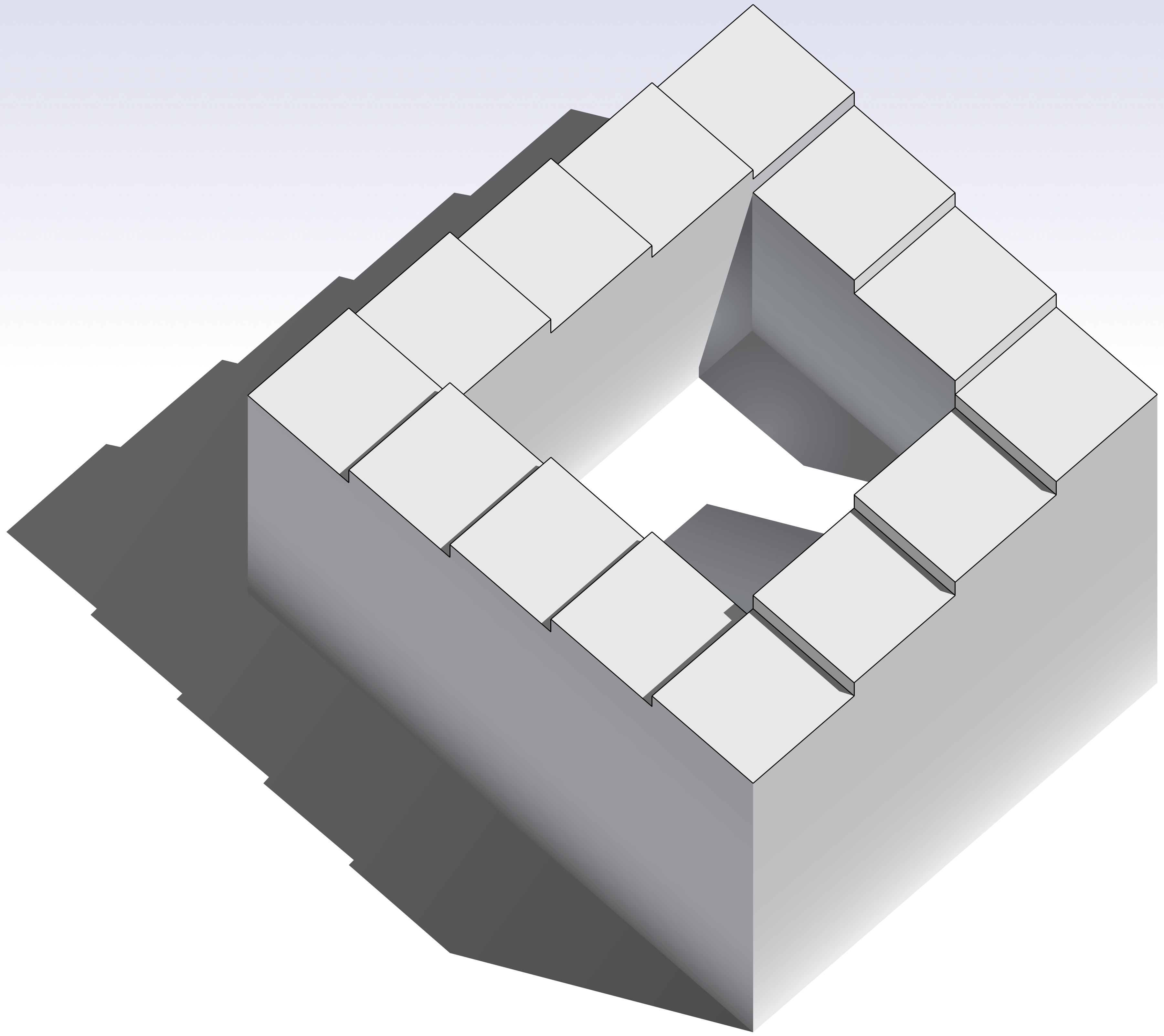
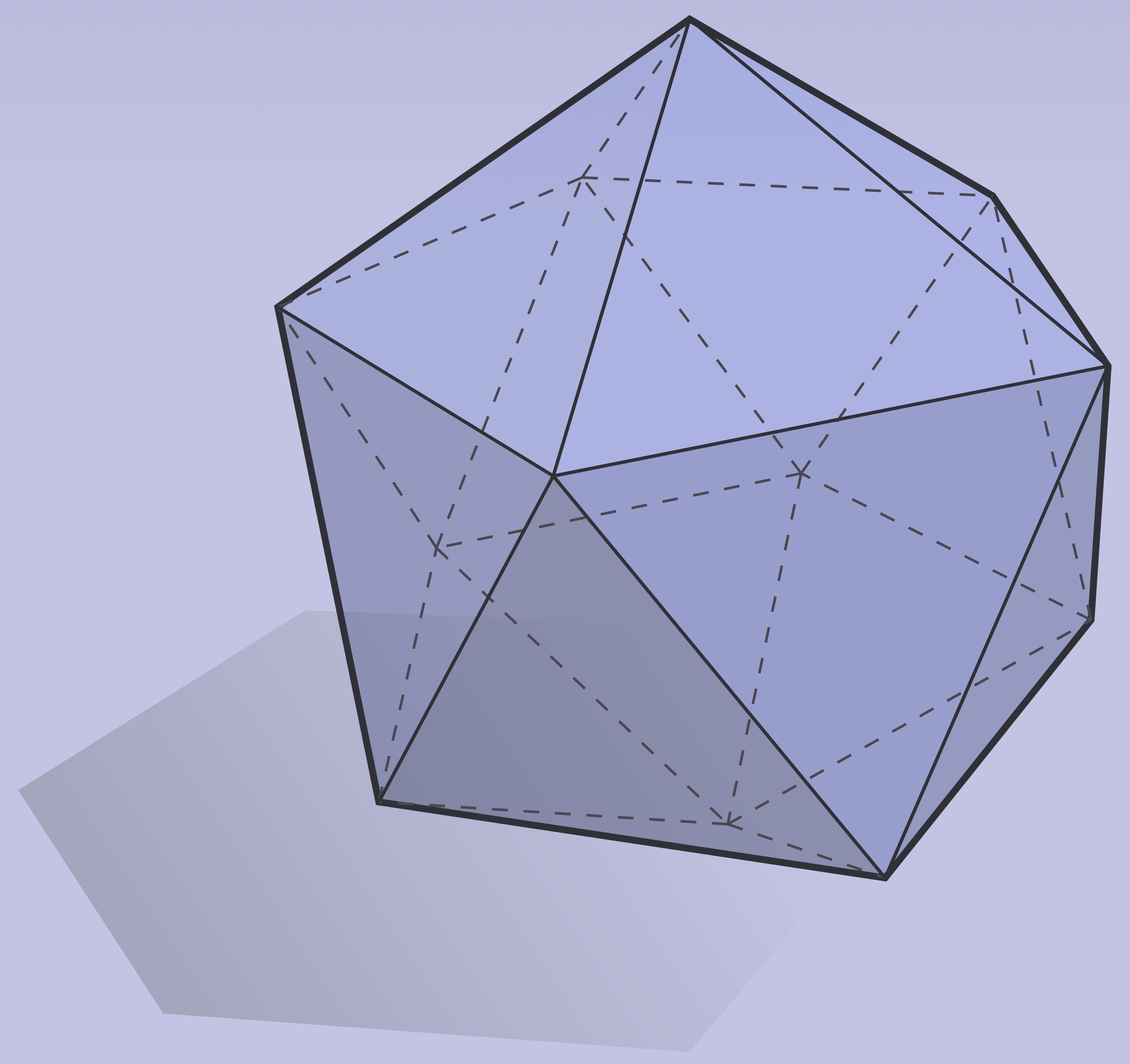


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 5:

DIFFERENTIAL FORMS IN $\mathbb{R}^n$

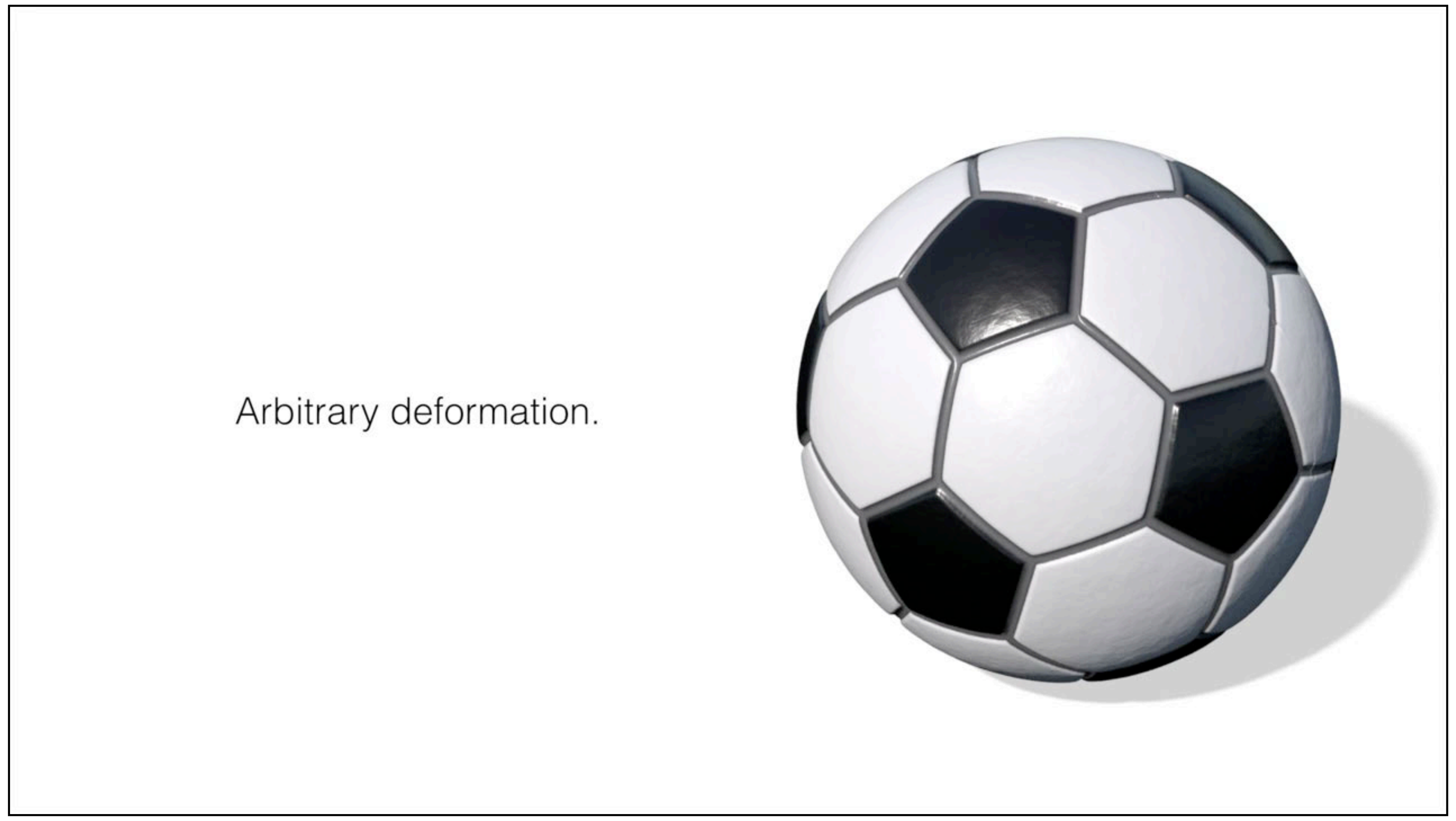


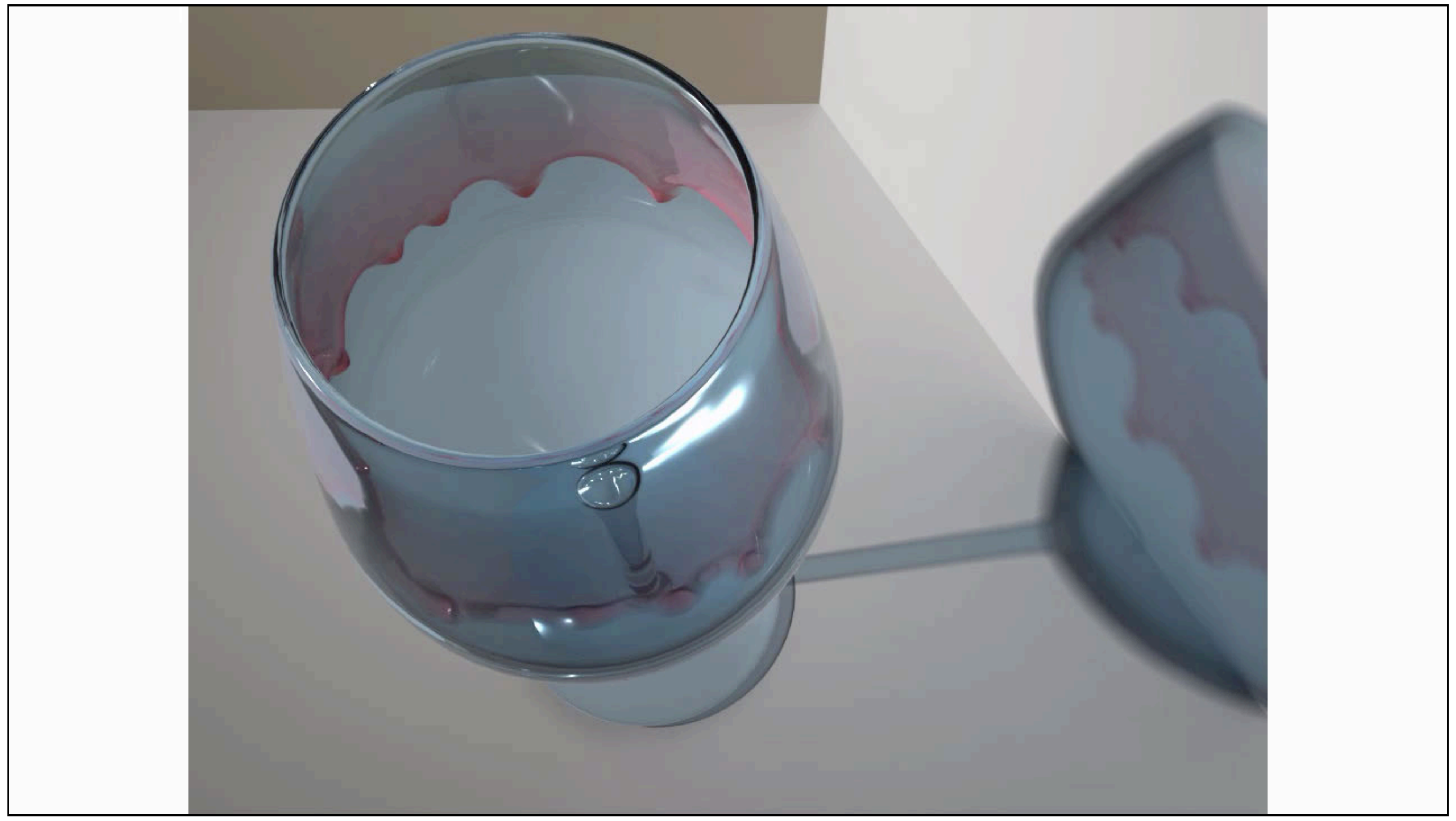
DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858

Motivation: Applications of Differential Forms

*Need to **measure** k-dimensional quantities that are changing in space & time!*





Where Are We Going Next?

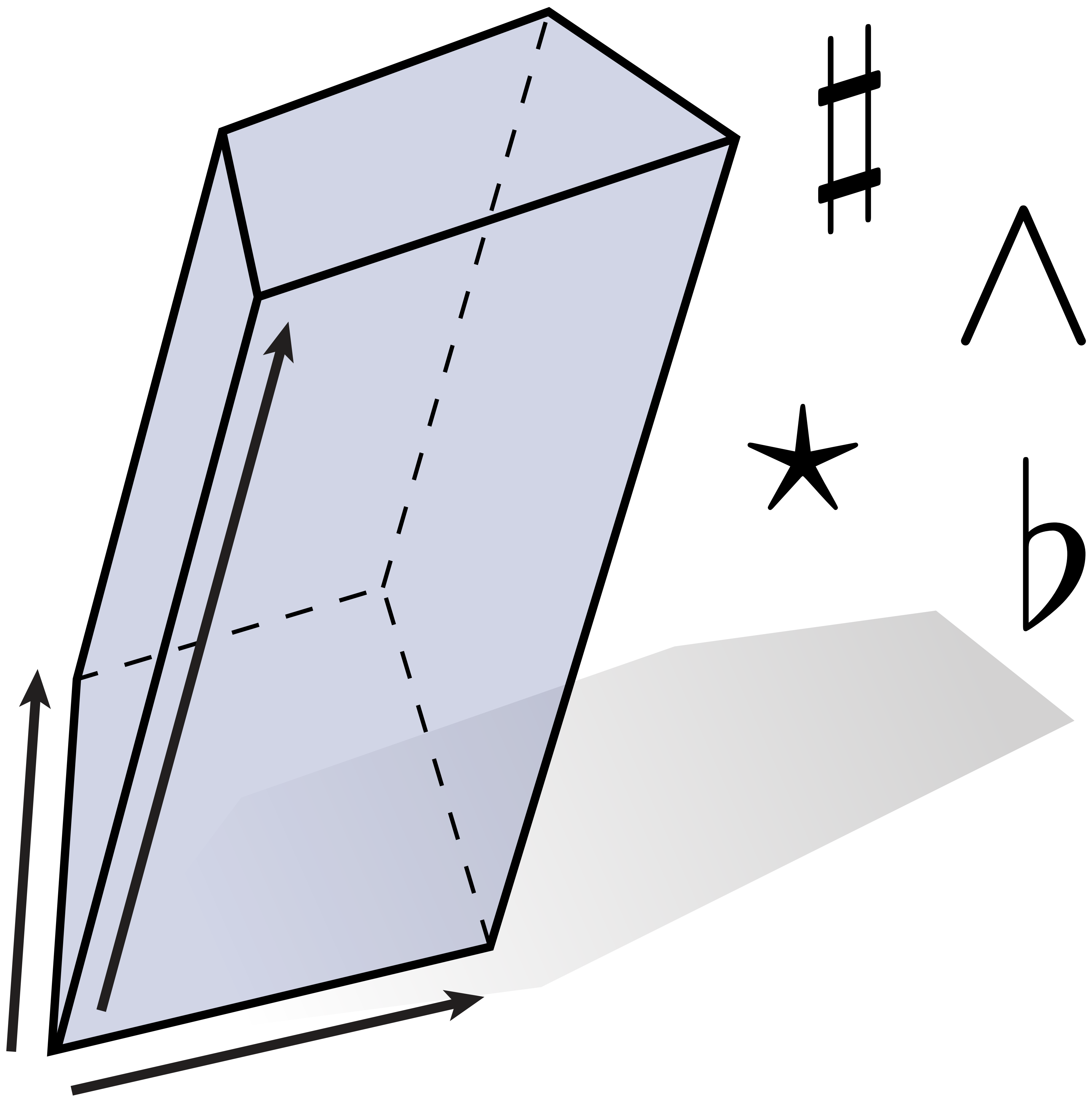
GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

- **Linear algebra:** "little arrows" (vectors)
- **Vector Calculus:** how do vectors *change?*

Next few lectures:

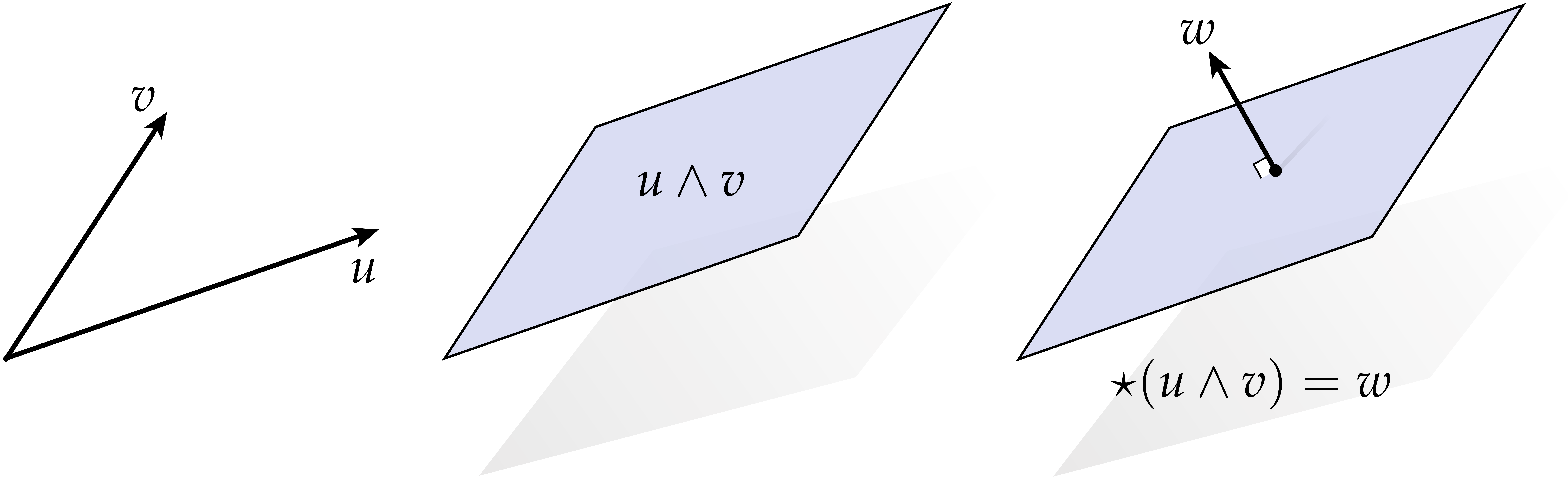
- **Exterior algebra**: "little volumes" (k-vectors)
- **Differential forms:** spatially-varying k-form
- **Exterior calculus**: how do k-vectors change?
- **DEC:** how do we do all of this on meshes?



