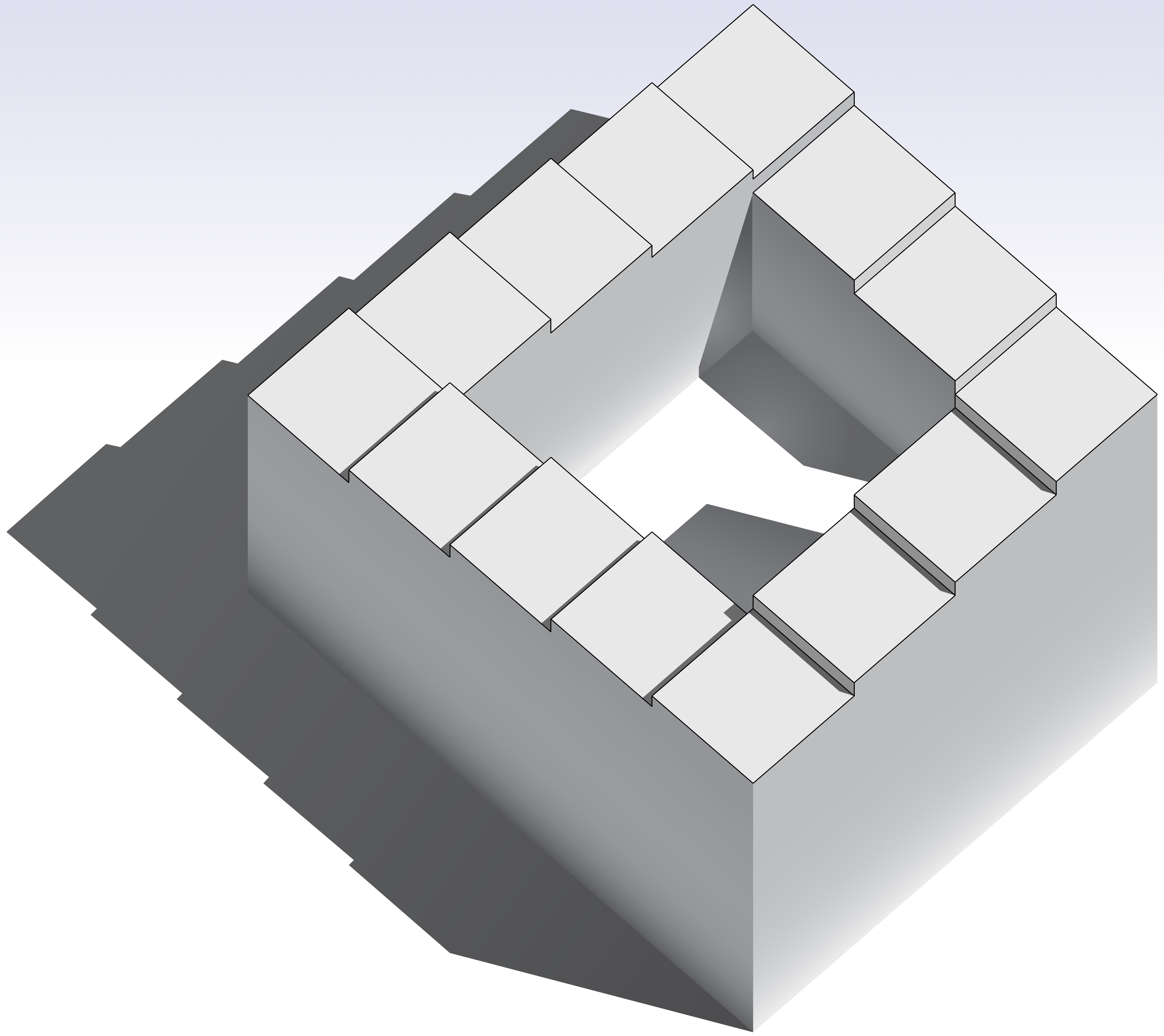
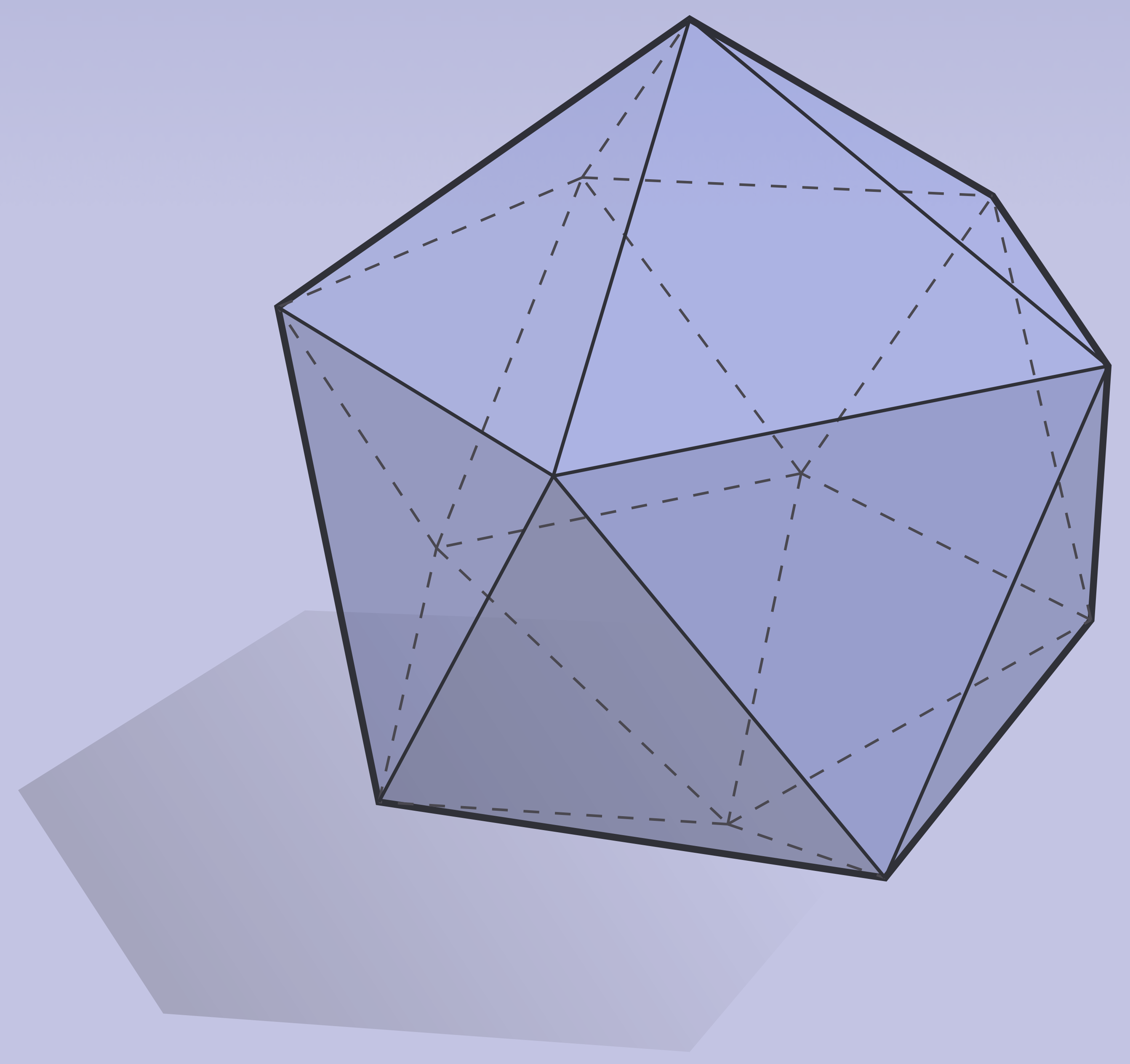


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 3:
EXTERIOR ALGEBRA



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858

Why Learn Exterior Calculus?

コンピュータサイエンスの建物
の地下では、金があります!

Translation: *"There is gold in the basement of the computer science building!"*

Key idea: language is important!

Not all languages are created equal…

$*54\cdot43$. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[*51·231] $\equiv . \iota`x \cap \iota`y = \Lambda .$

[*13·12] $\equiv . \alpha \cap \beta = \Lambda$ (1)

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda$ (2)

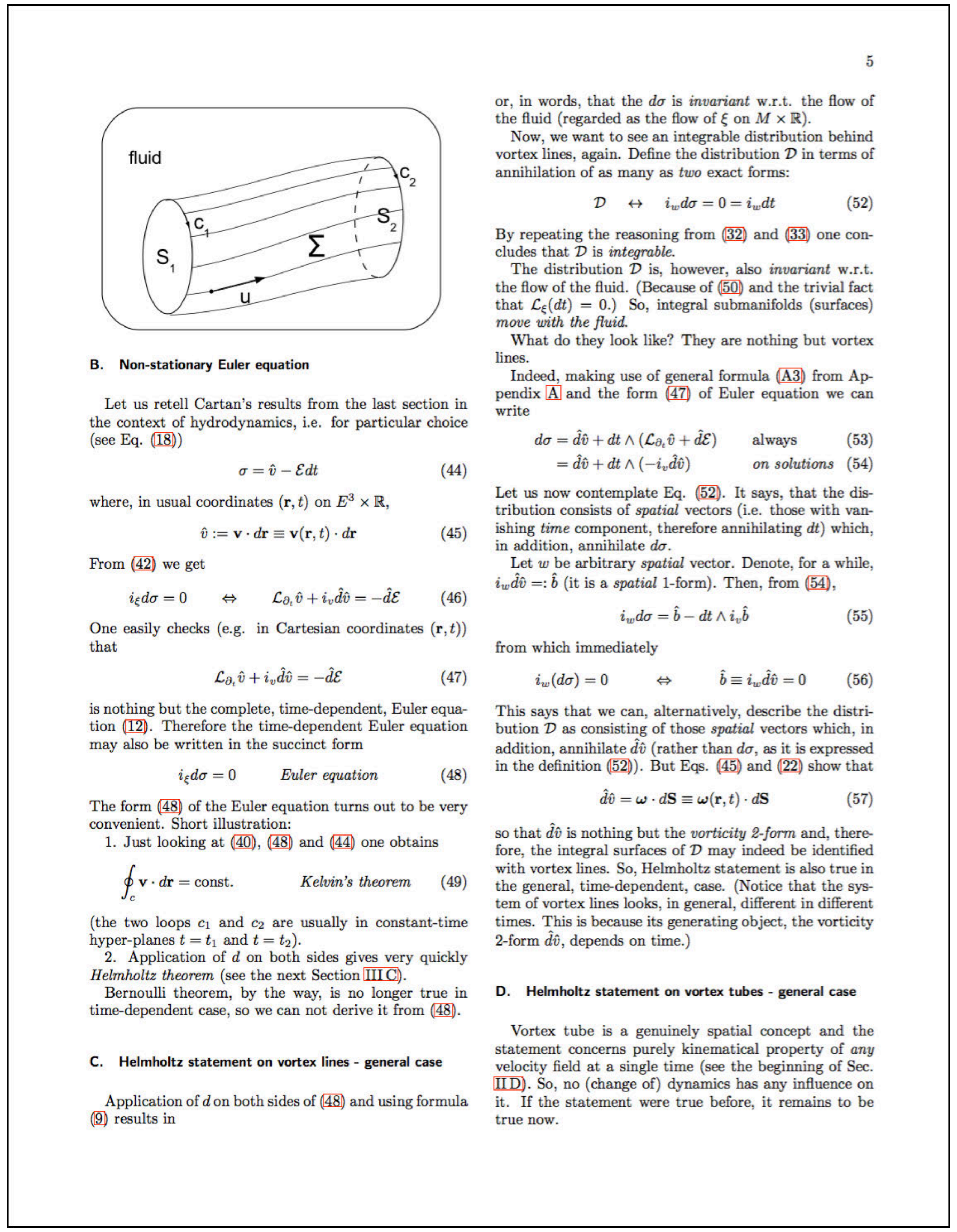
$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

(from Russel & Whitehead's *Principia Mathematica*, p. 379)

Why Learn Exterior Calculus?

- Natural language for talking about *signed volume*
  - facilitates communication w/ math, physics, …
  - provides new perspectives on computation
- **Geometry**
  - algebraic geometry
  - geometric algebra (Clifford algebra, spin physics)
- **Physics**
  - *"massless"* quantities are vectors (velocity, acceleration, …)
  - *"massive"* quantities are forms (momentum, force, …)
- **Computer Science** (*this class!*): geometric computation on meshes



Where Are We Going Next?

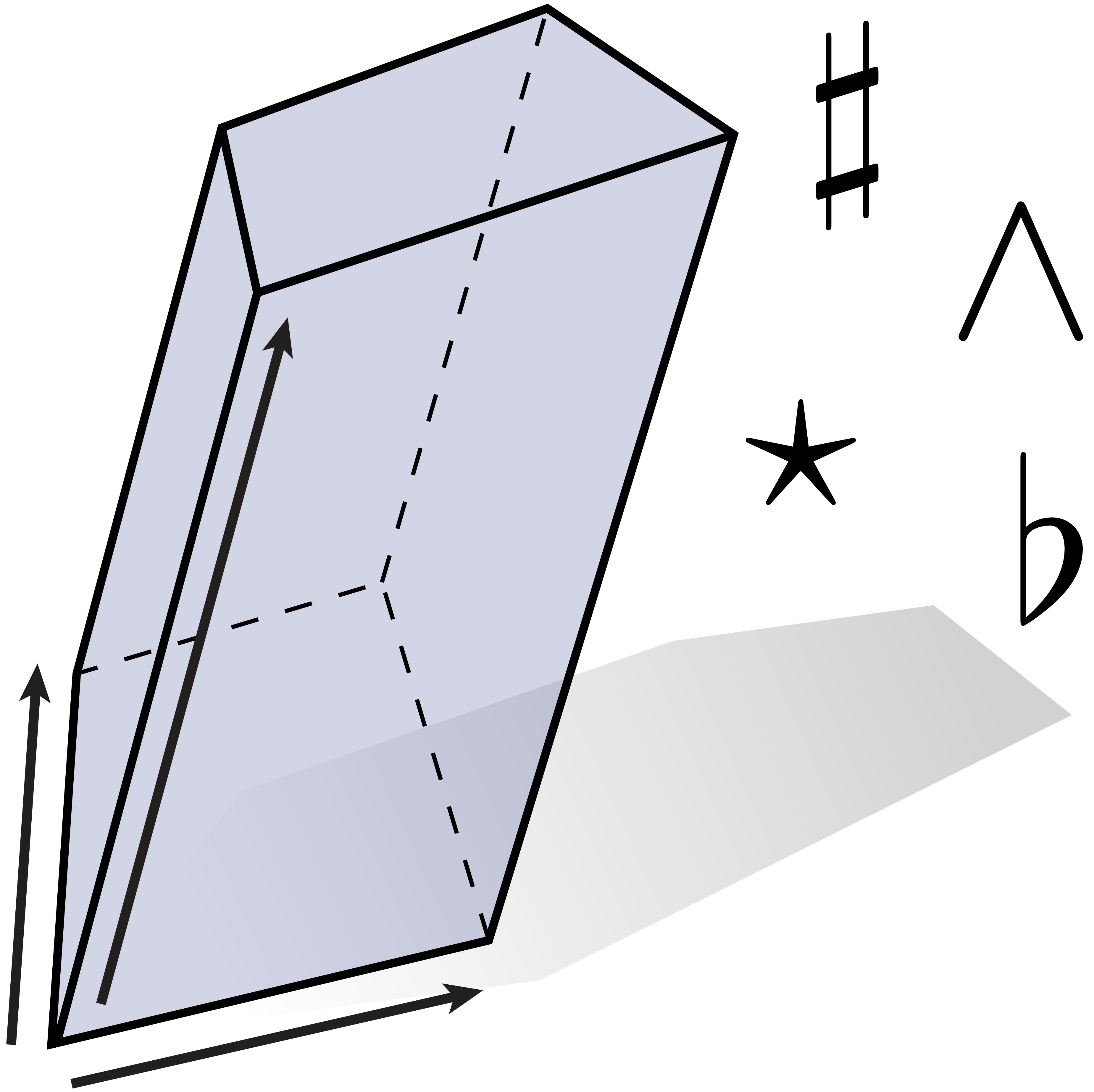
GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

- **Linear algebra:** "little arrows" (vectors)
- **Vector Calculus:** how do vectors *change?*

Next few lectures:

- **Exterior algebra**: "little volumes" (*k*-vectors)
- **Exterior calculus**: how do *k*-vectors change?
- **DEC:** how do we do all of this on meshes?

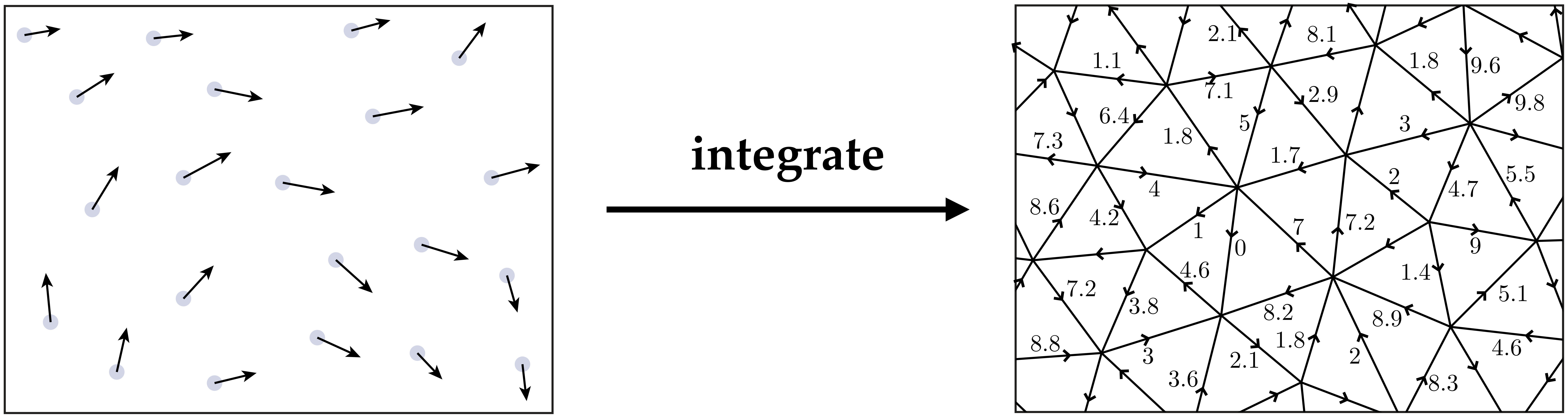


Basic idea: replace vector calculus with computation on meshes.

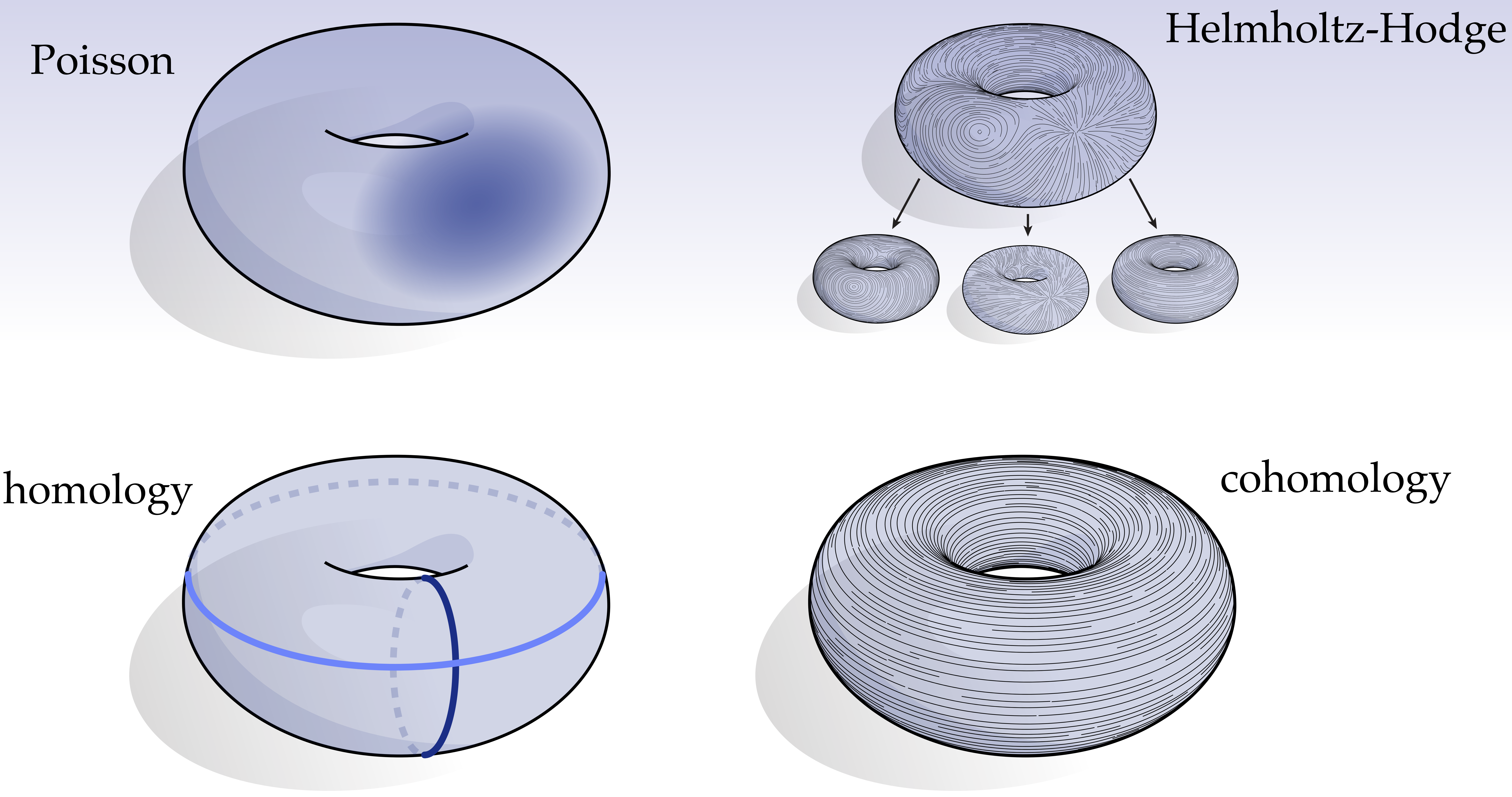
Why Are We Going There?

