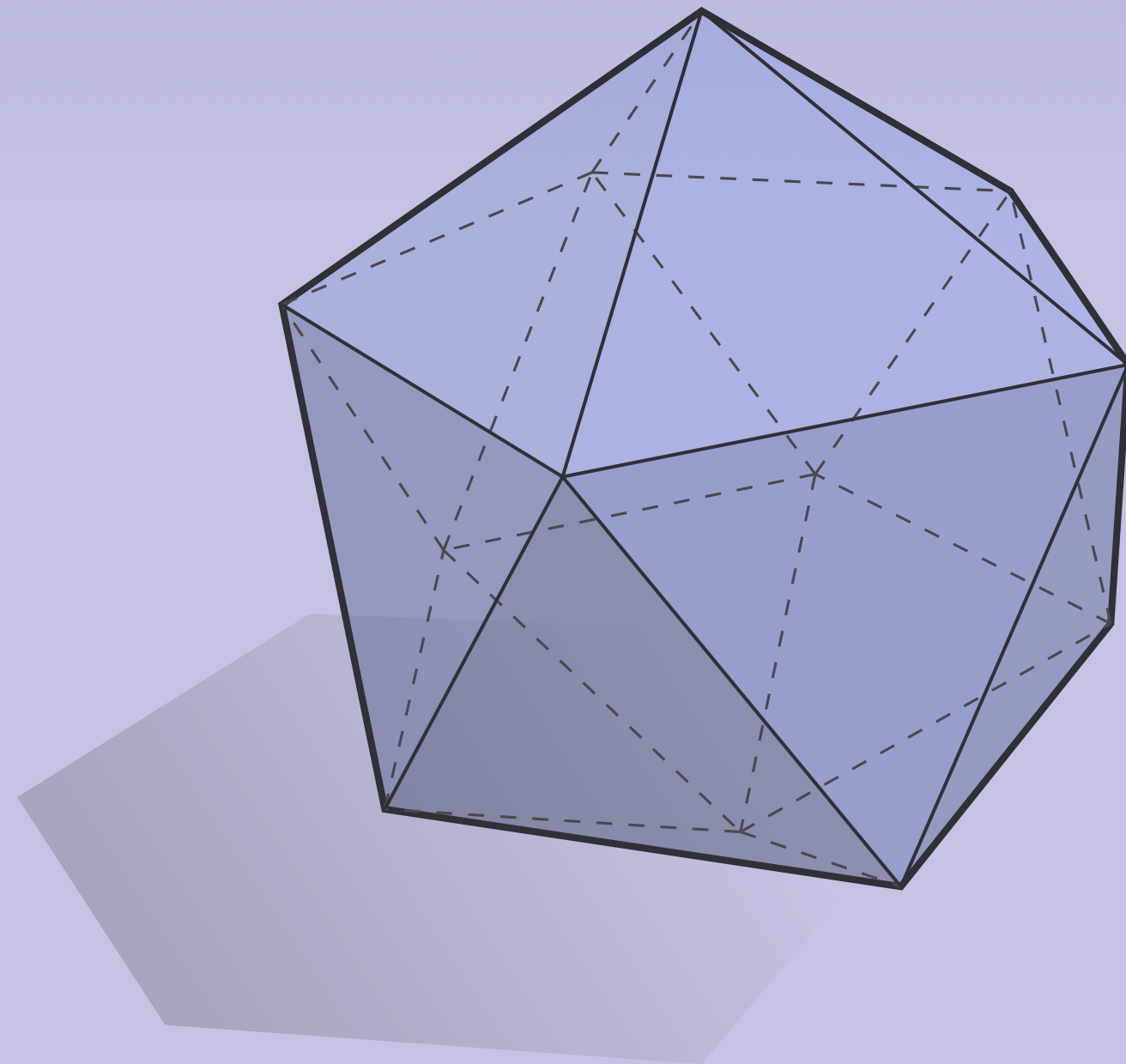


DISCRETE DIFFERENTIAL
GEOMETRY:
AN APPLIED INTRODUCTION
Keenan Crane • CMU 15-458/858

LECTURE 3:
EXTERIOR ALGEBRA



DISCRETE DIFFERENTIAL
GEOMETRY:
AN APPLIED INTRODUCTION

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Why Learn Exterior Calculus?

コンピュータサイエンスの建物
の地下では、金があります!

Translation: *“There is gold in the basement of the computer science building!”*

Key idea: language is important!

Not all languages are created equal...

*54·43. $\vdash :: \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54·26 . \supset \vdash :: \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[*51·231] $\equiv . \iota'x \cap \iota'y = \Lambda .$

[*13·12] $\equiv . \alpha \cap \beta = \Lambda \quad (1)$

$\vdash . (1) . *11·11·35 . \supset$

$\vdash :: (\exists x, y) . \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \quad (2)$

$\vdash . (2) . *11·54 . *52·1 . \supset \vdash . \text{Prop}$

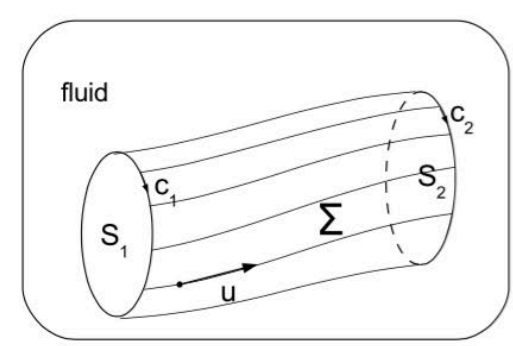
From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

(from Russel & Whitehead's *Principia Mathematica*, p. 379)

Why Learn Exterior Calculus?

- Natural language for talking about *signed volume*
 - facilitates communication w/ math, physics, ...
 - provides new perspectives on computation
- **Geometry**
 - algebraic geometry
 - geometric algebra (Clifford algebra, spin physics)
- **Physics**
 - “massless” quantities are vectors (velocity, acceleration, ...)
 - “massive” quantities are forms (momentum, force, ...)
- **Computer Science** (*this class!*): geometric computation on meshes

5



or, in words, that the $d\sigma$ is *invariant* w.r.t. the flow of the fluid (regarded as the flow of ξ on $M \times \mathbb{R}$).

Now, we want to see an integrable distribution behind vortex lines, again. Define the distribution \mathcal{D} in terms of annihilation of as many as *two* exact forms:

$$\mathcal{D} \leftrightarrow i_w d\sigma = 0 = i_w dt \quad (52)$$

By repeating the reasoning from (32) and (33) one concludes that \mathcal{D} is *integrable*.

The distribution \mathcal{D} is, however, also *invariant* w.r.t. the flow of the fluid. (Because of (50) and the trivial fact that $\mathcal{L}_\xi(dt) = 0$.) So, integral submanifolds (surfaces) *move with the fluid*.

What do they look like? They are nothing but vortex lines.

Indeed, making use of general formula (A3) from Appendix A and the form (47) of Euler equation we can write

$$d\sigma = \hat{d}\hat{v} + dt \wedge (\mathcal{L}_0 \hat{v} + \hat{d}\mathcal{E}) \quad \text{always} \quad (53)$$

$$= \hat{d}\hat{v} + dt \wedge (-i_w \hat{d}\hat{v}) \quad \text{on solutions} \quad (54)$$

Let us now contemplate Eq. (52). It says, that the distribution consists of *spatial* vectors (i.e. those with vanishing *time* component, therefore annihilating dt) which, in addition, annihilate $d\sigma$.

Let w be arbitrary *spatial* vector. Denote, for a while, $i_w \hat{d}\hat{v} =: \hat{b}$ (it is a *spatial* 1-form). Then, from (54),

$$i_w d\sigma = \hat{b} - dt \wedge i_w \hat{b} \quad (55)$$

from which immediately

$$i_w(d\sigma) = 0 \quad \Leftrightarrow \quad \hat{b} \equiv i_w \hat{d}\hat{v} = 0 \quad (56)$$

This says that we can, alternatively, describe the distribution \mathcal{D} as consisting of those *spatial* vectors which, in addition, annihilate $\hat{d}\hat{v}$ (rather than $d\sigma$, as it is expressed in the definition (52)). But Eqs. (45) and (22) show that

$$\hat{d}\hat{v} = \omega \cdot d\mathbf{S} \equiv \omega(\mathbf{r}, t) \cdot d\mathbf{S} \quad (57)$$

so that $\hat{d}\hat{v}$ is nothing but the *vorticity 2-form* and, therefore, the integral surfaces of \mathcal{D} may indeed be identified with vortex lines. So, Helmholtz statement is also true in the general, time-dependent, case. (Notice that the system of vortex lines looks, in general, different in different times. This is because its generating object, the vorticity 2-form $\hat{d}\hat{v}$, depends on time.)

D. Helmholtz statement on vortex tubes - general case

Vortex tube is a genuinely spatial concept and the statement concerns purely kinematical property of *any* velocity field at a single time (see the beginning of Sec. III D). So, no (change of) dynamics has any influence on it. If the statement were true before, it remains to be true now.

B. Non-stationary Euler equation

Let us retell Cartan's results from the last section in the context of hydrodynamics, i.e. for particular choice (see Eq. (18))

$$\sigma = \hat{v} - \mathcal{E} dt \quad (44)$$

where, in usual coordinates (\mathbf{r}, t) on $E^3 \times \mathbb{R}$,

$$\hat{v} := \mathbf{v} \cdot d\mathbf{r} \equiv \mathbf{v}(\mathbf{r}, t) \cdot d\mathbf{r} \quad (45)$$

From (42) we get

$$i_\xi d\sigma = 0 \quad \Leftrightarrow \quad \mathcal{L}_0 \hat{v} + i_w \hat{d}\hat{v} = -\hat{d}\mathcal{E} \quad (46)$$

One easily checks (e.g. in Cartesian coordinates (\mathbf{r}, t)) that

$$\mathcal{L}_0 \hat{v} + i_w \hat{d}\hat{v} = -\hat{d}\mathcal{E} \quad (47)$$

is nothing but the complete, time-dependent, Euler equation (12). Therefore the time-dependent Euler equation may also be written in the succinct form

$$i_\xi d\sigma = 0 \quad \text{Euler equation} \quad (48)$$

The form (48) of the Euler equation turns out to be very convenient. Short illustration:

1. Just looking at (40), (48) and (44) one obtains

$$\oint_c \mathbf{v} \cdot d\mathbf{r} = \text{const.} \quad \text{Kelvin's theorem} \quad (49)$$

(the two loops c_1 and c_2 are usually in constant-time hyper-planes $t = t_1$ and $t = t_2$).

2. Application of d on both sides gives very quickly *Helmholtz theorem* (see the next Section III C). Bernoulli theorem, by the way, is no longer true in time-dependent case, so we can not derive it from (48).

C. Helmholtz statement on vortex lines - general case

Application of d on both sides of (48) and using formula (9) results in

Where Are We Going Next?

GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

Linear algebra: “little arrows” (vectors)

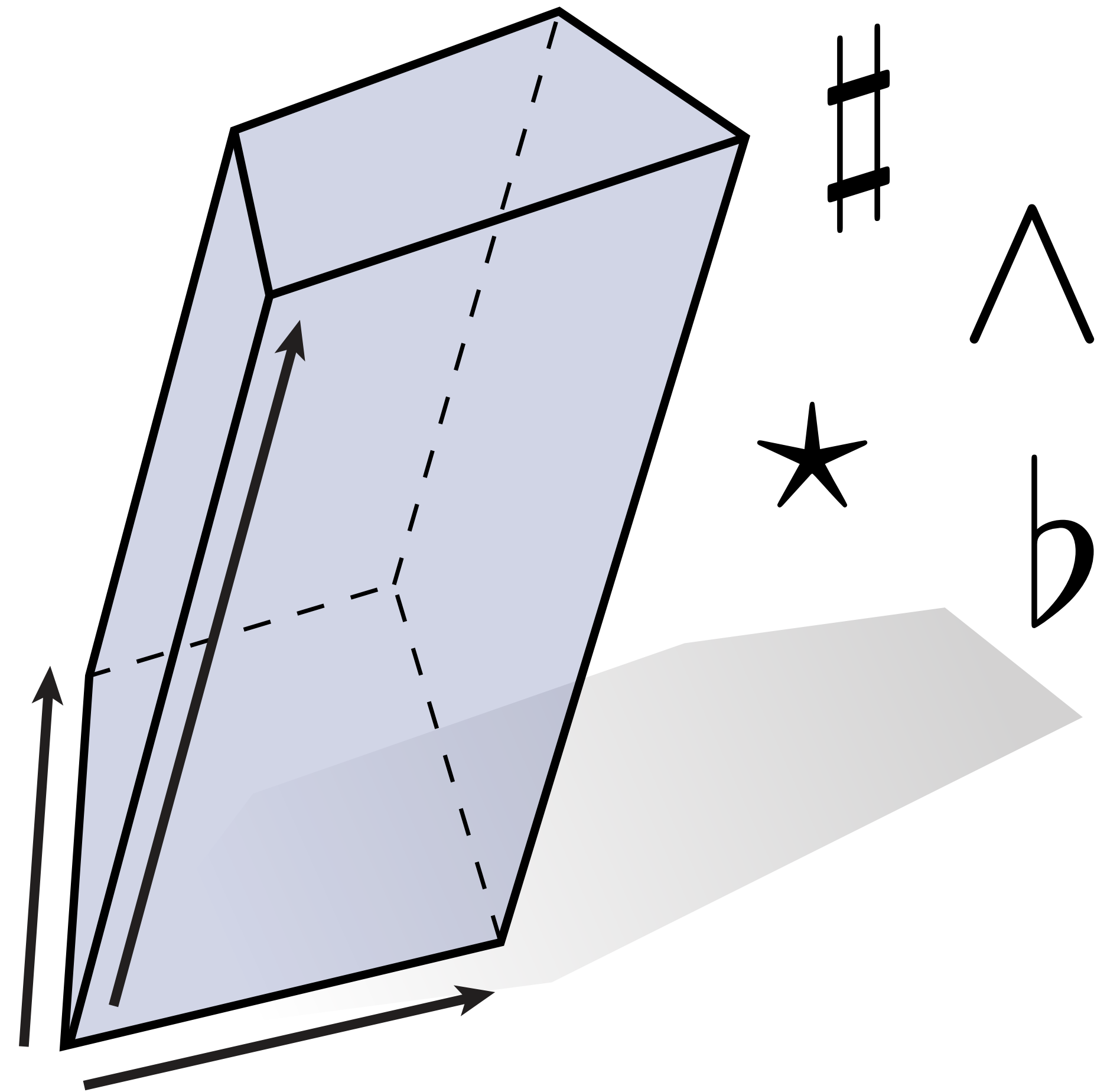
Vector Calculus: how do vectors *change*?

Next few lectures:

Exterior algebra: “little volumes” (k -vectors)

Exterior calculus: how do k -vectors change?

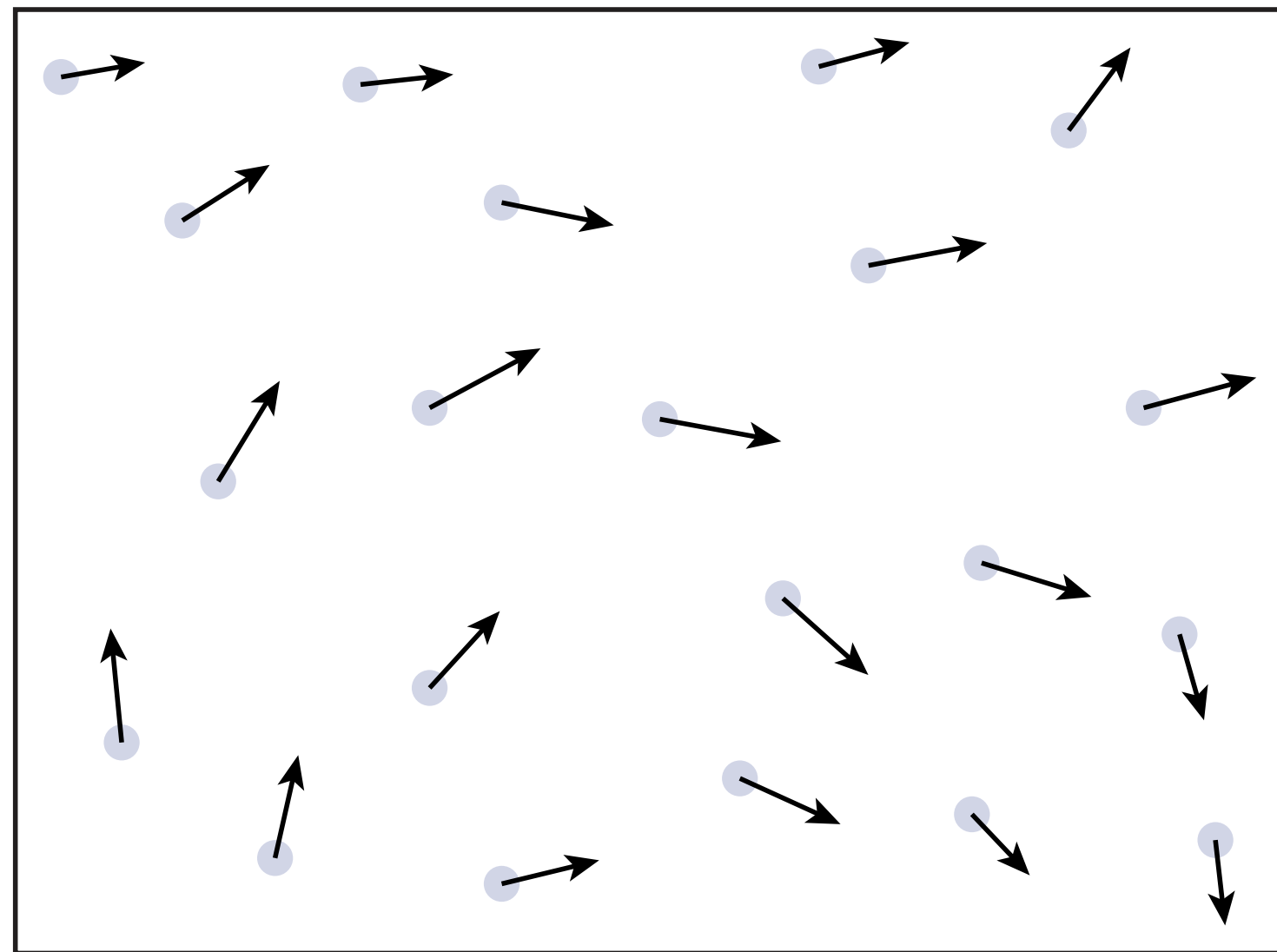
DEC: how do we do all of this on meshes?



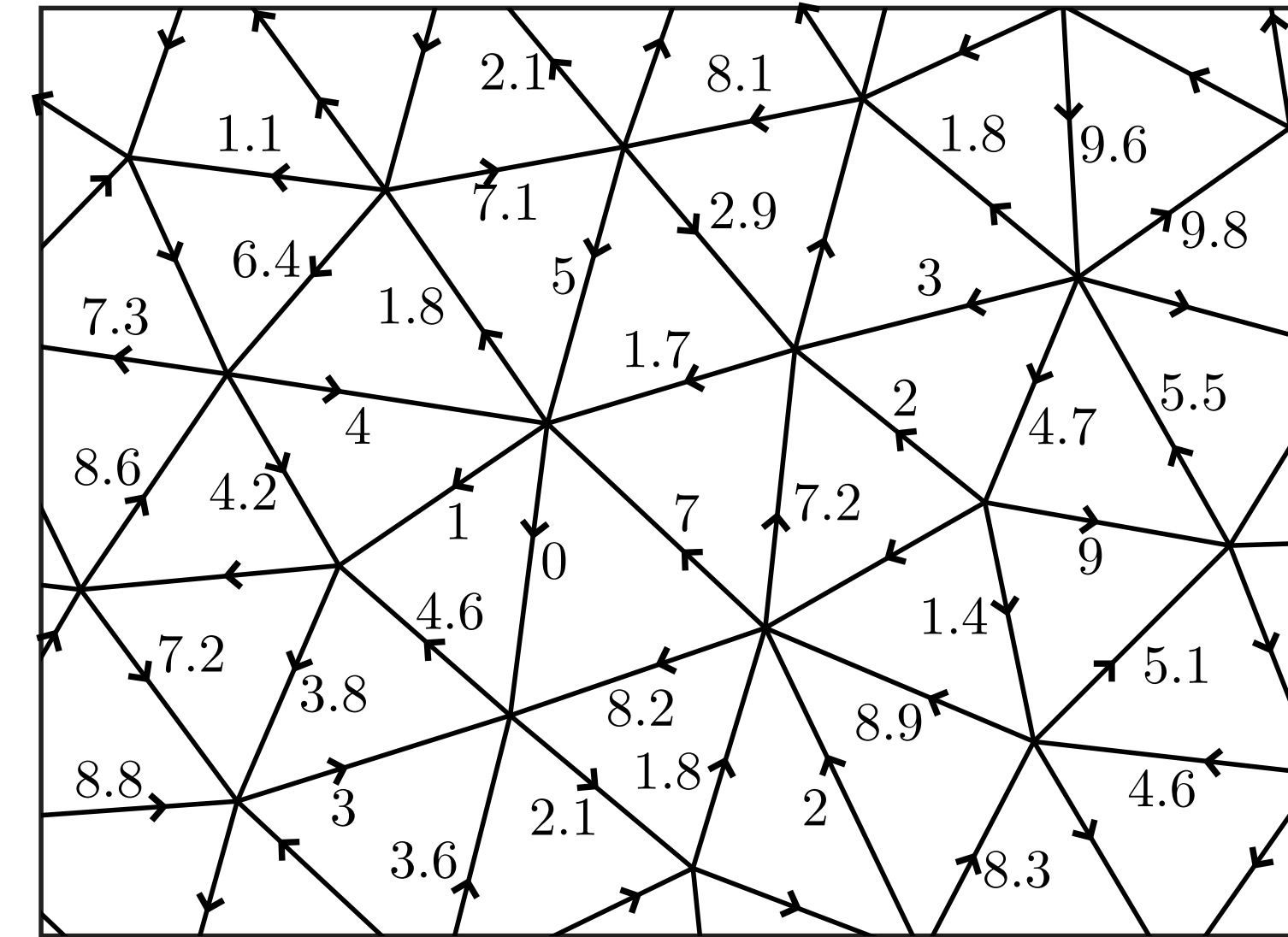
Basic idea: replace vector calculus with computation on meshes.

Why Are We Going There?