Basic idea: replace vector calculus with computation on meshes.

Recap: Exterior Algebra

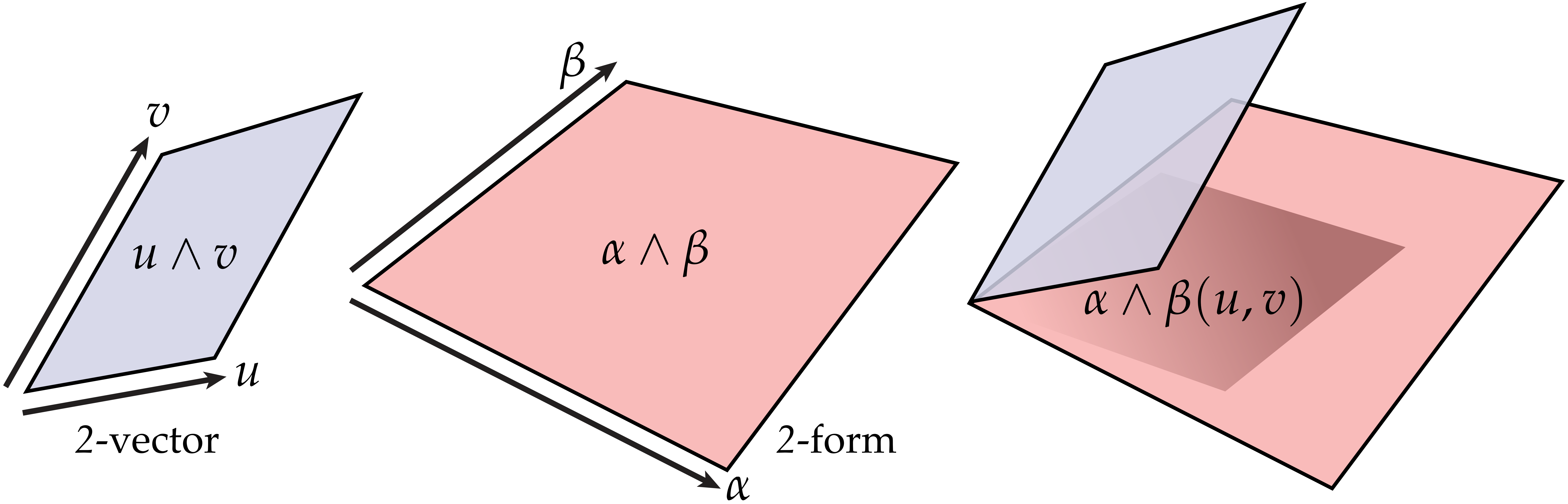
- Used *wedge product* to build up "little volumes" (*k-vectors*) from ordinary vectors



- Like linear subspaces, but have *magnitude* and *orientation*
- Use Hodge star to describe complementary volumes

Recap: k-Forms

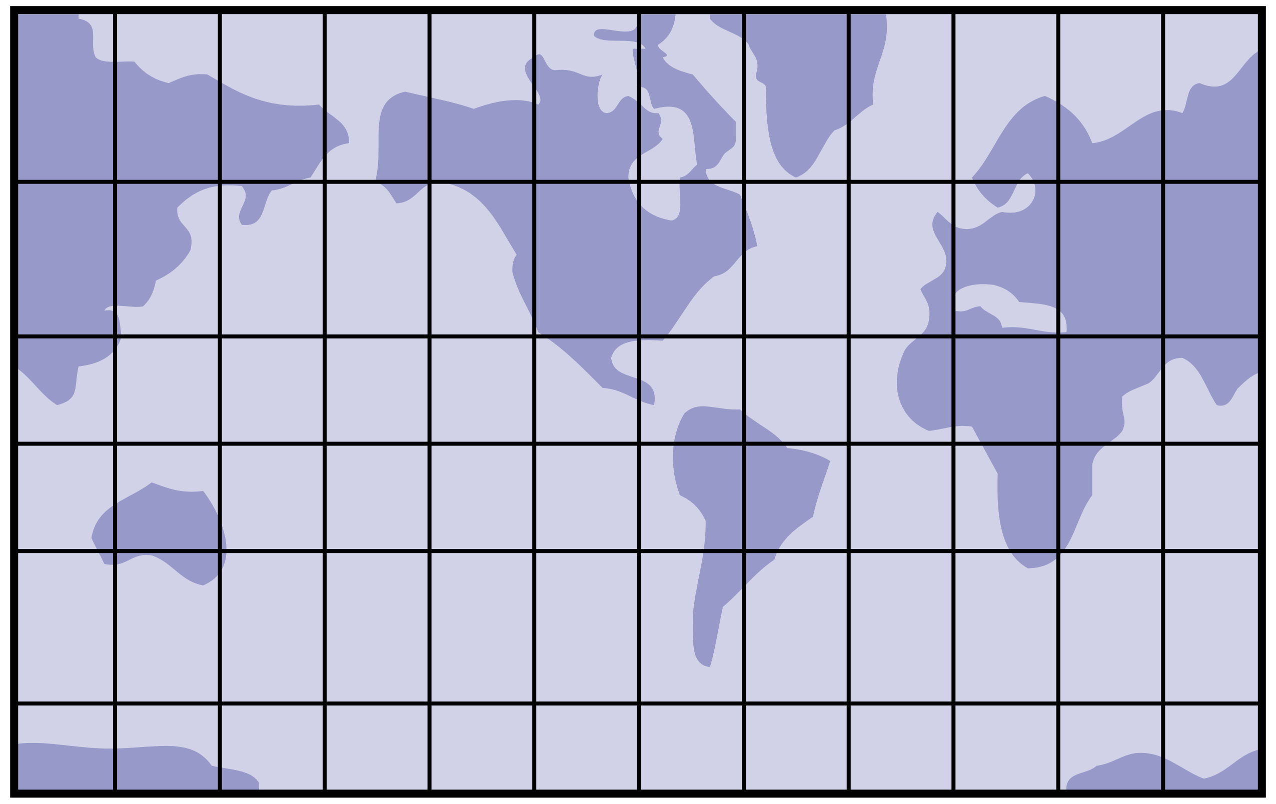
- Can measure a vector with a *covector;* can measure a k-vector with a *k-form*



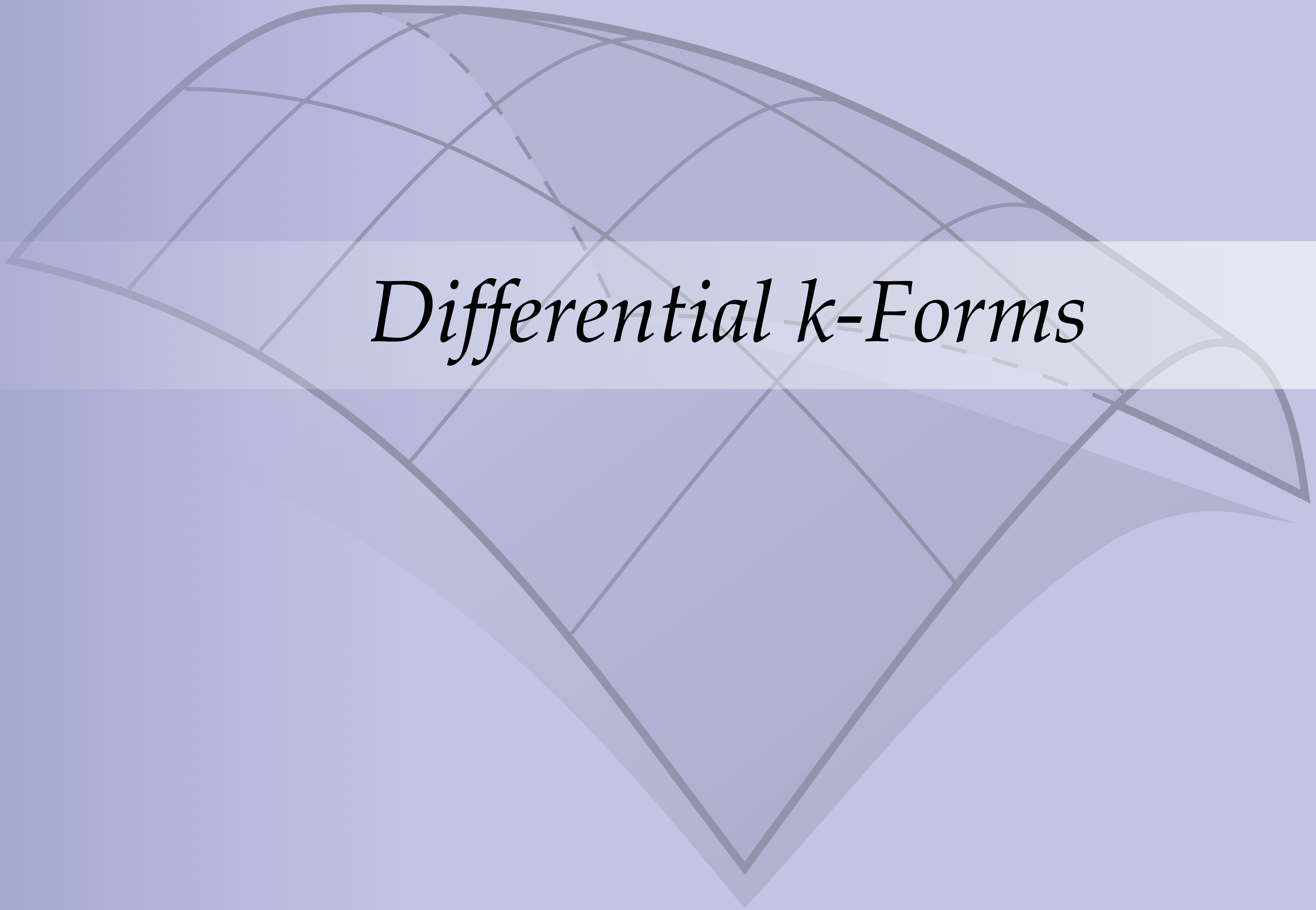
Key idea: project k-vector onto k-form and compute volume (*e.g.*, via determinant)

Exterior Calculus: Flat vs. Curved Spaces

- For now, we'll only consider *flat* spaces like the 2D plane
  - Keeps all our calculations simple
  - Don't have to deal with *Riemannian metric* (yet!)
- True power of exterior calculus revealed on *curved* spaces
  - In flat spaces, vectors and forms look nearly identical
  - Difference is no longer superficial on curved spaces
  - Close relationship to *curvature* (geometry)
  - Also close relationship to *mass* (physics)

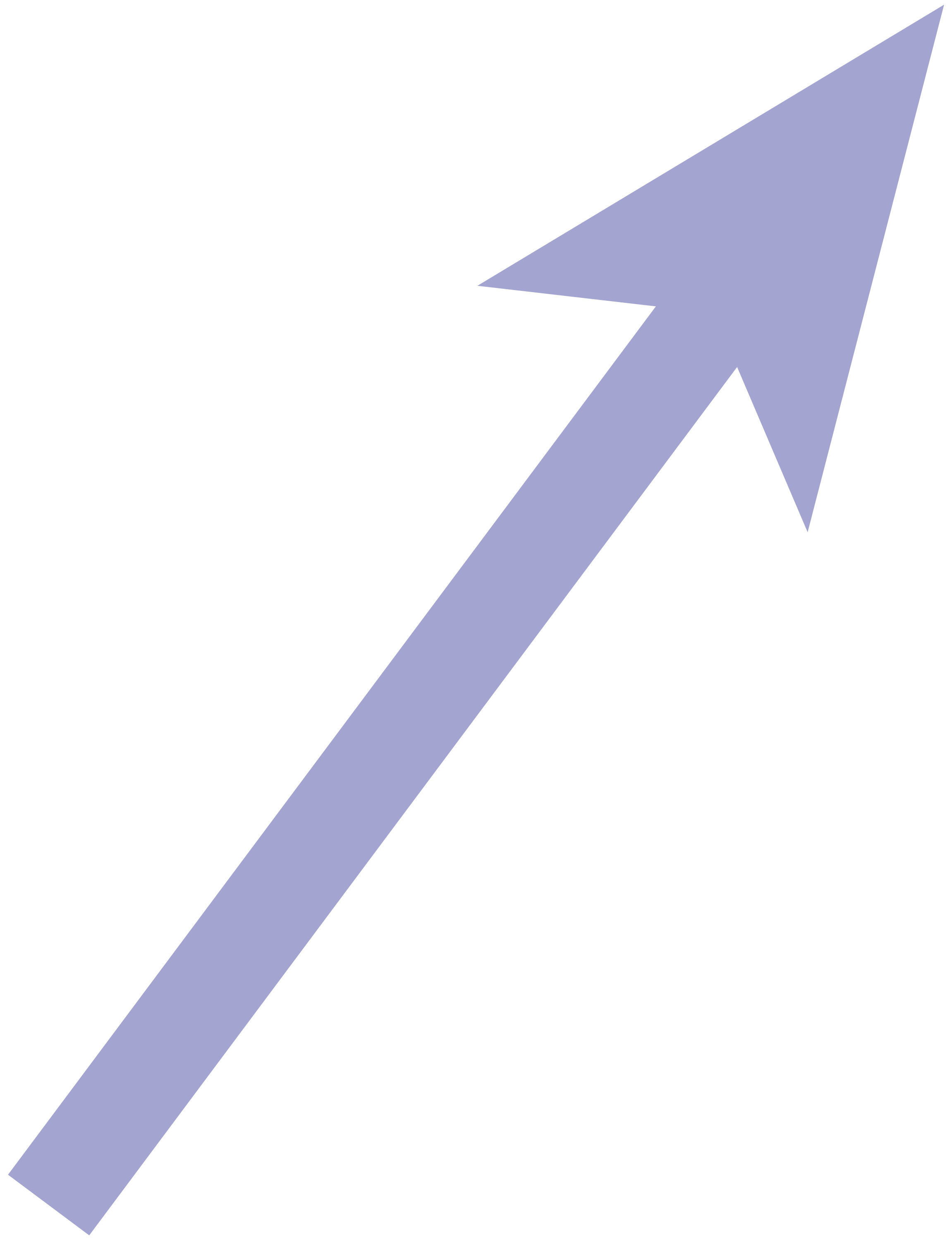


Differential k-Forms

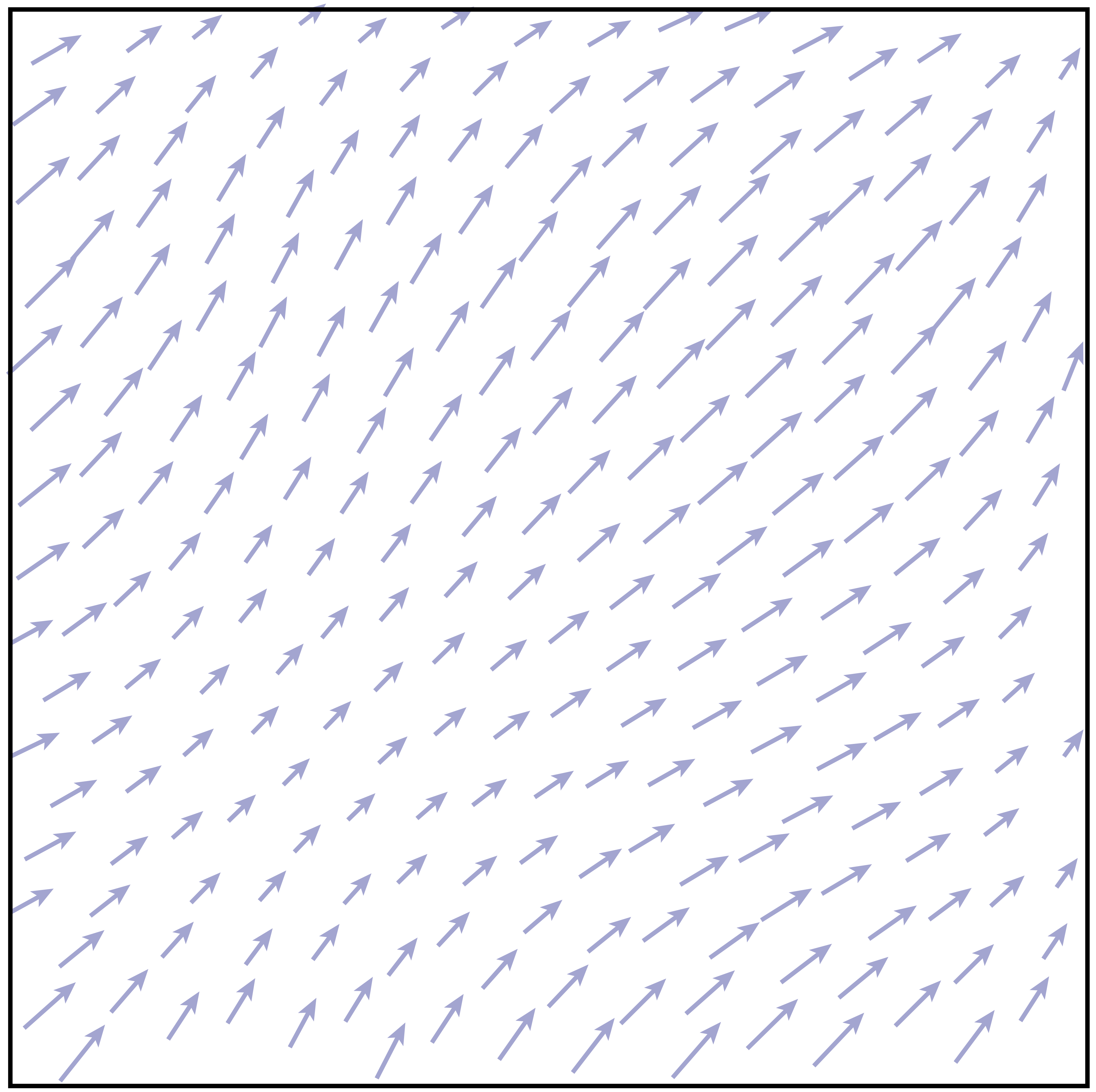


Review: Vector vs. Vector Field

- Recall that a *vector field* is an assignment of a vector to each point of space:



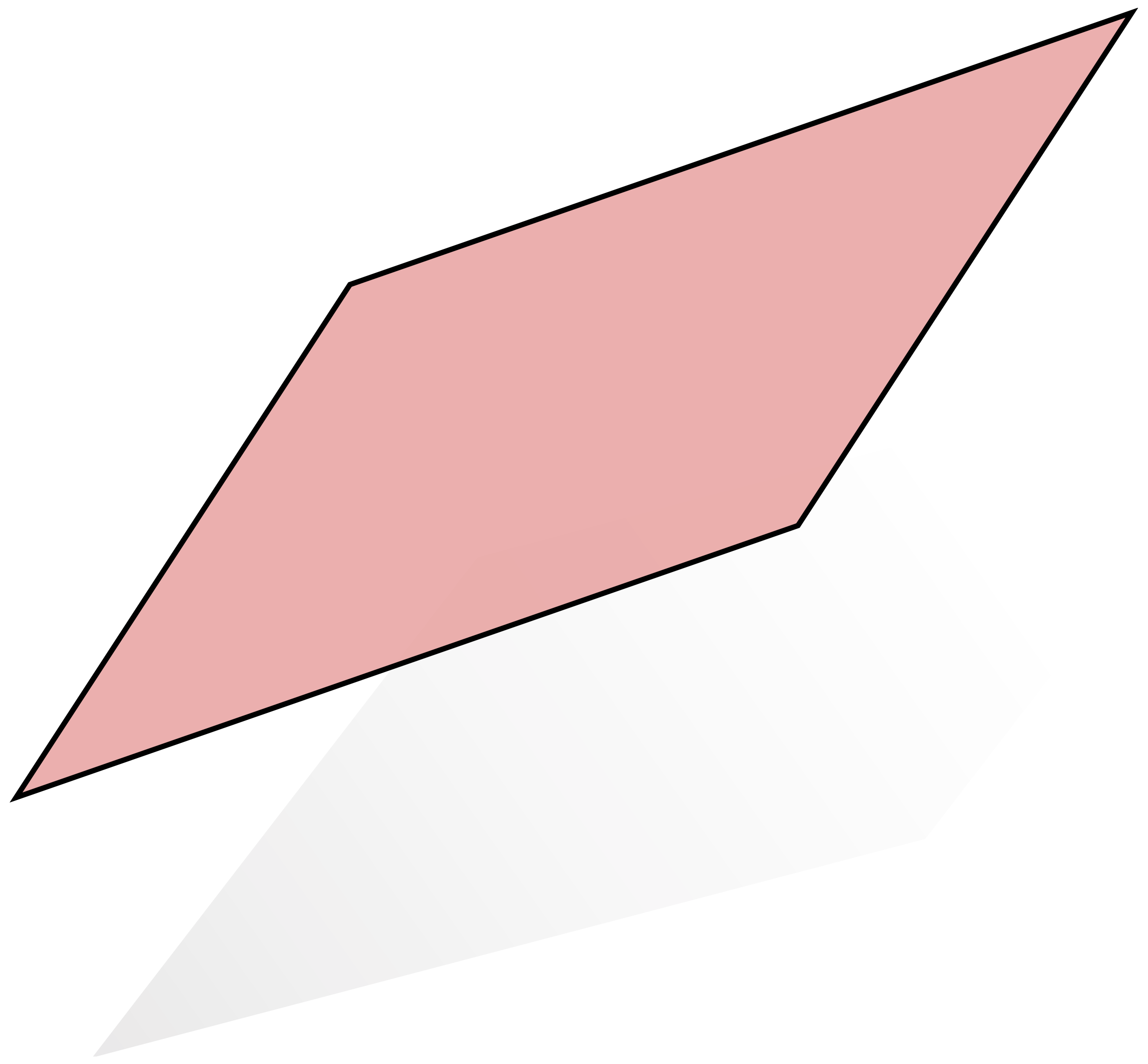
vector



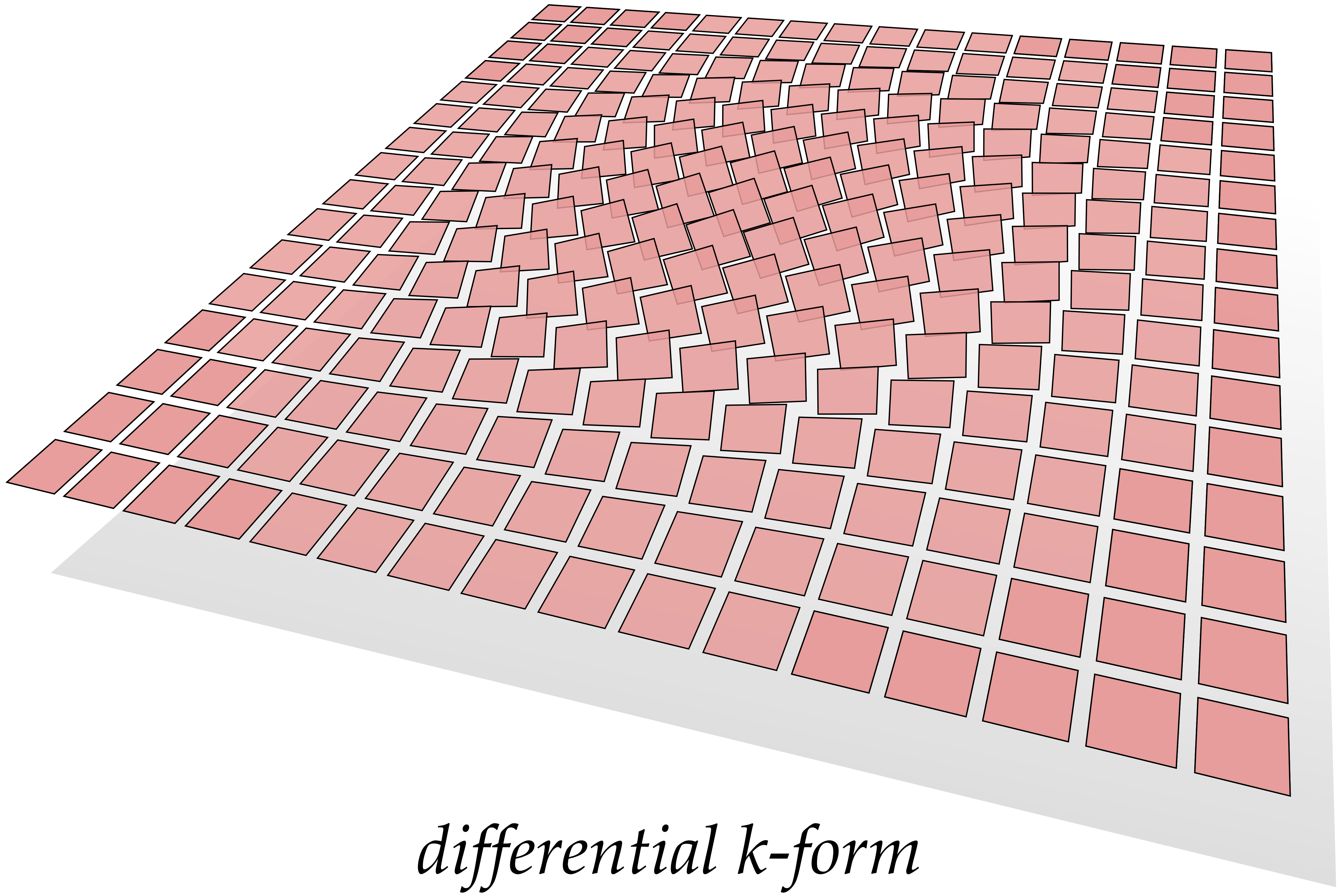
vector field

Differential Form

- A *differential k-form* is likewise an assignment of a *k*-form to each point*:



k-form

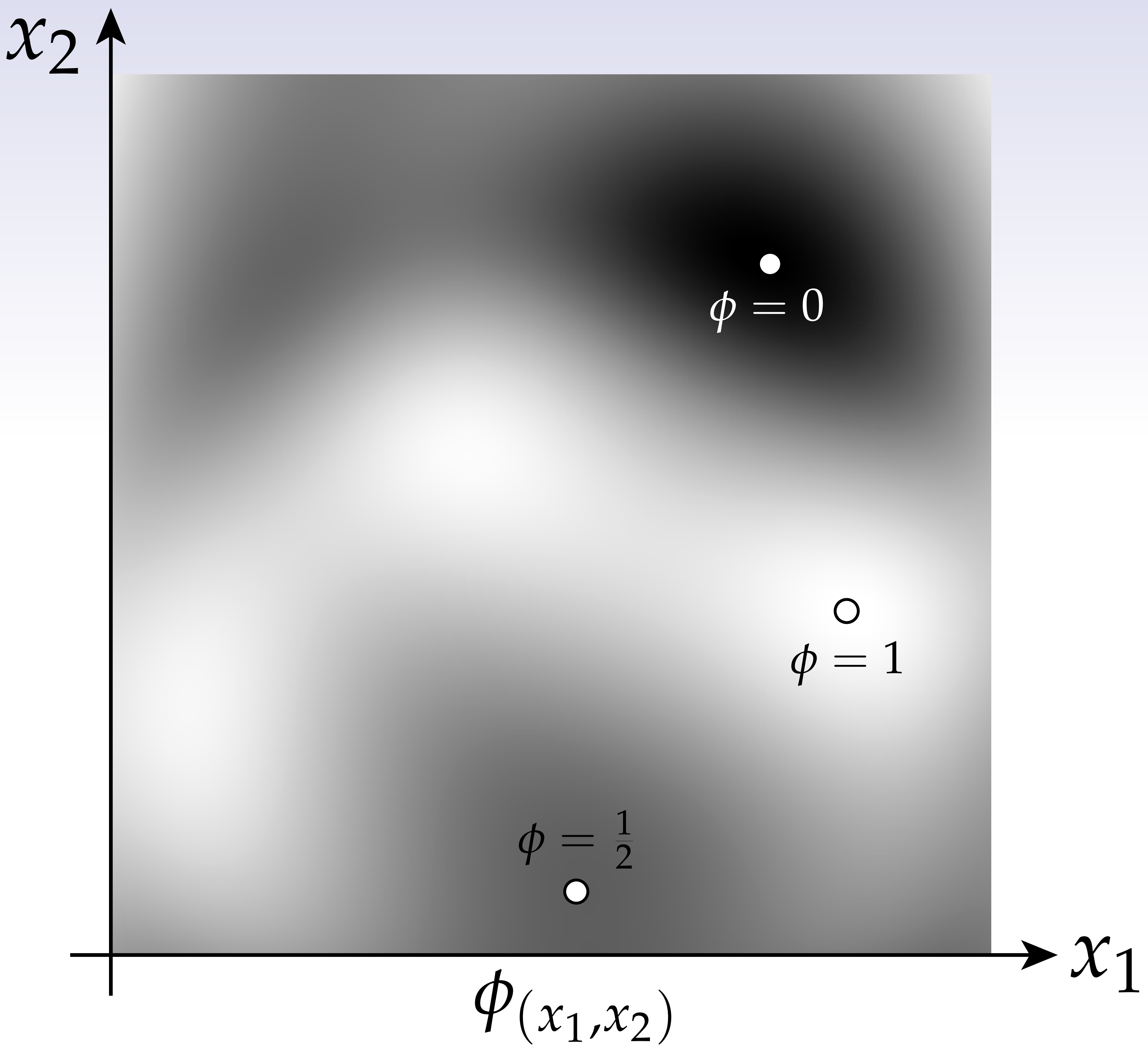


differential k-form

*Common (and confusing!) abbreviation: shorten "differential *k*-form" to just "*k*-form"!

Differential 0-Form

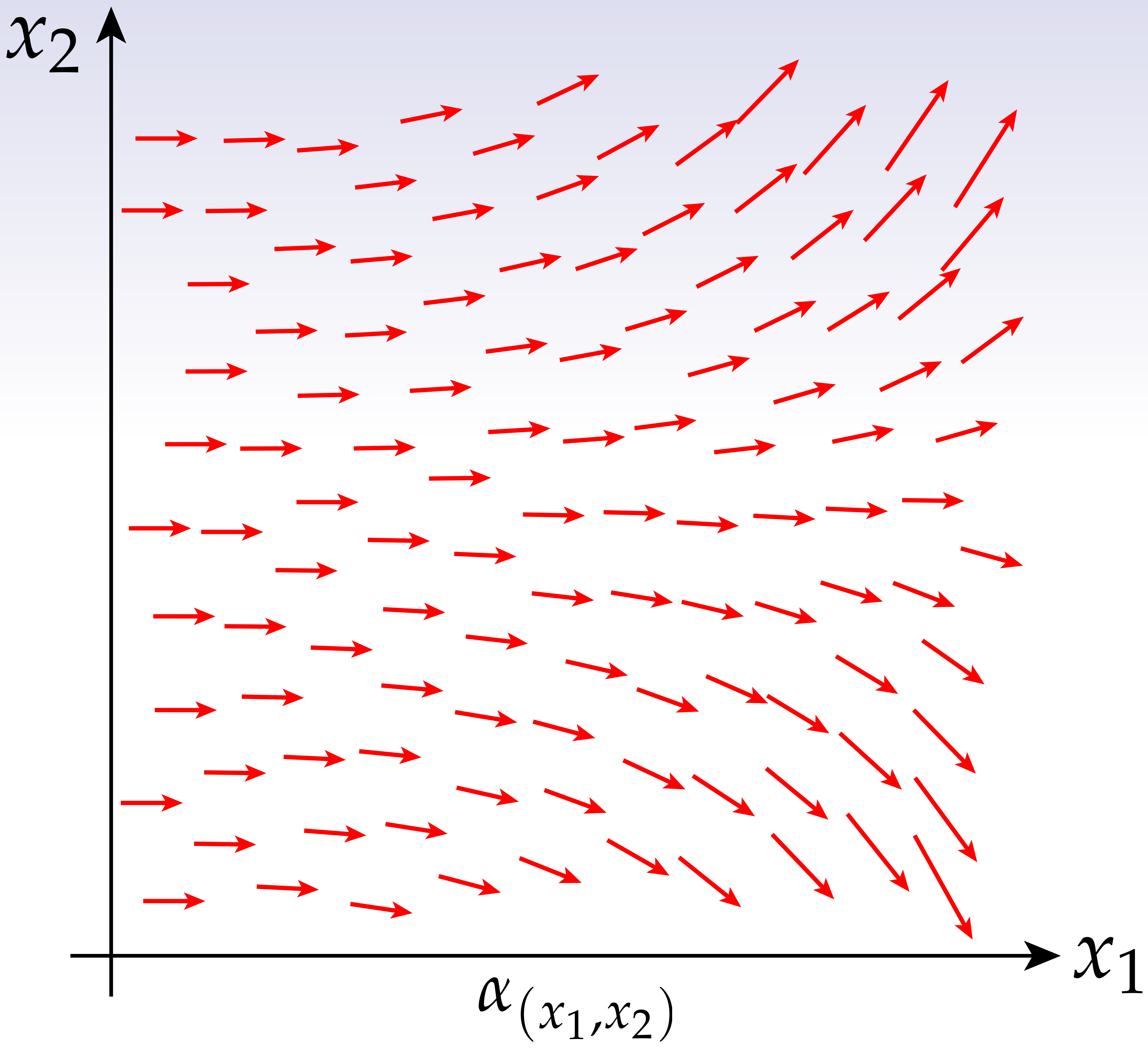
Assigns a scalar to each point. *E.g.*, in 2D we have a value at each point (x_1, x_2):



Note: exactly the same thing as a *scalar function!*

Differential 1-Form

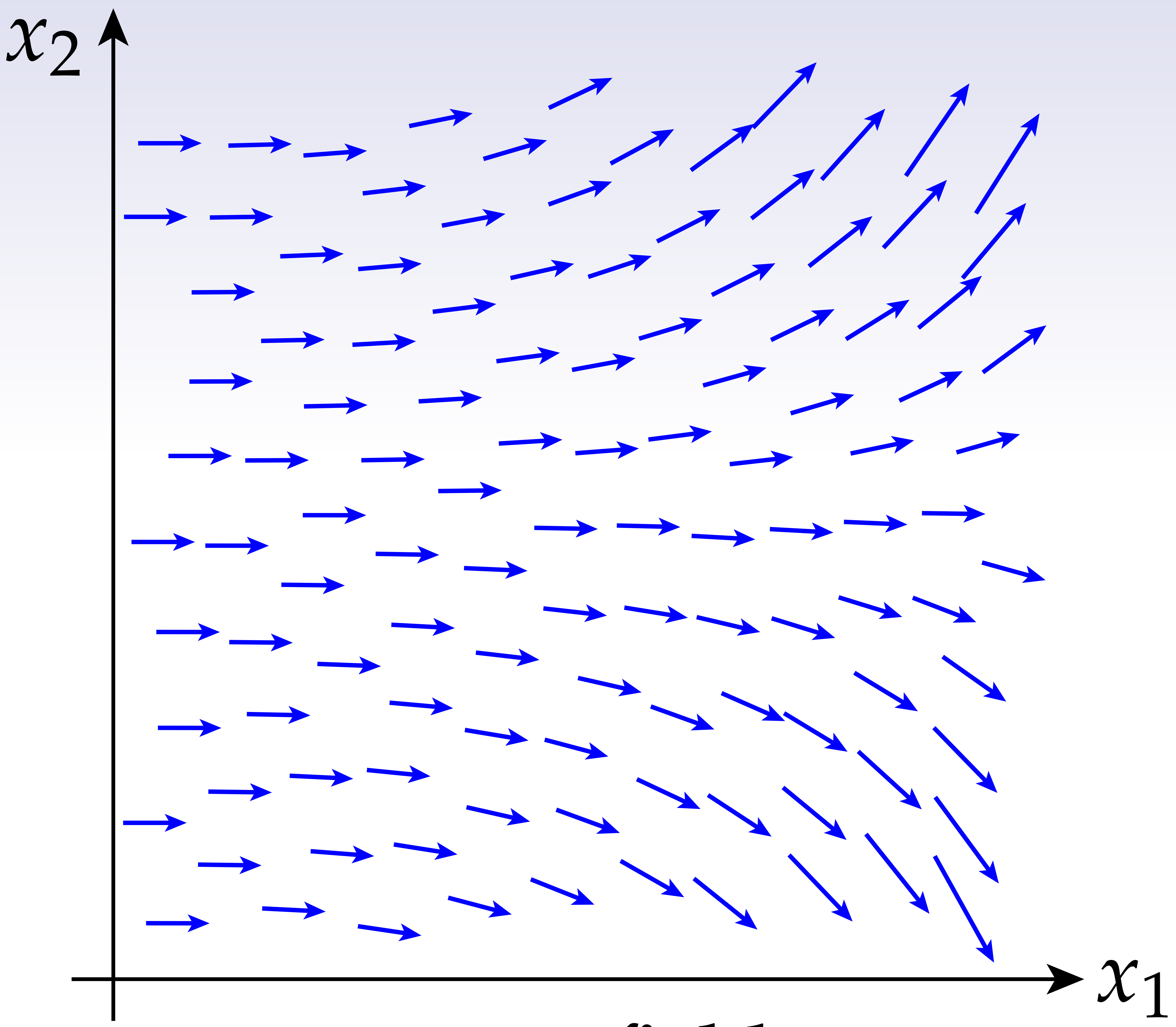
Assigns a 1-form each point. *E.g.*, in 2D we have a 1-form at each point (x_1, x_2):



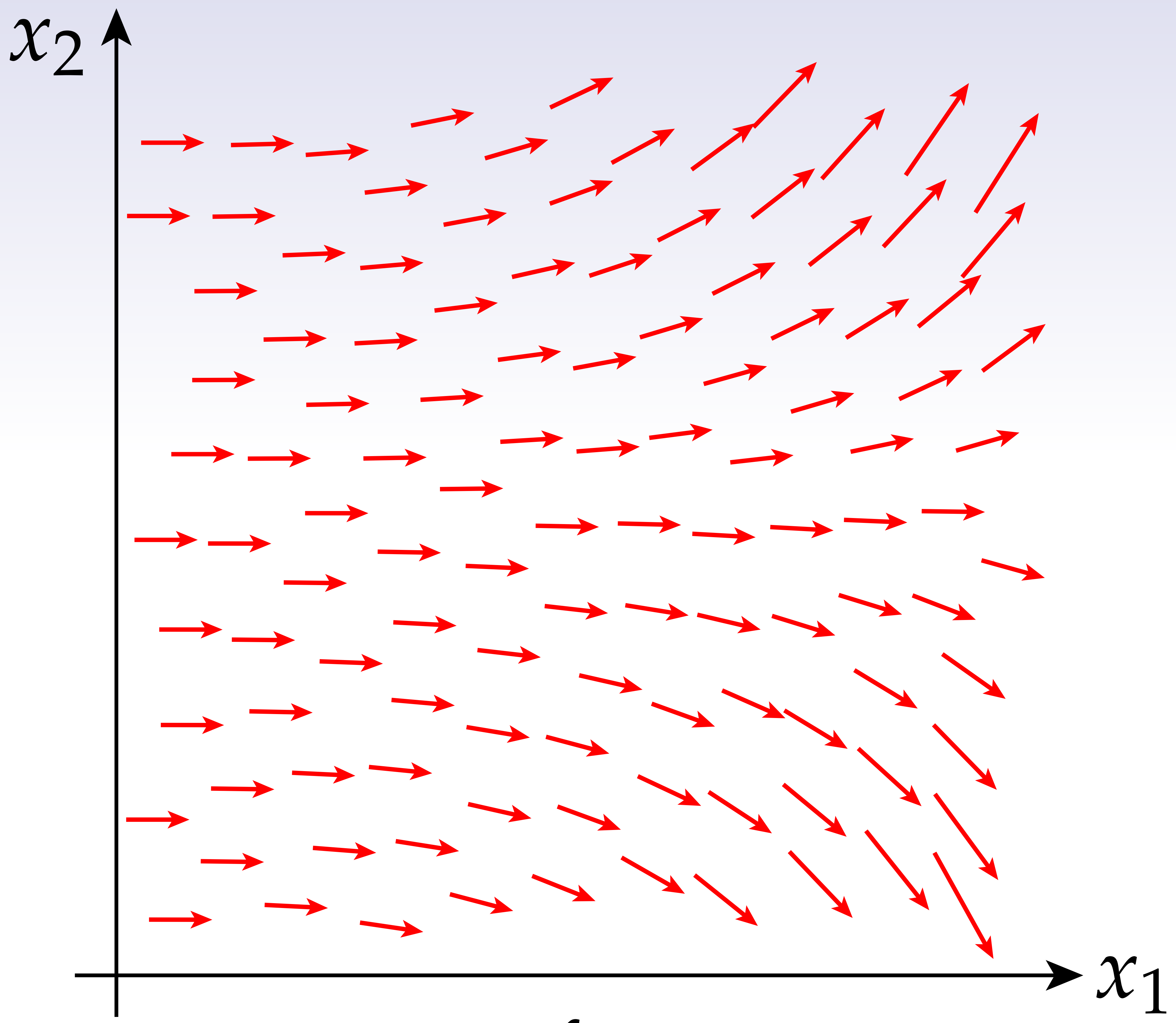
Note: <u>not</u> the same thing as a vector field!