- **Motivation:** *Do cool and useful stuff with meshes!*
- Geometry processing algorithms must solve *equations* on meshes (PDEs)
- Meshes are made up of little *volumes*

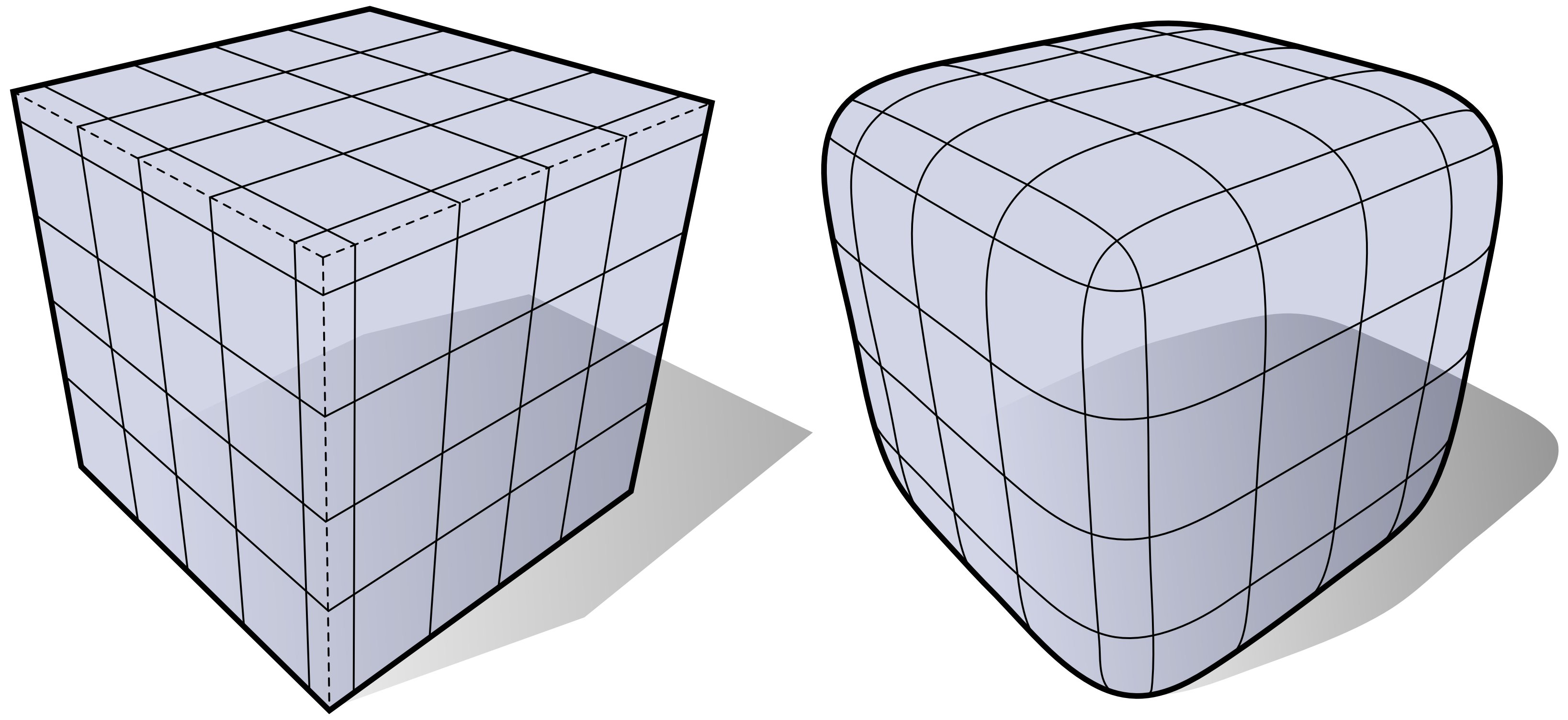
⇒ Need to learn to *integrate* equations over little volumes to do computation



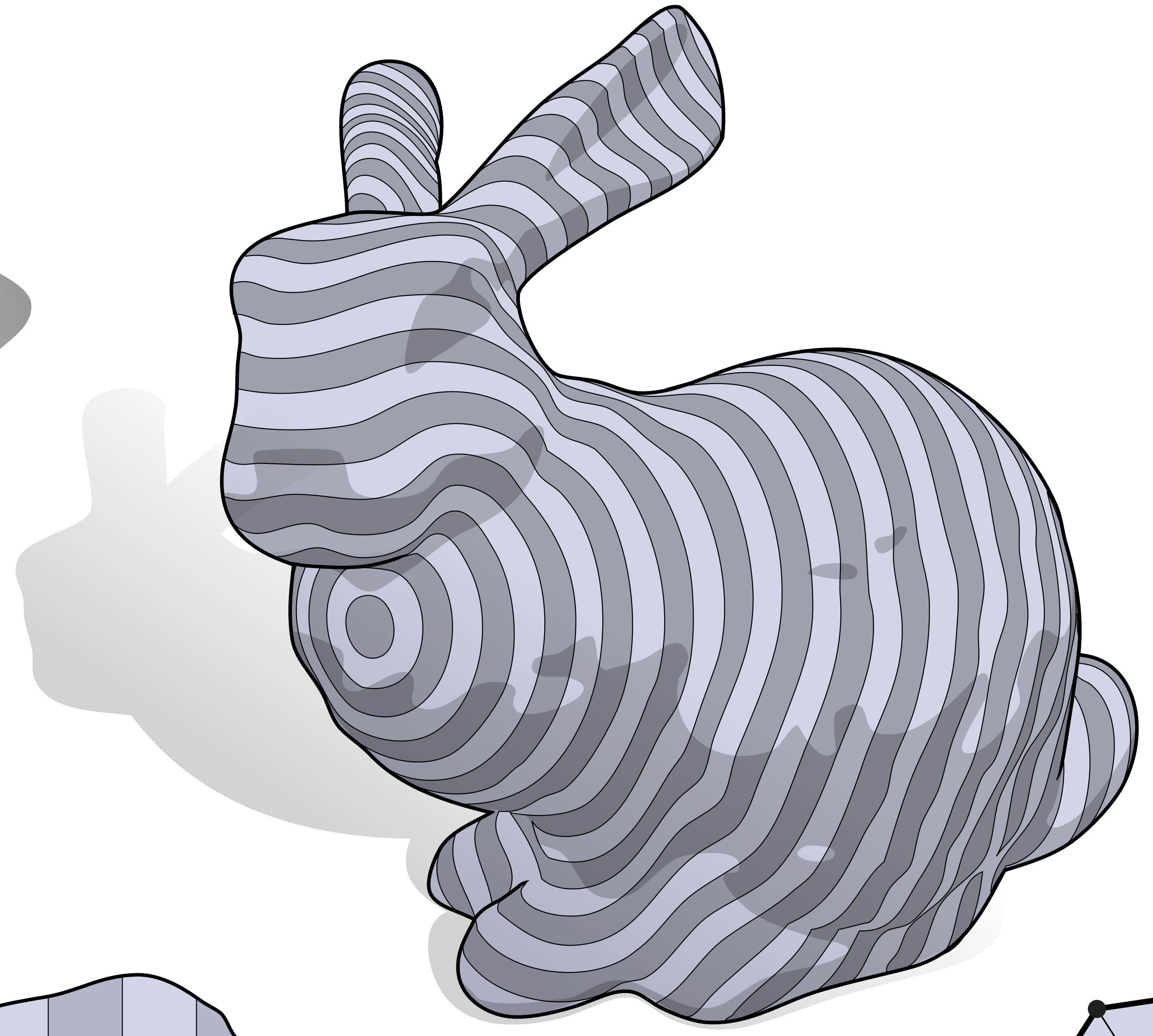
Basic Computational Tools



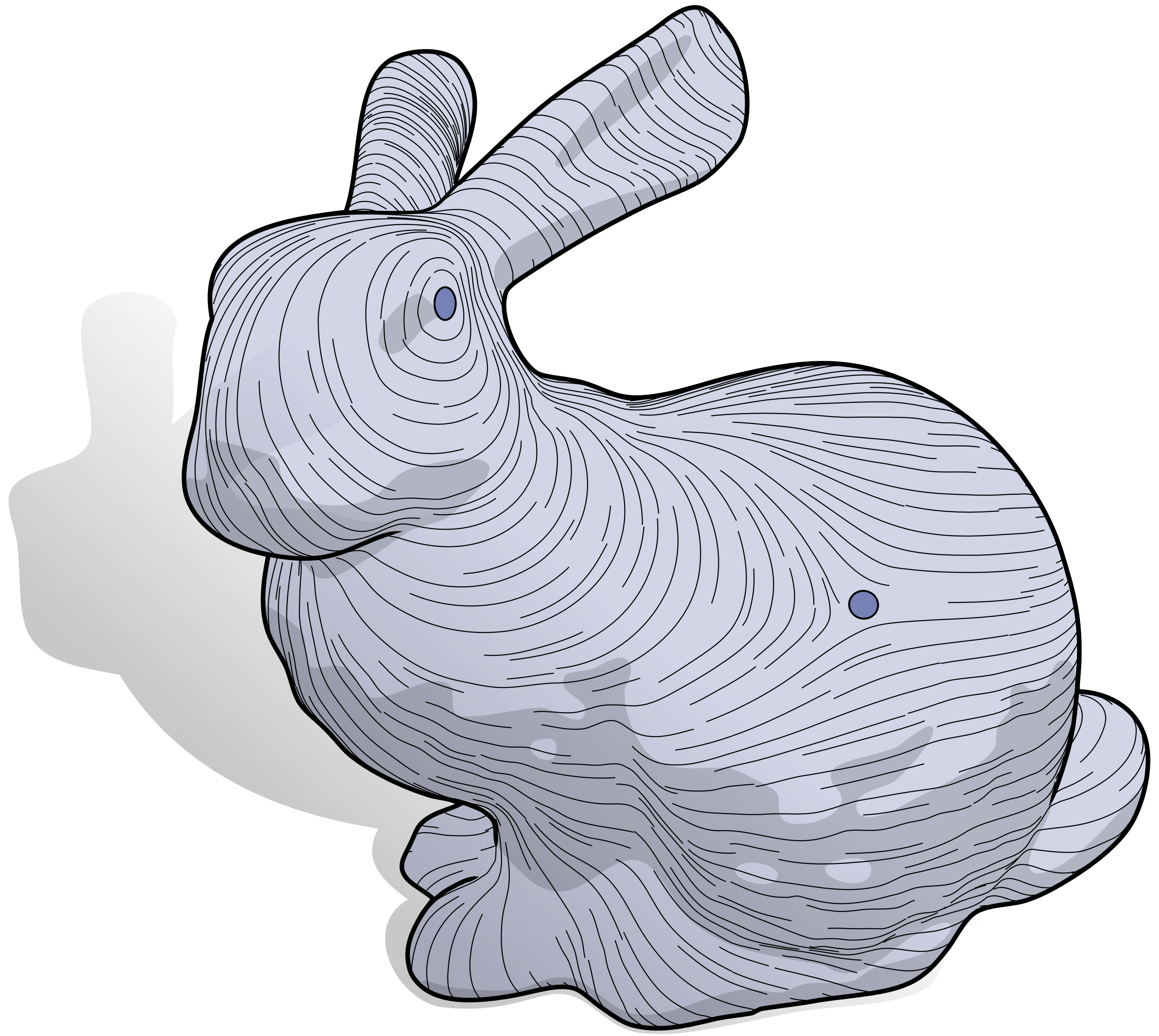
Applications



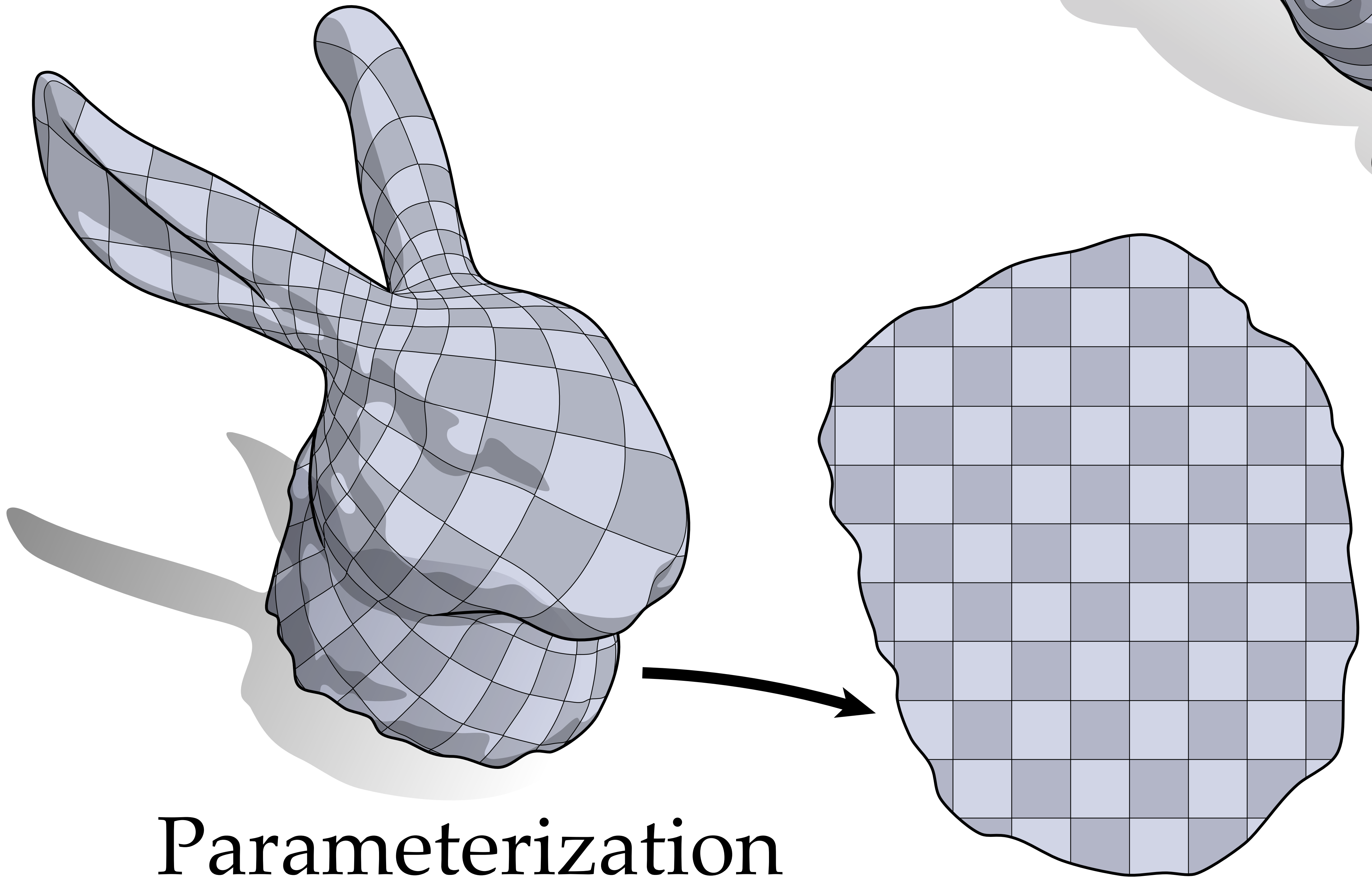
Smoothing



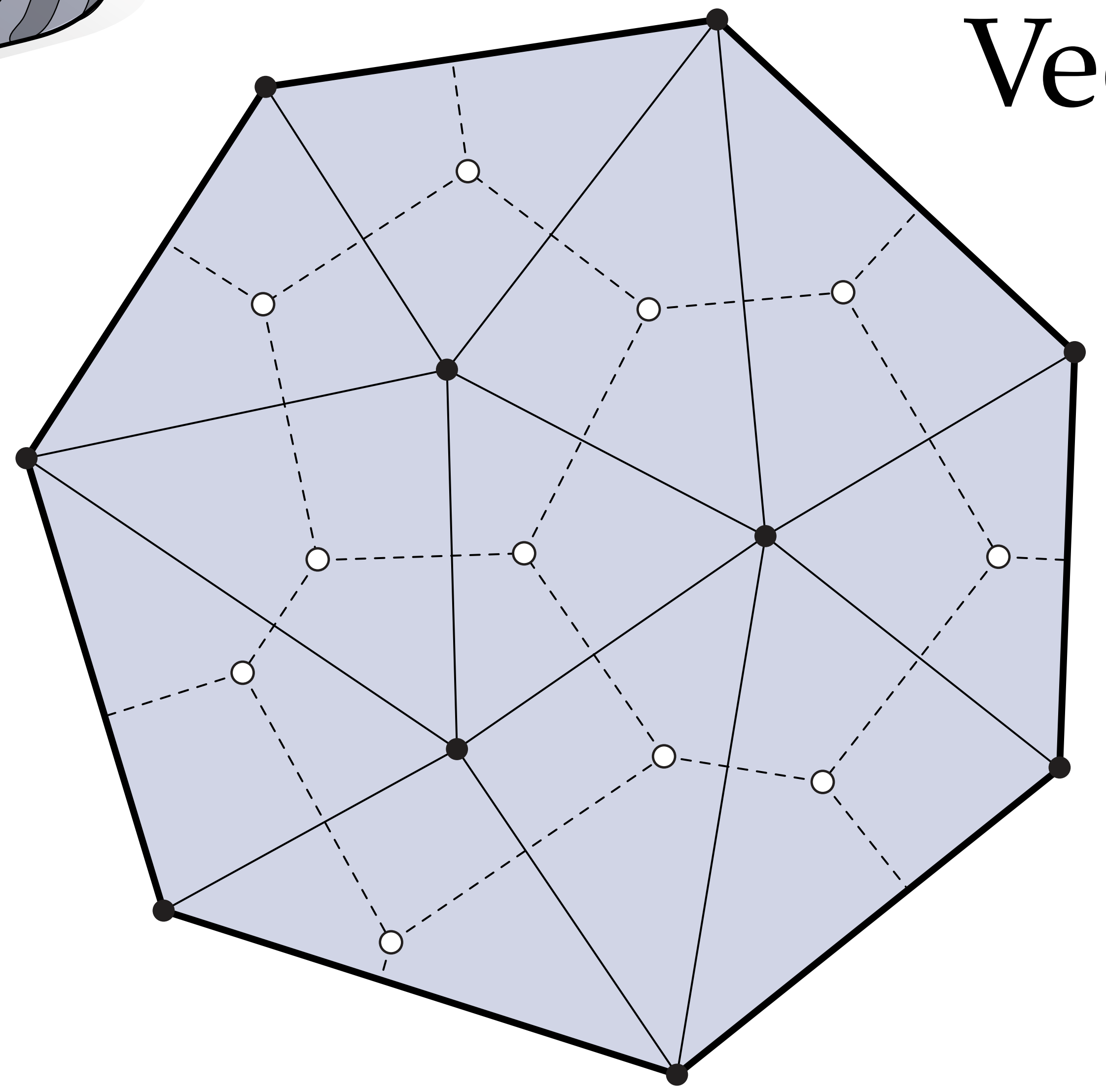
Distance



Vector Field Design



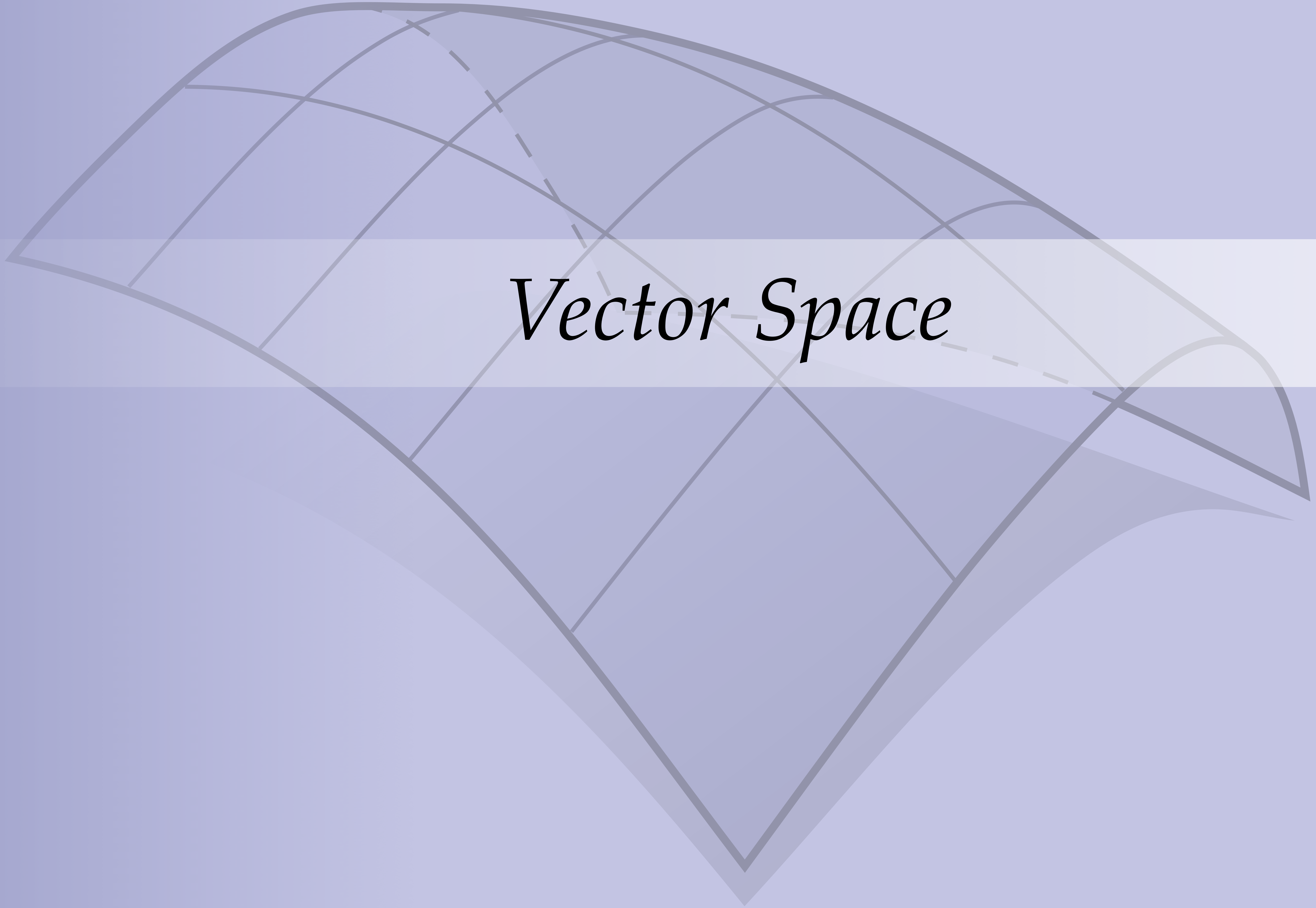
Parameterization



Meshing

…and more!