- **Motivation:** *Do cool and useful stuff with meshes!*
 - Geometry processing algorithms must solve *equations* on meshes (PDEs)
 - Meshes are made up of little *volumes*
- ⇒ Need to learn to *integrate* equations over little volumes to do computation

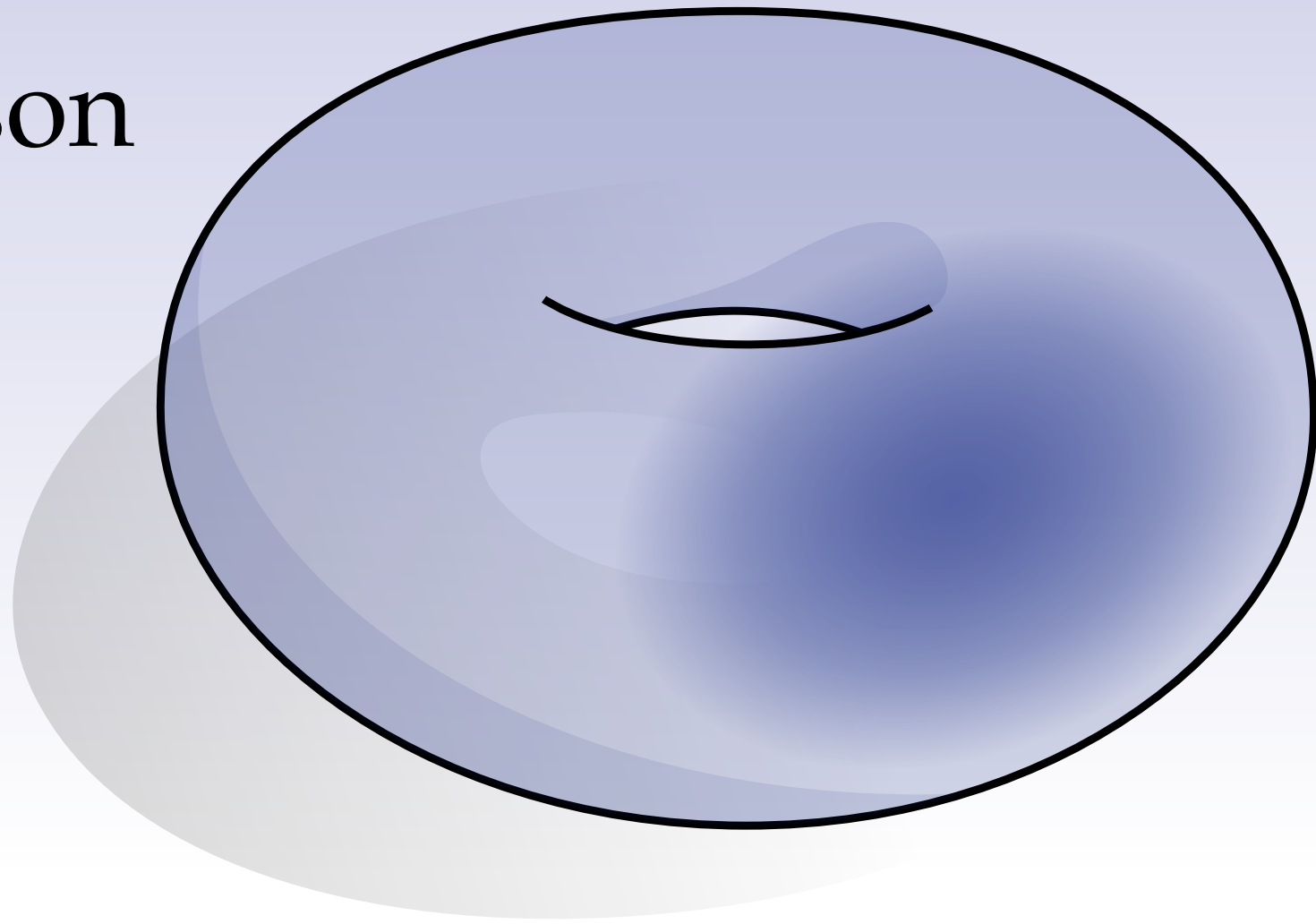


integrate

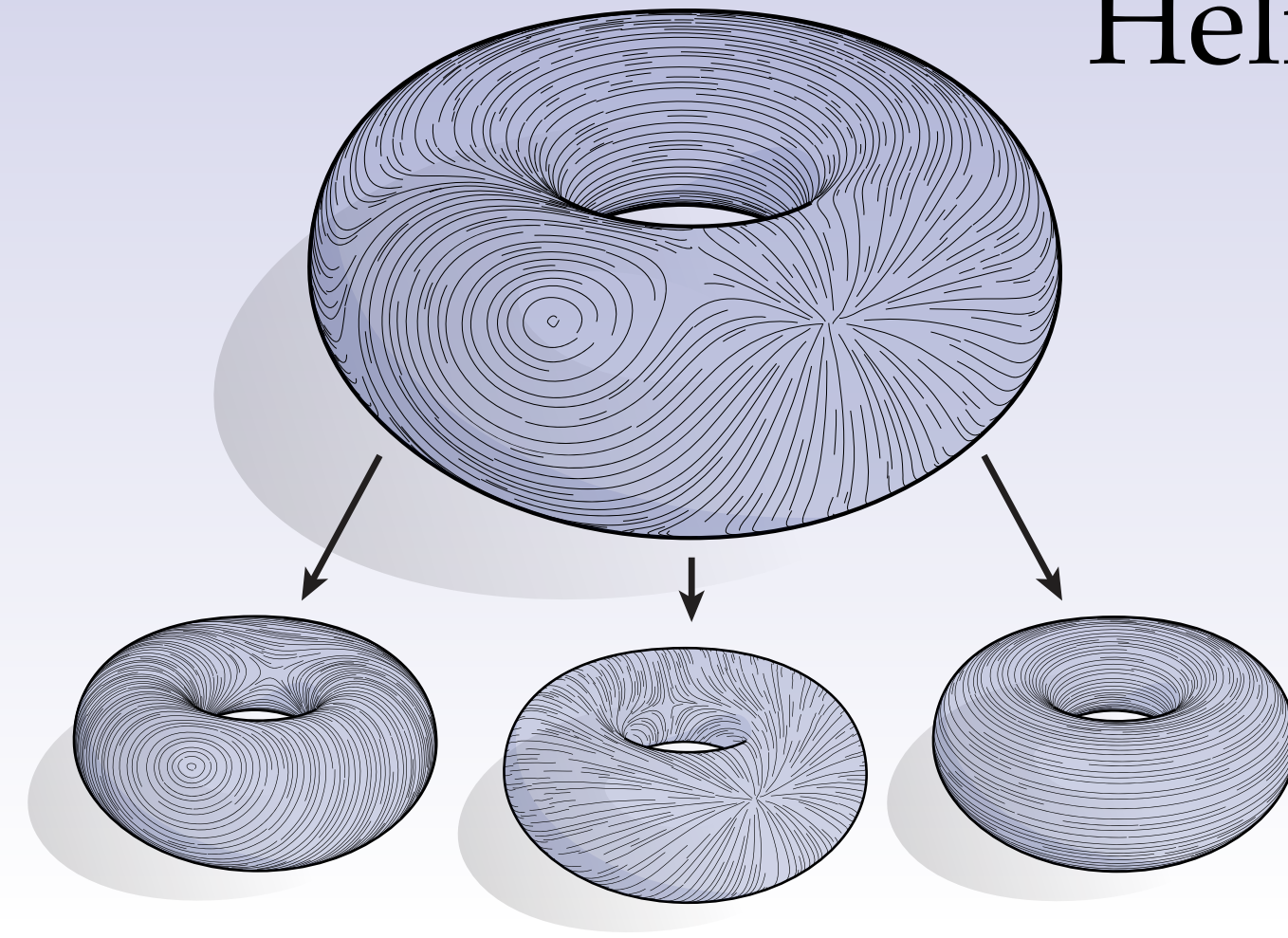


Basic Computational Tools

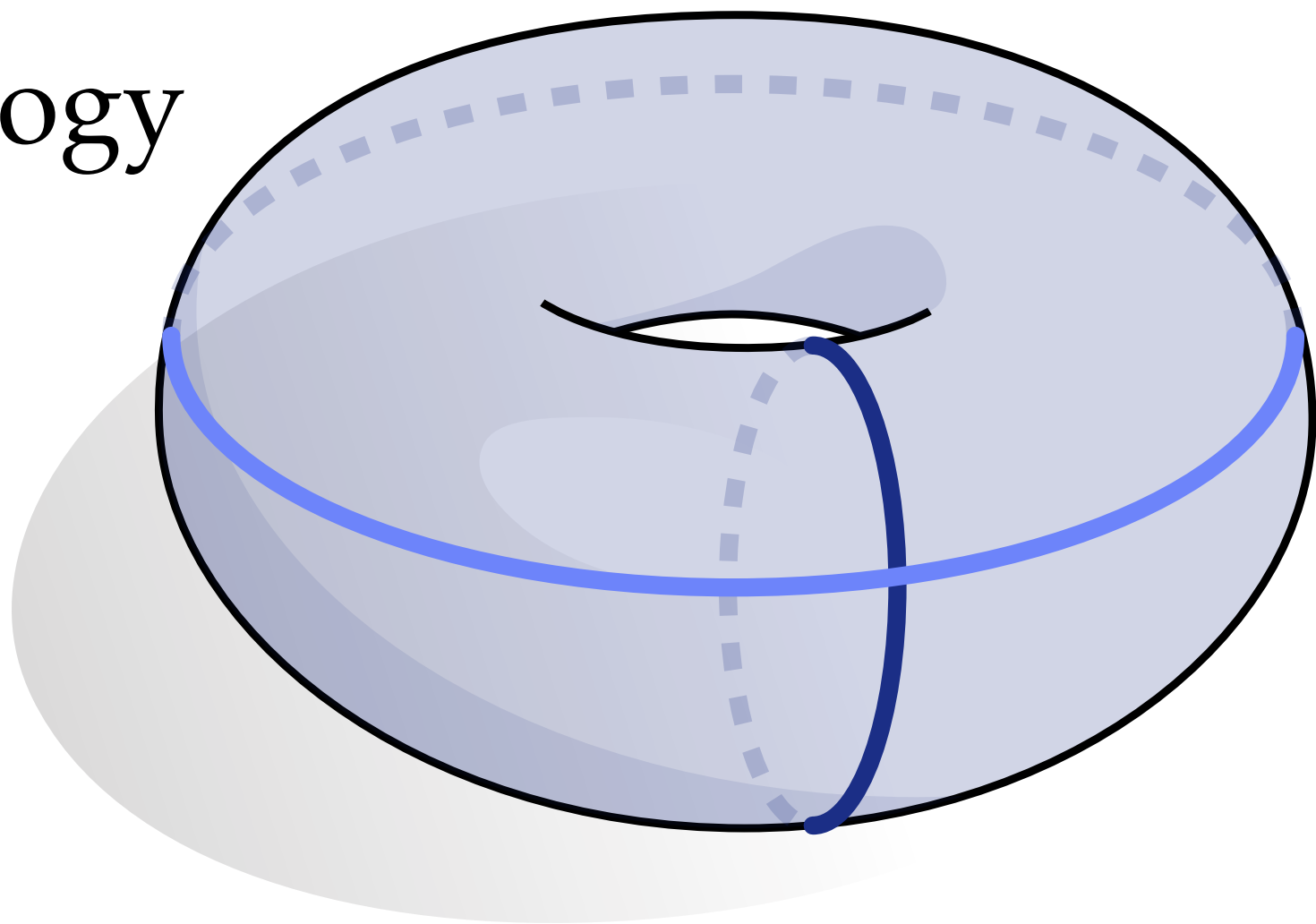
Poisson



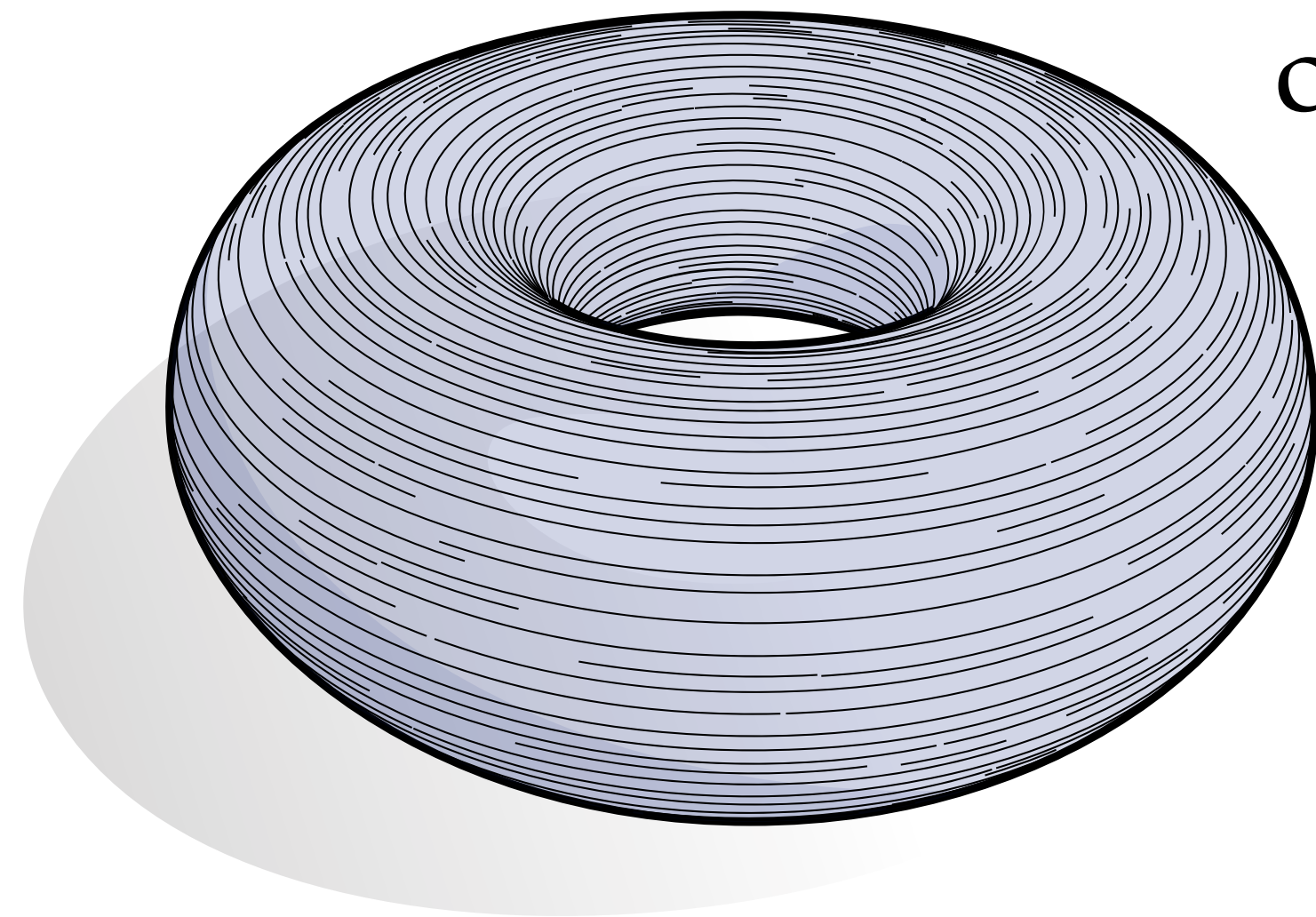
Helmholtz-Hodge



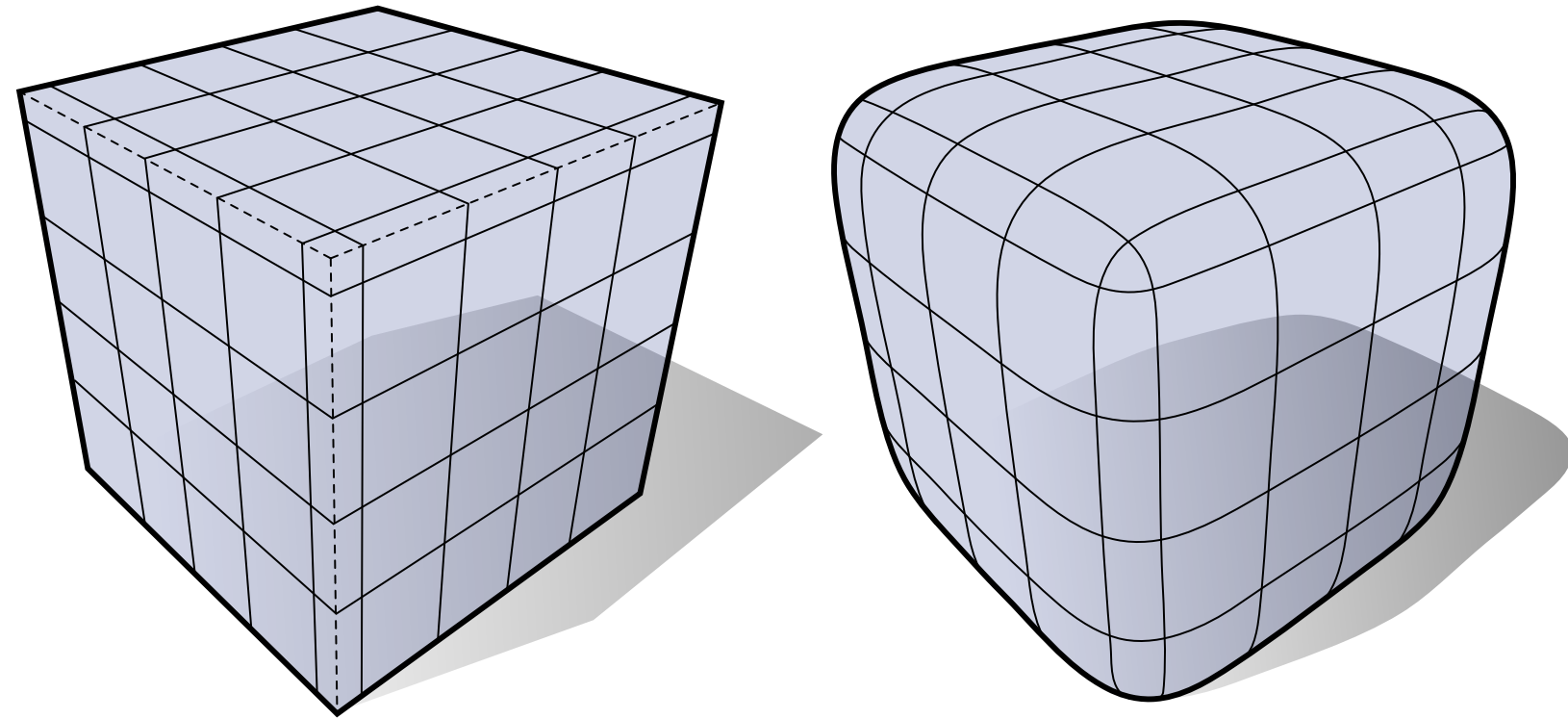
homology



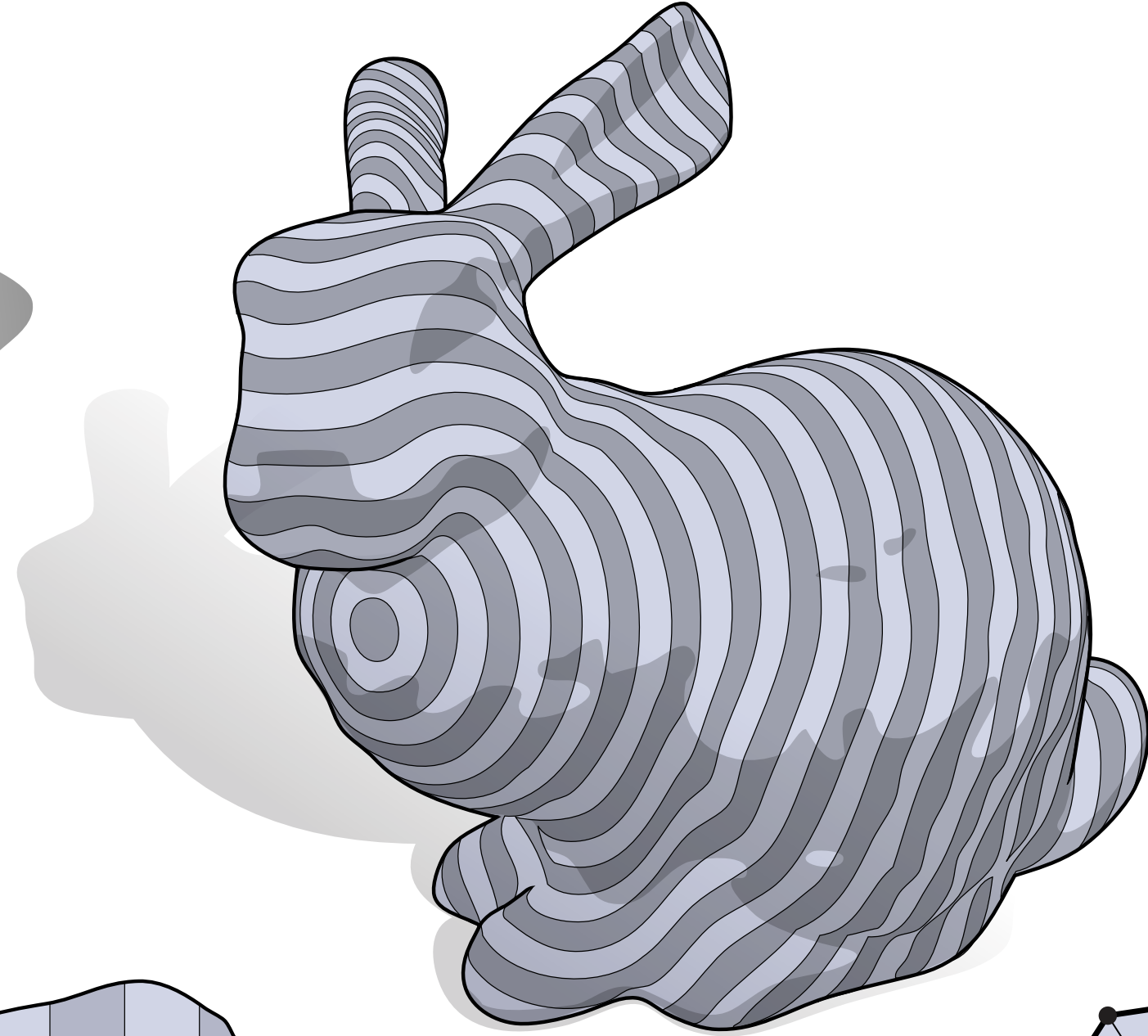
cohomology



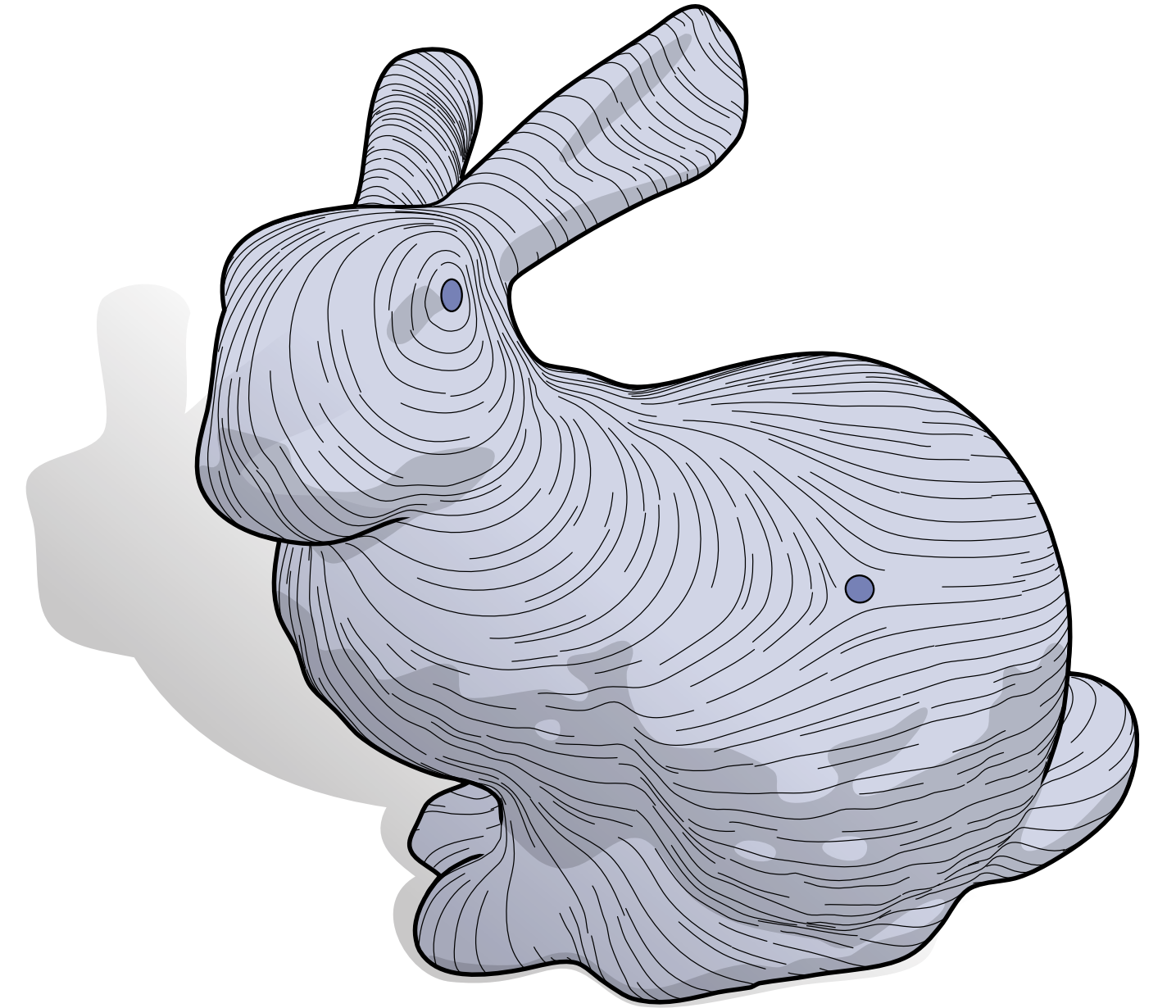
Applications



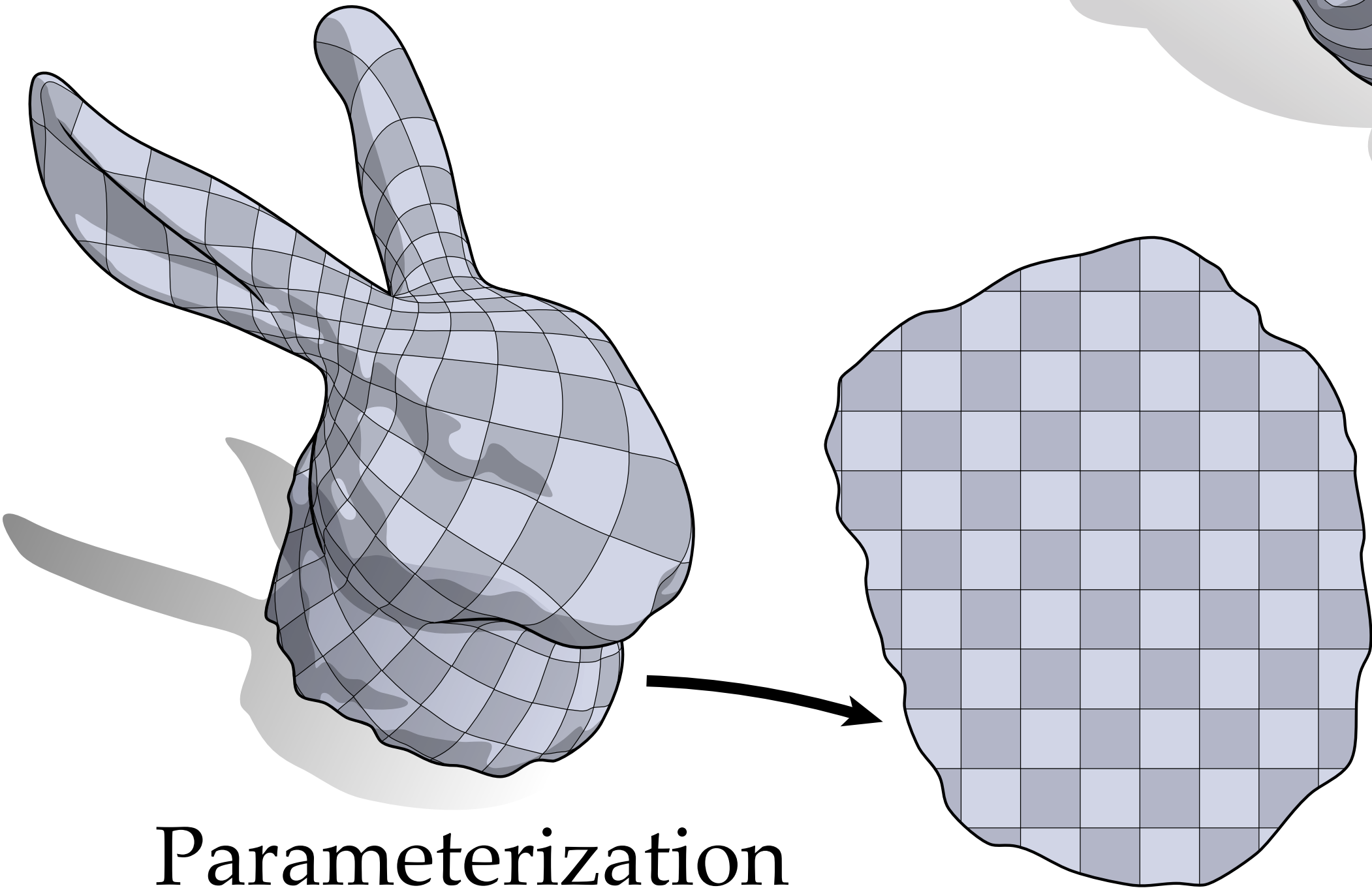
Smoothing



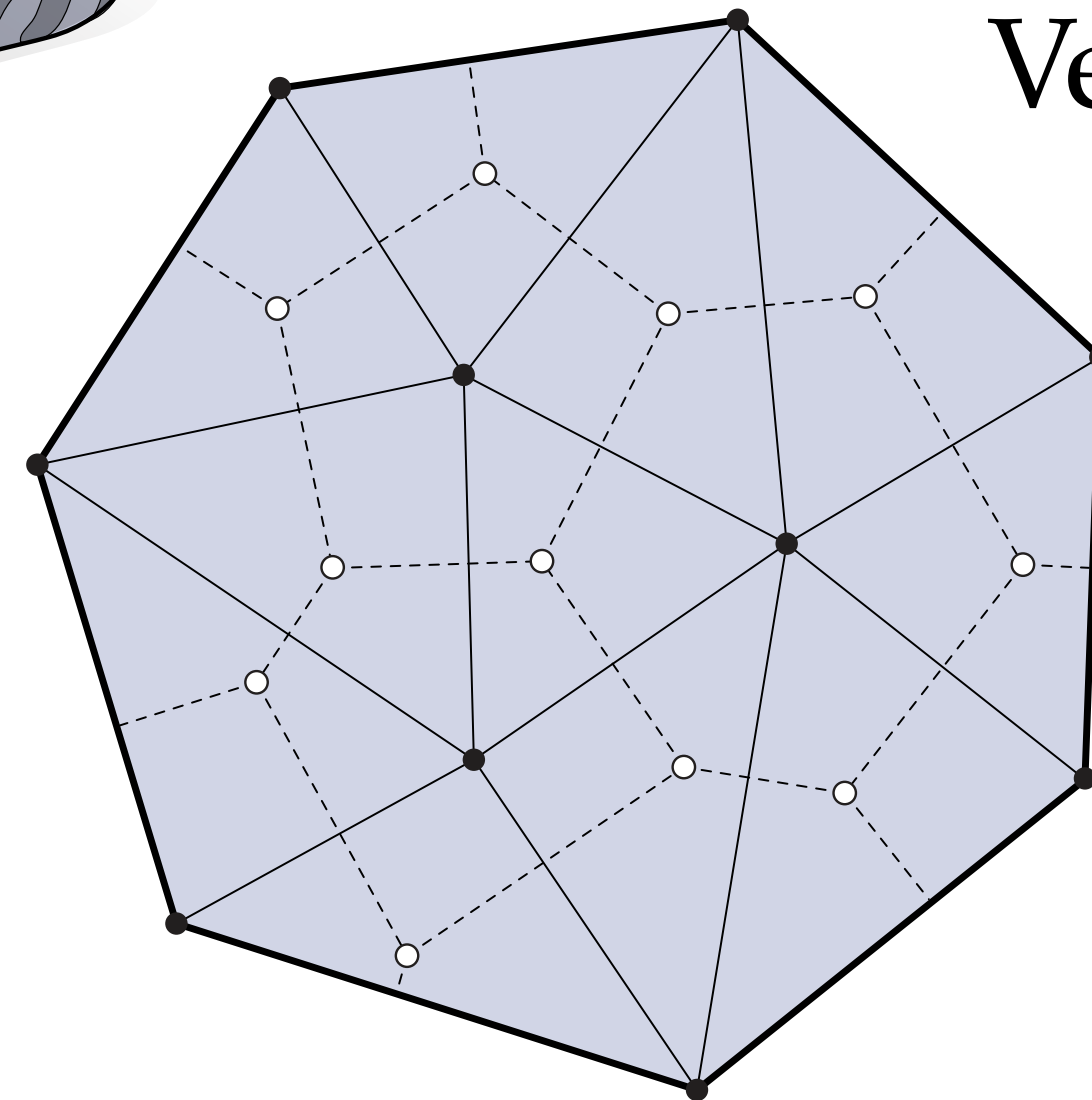
Distance



Vector Field Design

