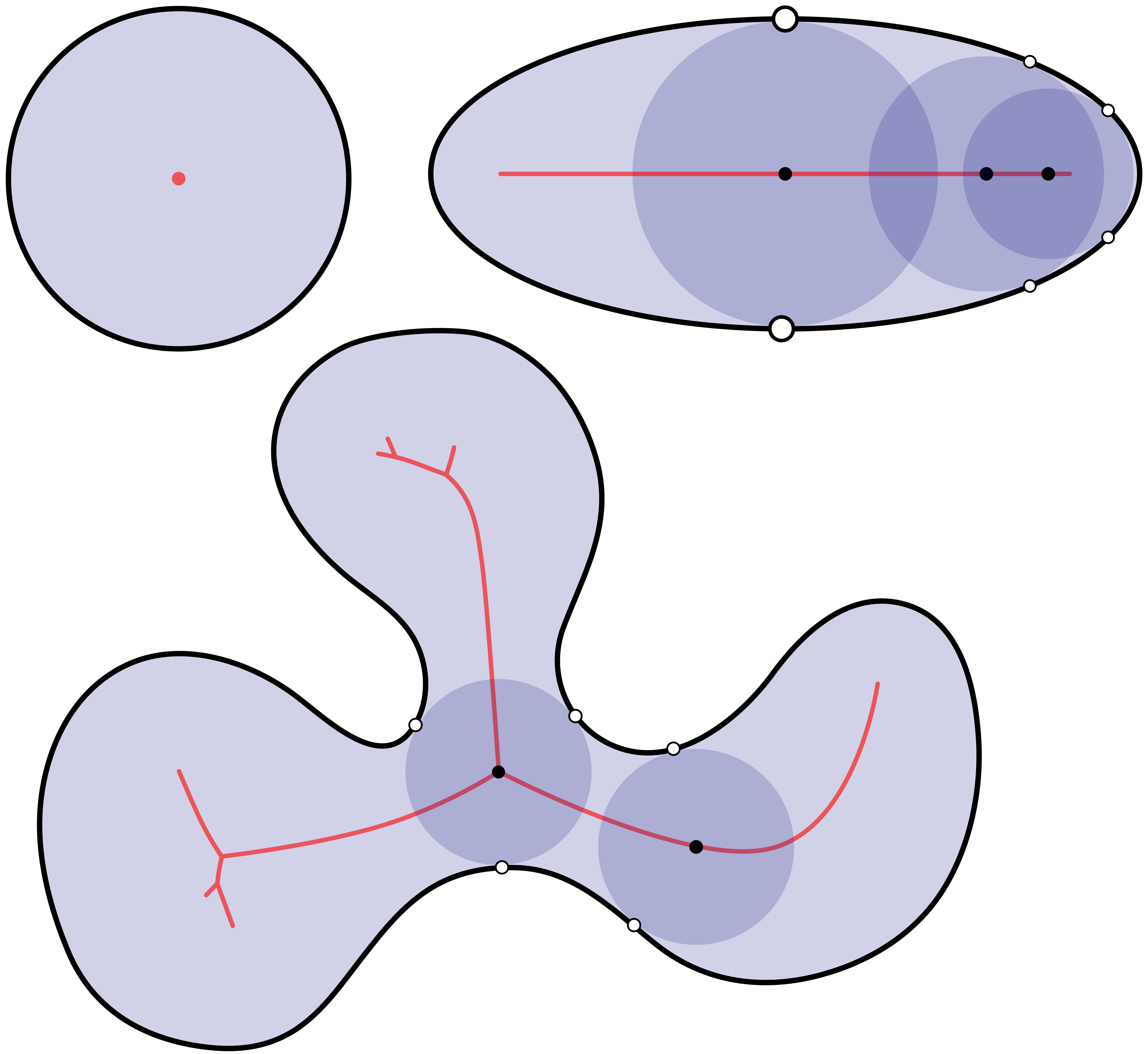
- What does cut locus look like for polyhedral surfaces?
- Recall that it's always shorter to go "around" a cone-like vertex (i.e., vertex with positive curvature $\Omega_i > 0$)
- Hence, polyhedral cut locus will contain <u>every</u> cone vertex in the entire surface
- Can look *very* different from the smooth cut locus!



Polyhedron image adapted from Itoh & Sinclair, "Thaw: A Tool for Approximating Cut Loci on a Triangulation of a Surface"

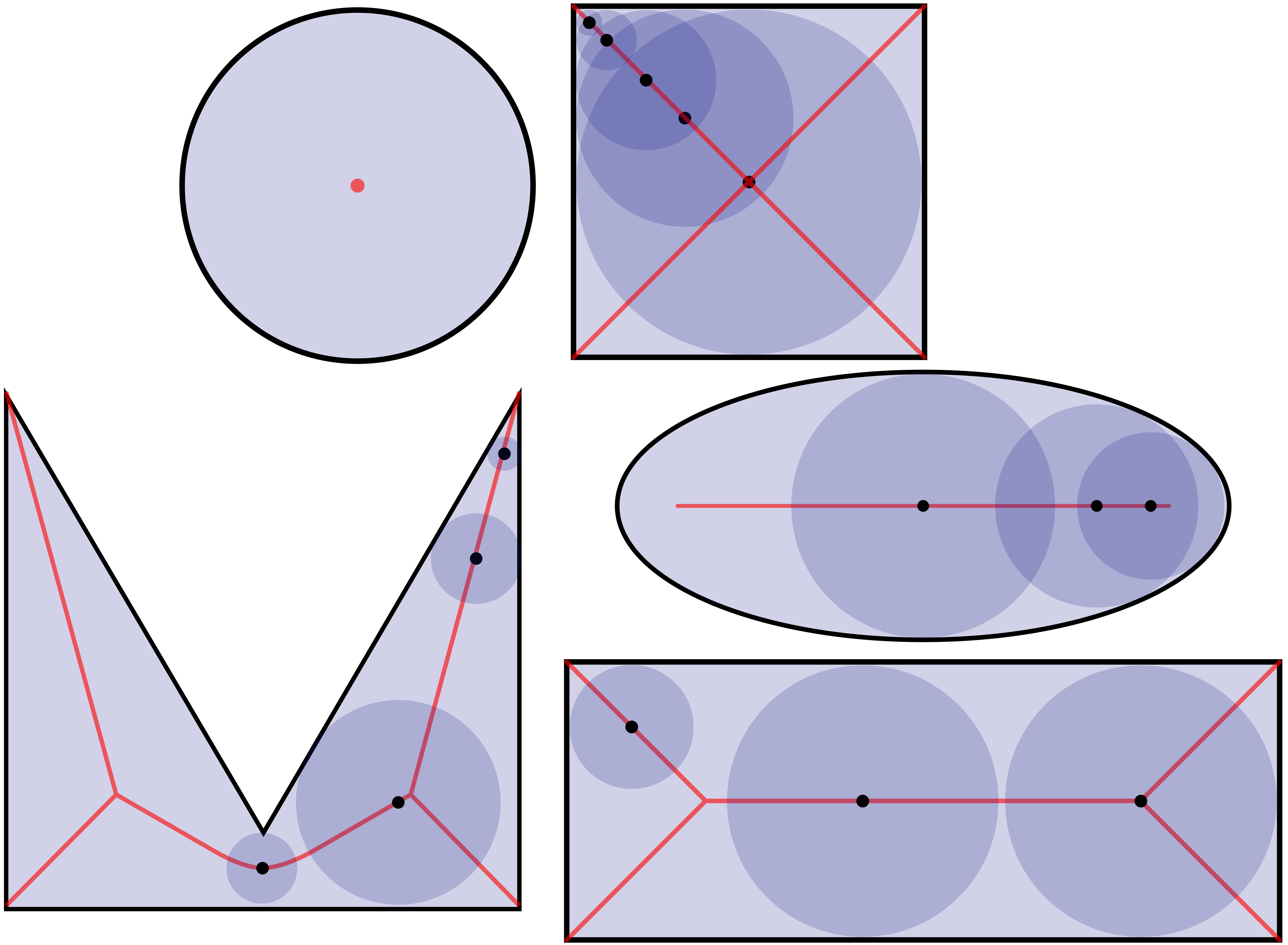
Medial Axis

- Similar to the cut locus, the *medial axis* of a surface or region is the set of all points p that do not have a unique closest point on the <u>boundary</u>
- A *medial ball* is a ball with center on the medial axis, and radius given by the distance to the closest point
- Like cut locus, can get quite complicated!
- Typically three branches (*why?*)
- Provides a "dual" representation: can recover original shape from
  - medial axis
  - radius function



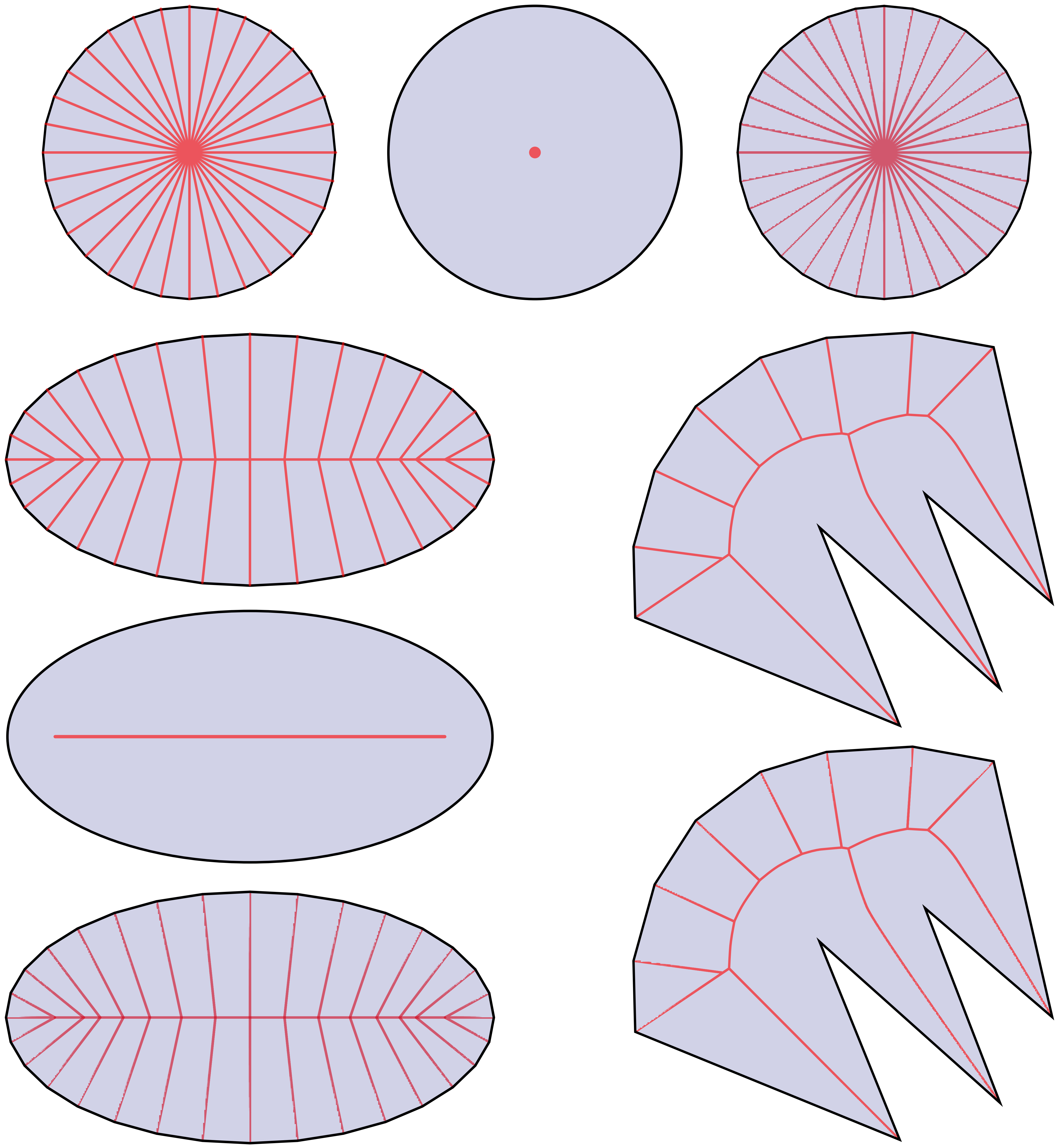
Discrete Medial Axis

- What does the medial axis of a discrete domain look like?
- Let's start with a square. (What did the medial axis for a circle look like?)
- What about a rectangle? (What happened with an ellipse?)
- How about a nonconvex polygon?
  - *surprise:* no longer just straight edges!



Discrete Medial Axis

- In general, medial axis touches *every* convex vertex
- May not look much like true (smooth) medial axis!
- One idea: "filter" using radius function…
  - still hard to say exactly which pieces should remain
- Lots of work on alternative *"shape skeletons"* for discrete curves & surfaces



Medial Axis in 3D

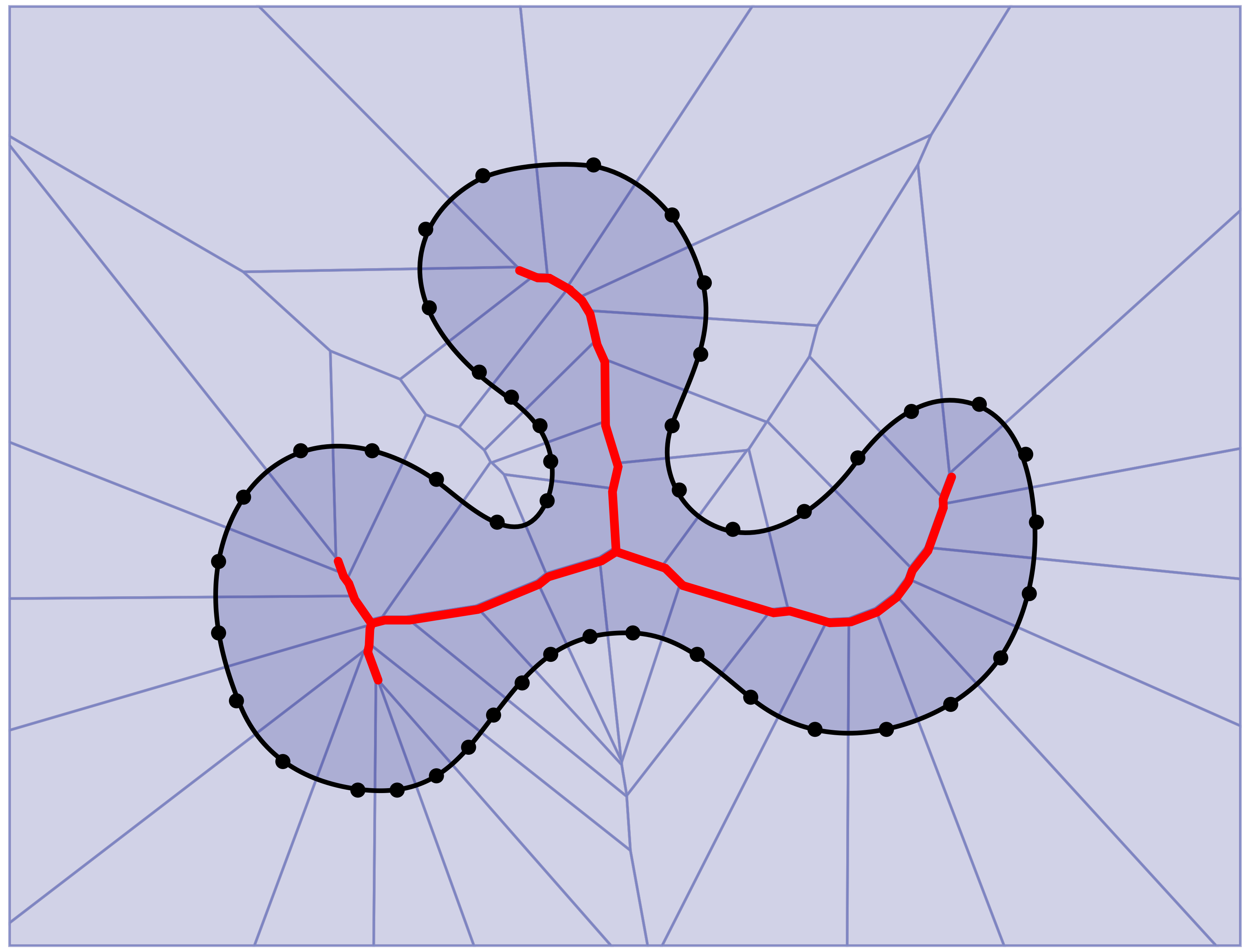
Same definition applies in any dimension—provides notion of "skeleton" for a shape:

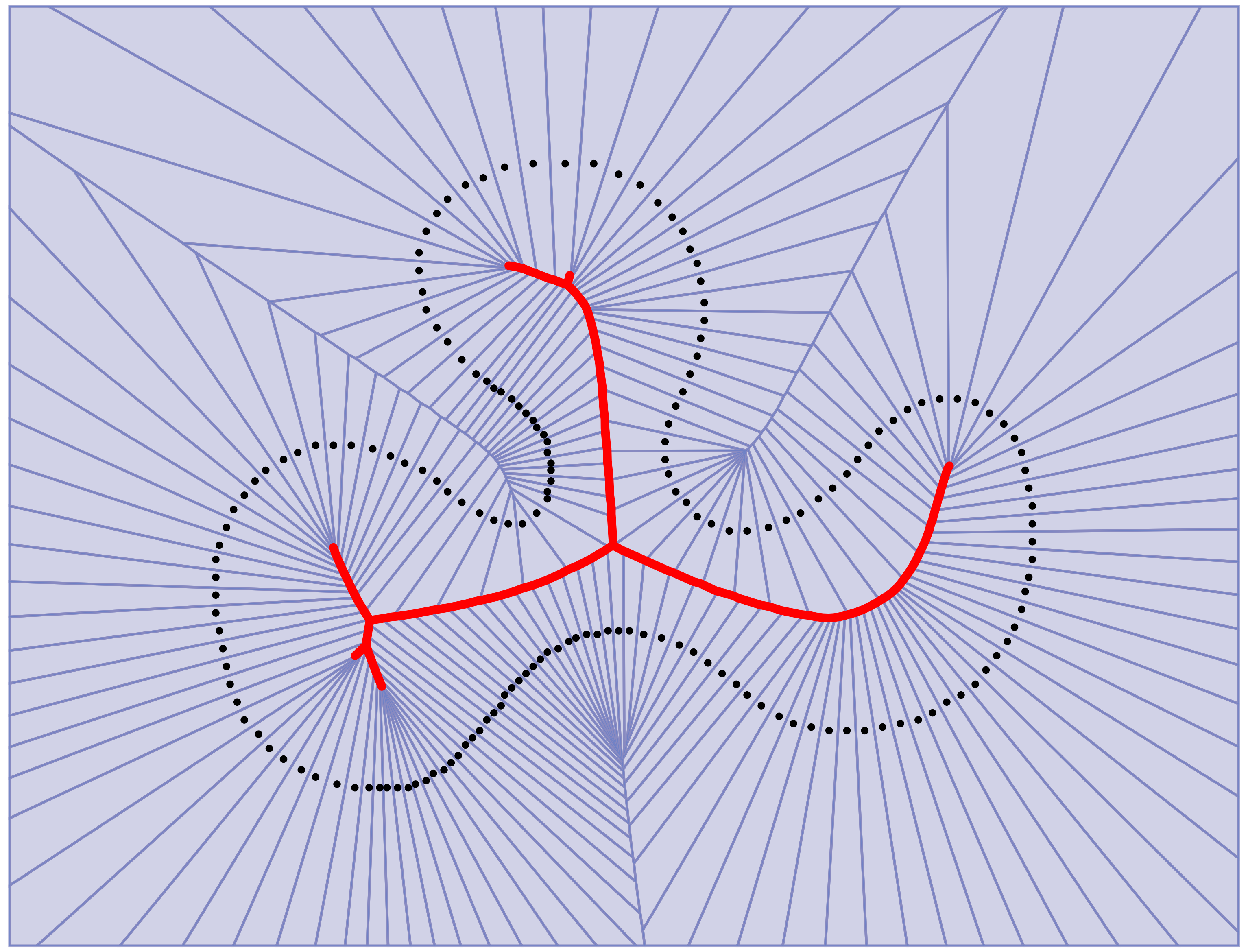
Hard to compute exactly (*e.g.*, quadratic pieces); often approximate by simplicial complex.