Vector Space

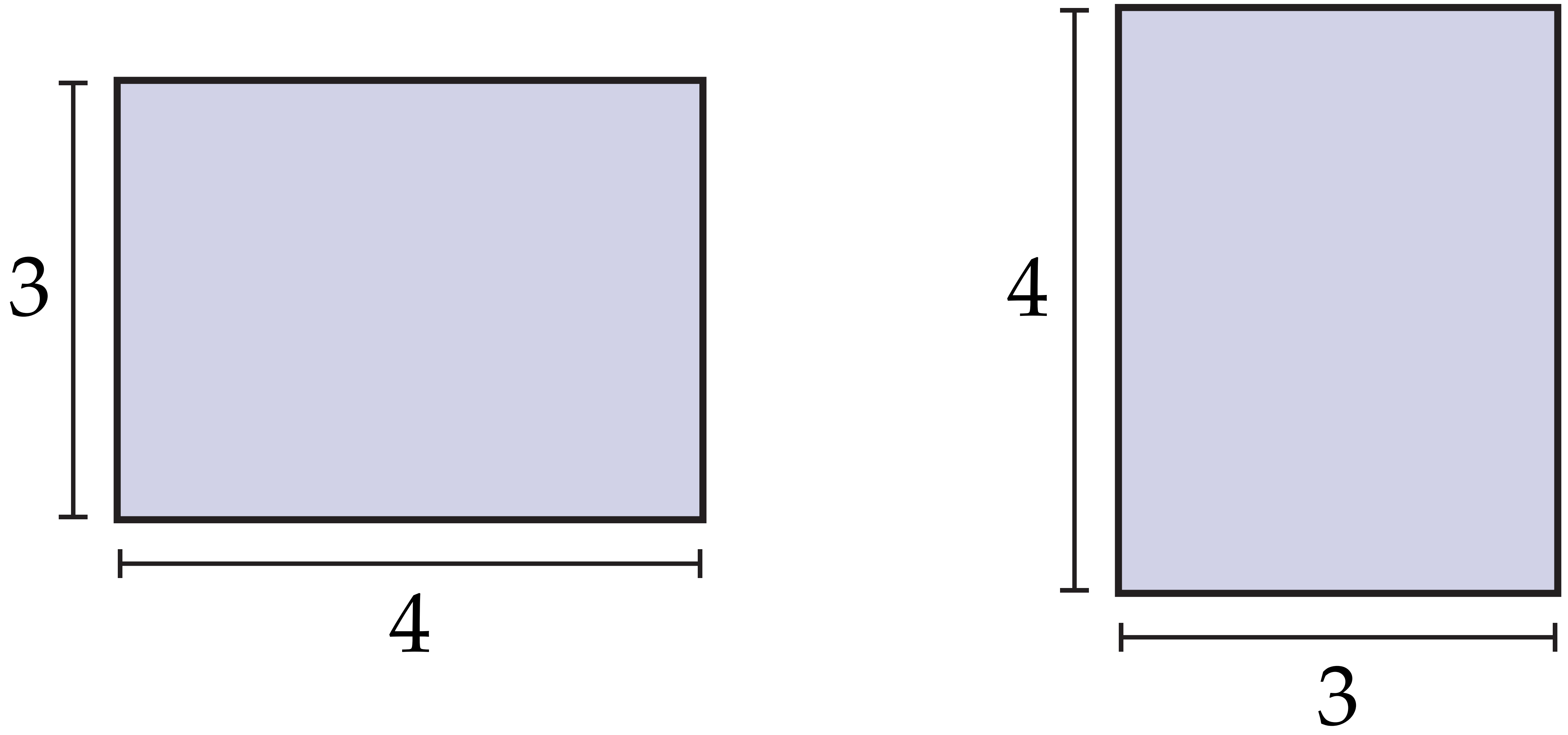


Warm Up: Multiplication

Question: why does $3 \times 4 = 4 \times 3$?

Answer: <u>not</u> just because *"that's the rule!"*

There is a very good geometric reason:

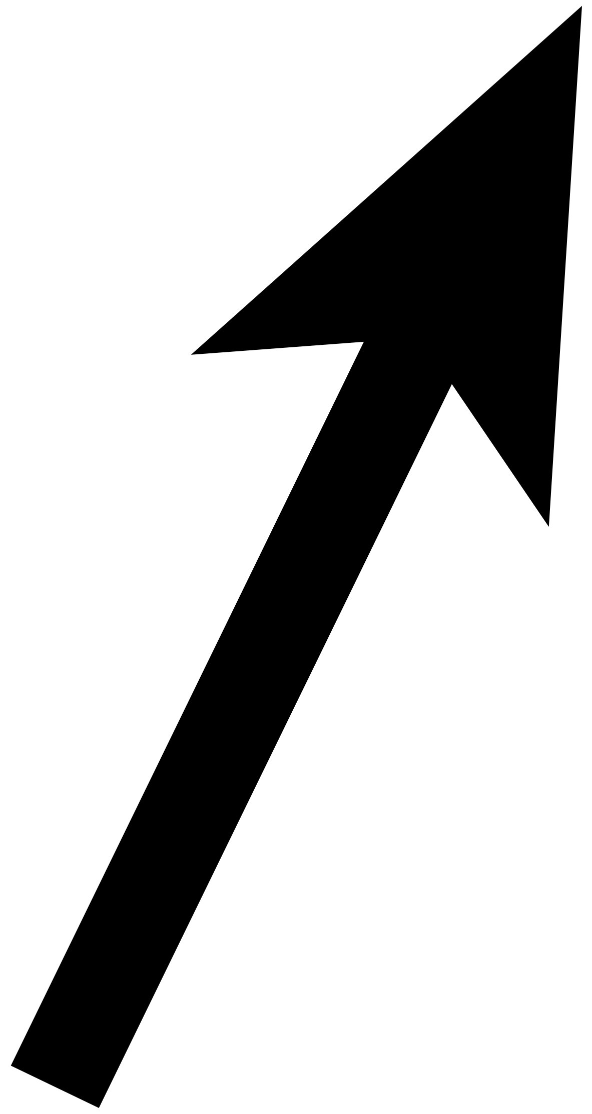


We didn't have to adopt this rule! We chose it because it captures natural behavior.

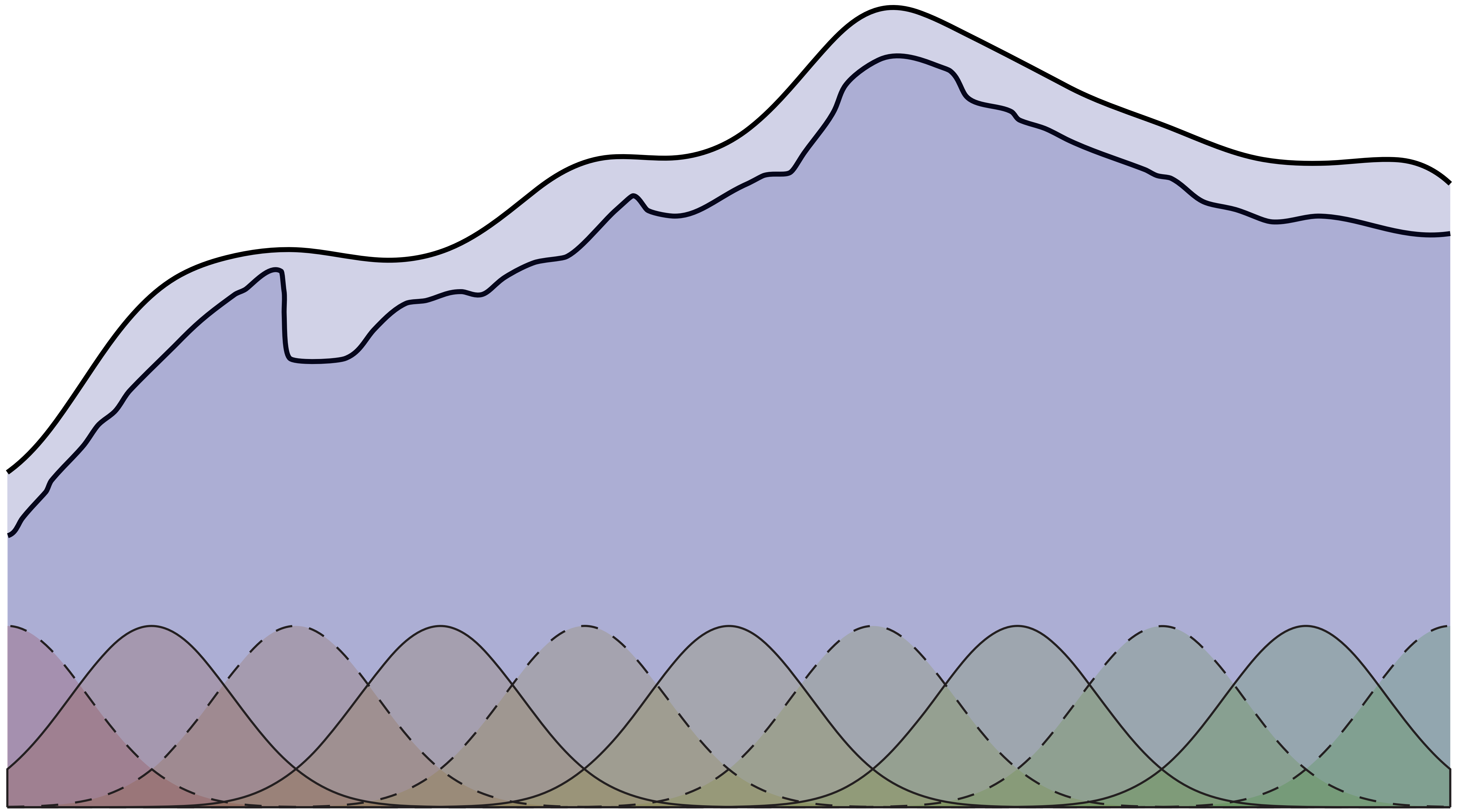
You should <u>never</u> accept a rule purely on faith. Always ask, *"<u>why</u> is this the rule?"*

Review: Vector Spaces

- What is a vector? *(Geometrically?)*



finite-dimensional



infinite-dimensional

For geometric computing, often care most about dimensions 1, 2, 3, …and ∞!

Review: Vector Spaces

- Formally, a *vector space* is a set V together with the operations*

$$+ : V \times V \to V \qquad \text{“addition”}$$

$$\cdot : \mathbb{R} \times V \to V \qquad \text{“scalar multiplication”}$$

- Must satisfy the following rules for all vectors *x,y,z* and scalars *a,b*:

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

$$\exists 0 \in V \text{ s.t. } x + 0 = 0 + x = x$$

$$\forall\, x, \exists \tilde{x} \in V \text{ s.t. } x + \tilde{x} = 0$$

$$(ab)x = a(bx)$$

$$1x = x$$

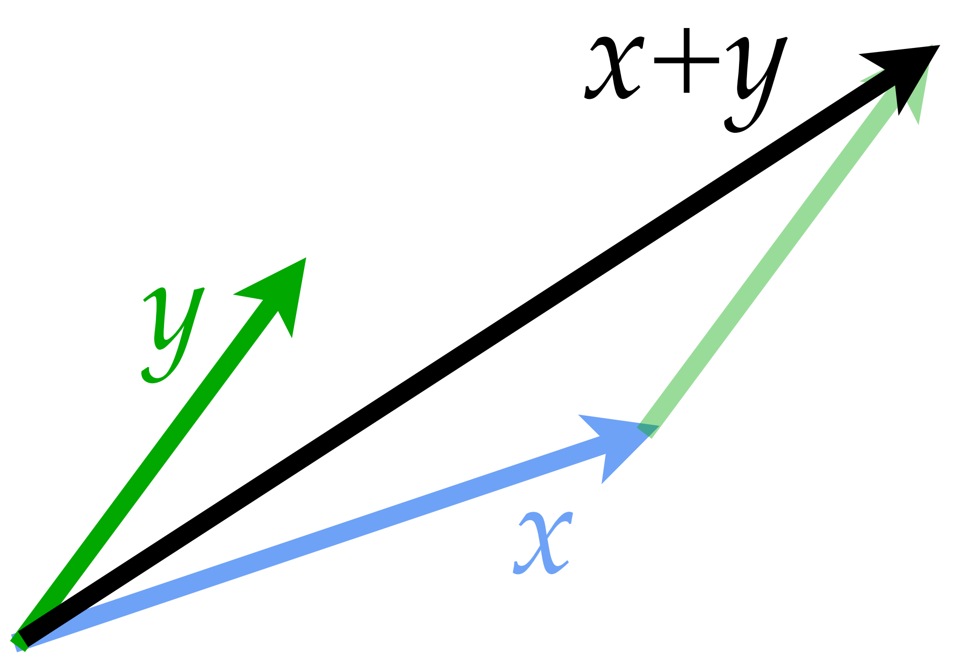
$$a(x + y) = ax + ay$$

$$(a + b)x = ax + bx$$

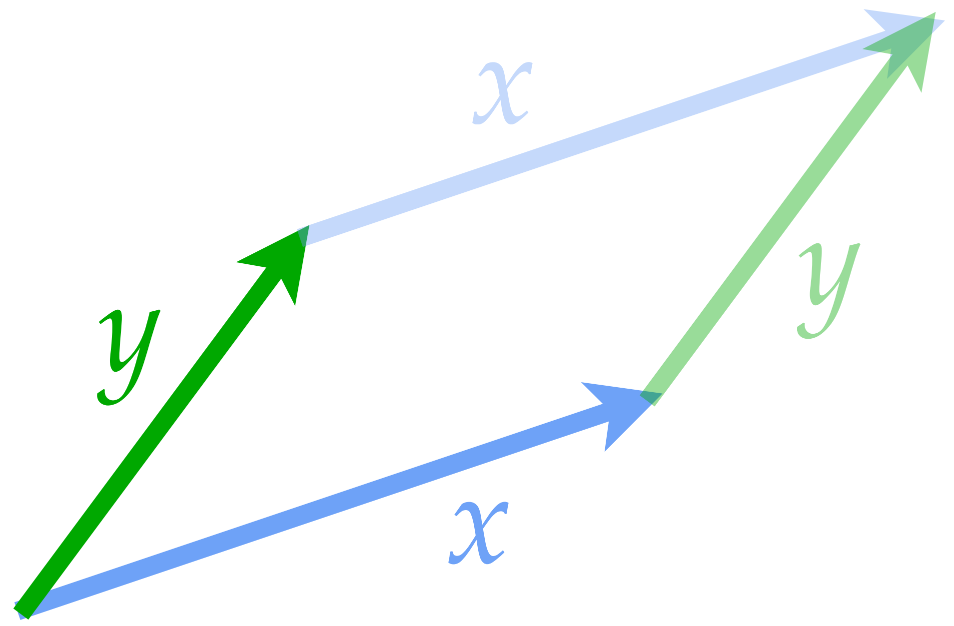
*Note: in general, could use something other than *reals* here.

Vector Spaces—Geometric Reasoning

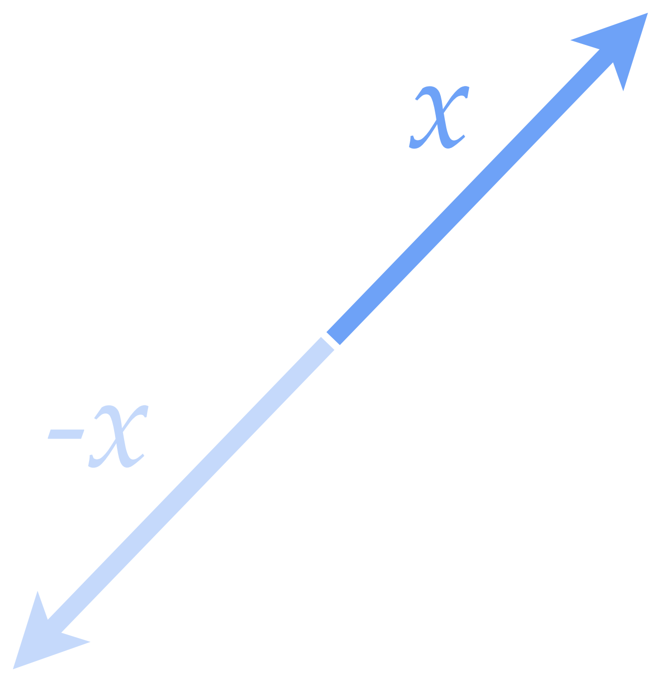
- Where do these rules come from?
- As with numbers, reflect how *oriented lengths* (vectors) behave in nature:



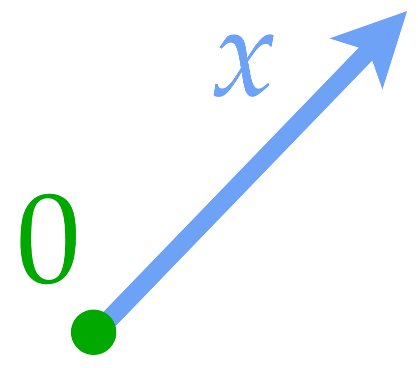
$\boxed{x + y \in V}$



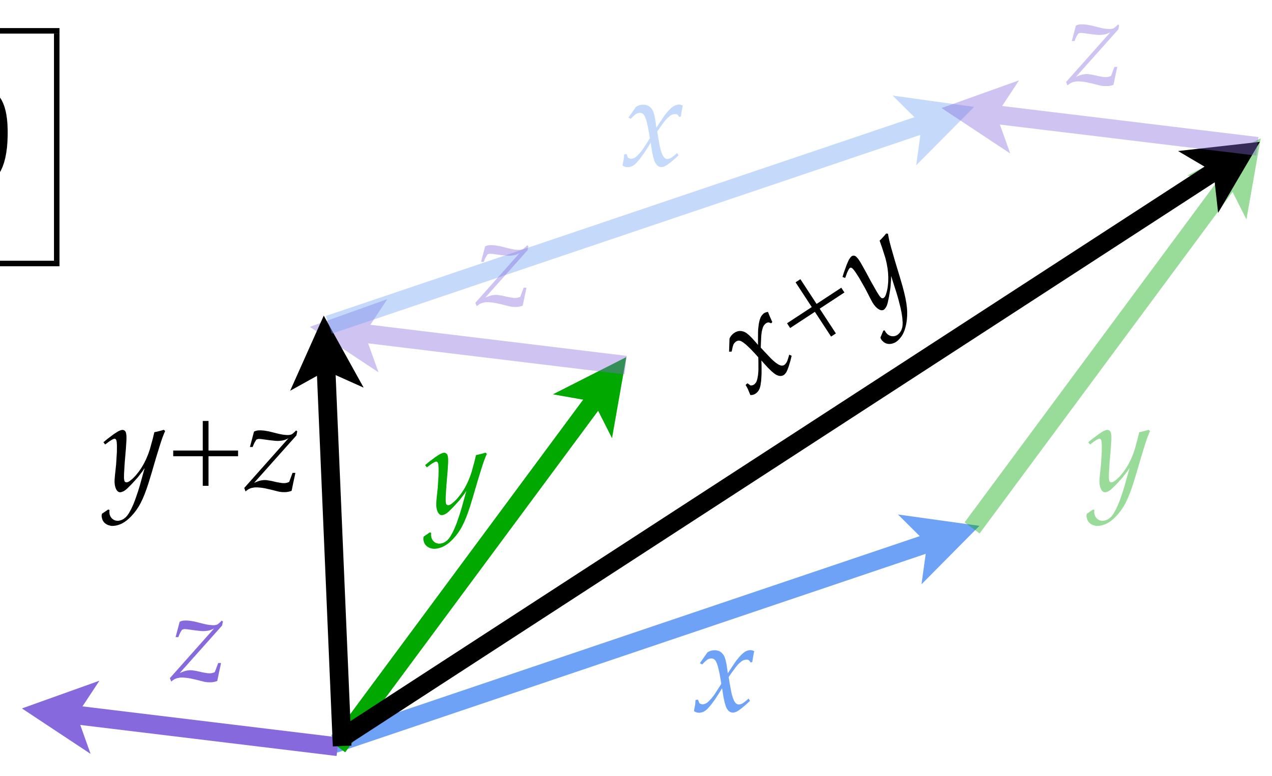
$\boxed{x + y = y + x}$



$\boxed{x + (-x) = 0}$



$\boxed{x + 0 = x}$



$\boxed{(x + y) + z = x + (y + z)}$

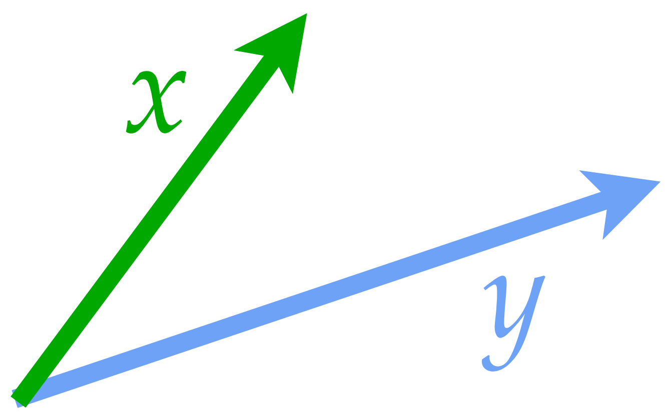
…but the algebra makes it easier to manage complexity!