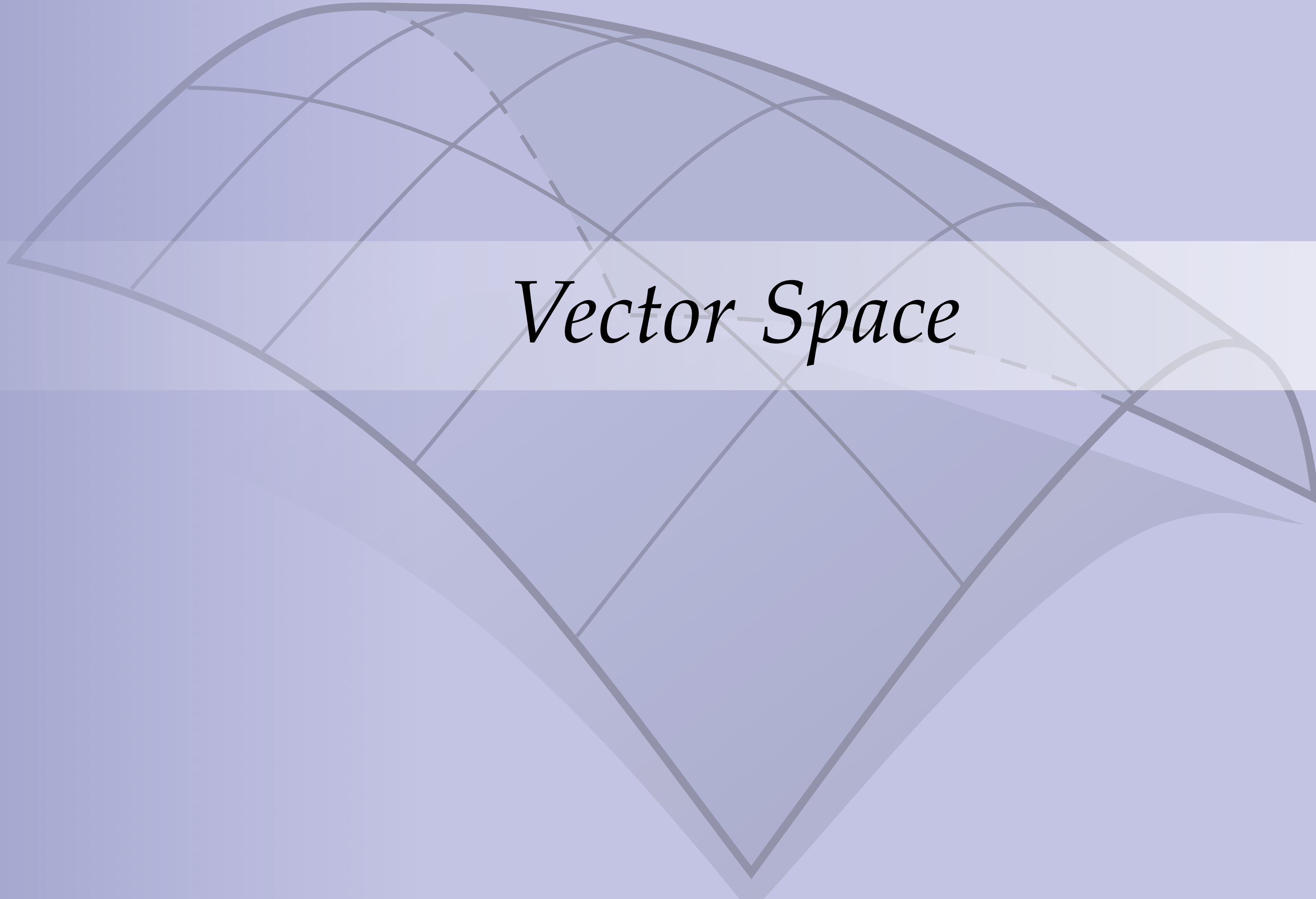


Parameterization



Meshing

...and more!



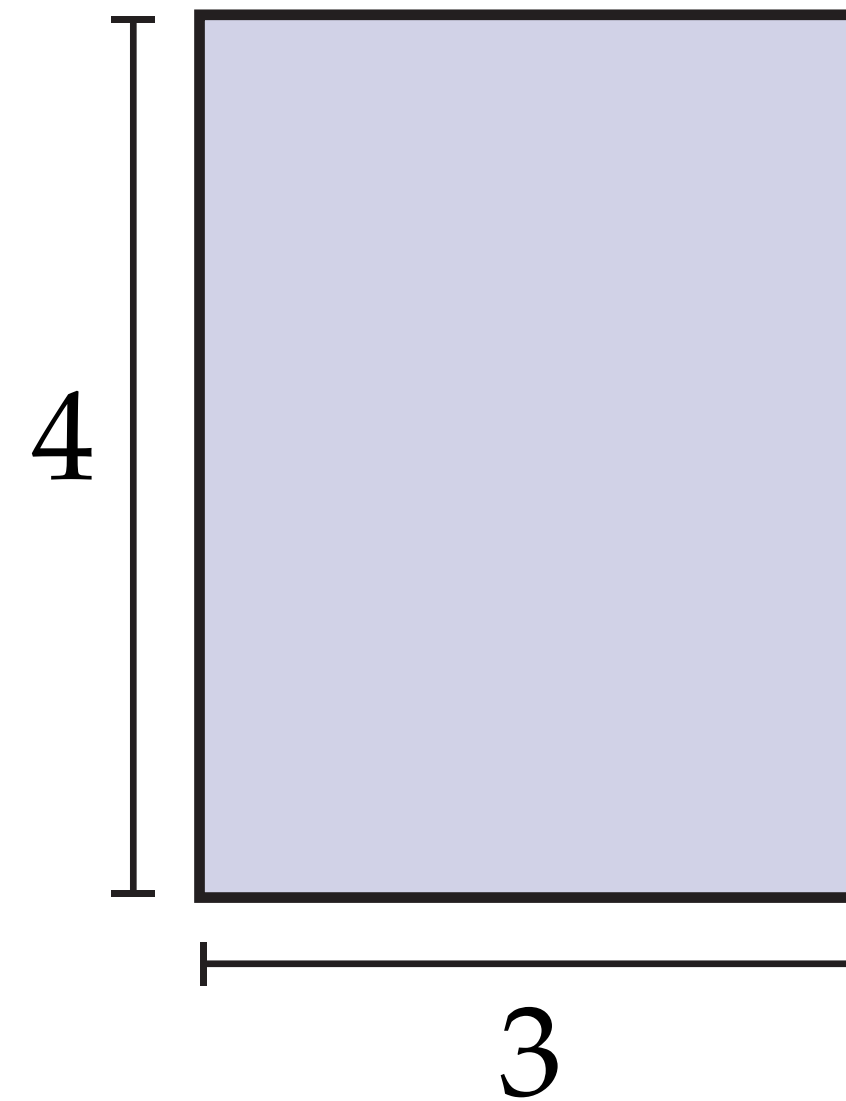
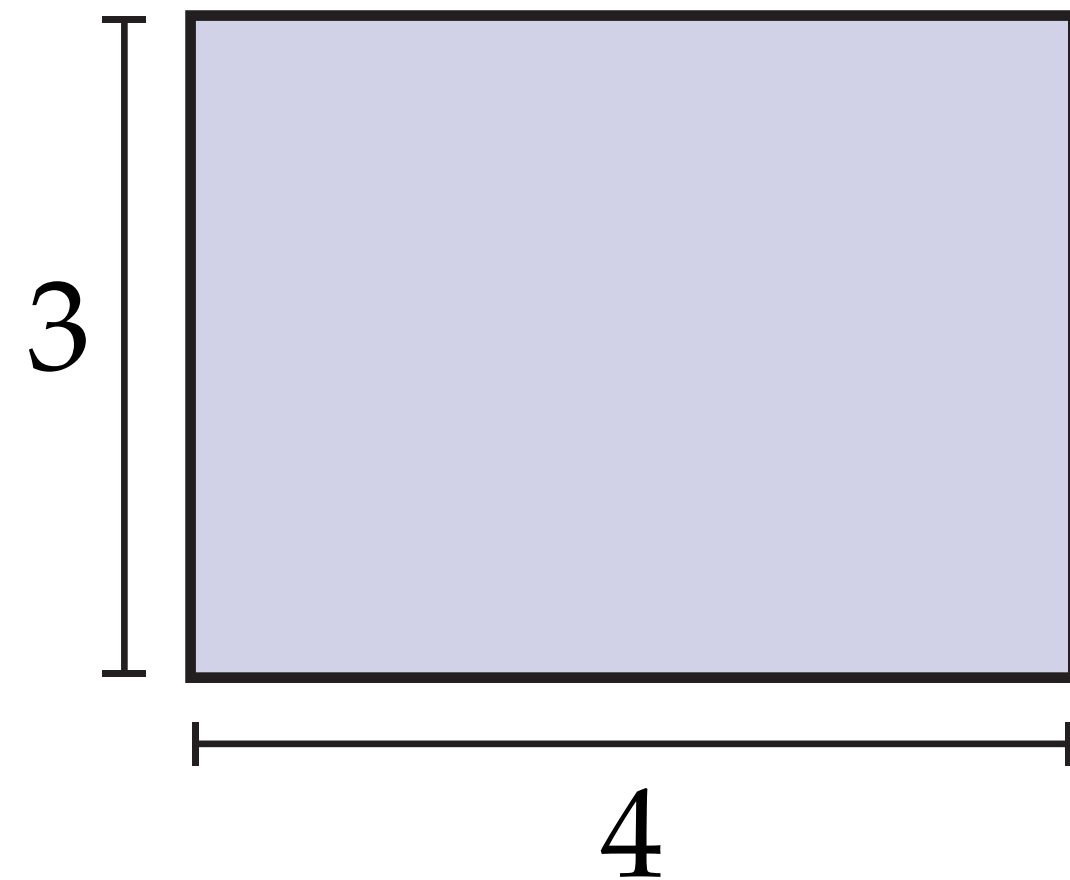
Vector Space

Warm Up: Multiplication

Question: why does $3 \times 4 = 4 \times 3$?

Answer: not just because “*that’s the rule!*”

There is a very good geometric reason:

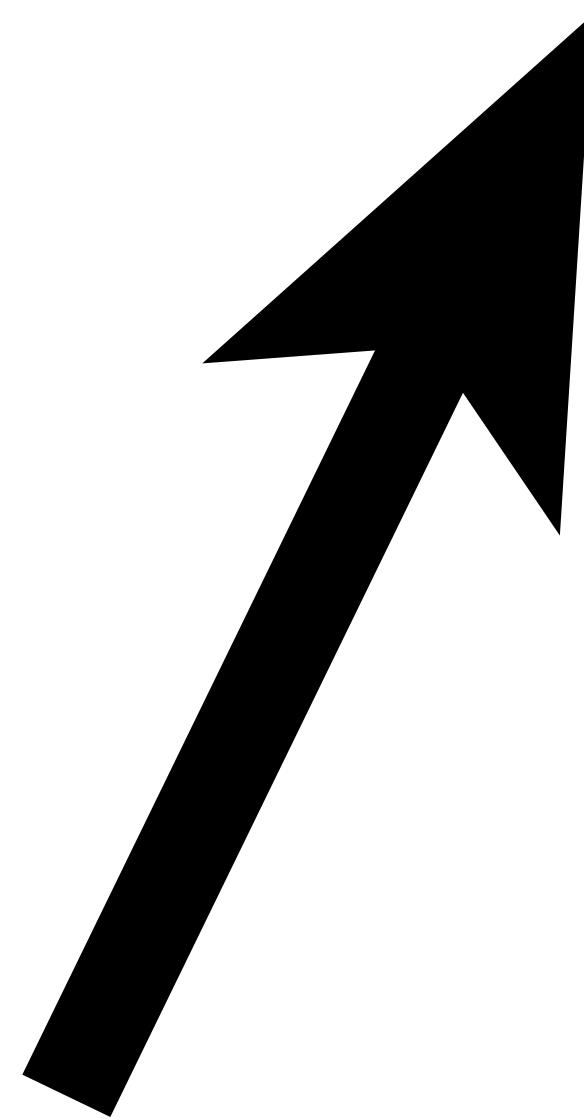


We didn't have to adopt this rule! We chose it because it captures natural behavior.

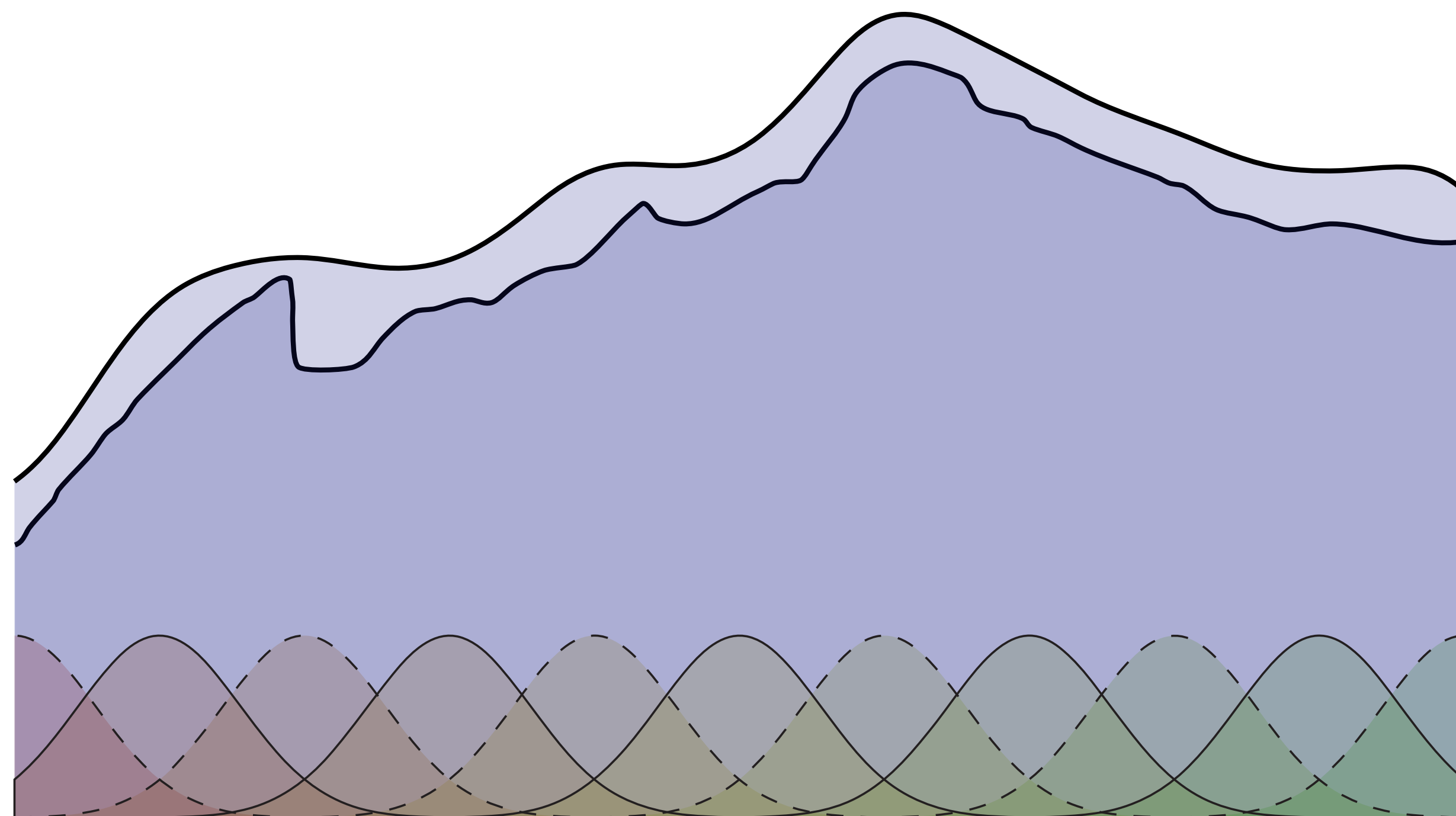
You should never accept a rule purely on faith. Always ask, “*why is this the rule?*”

Review: Vector Spaces

- What is a vector? (*Geometrically?*)



finite-dimensional



infinite-dimensional

For geometric computing, often care most about dimensions 1, 2, 3, ...and ∞ !