Vector Field vs. Differential 1-Form

Superficially, vector fields and differential 1-forms look the same (in $\mathbb{R}^n$):



vector field



1-form

But recall that a 1-form is a *linear function* from a vector to a scalar (here, at each point.)

Applying a Differential 1-Form to a Vector Field

Hence, we can use a differential 1-form to *measure* a vector field:

vector field $X_{(x_1,x_2)}$

differential 1-form $\alpha_{(x_1,x_2)}$

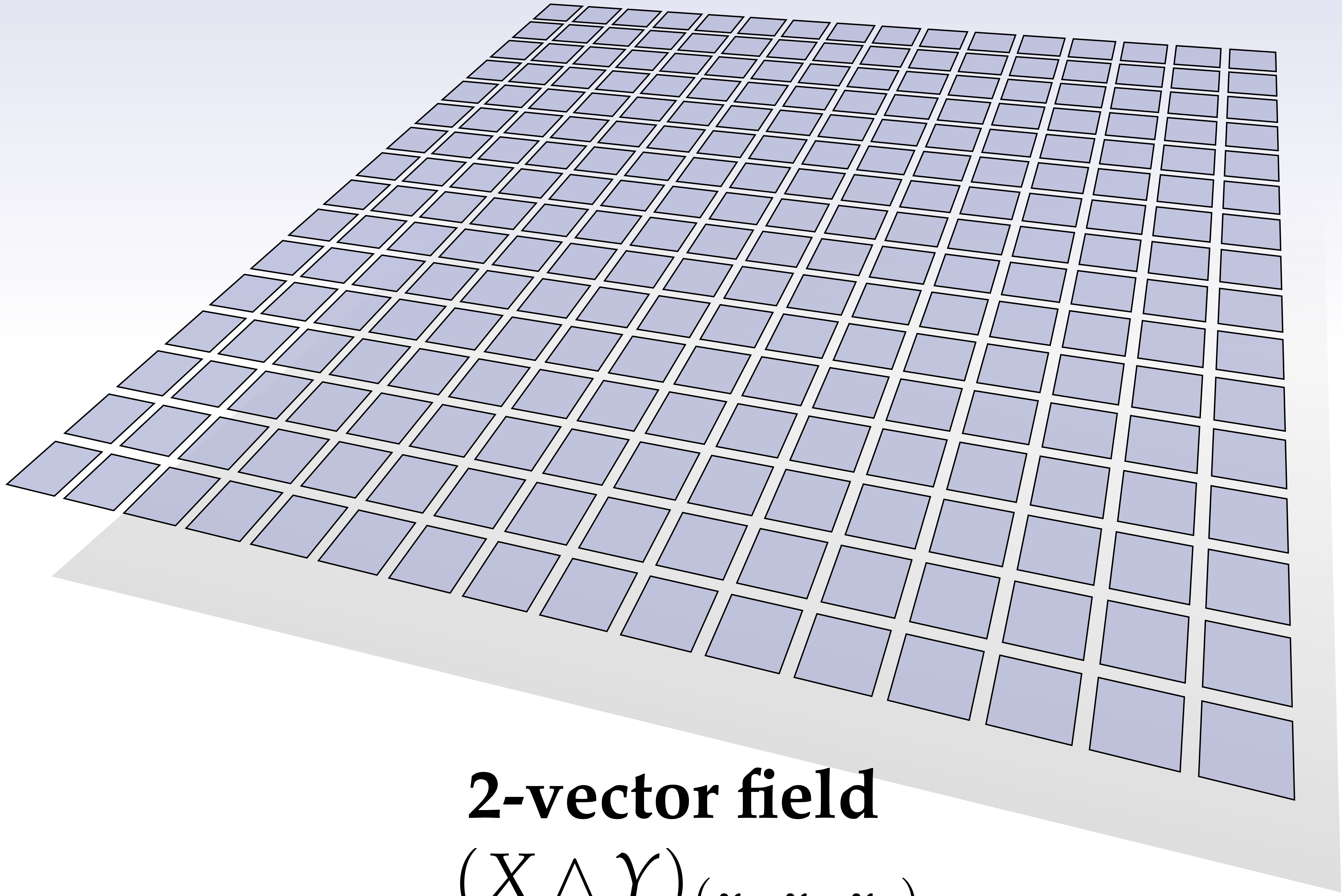
(superimposed)

scalar function $\alpha(X)_{(x_1,x_2)}$

Intuition: at each point (x_1,x_2) we see "how strong" the flow of X is along direction α.

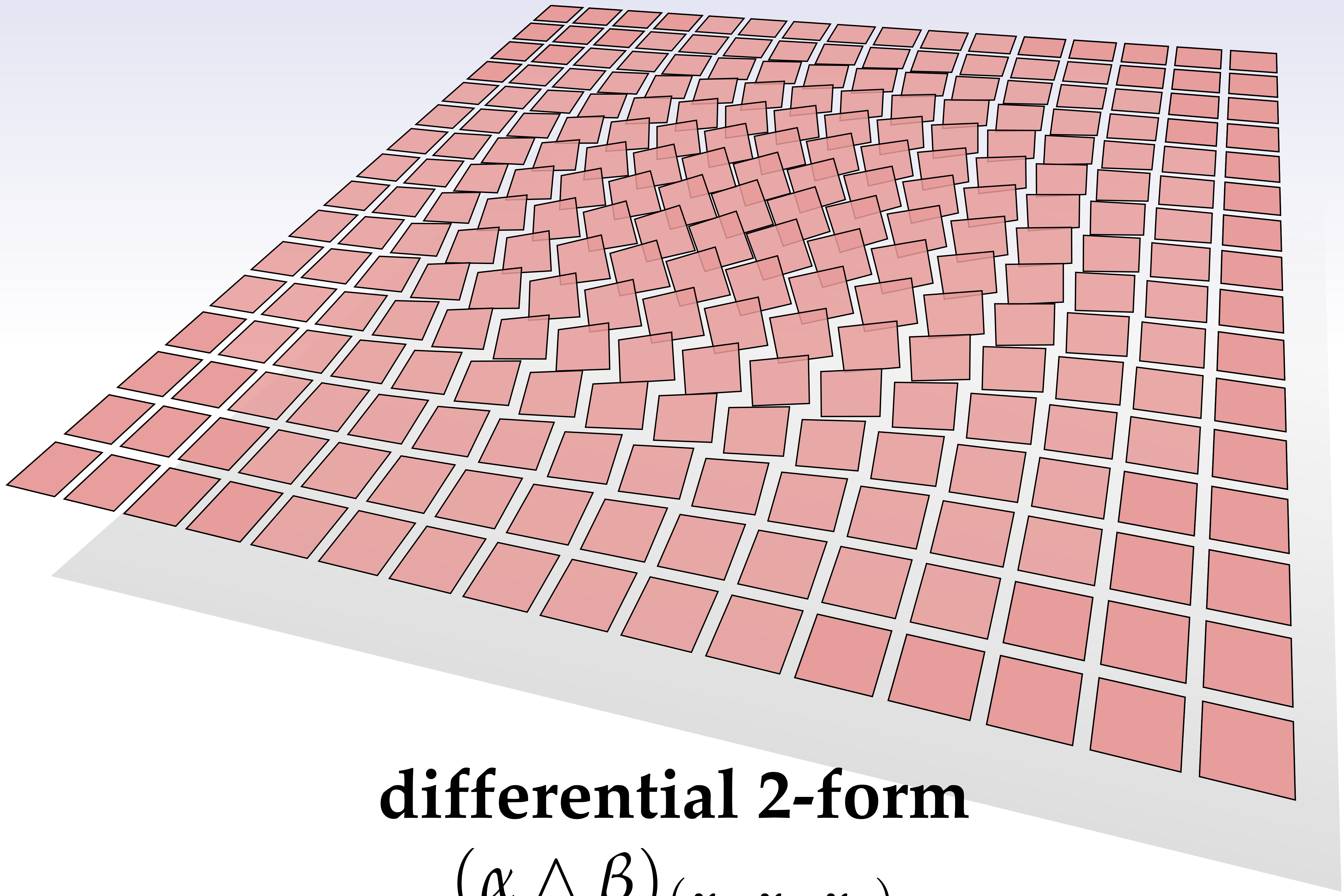
Differential 2-Forms

Likewise, a differential 2-form is an area measurement at each point (x_1,x_2,x_3):



2-vector field

$(X \wedge Y)_{(x_1,x_2,x_3)}$

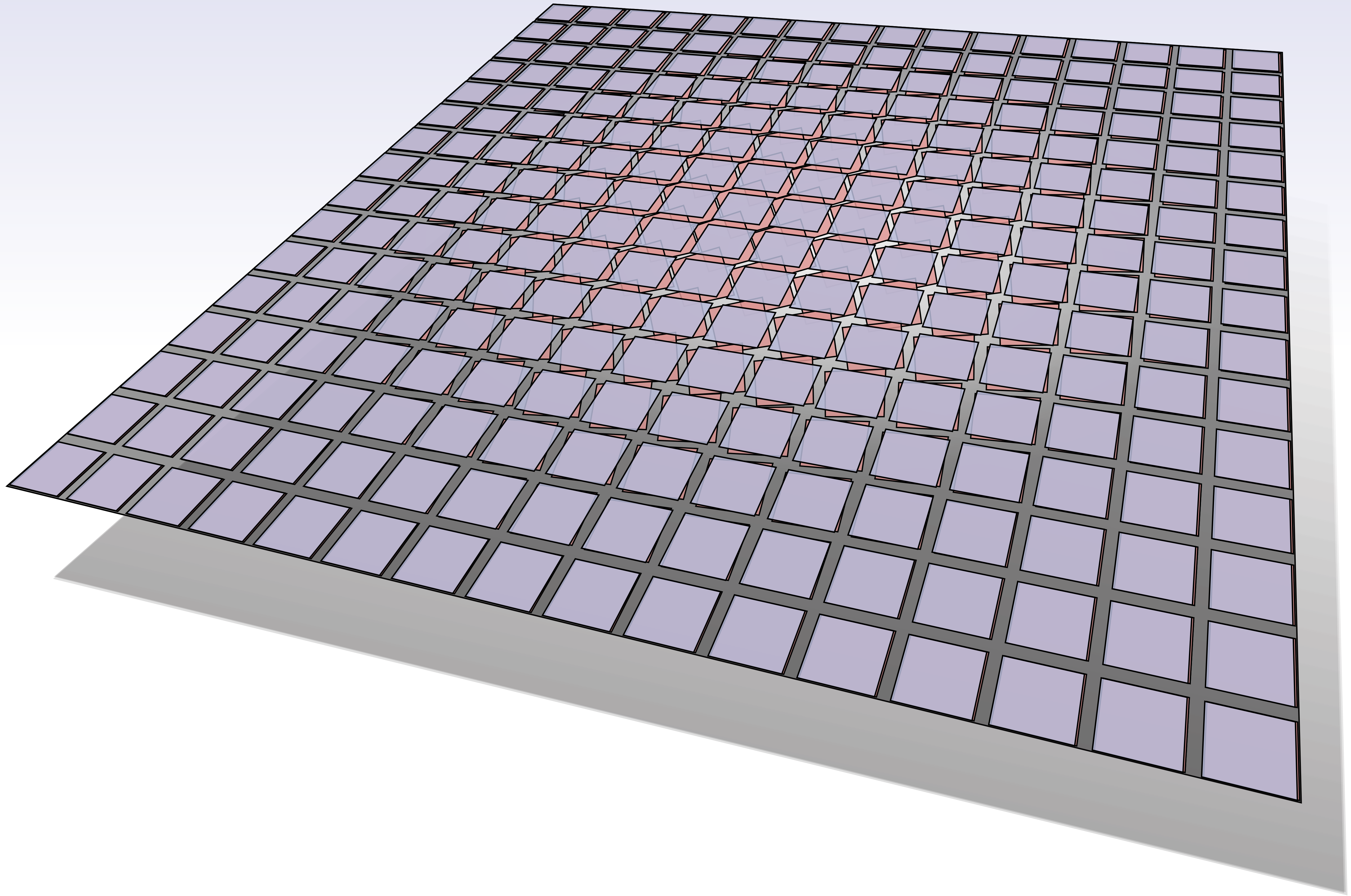


differential 2-form

$(\alpha \wedge \beta)_{(x_1,x_2,x_3)}$

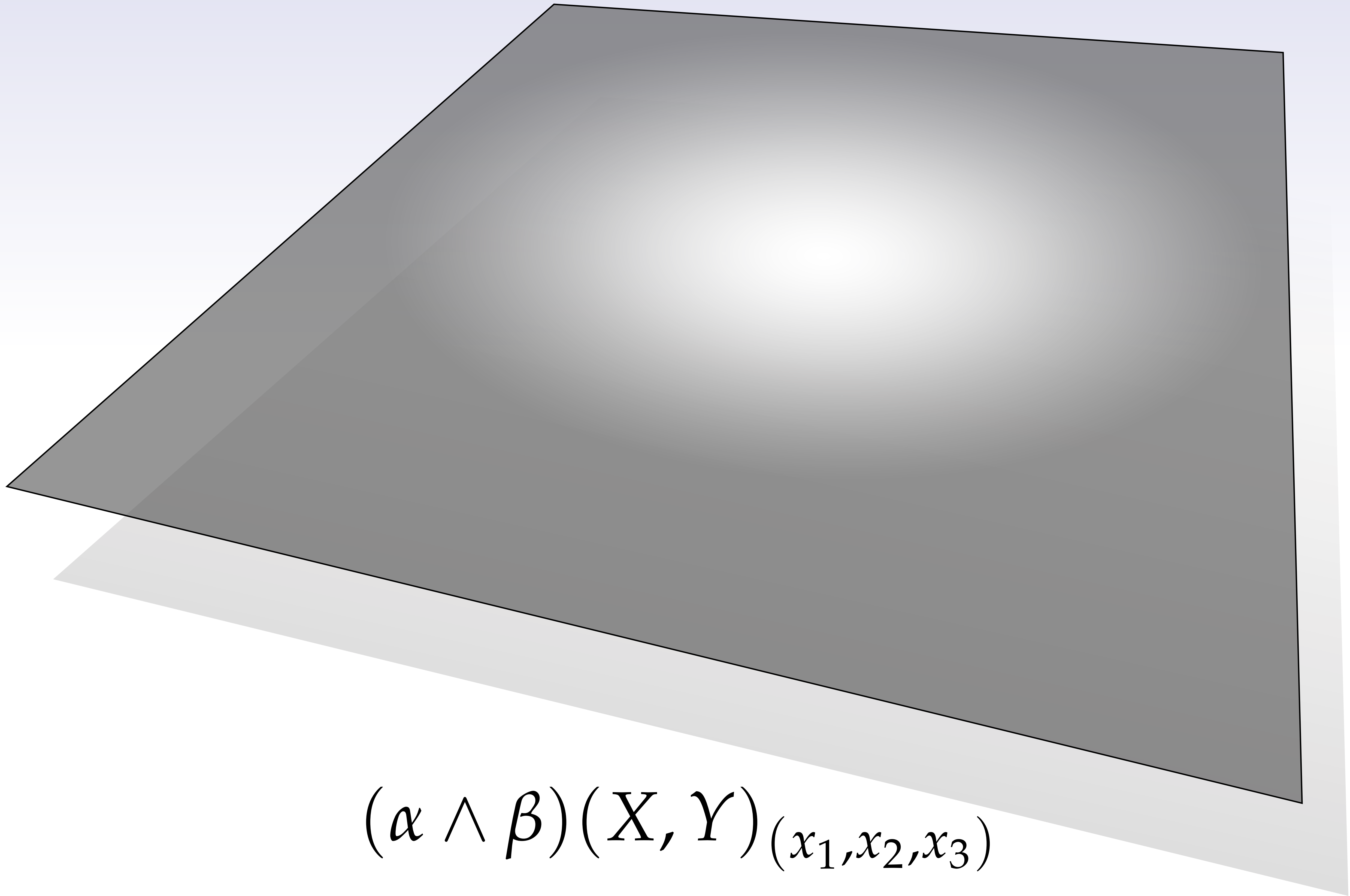
Note: only drawing a "slice" here.

Differential 2-Forms



Differential 2-Forms

Resulting function says how much 2-vector field "lines up" with differential 2-form.



$$(\alpha \wedge \beta)(X, Y)_{(x_1,x_2,x_3)}$$

Pointwise Operations on Differential k-Forms

- Most operations on differential k-forms simply apply that operation at each point.
- *E.g.*, consider two differential forms α, β on $\mathbb{R}^n$. At each point $p := (x_1,\ldots,x_n)$,

$$(\star\alpha)_p := \star(\alpha_p)$$

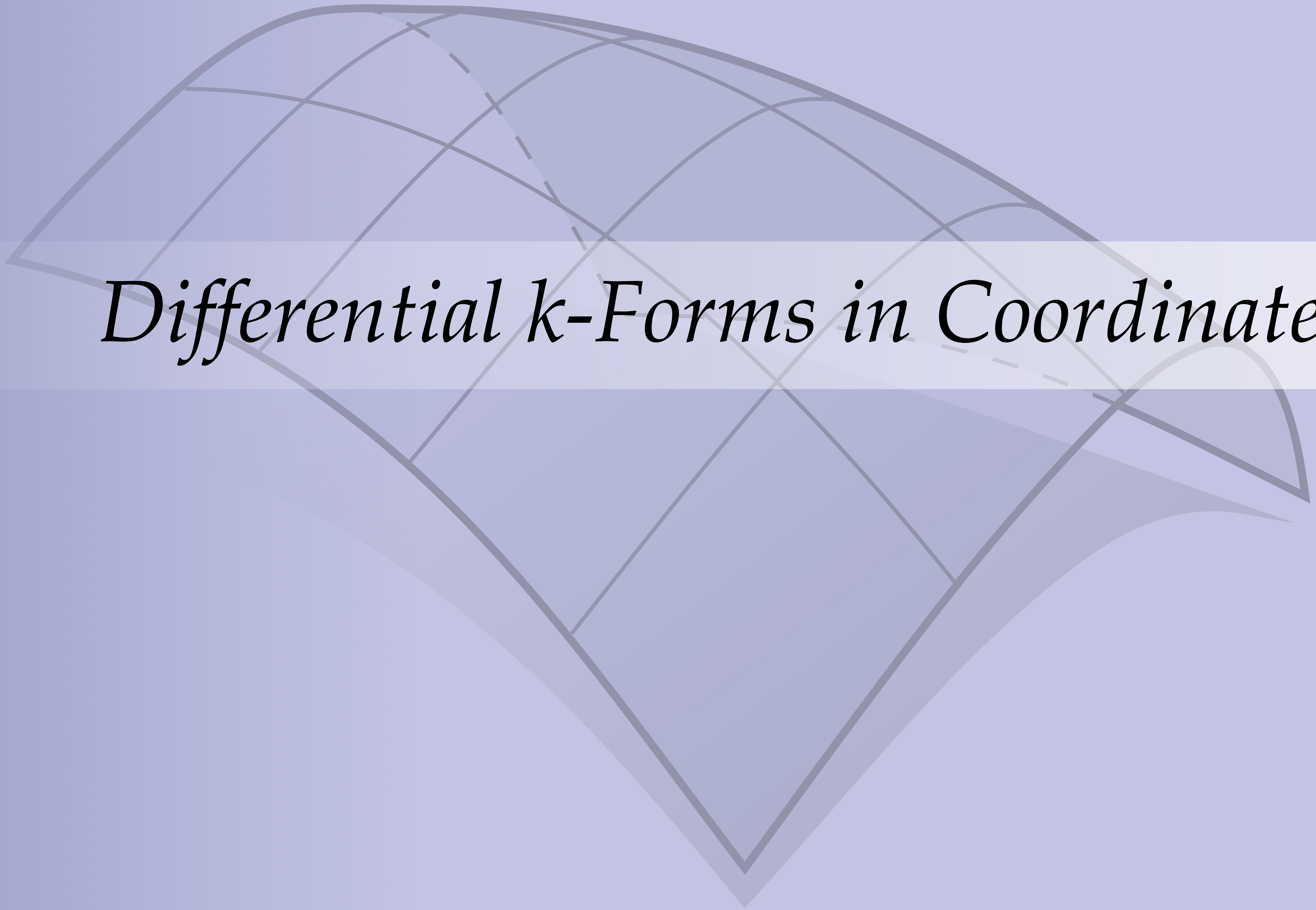
$$(\alpha \wedge \beta)_p := (\alpha_p) \wedge (\beta_p)$$