Review: Inner Product

- We can also associate a vector space with an *inner product*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

- The quantity $\langle x,y \rangle$ captures how well two vectors x, y in V “line up”
- For all vectors x, y, z in V, real numbers a, any *(real) inner product* must satisfy



symmetry

$$\langle x,y \rangle = \langle y,x \rangle$$

linearity

$$\langle ax,y \rangle = a\langle x,y \rangle$$

$$\langle x,y+z \rangle = \langle x,y \rangle + \langle x,z \rangle$$

positivity

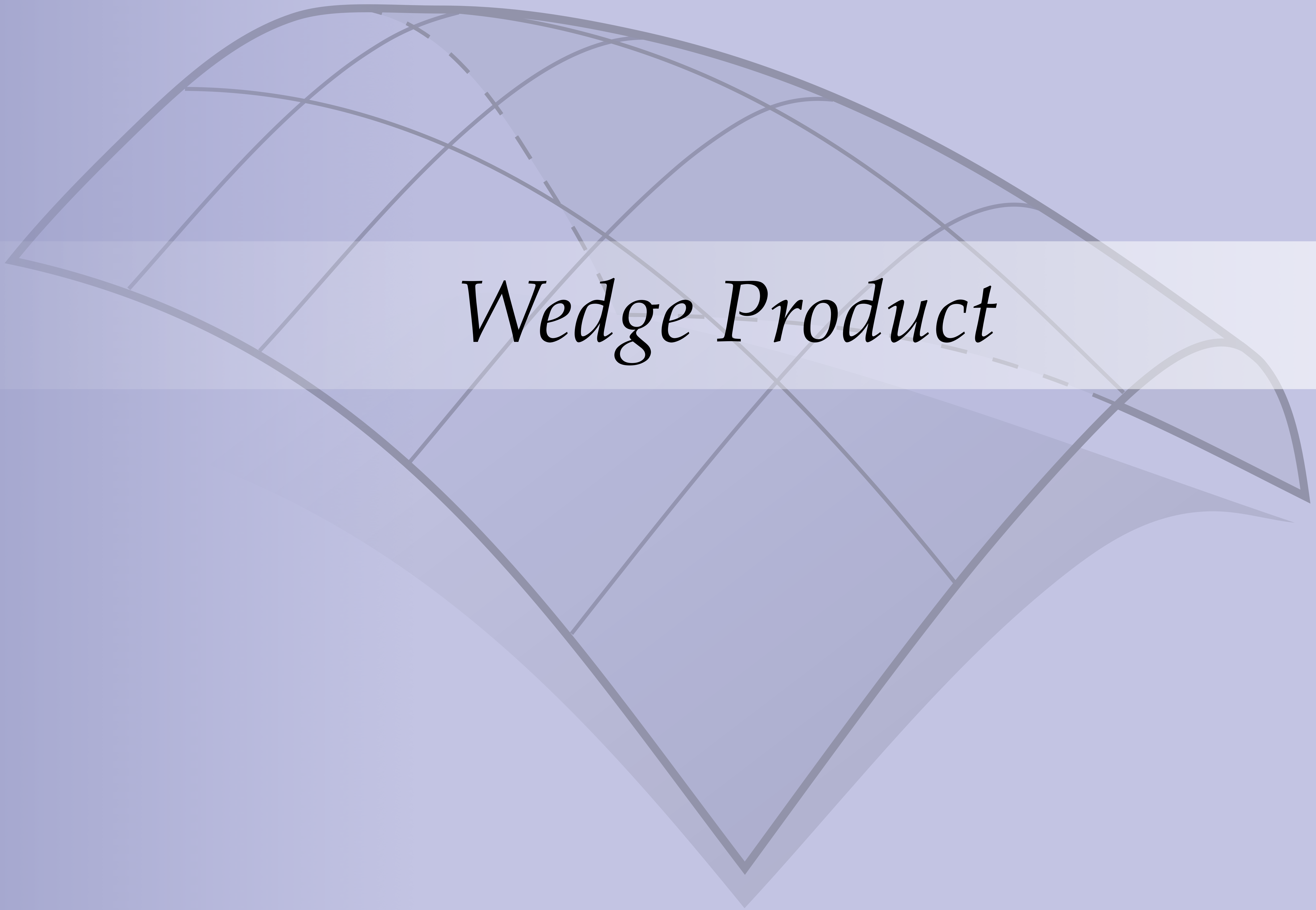
$$\langle x,x \rangle > 0, \; x \neq 0$$

$$\langle x,x \rangle = 0, \; x = 0$$

Example. *Euclidean* inner product $\langle x,y \rangle = \sum_{i=1}^{n} x_i y_i$

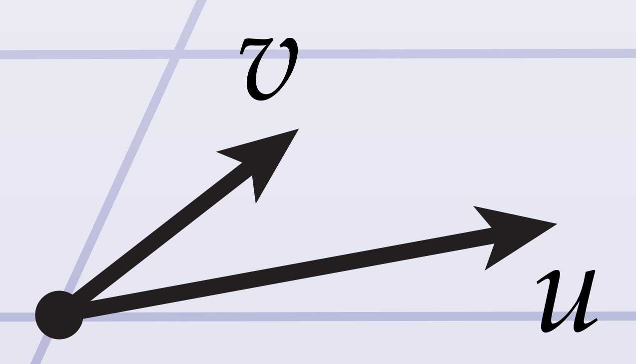
(Where do these “rules” come from? Why might they be natural?)

Wedge Product



Review: Span

Q: Geometrically, what is the *span* of two vectors?



$$u, v \in V, \quad \mathrm{span}(\{u, v\}) := \{x \in V \mid x = au + bv,\ a, b \in \mathbb{R}\}$$

Span

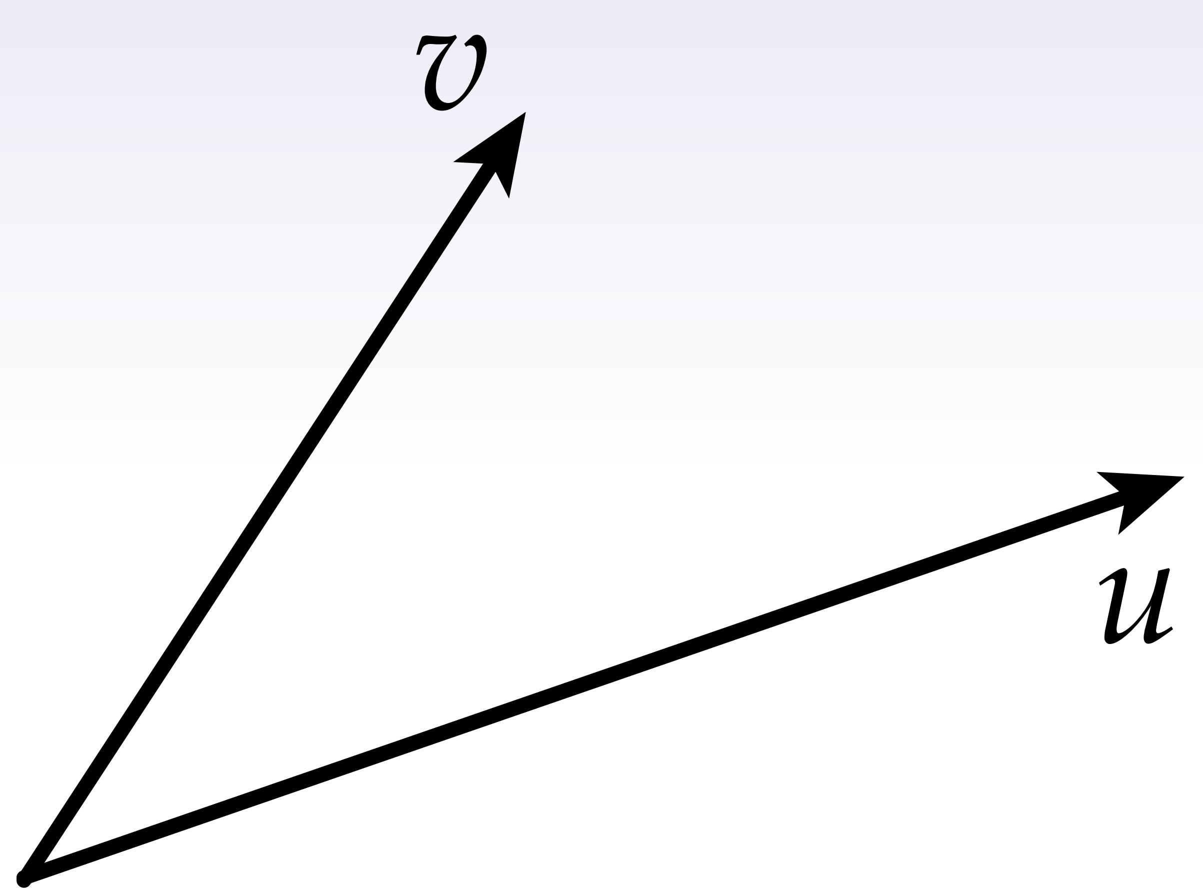
Definition. In any vector space V, the *span* of a finite* collection of vectors $\{ v_1, \ldots, v_k \}$ is the set of all possible linear combinations:

$$\text{span}(\{v_1, \ldots, v_n\}) := \left\{ x \in V \;\middle|\; x = \sum_{i=1}^{k} a_i v_i, \quad a_i \in \mathbb{R} \right\}.$$

The span of a collection of vectors is a *linear subspace*, *i.e.*, a subset that forms a vector space with respect to the original vector space operations.

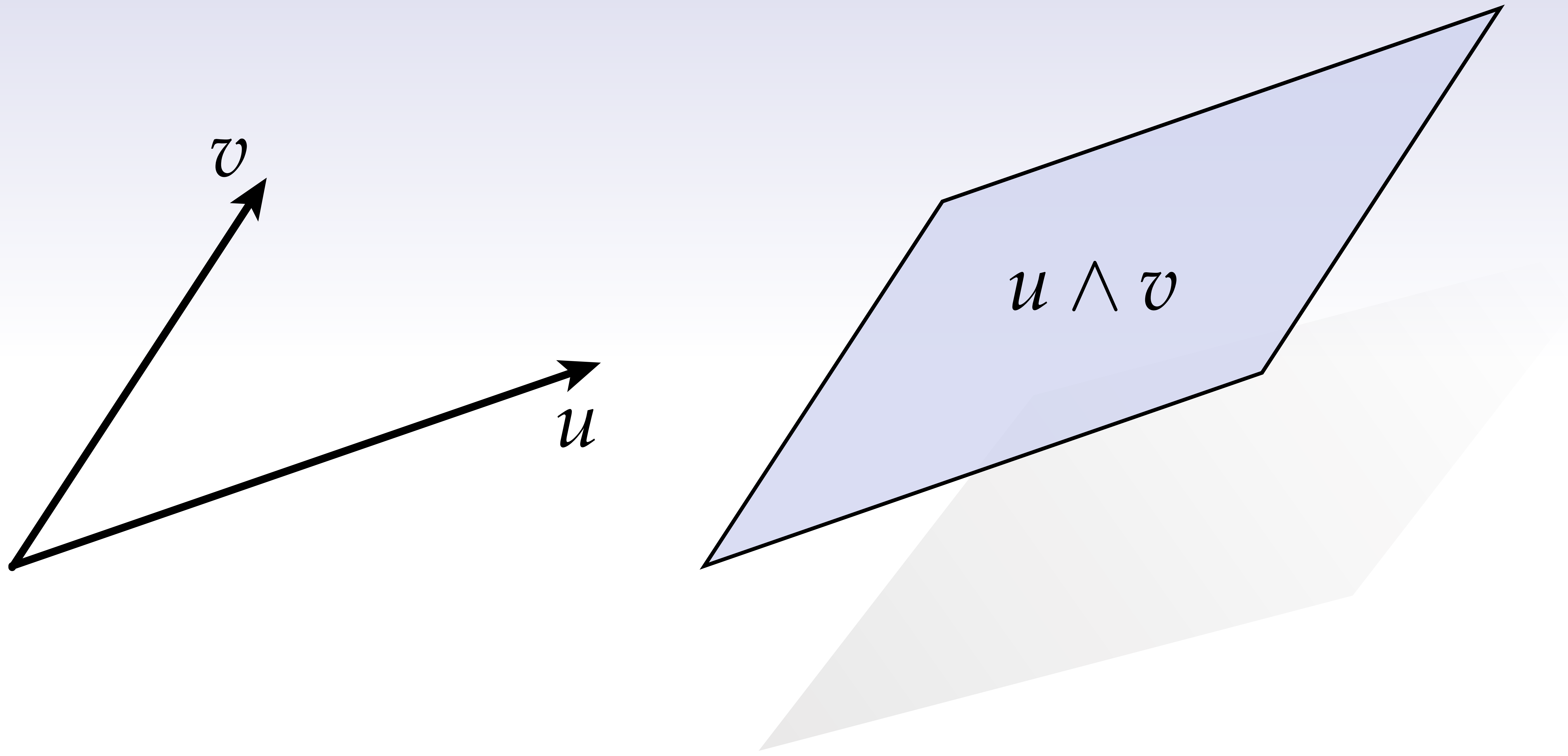
*Note: one can extend this definition to infinite sums, but only with additional assumptions about V.

Wedge Product $(\wedge)$



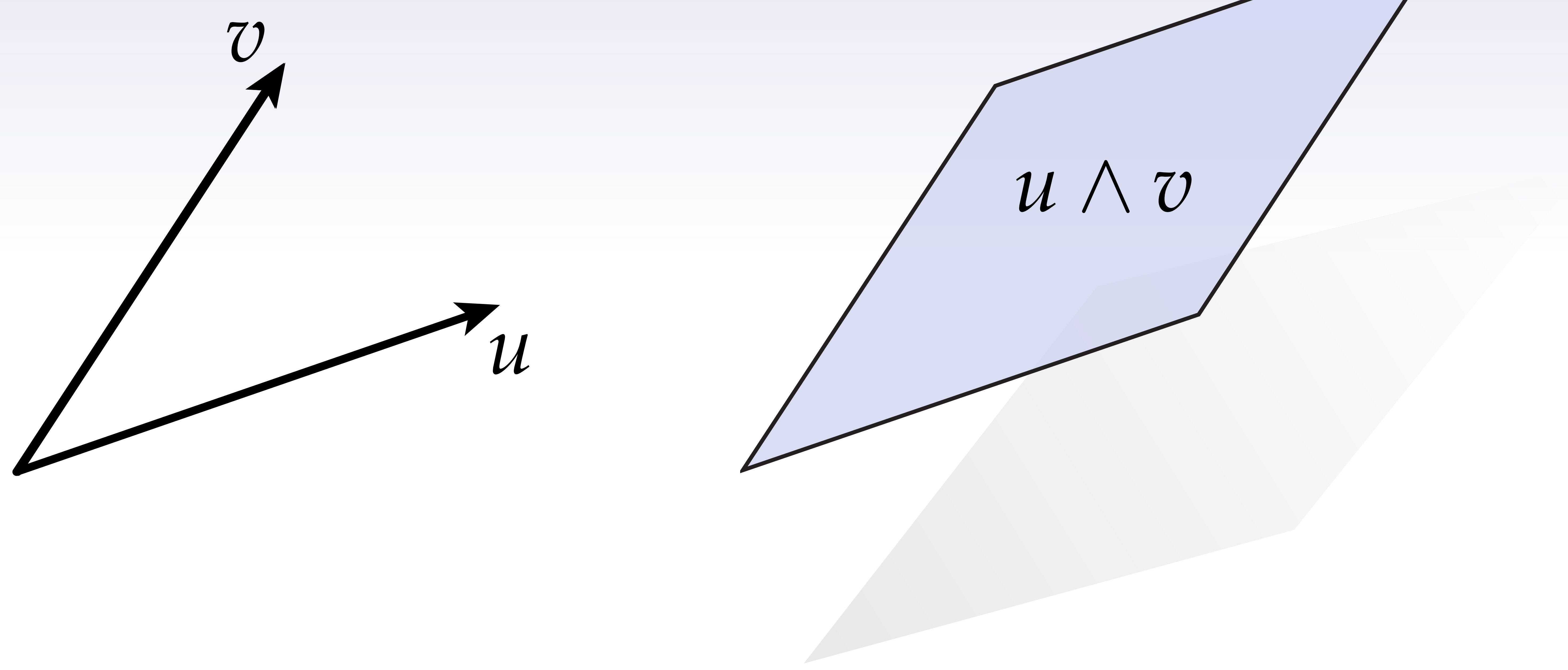
Analogy: *span*

Wedge Product $(\wedge)$



Analogy: *span*

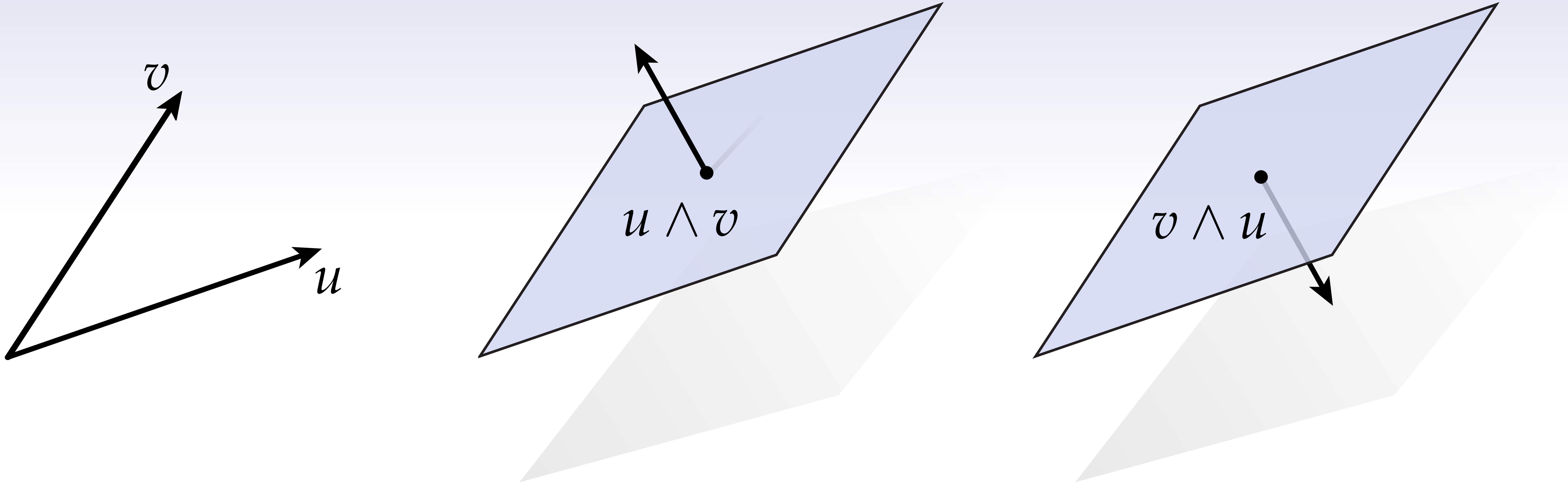
Wedge Product $(\wedge)$



Analogy: *span*

Wedge Product $(\wedge)$

$$u \wedge v = -v \wedge u$$



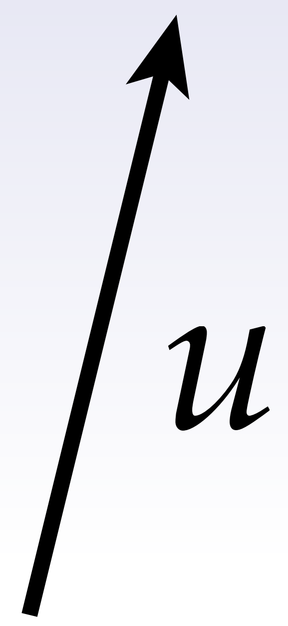
Analogy: *span*

Key differences: orientation & “finite extent”

Key property: *antisymmetry*

Wedge Product—Degeneracy

Q: What is the wedge product of a vector with itself?