Review: Vector Spaces

- Formally, a *vector space* is a set V together with the operations*

$$+ : V \times V \rightarrow V \quad \text{“addition”}$$

$$\cdot : \mathbb{R} \times V \rightarrow V \quad \text{“scalar multiplication”}$$

- Must satisfy the following rules for all vectors x, y, z and scalars a, b :

$$x + y = y + x$$

$$(ab)x = a(bx)$$

$$(x + y) + z = x + (y + z)$$

$$1x = x$$

$$\exists 0 \in V \text{ s.t. } x + 0 = 0 + x = x$$

$$a(x + y) = ax + ay$$

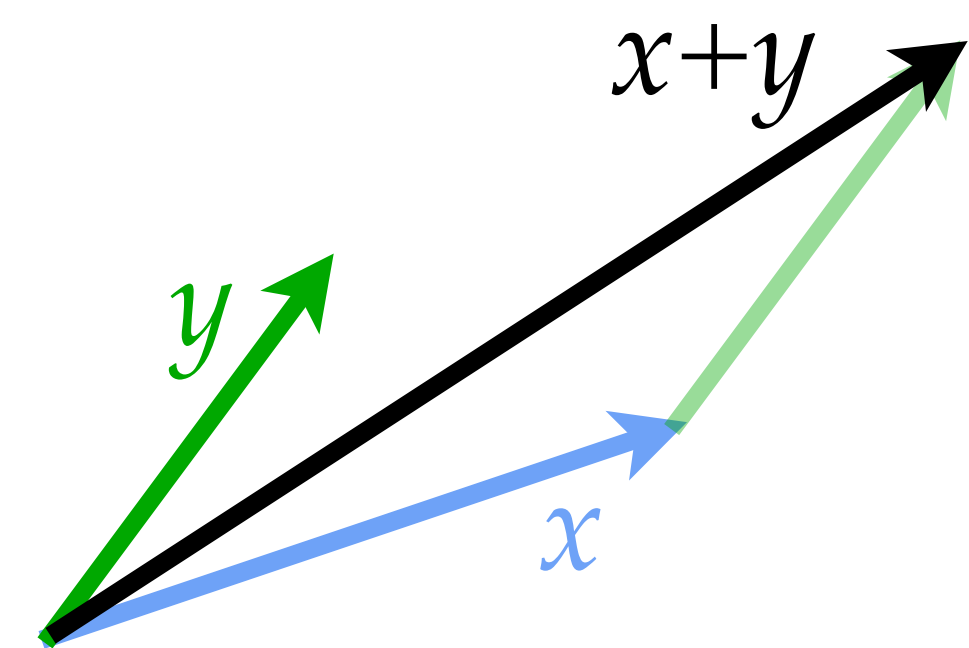
$$\forall x, \exists \tilde{x} \in V \text{ s.t. } x + \tilde{x} = 0$$

$$(a + b)x = ax + bx$$

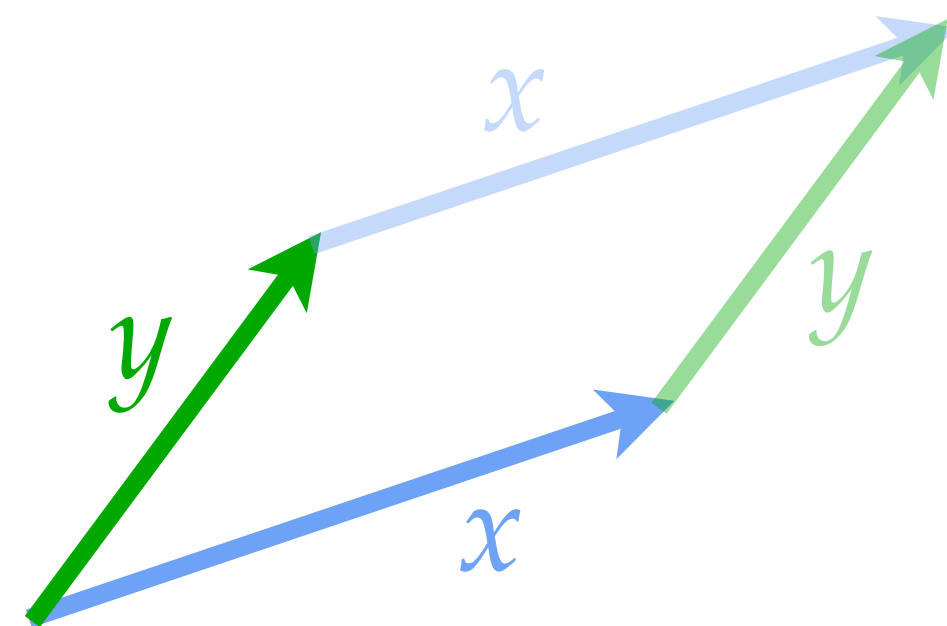
*Note: in general, could use something other than *reals* here.

Vector Spaces — Geometric Reasoning

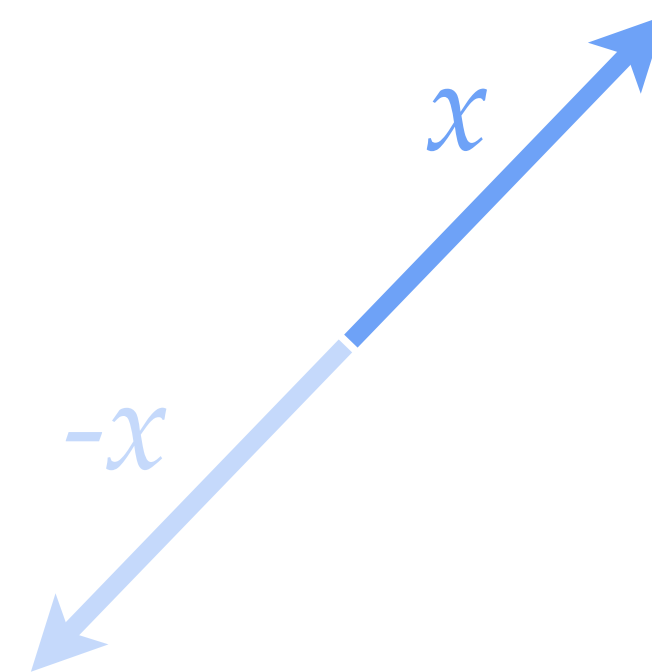
- Where do these rules come from?
- As with numbers, reflect how *oriented lengths* (vectors) behave in nature:



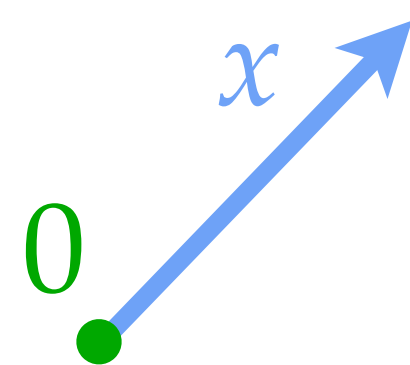
$$x + y \in V$$



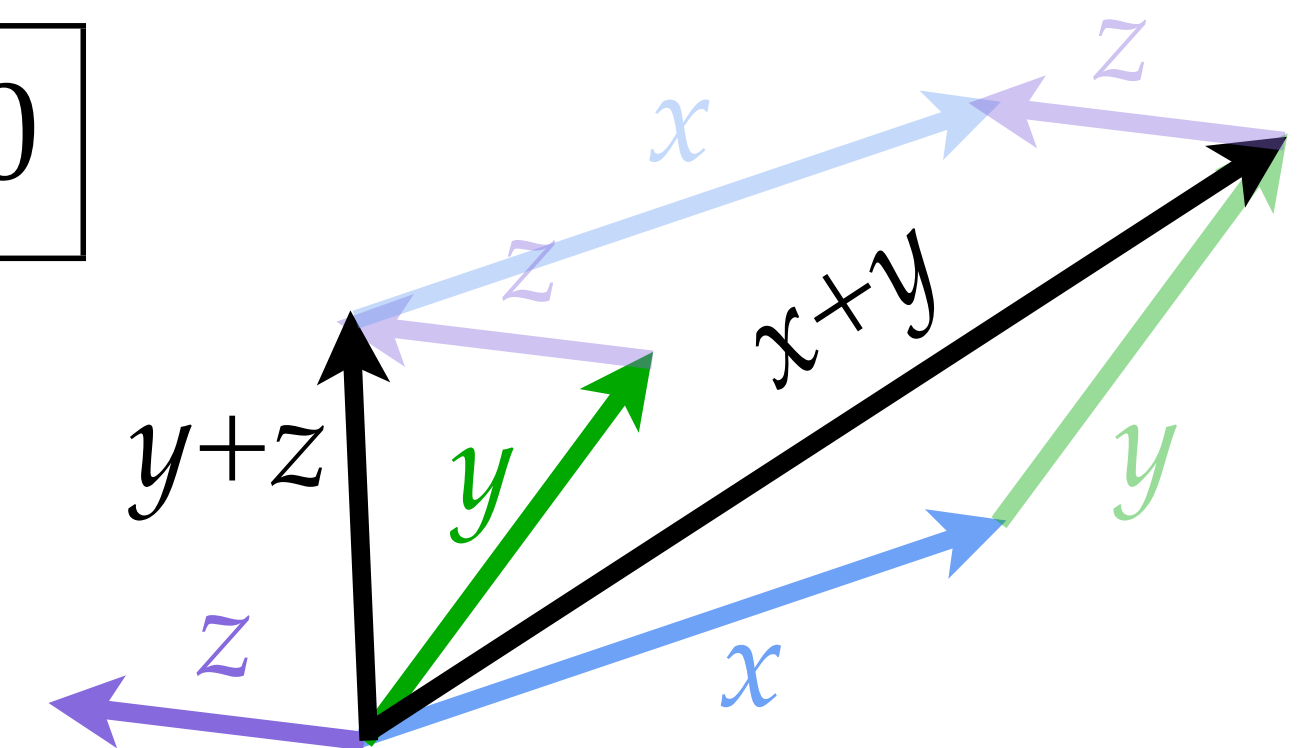
$$x + y = y + x$$



$$x + (-x) = 0$$



$$x + 0 = x$$



$$(x + y) + z = x + (y + z)$$

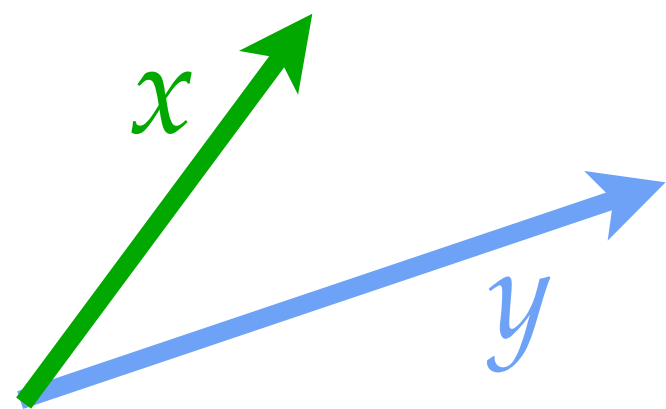
...but the algebra makes it easier to manage complexity!

Review: Inner Product

- We can also associate a vector space with an *inner product*

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

- The quantity $\langle x, y \rangle$ captures how well two vectors x, y in V “line up”
- For all vectors x, y, z in V , real numbers a , any (*real*) *inner product* must satisfy



symmetry

$$\langle x, y \rangle = \langle y, x \rangle$$

linearity

$$\langle ax, y \rangle = a \langle x, y \rangle$$

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

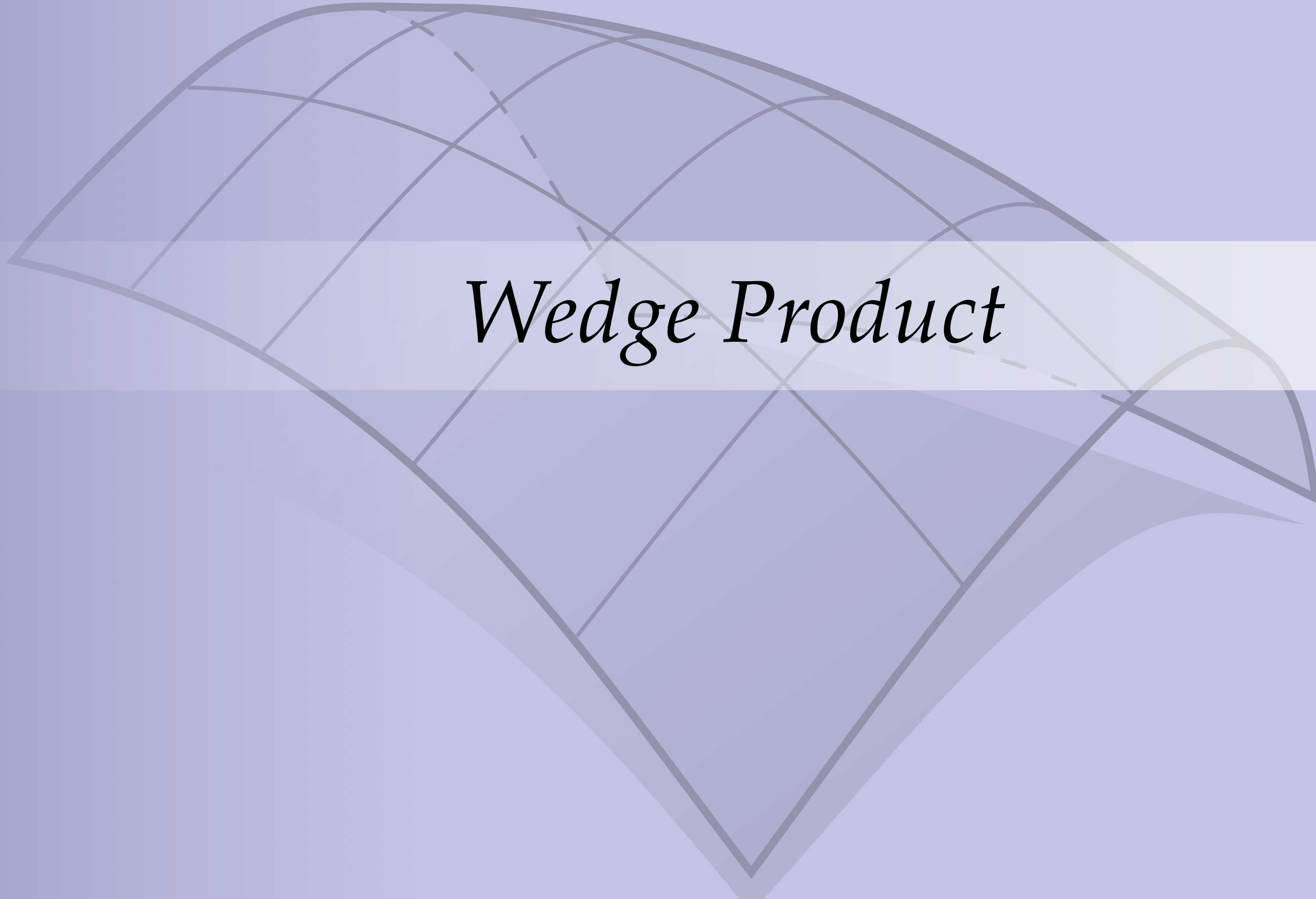
positivity

$$\langle x, x \rangle > 0, x \neq 0$$

$$\langle x, x \rangle = 0, x = 0$$

Example. Euclidean inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$

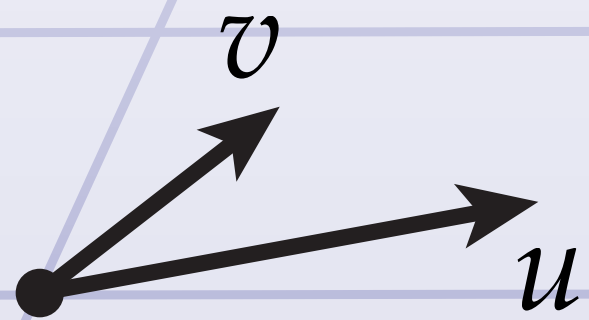
(Where do these “rules” come from? Why might they be natural?)



Wedge Product

Review: Span

Q: Geometrically, what is the *span* of two vectors?



$$u, v \in V, \quad \text{span}(\{u, v\}) := \{x \in V \mid x = au + bv, a, b \in \mathbb{R}\}$$

Span

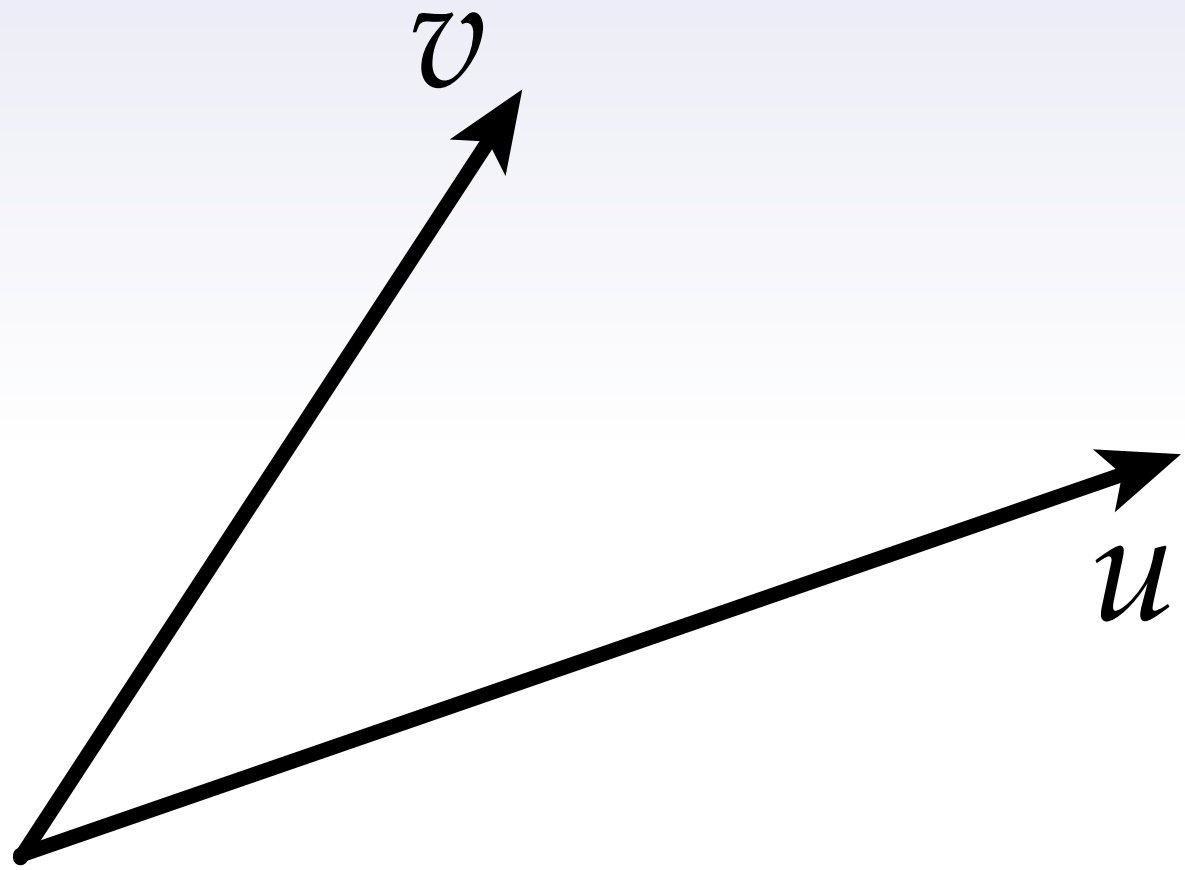
Definition. In any vector space V , the *span* of a finite* collection of vectors $\{v_1, \dots, v_k\}$ is the set of all possible linear combinations:

$$\text{span}(\{v_1, \dots, v_n\}) := \left\{ x \in V \mid x = \sum_{i=1}^k a_i v_i, \quad a_i \in \mathbb{R} \right\}.$$

The span of a collection of vectors is a *linear subspace*, i.e., a subset that forms a vector space with respect to the original vector space operations.

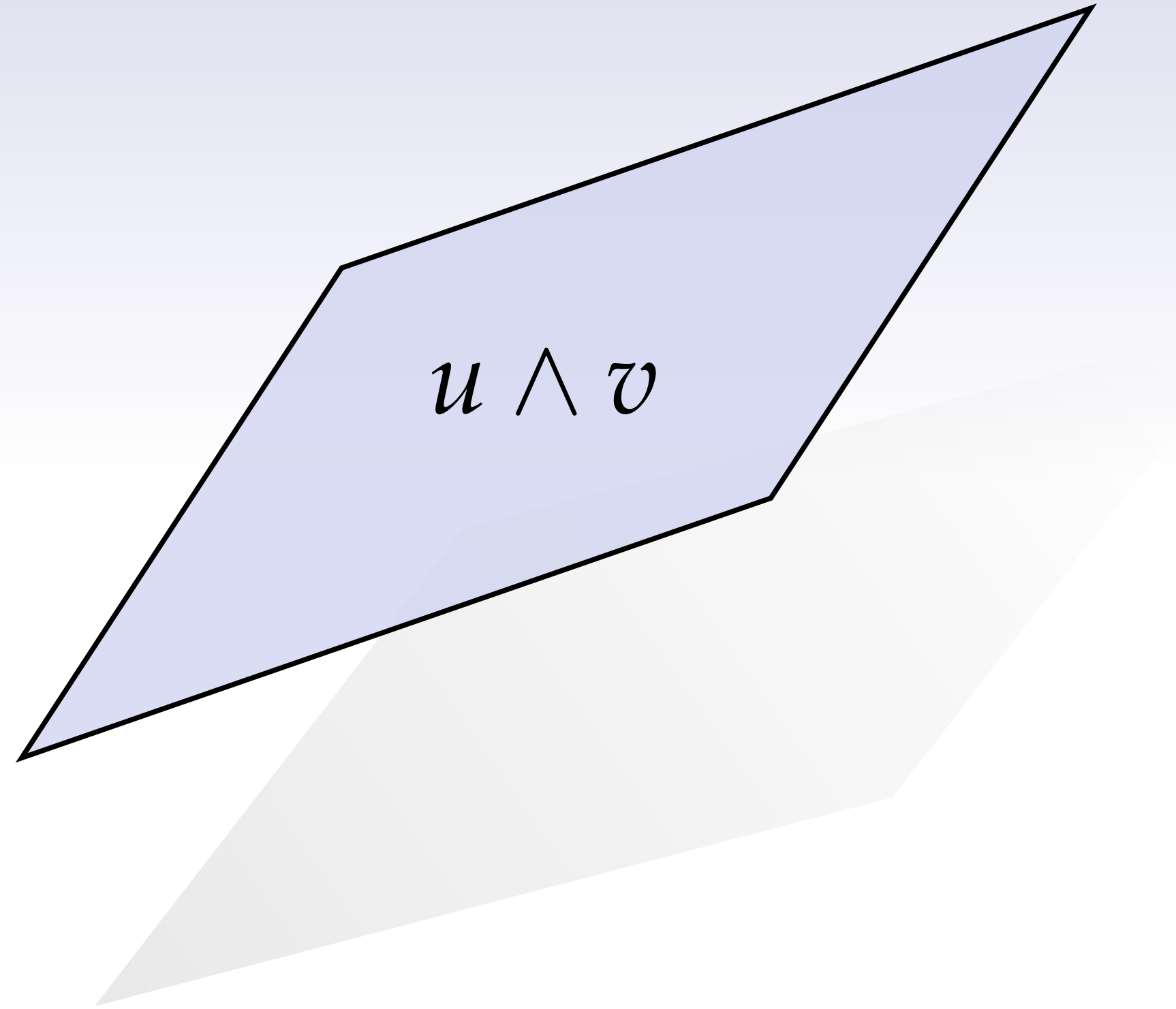
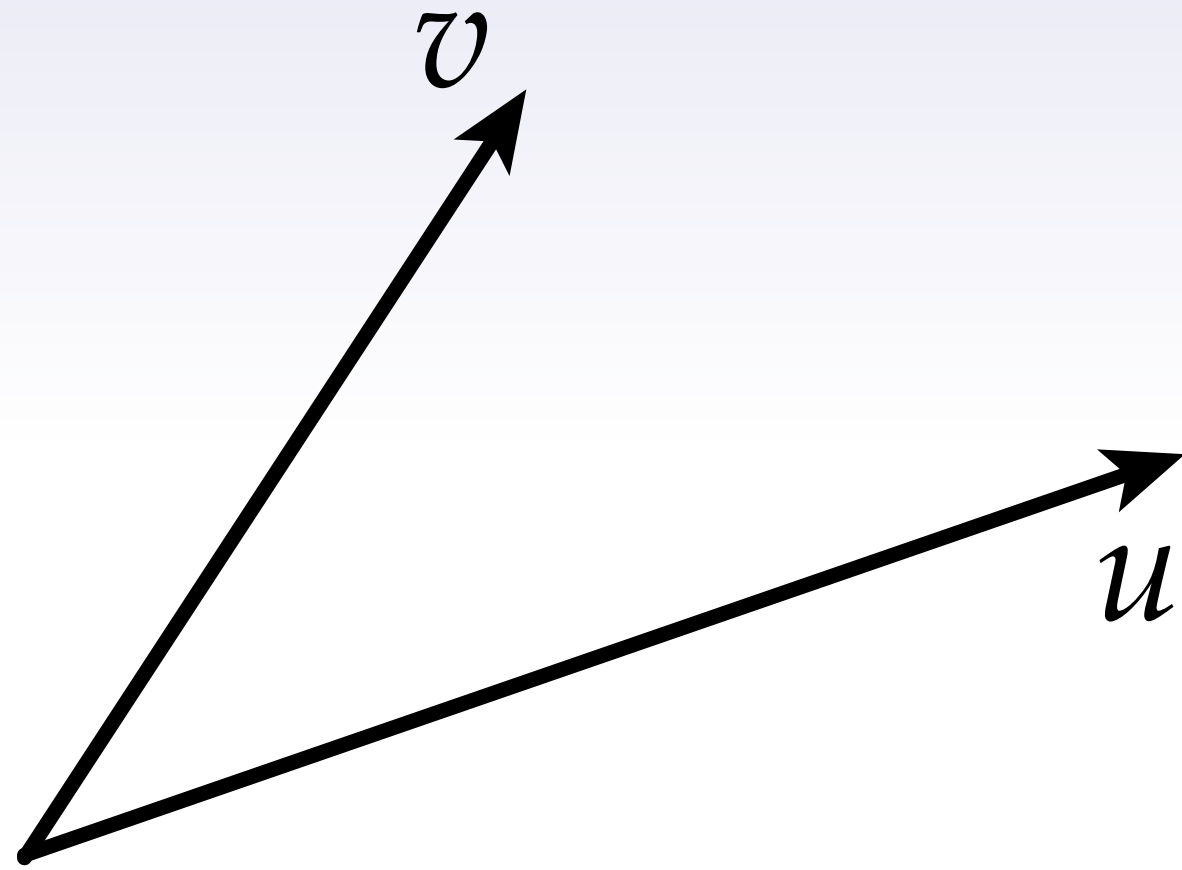
*Note: one can extend this definition to infinite sums, but only with additional assumptions about V .