image credit: S. Yoshizawa, A. Belyaev, & H-P. Seidel

Computing the Medial Axis

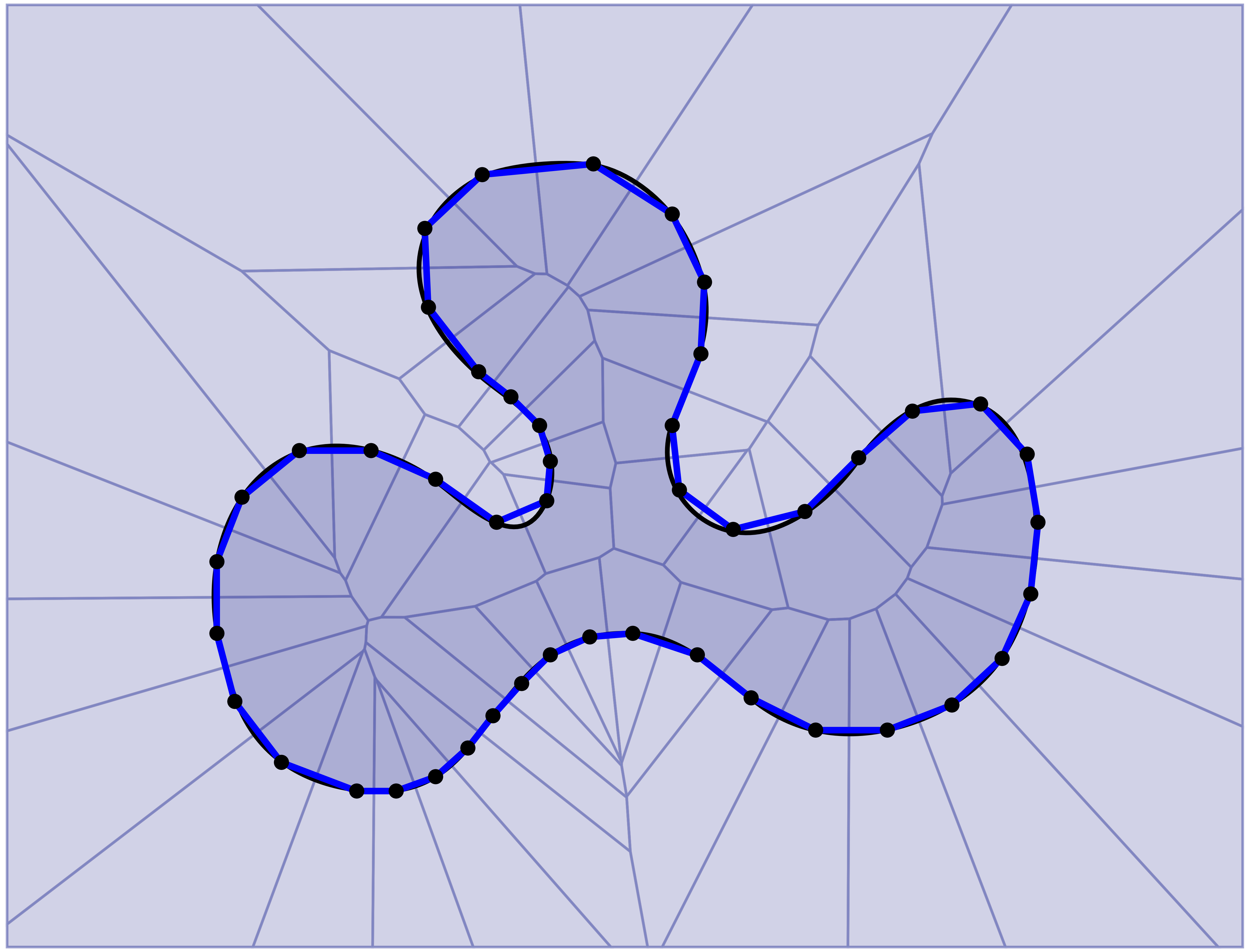
- **Many** algorithms for computing/approximating medial axis & other "shape skeletons"
- One line of thought: use *Voronoi diagram* as starting point:
  - sample points on boundary
  - compute Voronoi diagram
  - keep "short" facets of tall/skinny cells
- With enough points, get correct topology

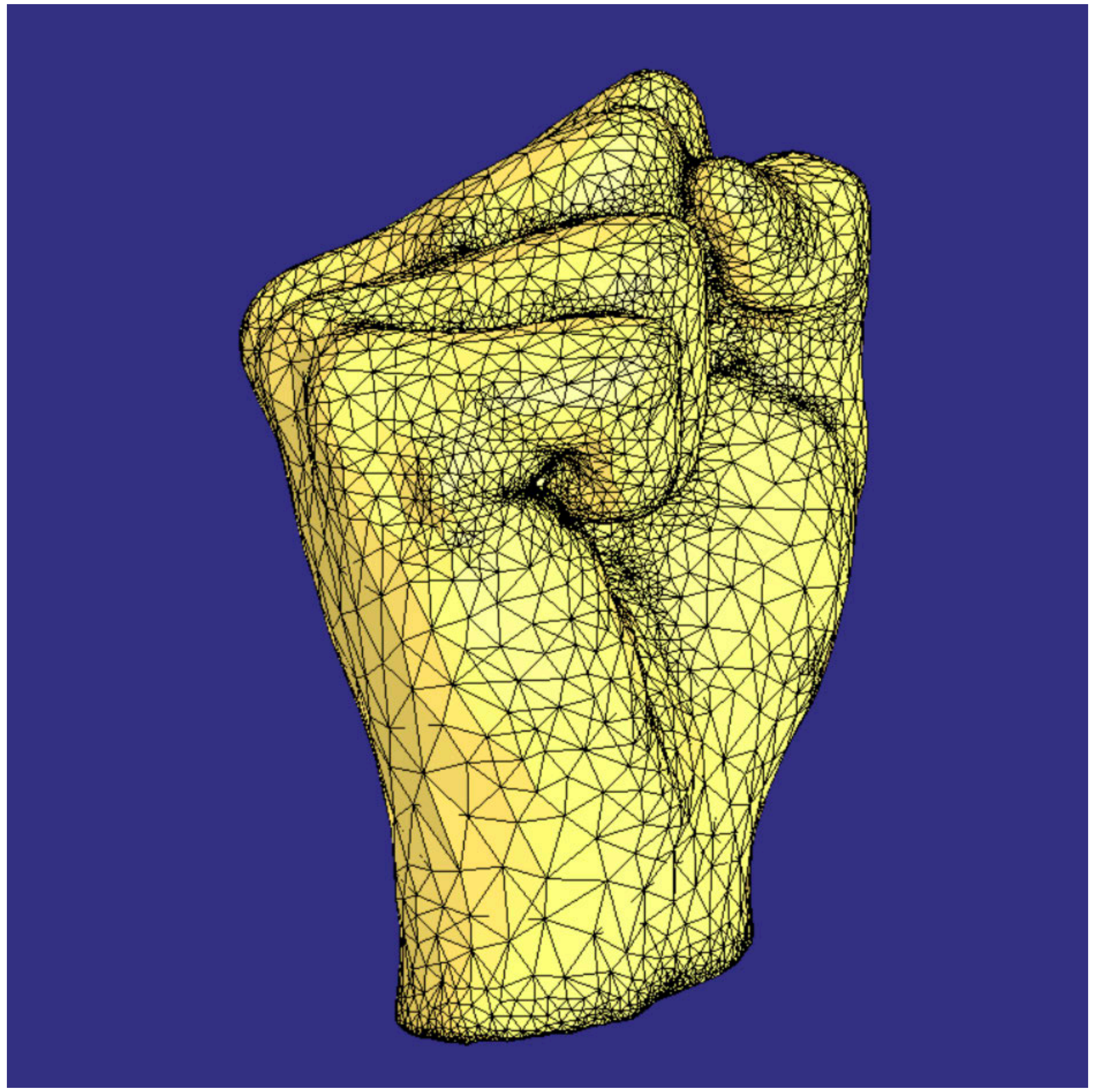




Medial Axis & Surface Reconstruction

- Can use similar approach for **surface reconstruction** from points
  - connect *centers* of skinny cells that meet along "long" edges
- In 3D, gives surface reconstruction with guarantees on topology (w/ enough points)



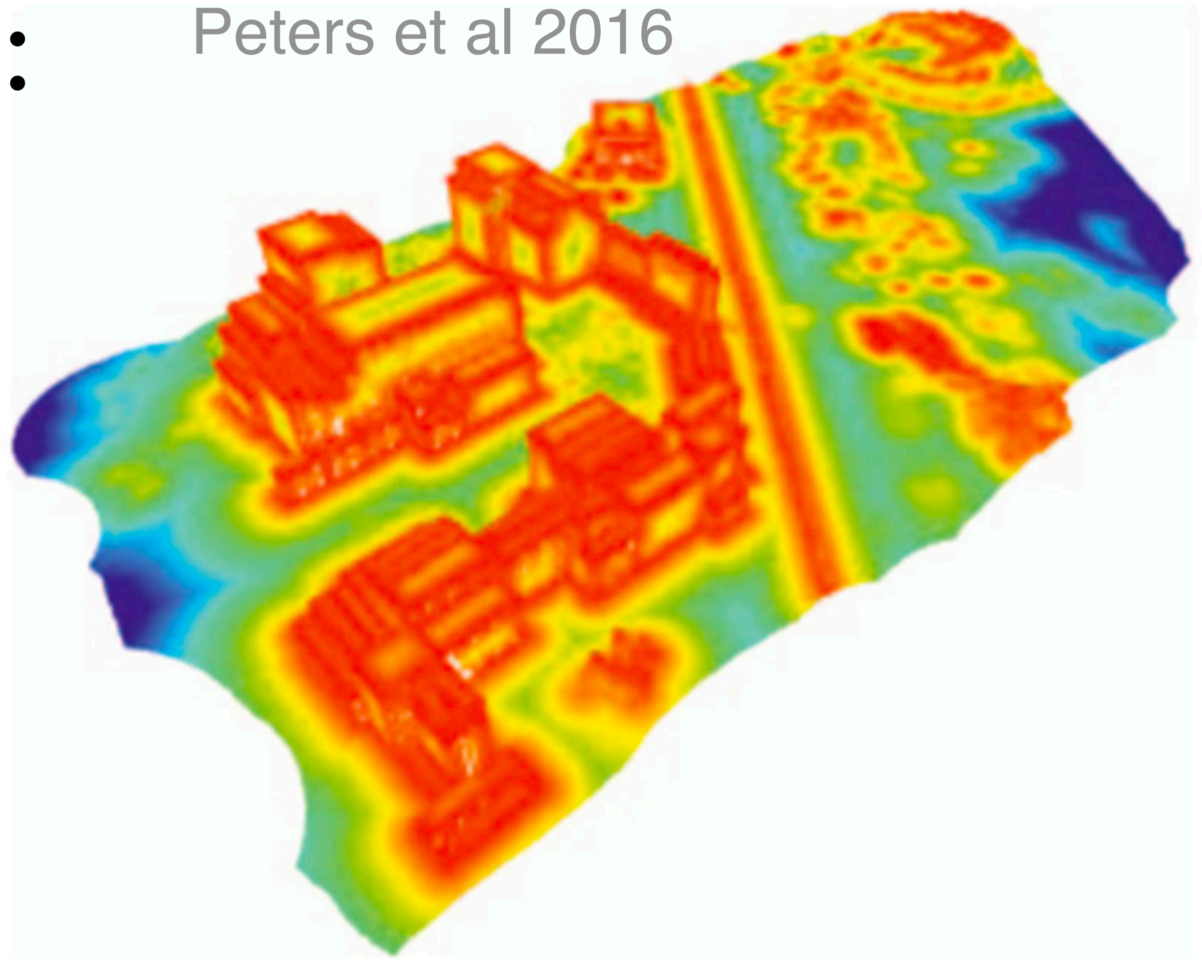


Amenta et al, "A New Voronoi-Based Surface Reconstruction Algorithm"

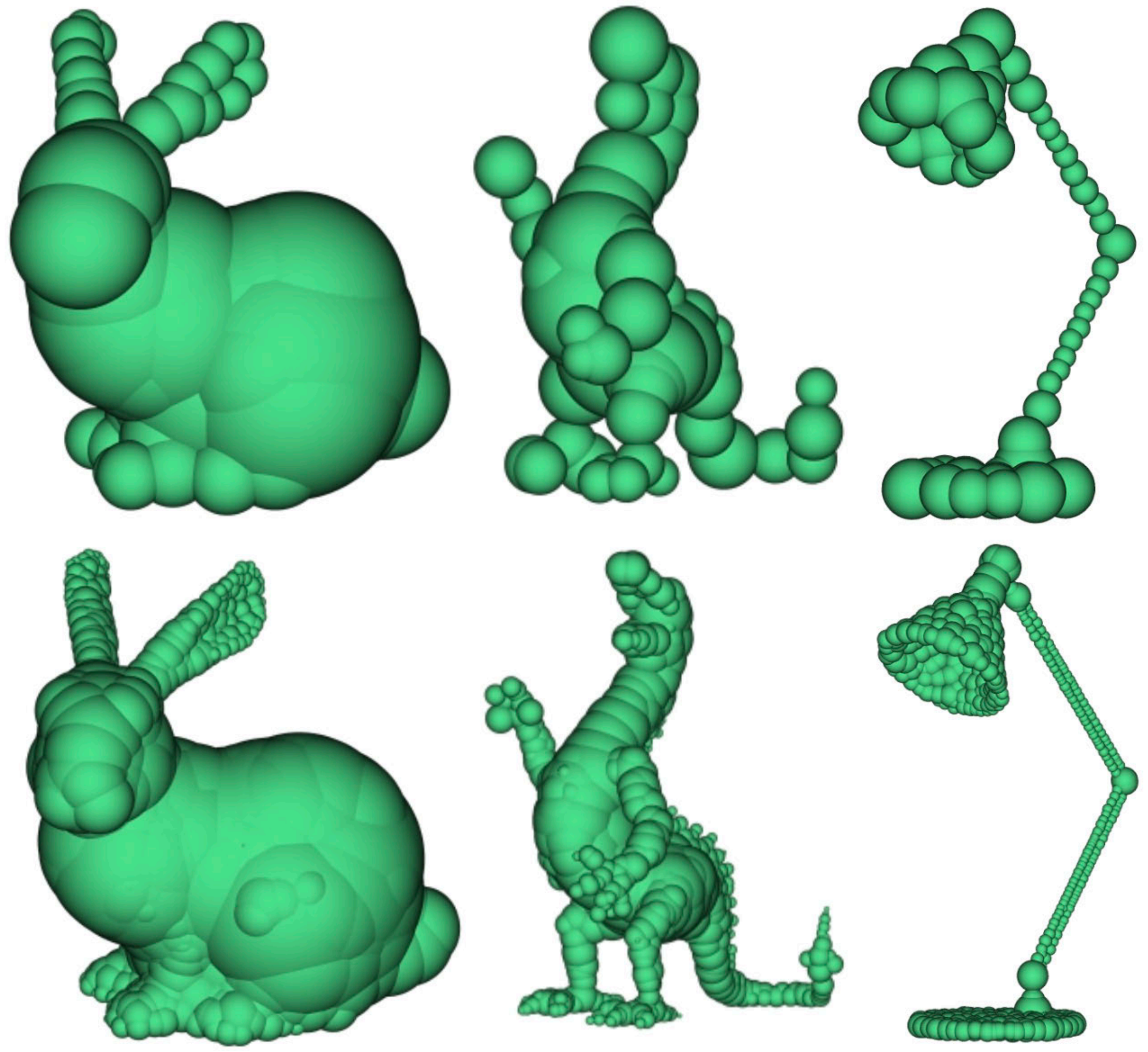
Medial Axis — Applications

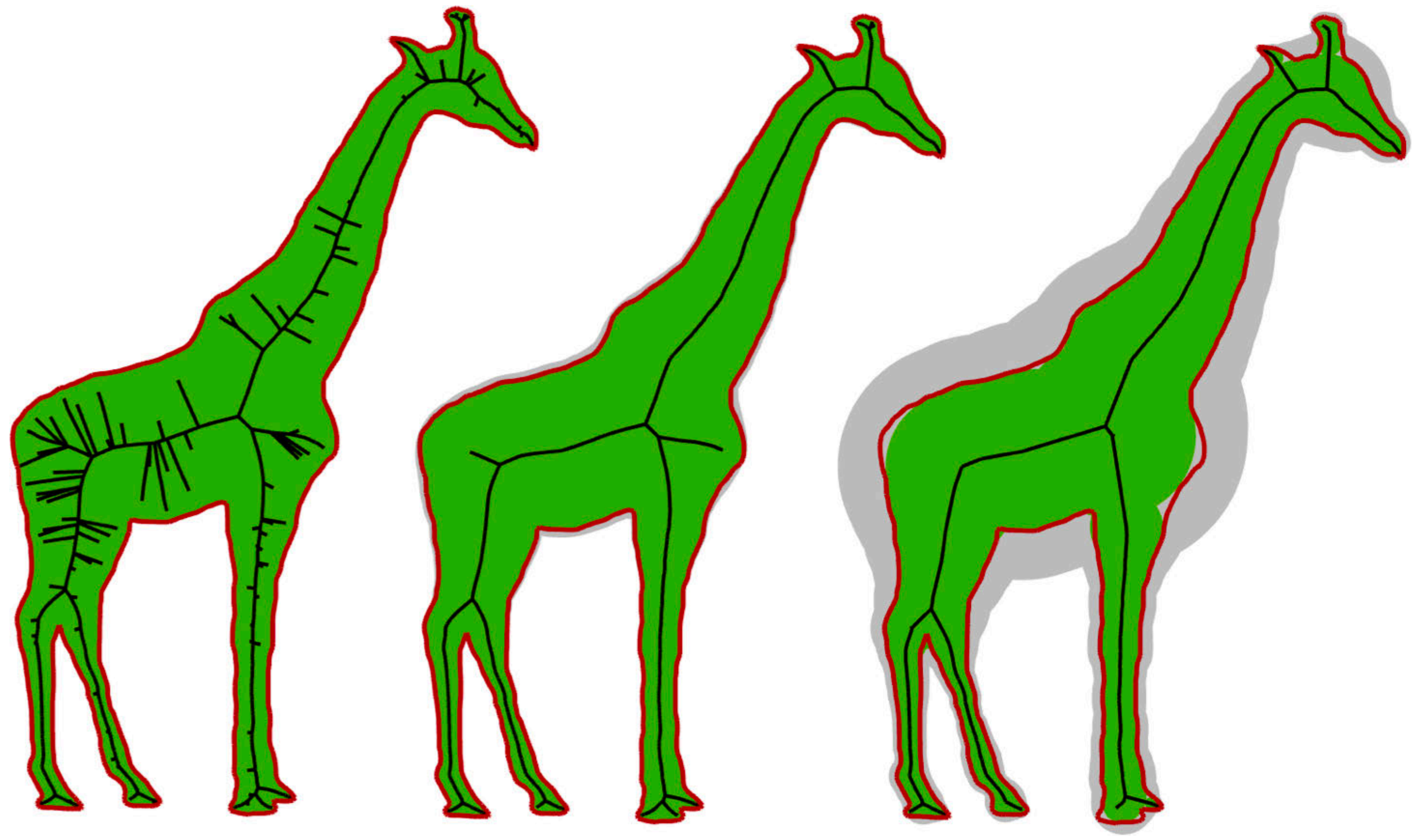
- Many applications of medial axis:
  - surface reconstruction
  - shape skeletons
  - local feature size
  - fast collision detection
  - fluid simulation
  - …