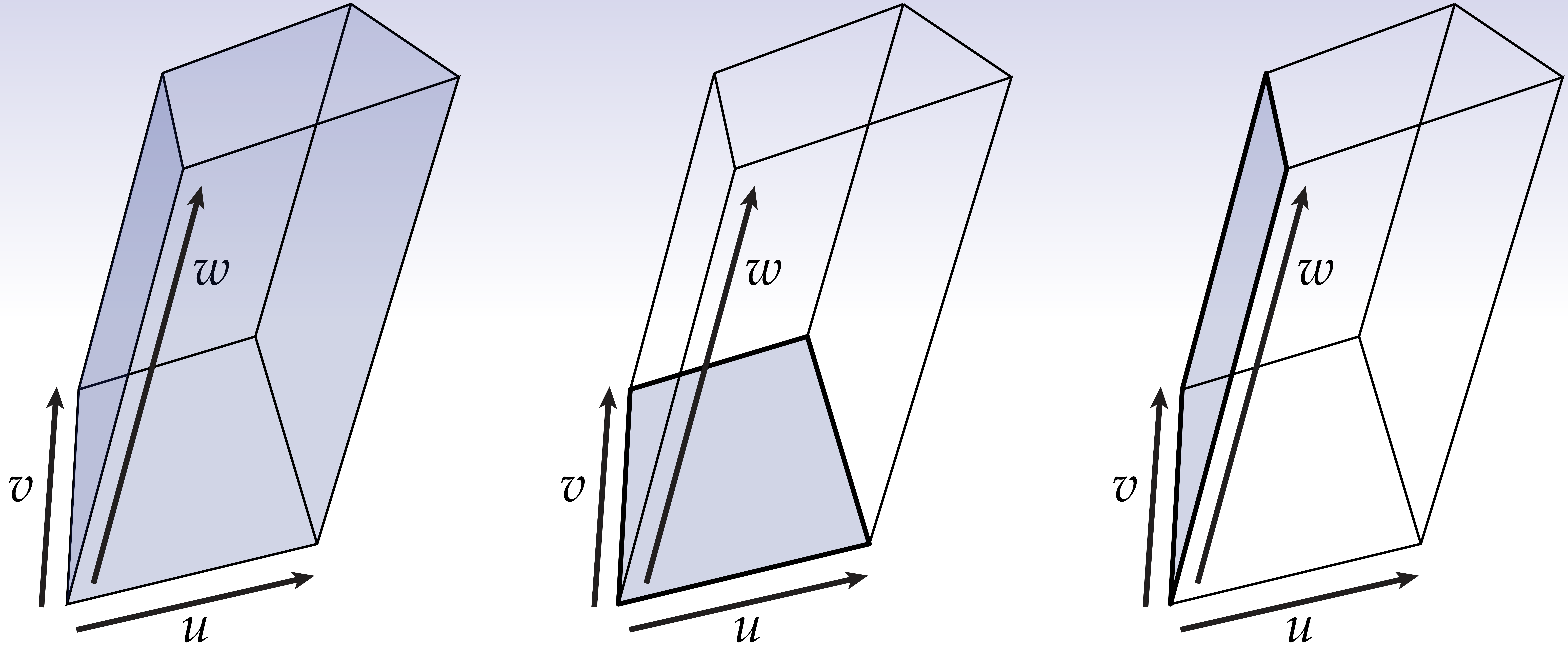


A: Geometrically, spans a region of *zero area*.

$$\boxed{u \wedge u = 0}$$

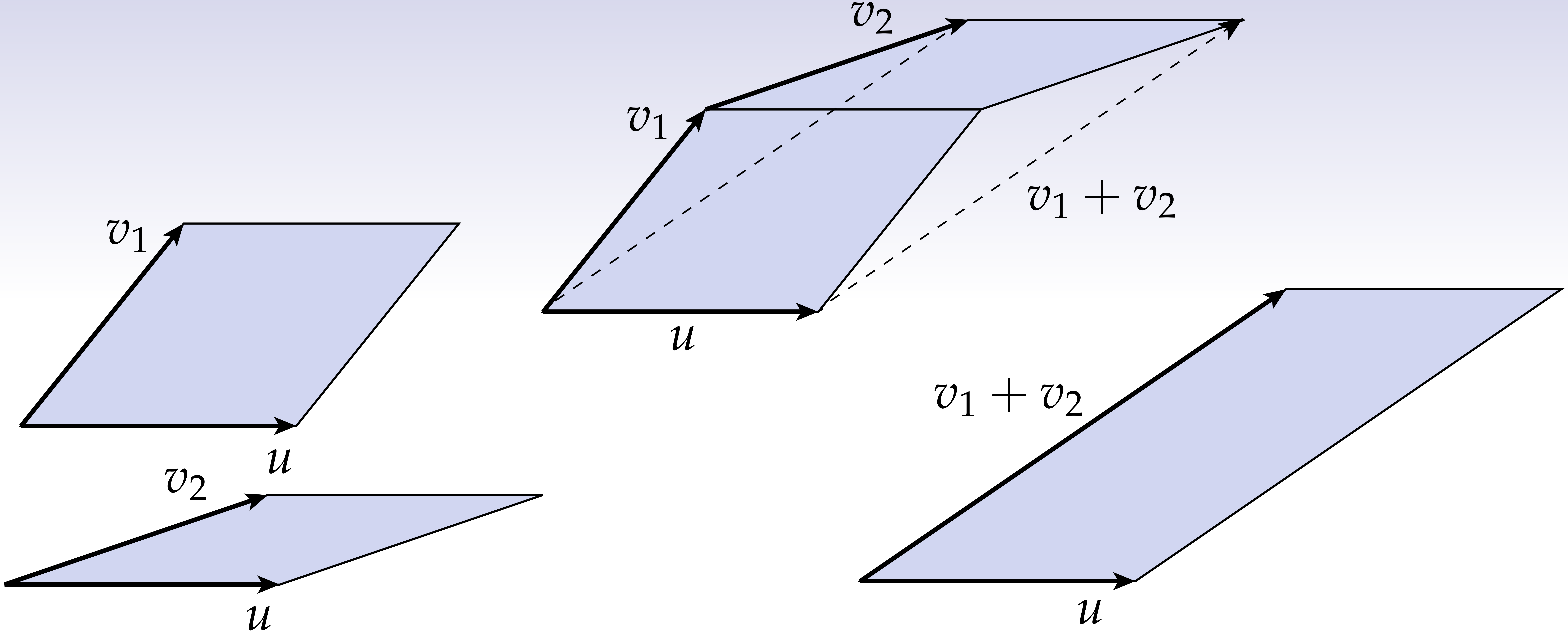
*May change when we generalize (later…)

Wedge Product - Associativity



$u \wedge v \wedge w$ $\qquad$ $(u \wedge v) \wedge w$ $\qquad$ $u \wedge (v \wedge w)$

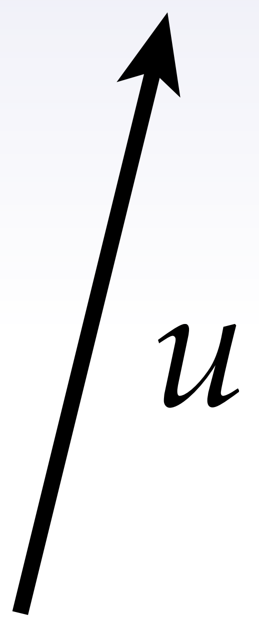
Wedge Product - Distributivity

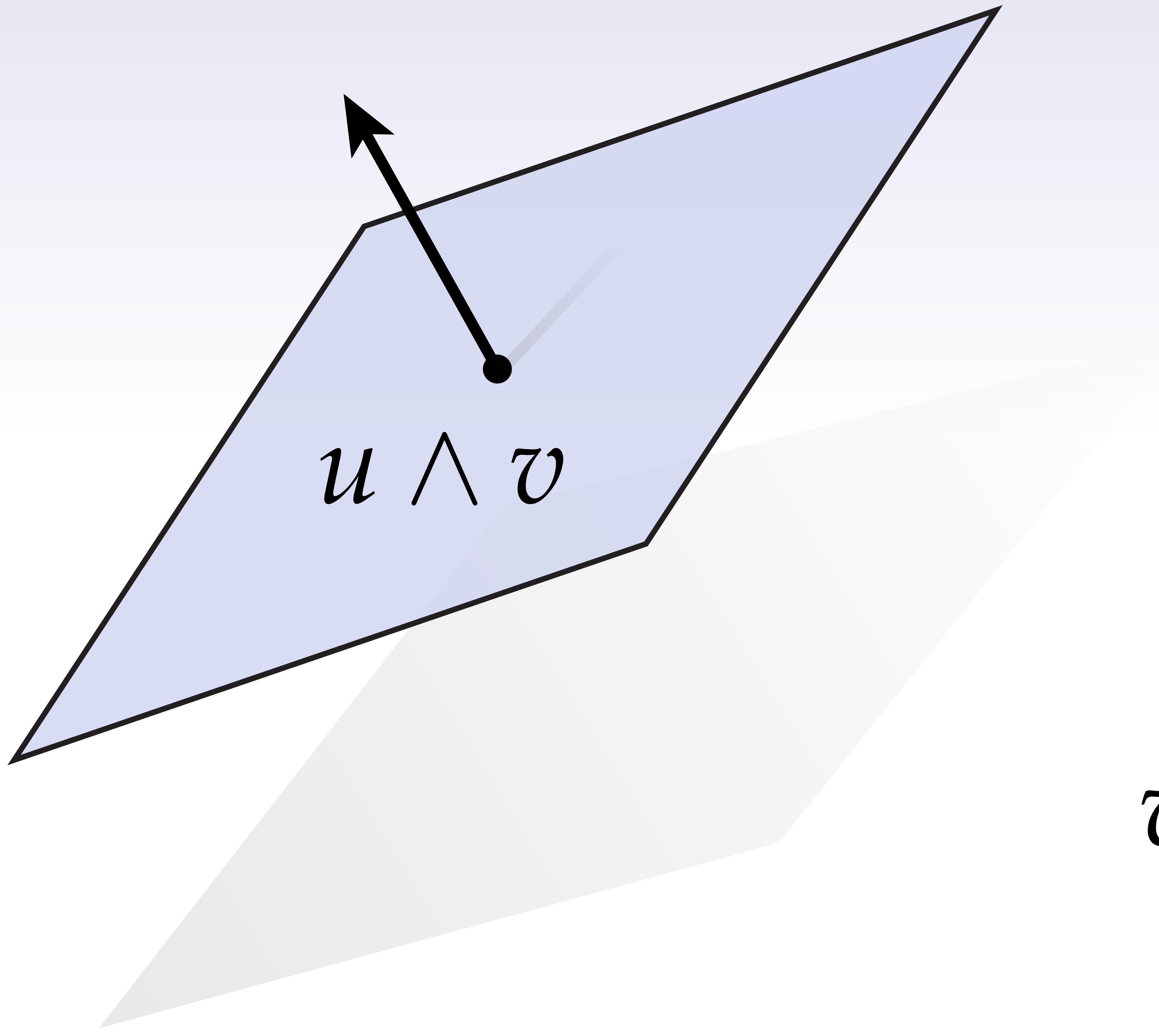


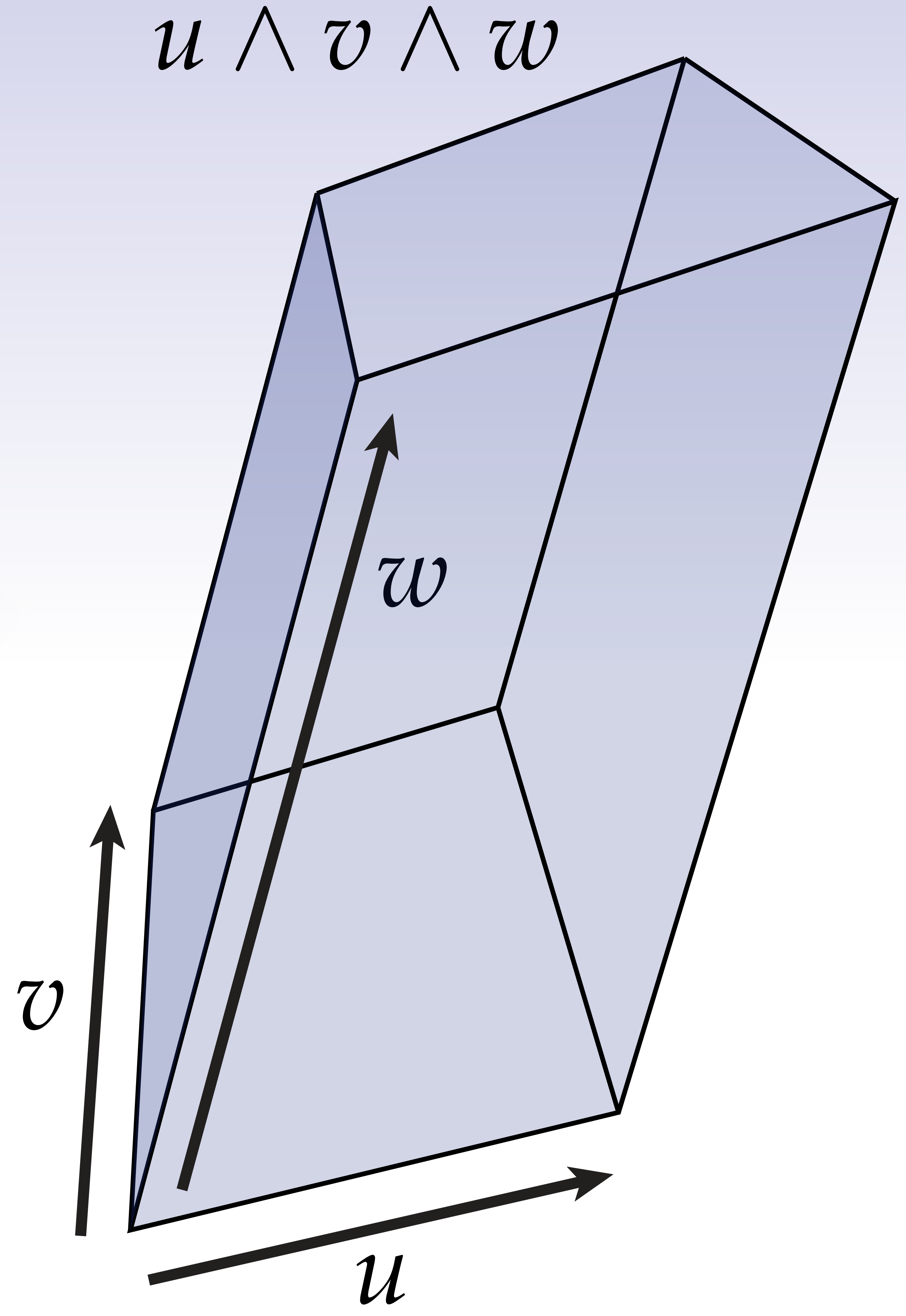
$$u \wedge v_1 + u \wedge v_2 = u \wedge (v_1 + v_2)$$

k-Vectors

The wedge of k vectors is called a *"k-vector."*





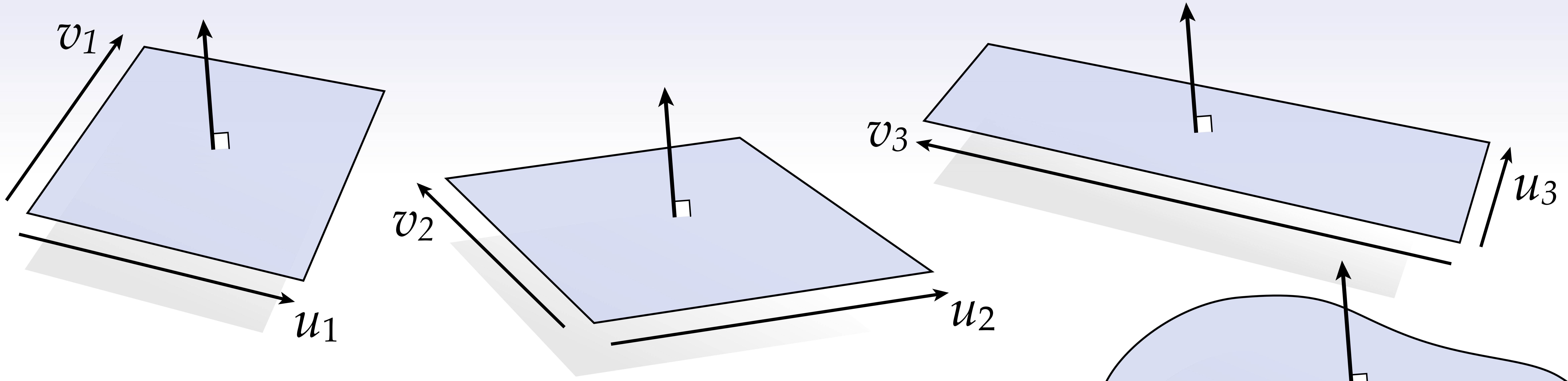


0-vector **1-vector** **2-vector** **3-vector**

Visualization of k-Vectors

Our visualization is a little misleading: k-vectors only have *direction* & *magnitude*.

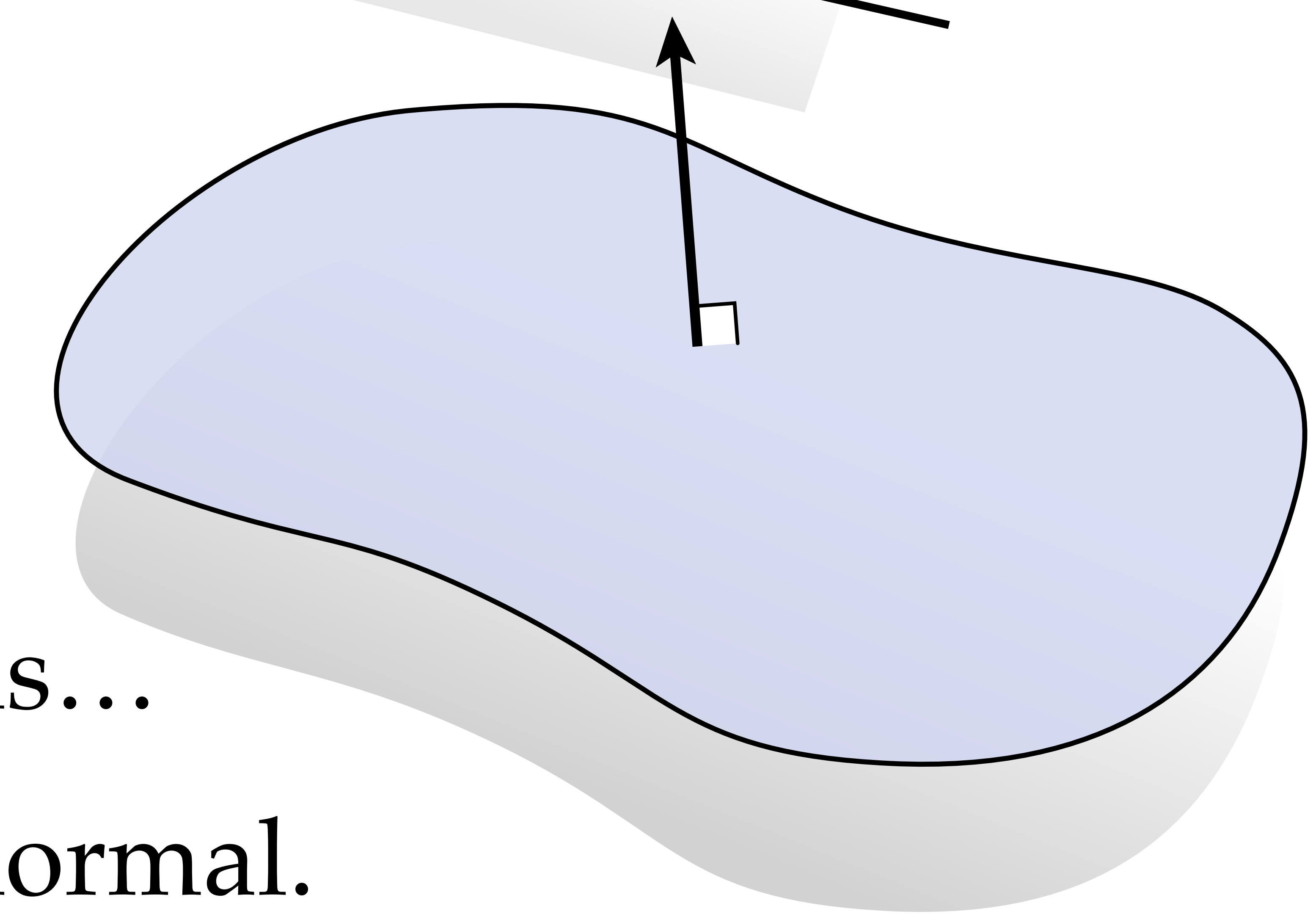
E.g., parallelograms w/ same plane, orientation, and area represent same 2-vector:



$$u_1 \wedge v_1 = u_2 \wedge v_2 = u_3 \wedge v_3$$

Might say a 2-vector is an *equivalence class* of parallelograms…

…or more generally *any* little patch with the same area & normal.