Wedge Product (\wedge)



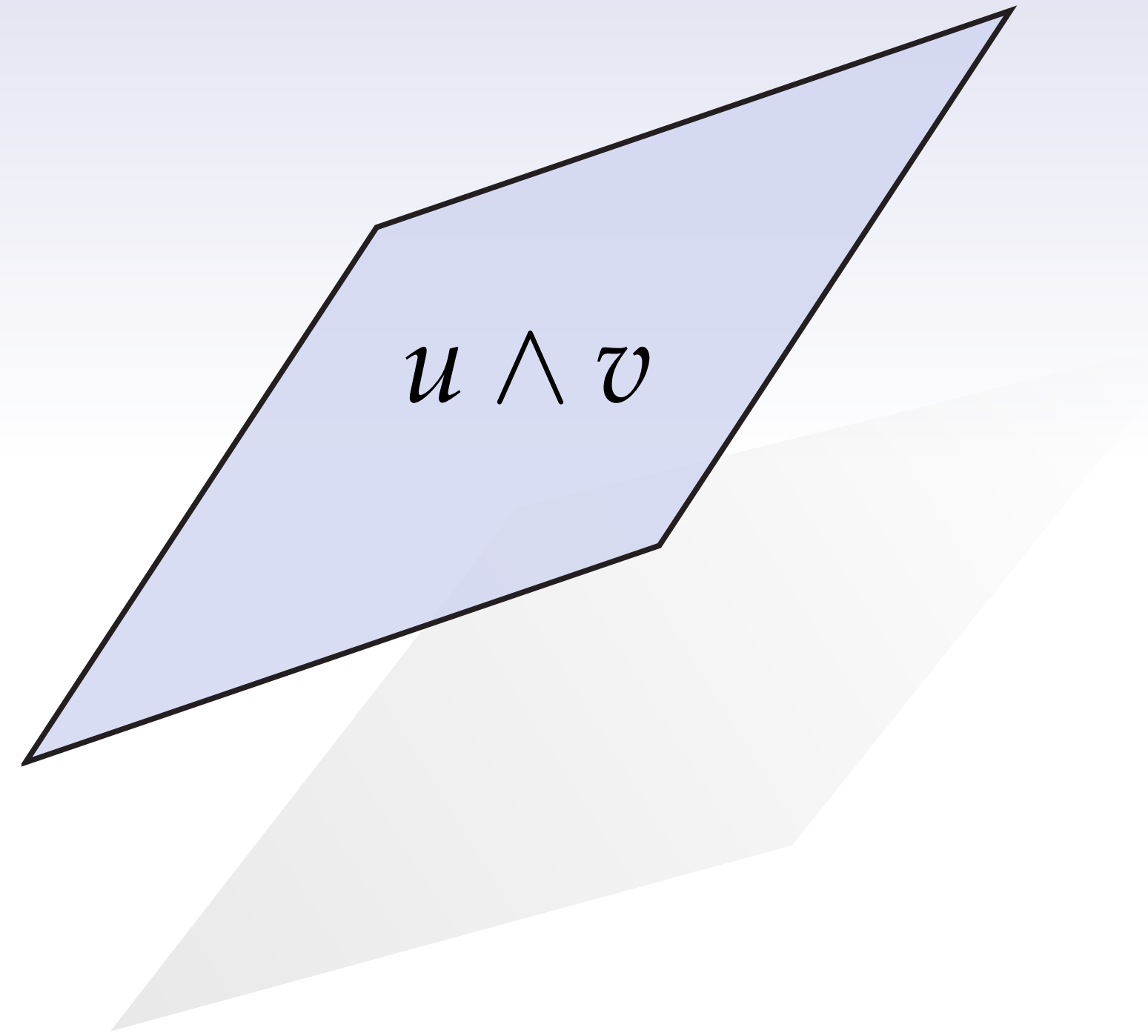
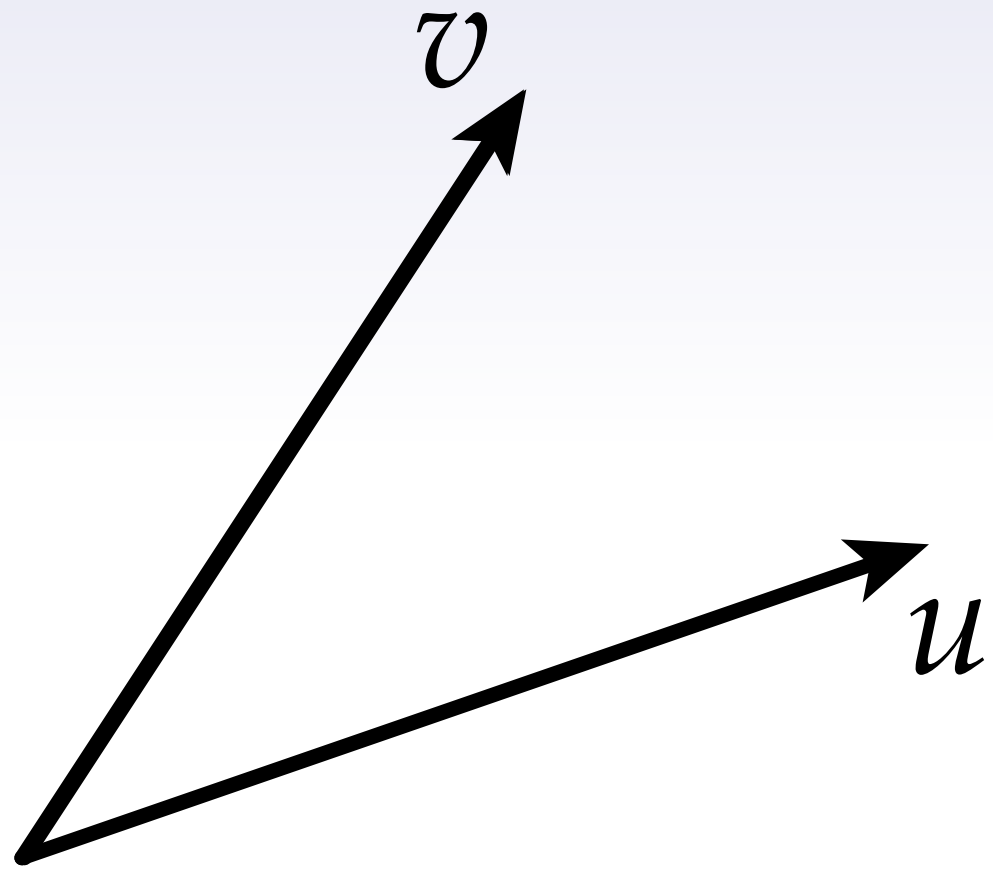
Analogy: *span*

Wedge Product (\wedge)



Analogy: *span*

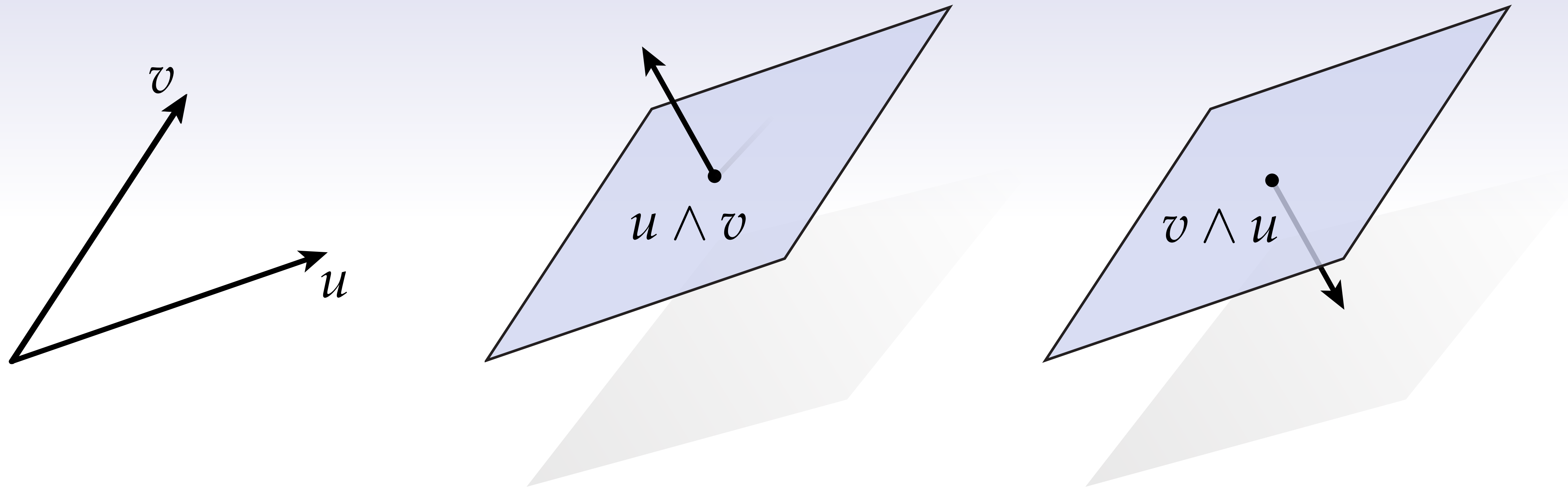
Wedge Product (\wedge)



Analogy: *span*

Wedge Product (\wedge)

$$u \wedge v = -v \wedge u$$



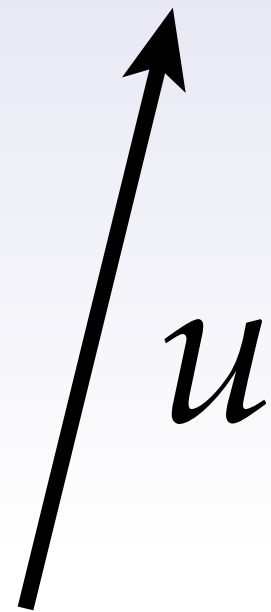
Analogy: *span*

Key differences: orientation & “finite extent”

Key property: *antisymmetry*

Wedge Product—Degeneracy

Q: What is the wedge product of a vector with itself?

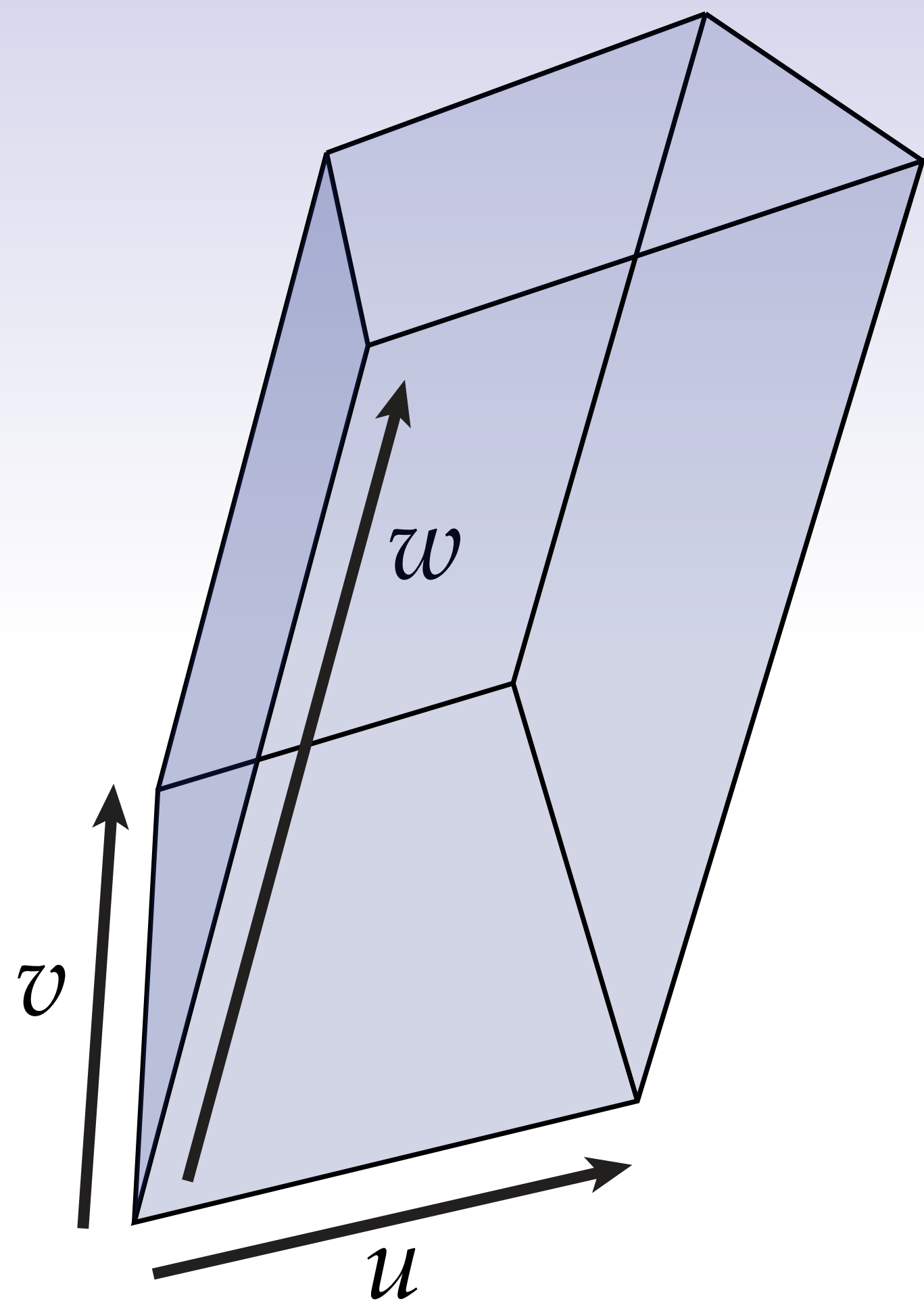


A: Geometrically, spans a region of *zero area*.

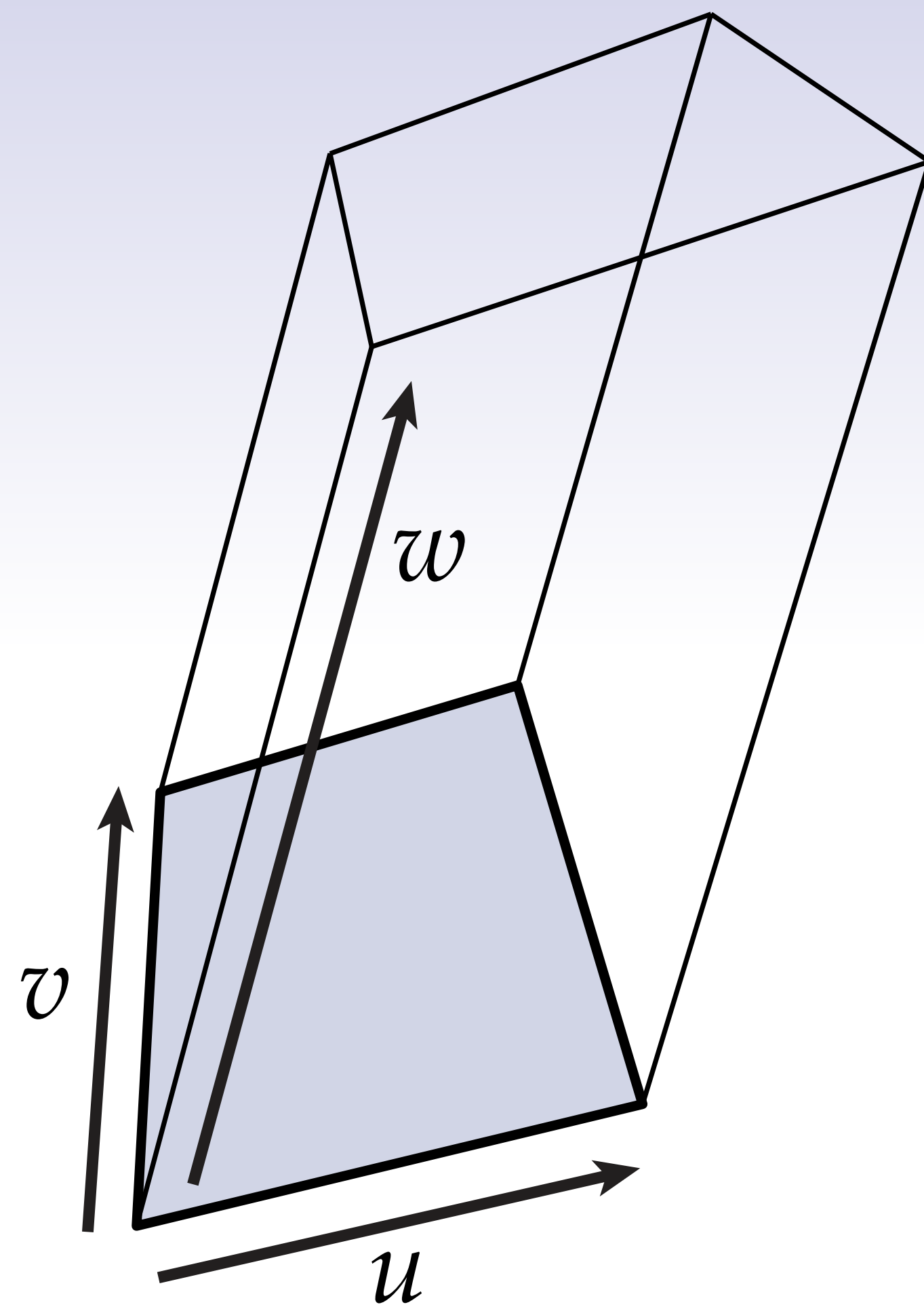
$$u \wedge u = 0$$

*May change when we generalize (later...)