- In other words, to get the Hodge star of the *differential* k-form, we just apply the Hodge star to the individual k forms at each point p; to take the wedge of two differential k-forms we just wedge their values at each point.
- Likewise, if X_1, …, X_k are vector fields on all of R^n, then

$$\alpha(X_1,\ldots,X_k)_p := (\alpha_p)((X_1)_p,\ldots,(X_k)_p)$$

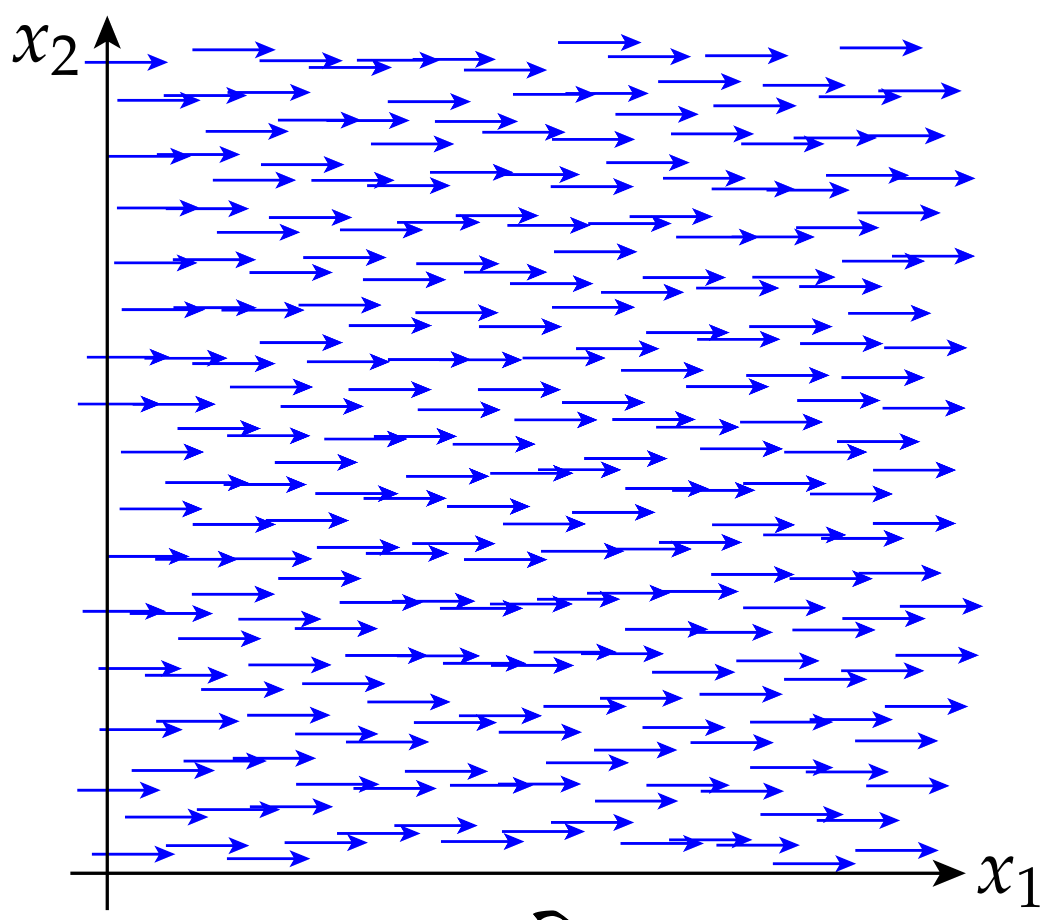
Typically we omit the p and just write $\star\alpha, \alpha \wedge \beta, \alpha(X,Y)$, etc.

Differential k-Forms in Coordinates

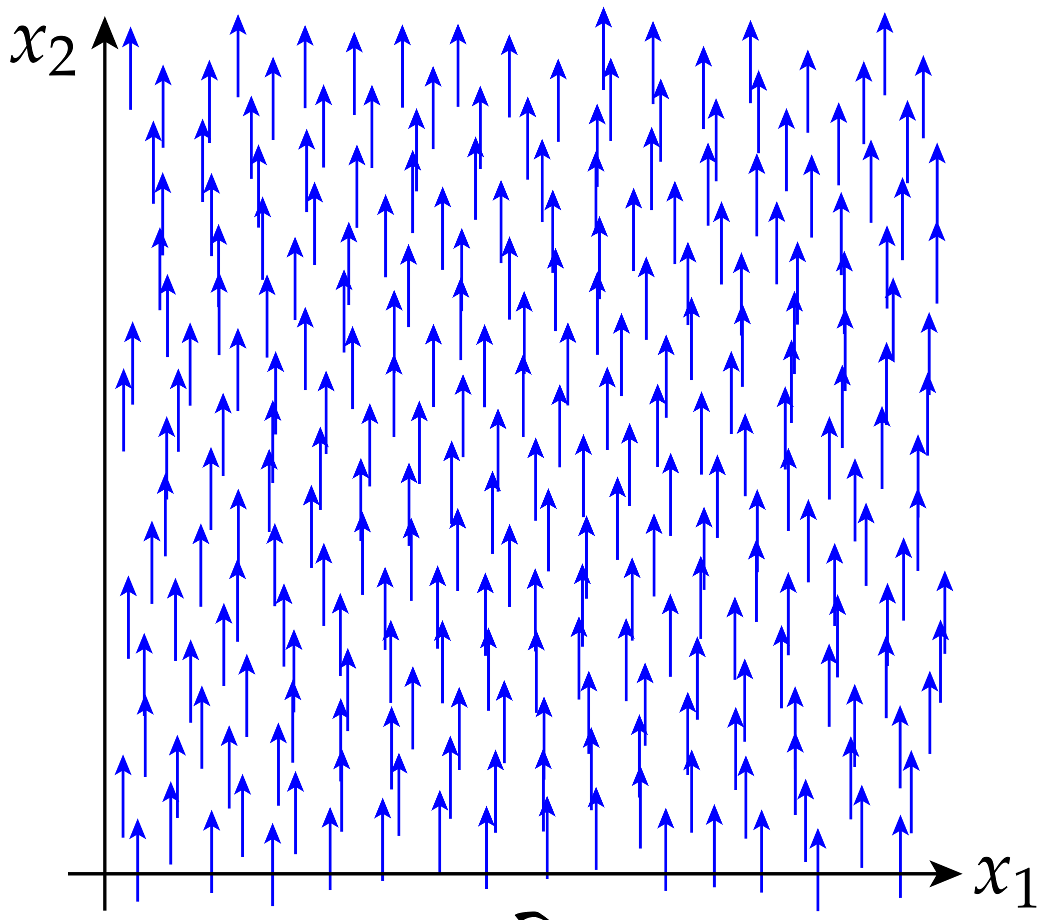


Basis Vector Fields

- Just as we can pick a basis for *vectors*, we can also pick a basis for *vector fields*
- The standard basis for vector fields on $\mathbb{R}^n$ are just **constant** vector fields of unit magnitude pointing along each of the coordinate axes:



$$\frac{\partial}{\partial x^1}$$

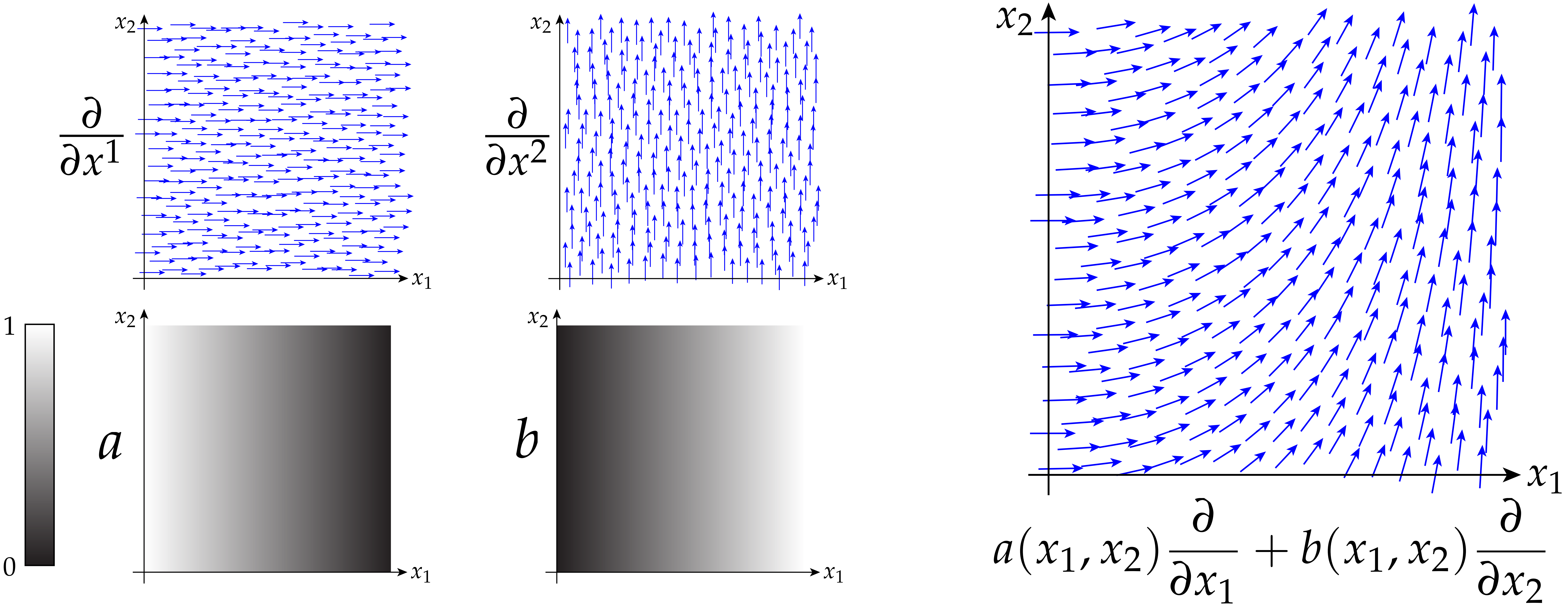


$$\frac{\partial}{\partial x^2}$$

- Notice that the basis vector fields have names that look like partial derivatives.
- You will do yourself a *huge* favor by **forgetting that they have anything at all to do with derivatives!** (For now…)

Basis Expansion of Vector Fields

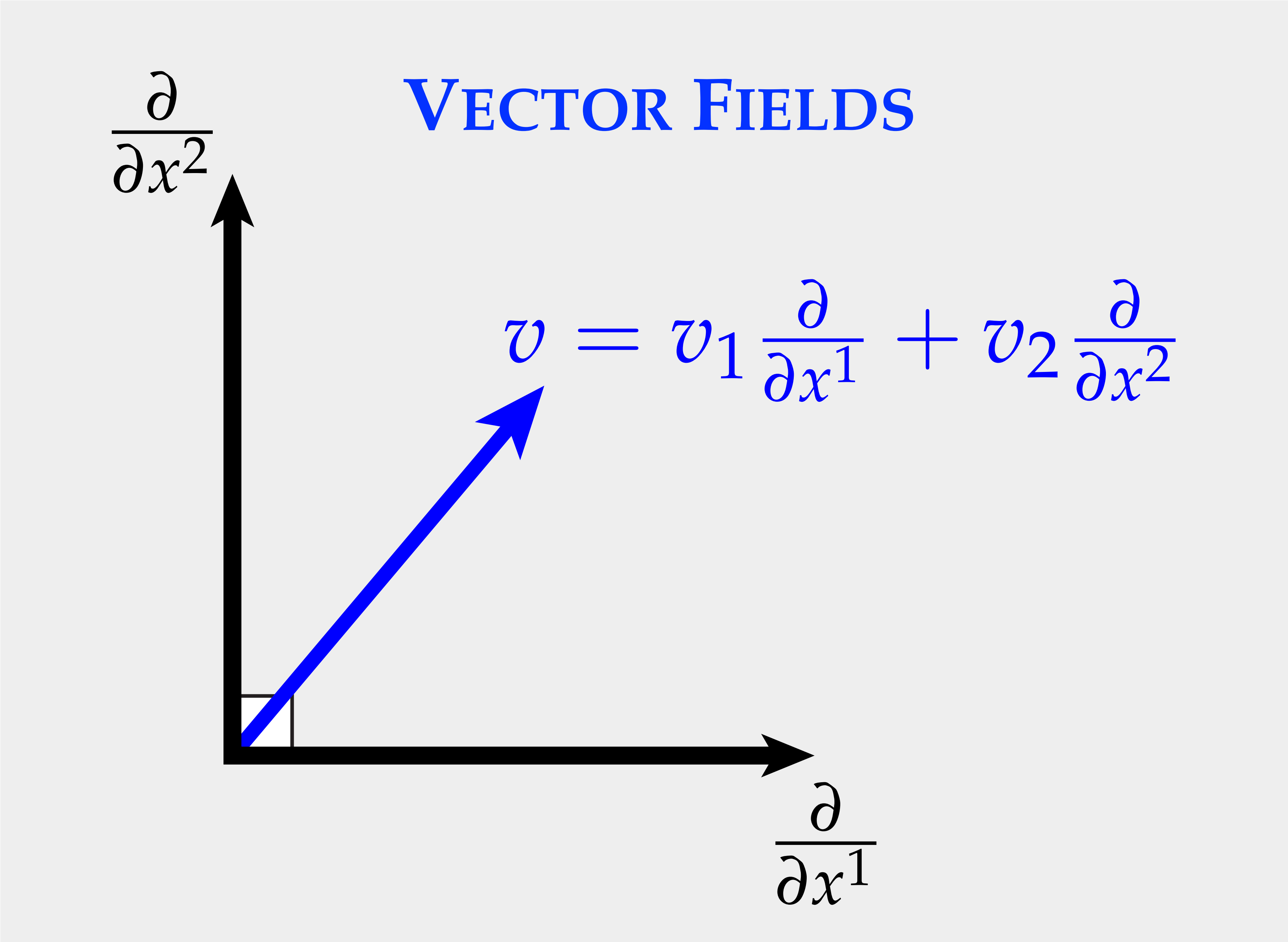
- Any other vector field is then a linear combination of the basis vector fields…
- …*but*, the coefficients of the linear combination may vary across the domain:

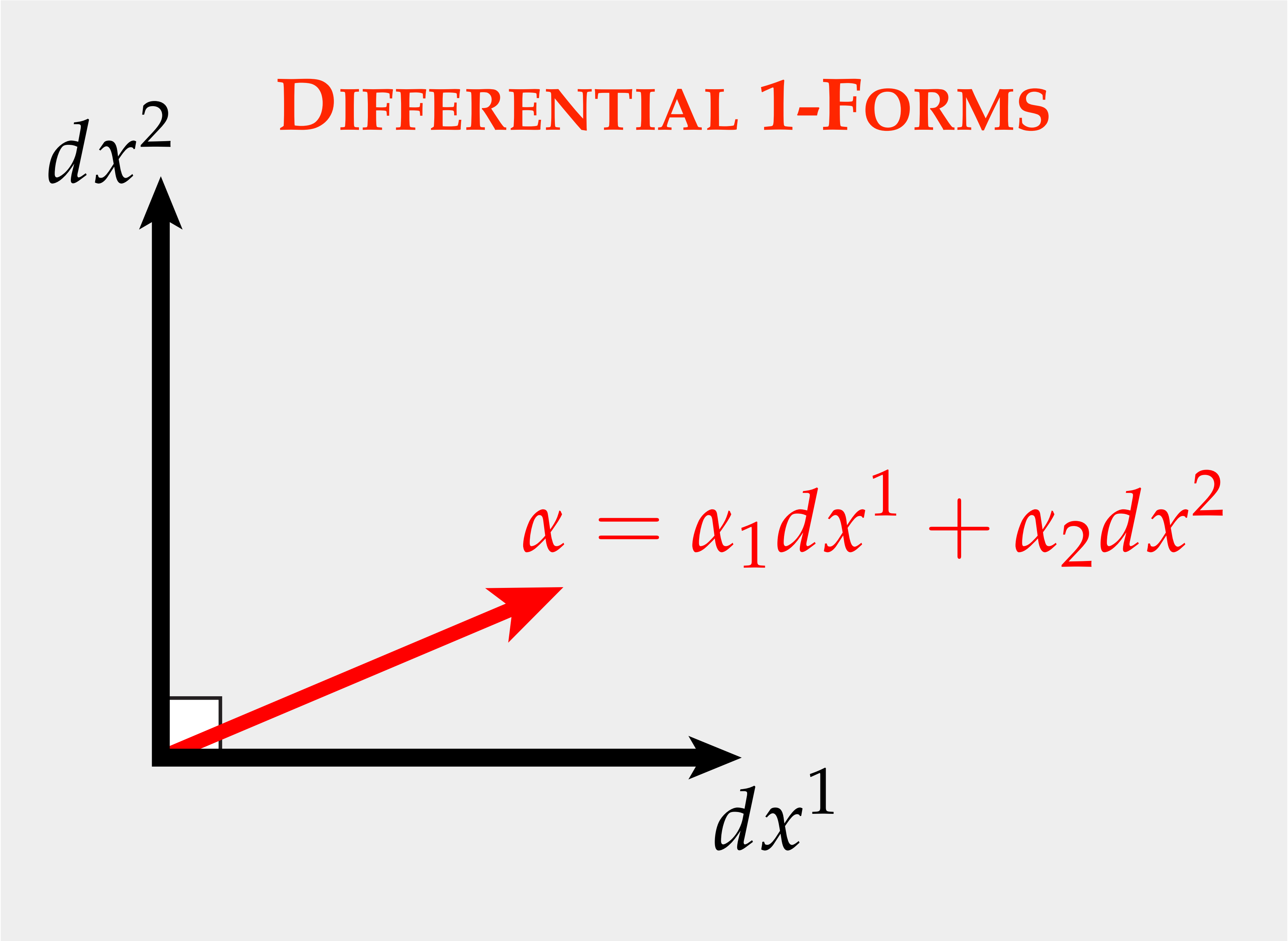


Q: What would happen if we didn't allow coefficients to vary?