0-vectors as Scalars

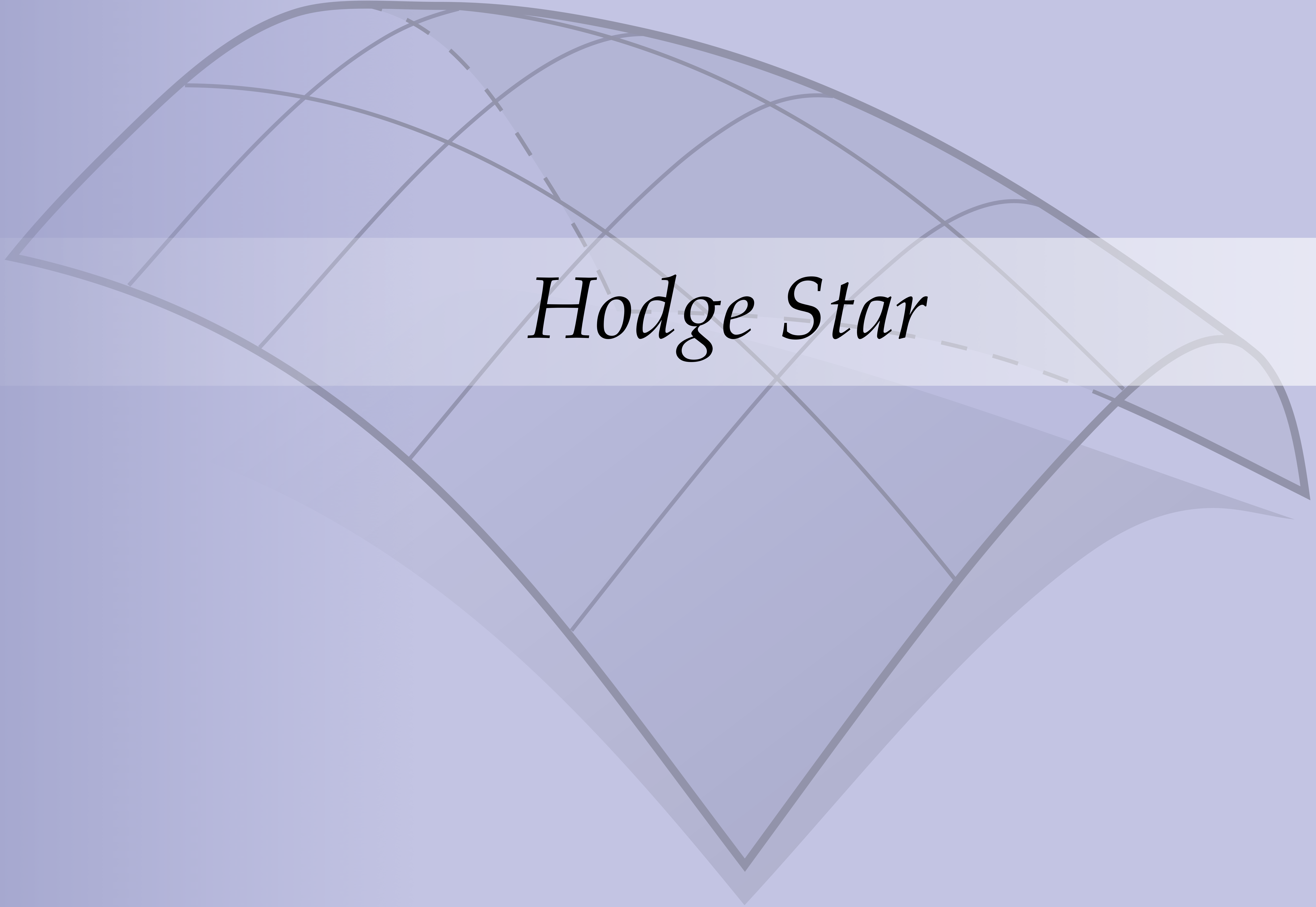
Q: What do you get when you wedge *zero* vectors together?

A: You get this:

For convenience, however, we will say that a "0-vector" is a *scalar value* (e.g., a real number). This treatment becomes extremely useful later on...

Key idea: *magnitude*, but no *direction* (scalar).

Hodge Star



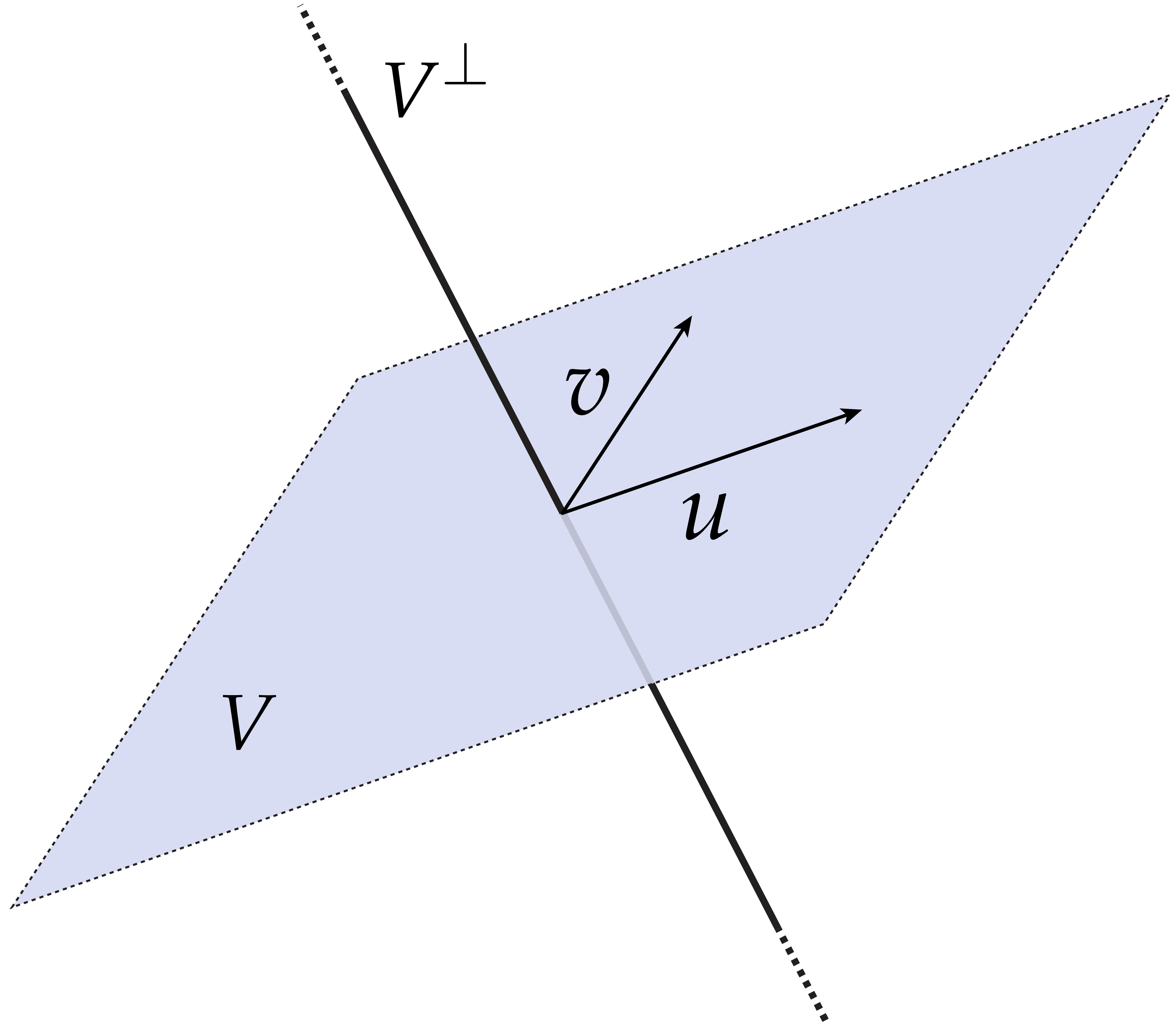
Review: Orthogonal Complement

Q: Geometrically, what is the *orthogonal complement* of a linear subspace?

Example: *orthogonal complement of a span*

$V := \mathrm{span}(\{u, v\})$

$V^\perp := \{x \in \mathbb{R}^n | \langle x, w \rangle = 0 \; \forall w \in V\}$

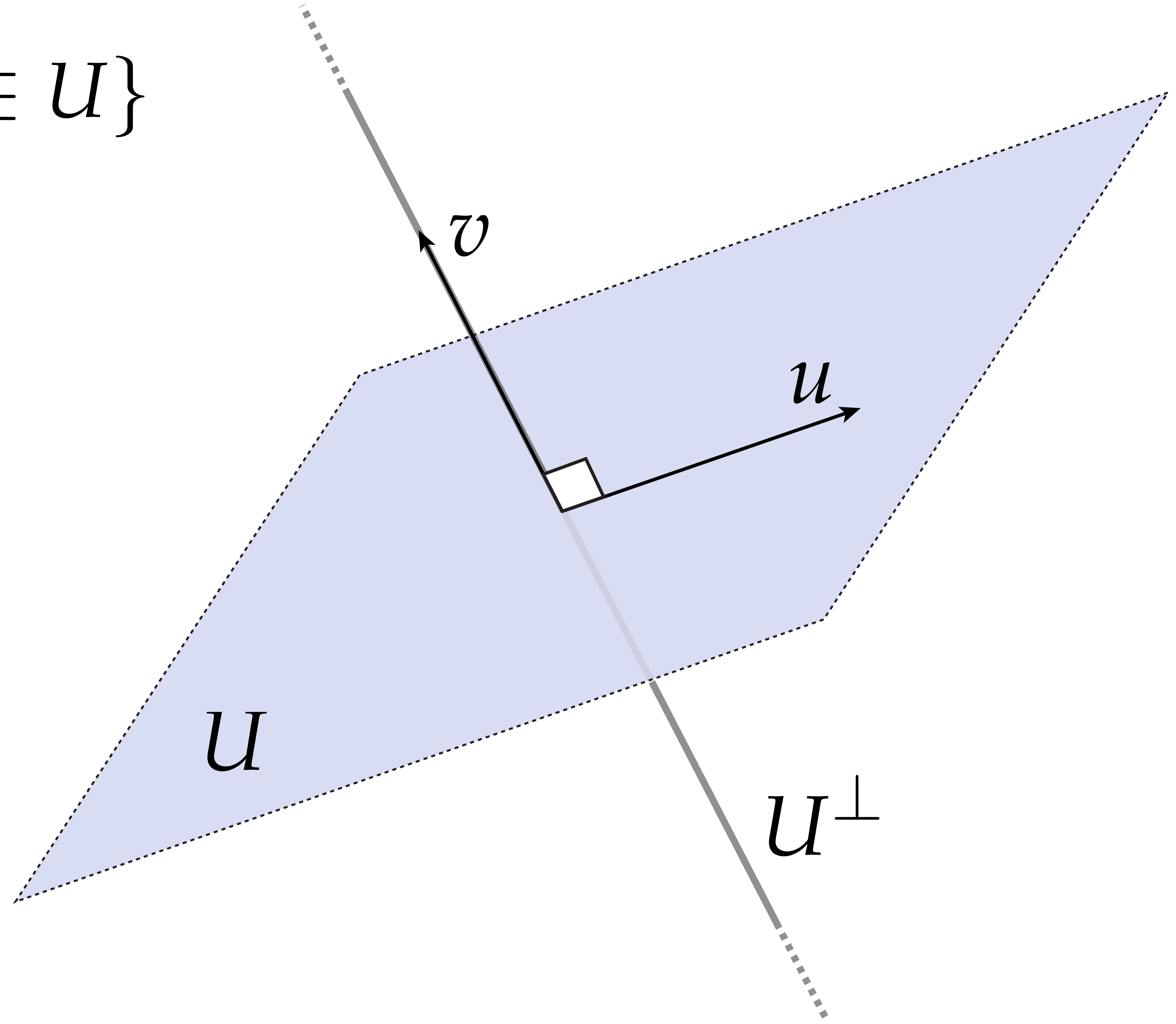


Notice: orthogonal complement meaningful only if we have an *inner product!*

Orthogonal Complement

Definition: Let $U \subseteq V$ be a linear subspace of a vector space V with an inner product $\langle\cdot,\cdot\rangle$. The *orthogonal complement* of U is the collection of vectors

$$U^\perp := \{v \in V | \langle u, v\rangle = 0,\ \forall u \in U\}$$



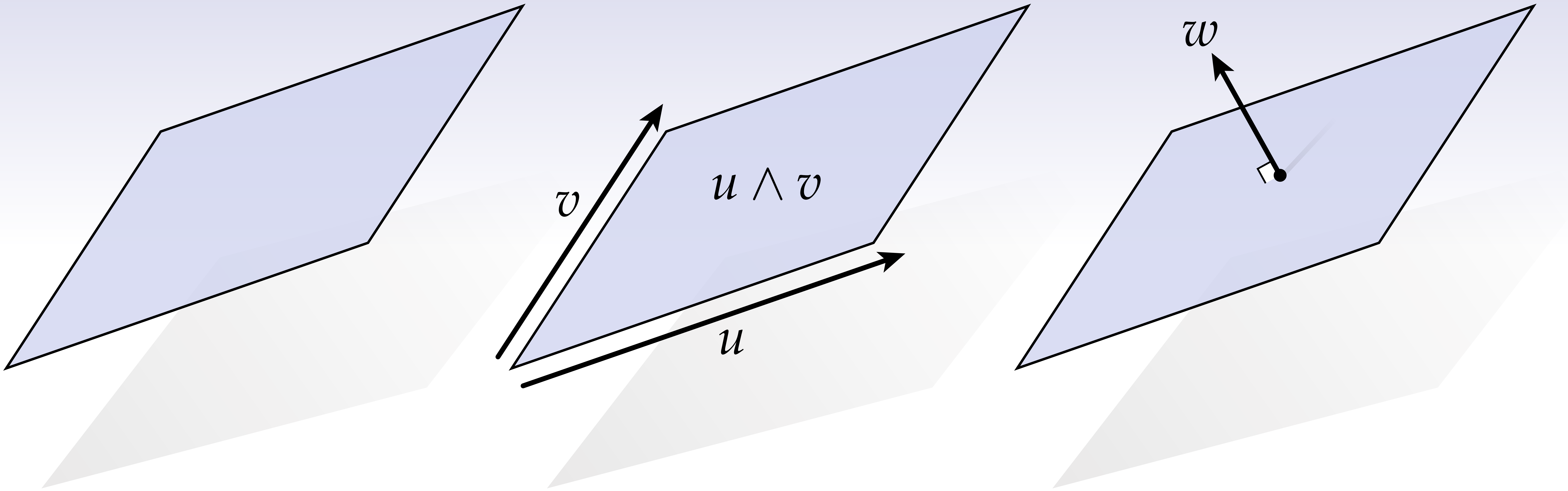
Why is it useful to talk about a *complement?*

Example. "What kind of cuisine do you like?"
Option 1: "I like Vietnamese, Italian, Ethiopian, ..."
Option 2: "I like everything but Bavarian food!"

Key idea: often it's easier to specify a set by saying what it *doesn't* contain.

Hodge Star $(\star)$

$$\star(u \wedge v) = w$$



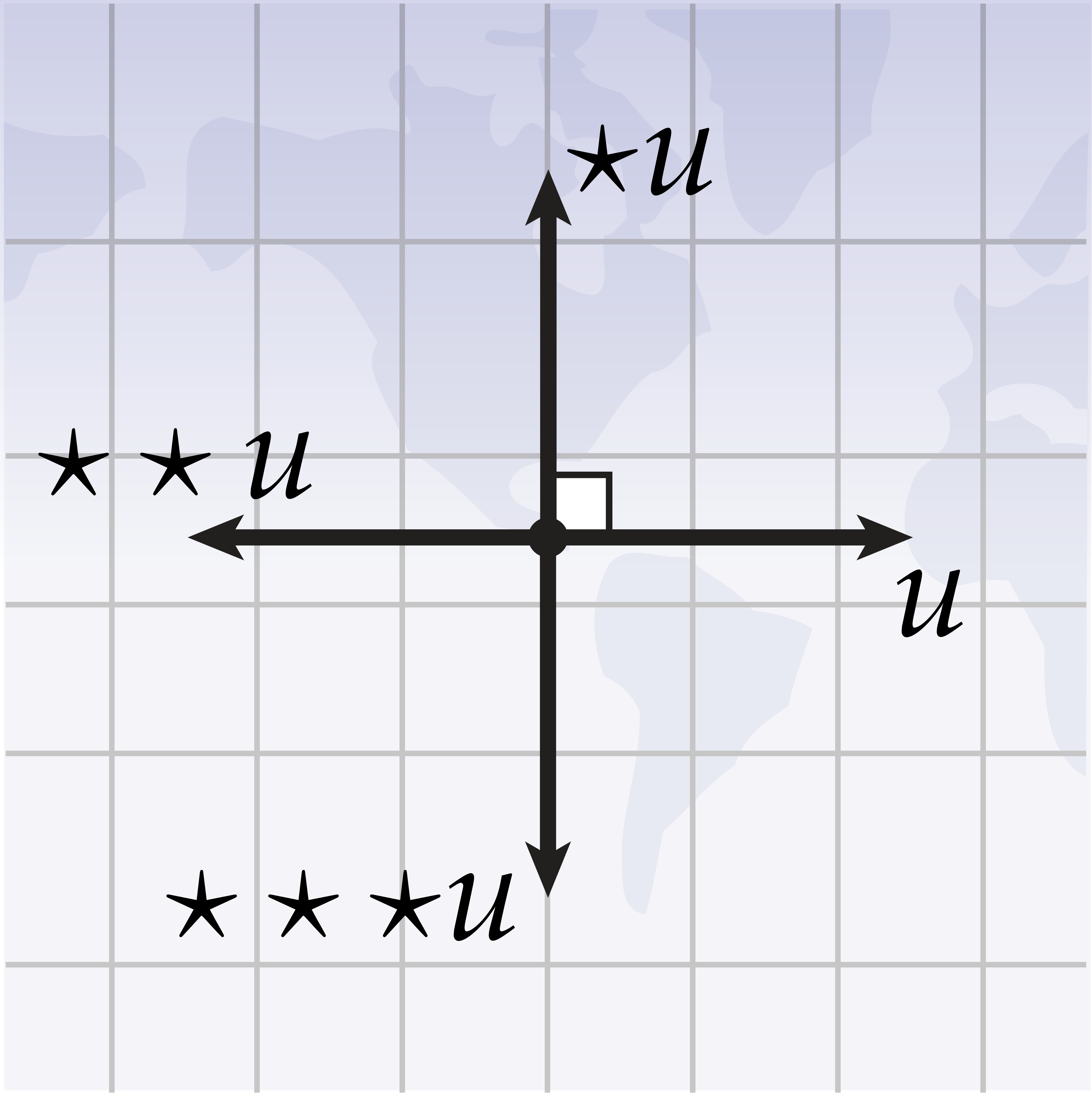
Analogy: *orthogonal complement*

Key differences: orientation & magnitude

Important detail: $z \wedge \star z$ is *positively oriented*

$$k \mapsto (n-k)$$

Hodge Star - 2D

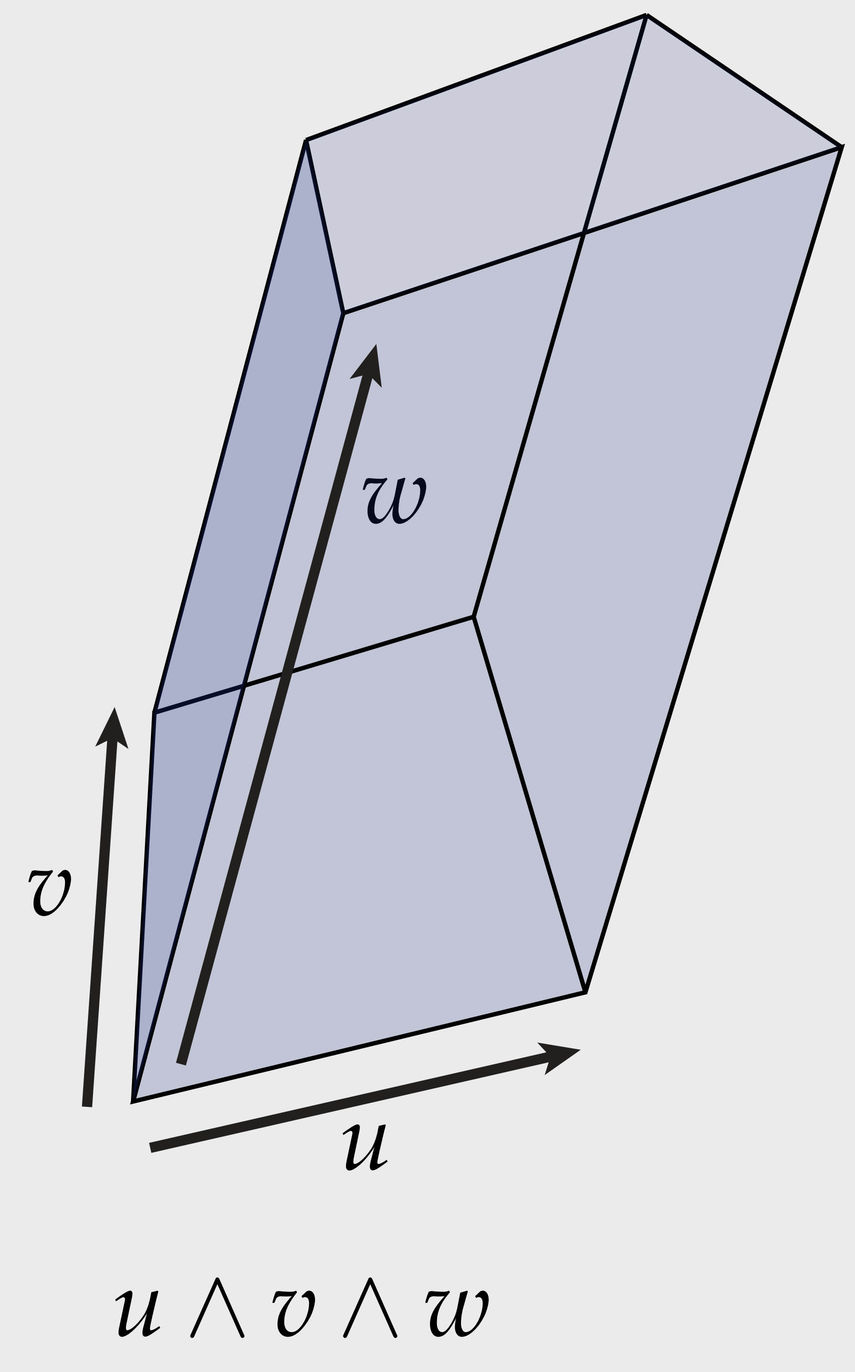


Analogy: *90-degree rotation*

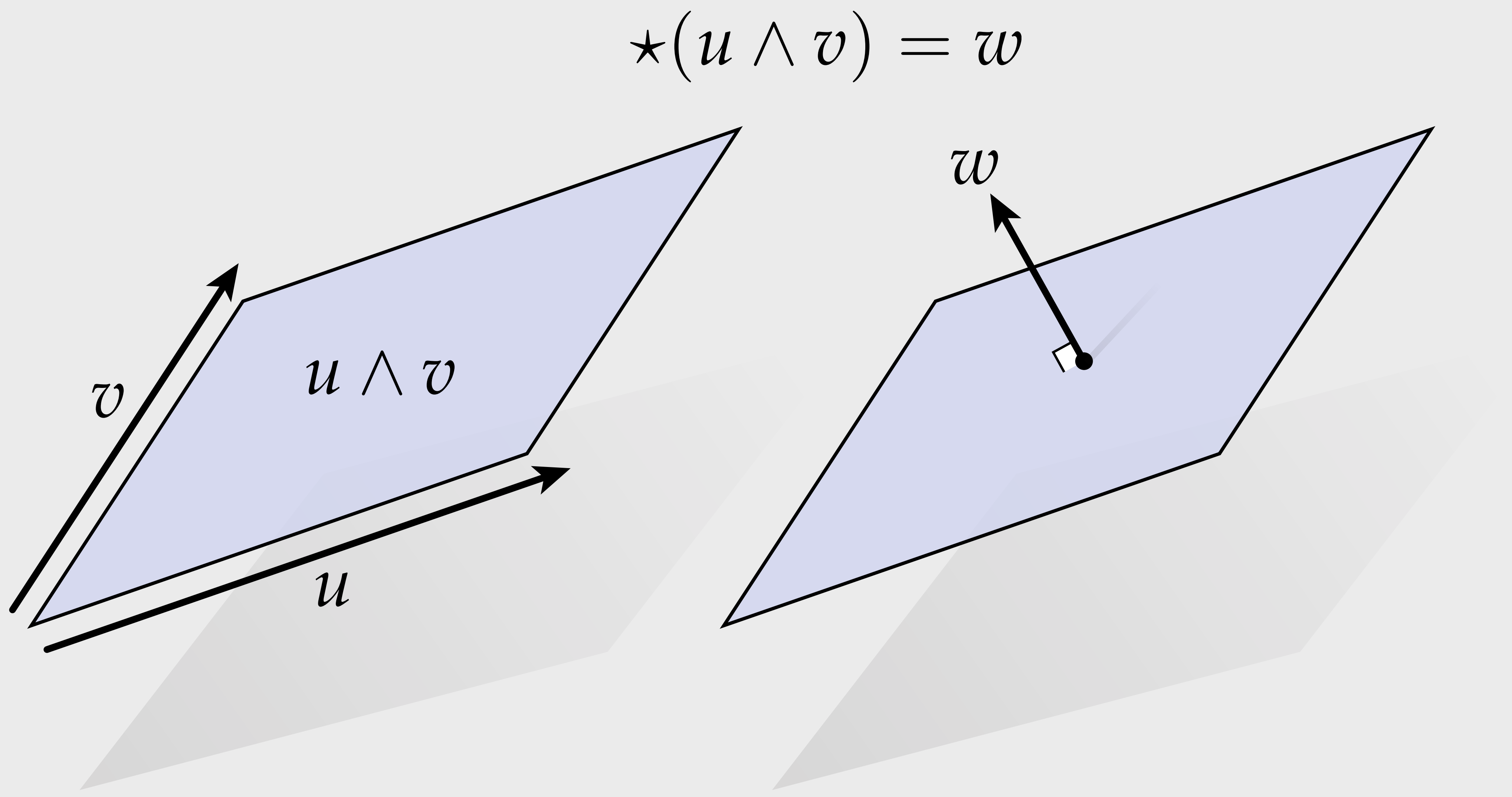
Exterior Algebra—Recap

Let V be an n-dimensional vector space, consisting of vectors or *1-vectors*.

Can "wedge together" k vectors to get a *k-vector* (signed volume).



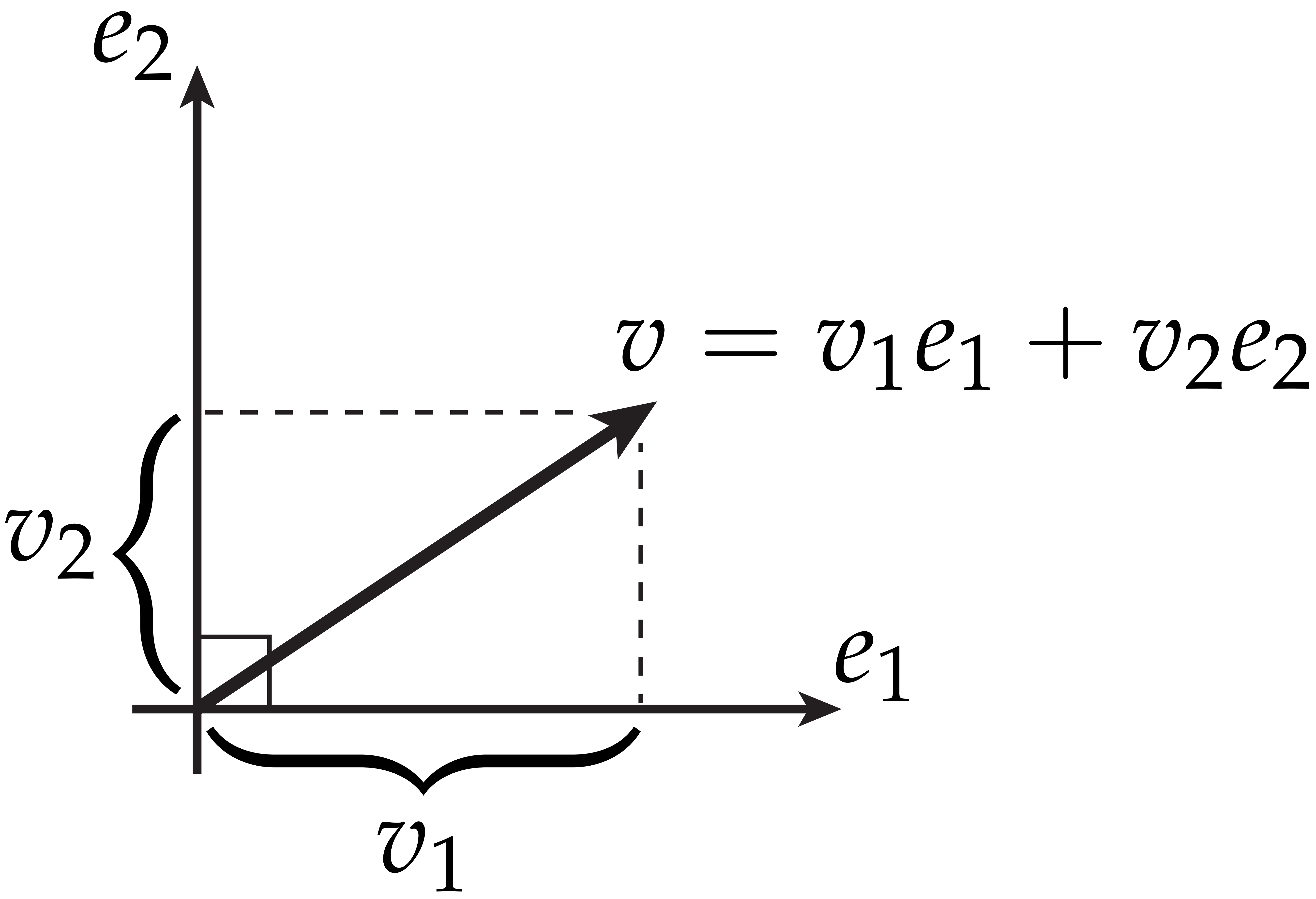
Can apply the Hodge star to get the "complementary" k-vector.



(Also have the usual vector space operations: sum, scalar multiplication, …)

Coordinate Representation

Basis — Visualized



Key idea: encode a vector by its extent along a collection of independent axes.

Basis & Dimension

Definition. Let V be a vector space. A collection of vectors is *linearly independent* if no vector in the collection can be expressed as a linear combination of the others. A linearly independent collection of vectors $\{e_1, \ldots, e_n\}$ is a *basis* for V if every vector $v \in V$ can be expressed as

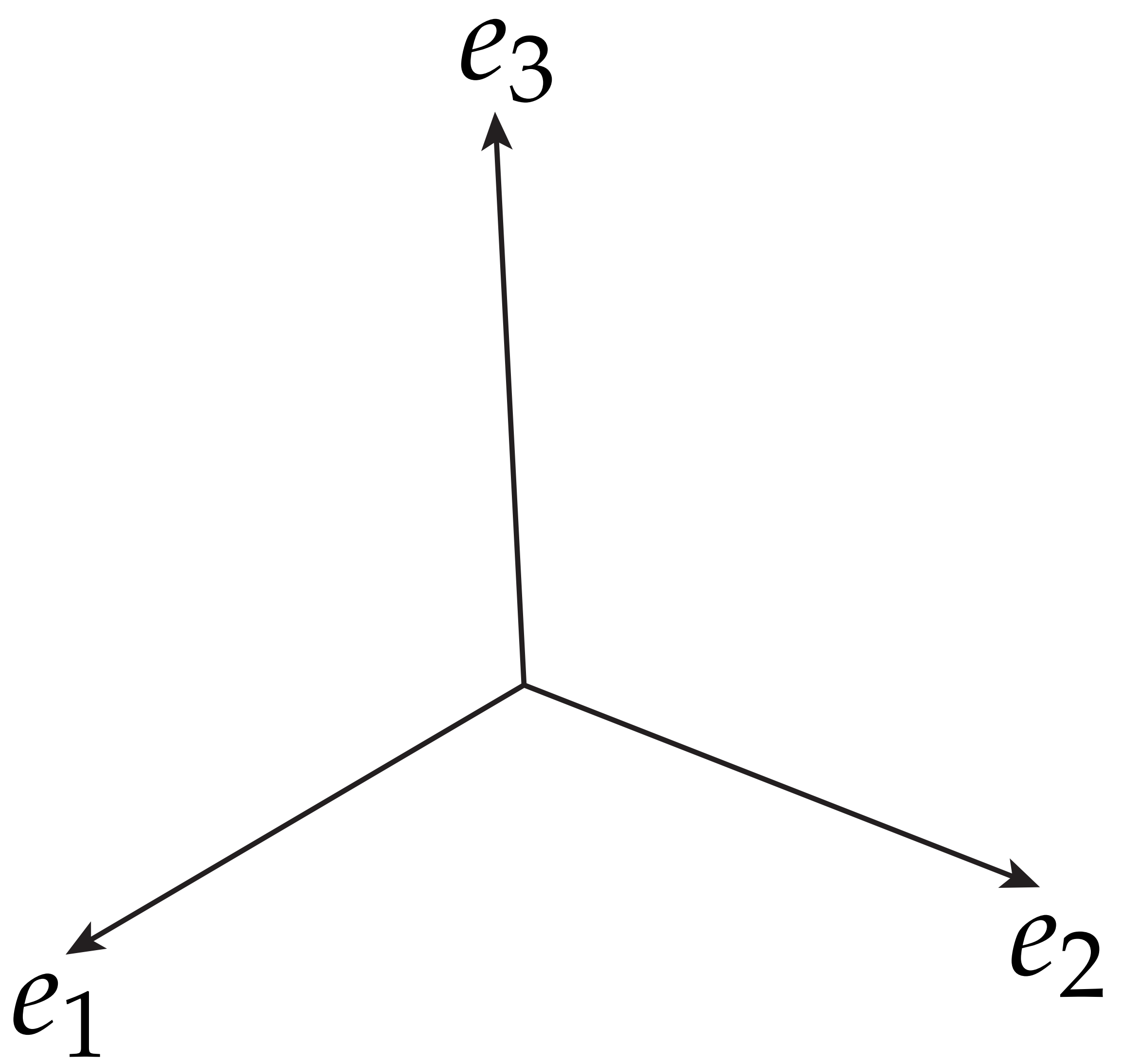
$$v = v_1 e_1 + \cdots + v_n e_n$$

for some collection of coefficients $v_1, \ldots, v_n \in \mathbb{R}$, i.e., if every vector can be uniquely expressed as a linear combination of the *basis vectors* e_i. In this case, we say that V is *finite dimensional*, with dimension n.

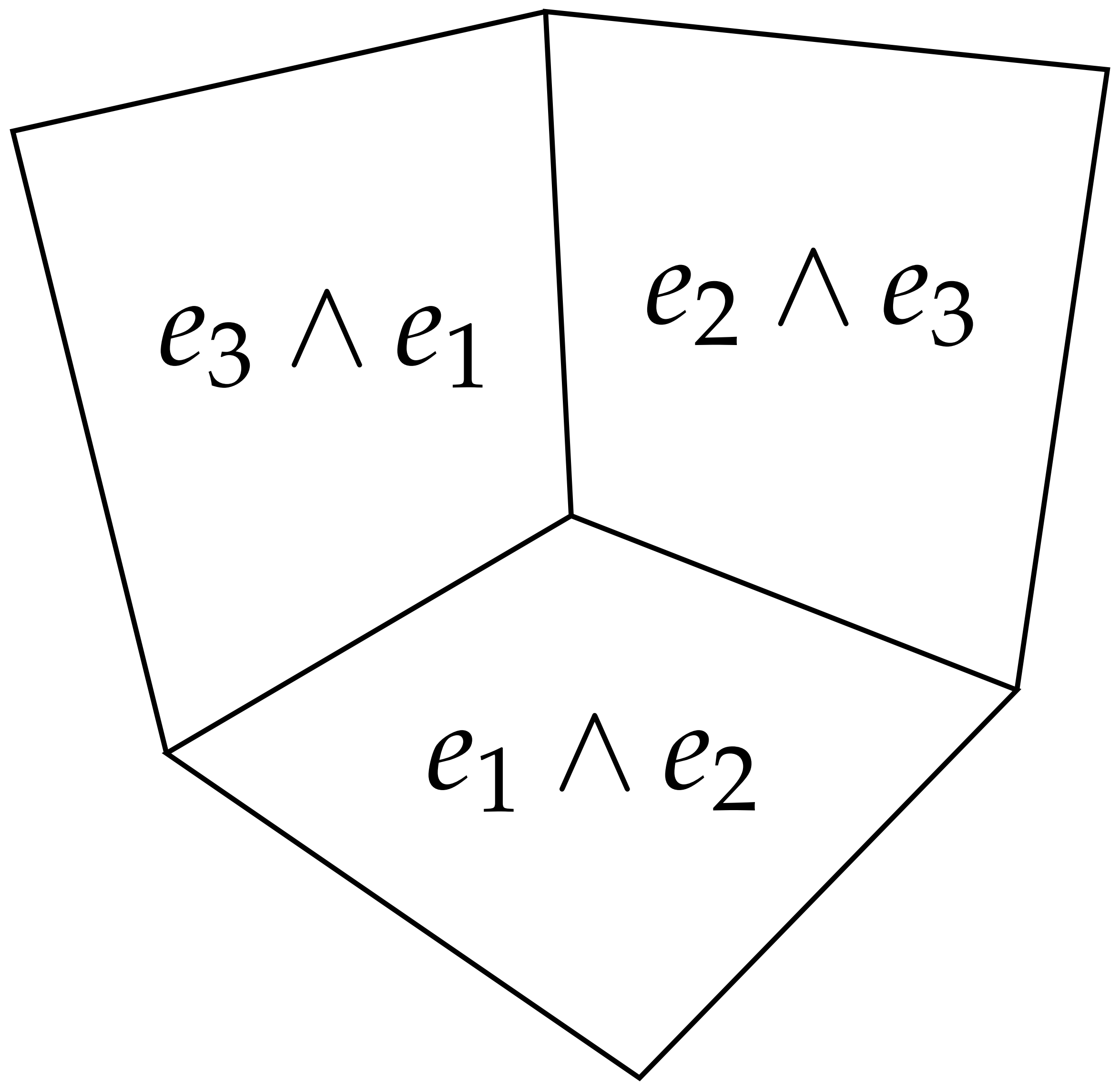
Basis k-Vectors — Visualized

$(V = \mathbb{R}^3)$

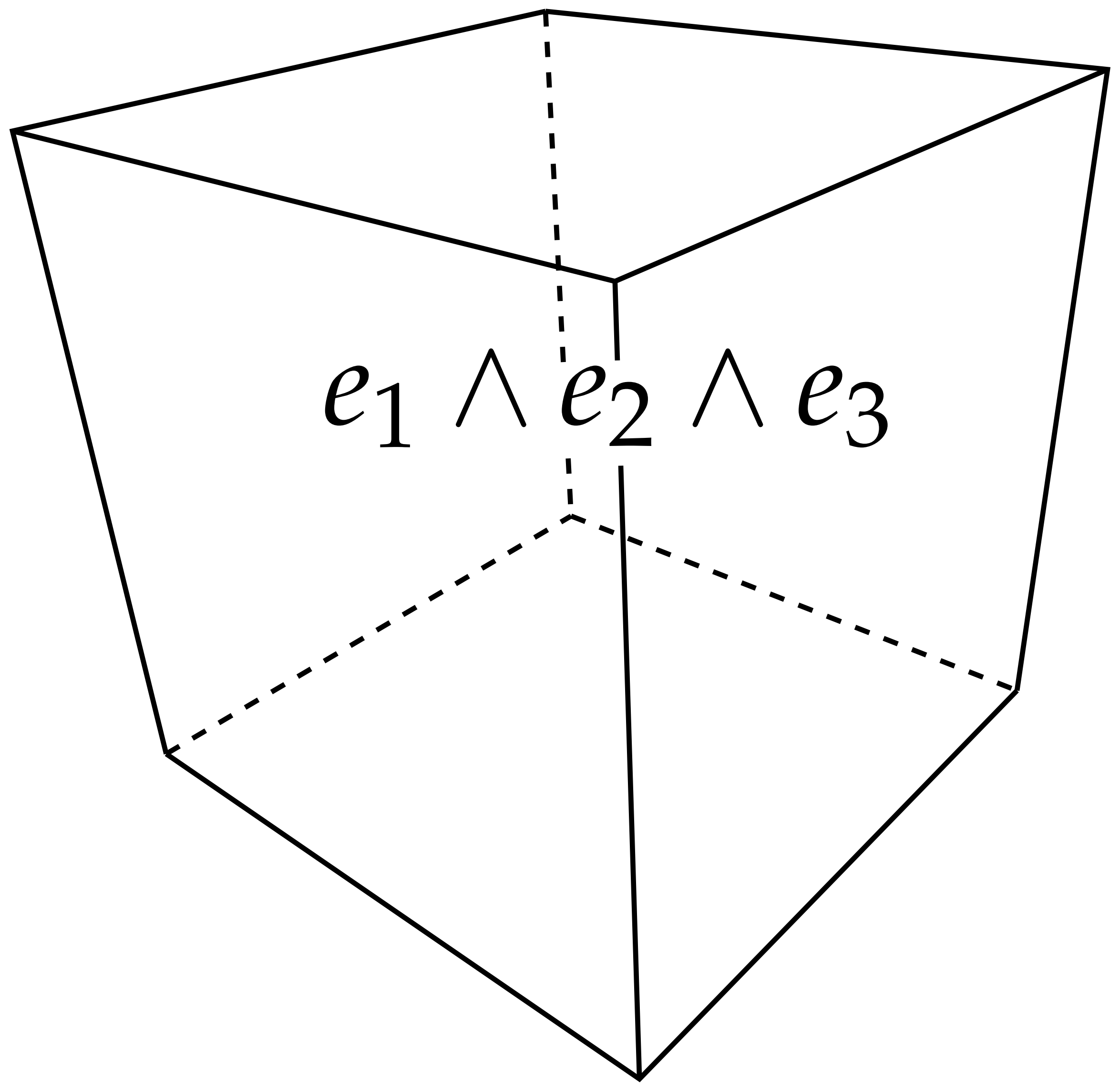
basis 1-vectors



basis 2-vectors



basis 3-vectors



Key idea: signed volumes can be expressed as linear combinations of "basis volumes" or basis *k-vectors*.

Basis k-Vectors — How Many?

Consider $V = \mathbb{R}^4$ with basis $\{e_1, e_2, e_3, e_4\}$.

Q: How many basis 2-vectors?

$$\begin{array}{lll} e_1 \wedge e_2 & & \\ e_1 \wedge e_3 & e_2 \wedge e_3 & \\ e_1 \wedge e_4 & e_2 \wedge e_4 & e_3 \wedge e_4 \end{array}$$

Why not $e_3 \wedge e_2$? $e_4 \wedge e_4$?

What do these bases represent ***geometrically?***

Q: How many basis 3-vectors?

$$\begin{array}{l} e_1 \wedge e_2 \wedge e_3 \\ e_1 \wedge e_2 \wedge e_4 \\ e_1 \wedge e_3 \wedge e_4 \\ e^2 \wedge e^3 \wedge e^4 \end{array}$$

$$\dim_{n,k} = \binom{n}{k}$$

Q: How many basis 4-vectors?

$$e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

Q: How many basis 1-vectors?

Q: How many basis 0-vectors?

Q: Notice a pattern?

$\mathbb{R}^3$	$\mathbb{R}^4$
1	1
3	4
3	6
1	4
	1

Hodge Star — Basis k-Vectors

Consider $V = \mathbb{R}^3$ with orthonormal basis $\{e_1, e_2, e_3\}$

Q: How does the Hodge star map basis k-vectors to basis $(n-k)$-vectors (n=3)?

A: For any basis k-vector $\alpha := e_{i_1} \wedge \cdots \wedge e_{i_k}$, we must have $\det(\alpha \wedge \star\alpha) = 1$.

In other words, if we start with a "unit volume," wedging with its Hodge star must also give a unit, positively-oriented unit volume. For example:

Given $\alpha := e_2$, find $\star\alpha$ such that $\det(e_2 \wedge \star e_2) = 1$.

$\Rightarrow$ Must have $\star\alpha = e_3 \wedge e_1$, since then

$$e_2 \wedge \star e_2 = e_2 \wedge e_3 \wedge e_1,$$

which is an even permutation of $e_1 \wedge e_2 \wedge e_3$.

$$\begin{aligned}
\star 1 &= e_1 \wedge e_2 \wedge e_3 \\
\star e_1 &= e_2 \wedge e_3 \\
\star e_2 &= e_3 \wedge e_1 \\
\star e_3 &= e_1 \wedge e_2 \\
\star(e_2 \wedge e_3) &= e_1 \\
\star(e_3 \wedge e_1) &= e_2 \\
\star(e_1 \wedge e_2) &= e_3 \\
\star(e_1 \wedge e_2 \wedge e_3) &= 1
\end{aligned}$$

Exterior Algebra—Formal Definition

Definition. Let $e_1, \ldots, e_n$ be the basis for an n-dimensional inner product space V. For each integer $0 \leq k \leq n$, let $\bigwedge^k$ denote an $\binom{n}{k}$-dimensional vector space with basis elements denoted by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for all possible sequences of indices $1 \leq i_1 < \cdots < i_k \leq n$, corresponding to all possible "axis-aligned" k-dimensional volumes. Elements of $\bigwedge^k$ are called *k-vectors*. The *wedge product* is a bilinear map

$$\wedge_{k,l} : \bigwedge^k \times \bigwedge^l \to \bigwedge^{k+l}$$

uniquely determined by its action on basis elements; in particular, for any collection of *distinct* indices $i_1, \ldots, i_{k+l}$,

$$(e_{i_1} \wedge \cdots \wedge e_{i_k}) \wedge_{k,l} (e_{i_{k+1}} \wedge \cdots \wedge e_{i_{k+l}}) := \operatorname{sgn}(\sigma)\, e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_{k+l})},$$

where σ is a permutation that puts the indices of the two arguments in canonical (lexicographic) order. Arguments with repeated indices are mapped to $0 \in \bigwedge^{k+l}$. For brevity, one typically drops the subscript on $\wedge_{k,l}$. Finally, the *Hodge star on k-vectors* is a linear isomorphism

$$\star : \bigwedge^k \to \bigwedge^{n-k}$$

uniquely determined by the relationship

$$\det(\alpha \wedge \star\alpha) = 1$$

where α is any k-vector of the form $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$ and det denotes the determinant of the constituent 1-vectors (treated as column vectors) with respect to the inner product on V. The collection of vector spaces $\bigwedge^k$ together with the maps $\wedge$ and $\star$ define an *exterior algebra* on V, sometimes known as a *graded algebra*.

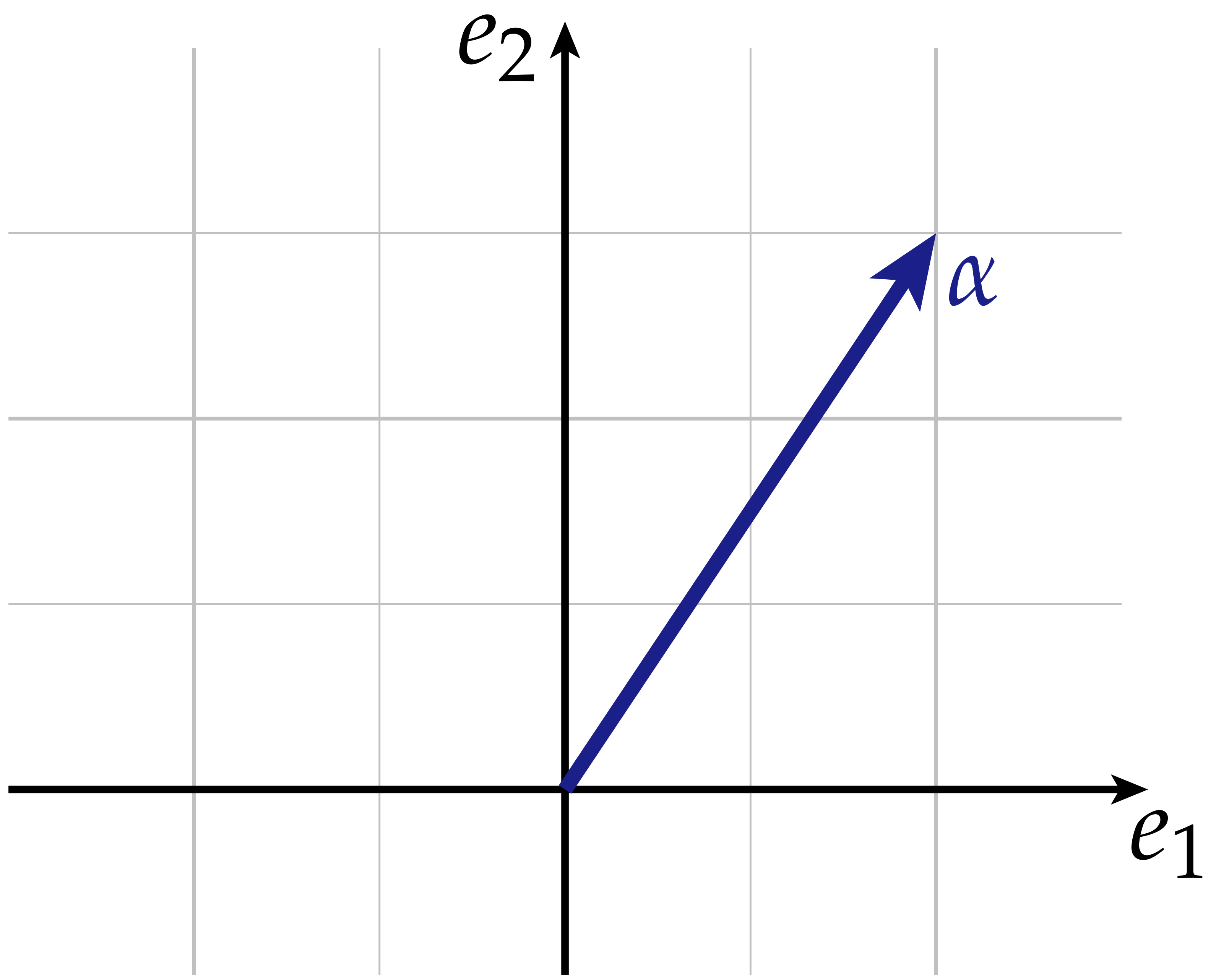
Don't worry about this unless you really want to! Concepts & mechanics more important.

Sanity Check

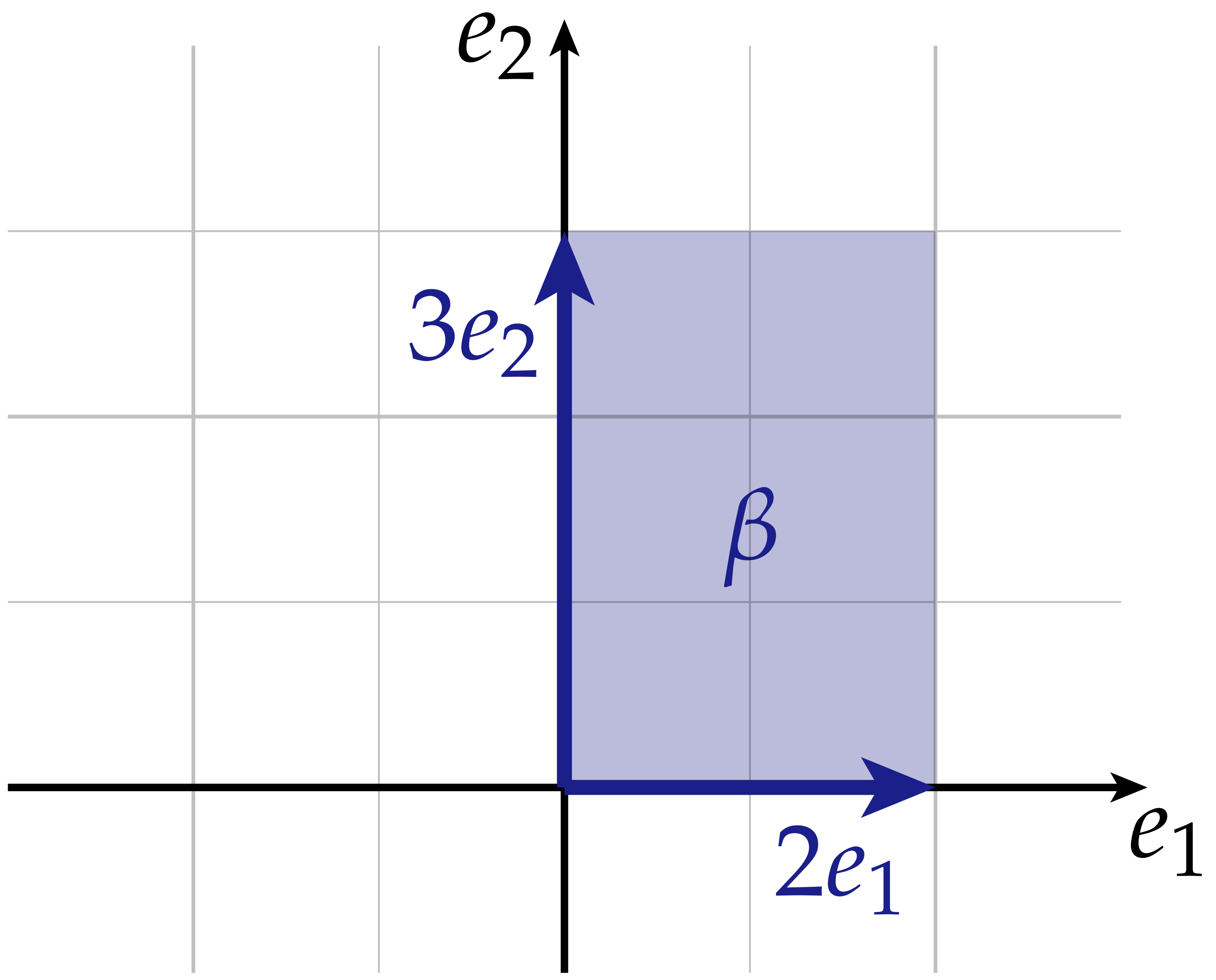
Q: What's the difference between

$$\alpha = 2e_1 + 3e_2 \qquad \text{and} \qquad \beta = 2e_1 \wedge 3e_2?$$

A:



(vector)



(2-vector)

Exterior Algebra—Example

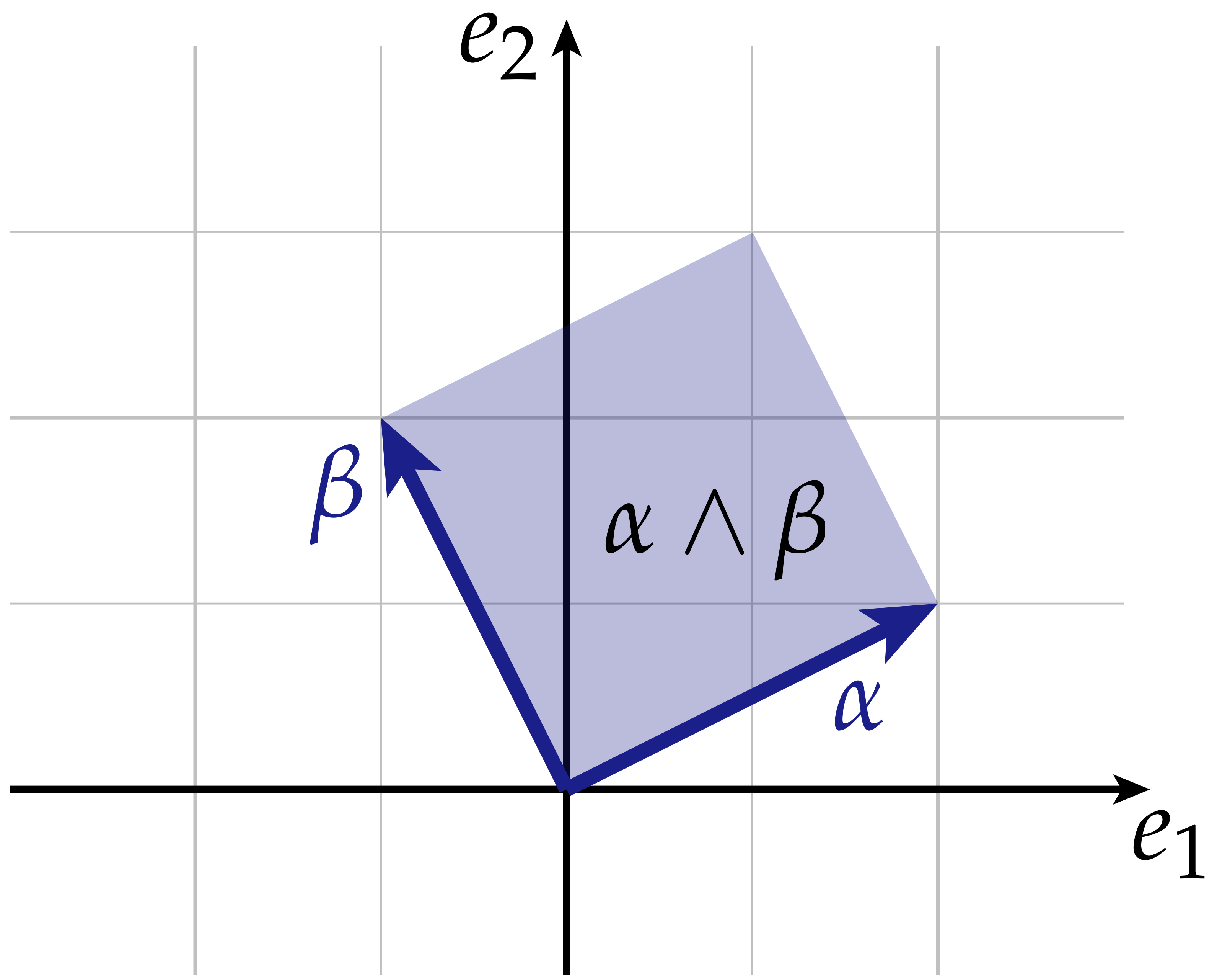
$V = \mathbb{R}^2$

$\alpha = 2e_1 + e_2$

$\beta = -e_1 + 2e_2$

Q: What is the value of $\alpha \wedge \beta$?

A:
$$\begin{aligned}
\alpha \wedge \beta &= (2e_1 + e_2) \wedge (-e_1 + 2e_2) \\
&= (2e_1 + e_2) \wedge (-e_1) + (2e_1 + e_2) \wedge (2e_2) \\
&= -\cancelto{0}{2e_1 \wedge e_1} - e_2 \wedge e_1 + 4e_1 \wedge e_2 + \cancelto{0}{2e_2 \wedge e_2} \\
&= e_1 \wedge e_2 + 4e_1 \wedge e_2 \\
&= \boxed{5e_1 \wedge e_2}
\end{aligned}$$



Q: What does the result *mean*, geometrically?

Exterior Algebra—Example

$V = \mathbb{R}^3$

$\alpha = 2e_1 \wedge e_2$

$\beta = 3e_3$

$\gamma = e_2 \wedge e_1$

Q: What is $\star(\alpha \wedge \beta + \beta \wedge \gamma)$?

A:

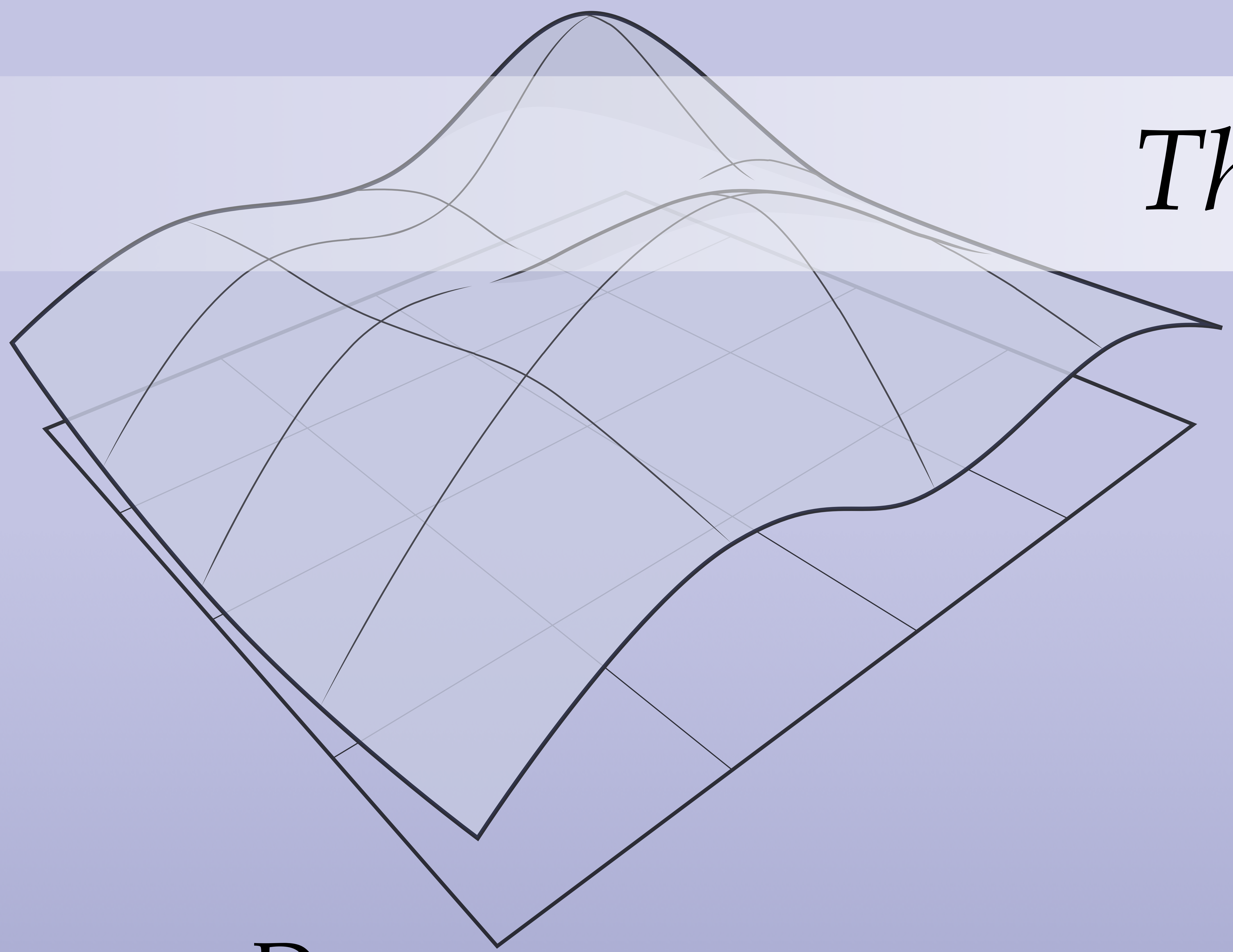
$$\begin{aligned}\star(\alpha \wedge \beta + \beta \wedge \gamma) &= \star((2e_1 \wedge e_2) \wedge 3e_3 + 3e_3 \wedge (e_2 \wedge e_1)) \\ &= \star(6e_1 \wedge e_2 \wedge e_3 + 3e_3 \wedge e_2 \wedge e_1) \\ &= \star(6e_1 \wedge e_2 \wedge e_3 - 3e_1 \wedge e_2 \wedge e_3) \\ &= \star(3e_1 \wedge e_2 \wedge e_3) \\ &= 3.\end{aligned}$$

Key idea: in this example, it would have been fairly hard to reason about the answer geometrically. Sometimes the algebraic approach is (*incredibly!*) useful.

Exterior Algebra - Summary

- **Exterior algebra**
  - language for manipulating *signed volumes*
  - length matters (magnitude)
  - order matters (orientation)
  - behaves like a vector space (e.g., can add two volumes, scale a volume, ...)
- **Wedge product**—analogous to *span* of vectors
- **Hodge star**—analogous to *orthogonal complement* (in 2D: 90-degree rotation)
- Coordinate representation—encode vectors in a *basis*
  - Basis *k*-vectors are all possible wedges of basis 1-vectors

Thanks!



Discrete Differential Geometry: An Applied Introduction

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