Peters et al 2016

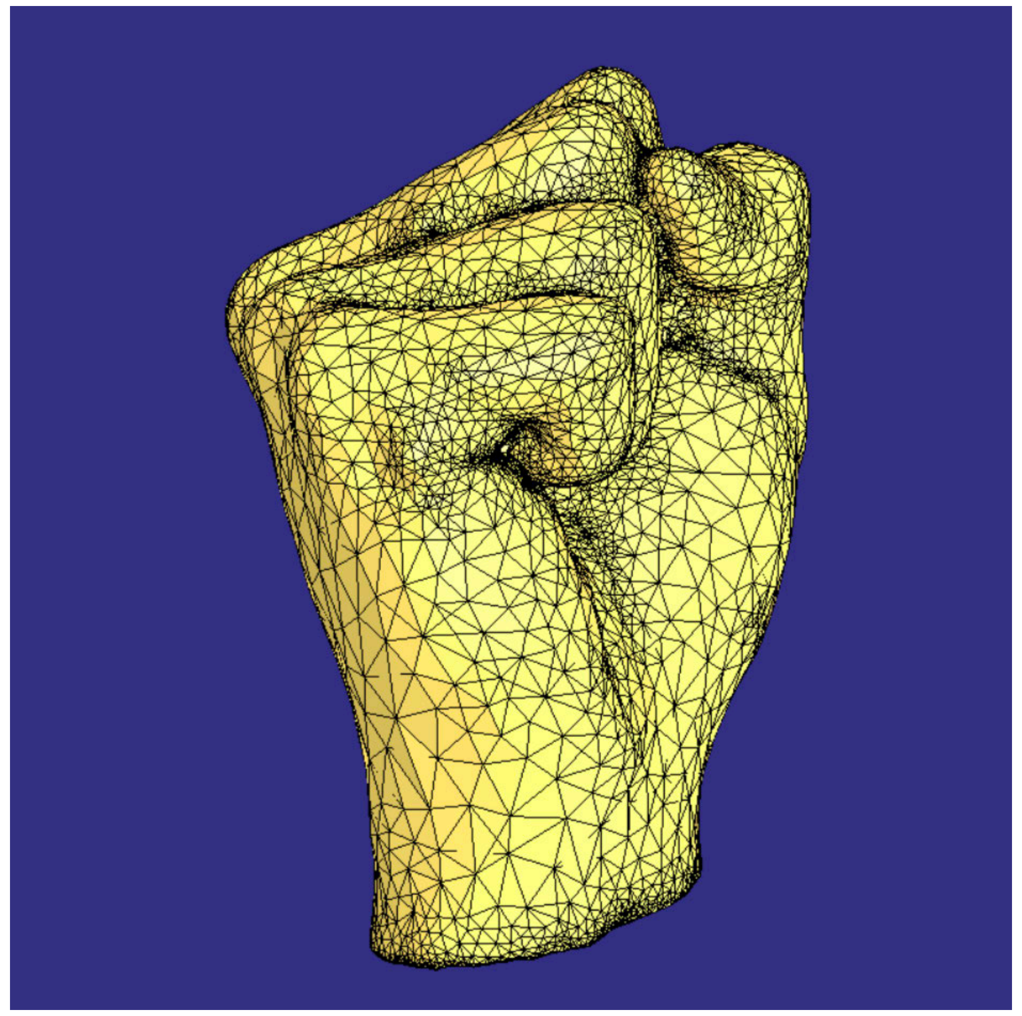


Bradshaw & Sullivan 2004

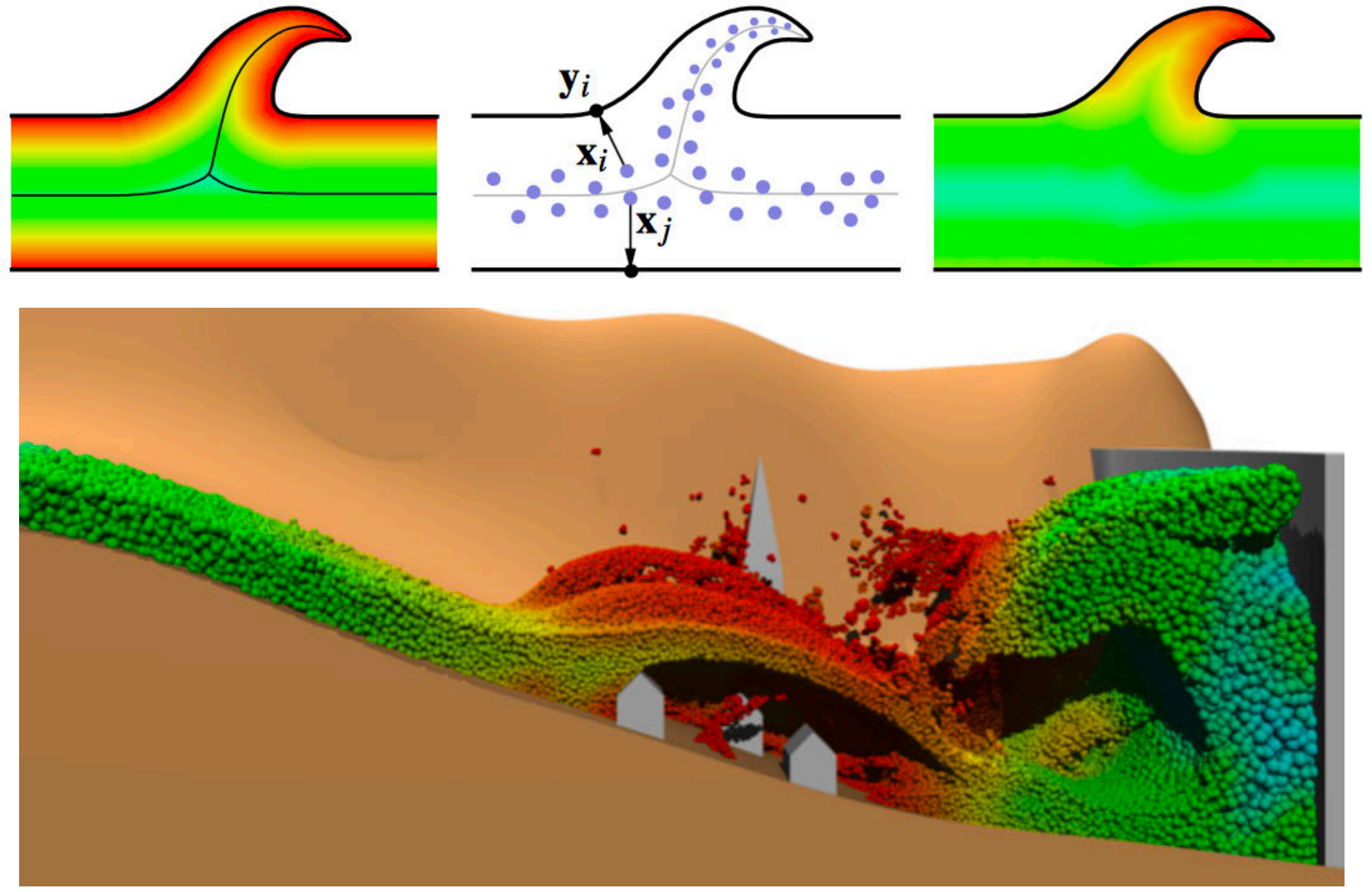




Giesen et al 2009

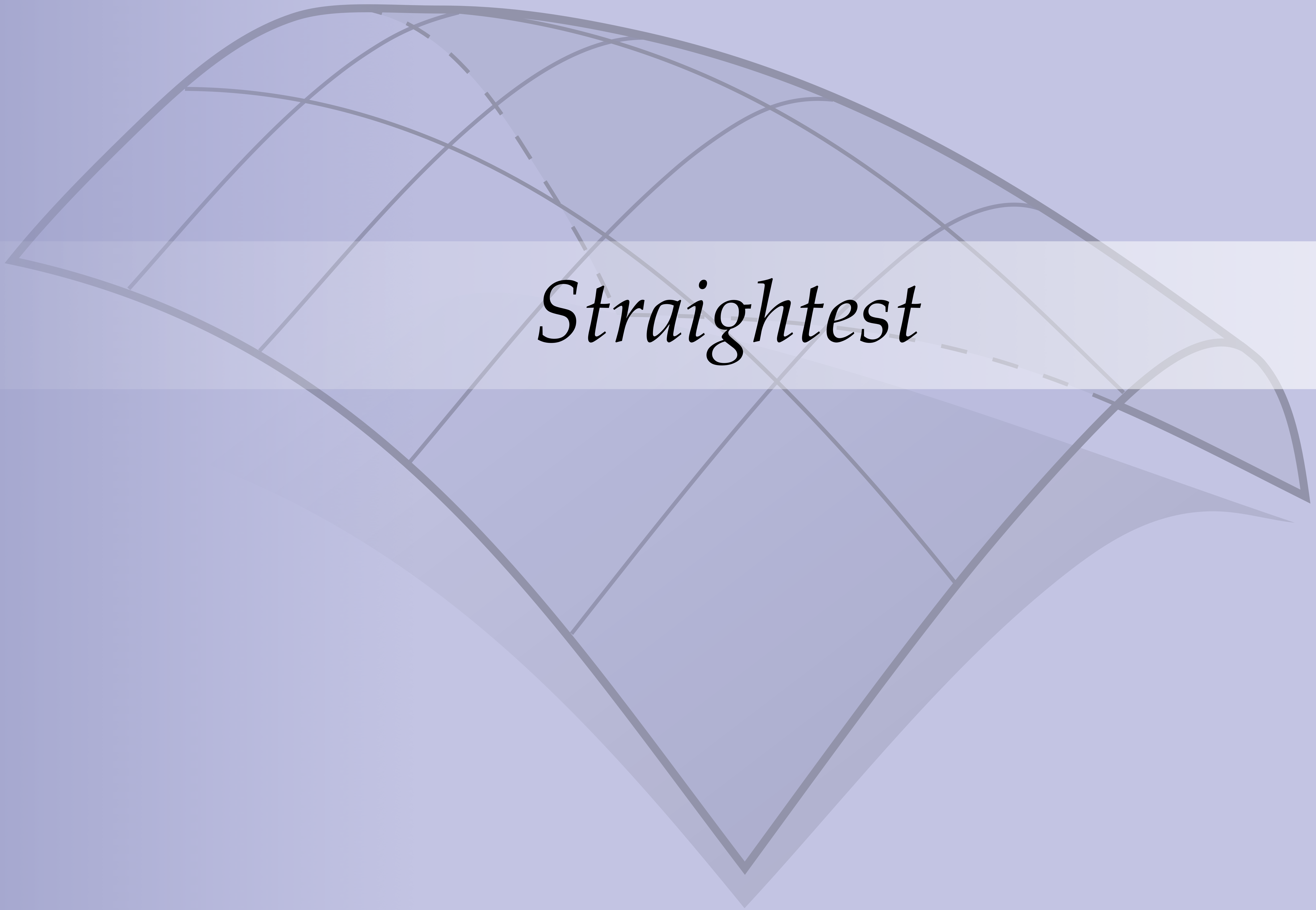


Amenta et al 1998



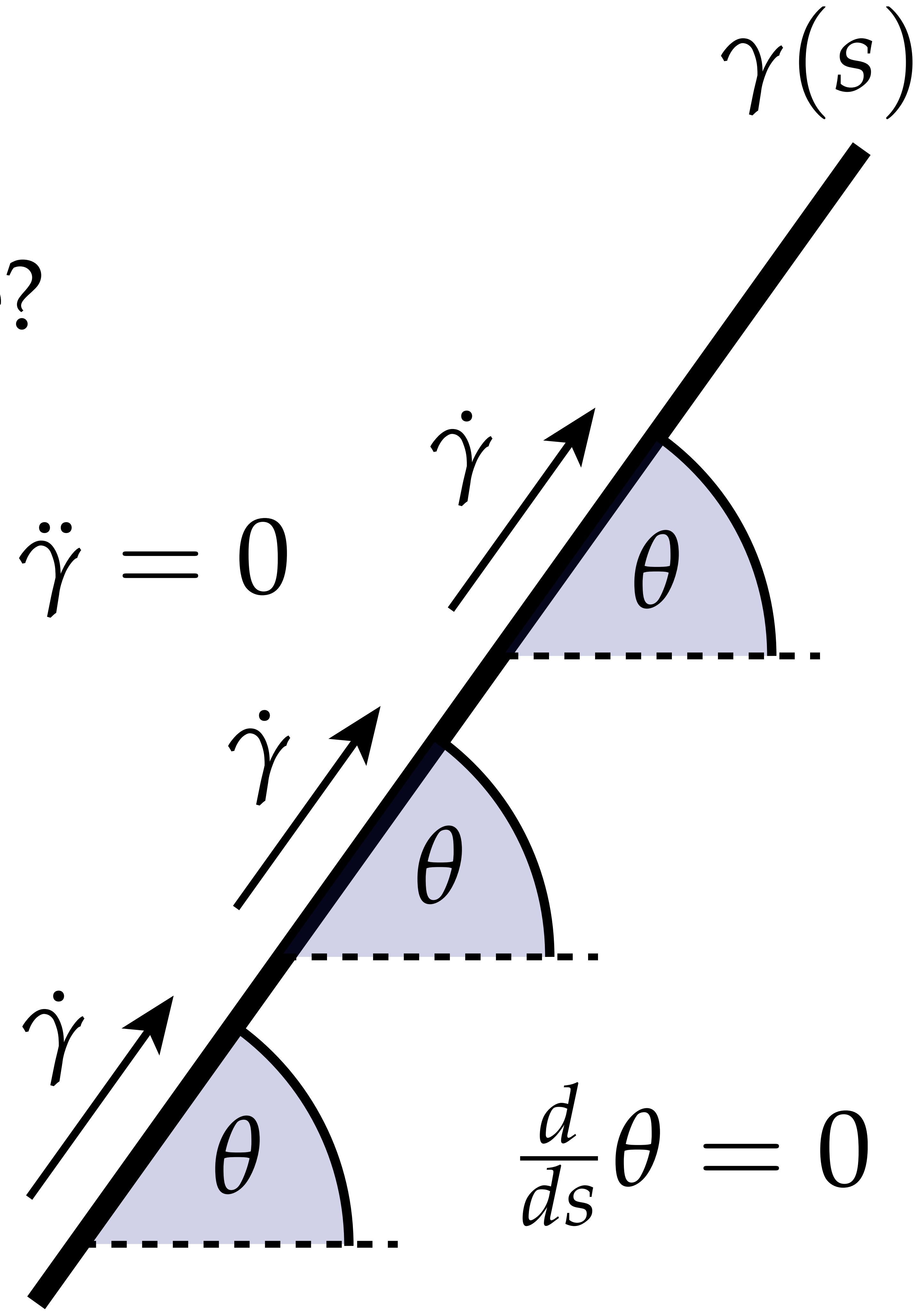
Adams et al 2007

Straightest



Straightest Paths

- A Euclidean line can be characterized as a curve that is "as straight as possible"
- **Q:** How can we make this statement more precise?
  - **geometrically**: no curvature
  - **dynamically**: no acceleration
- How can we generalize to curves in manifolds?
  - **geometrically**: no *geodesic curvature*
  - **dynamically**: zero *covariant derivative*



Straightness—Geometric Perspective

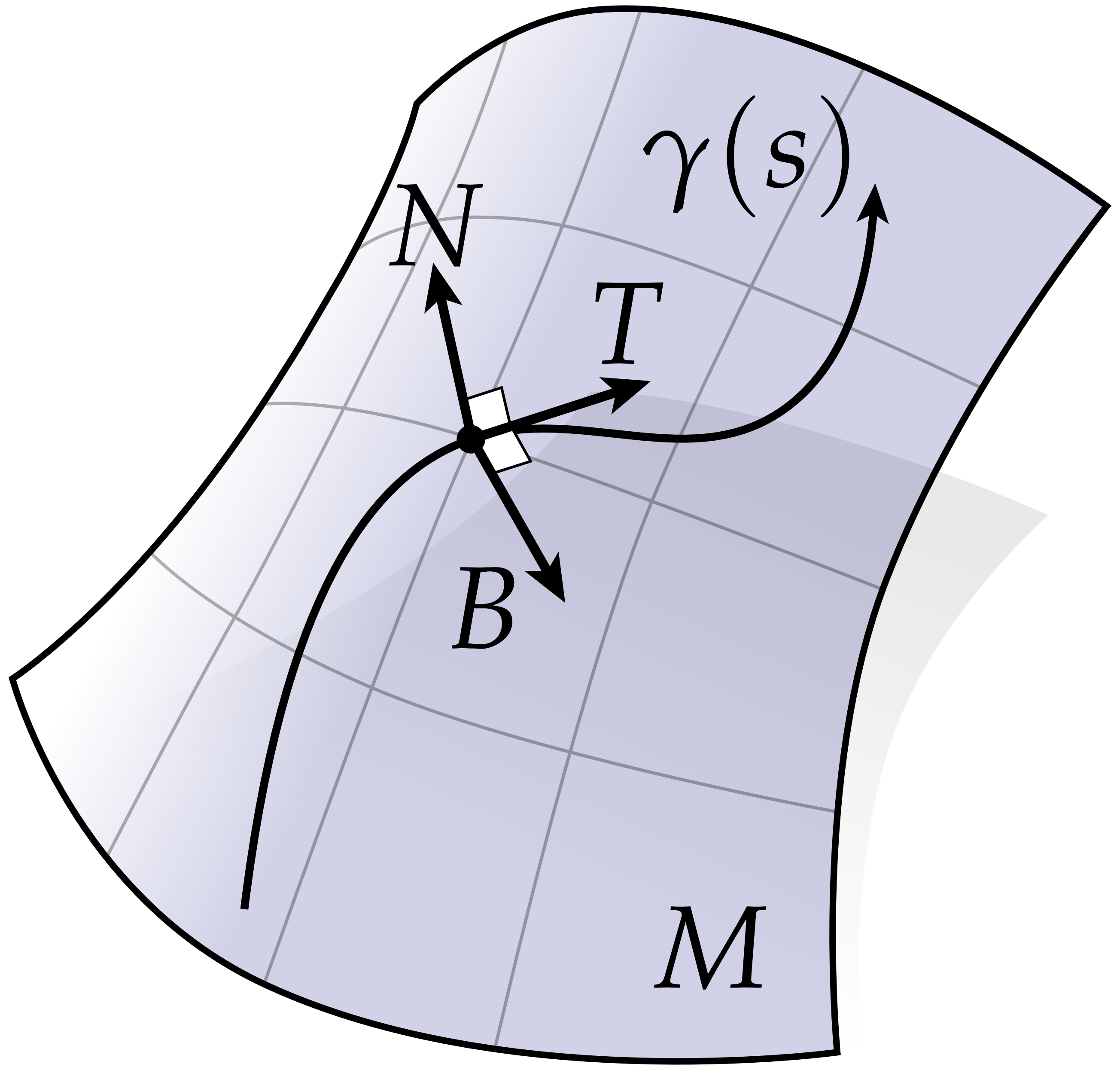
- Consider a curve $\gamma(s)$ with tangent T in a surface with normal N, and let $B := T \times N$.
- Can decompose "bending" into two pieces:

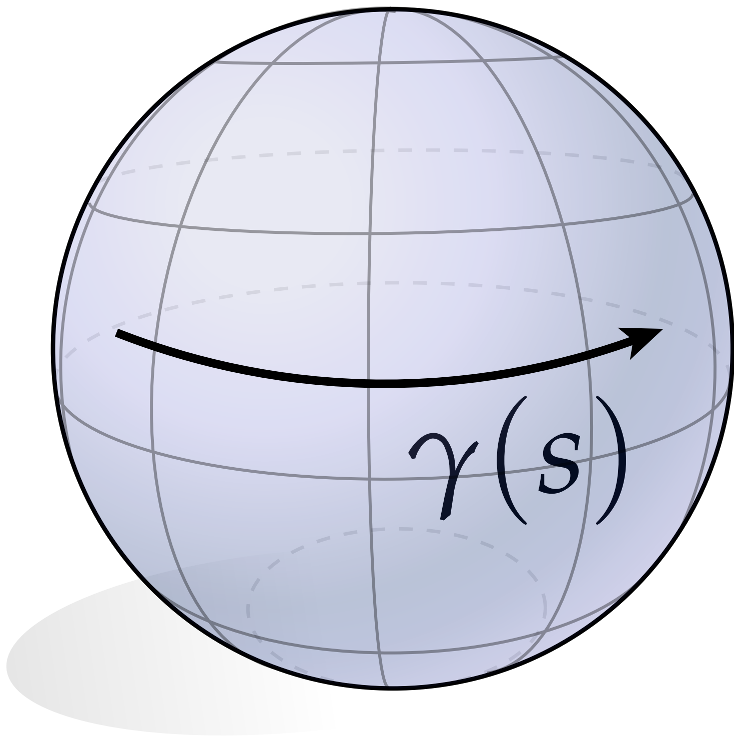
$$\kappa_n := \langle N, \tfrac{d}{ds}T \rangle \qquad \textbf{normal curvature}$$

$$\kappa_g := \langle B, \tfrac{d}{ds}T \rangle \qquad \textbf{geodesic curvature}$$

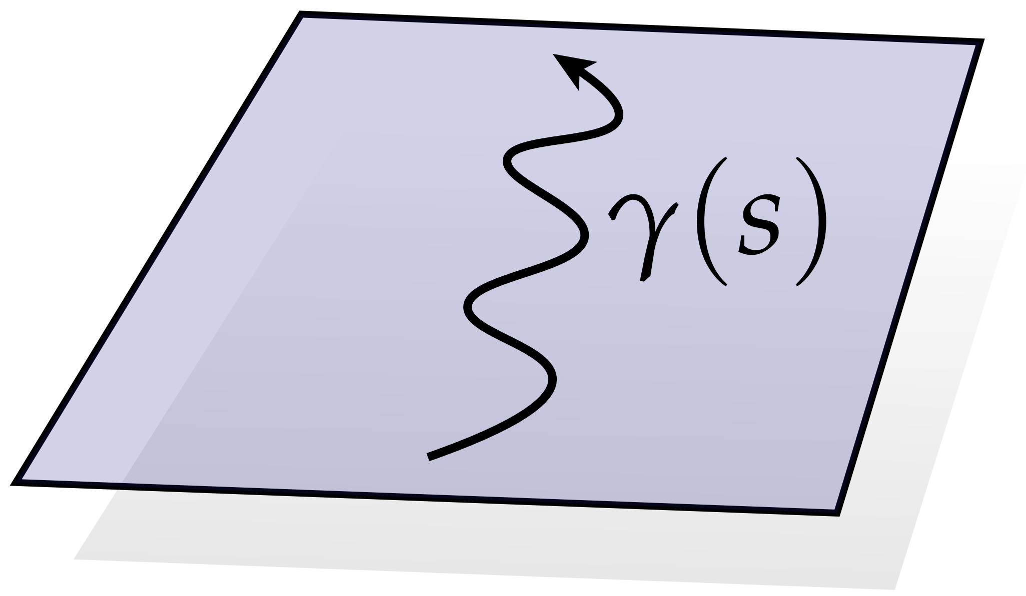
- Curve is "forced" to have normal curvature due to curvature of M
- Any additional bending beyond this minimal amount is geodesic curvature