Bases for Vector Fields and Differential 1-forms

The story is nearly identical for differential 1-forms, but with different, *dual* bases:



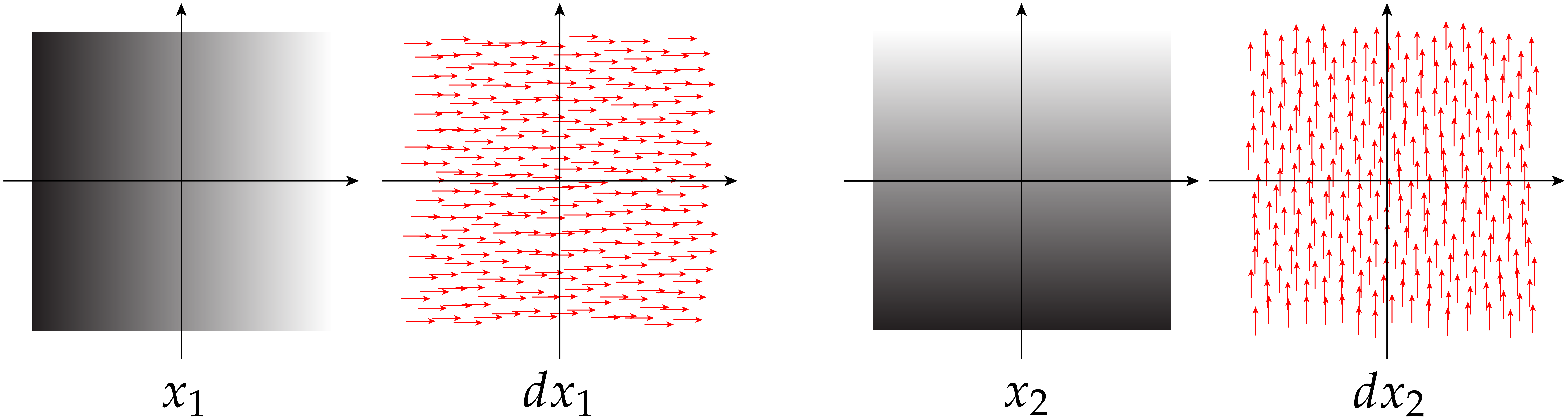


$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

Stay sane: think of these symbols as *bases*; forget they look like *derivatives!*

Coordinate Bases as Derivatives

Q: Ok, but why the heck do we use symbols that look like *derivatives?*

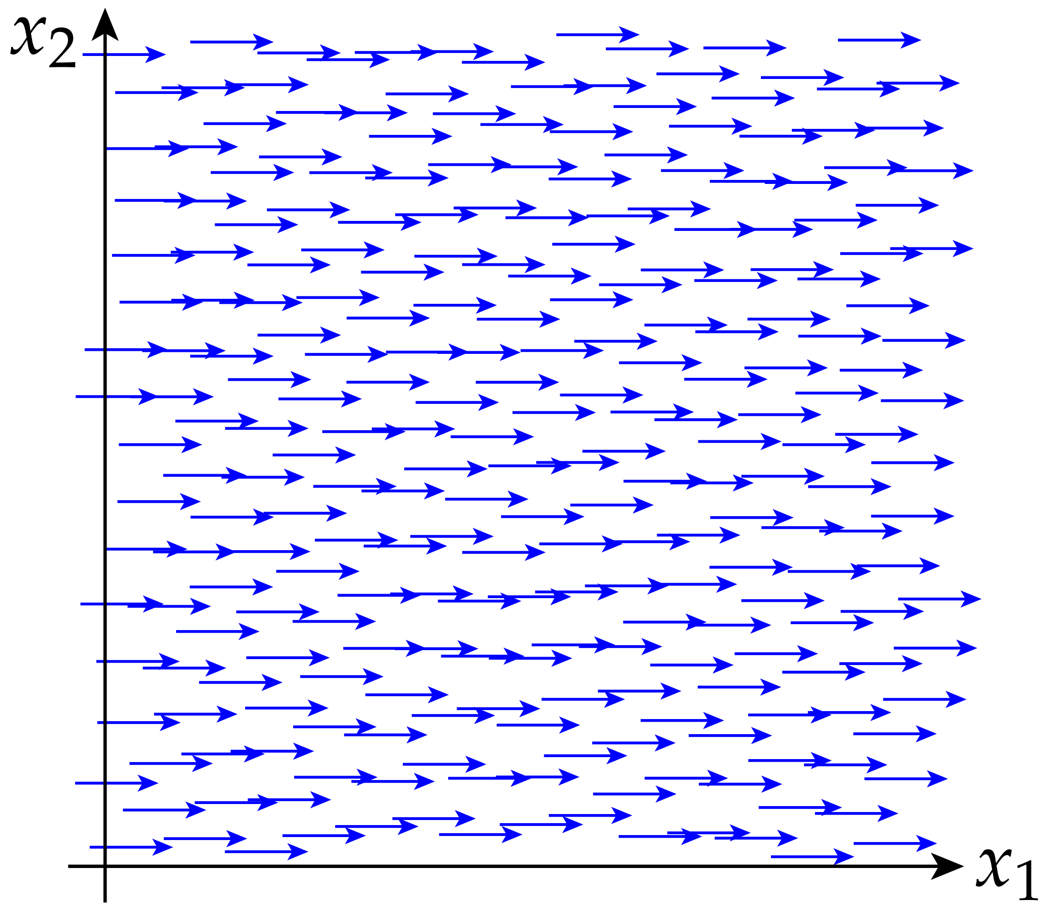


Key idea: derivative of each coordinate function yields a constant basis field.

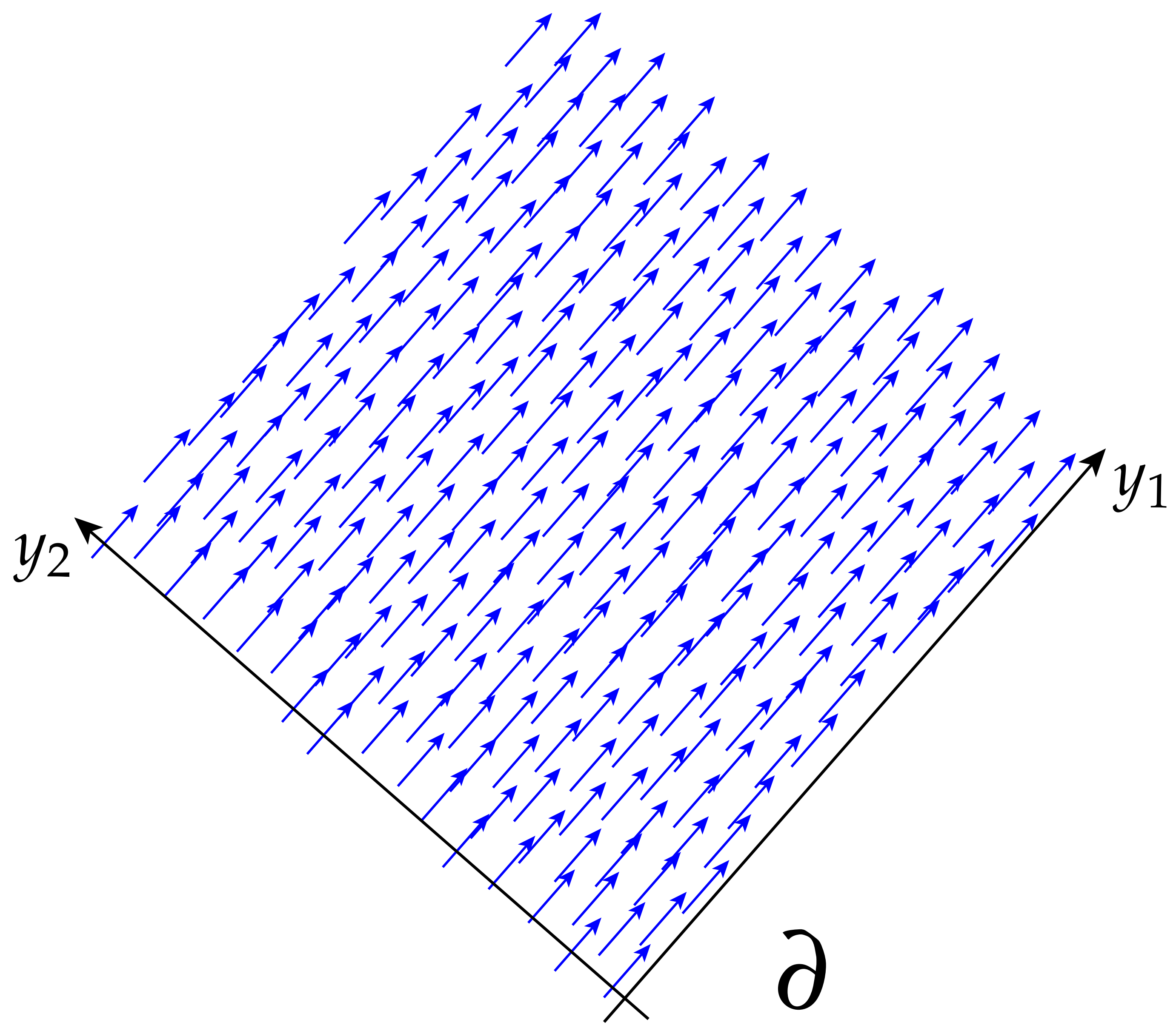
*We'll give a more precise meaning to "d" in a little bit.

Coordinate Notation—Further Apologies

- One very good reason for adopting this notation—consider a situation where we want to work with two different coordinate systems:



$$\frac{\partial}{\partial x^1}$$



$$\frac{\partial}{\partial y^1}$$

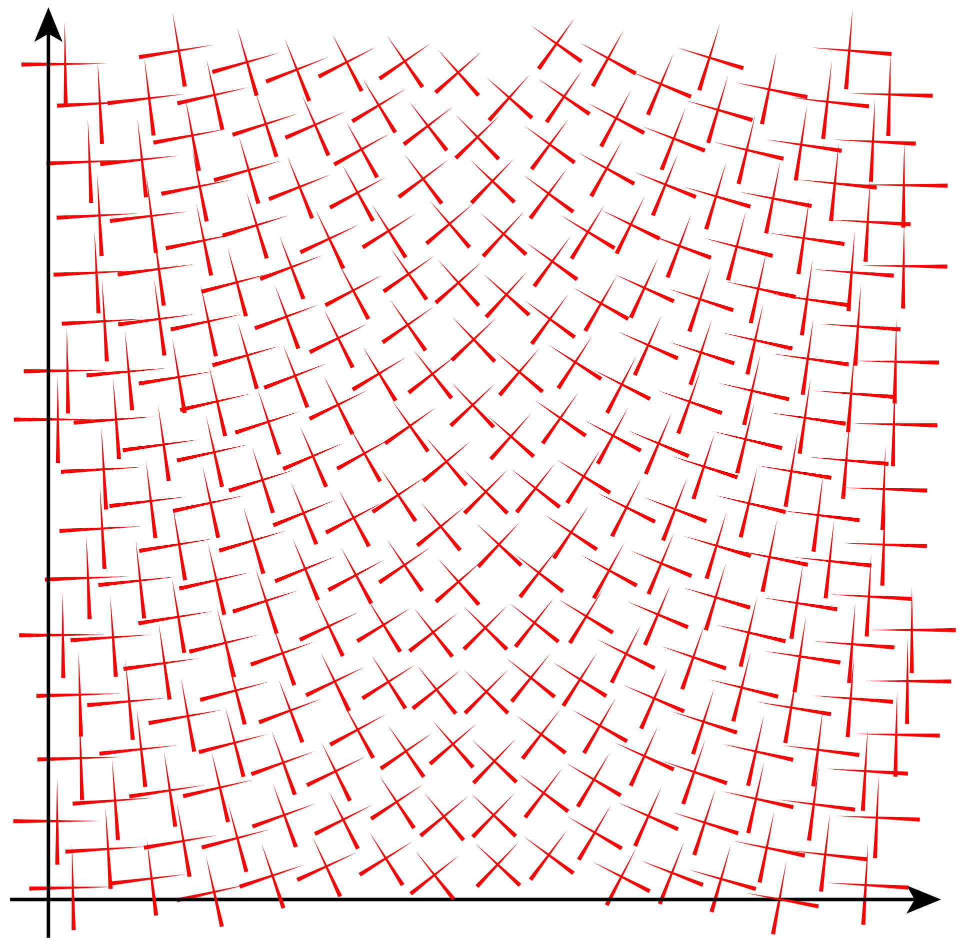
- Including the name of the coordinate system, in our name for the basis fields makes it clear which one we mean. (Not true with e_i, X_i, etc.)

Example: Hodge Star of Differential 1-form

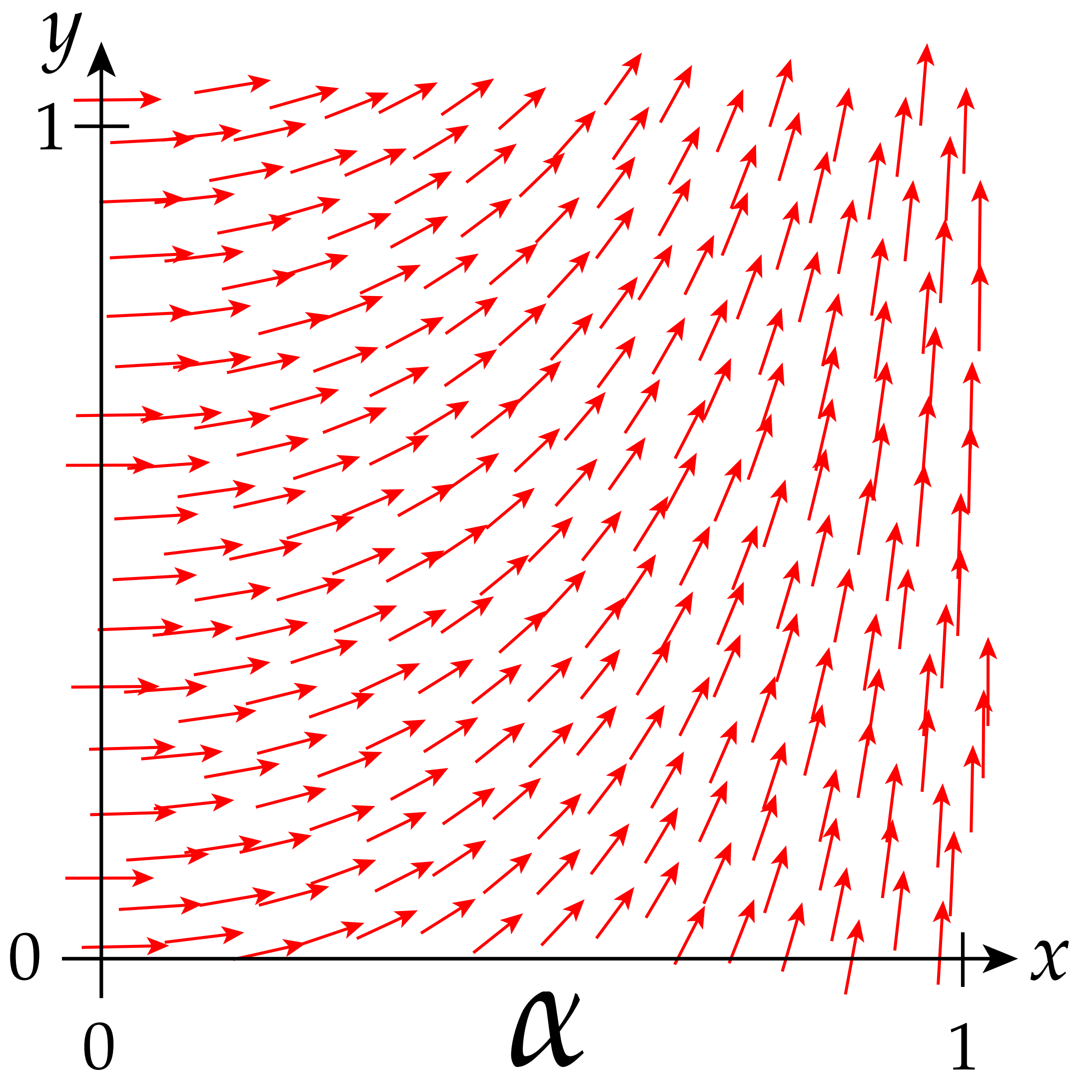
- Consider the differential 1-form $\alpha := (1-x)dx + xdy$
  - Use coordinates (x,y) instead of (x_1,x_2)
  - Notice this expression varies over space

Q: What's its Hodge star?

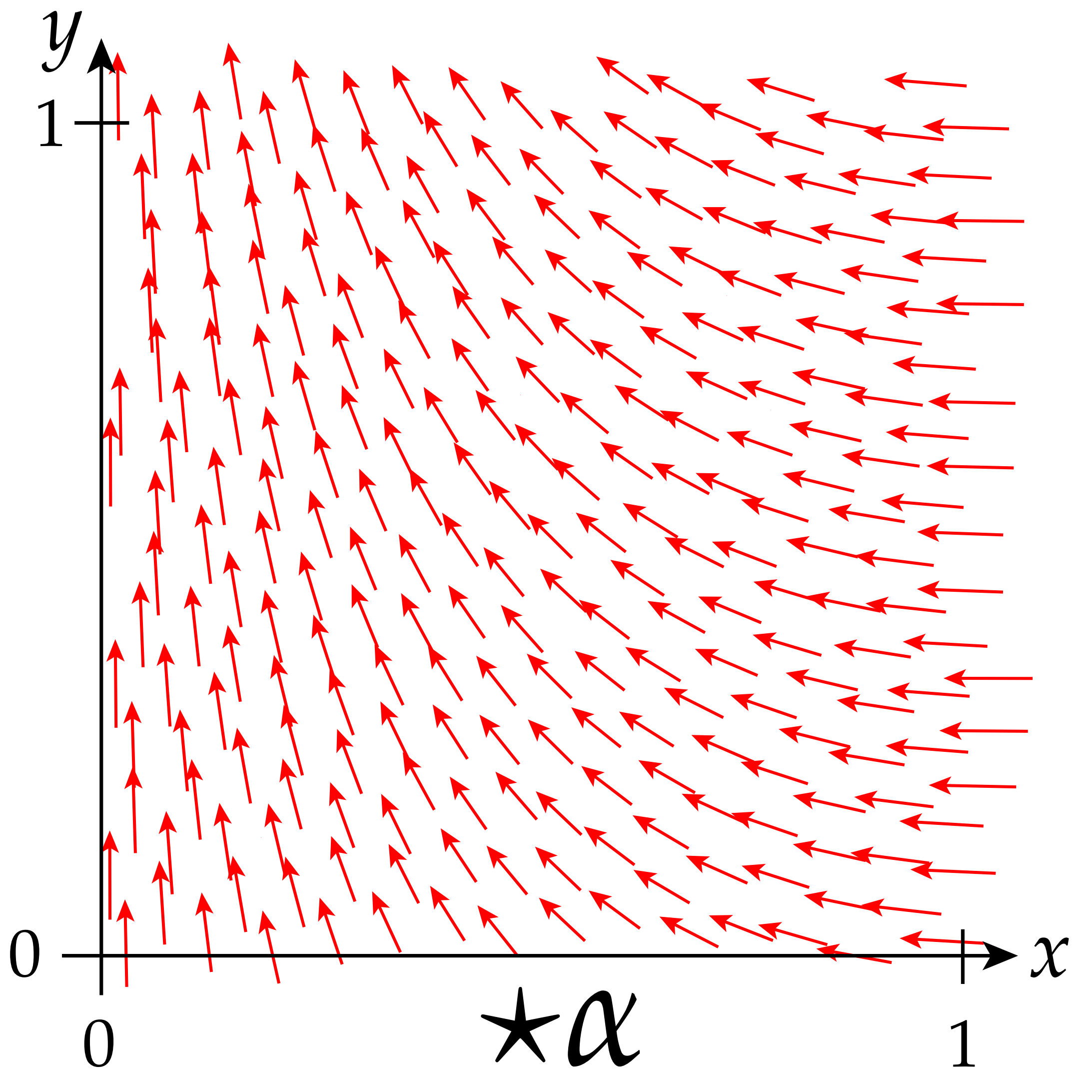
$$\begin{aligned}\star\alpha &= \star((1-x)dx) + \star(xdy)\\ &= (1-x)(\star dx) + x(\star dy)\\ &= (1-x)dy + -xdx\end{aligned}$$



$\alpha, \star\alpha$



α



$\star\alpha$

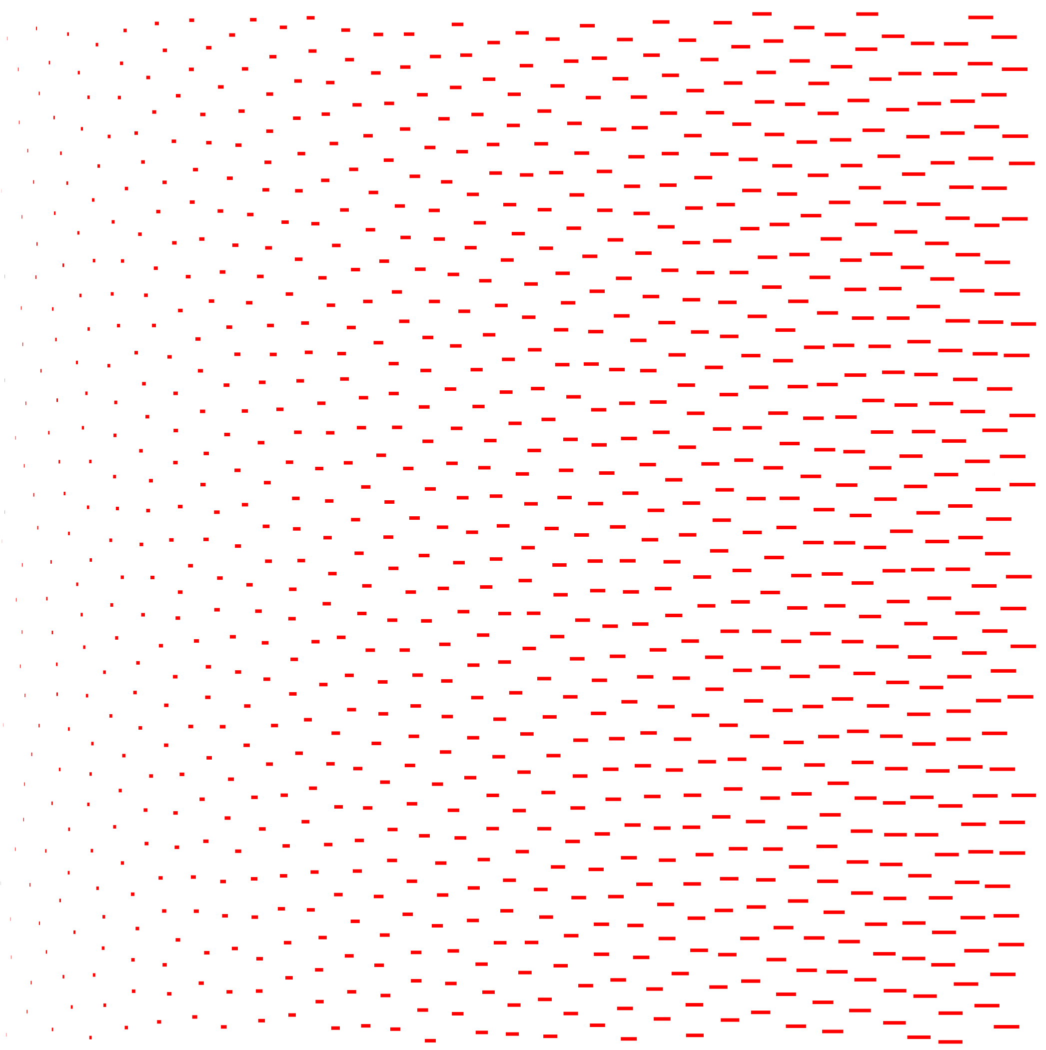
Recall that in 2D, 1-form Hodge star is quarter-turn.