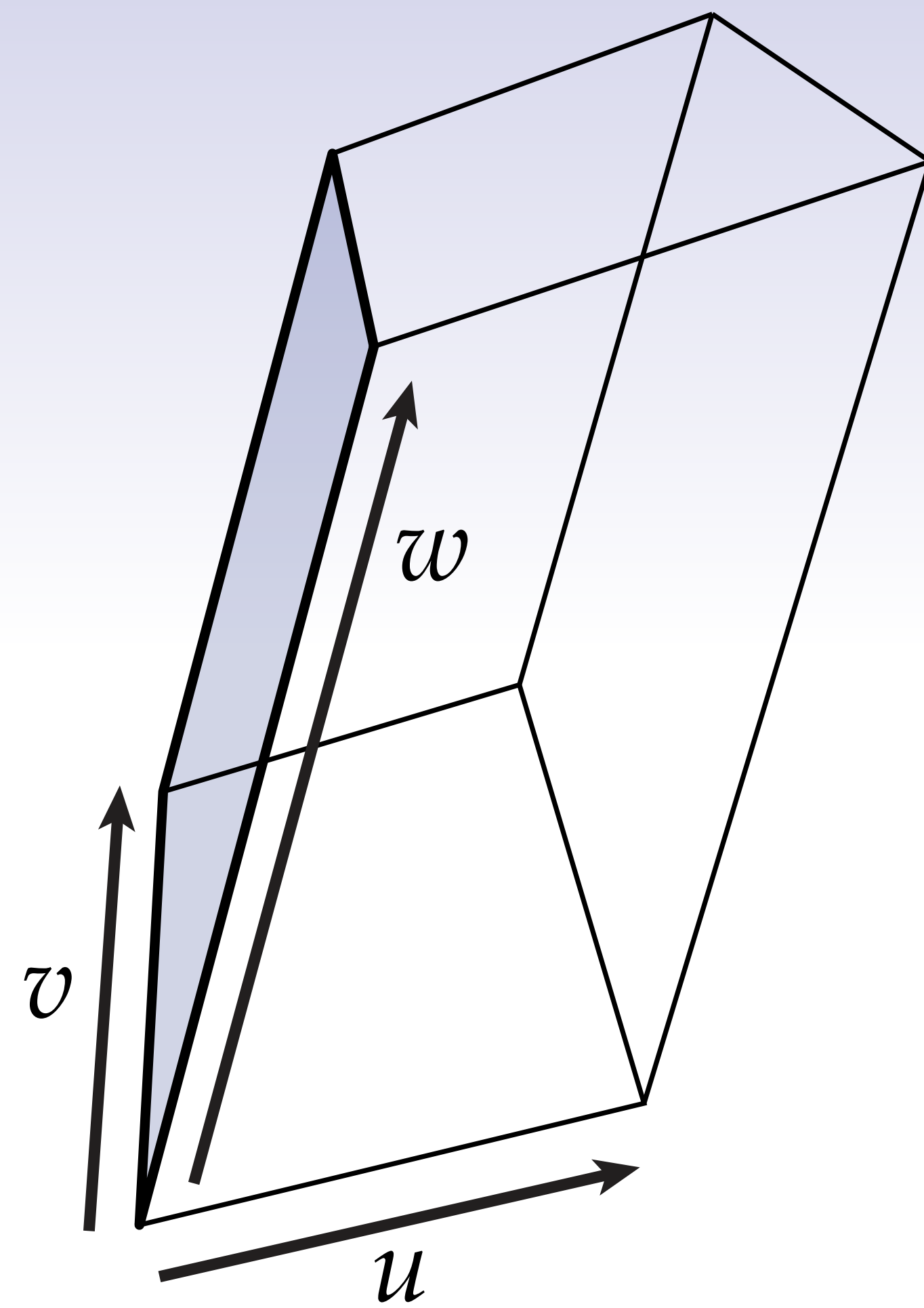
Wedge Product - Associativity



$$u \wedge v \wedge w$$

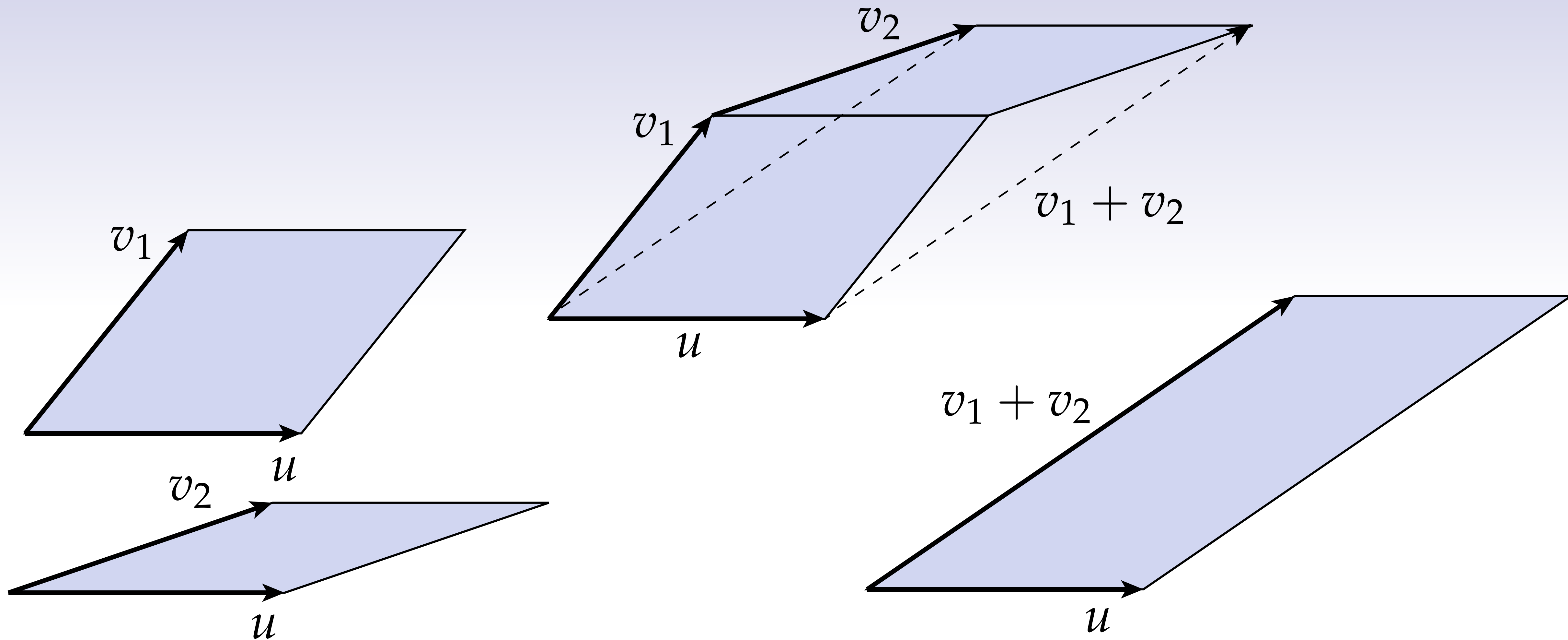


$$(u \wedge v) \wedge w$$



$$u \wedge (v \wedge w)$$

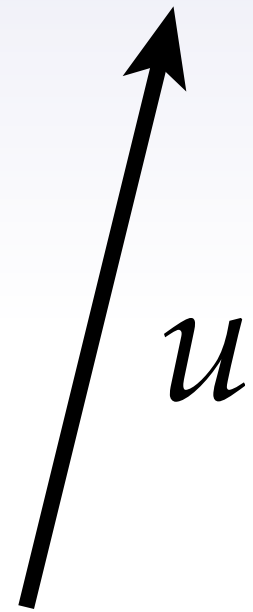
Wedge Product - Distributivity



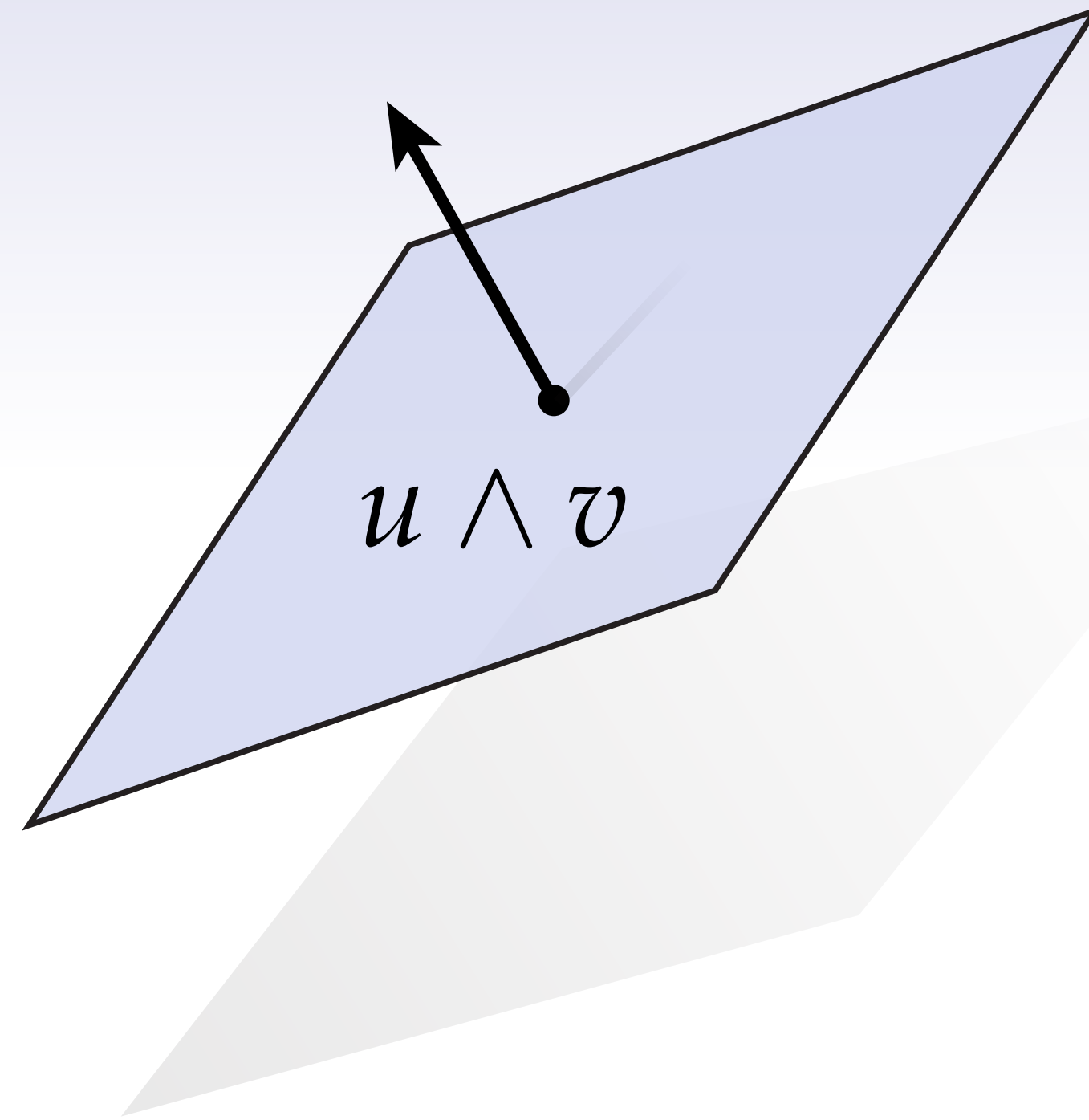
$$u \wedge v_1 + u \wedge v_2 = u \wedge (v_1 + v_2)$$

k-Vectors

The wedge of k vectors is called a "*k*-vector."

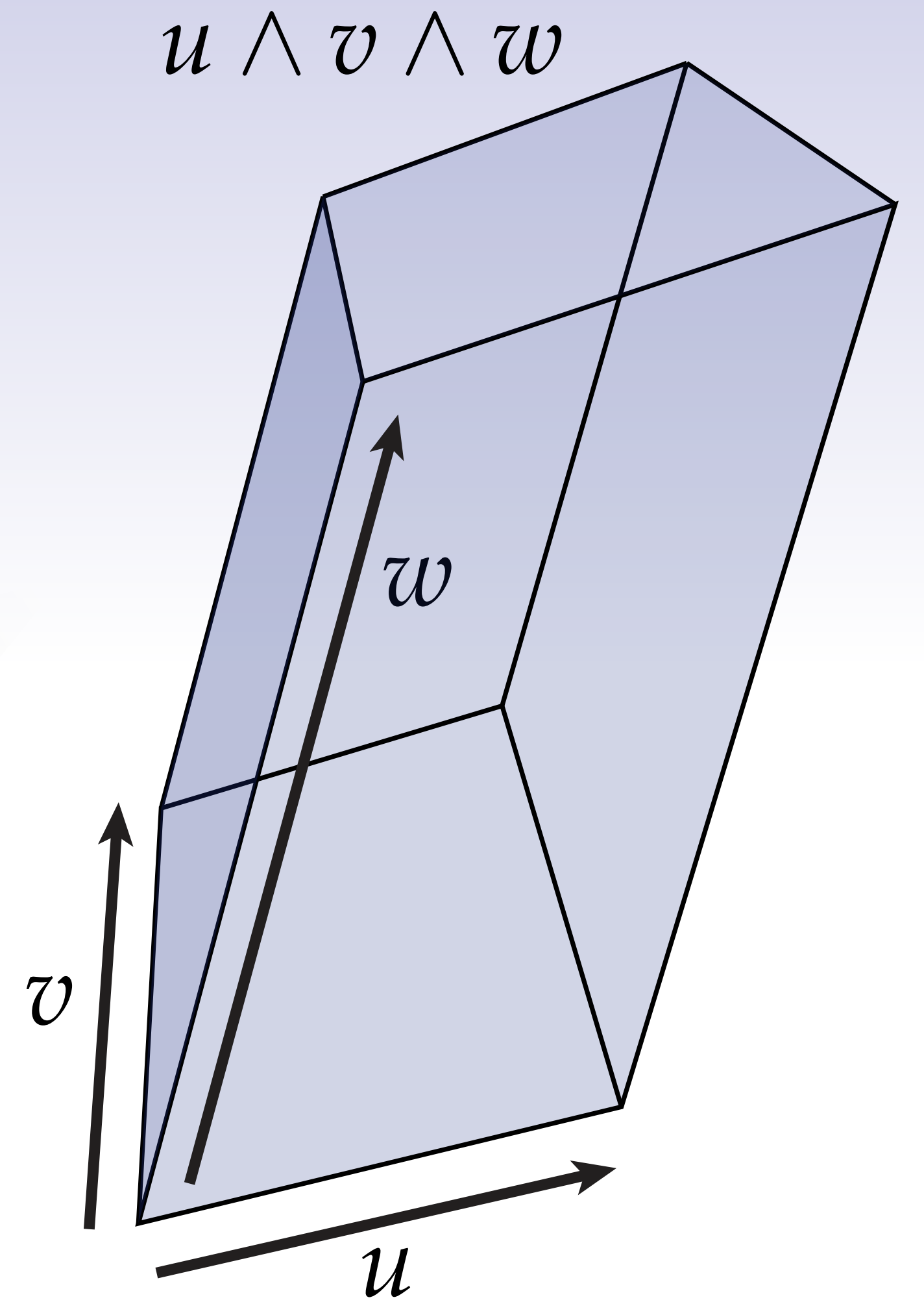


0-vector



1-vector

2-vector

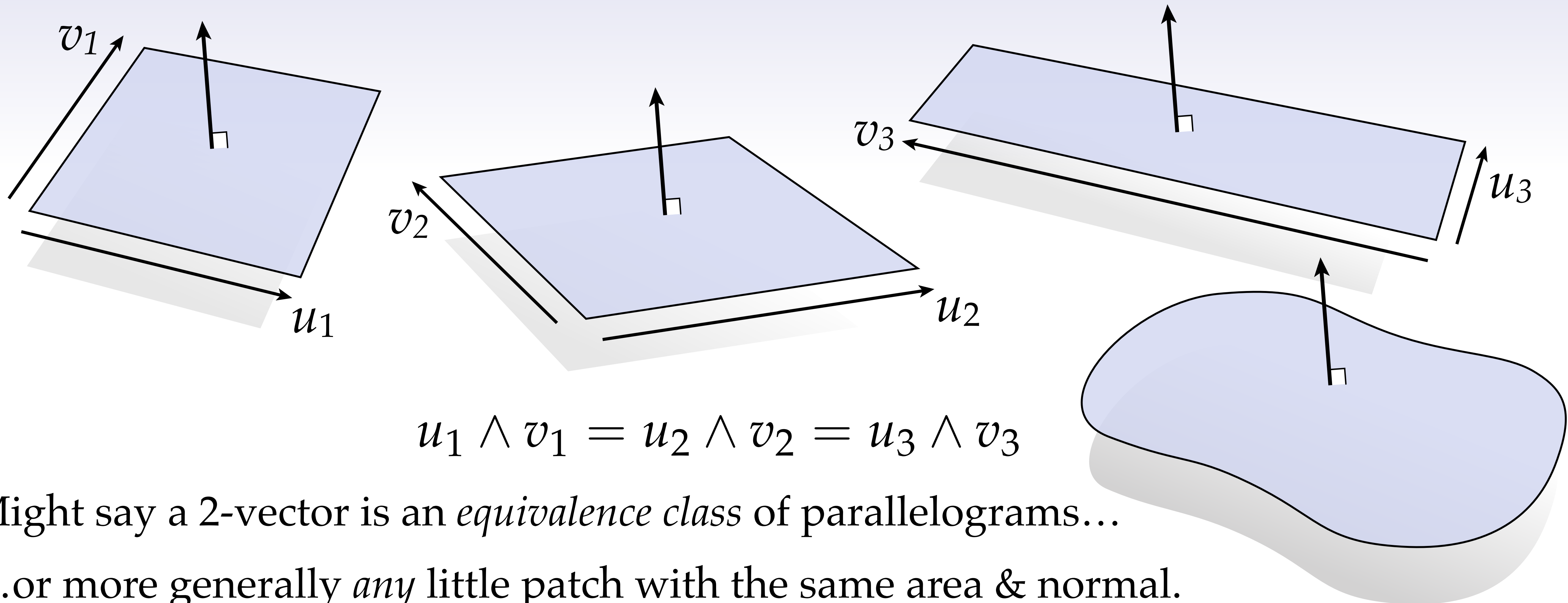


3-vector

Visualization of k -Vectors

Our visualization is a little misleading: k -vectors only have *direction & magnitude*.

E.g., parallelograms w/ same plane, orientation, and area represent same 2-vector:



Might say a 2-vector is an *equivalence class* of parallelograms...

...or more generally *any* little patch with the same area & normal.

0-vectors as Scalars

Q: What do you get when you wedge *zero* vectors together?

A: You get this:

For convenience, however, we will say that a “0-vector” is a *scalar value* (e.g., a real number). This treatment becomes extremely useful later on...

Key idea: *magnitude*, but no *direction* (scalar).



Hodge Star

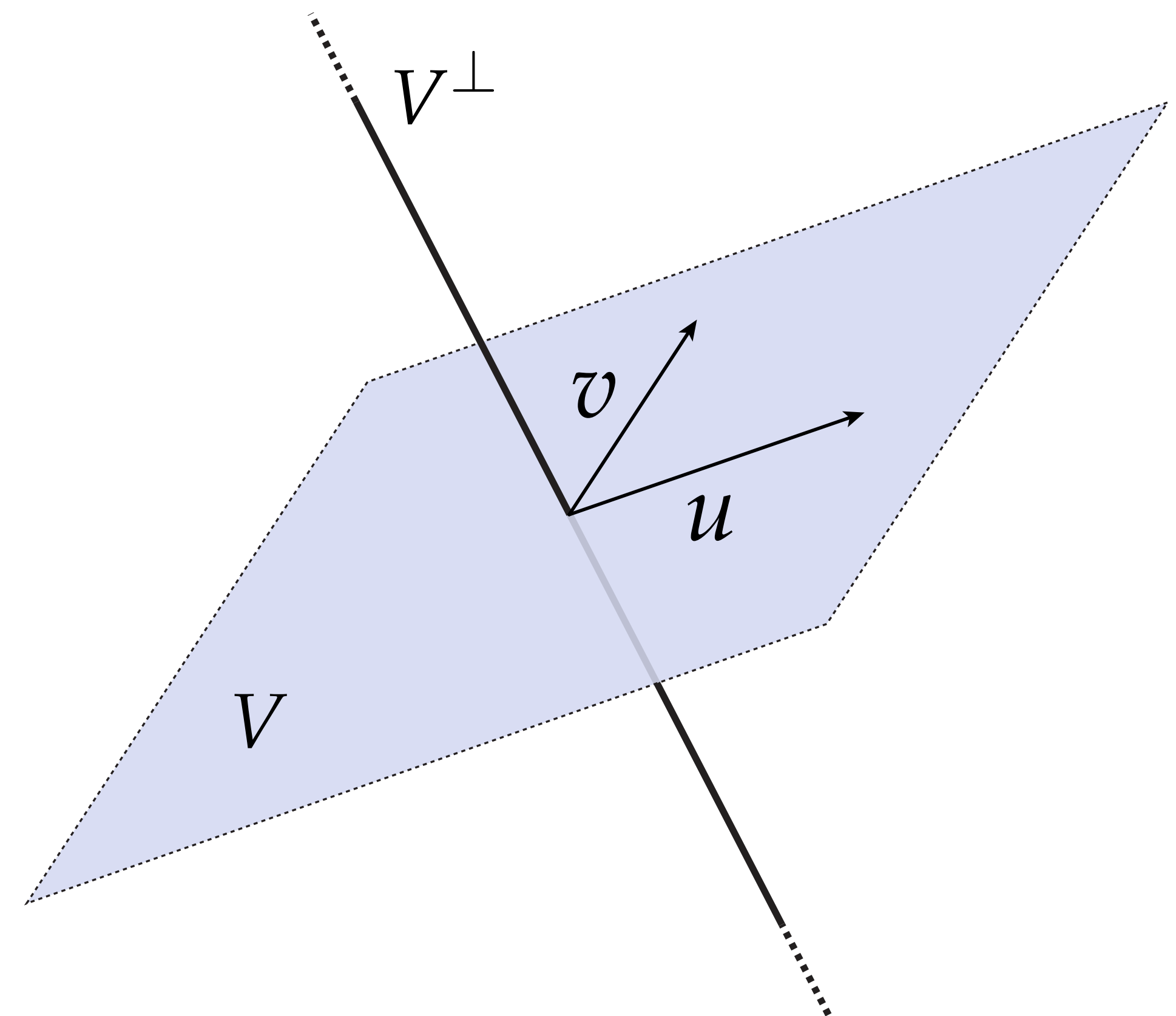
Review: Orthogonal Complement

Q: Geometrically, what is the *orthogonal complement* of a linear subspace?

Example: *orthogonal complement of a span*

$$V := \text{span}(\{u, v\})$$

$$V^\perp := \{x \in \mathbb{R}^n \mid \langle x, w \rangle = 0 \ \forall w \in V\}$$



Notice: orthogonal complement meaningful only if we have an *inner product*!

Orthogonal Complement

Definition: Let $U \subseteq V$ be a linear subspace of a vector space V with an inner product $\langle \cdot, \cdot \rangle$. The *orthogonal complement* of U is the collection of vectors

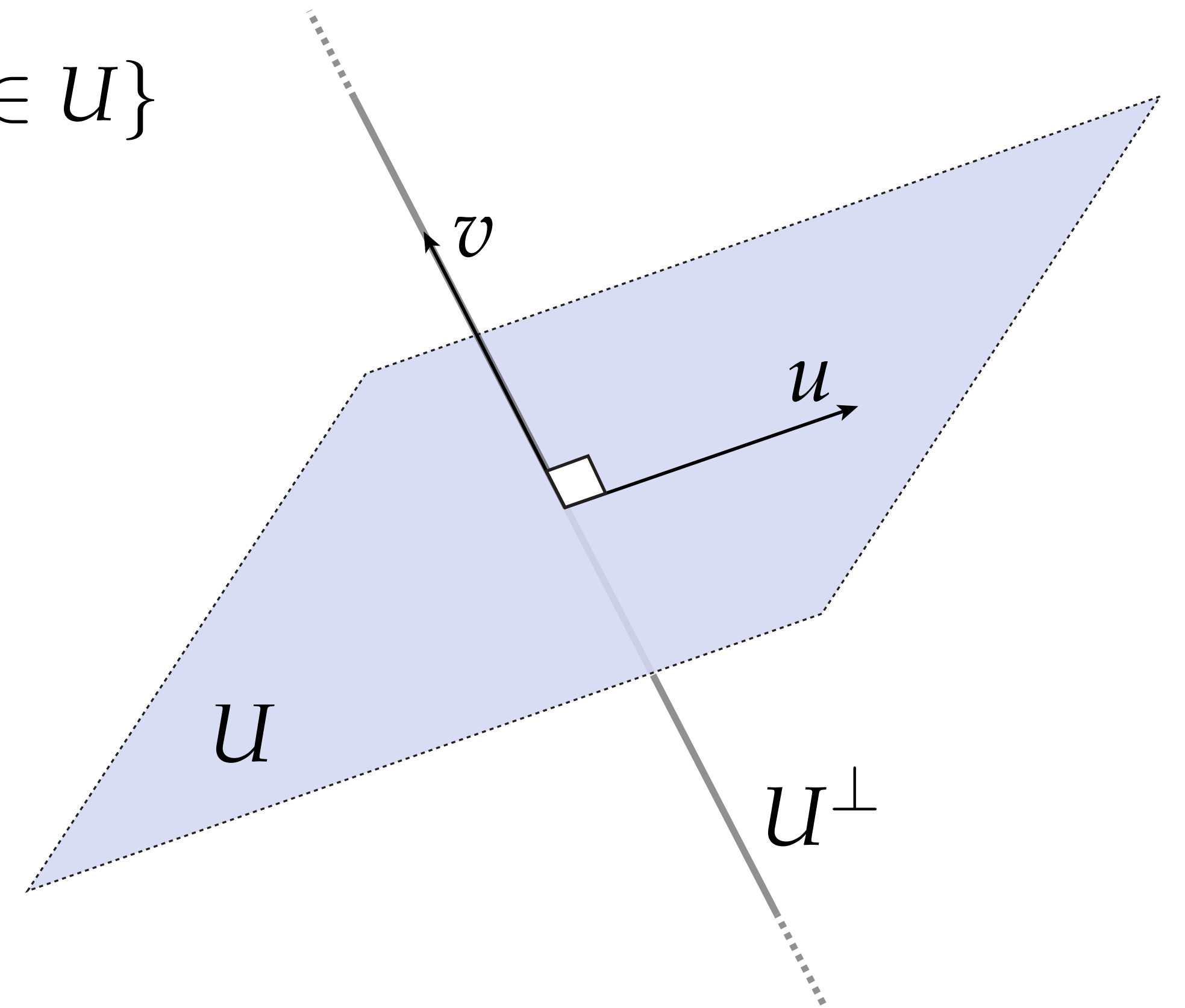
$$U^\perp := \{v \in V \mid \langle u, v \rangle = 0, \forall u \in U\}$$

Why is it useful to talk about a *complement*?

Example. “What kind of cuisine do you like?”

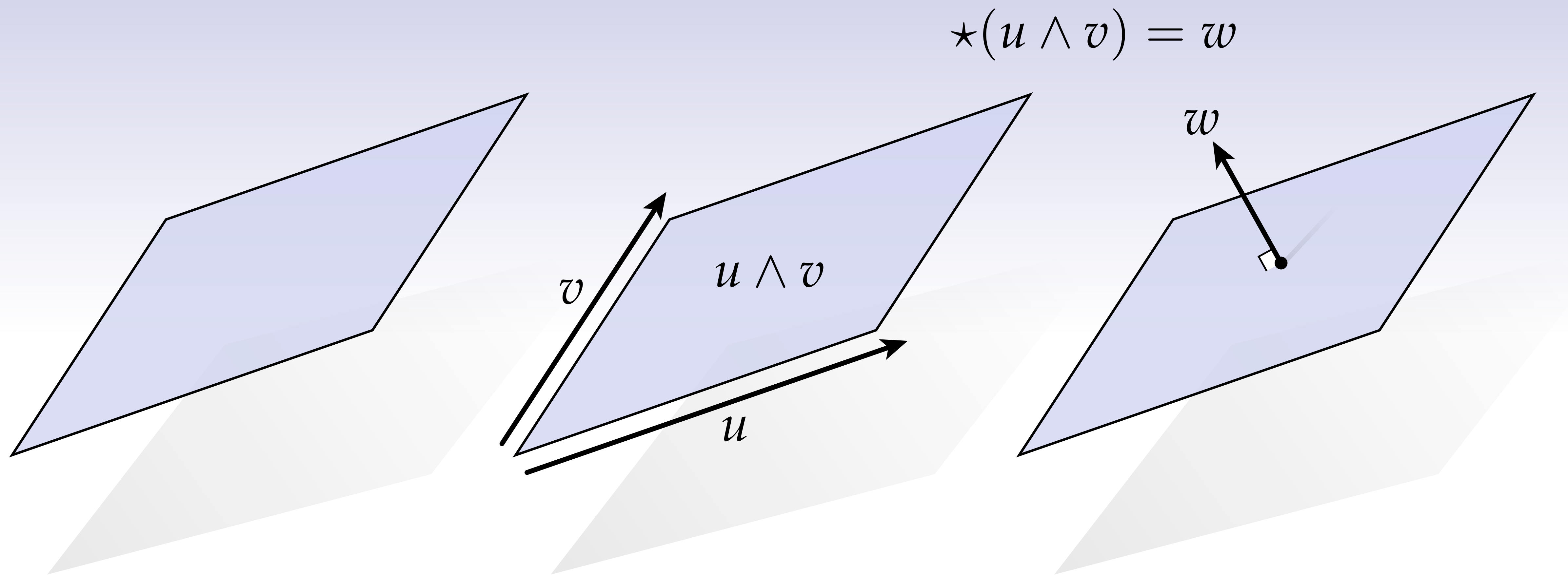
Option 1: “I like Vietnamese, Italian, Ethiopian, ...”

Option 2: “I like everything but Bavarian food!”



Key idea: often it's easier to specify a set by saying what it *doesn't* contain.

Hodge Star (\star)



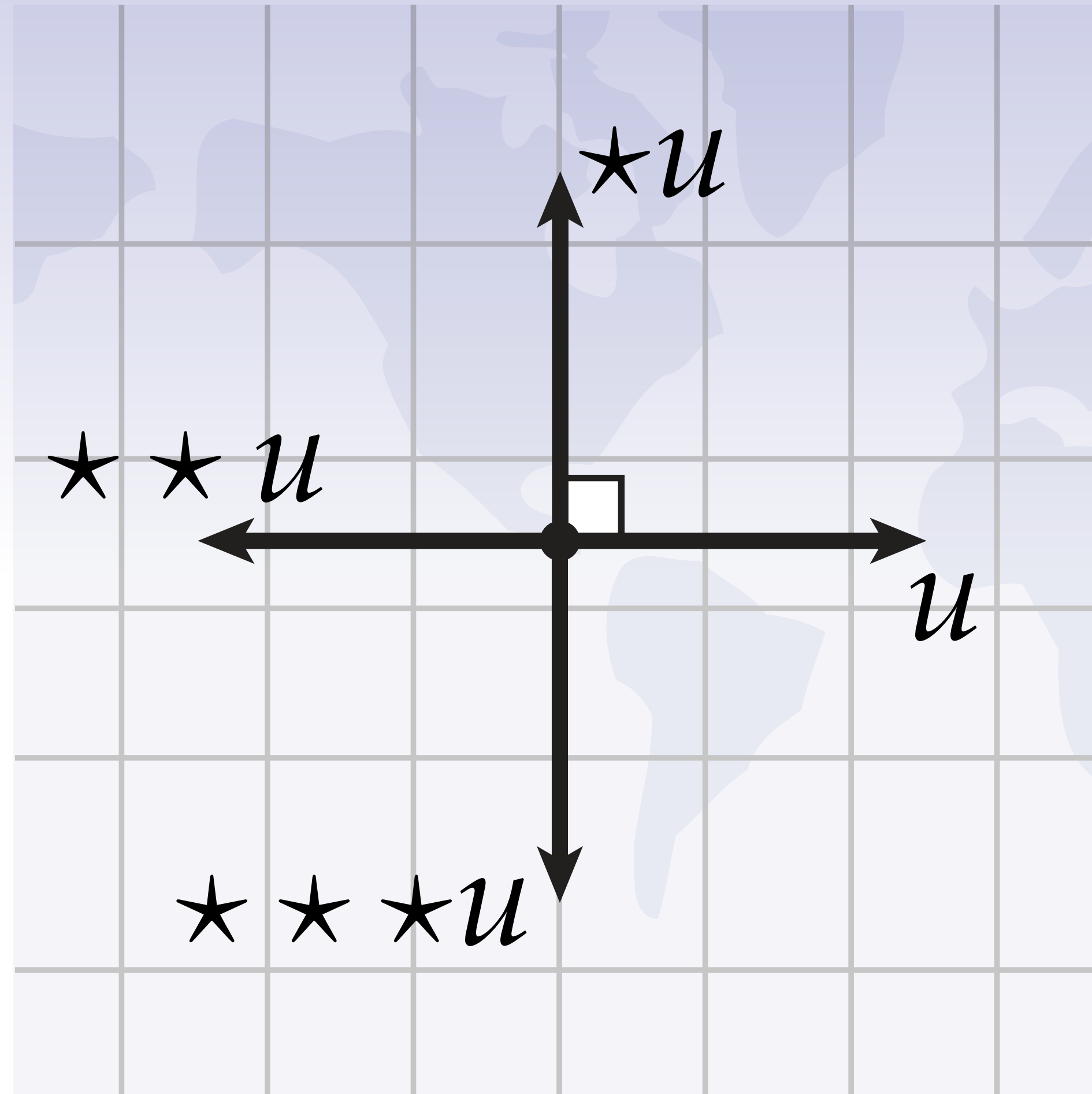
Analogy: *orthogonal complement*

Key differences: orientation & magnitude

Important detail: $z \wedge \star z$ is *positively oriented*

$$k \mapsto (n - k)$$

Hodge Star - 2D

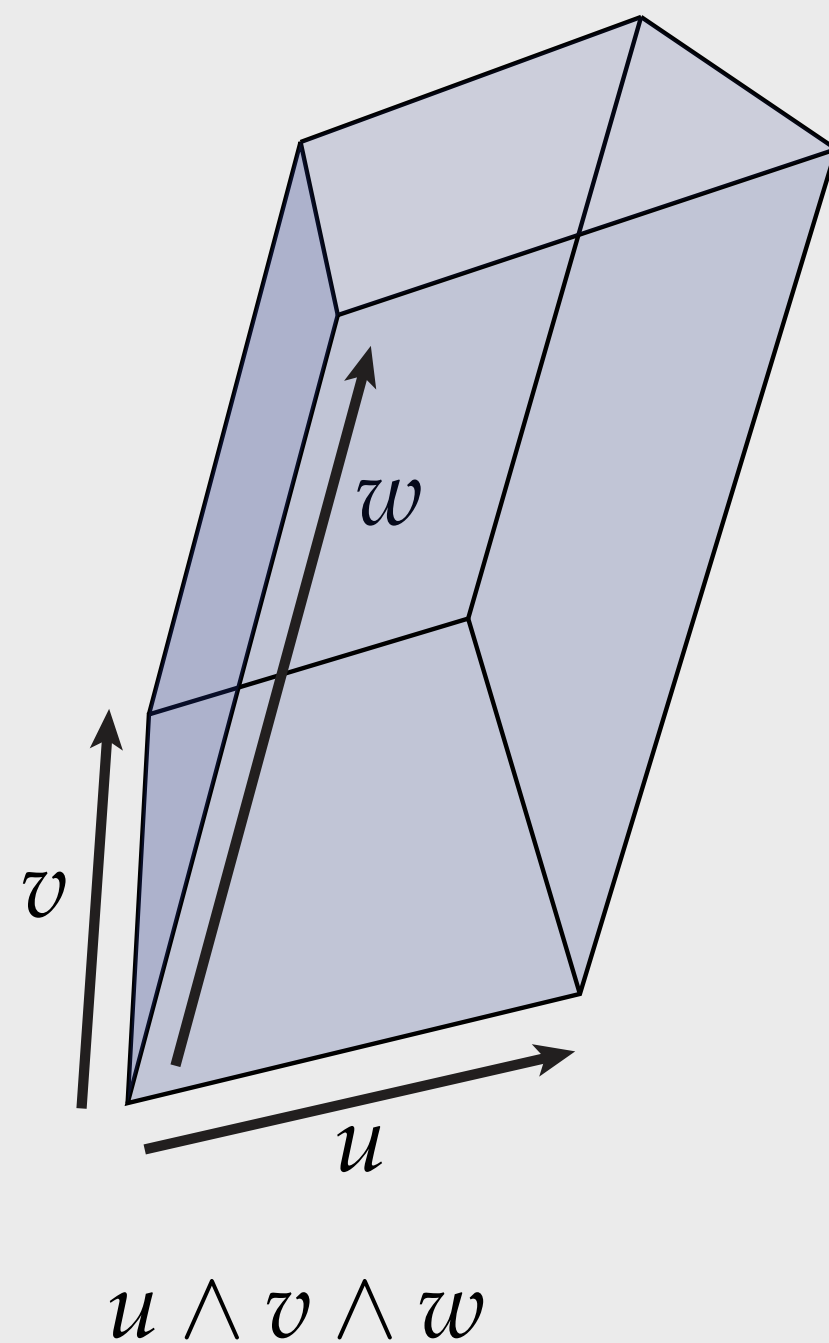


Analogy: 90-degree rotation

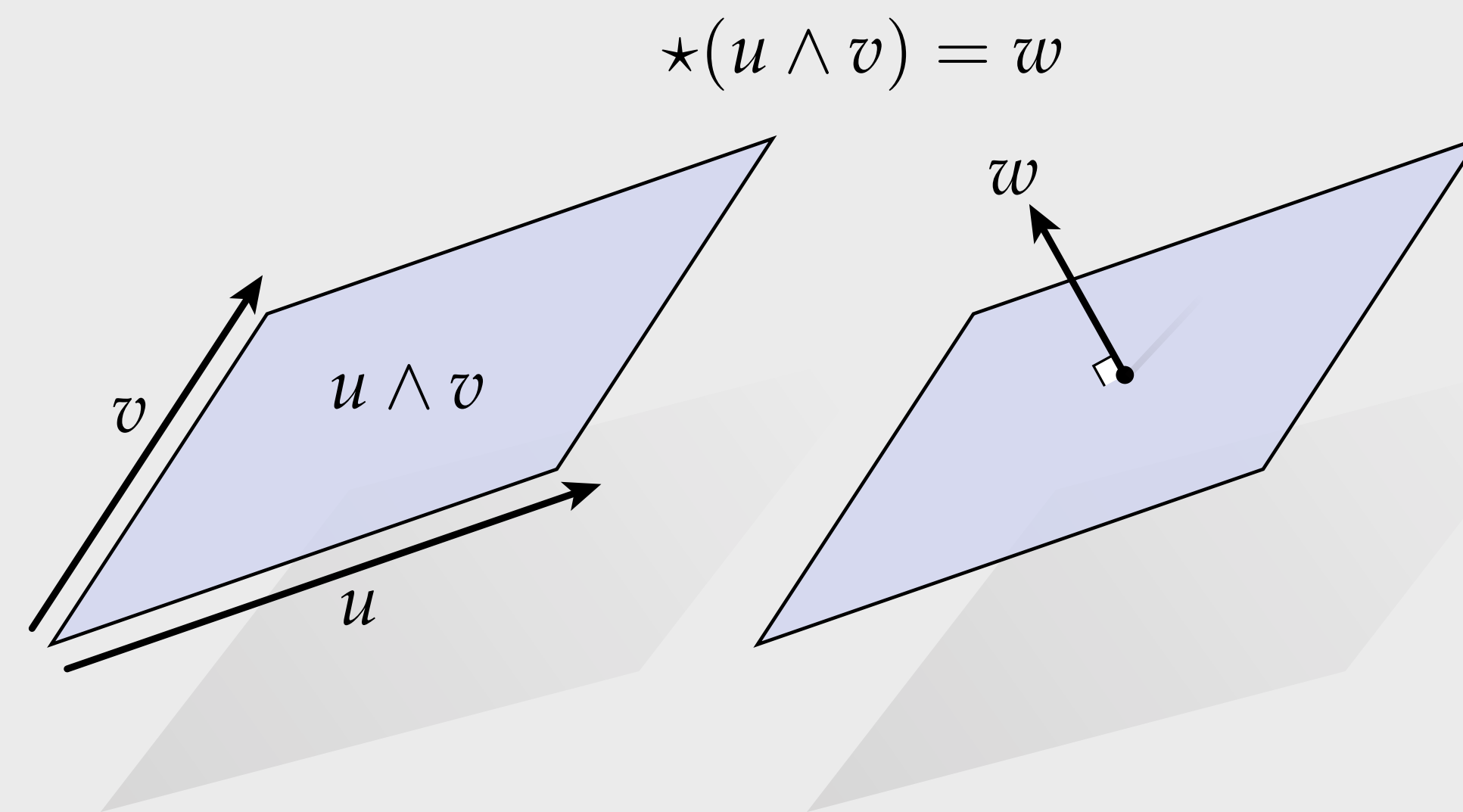
Exterior Algebra—Recap

Let V be an n -dimensional vector space, consisting of vectors or 1 -vectors.

Can “wedge together” k vectors to get a k -vector (signed volume).



Can apply the Hodge star to get the “complementary” k -vector.

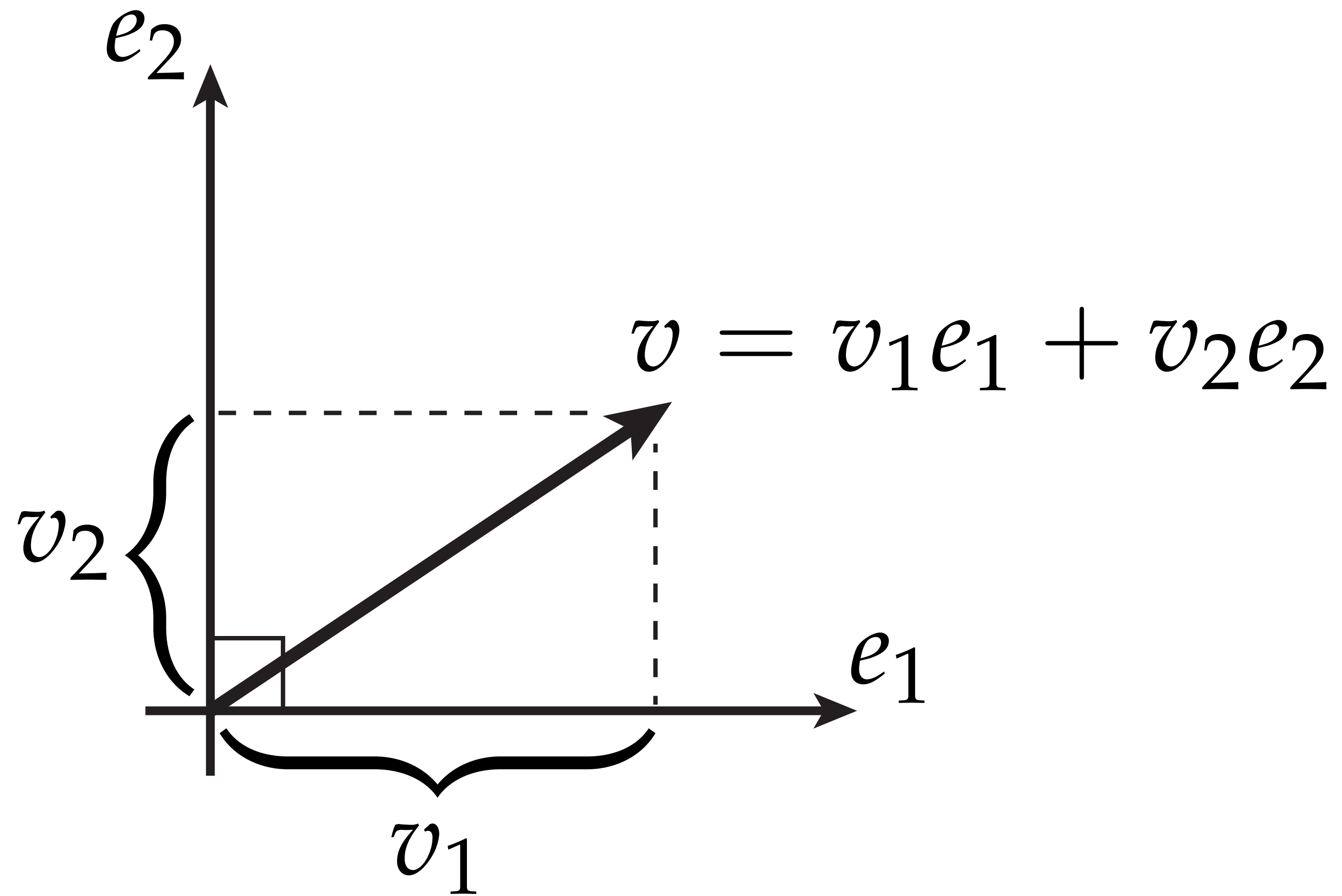


(Also have the usual vector space operations: sum, scalar multiplication, ...)



Coordinate Representation

Basis — Visualized



Key idea: encode a vector by its extent along a collection of independent axes.

Basis & Dimension

Definition. Let V be a vector space. A collection of vectors is *linearly independent* if no vector in the collection can be expressed as a linear combination of the others. A linearly independent collection of vectors $\{e_1, \dots, e_n\}$ is a *basis* for V if every vector $v \in V$ can be expressed as

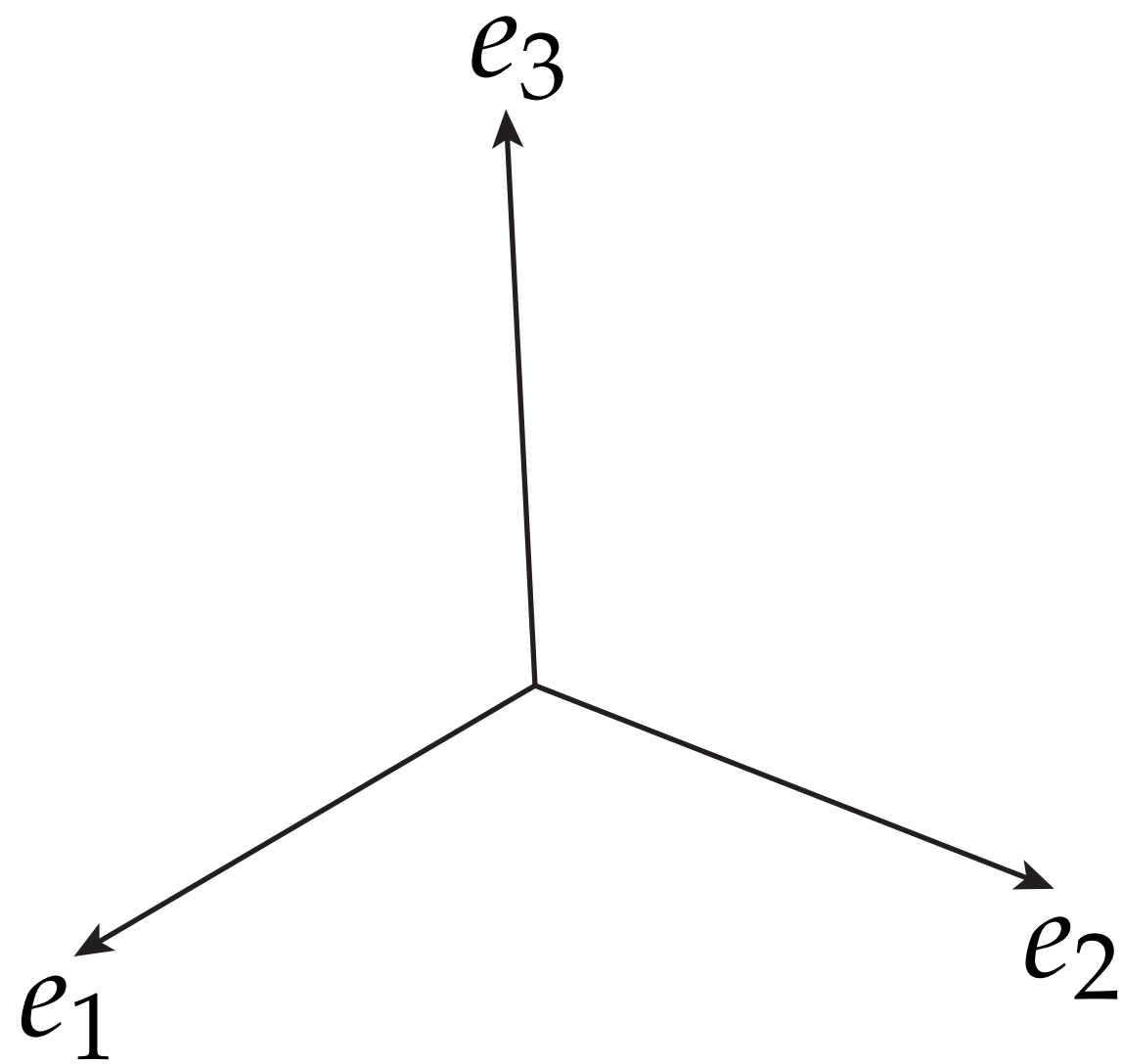
$$v = v_1 e_1 + \dots + v_n e_n$$

for some collection of coefficients $v_1, \dots, v_n \in \mathbb{R}$, i.e., if every vector can be uniquely expressed as a linear combination of the *basis vectors* e_i . In this case, we say that V is *finite dimensional*, with dimension n .

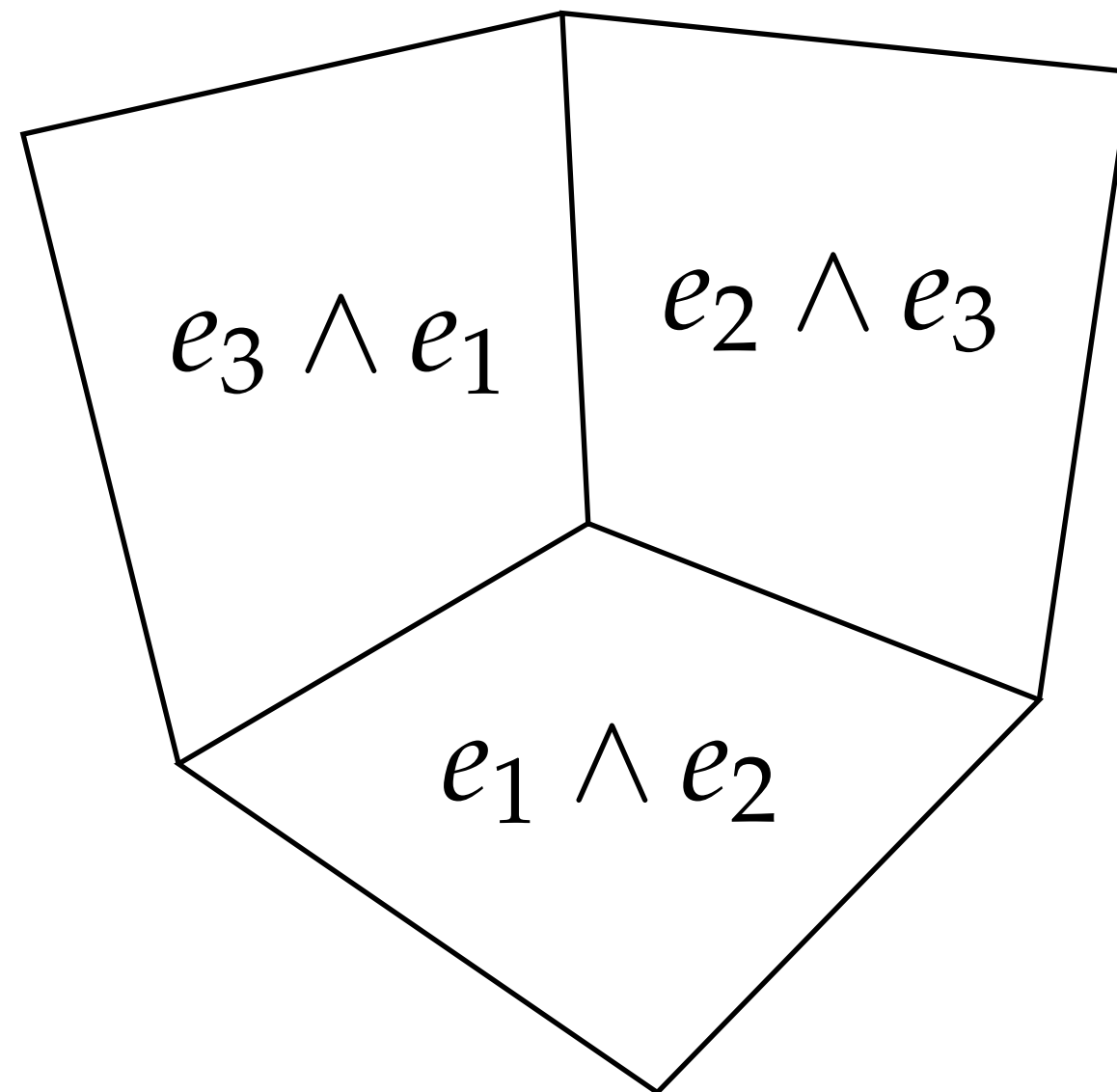
Basis k -Vectors — Visualized

$$(V = \mathbb{R}^3)$$

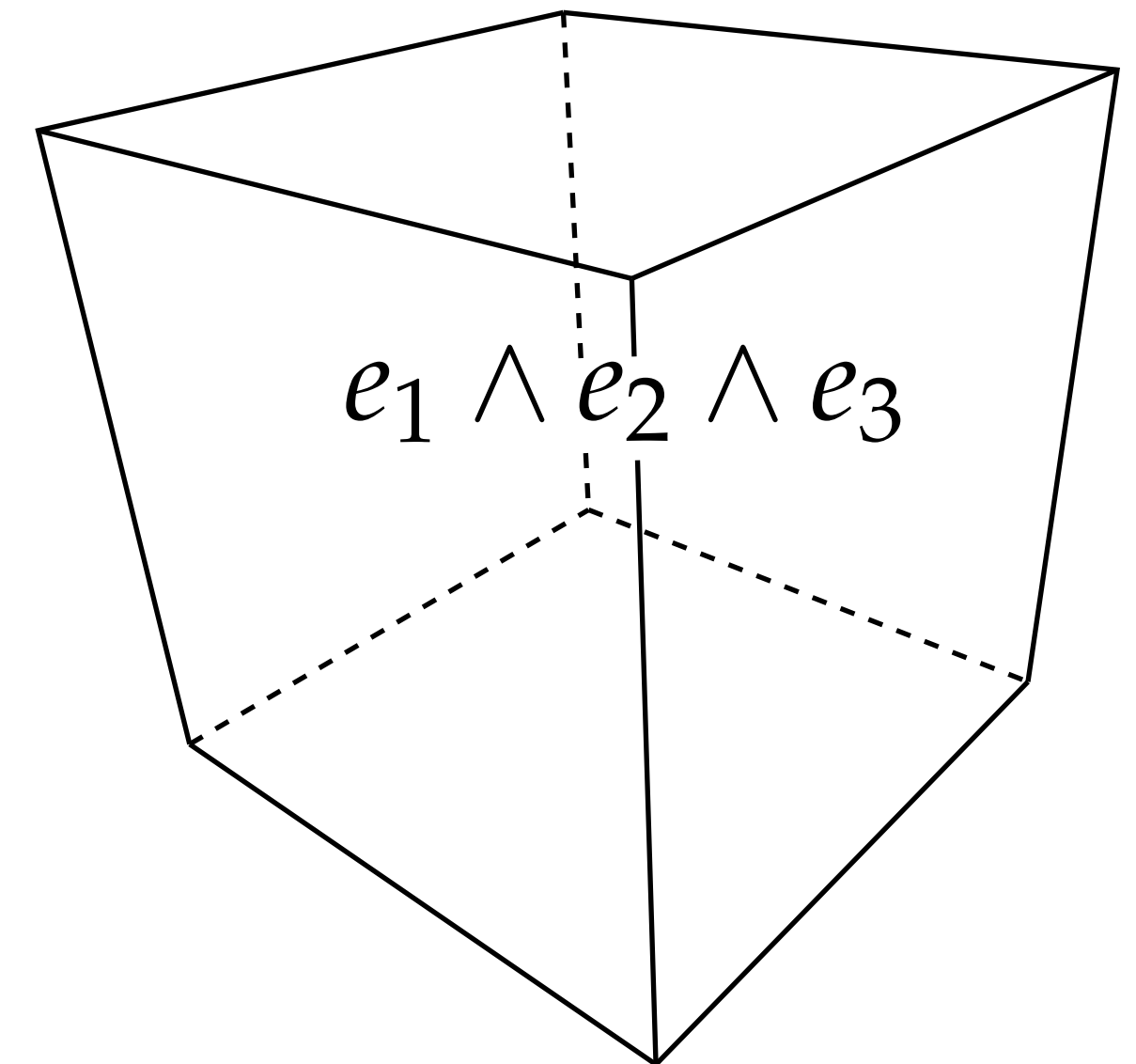
basis 1-vectors



basis 2-vectors



basis 3-vectors



Key idea: signed volumes can be expressed as linear combinations of “basis volumes” or basis k -vectors.

Basis k -Vectors — How Many?

Consider $V = \mathbb{R}^4$ with basis $\{e_1, e_2, e_3, e_4\}$.

Q: How many basis 2-vectors?

$$\begin{array}{l} e_1 \wedge e_2 \\ e_1 \wedge e_3 \quad e_2 \wedge e_3 \\ e_1 \wedge e_4 \quad e_2 \wedge e_4 \quad e_3 \wedge e_4 \end{array}$$

Q: How many basis 3-vectors?

$$\begin{array}{l} e_1 \wedge e_2 \wedge e_3 \\ e_1 \wedge e_2 \wedge e_4 \\ e_1 \wedge e_3 \wedge e_4 \\ e_2 \wedge e_3 \wedge e_4 \end{array}$$

$$\dim_{n,k} = \binom{n}{k}$$

Why not $e_3 \wedge e_2$? $e_4 \wedge e_4$?

What do these bases represent geometrically?

Q: How many basis 4-vectors?

$$e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

Q: How many basis 1-vectors?

Q: How many basis 0-vectors?

Q: Notice a pattern?

\mathbb{R}^3	\mathbb{R}^4
1	1
3	4
3	6
1	4
	1

Hodge Star — Basis k -Vectors

Consider $V = \mathbb{R}^3$ with orthonormal basis $\{e_1, e_2, e_3\}$

Q: How does the Hodge star map basis k -vectors to basis $(n - k)$ -vectors ($n=3$)?

A: For any basis k -vector $\alpha := e_{i_1} \wedge \cdots \wedge e_{i_k}$, we must have $\det(\alpha \wedge \star\alpha) = 1$.

In other words, if we start with a “unit volume,” wedging with its Hodge star must also give a unit, positively-oriented unit volume. For example:

Given $\alpha := e_2$, find $\star\alpha$ such that $\det(e_2 \wedge \star e_2) = 1$.

\Rightarrow Must have $\star\alpha = e_3 \wedge e_1$, since then

$$e_2 \wedge \star e_2 = e_2 \wedge e_3 \wedge e_1,$$

which is an even permutation of $e_1 \wedge e_2 \wedge e_3$.

$$\begin{aligned}\star 1 &= e_1 \wedge e_2 \wedge e_3 \\ \star e_1 &= e_2 \wedge e_3 \\ \star e_2 &= e_3 \wedge e_1 \\ \star e_3 &= e_1 \wedge e_2 \\ \star(e_2 \wedge e_3) &= e_1 \\ \star(e_3 \wedge e_1) &= e_2 \\ \star(e_1 \wedge e_2) &= e_3 \\ \star(e_1 \wedge e_2 \wedge e_3) &= 1\end{aligned}$$

Exterior Algebra—Formal Definition

Definition. Let e_1, \dots, e_n be the basis for an n -dimensional inner product space V . For each integer $0 \leq k \leq n$, let Λ^k denote an $\binom{n}{k}$ -dimensional vector space with basis elements denoted by $e_{i_1} \wedge \dots \wedge e_{i_k}$ for all possible sequences of indices $1 \leq i_1 < \dots < i_k \leq n$, corresponding to all possible “axis-aligned” k -dimensional volumes. Elements of Λ^k are called k -vectors. The *wedge product* is a bilinear map

$$\wedge_{k,l} : \Lambda^k \times \Lambda^l \rightarrow \Lambda^{k+l}$$

uniquely determined by its action on basis elements; in particular, for any collection of *distinct* indices i_1, \dots, i_{k+l} ,

$$(e_{i_1} \wedge \dots \wedge e_{i_k}) \wedge_{k,l} (e_{i_{k+1}} \wedge \dots \wedge e_{i_{k+l}}) := \text{sgn}(\sigma) e_{\sigma(i_1)} \wedge \dots \wedge e_{\sigma(i_{k+l})},$$

where σ is a permutation that puts the indices of the two arguments in canonical (lexicographic) order. Arguments with repeated indices are mapped to $0 \in \Lambda^{k+l}$. For brevity, one typically drops the subscript on $\wedge_{k,l}$. Finally, the *Hodge star on k -vectors* is a linear isomorphism

$$\star : \Lambda^k \rightarrow \Lambda^{n-k}$$

uniquely determined by the relationship

$$\det(\alpha \wedge \star \alpha) = 1$$

where α is any k -vector of the form $\alpha = e_{i_1} \wedge \dots \wedge e_{i_k}$ and \det denotes the determinant of the constituent 1-vectors (treated as column vectors) with respect to the inner product on V . The collection of vector spaces Λ^k together with the maps \wedge and \star define an *exterior algebra* on V , sometimes known as a *graded algebra*.

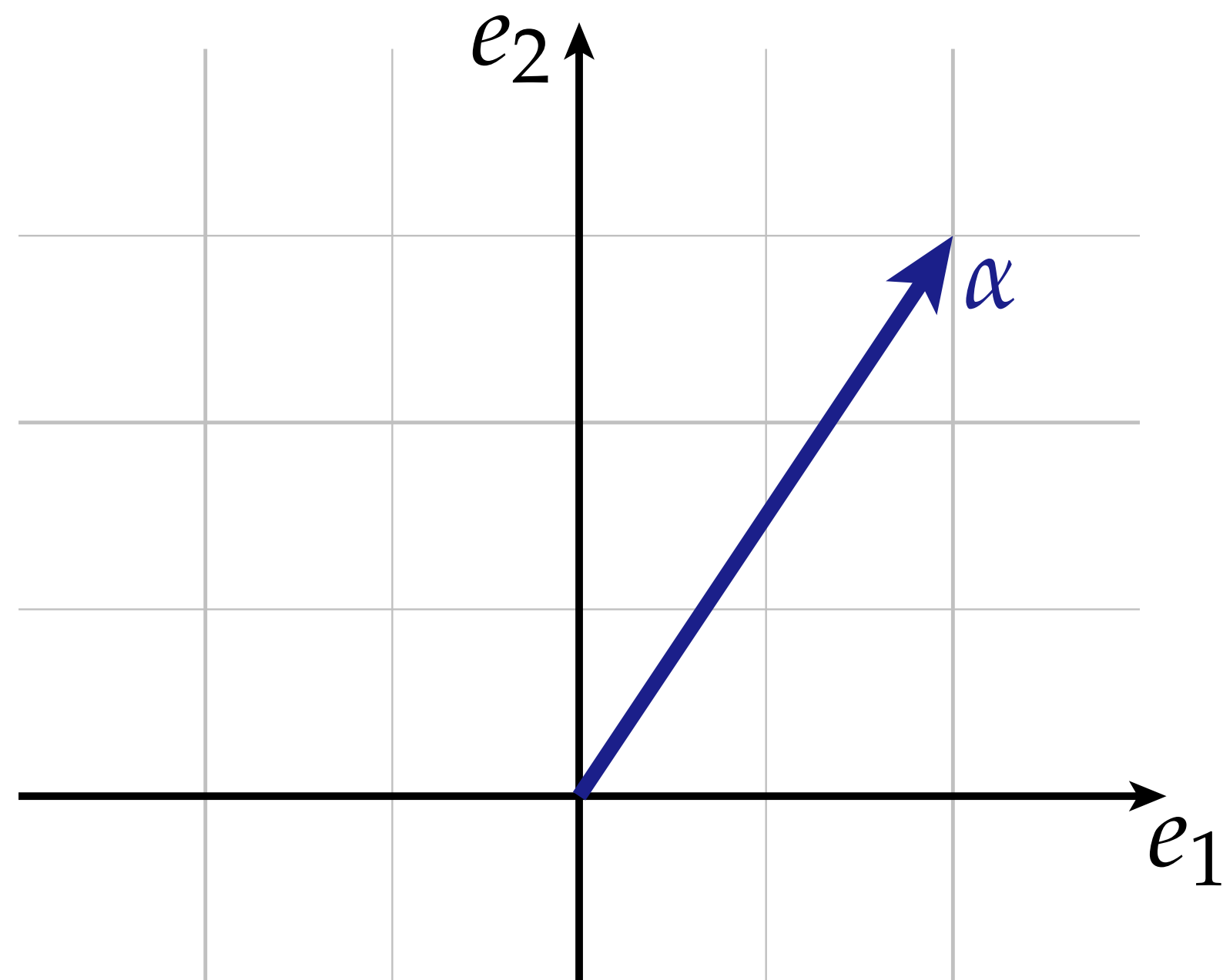
Don't worry about this unless you really want to! Concepts & mechanics more important.

Sanity Check

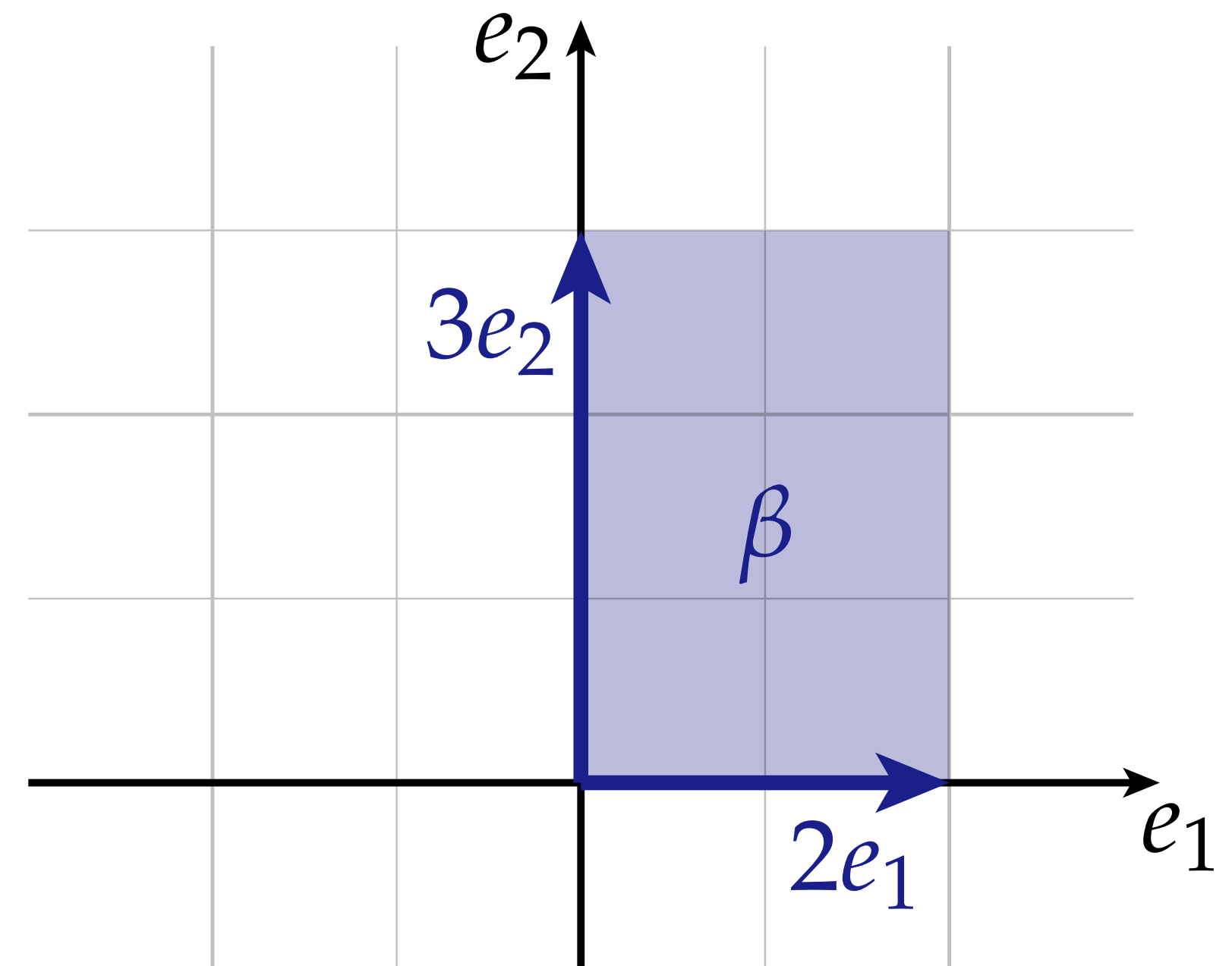
Q: What's the difference between

$$\alpha = 2e_1 + 3e_2 \quad \text{and} \quad \beta = 2e_1 \wedge 3e_2?$$

A:



(vector)



(2-vector)

Exterior Algebra—Example

$$V = \mathbb{R}^2$$

$$\alpha = 2e_1 + e_2$$

$$\beta = -e_1 + 2e_2$$

Q: What is the value of $\alpha \wedge \beta$?

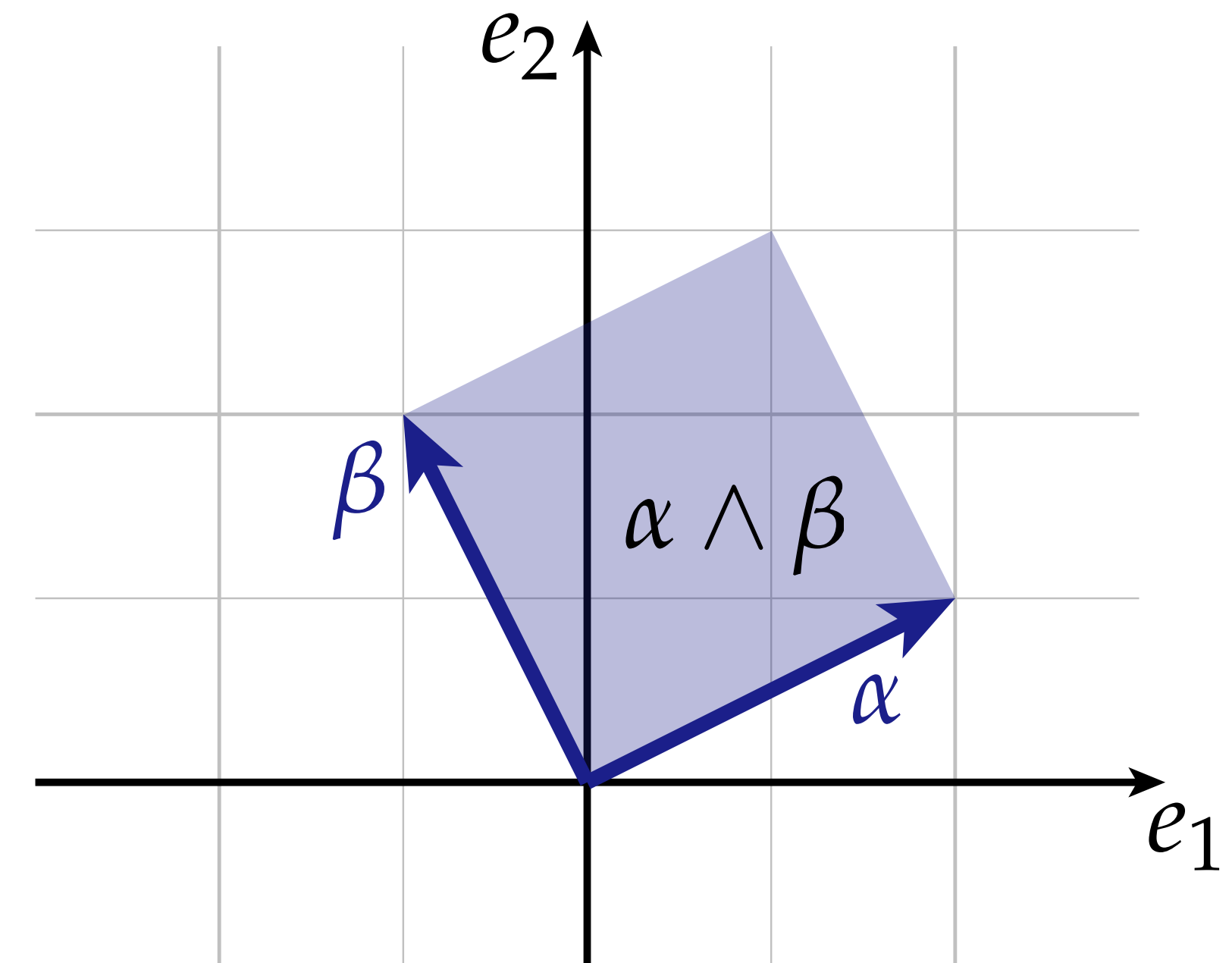
A: $\alpha \wedge \beta = (2e_1 + e_2) \wedge (-e_1 + 2e_2)$

$$= (2e_1 + e_2) \wedge (-e_1) + (2e_1 + e_2) \wedge (2e_2)$$

$$= \cancel{-2e_1 \wedge e_1}^0 - e_2 \wedge e_1 + 4e_1 \wedge e_2 + \cancel{2e_2 \wedge e_2}^0$$

$$= e_1 \wedge e_2 + 4e_1 \wedge e_2$$

$$= \boxed{5e_1 \wedge e_2}$$



Q: What does the result *mean*, geometrically?

Exterior Algebra—Example

$$V = \mathbb{R}^3$$

Q: What is $\star(\alpha \wedge \beta + \beta \wedge \gamma)$?

$$\alpha = 2e_1 \wedge e_2$$

$$\beta = 3e_3$$

$$\gamma = e_2 \wedge e_1$$

$$\begin{aligned} \mathbf{A:} \star(\alpha \wedge \beta + \beta \wedge \gamma) &= \star((2e_1 \wedge e_2) \wedge 3e_3 + 3e_3 \wedge (e_2 \wedge e_1)) \\ &= \star(6e_1 \wedge e_2 \wedge e_3 + 3e_3 \wedge e_2 \wedge e_1) \\ &= \star(6e_1 \wedge e_2 \wedge e_3 - 3e_1 \wedge e_2 \wedge e_3) \\ &= \star(3e_1 \wedge e_2 \wedge e_3) \\ &= 3. \end{aligned}$$

Key idea: in this example, it would have been fairly hard to reason about the answer geometrically. Sometimes the algebraic approach is (*incredibly!*) useful.

Exterior Algebra - Summary

- **Exterior algebra**

- language for manipulating *signed volumes*

- length matters (magnitude)

- order matters (orientation)

- behaves like a vector space (e.g., can add two volumes, scale a volume, ...)

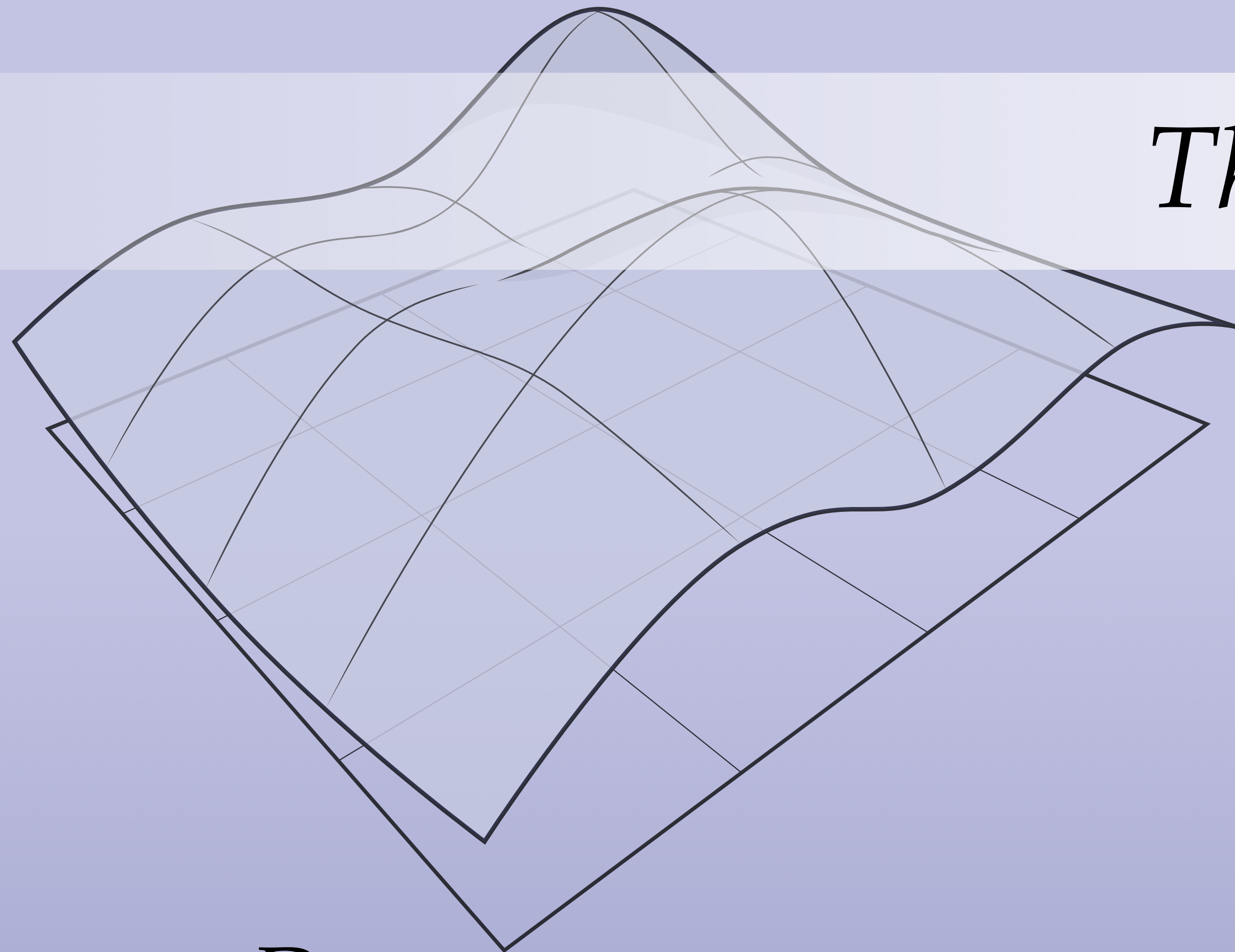
- **Wedge product**—analogous to *span* of vectors

- **Hodge star**—analogous to *orthogonal complement* (in 2D: 90-degree rotation)

- Coordinate representation—encode vectors in a *basis*

- Basis k -vectors are all possible wedges of basis 1-vectors

Thanks!



DISCRETE DIFFERENTIAL

GEOMETRY:

AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858