Key idea: geodesic is curve where $\kappa_g = 0$





large κ_n;
small κ_g

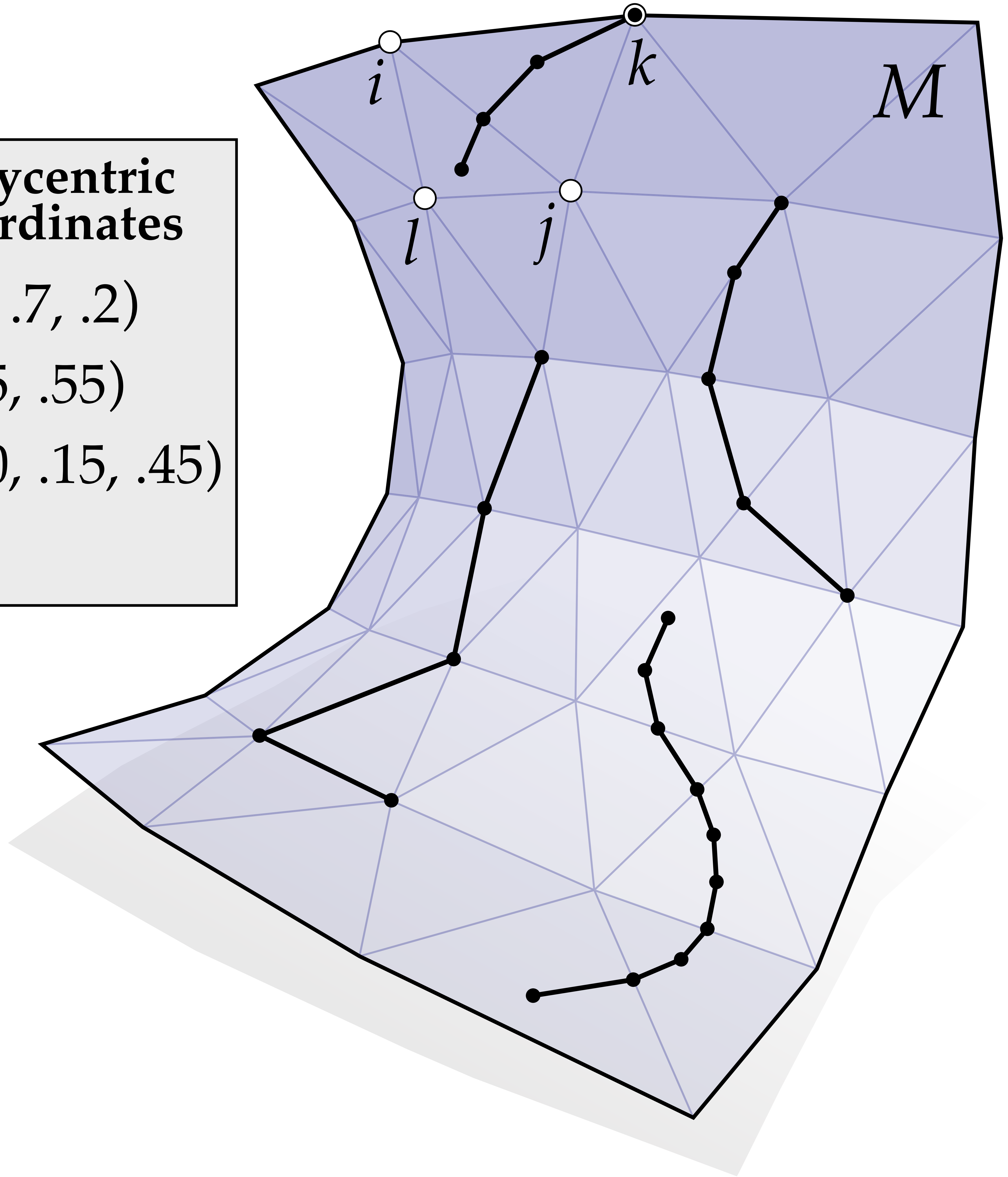


large κ_g;
small κ_n

Discrete Curves on Discrete Surfaces

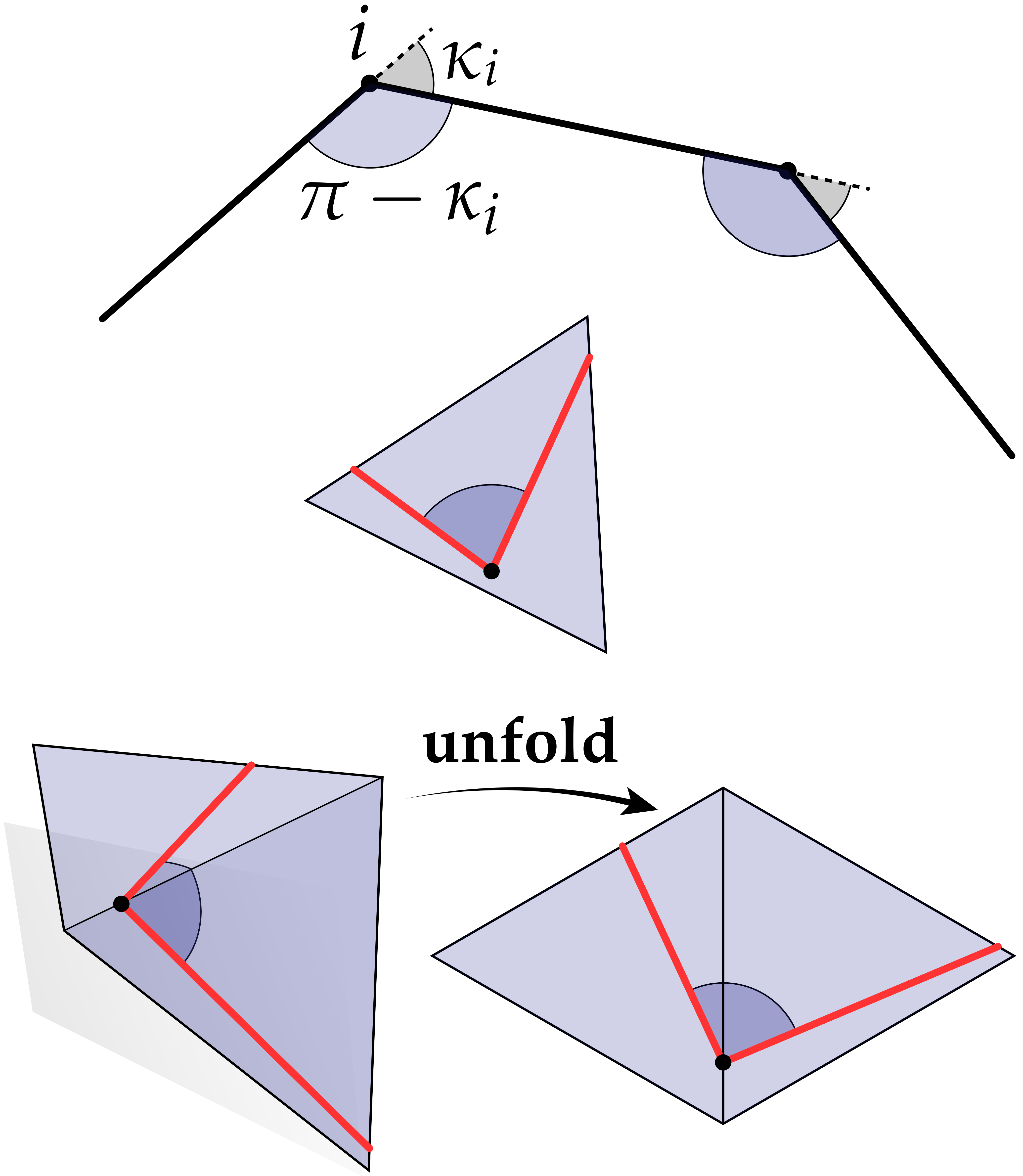
- To understand straightest curves on discrete surfaces, first have to define what we mean by a *discrete curve*
- One definition: a discrete curve in a simplicial surface M is any continuous curve γ that is piecewise linear in each simplex
- Doesn't have to be a path of edges: could pass through faces, have multiple vertices in one face, …
- Encode as sequence of simplices (not all same degree), and barycentric coordinates for each simplex

simplex	**barycentric coordinates**
ilj	(.1, .7, .2)
ij	(.45, .55)
ijk	(.40, .15, .45)
k	(1)



Discrete Geodesic Curvature

- For planar curve, one definition of discrete curvature was *turning angle* κ_i
- Since most points of a simplicial surface are *intrinsically* flat, can adopt this same definition for discrete geodesic curvature
- *Faces*: just measure angle between segments
- *Edges*: “unfold” and measure angle
- *Vertices*: not as simple—can’t unfold!
  - Recall trouble w/ **shortest** geodesics…



Discrete Straightest Geodesics

- In the smooth setting, characterized geodesics as curves with zero geodesic curvature
- In the discrete setting, have a hard time at vertices: can't unfold, no *shortest* paths through some vertices…
- Alternative smooth characterization: just have <u>same</u> angle on either side of the curve
- Translates naturally to the discrete setting: equal angle sum on either side of the curve
- Provides definition of discrete **straightest** geodesics (Polthier & Schmies 1998)

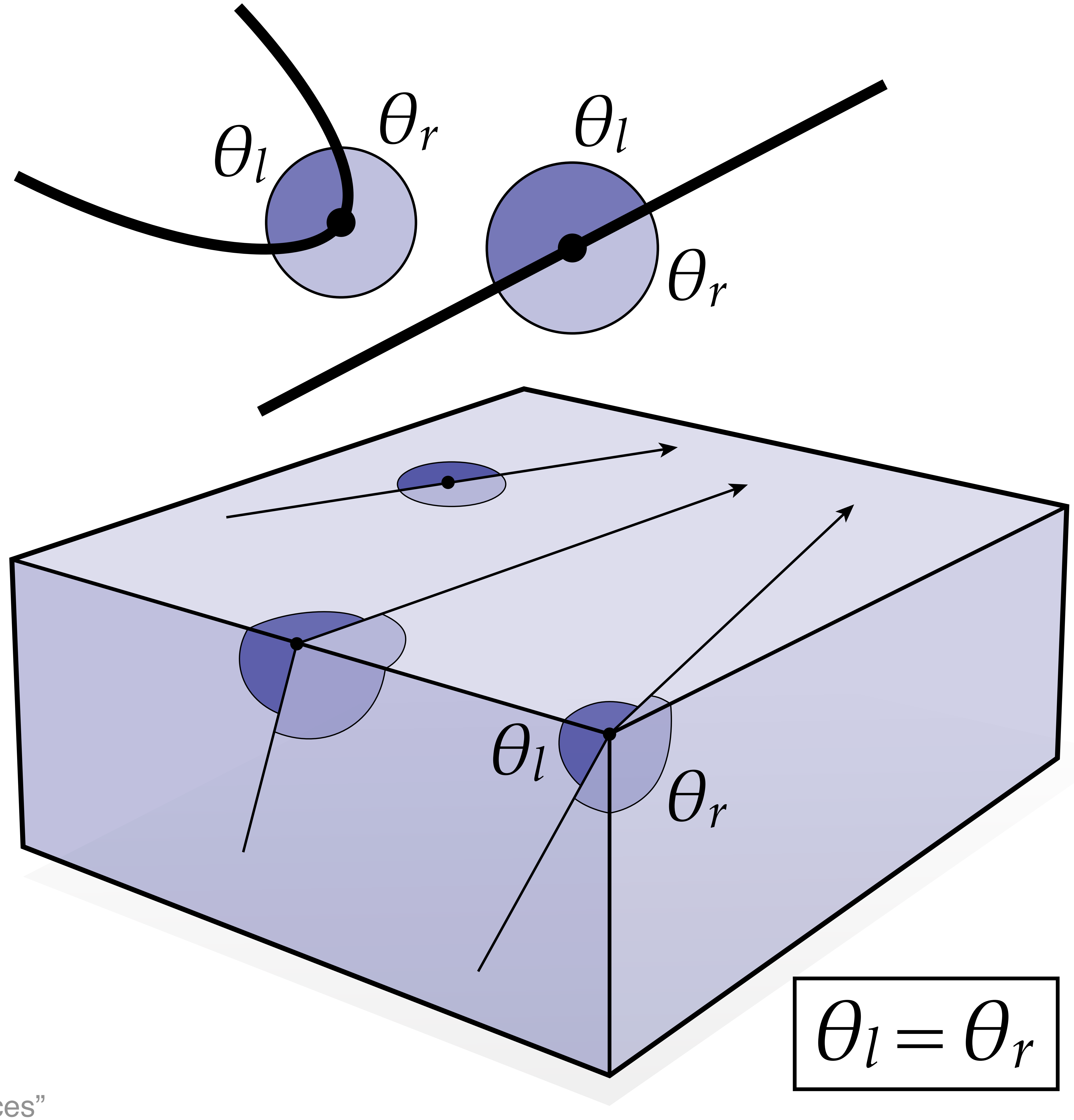
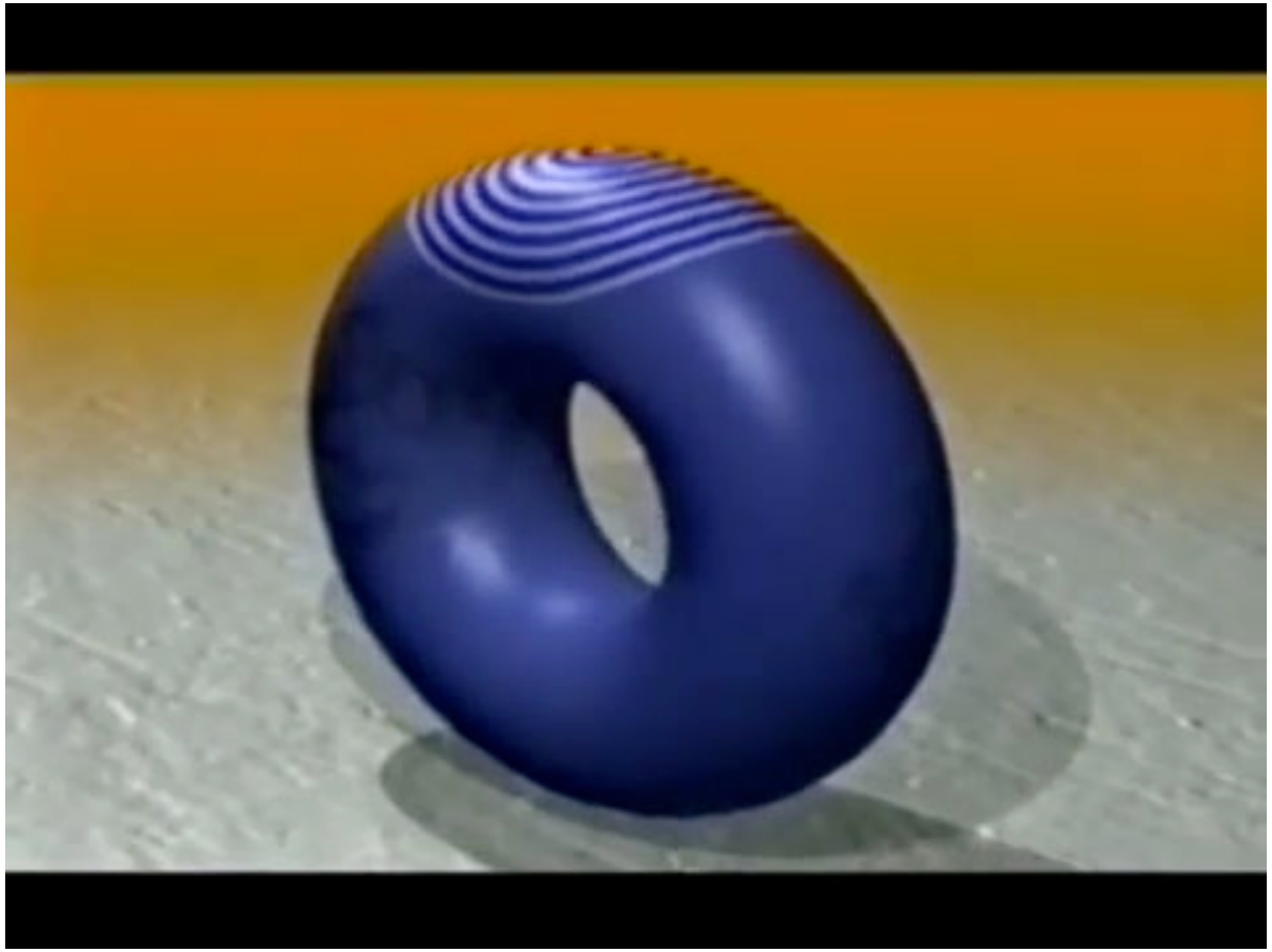
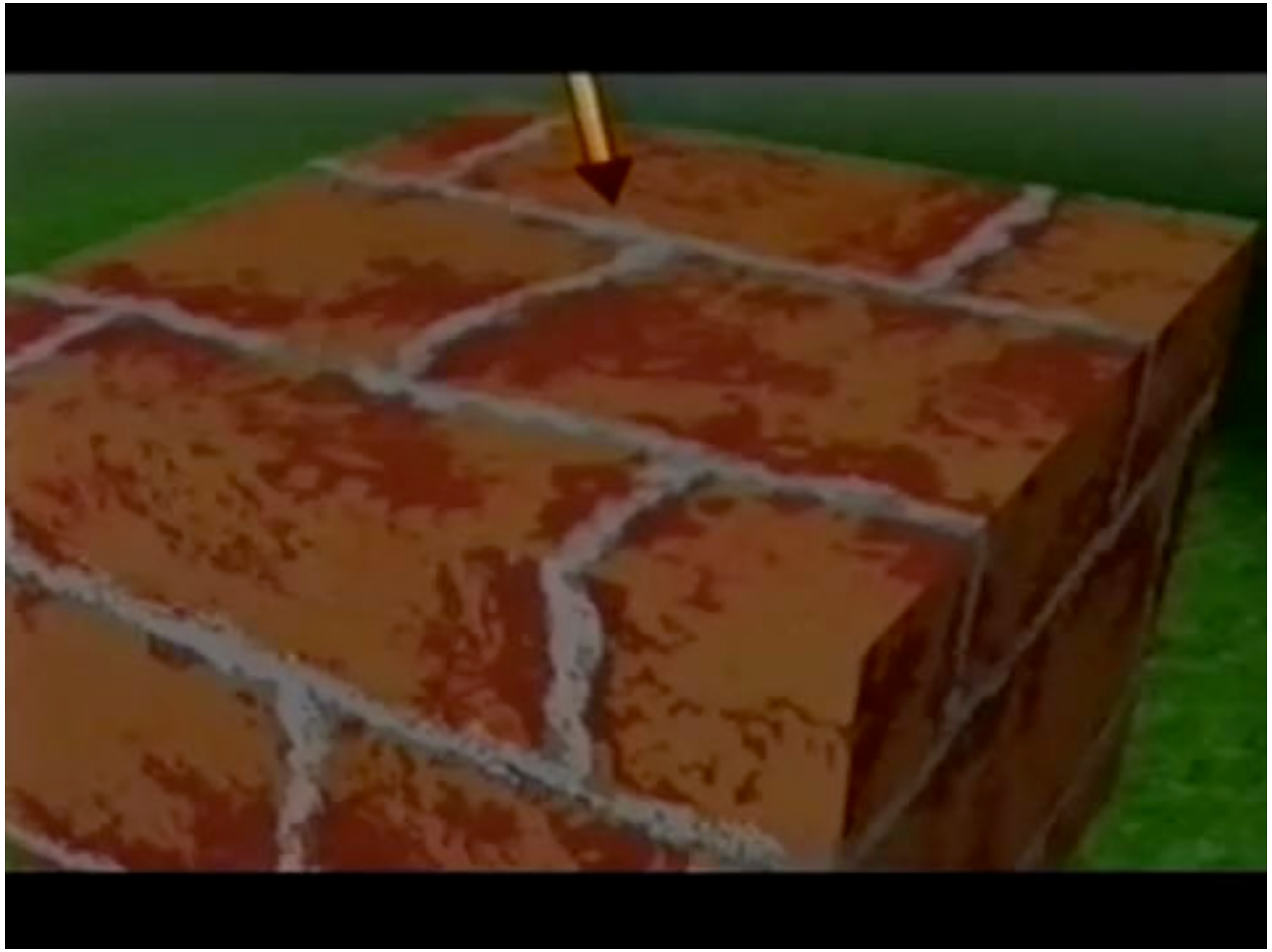


image adapted from Radeschi, Polthier & Schmies, "Straightest Geodesics on Polyhedral Surfaces"

Geodesics and Waves

Might seem that geodesics are “unlikely” to pass exactly through a vertex, but consider simulating a continuous wavefront—how should it behave when it hits a vertex?





video from Polthier, Schmies, Steffens & Teitzel, “Geodesics and Waves” (1997)

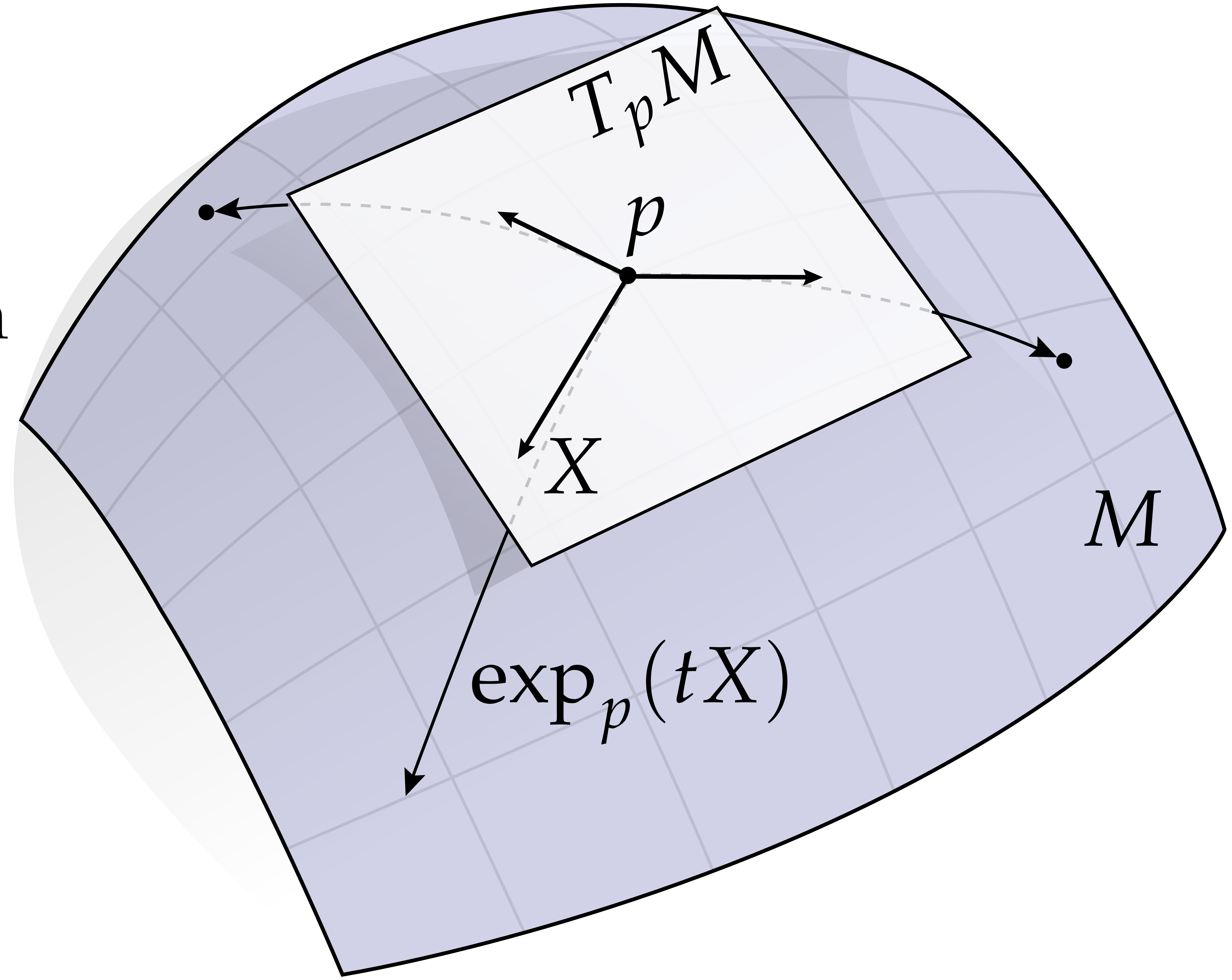
Exponential Map

- At a point p of a smooth surface M, the *exponential map* $\exp_p\colon T_pM \to M$ takes a tangent vector X to the point reached by walking along a geodesic in the direction $X/|X|$ for distance $|X|$

exponential map at p — tangent vectors — points

$$\exp_p\colon\ T_pM \to M$$

- Can also imagine that exp "wraps" the tangent plane around the surface

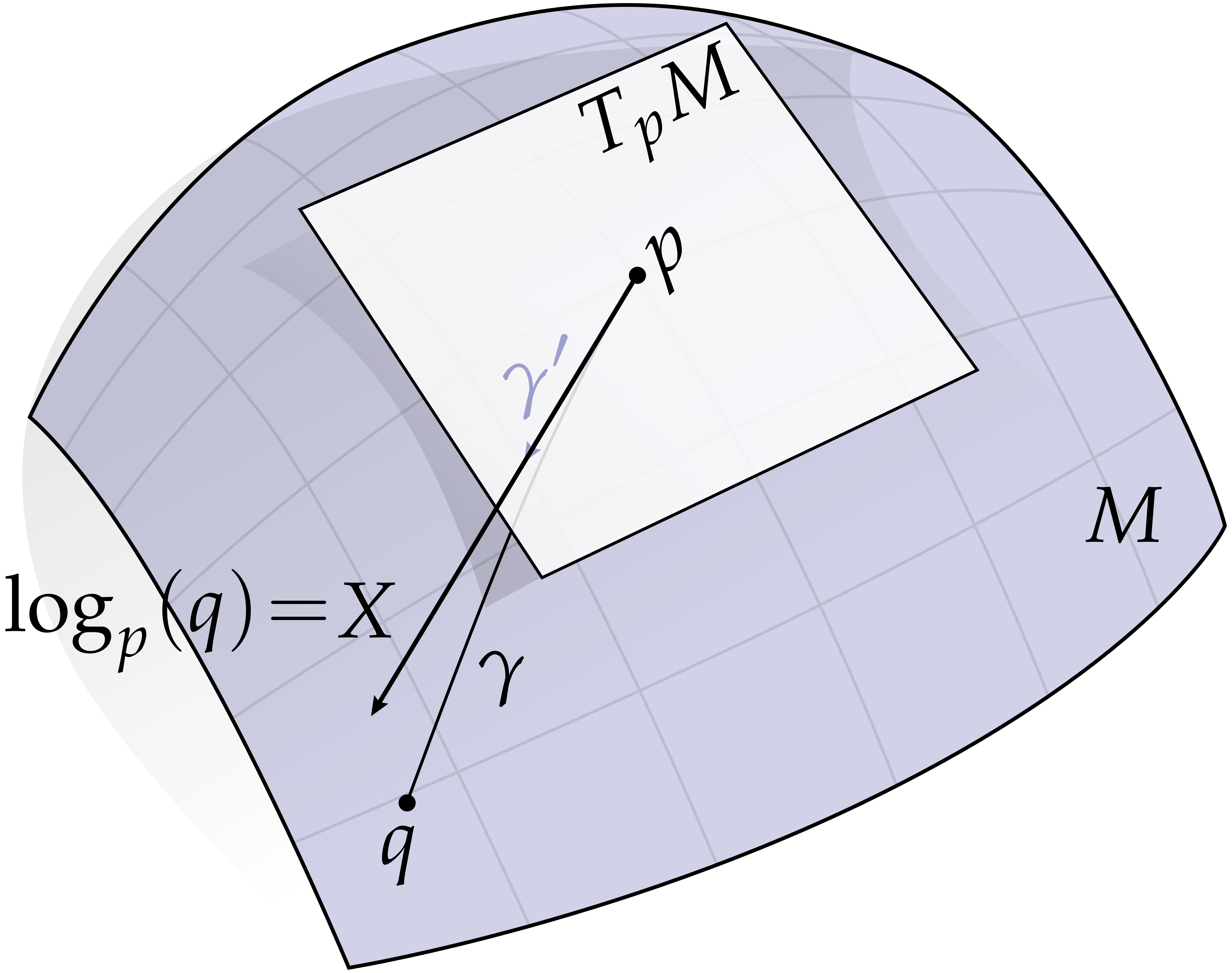


Key idea: provides notion of "translation" for curved domains

Logarithmic Map

- **Q:** Is the exponential map *surjective? I.e.*, can we reach every point q from p?
- **A:** Yes (Hopf-Rinow): Consider a smooth surface M without boundary. Then
  - find the shortest geodesic γ from p to q
  - let X be a vector in direction γ' w/ length $|\gamma|$
  - then by construction, $\exp_p(X) = q$
  - Can also write $\log_p(q) = X$
- Map from q to X is called the *log map*

Q: Is the log map uniquely determined?

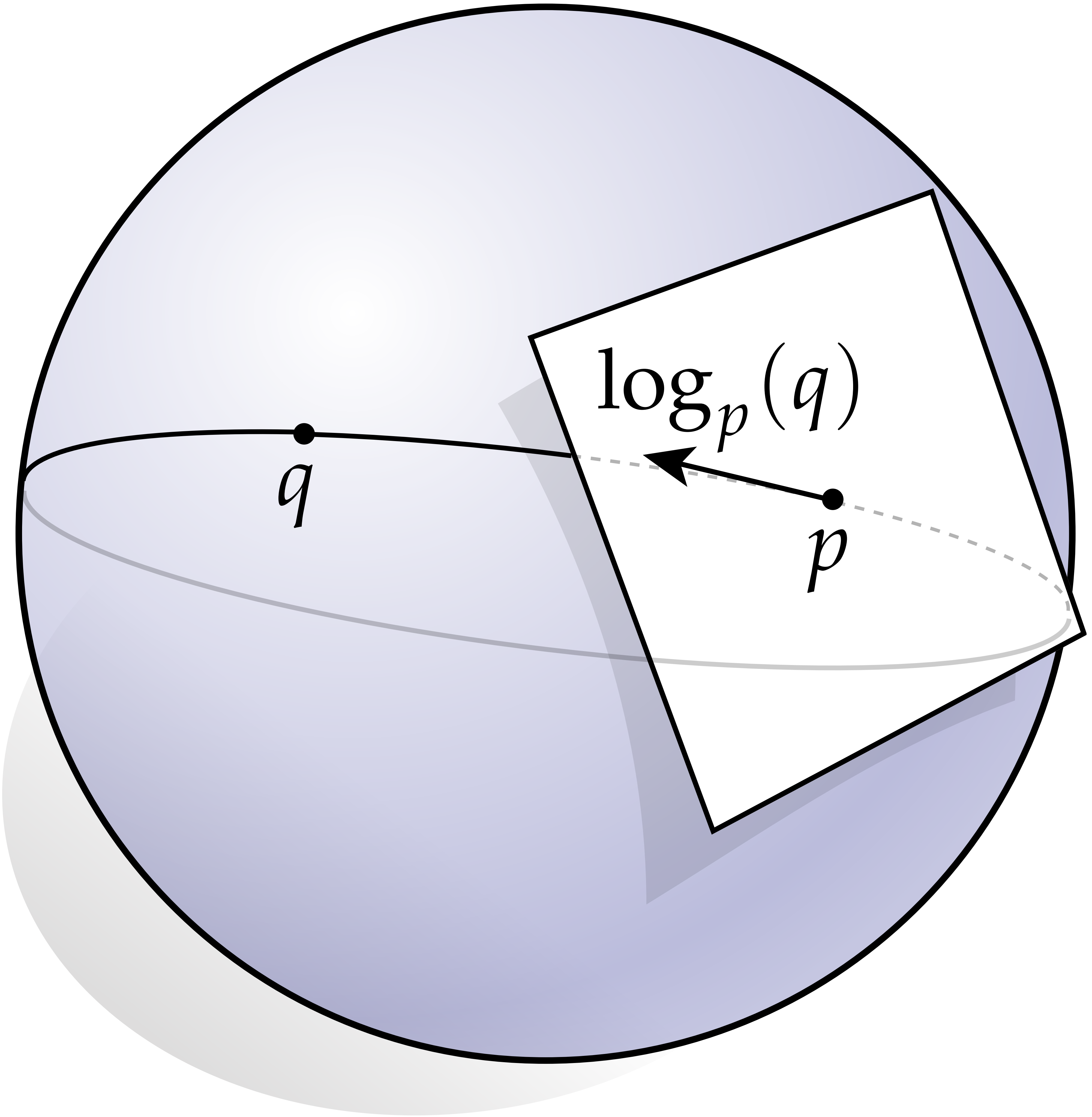


log map at p points   tangent vectors

$$\log_p : M \to T_pM$$

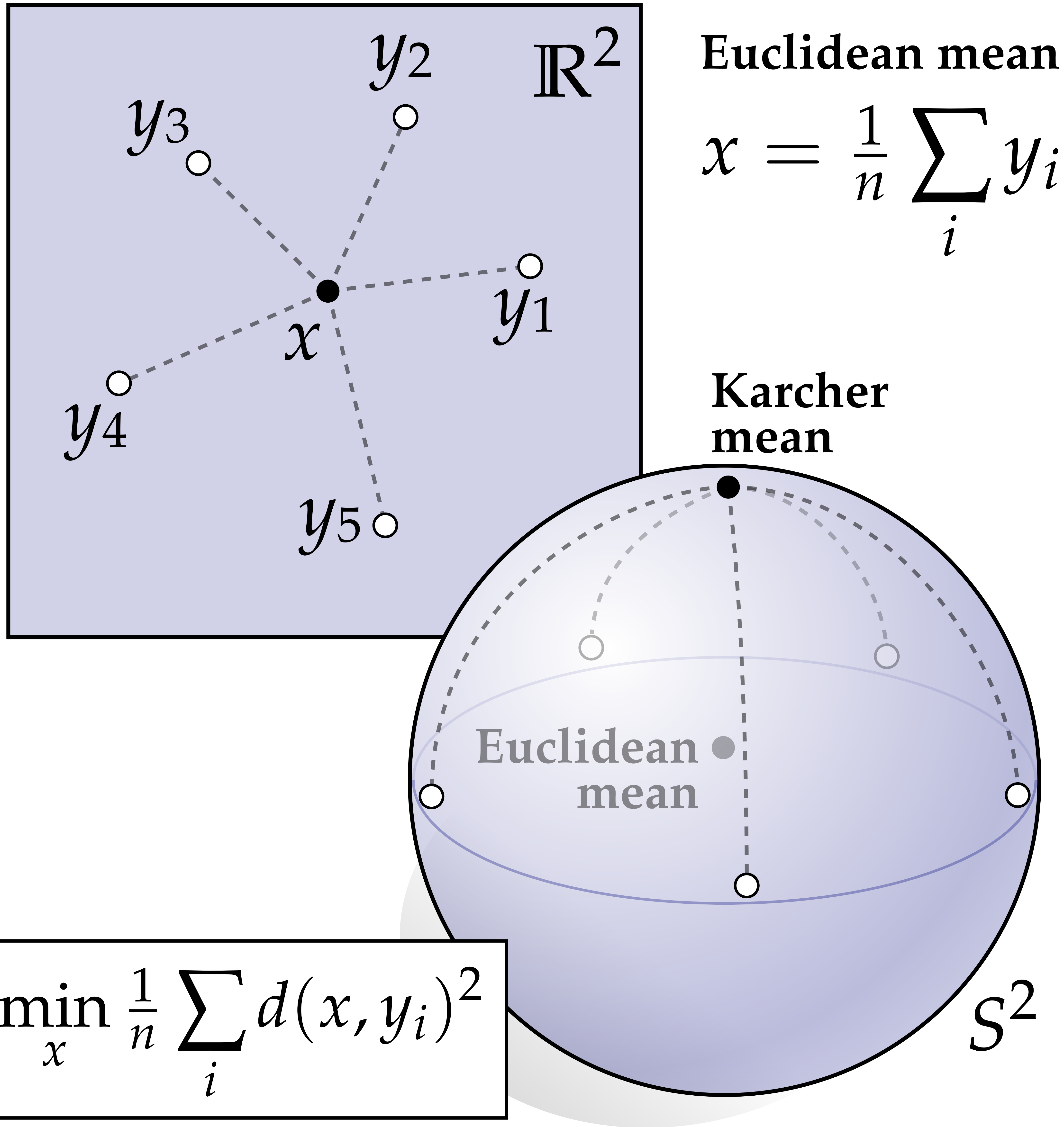
Exponential Map—Injectivity

- Equivalently, is the exponential map always *injective?* (*I.e.*, is there a <u>unique</u> geodesic that takes us from p to q?)
- No! Consider the exponential map on the sphere…
- By convention, log map therefore gives the *smallest* vector X such that $\exp_p(X) = q$
- **Q:** Why are exp/log map <u>useful</u>?
- **A:** Allows us to *locally* work with points on curved spaces as though they are just vectors in a flat space



Averages on Surfaces

- Average of points in the plane is easy: just add up coordinates, divide by number of points
- How do we talk about an average of points on a curved surface?
  - average of coordinates may no longer be on the surface
  - might not even know how surface is embedded into space…
- Motivates idea of *Karcher mean:*
  - average is point that minimizes sum of squared geodesic distances to all points
  - in the plane, agrees with usual notion of “average” in the plane (why?)



Euclidean mean

$$x = \frac{1}{n}\sum_i y_i$$

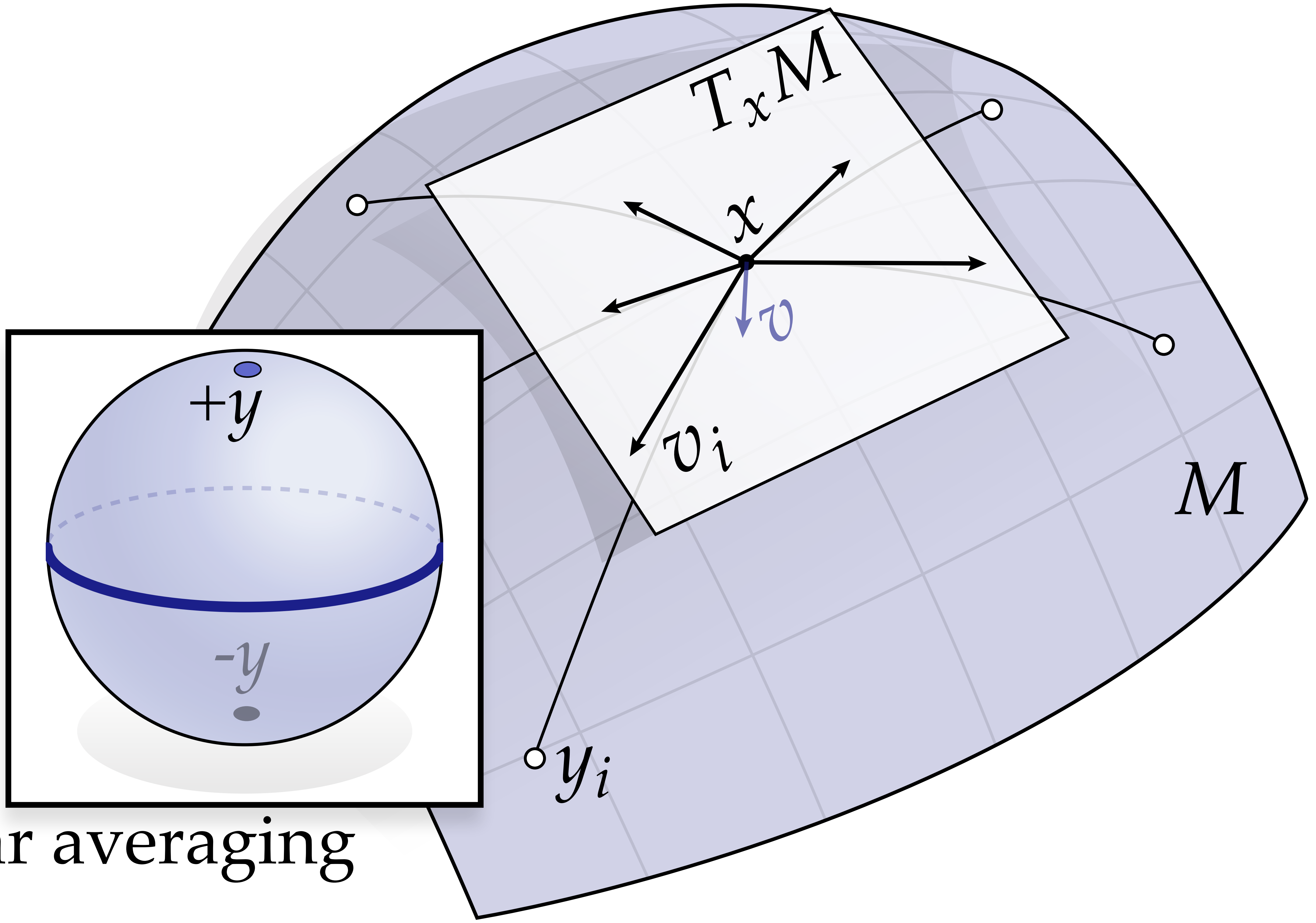
$$\min_x \frac{1}{n}\sum_i d(x, y_i)^2$$

Karcher Mean via Log Map

- Want to compute mean of points y_i
- Iterative algorithm:
  - pick a random initial starting point x
  - compute the log v_i of all points y_i
  - compute the mean v of all the vectors v_i
  - move x to $\exp_x(v)$ and repeat
- Will quickly converge to *some* Karcher mean
  - in general may not be unique—consider two points $y_1 = -y_2$ on the sphere
- Can also be used to average, *e.g.*, rotations

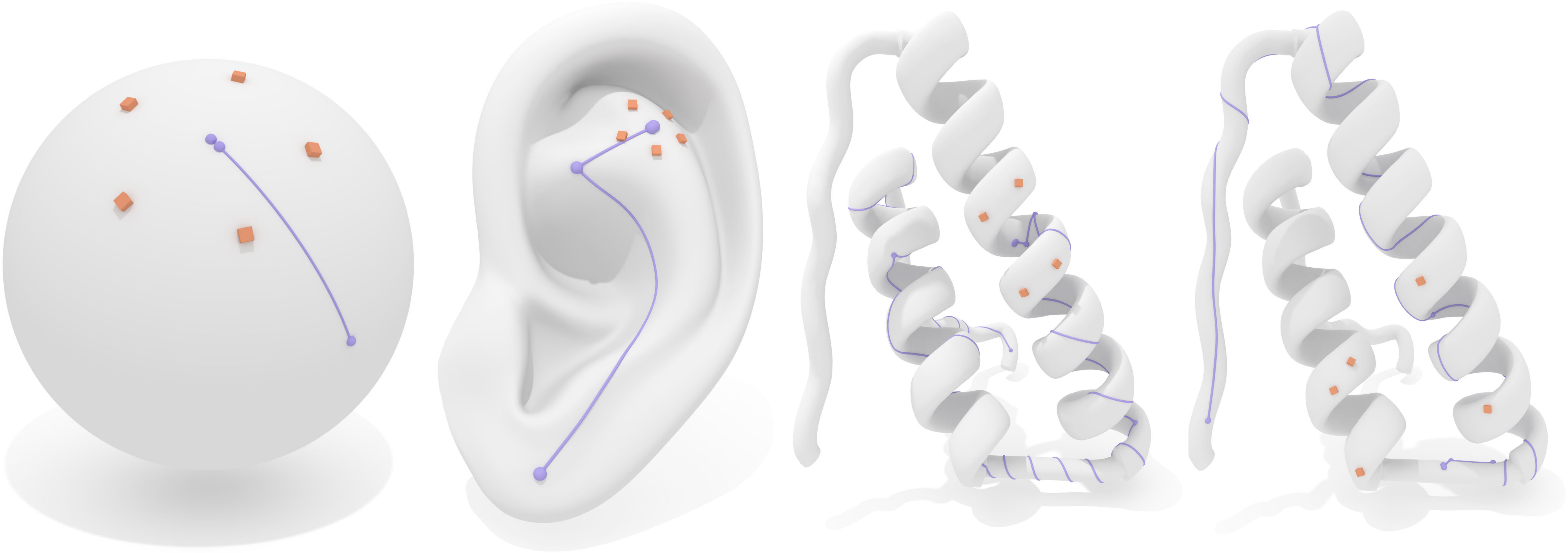
$$v \leftarrow \frac{1}{n}\sum_i \log_x(y_i)$$

$$x \leftarrow \exp_x(v)$$



Key idea: turn “curved averaging” into linear averaging

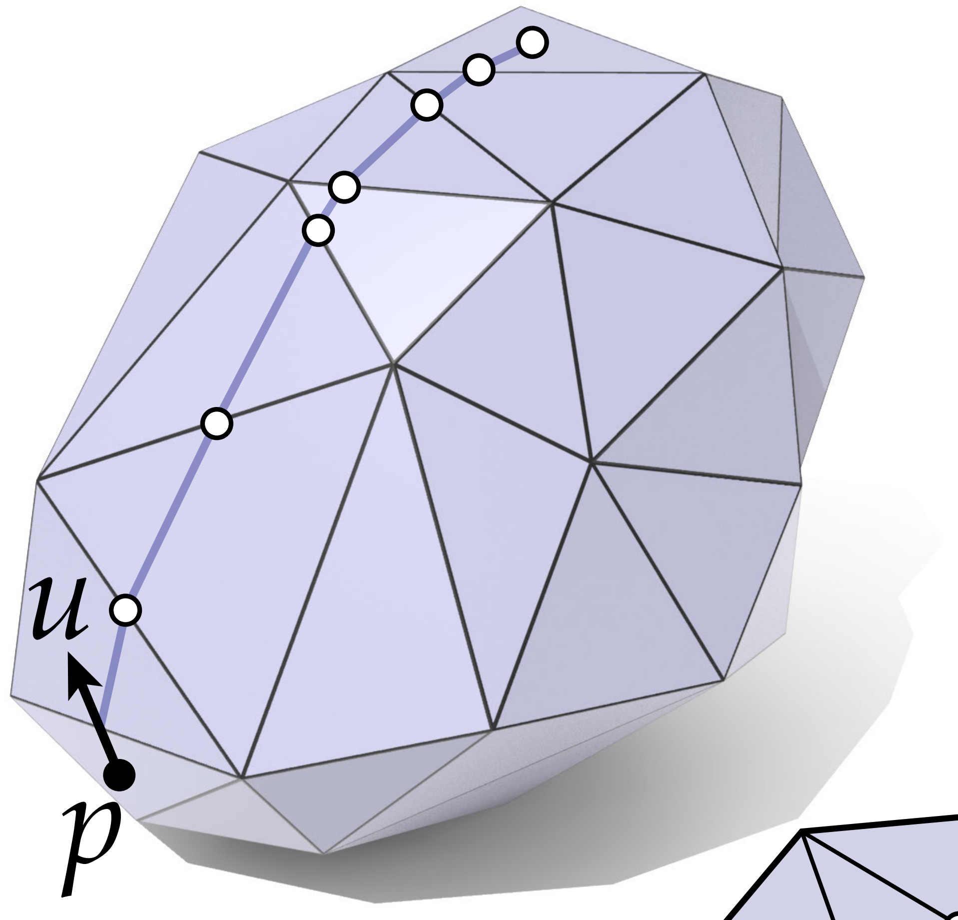
Karcher Mean — Examples

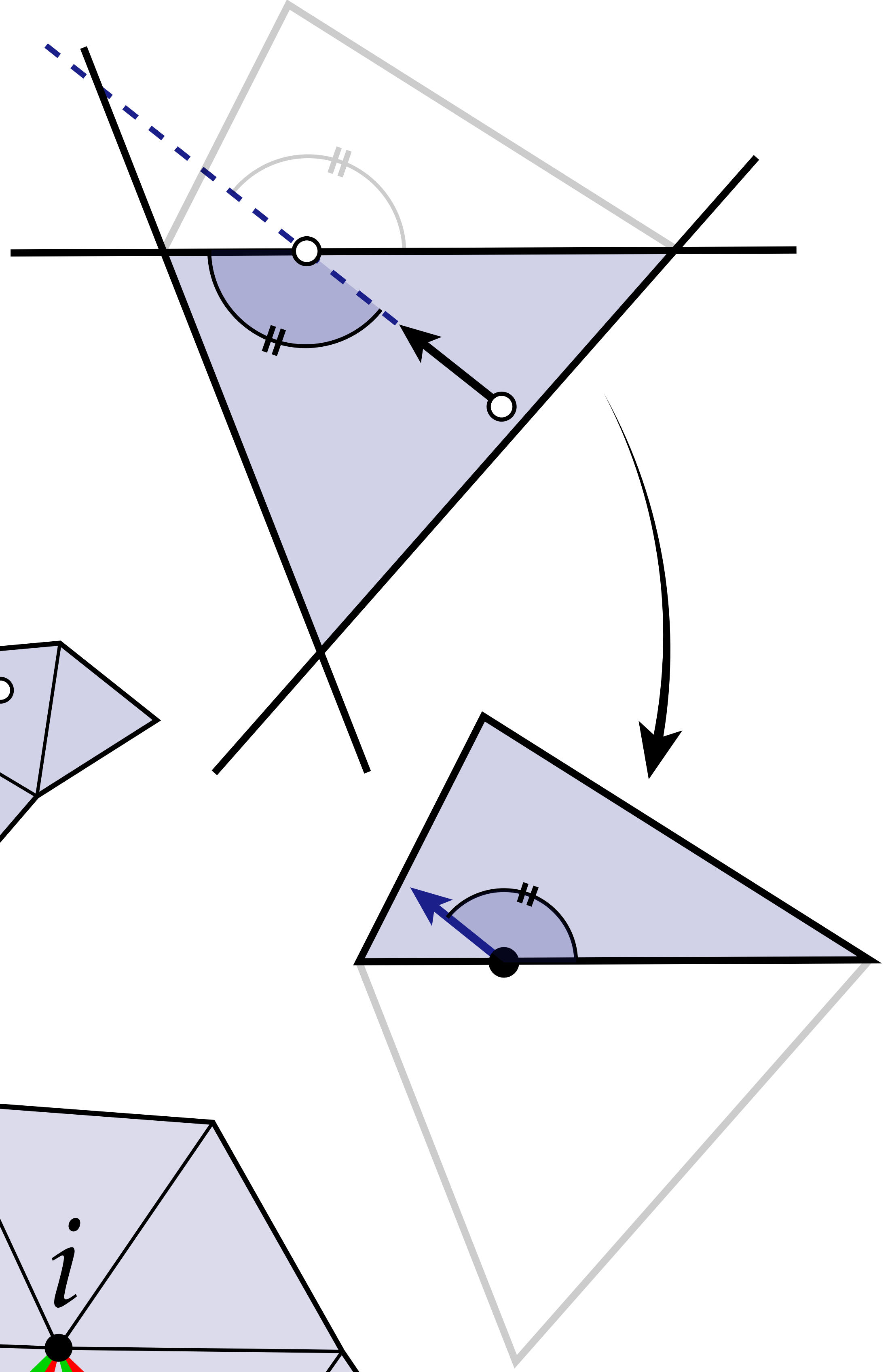


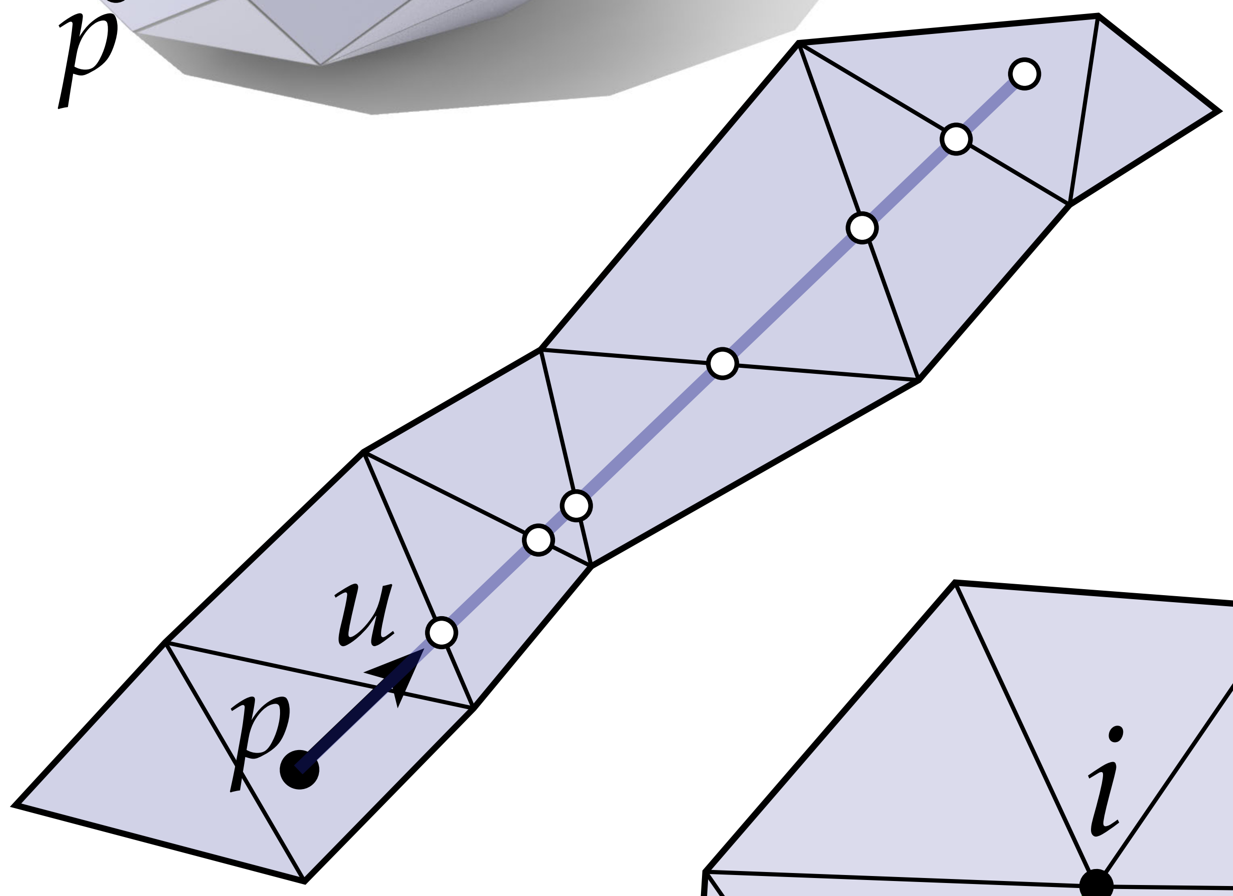
Notice: not always as easy as taking Euclidean average & projecting onto surface!

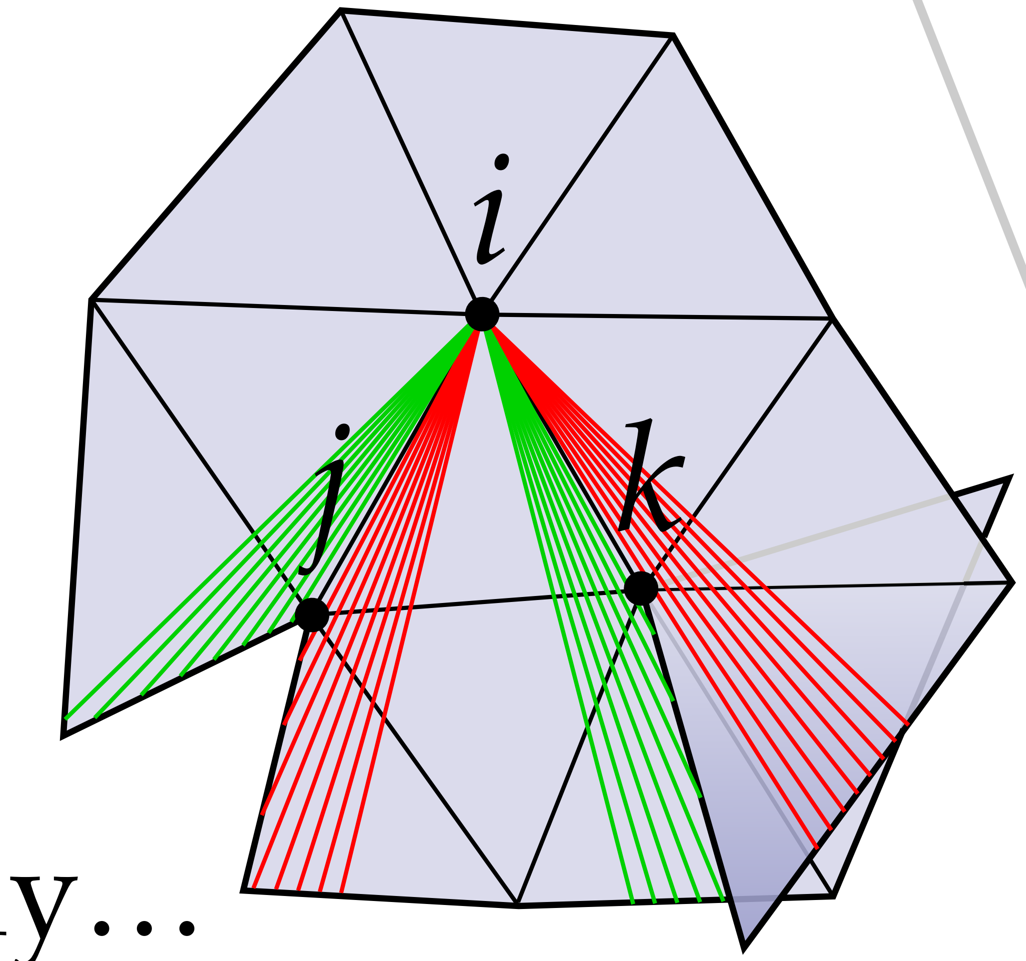
Discrete Exponential Map

- Easy to evaluate exp map on discrete surfaces
- Given point p and vector u, start walking along u
  - *i.e.*, just intersect ray with edges of triangle
  - continue w/ same angle in next triangle
  - if we hit a <u>vertex</u>, continue in direction that makes equal angles (*straightest*)
- **Q:** How big is the injectivity radius?
- **A:** Distance to the closest cone vertex ($\Omega > 0$)
- **Q:** Is the discrete exponential map surjective?
- **A:** No! Consider a saddle vertex ($\Omega < 0$)





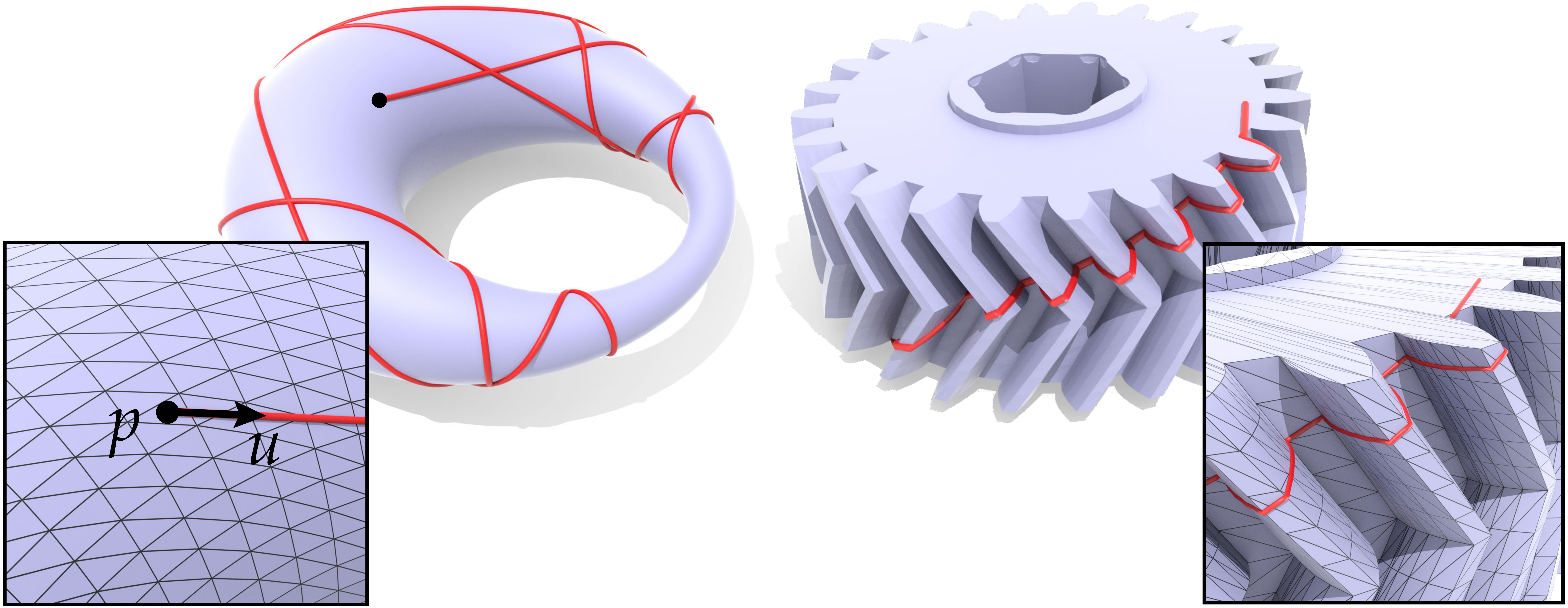




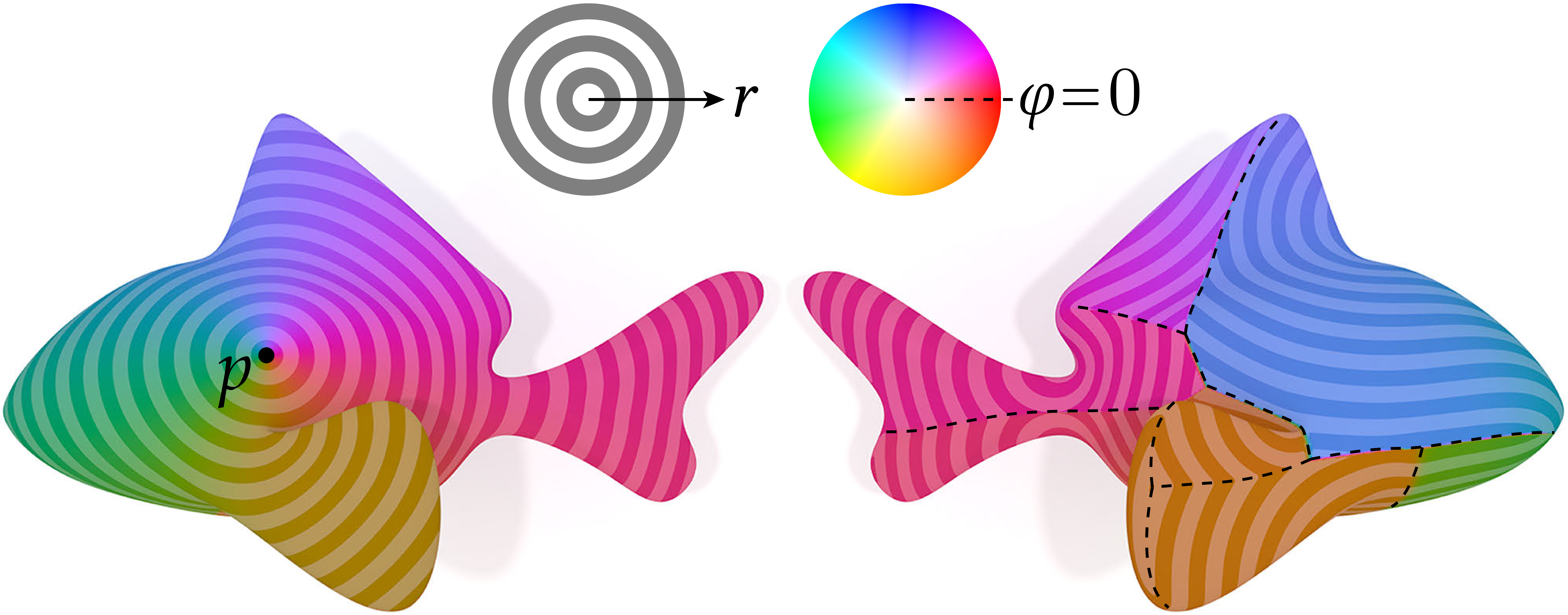
Notice: like "shortest", "straightest" doesn't work out perfectly…

Discrete Exponential Map—Examples

- Discrete exponential map provides a practical way to *approximate* geodesics on smooth surfaces (by triangulating them), and gives *exact* geodesics on discrete surfaces



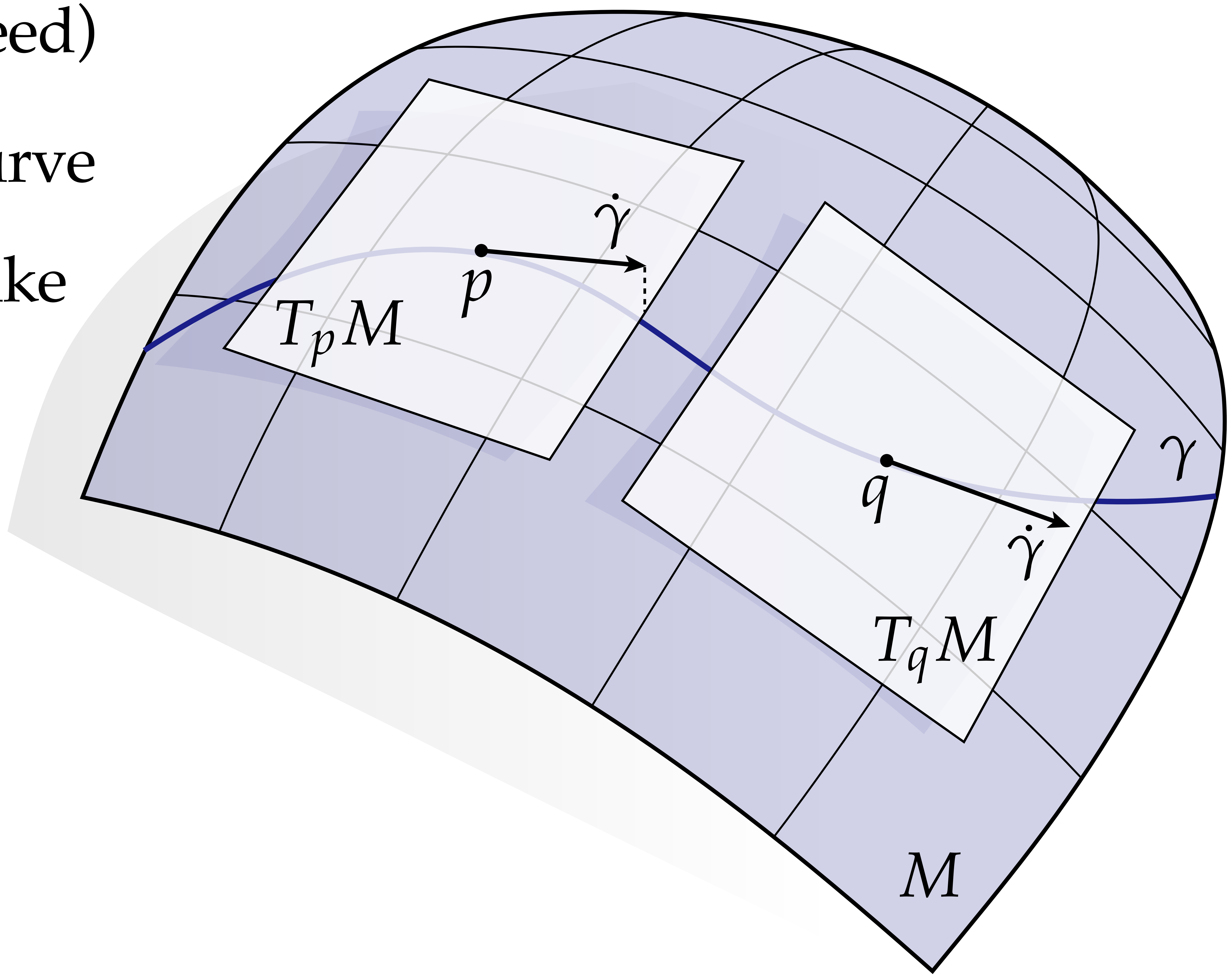
Computing the Log Map



Sharp, Soliman, Crane, "The Vector Heat Method" (2019)

Straightness — Dynamic Perspective

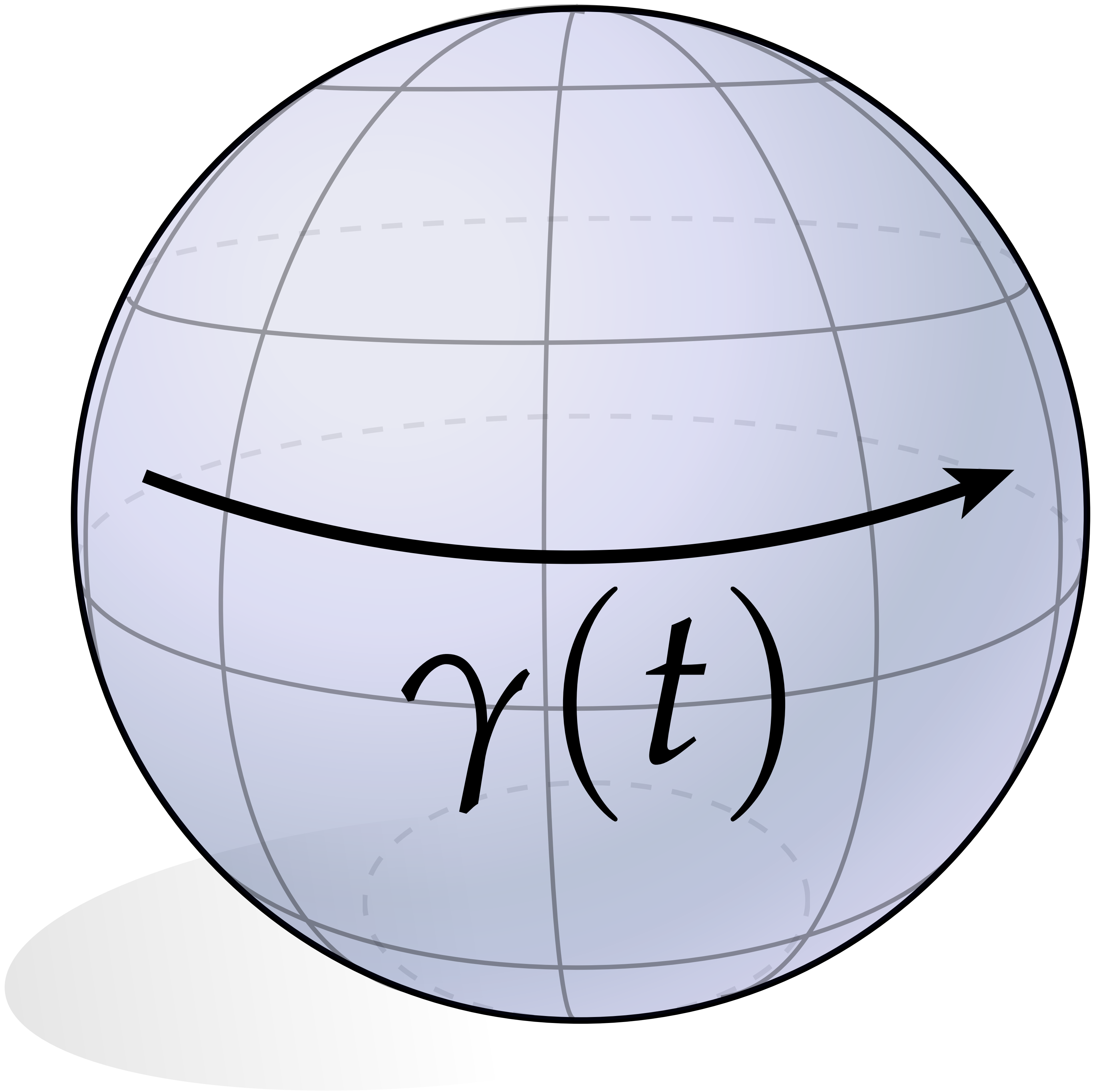
- Dynamic perspective: geodesic has zero tangential acceleration
- Consider curve $\gamma(t)$: $[a,b] \longrightarrow M$ (*not* unit speed)
- *Tangential velocity* is just the tangent to the curve
- *Tangential acceleration* should be something like the "tangential change in the tangent," but:
  - **extrinsically**, change in tangent is not a tangent vector
  - **intrinsically**, tangents belong to different vector spaces
- So, how do we measure acceleration?



Geodesic Equation

The *covariant derivative* ∇ provides another characterization of geodesics:

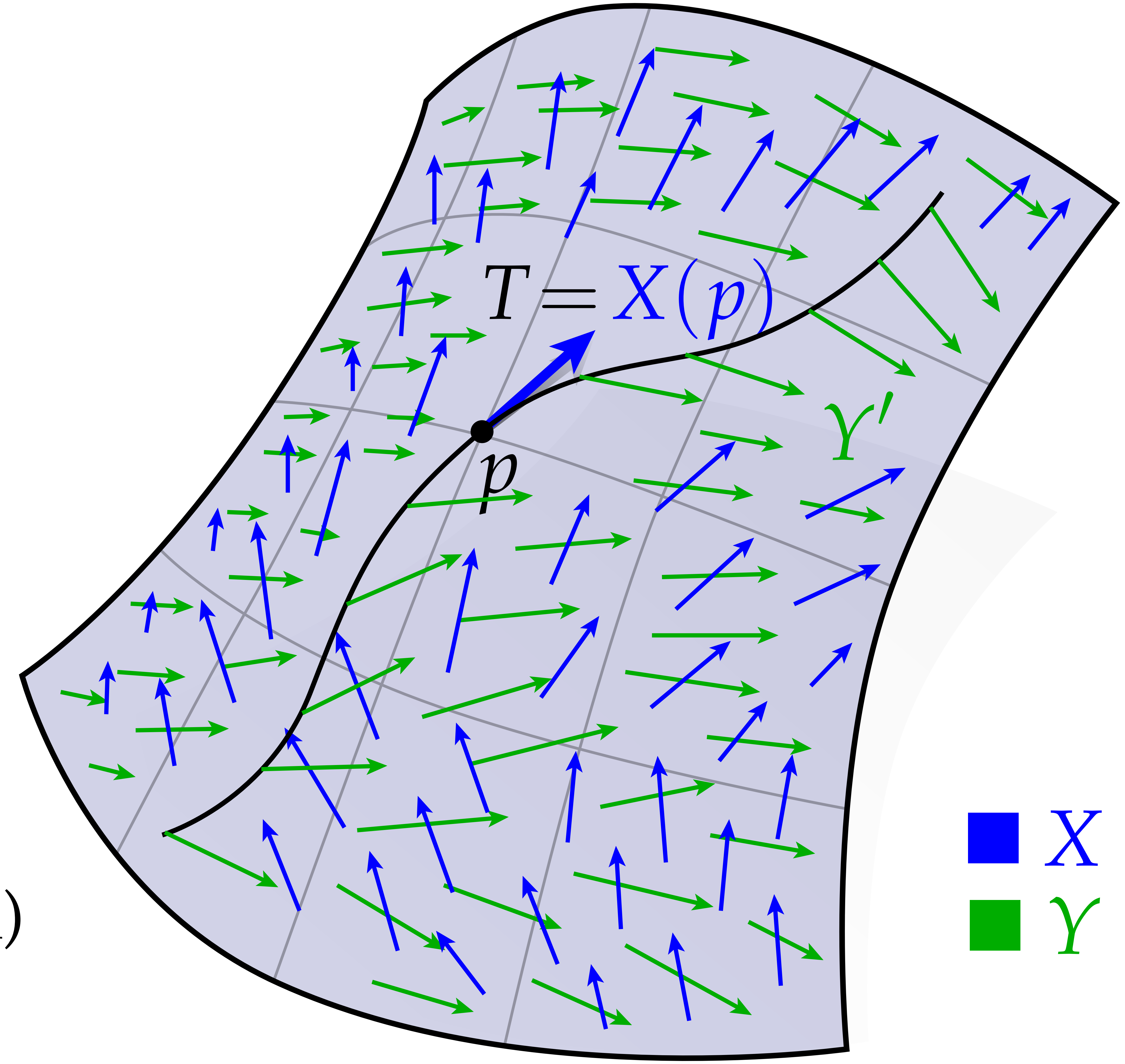
$$\underbrace{\nabla_{\dot{\gamma}}}_{\text{change along tangent direction}} \overbrace{\dot{\gamma}}^{\text{tangent to curve}} = 0$$



Intuition: no “in-plane turning” as we move along the curve.

Covariant Derivative—Extrinsic

- Suppose we want to measure how fast a vector field Y is changing along another vector field X at a point p
- Find a curve $\gamma(t)$ with tangent $X(p)$ at p
- Restrict Y to a vector field $Y'(\mathrm{t}) := Y(\gamma(t))$
- Take the derivative dY'/dt
- Removing the normal component gives the *covariant derivative* $\nabla_X Y$ of Y along X
- Sound familiar?
  - not so different from how we defined *geodesic curvature* (change of T in B direction)
  - which explains geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$



Key idea: covariant derivative gives change in one vector field along another.

Covariant Derivative—Intrinsic Definition

- Since geodesics are intrinsic, can also define "straightness" using only the metric g
- For any function ϕ, tangent vector fields X, Y, Z, operator ∇ uniquely determined by

$$\begin{aligned}
\nabla_Z(X+Y) &= \nabla_Z X + \nabla_Z Y \\
\nabla_{X+Y} Z &= \nabla_X Z + \nabla_Y Z \\
\nabla_{\phi X} Y &= \phi \nabla_X Y \\
\nabla_X(\phi Y) &= (D_X \phi) Y + \phi \nabla_X Y
\end{aligned}$$

(linearity)

(product rule)

$$D_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

"compatible w/ metric"

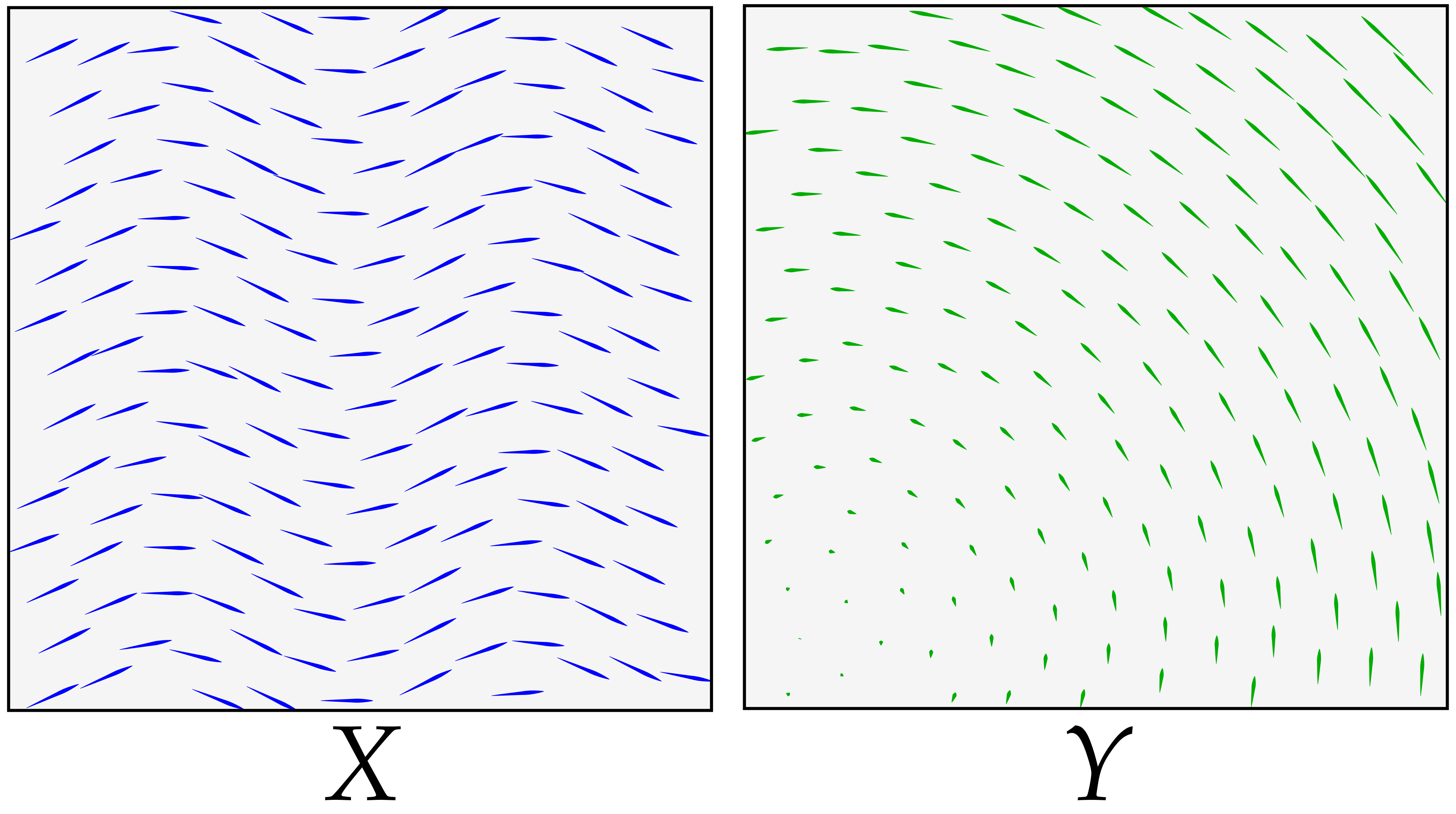
$$\nabla_X Y - \nabla_Y X = [X,Y]$$

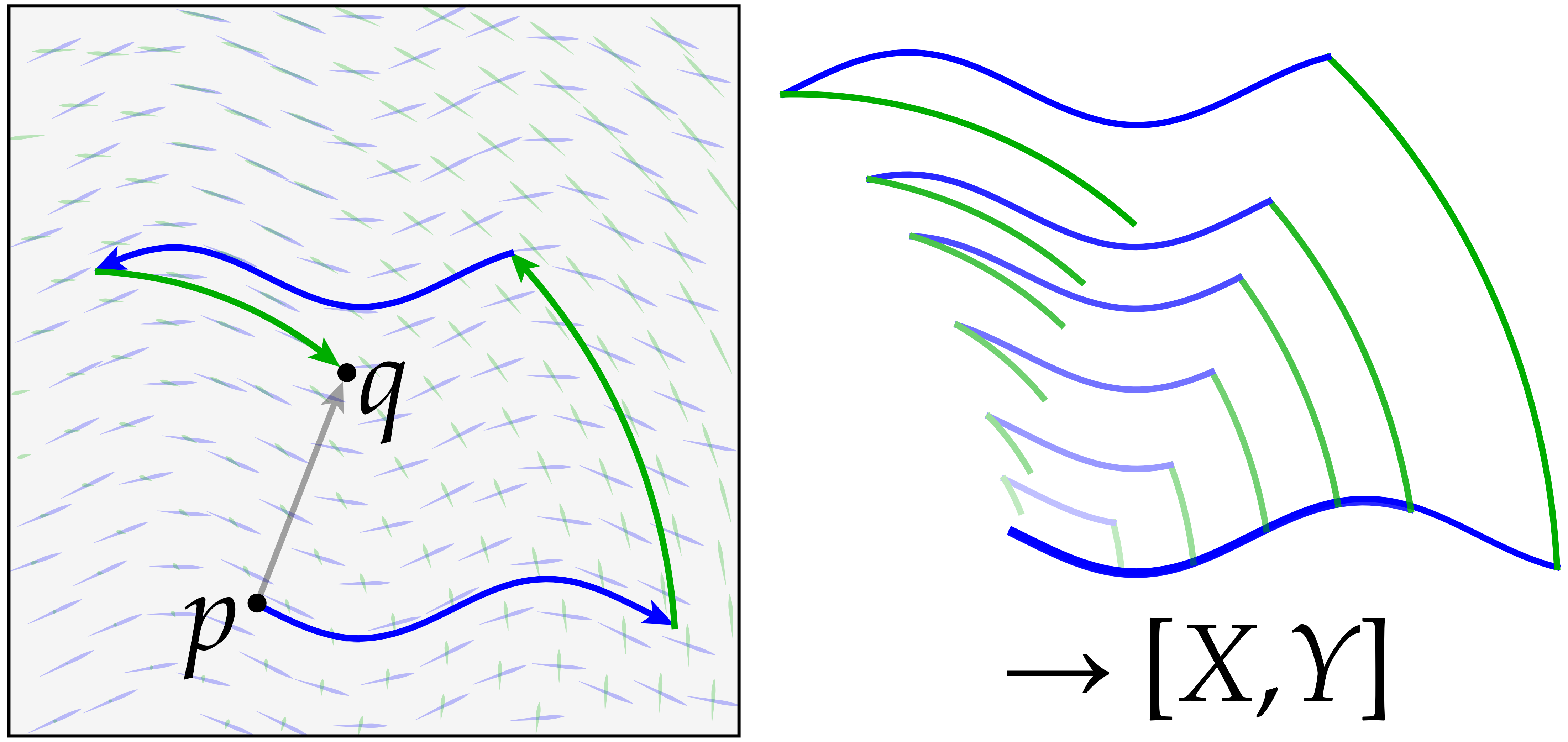
"torsion free"

Lie Bracket of Vector Fields

- The *Lie bracket* $[X,Y]$ measures failure of flows along two vector fields X, Y to commute
- Starting at any point p, follow X for time $\tau > 0$, then Y, then $-X$, then $-Y$ to arrive at a point q
- Lie bracket at p is vector given by limit of $(q\text{-}p)/\tau$ as $\tau \to 0$
- For vector fields expressed in local coordinates $u_1, \ldots, u_n$, can write as

$$[X,Y] = \sum_{i,j=1}^{n} \left(X^j \frac{\partial}{\partial u_j} Y^i - Y^j \frac{\partial}{\partial u_j} X^i \right) \frac{\partial}{\partial u^i}$$





Covariant Derivative from Metric

Claim. Covariant derivative is uniquely determined by the Riemannian metric g.

Proof. For any three vector fields U, V, W, we have

$$D_U g(V, W) = g(\nabla_U V, W) + g(V, \nabla_U W) \tag{1}$$
$$D_V g(W, U) = g(\nabla_V W, U) + g(W, \nabla_V U) \tag{2}$$
$$D_W g(U, V) = g(\nabla_W U, V) + g(U, \nabla_W V) \tag{3}$$

By symmetry and bilinearity of the metric g, adding (1) and (2) and subtracting (3) gives

$$\begin{aligned} &D_U g(V, W) + D_V g(W, U) - D_W g(U, V) = \\ &g(\nabla_U V + \nabla_V U) + g([U, W], V) + g([V, W], U) = \\ &2g(\nabla_V U, W) + g([U, V], W) + g([V, W], U) + g([U, W], V). \end{aligned}$$

Hence,

$$g(\nabla_V U, W) = \frac{1}{2}\left(D_U g(V, W) + D_V g(W, U) - D_W g(U, V) - g([U, V], W) - g([V, W], U) - g([U, W], V)\right).$$

Key observation: can *solve* for covariant derivative in terms of data we know (metric g).

Christoffel Symbols

- Let $X_1, \ldots X_n$ be our usual basis vector fields (in local coordinates)
- *Christoffel symbols* tell us how to differentiate one basis along another: $\nabla_{X_j} X_i = \Gamma^k_{ij} X_k$
- By linearity, we then know how to take *any* covariant derivative

Recall the expression

$$g(\nabla_V U, W) = \frac{1}{2}\left(D_U g(V,W) + D_V g(W,U) - D_W g(U,V) - g([U,V],W) - g([V,W],U) - g([U,W],V)\right).$$

Since $[X_i, X_j] = 0$ for any two coordinate vector fields, we get

$$2g(\nabla_{X_k} X_i, X_j) = D_{X_i} g(X_k, X_j) + D_{X_k} g(X_j, X_i) - D_{X_j} g(X_i, X_k).$$

In terms of Christoffel symbols, the left-hand side is

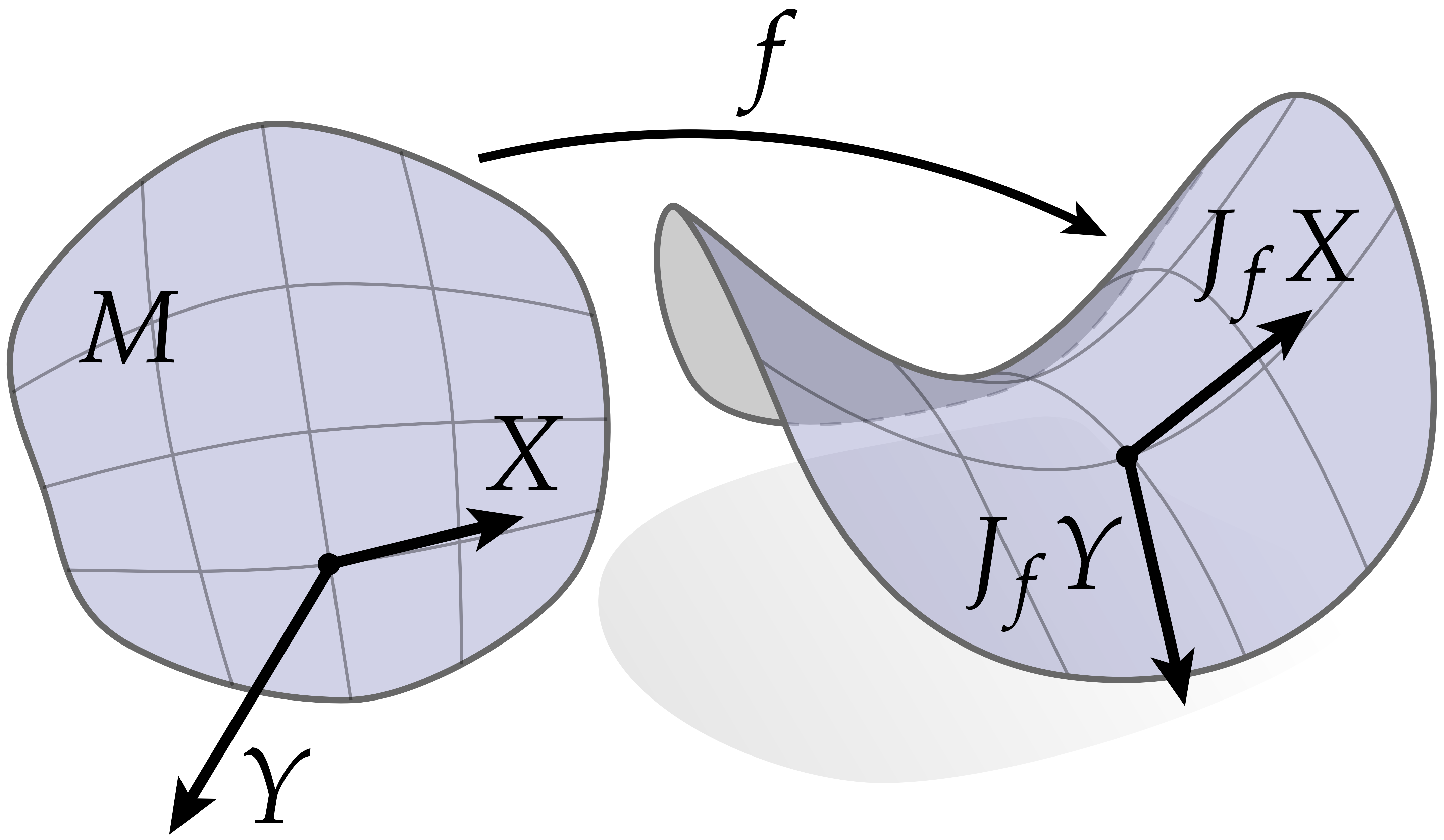
$$2g(\Gamma^p_{ik} X_p, X_j) = 2\Gamma^p_{ik} g(X_p, X_j) = 2\Gamma^p_{ik} g_{pj}$$

and we can write the right-hand side as $g_{kj,i} + g_{ji,k} - g_{ik,j}$.

Hence, our final expression for the Christoffel symbols is $\boxed{\Gamma^p_{ik} = \tfrac{1}{2} g^{pj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)}$

Solving the Geodesic Equation

- Can use Christroffel symbols to numerically compute geodesics on <u>smooth</u> surfaces
- Given surface $f : M \to \mathbb{R}^3$
  - write out Jacobian J_f
  - write out metric $g = J_f^\top J_f$ and its inverse g^{ij}
  - write out Christoffel symbols Γ
  - express geodesic equation via Γ
- From here, can use any standard numerical integrator (e.g., Runge-Kutta) to step an initial position/direction forward in "time"

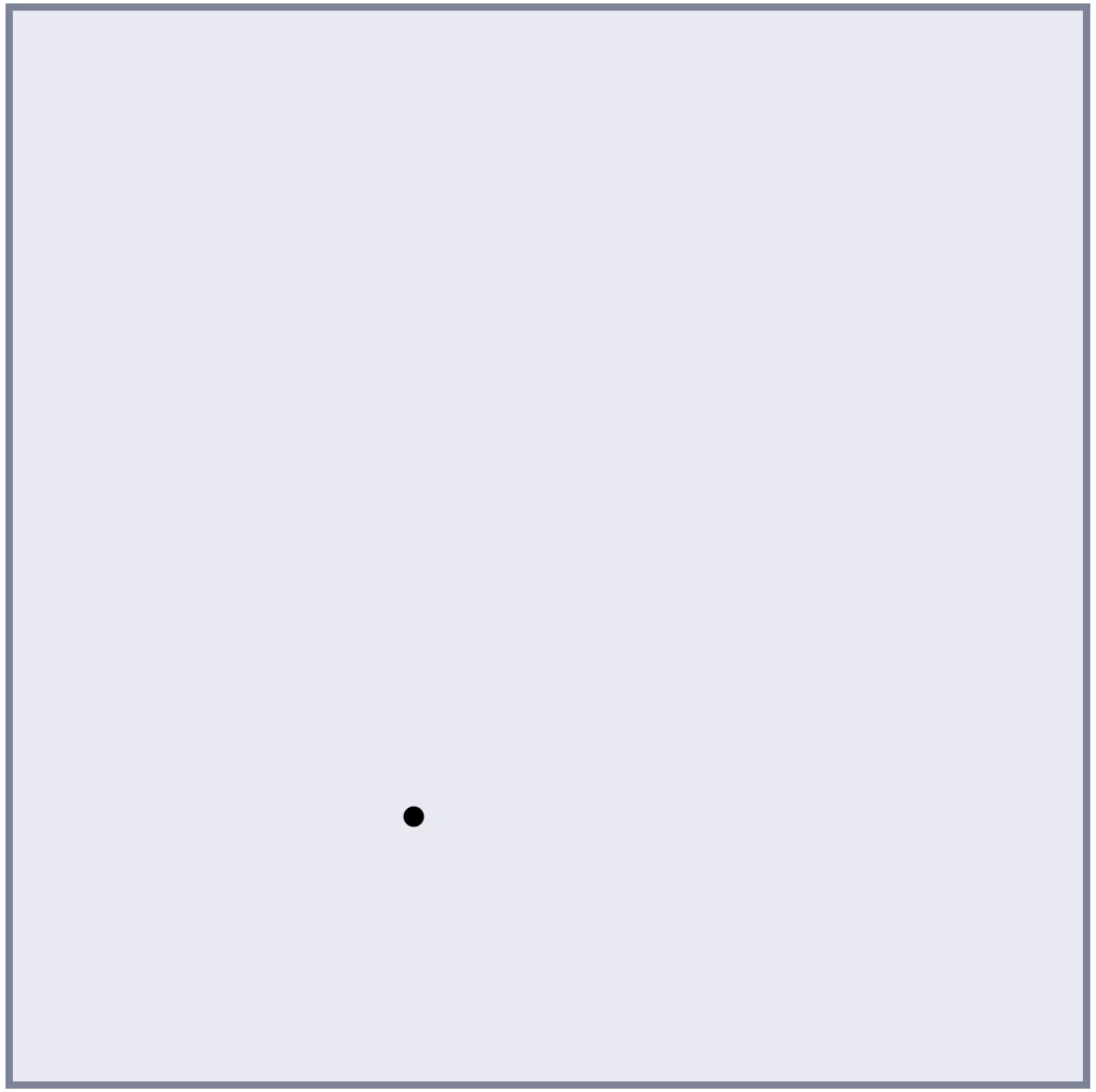


$$\Gamma^p_{ik} = \frac{1}{2} g^{pj} (g_{ij,k} + g_{jk,i} - g_{ki,j})$$

$$\nabla_{\dot\gamma} \dot\gamma = 0 \qquad \nabla_{X_j} X_i = \Gamma^k_{ij} X_k$$

$$\Rightarrow \boxed{\ddot\gamma^k + \Gamma^k_{ij} \dot\gamma^i \dot\gamma^j = 0}$$

Solving the Geodesic Equation



- Apply f to resulting curve in parameter domain to get a geodesic on the surface

$$\Rightarrow \boxed{\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0}$$

Computing Geodesics on a Parametrized Surface

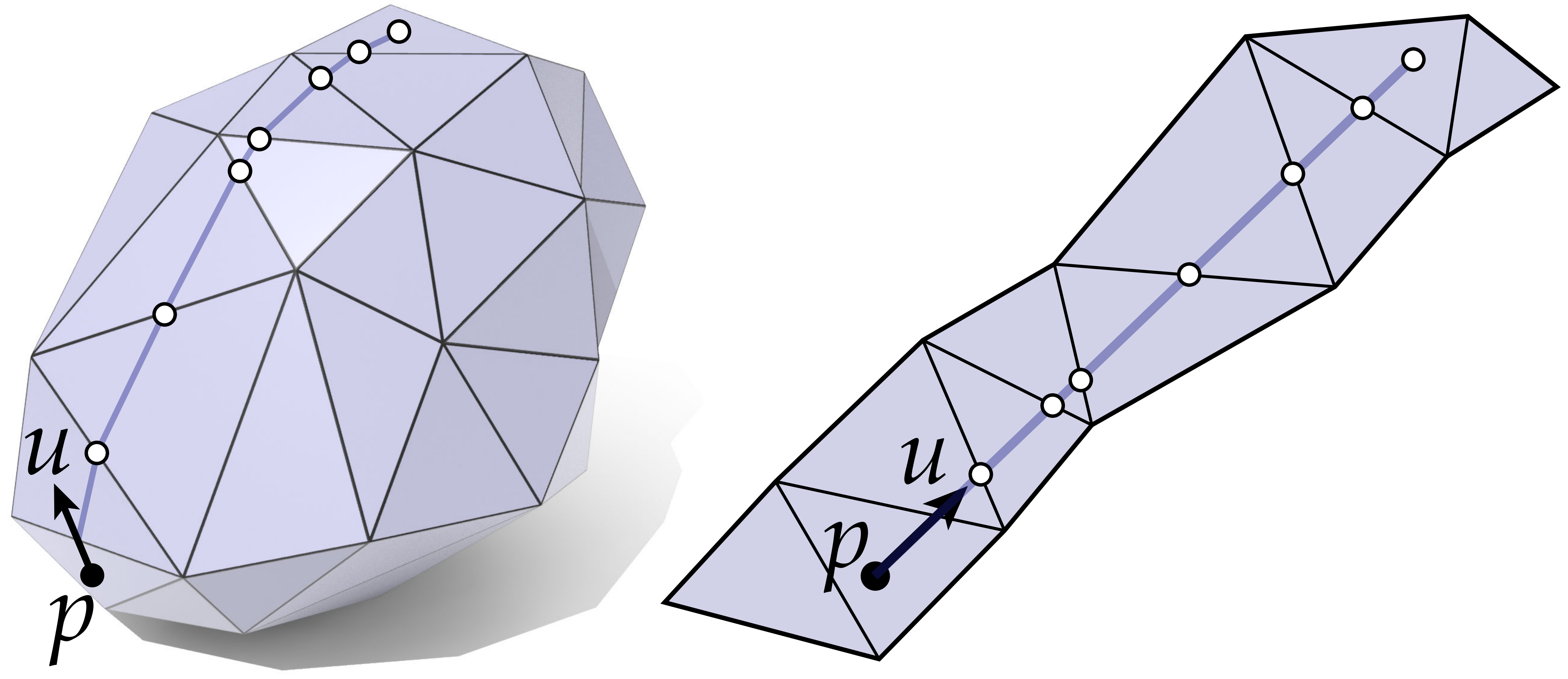
Now have two ways to solve initial value problem for a smooth parameterized surface f:

- <u>Discretization</u>
  - triangulate the surface f
  - trace rays along discrete surface
- <u>ODE integration</u>
  - write metric g in terms of f
  - write Christoffel symbols Γ in terms of g
  - solve geodesic equation via ODE solver

Q: What are the pros/cons?

– speed, memory, accuracy, *simplicity*…

– generality (smooth *and* discrete)



$$g = J_f^\top J_f$$

$$\underbrace{\tfrac{1}{2} g^{pj} (g_{ij,k} + g_{jk,i} - g_{ki,j})}_{\Gamma^p_{ik}}$$

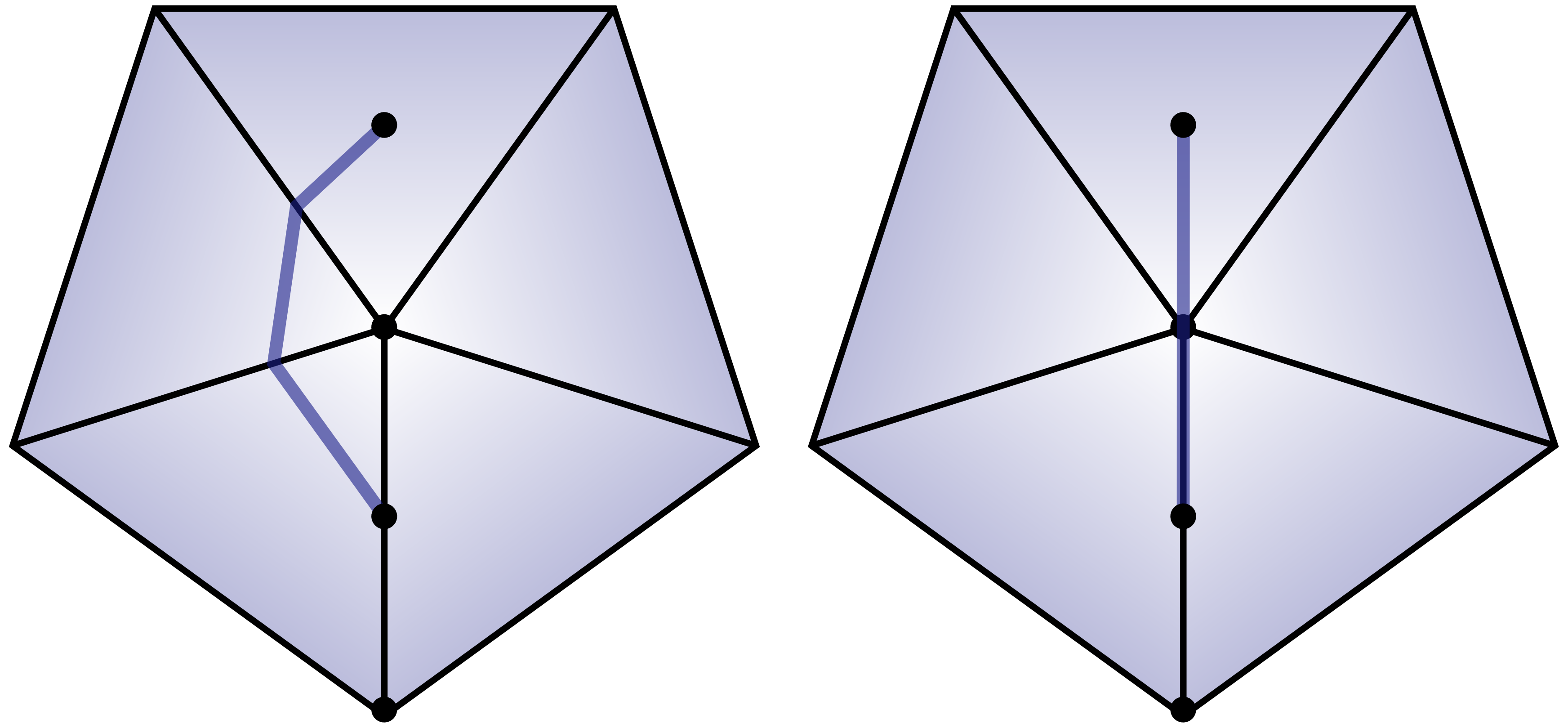
$$\boxed{\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0}$$

Summary

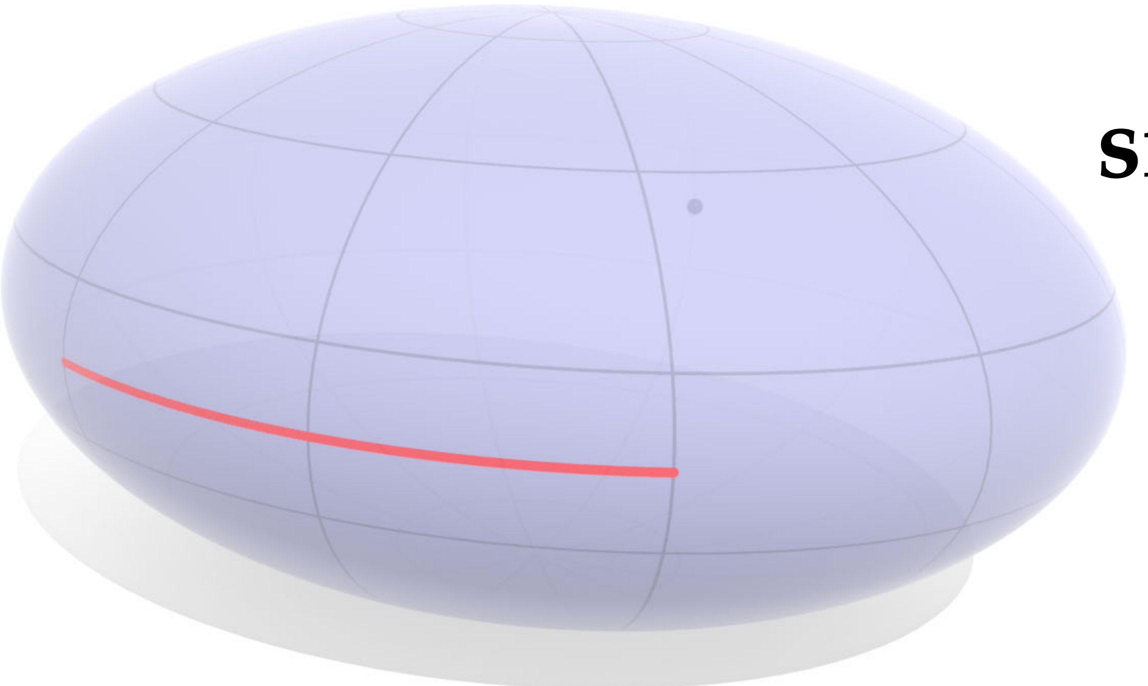
Geodesics—Shortest vs. Straightest, Smooth vs. Discrete

- In smooth setting, several equivalent characterizations:
  - shortest (harmonic)
  - straightest (zero curvature, zero acceleration)
- In discrete setting, characterizations no longer agree!
  - **shortest** natural for boundary value problem
  - **straightest** natural for initial value problem
  - *convex:* shortest paths are straightest (but not vice versa)
  - *nonconvex:* shortest may not even be straightest! (saddles)
- *Neither* definition faithfully captures all smooth behavior:
  - (shortest) cut locus/medial axis touches *every* convex vertex
  - (straightest) exponential map is not surjective
- Use the right tool for the job (*and look for other definitions!*)

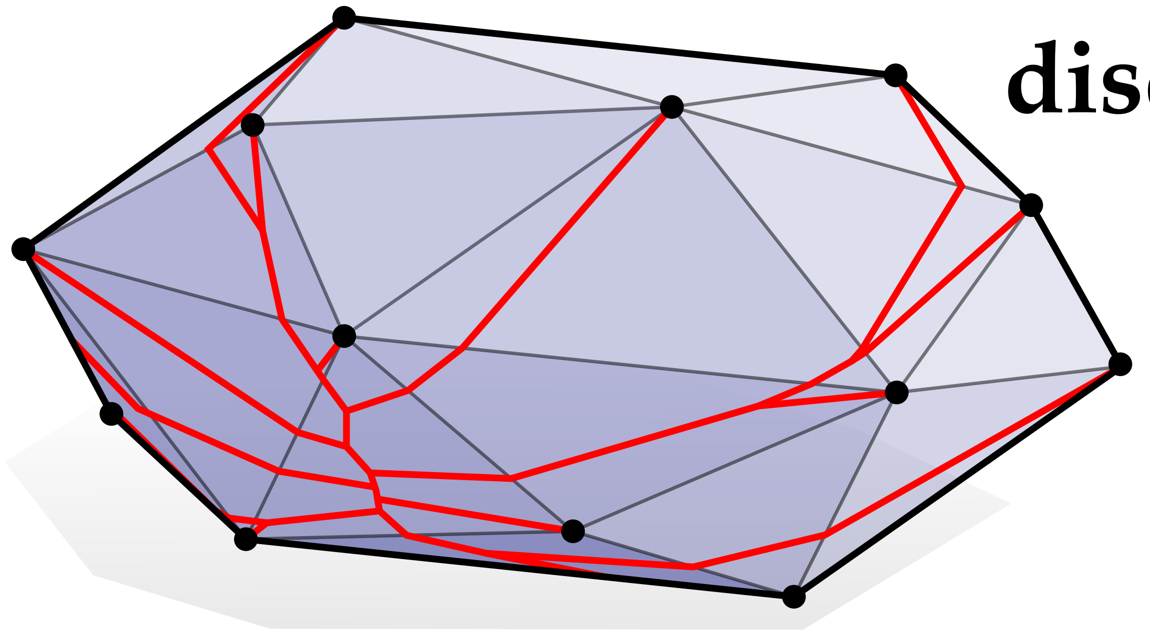
shortest **straightest**



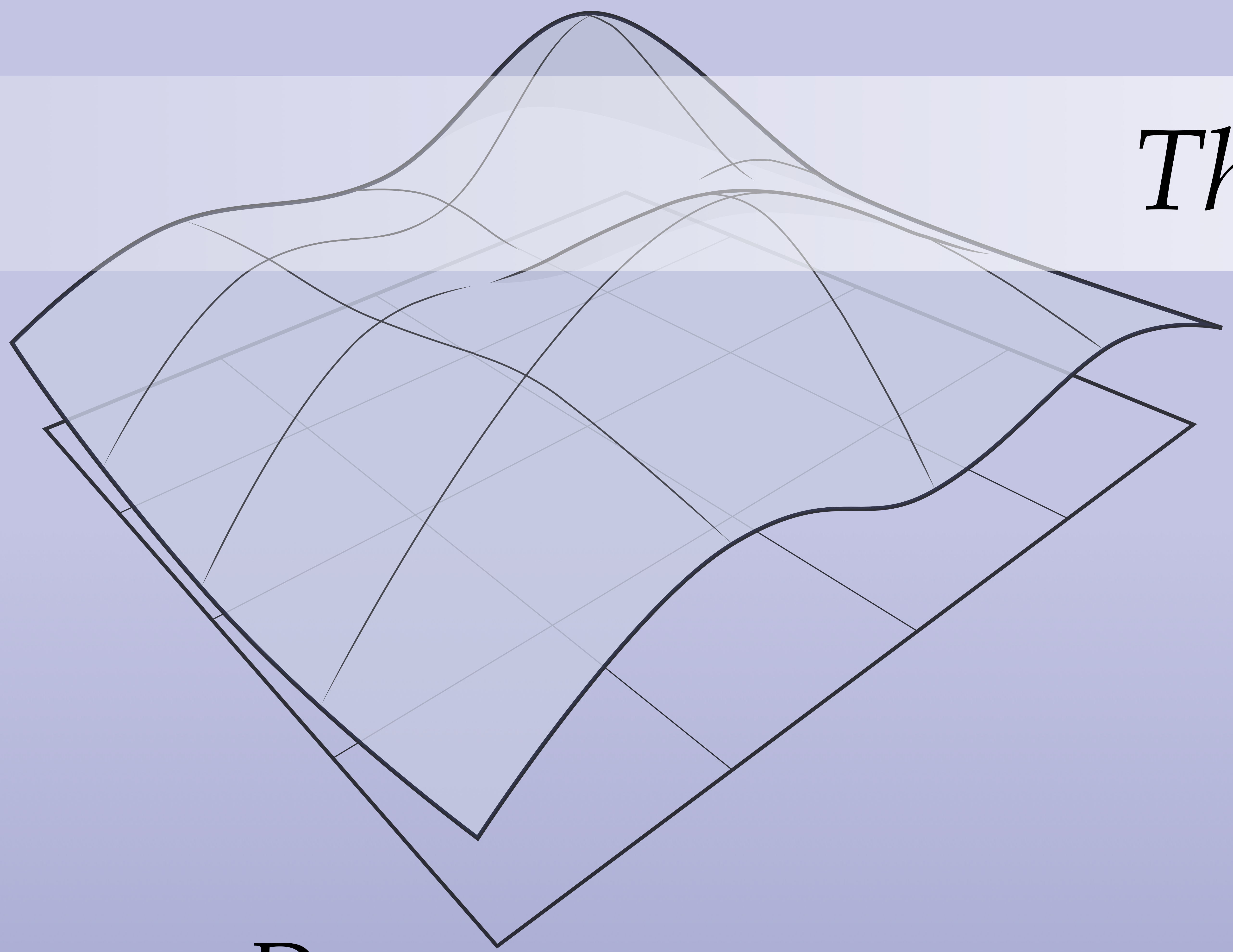
smooth



discrete



Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

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