When we overlay the two we get little crosses
(almost look like little areas…)

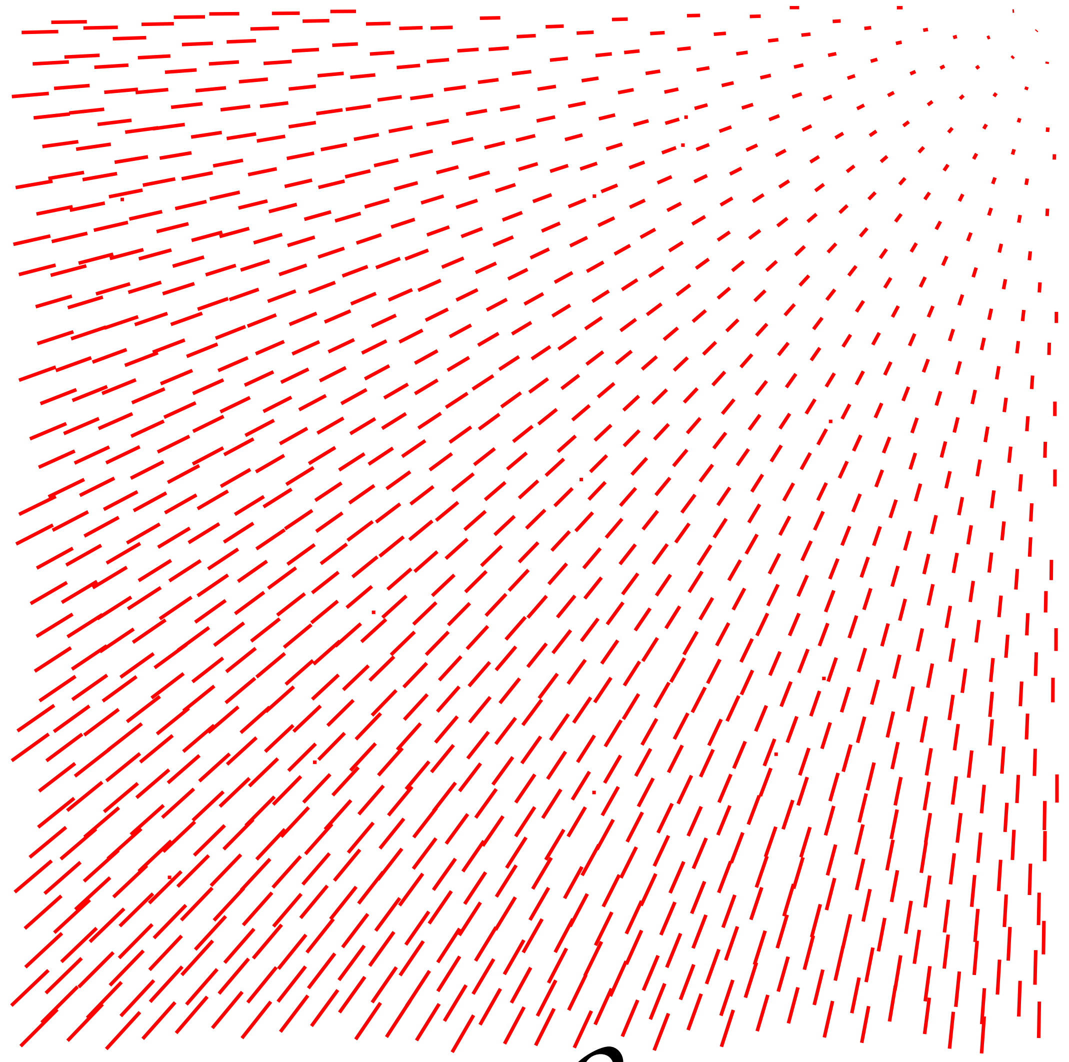
Example: Wedge of Differential 1-Forms

Consider the differential 1-forms*

$$\alpha := x dx, \qquad \beta := (1-x)dx + (1-y)dy$$



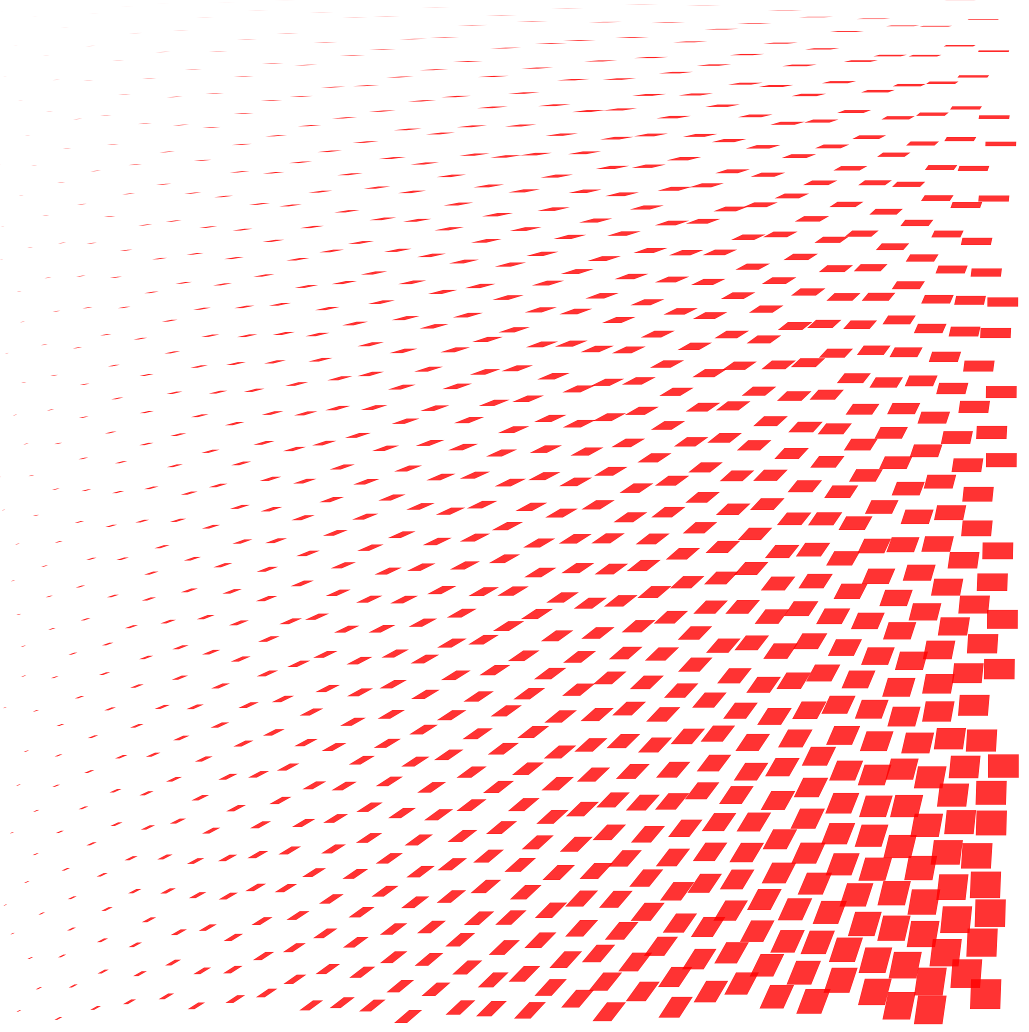
α



β

Q: What's their wedge product?

$$\begin{aligned}\alpha \wedge \beta &= (x dx) \wedge ((1-x)dx + (1-y)dy) \\ &= (x dx) \wedge ((1-x)dx) + (x dx) \wedge ((1-y)dy) \\ &= x(1-x)\cancelto{0}{dx \wedge dx} + x(1-y)dx \wedge dy \\ &= (x - xy) dx \wedge dy\end{aligned}$$



$\alpha \wedge \beta$

(What does the result **look** like?)

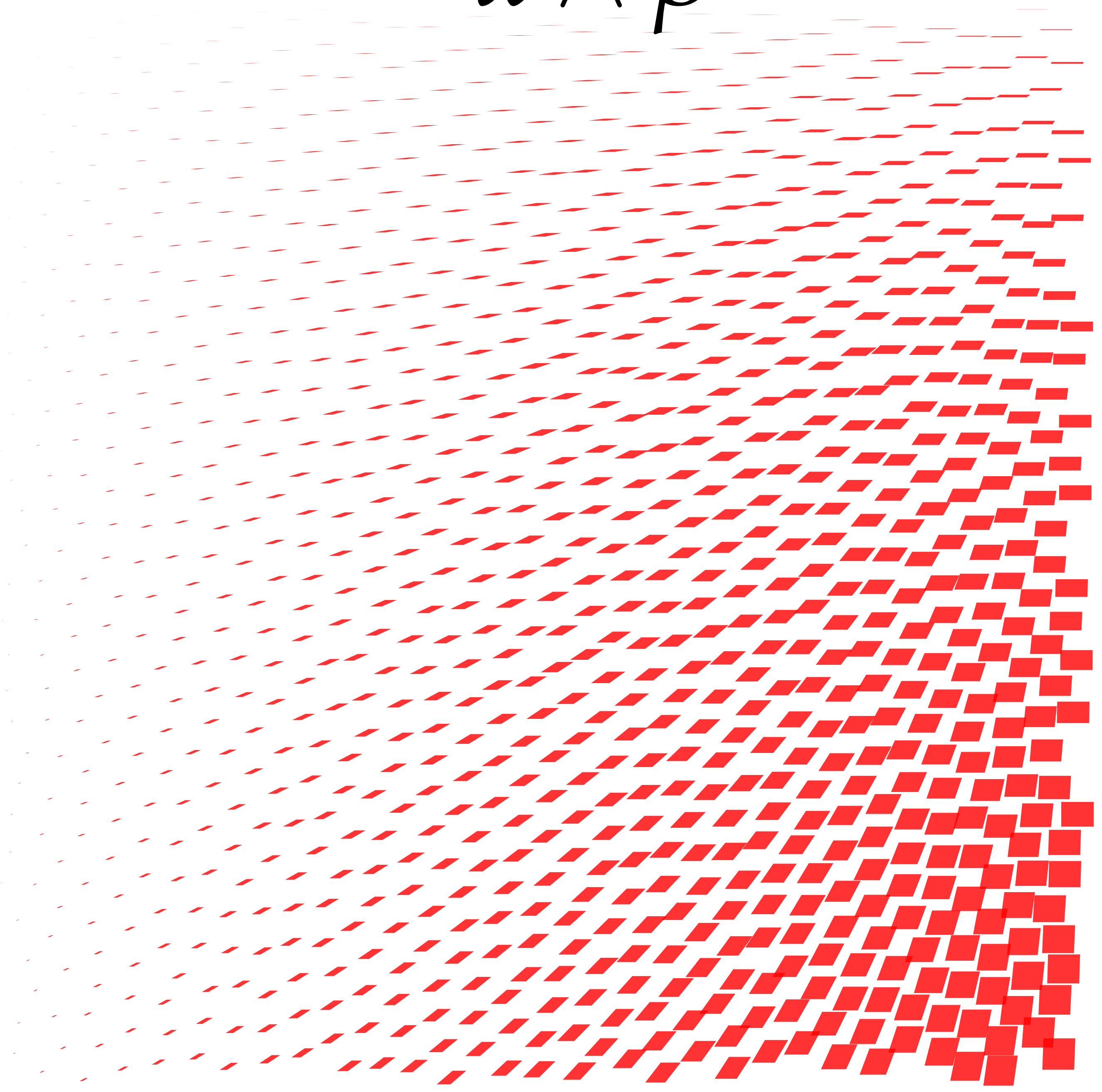
*All plots in this slide (and the next few slides) are over the unit square [0,1] x [0,1].

Volume Form / Differential n-form

- Our picture has little parallelograms
- But what information does our differential 2-form actually encode?

$$\alpha \wedge \beta = (x - xy)dx \wedge dy$$

- Magnitude (x-xy), and "direction" $dx \wedge dy$



- But in the plane, *every* differential 2-form will be a multiple of $dx \wedge dy$!
  - More precisely, some scalar function times $dx \wedge dy$, which measures *unit area*
- In n-dimensions, any *positive* multiple of $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is called a *volume form*
  - Provides some meaningful (i.e., nonzero, nonnegative) notion of volume

Applying a Differential 1-Form to a Vector Field

- The whole point of a differential 1-form is to measure vector fields. So let's do it!

$$\alpha := x\,dx$$

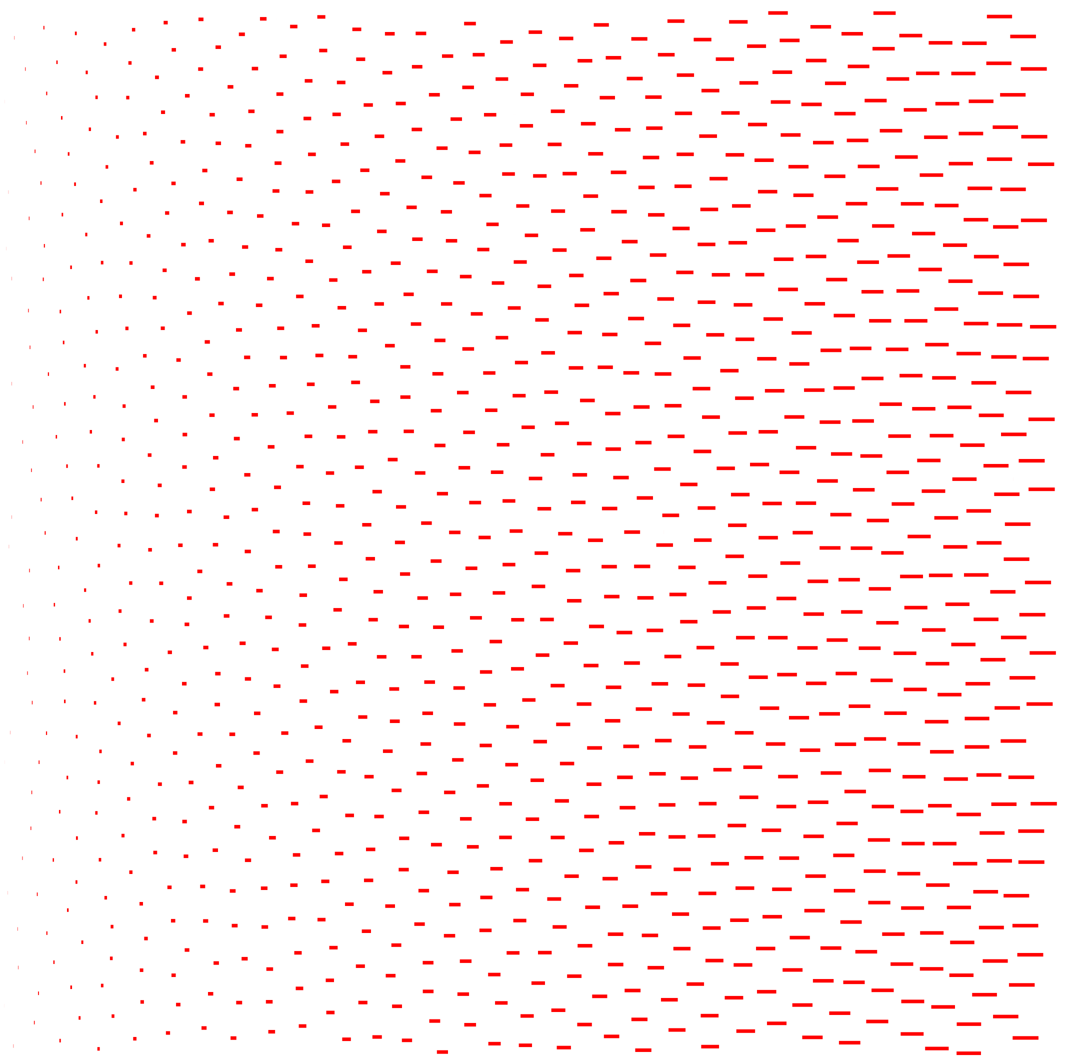
$$X := (1-x)\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y}$$

$$\alpha(X) = (x\,dx)\left((1-x)\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y}\right)$$

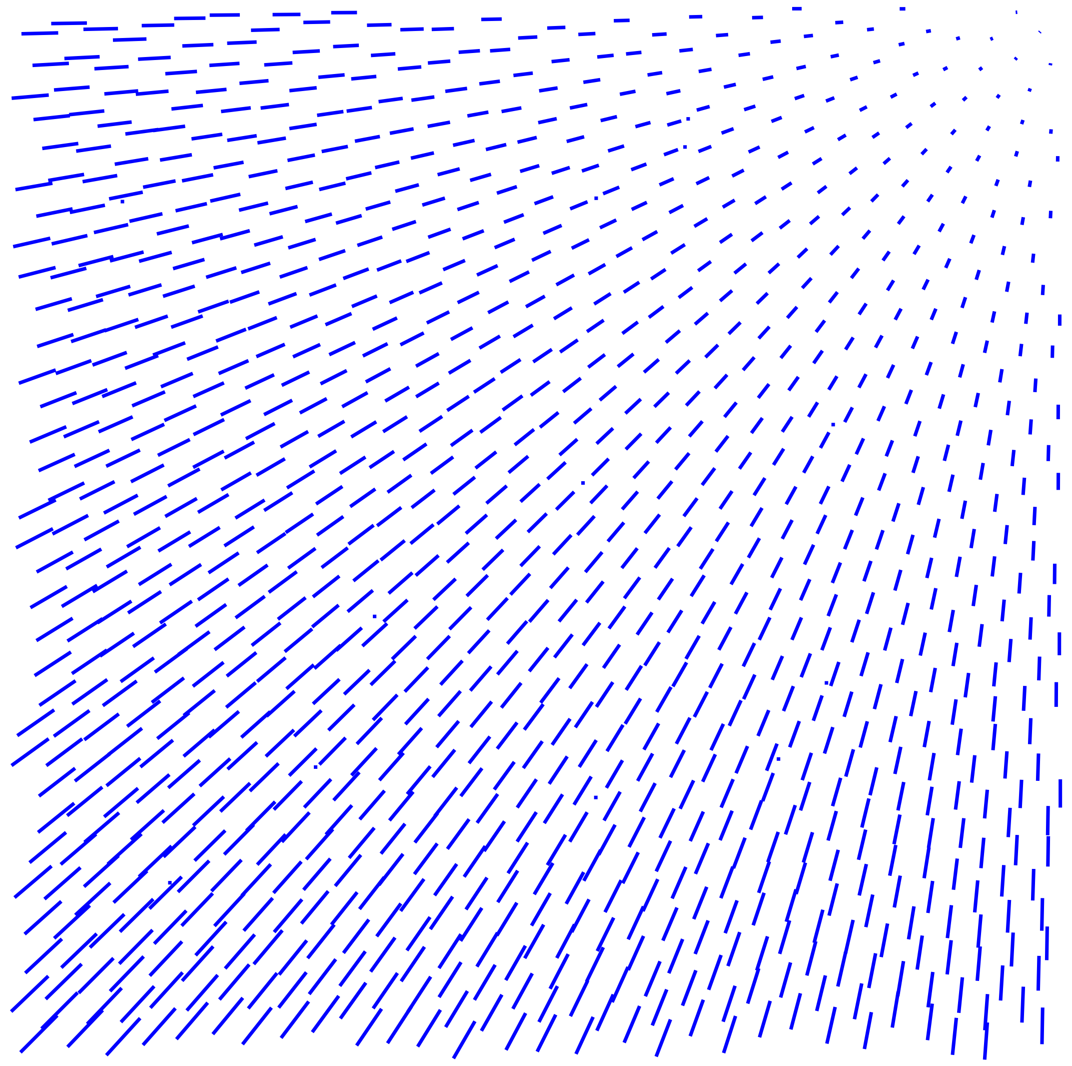
$$= (x\,dx)\left((1-x)\frac{\partial}{\partial x}\right) + (x\,dx)\left((1-y)\frac{\partial}{\partial y}\right)$$

$$= (x-x^2)\cancelto{1}{dx(\frac{\partial}{\partial x})} + (x-xy)\cancelto{0}{dx(\frac{\partial}{\partial y})}$$

$$= x - x^2$$



α



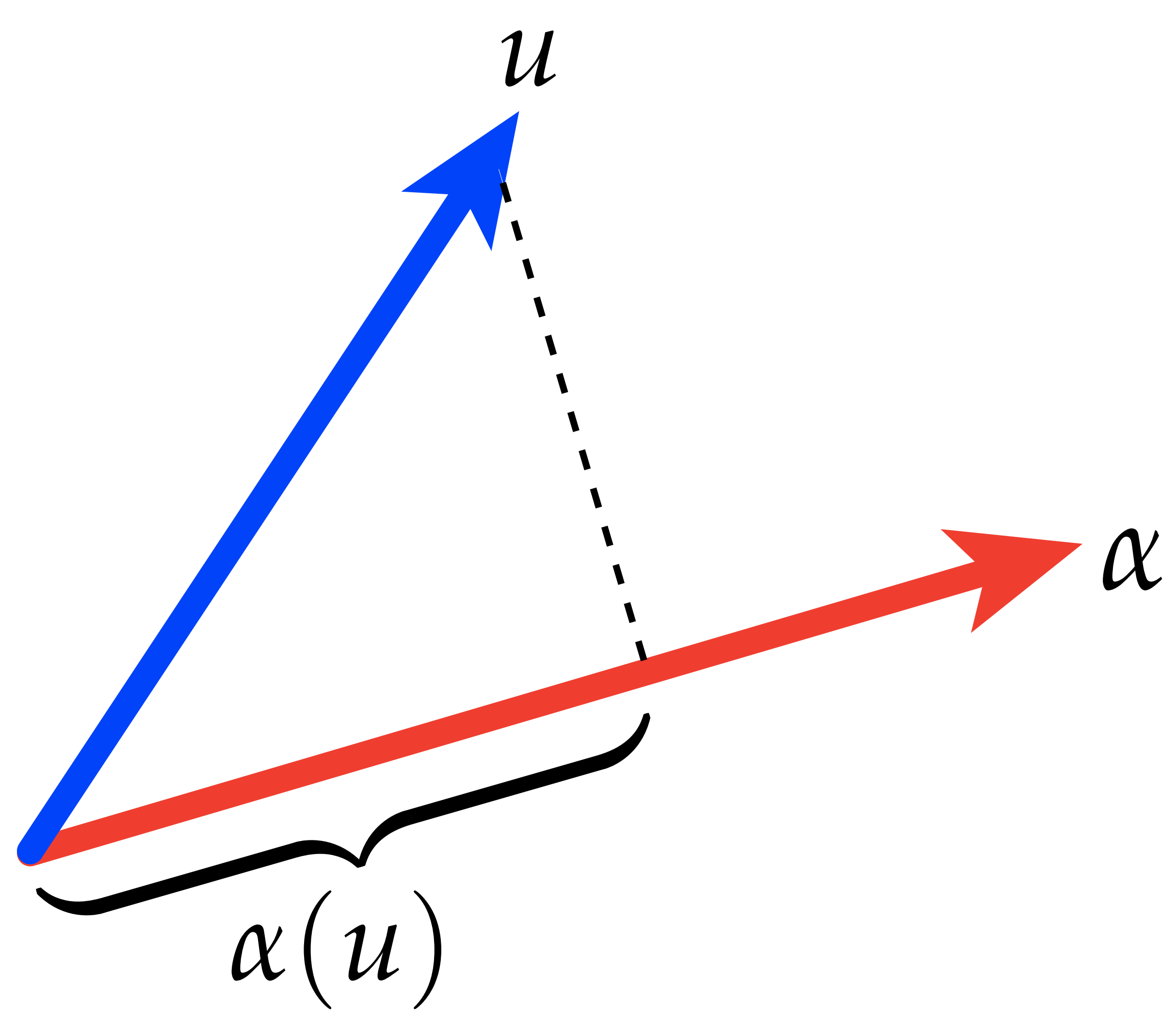
X

$\alpha(X)$

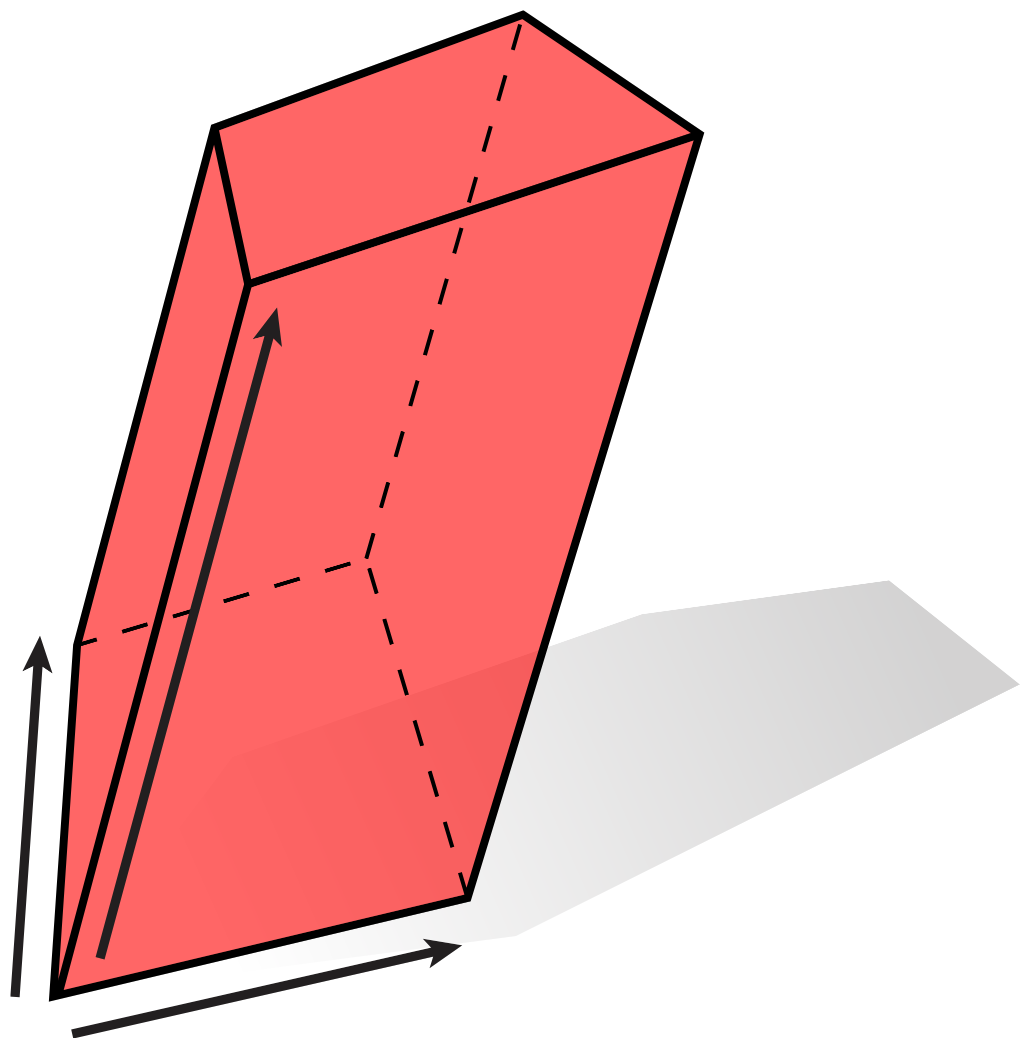
(Kind of like a dot product…)

Differential Forms in R^n - Summary

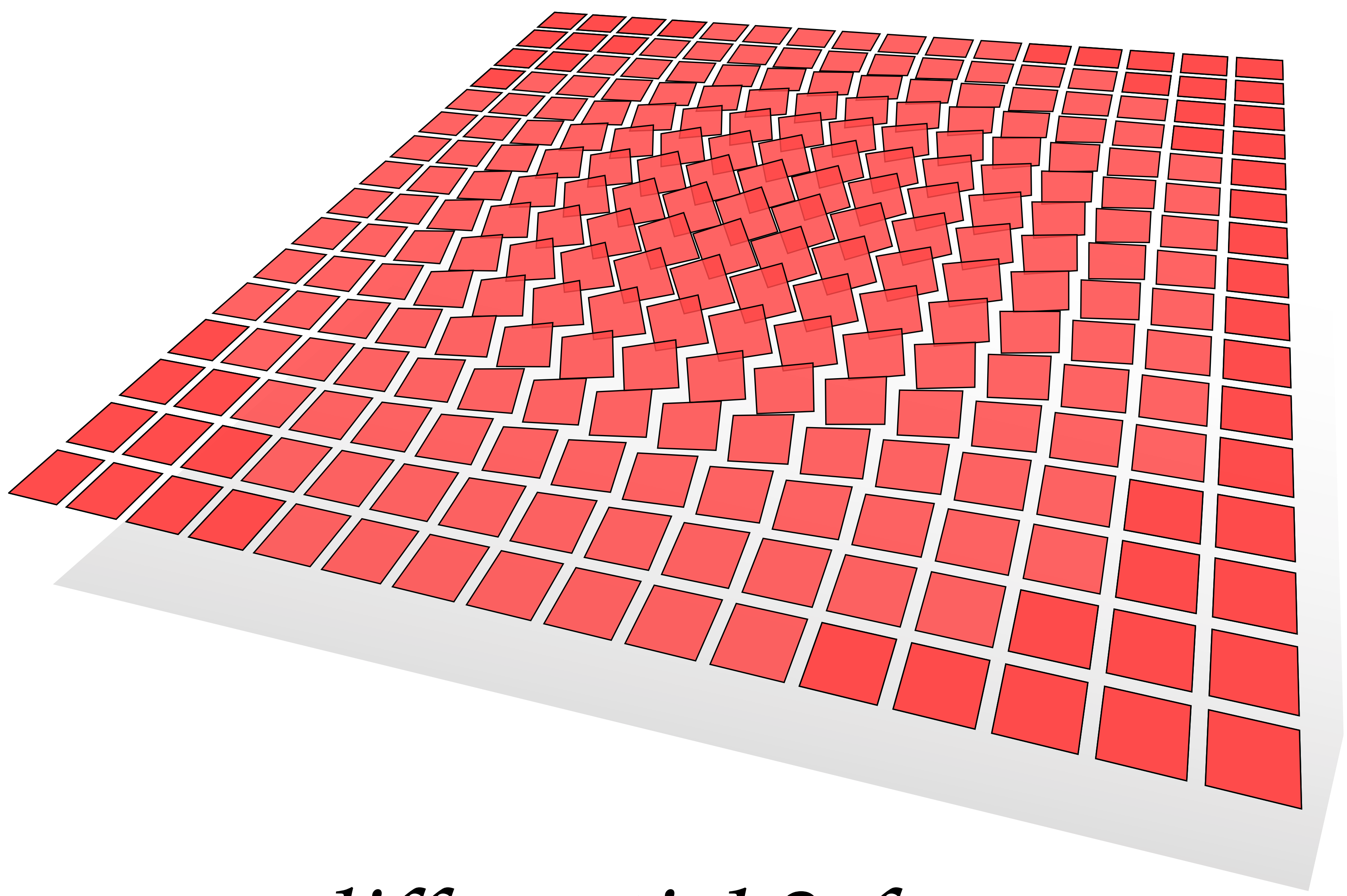
- Started with a vector space V (*e.g.*, $\mathbb{R}^n$)
  - ***(1-forms)*** Dual space V^* of covectors, i.e., linear measurements of vectors
  - ***(k-forms)*** Wedge together k covectors to get a measurement of k-dim. volumes
  - ***(differential k-forms)*** Put a k-form at each point of space



1-form



3-form



differential 2-form

Exterior Algebra & Differential Forms—Summary

	primal	**dual**
linear algebra	vectors	covectors
exterior algebra	k-vectors	k-forms
spatially-varying	k-vector fields	differential k-forms

Where Are We Going Next?

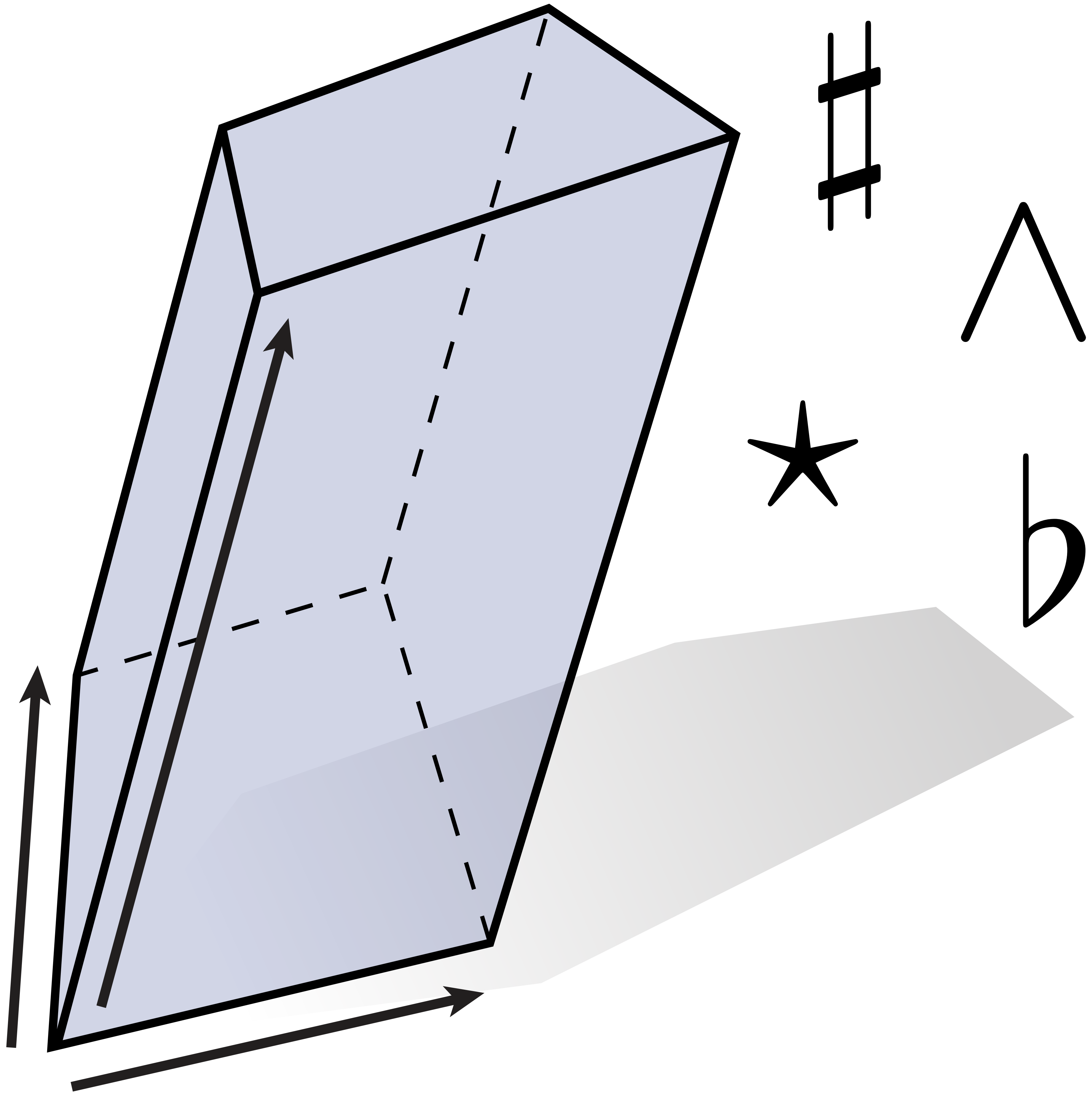
GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

- **Linear algebra:** "little arrows" (vectors)
- **Vector Calculus:** how do vectors *change?*

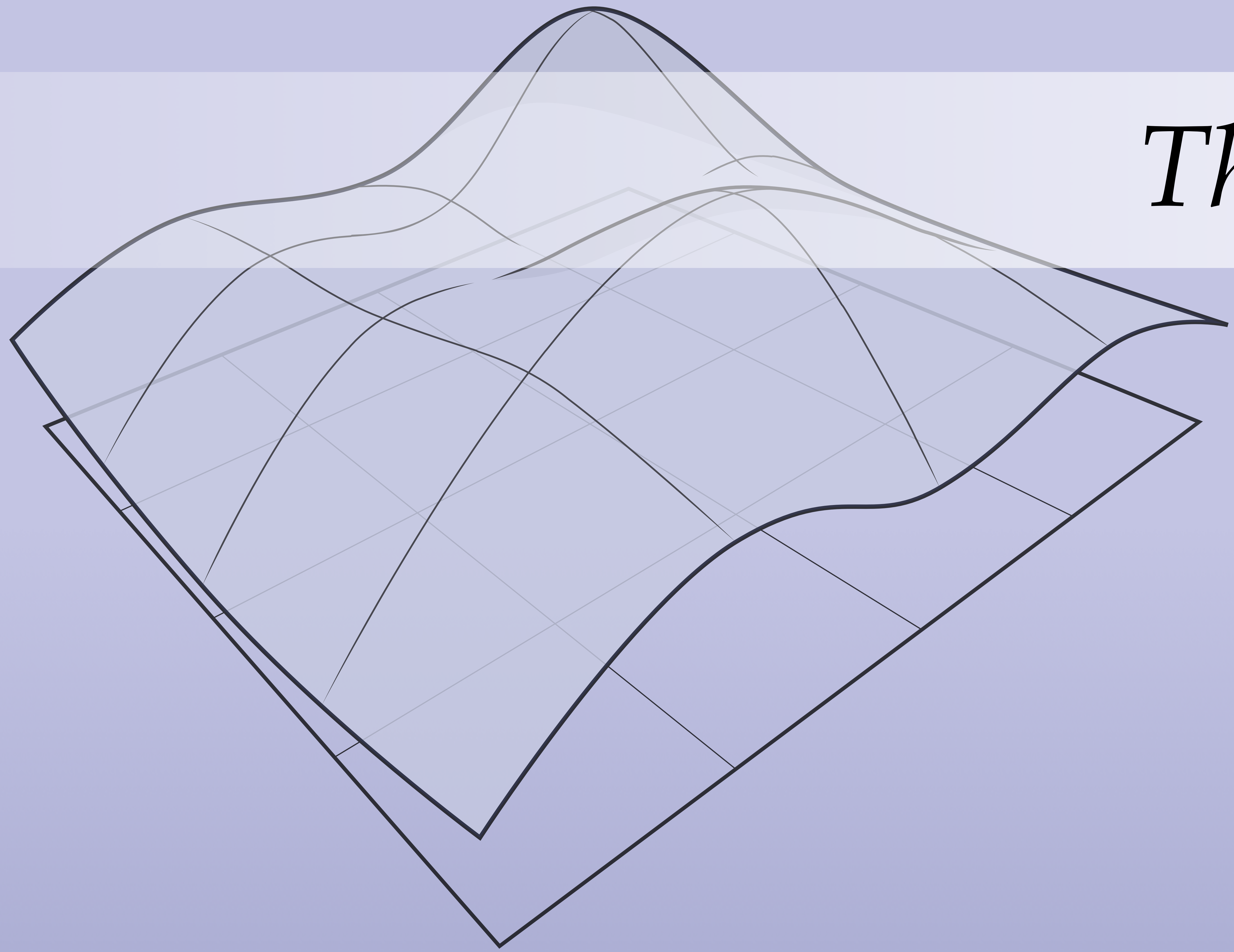
Next few lectures:

- **Exterior algebra**: "little volumes" (k-vectors)
- **Exterior calculus**: how do k-vectors change?
- **DEC:** how do we do all of this on meshes?



Basic idea: replace vector calculus with computation on meshes.

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION