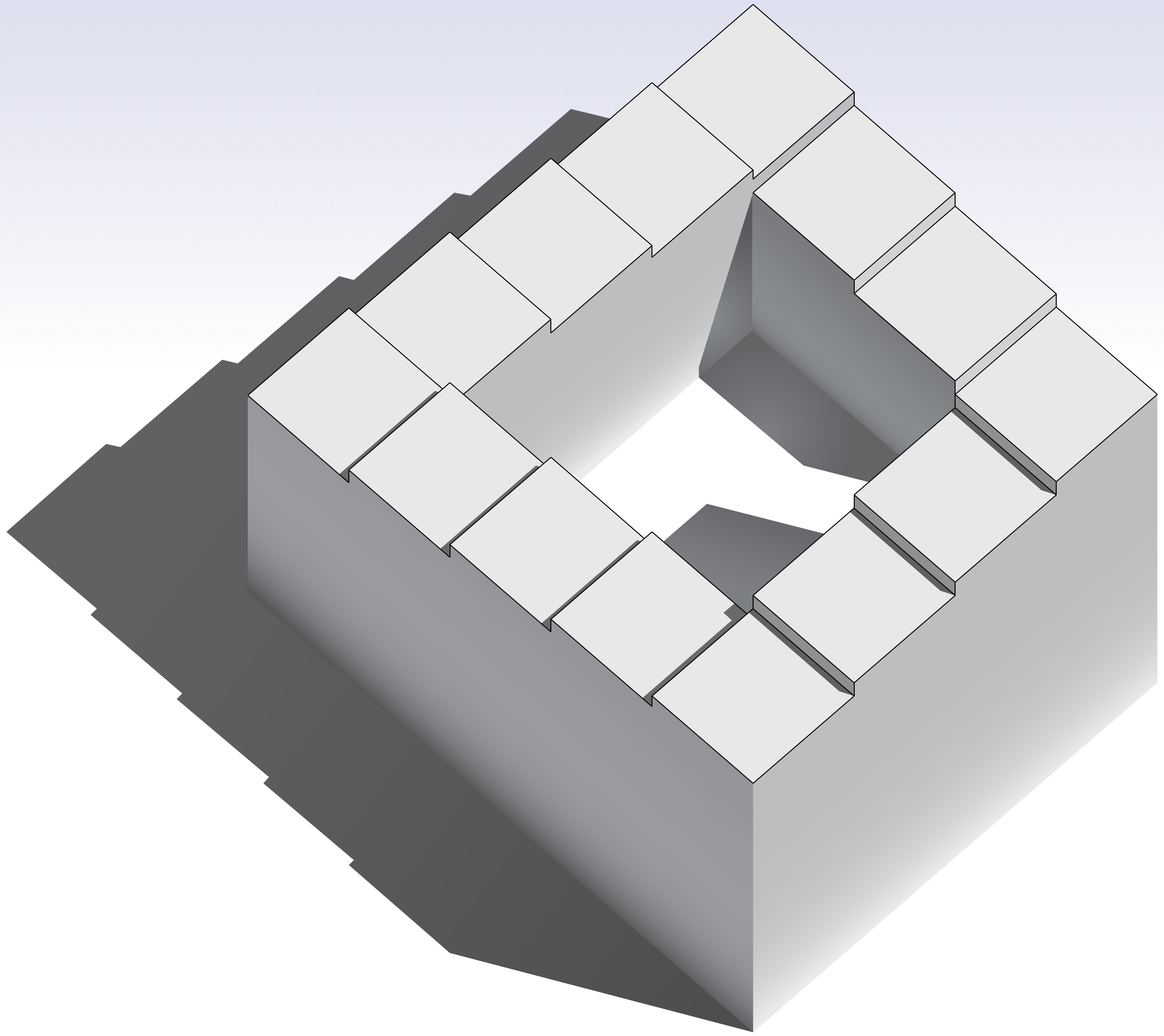
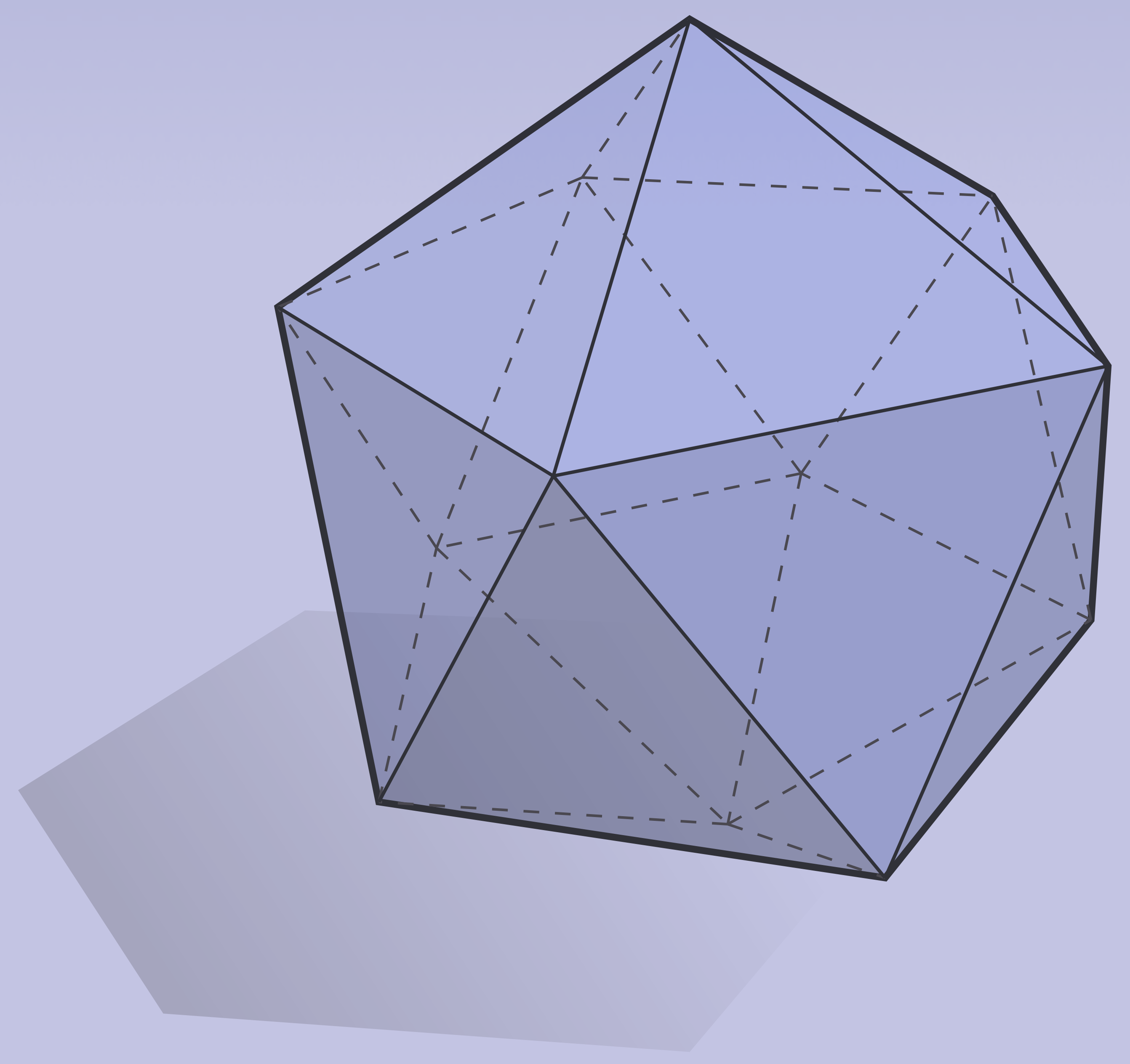


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 9:
Discrete Exterior Calculus



Discrete Differential Geometry: An Applied Introduction

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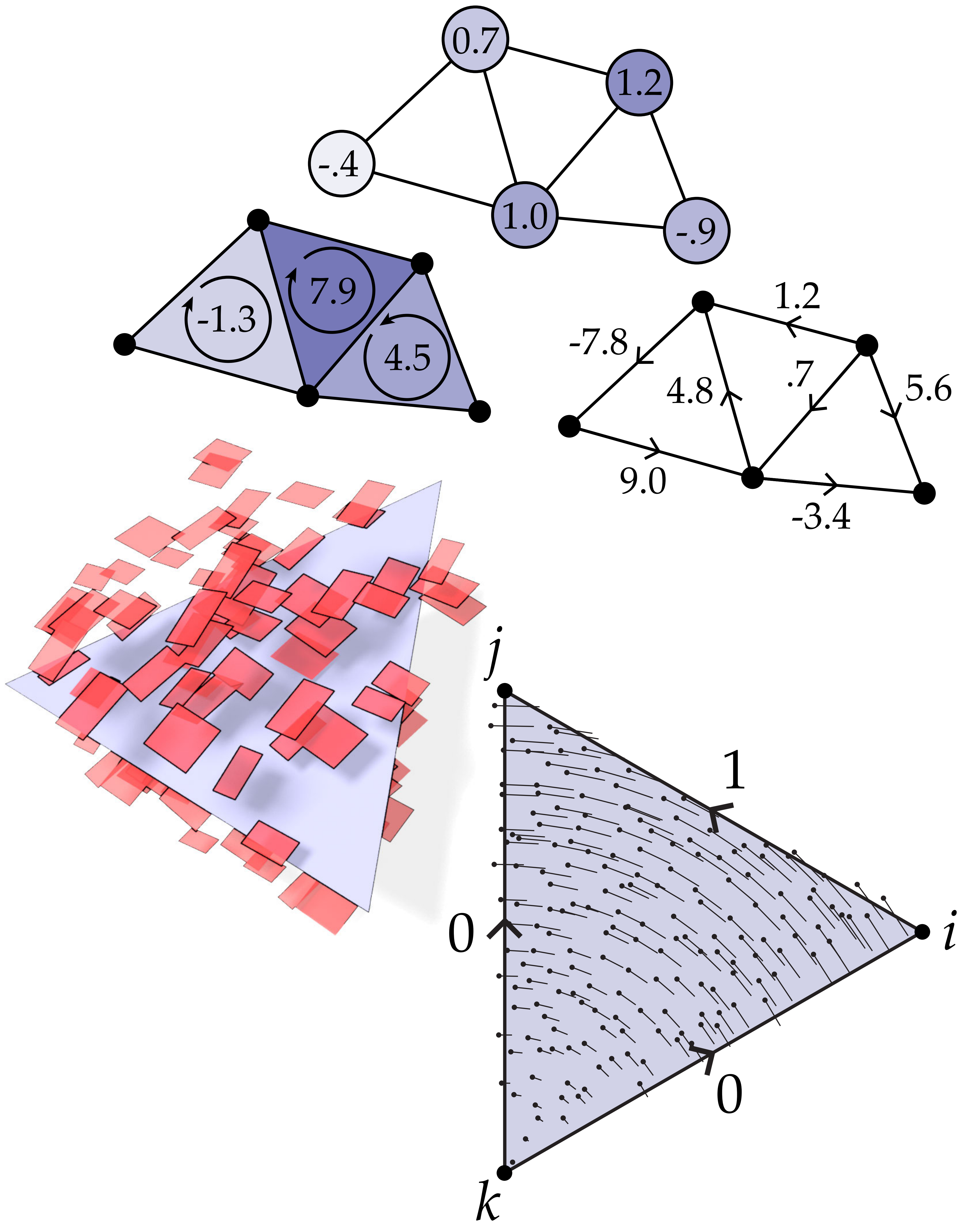
Review—Discrete Differential Forms

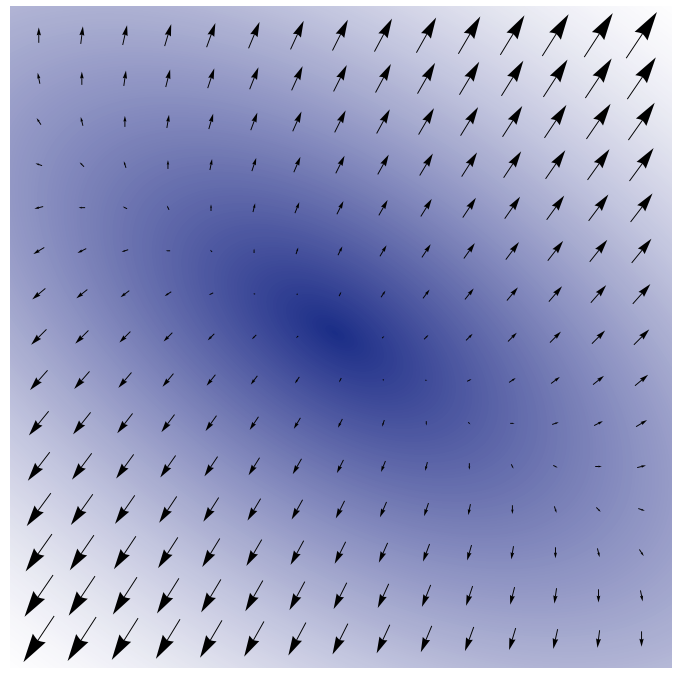
- A *discrete differential k-form* amounts to a value stored on each oriented *k*-simplex
- **Discretization:** integrate (continuous) differential *k*-form over each oriented *k*-simplex
- **Interpolation:** take linear combinations of *Whitney bases* to get continuous differential *k*-form
- How do we actually "do stuff" with this data?
- This lecture: **calculus** on discrete differential forms
  - differentiation—*discrete* exterior derivative
    - key tool: Stokes' theorem
  - integration—just take sums!
  - Hodge star—approximate integral over dual cells



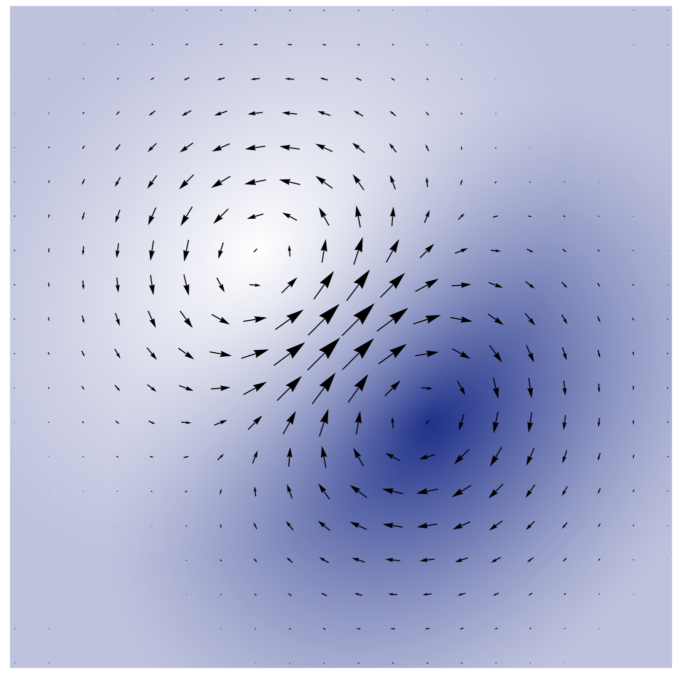
Discrete Exterior Derivative

Reminder: Exterior Derivative

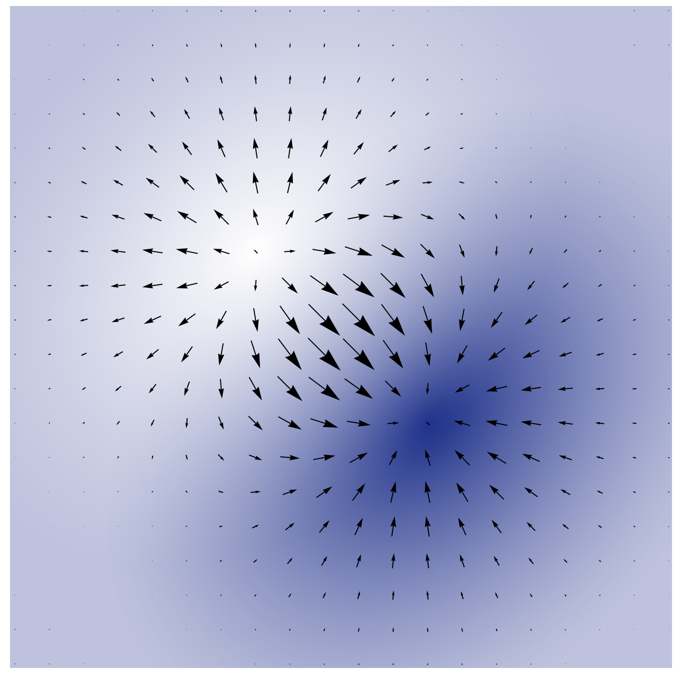
- Recall that in the smooth setting, the exterior derivative:
  - maps differential k-forms to differential $(k+1)$-forms
  - satisfies a product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
  - yields zero when you apply it twice: $d \circ d = 0$
  - is similar to the *gradient* when applied to a 0-form
  - is similar to *curl* when applied to a 1-form
  - is similar to *divergence* when composed w/ Hodge star
- To get **discrete** exterior derivative, we will imagine that we apply the exterior derivative to a continuous k-form and integrate the result over (oriented) simplices



$d\phi$



$d\alpha$

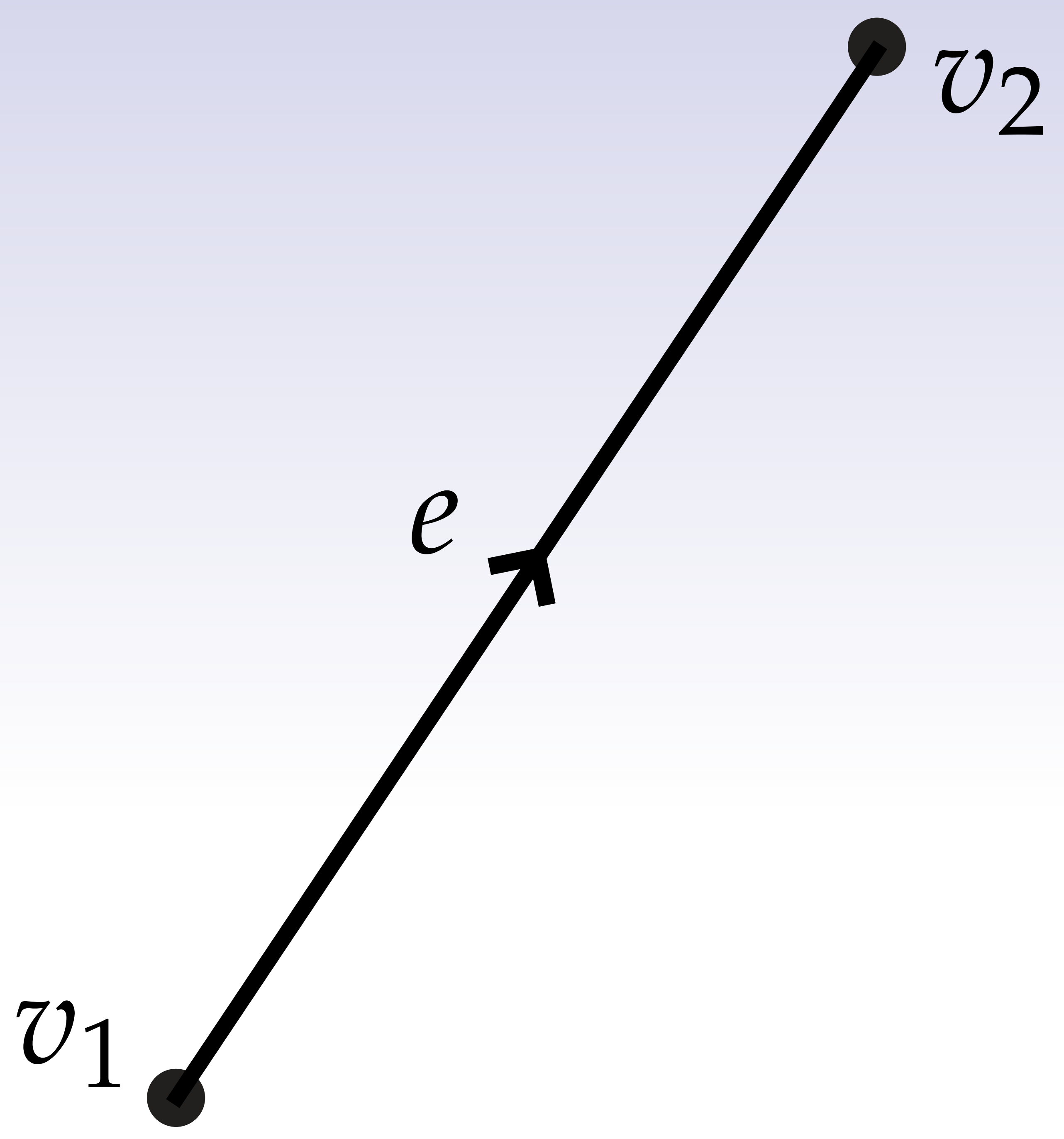


$d \star \alpha$

Discrete Exterior Derivative (0-Forms)

$\hat{\phi}$ — discrete 0-form (*values of $\hat{\phi}$ at vertices*)

$\widehat{d\phi}$ — discrete 1-form (*integrals of $d\phi$ along edges*)



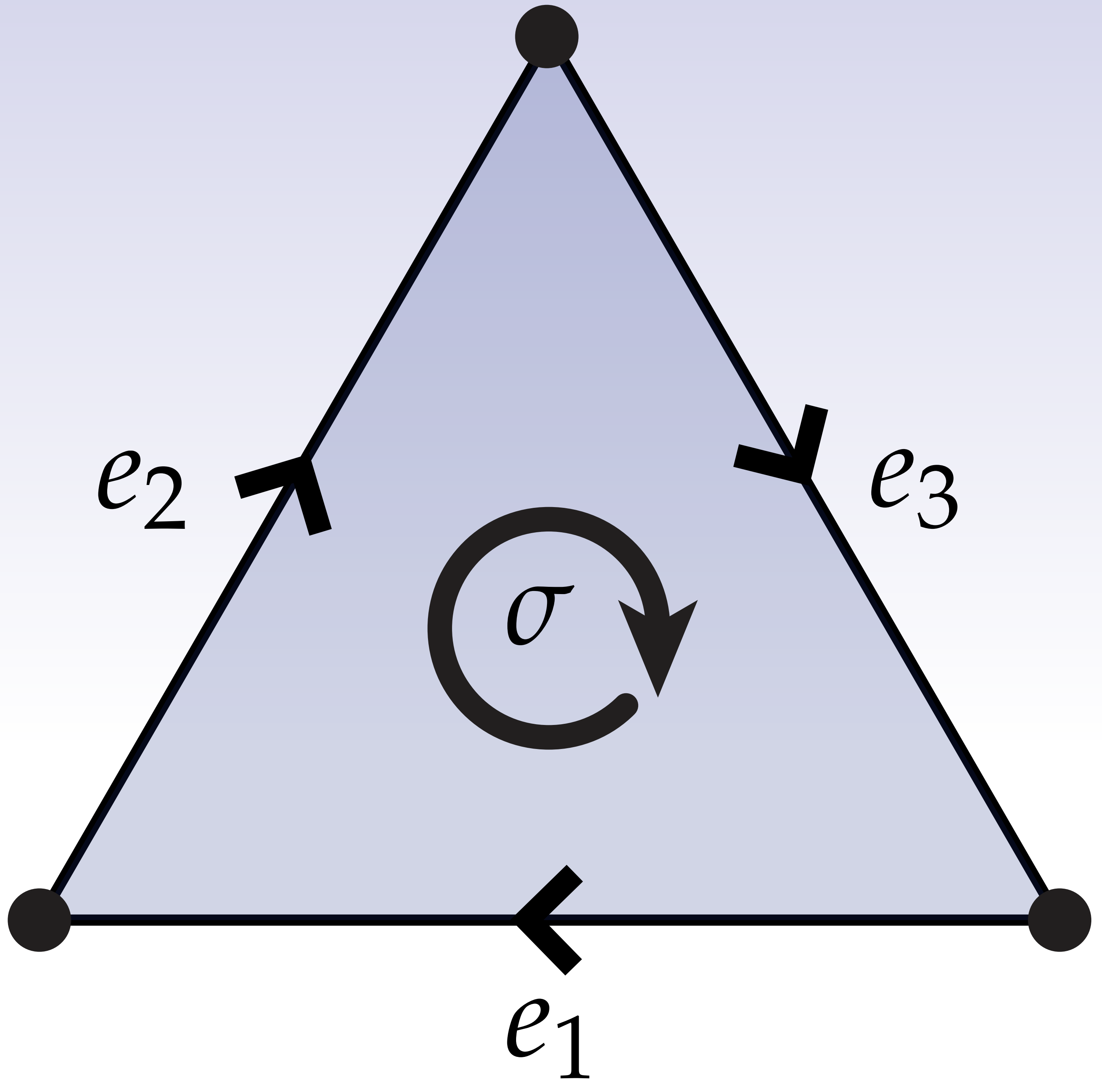
$$(\widehat{d\phi})_e = \int_e d\phi \; = \int_{\partial e} \phi \; = \hat{\phi}_2 - \hat{\phi}_1$$

Key idea: even if we only know ϕ at endpoints, can <u>exactly</u> integrate derivative along whole edge

Discrete Exterior Derivative (1-Forms)

$\hat{\alpha}$ — primal 1-form (*integrals of α along edges*)

$\widehat{d\alpha}$ — primal 2-form (*integrals of $d\alpha$ over triangles*)



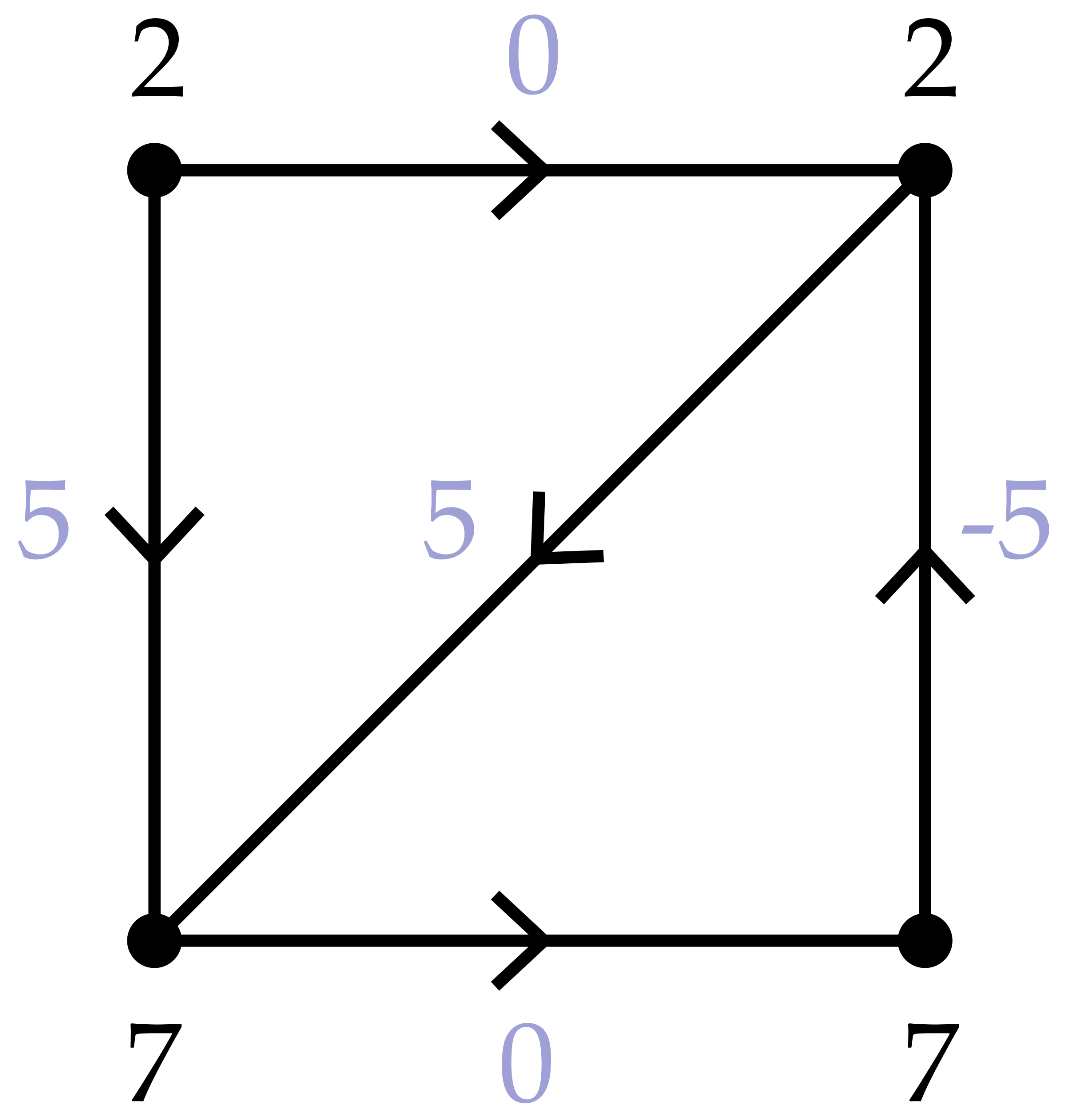
$$(\widehat{d\alpha})_\sigma = \int_\sigma d\alpha = \int_{\partial\sigma} \alpha = \sum_{i=1}^{3} \int_{e_i} \alpha = \sum_{i=1}^{3} \hat{\alpha}_i$$

In general: discrete exterior derivative is *coboundary* operator for *cochains*.

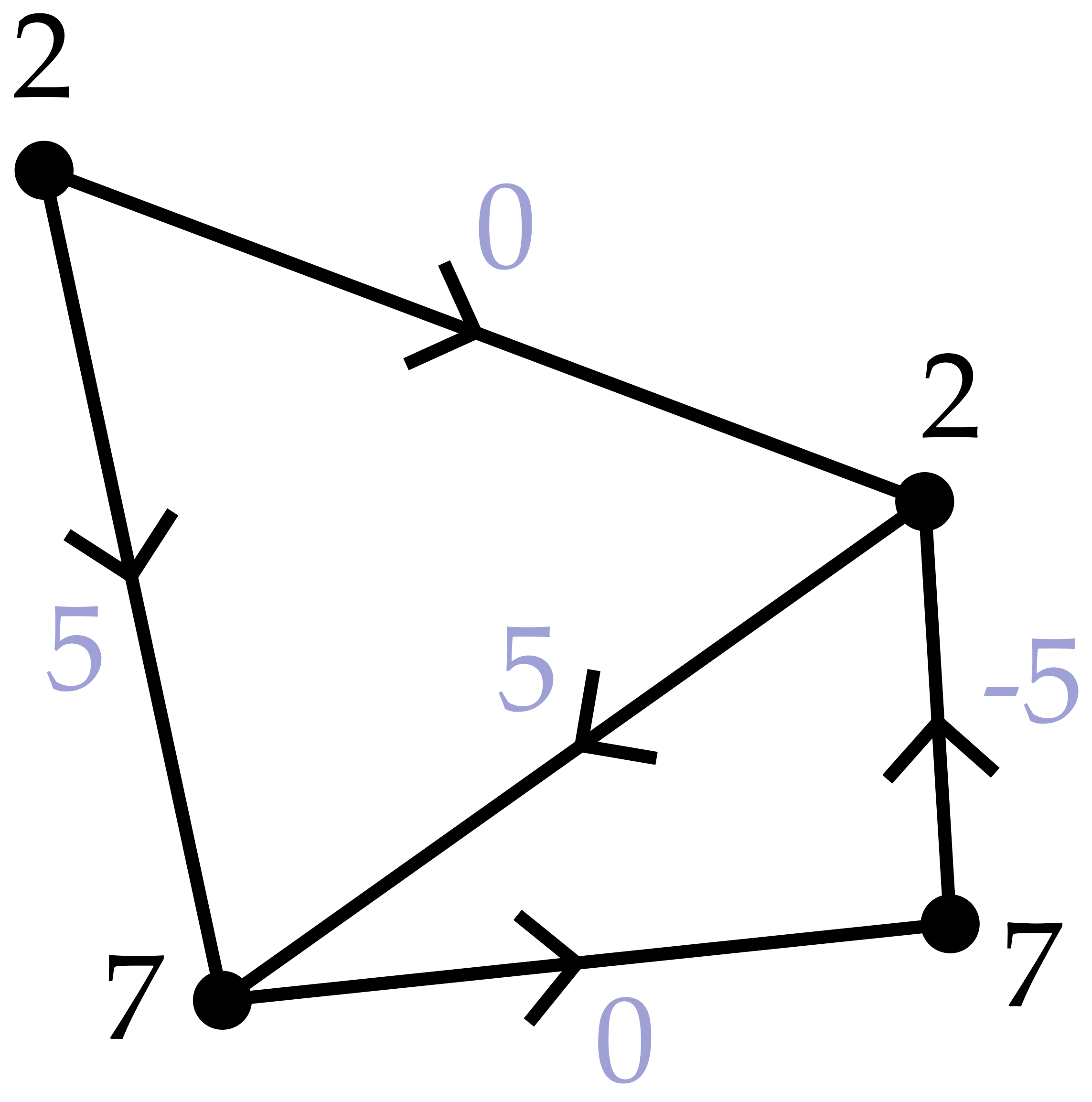
Discrete Exterior Derivative—Examples

When applying discrete exterior derivative, must carefully consider *orientation*

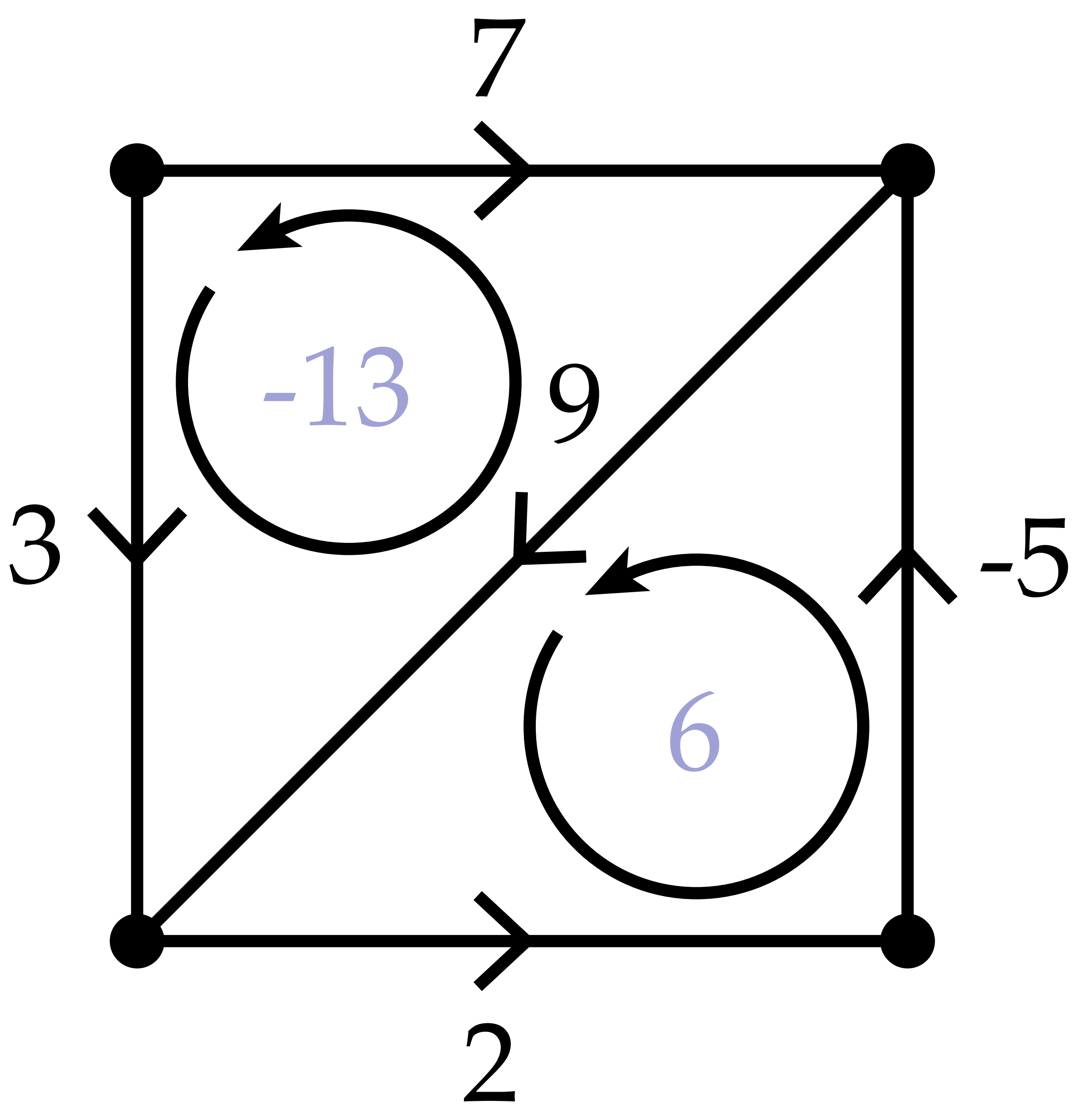
Example (0-form)



Note: exterior derivative has *nothing* to do with geometry!



Example (1-form)

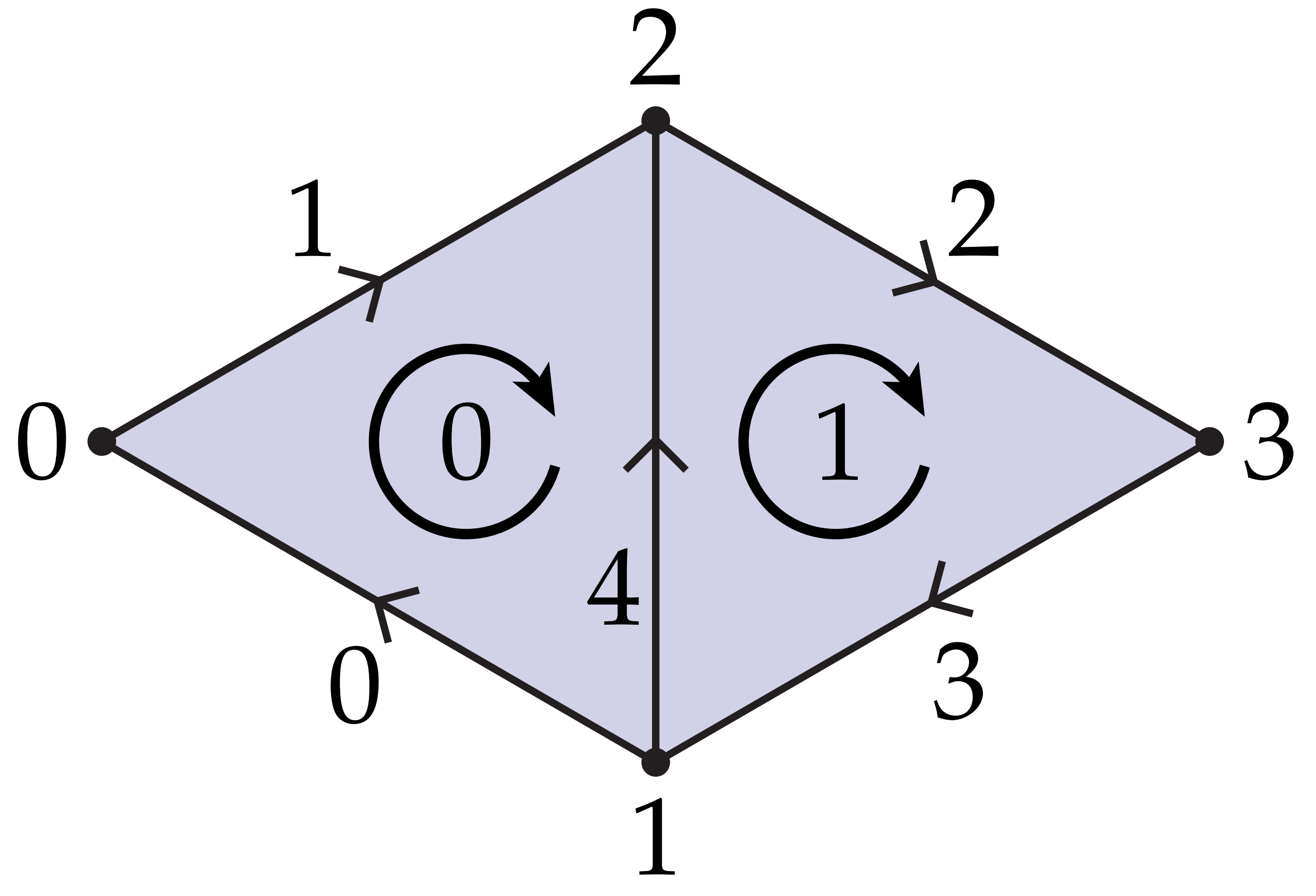


$3 - 9 - 7 = -13$

$9 + 2 + (-5) = 6$

Discrete Exterior Derivative—Matrix Representation

- The discrete exterior derivative on discrete k-forms, denoted by d_k, is a linear map from values on k-simplices to values on $(k+1)$-simplices:
  - d_0 maps values on vertices to values on edges
  - d_1 maps values on edges to values on triangles
  - d_2 maps values on triangles to values on tetrahedra
  - …
  - stops at $k = n-1$ (where n is dimension)
- Can encode each operator as a matrix, by assigning indices to mesh elements
- Matrix representations of exterior derivatives are then just the *signed incidence matrices*



$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

$$E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

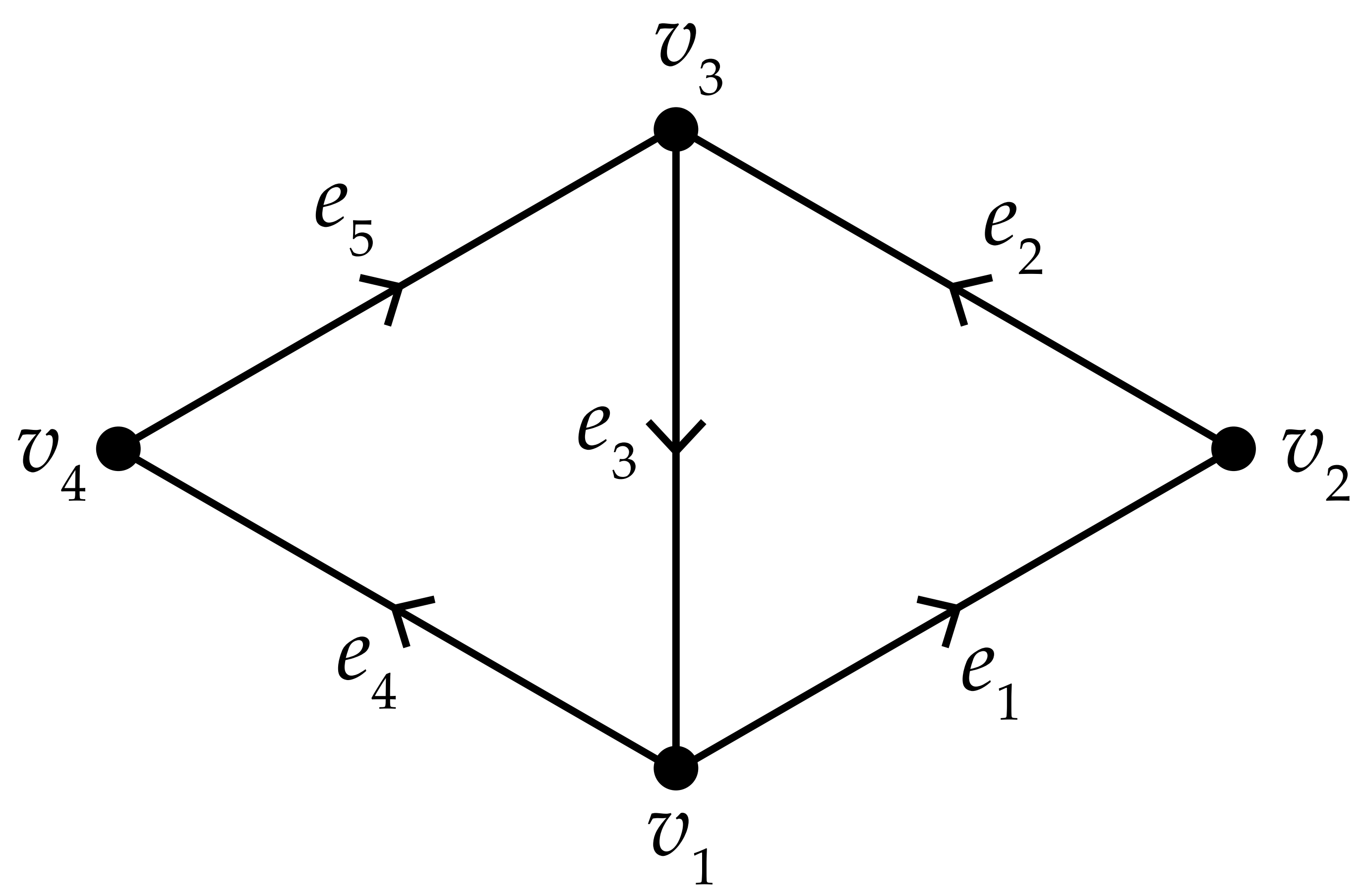
Discrete Exterior Derivative d_0 *—Example*

- To build the exterior derivative on 0-forms, we first need to assign an index to each *vertex* and each *edge*
  - A discrete 0-form is a vector of $|V|$ values (one per vertex)
  - A discrete 1-form is a vector of $|E|$ values (one per edge)
- The discrete exterior derivative d_0 is therefore a $|E| \times |V|$ matrix, taking values at vertices to values at edges

$$\phi \in \mathbb{R}^{|V|}$$

$$\alpha \in \mathbb{R}^{|E|}$$

$$d_0 \in \mathbb{R}^{|E| \times |V|}$$



$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{d_0} \end{array} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}}_{\alpha}$$

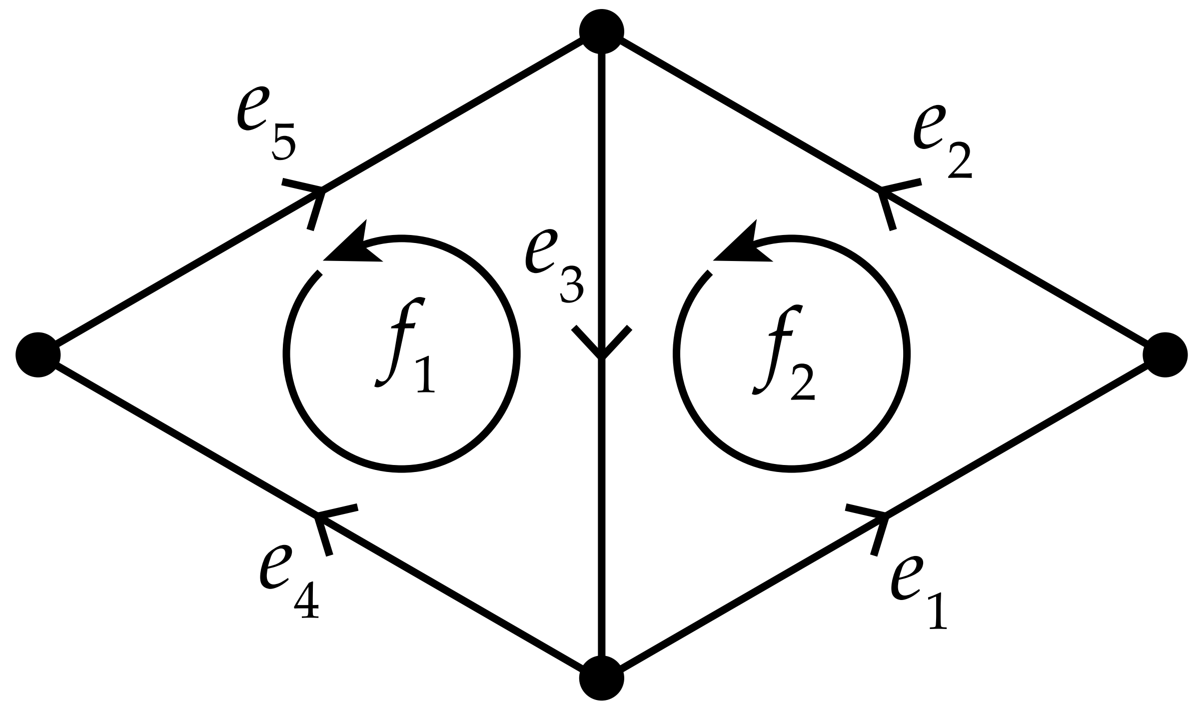
Discrete Exterior Derivative d_1 — Example

- To build the exterior derivative on 1-forms, we first need to assign an index to each *edge* and each *face*
  - A discrete 1-form is a list of $|E|$ values (one per edge)
  - A discrete 2-form is a list of $|F|$ values (one per face)
- The discrete exterior derivative d_1 is therefore a $|F|$x$|E|$ matrix, taking values at edges to values at faces

$$\alpha \in \mathbb{R}^{|E|}$$

$$\omega \in \mathbb{R}^{|F|}$$

$$d_1 \in \mathbb{R}^{|F|\times|E|}$$

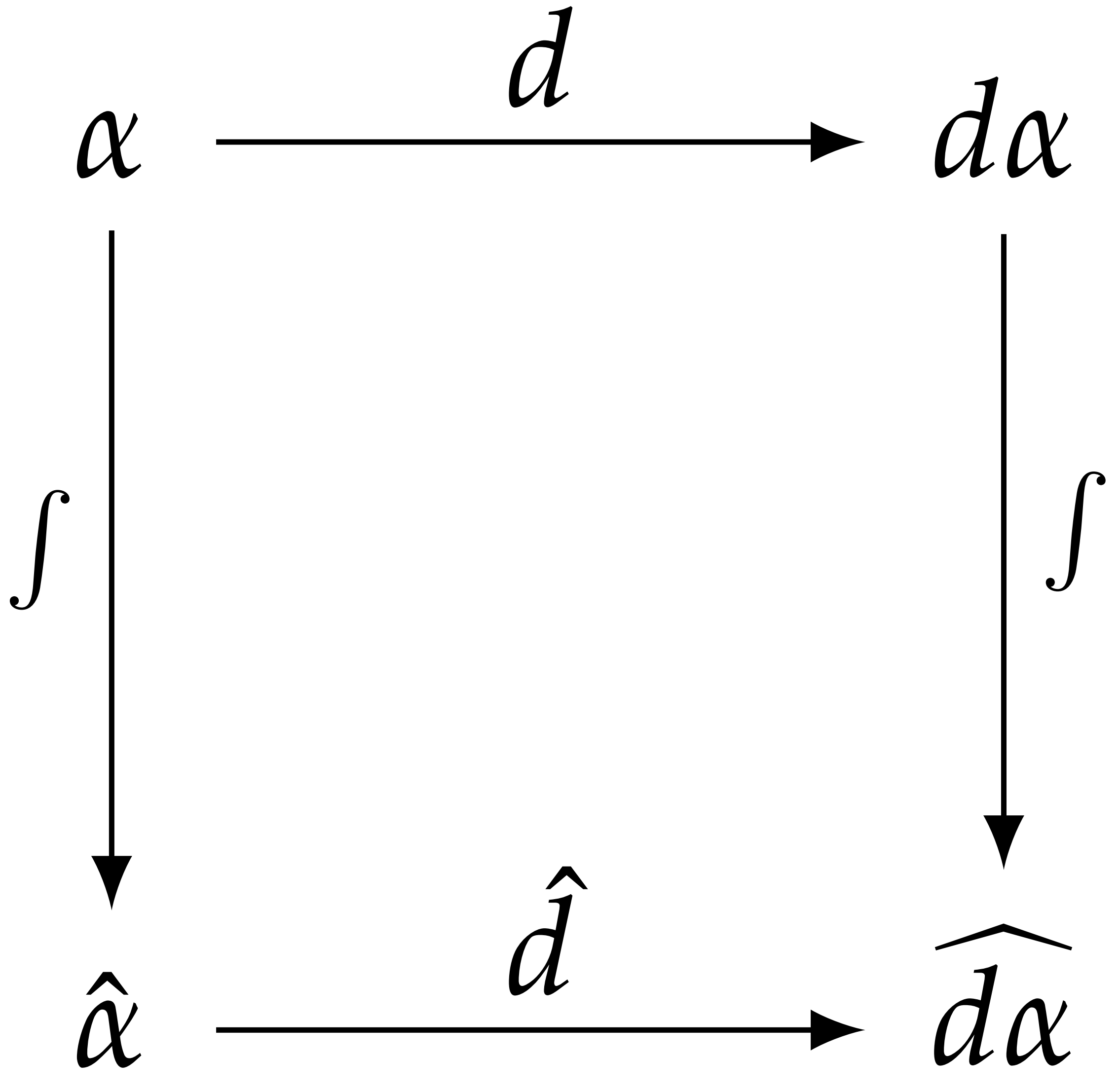


$$\begin{array}{c} \\ f_1 \\ f_2 \end{array}\overset{\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array}}{\underbrace{\begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}}_{d_1}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}}_{\omega}$$

Exterior Derivative Commutes w/ Discretization

By construction, discrete exterior derivative satisfies an important property:

> Taking the **smooth** exterior derivative and then discretizing yields the same result as *discretizing* and then applying the **discrete** exterior derivative.



d	—	smooth exterior derivative
$\hat{d}$	—	discrete exterior derivative
$\int$	—	de Rham map (discretization)
α	—	smooth k-form
$\hat{\alpha}$	—	discrete k-form
$d\alpha$	—	smooth $(k+1)$-form
$\widehat{d\alpha}$	—	discrete $(k+1)$-form

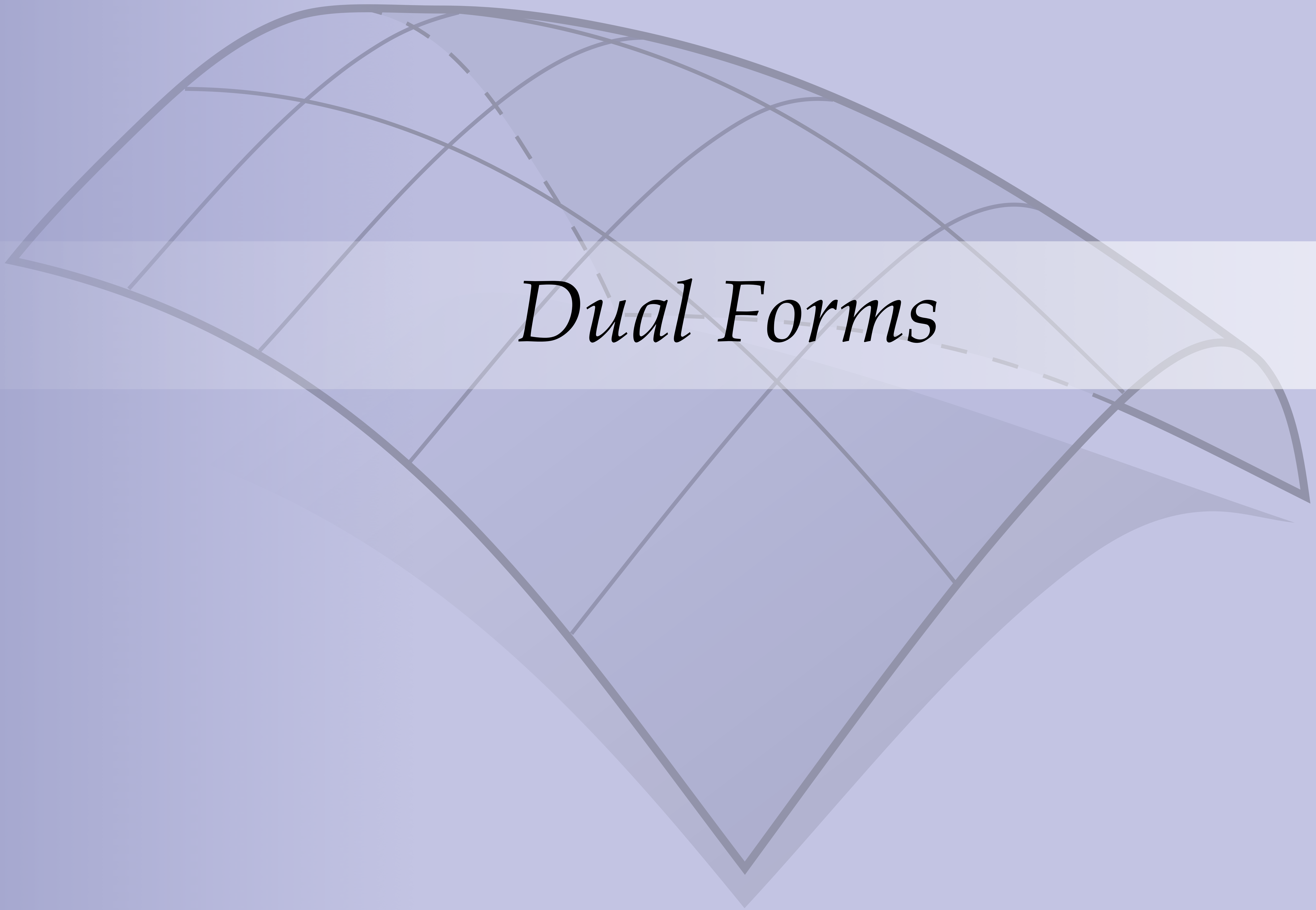
Corollary: applying discrete d twice yields zero (why?)

Exactness of Discrete Exterior Derivative—Example

To verify that applying discrete exterior derivative twice yields zero, could also just multiply exterior derivative matrices for 0- and 1-forms:

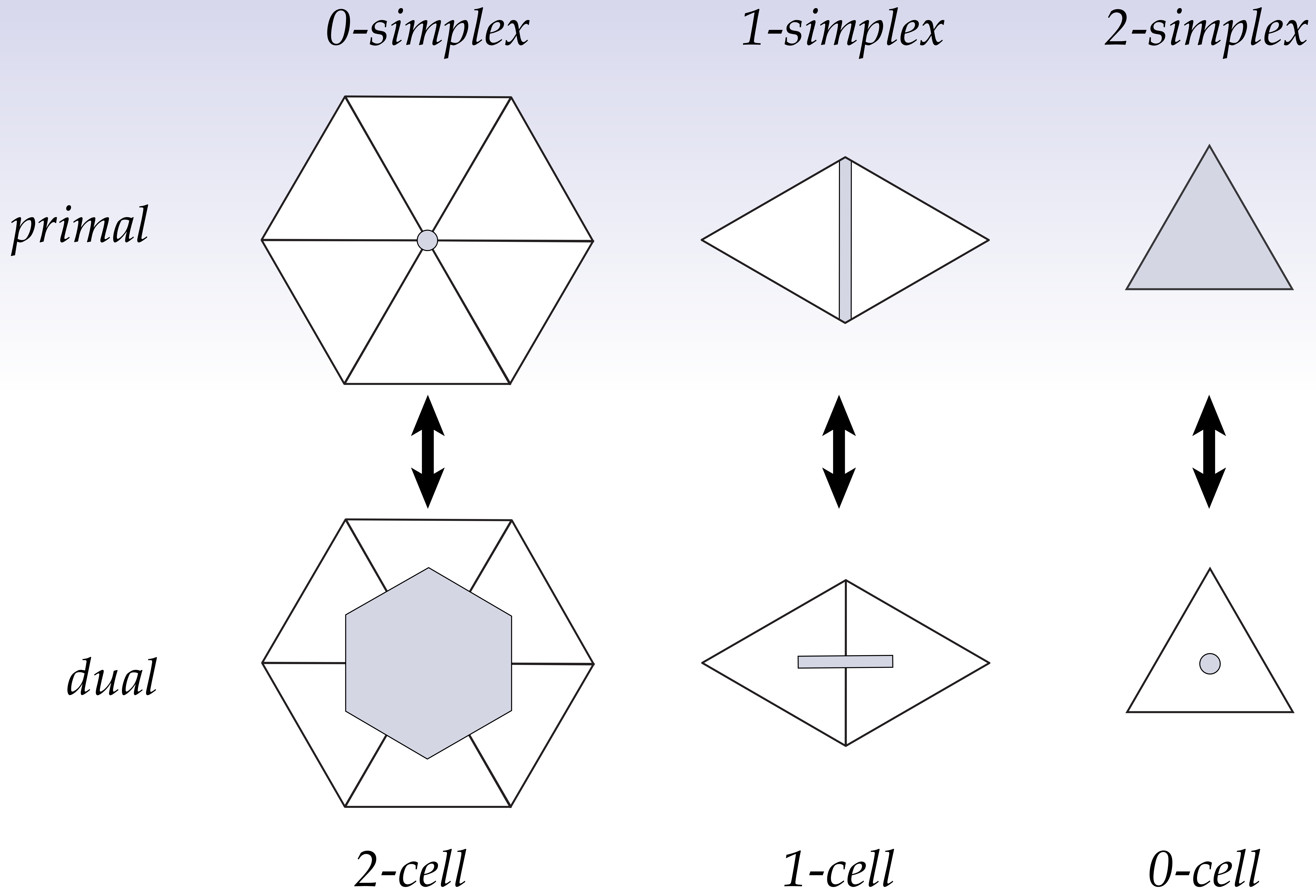
$$d_1 d_0 = \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Another interpretation: <u>co</u>boundary of <u>co</u>boundary is always zero!



Dual Forms

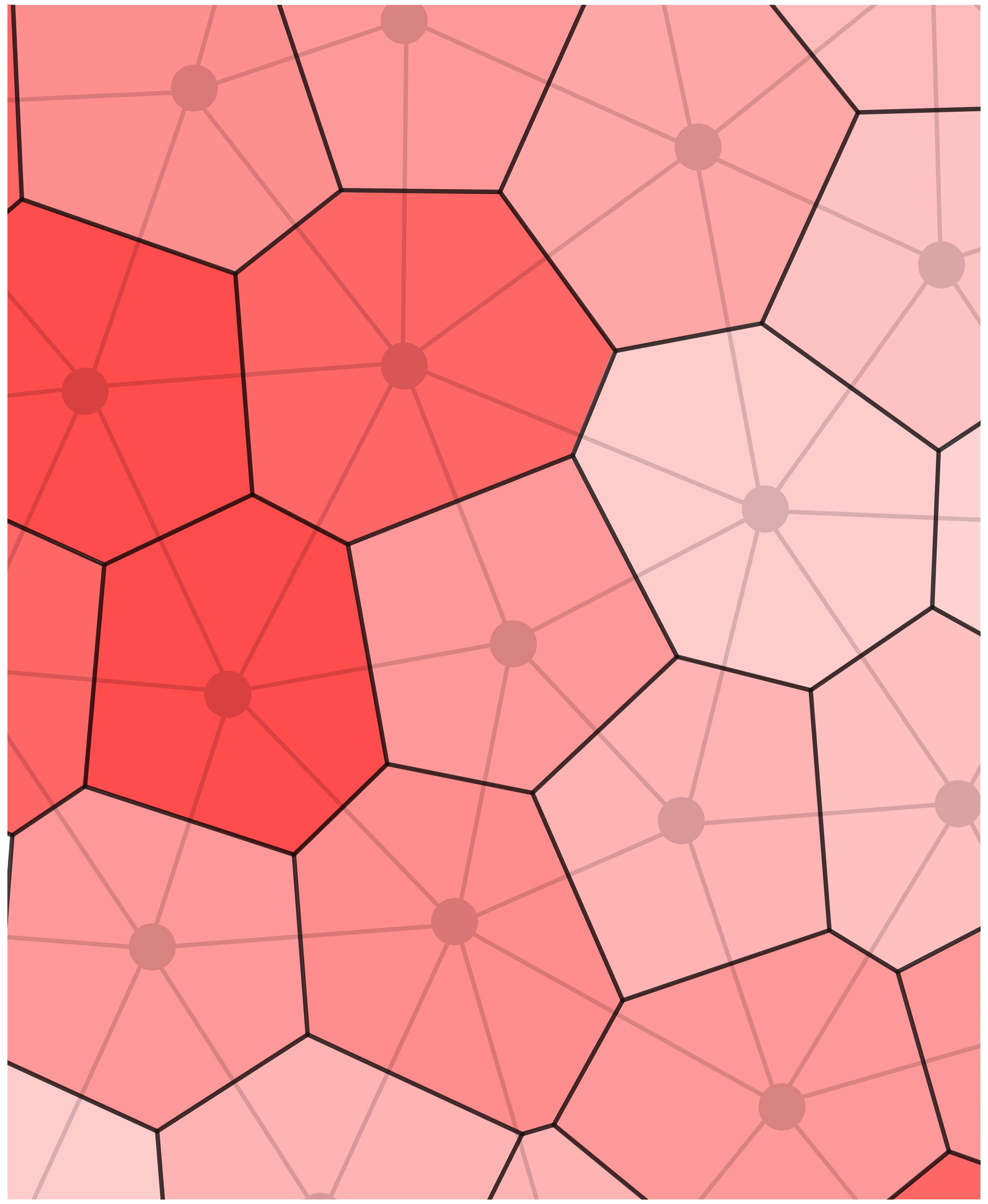
Reminder: Poincaré Duality



Dual Discrete Differential k-Form

Just as a discrete differential k-form was a value per k-simplex, a *dual discrete differential k-form* is a value per **dual** k-cell:

- a *dual 0-form* is a value per **dual** vertex
- a *dual 1-form* is a value per **dual** edge
- a *dual 2-form* is a value per **dual** cell



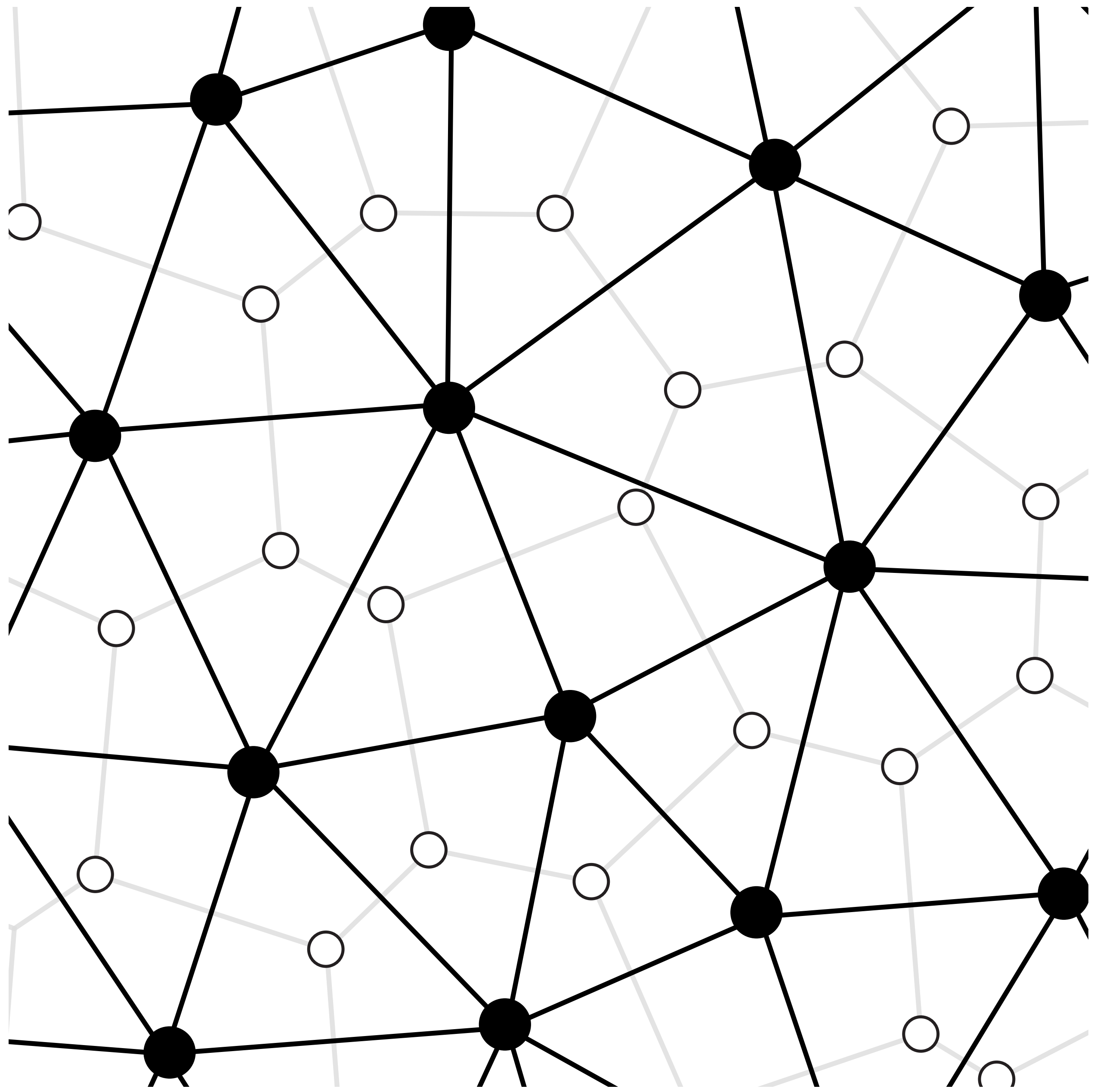
dual 2-form

(Can also formalize via dual chains, dual cochains…)

Primal vs. Dual Discrete Differential k-Forms

Let's compare primal and dual discrete *k*-forms on a triangle mesh (n=2):

	primal	**dual**
0-forms	vertices	dual vertices (*triangles*)
1-forms	edges	dual edges (*edges*)
2-forms	triangle	dual cells (*vertices*)



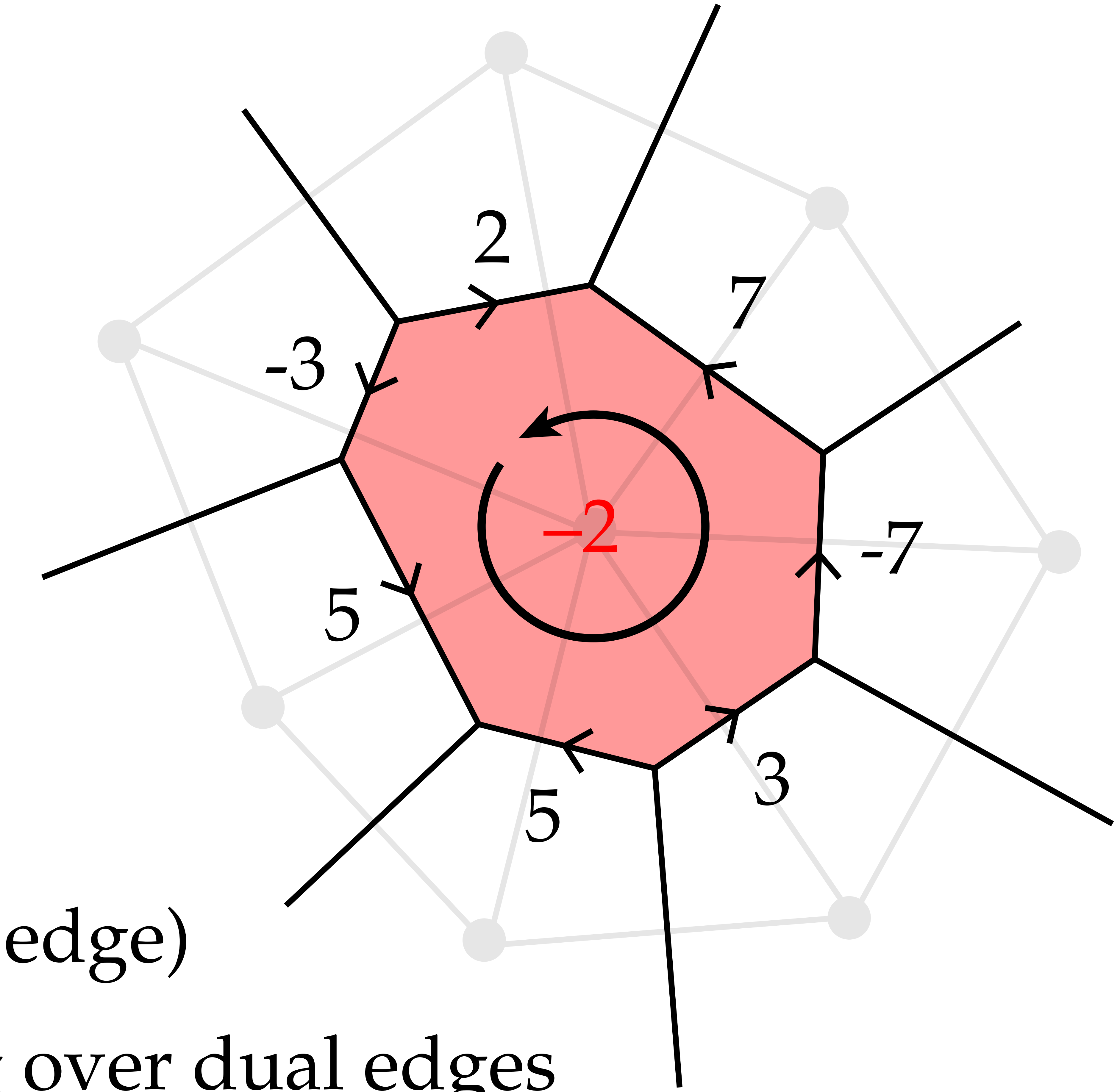
Note: no such thing as "primal" and "dual" forms in smooth setting!

Q: Is the number of values stored for a primal and dual *k*-form always the same?

A: No! In practice, store dual values on primal mesh (*e.g.*, dual 0-forms on triangles)

Dual Exterior Derivative

- Discrete exterior derivative on *dual* k-forms works in essentially the same way as for primal forms:
  - To get the derivative on a (k+1)-cell, sum up values on each k-cell along its boundary
  - Sign of each term in the sum is determined by relative orientation of (k+1)-cell and k-cell



<u>Example.</u>

Let α be a dual discrete 1-form (one value per dual edge)

Then $d\alpha$ is a value per 2-cell, obtained by summing over dual edges

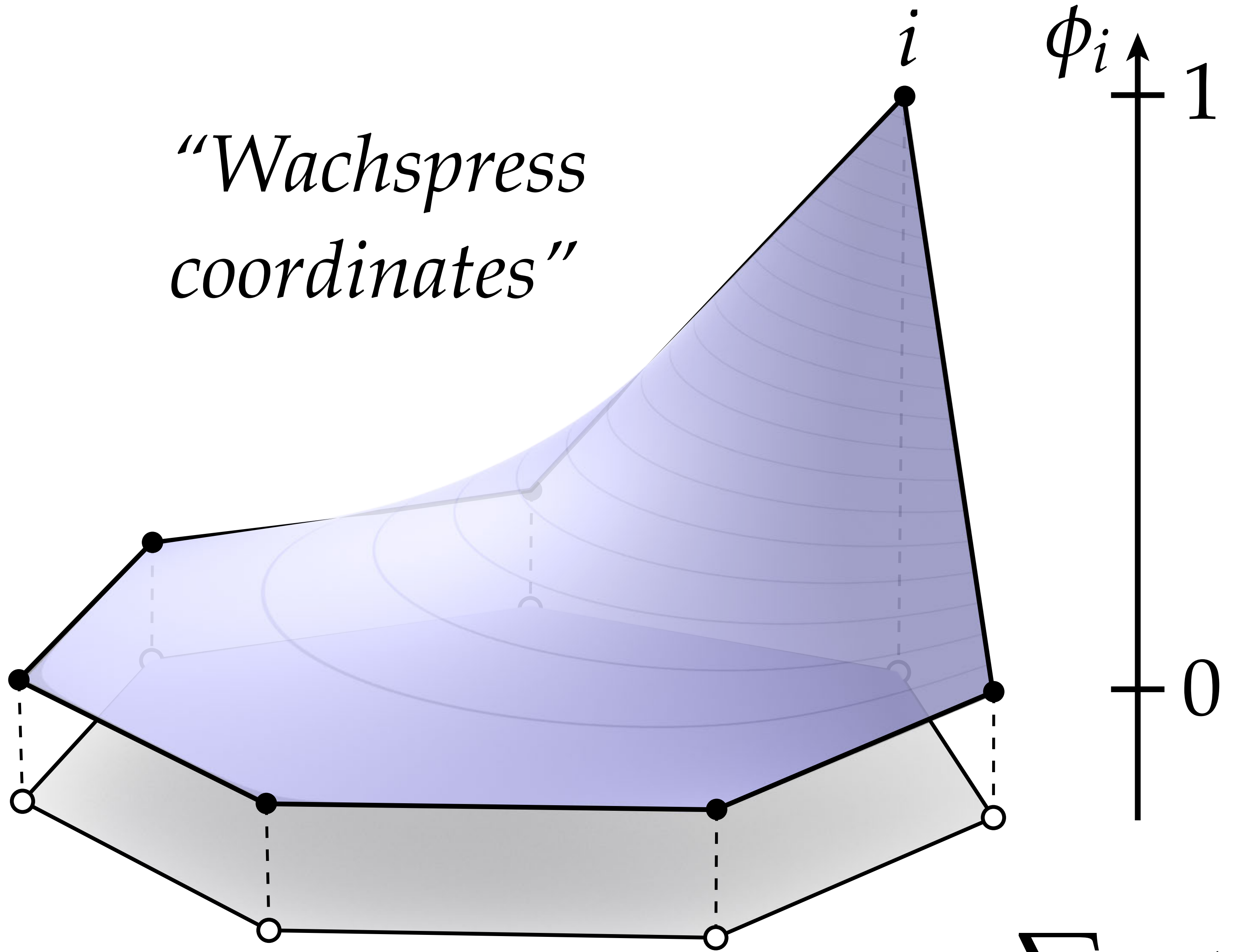
(As usual, relative orientation matters!)

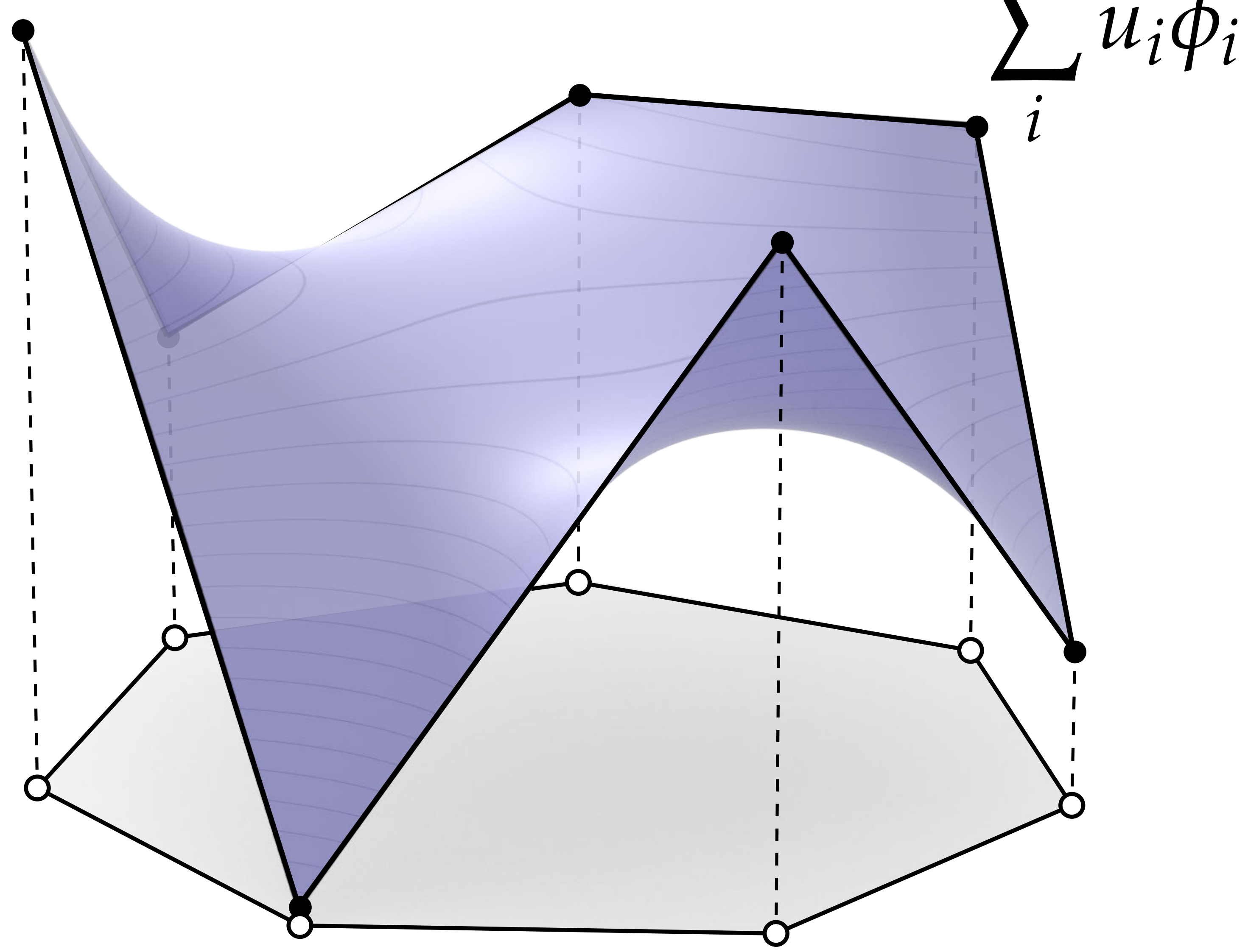
$-7 + 7 - 2 + (-3) + 5 - 5 + 3 = -2$

Notice: as with primal d, we don't need lengths, areas, …

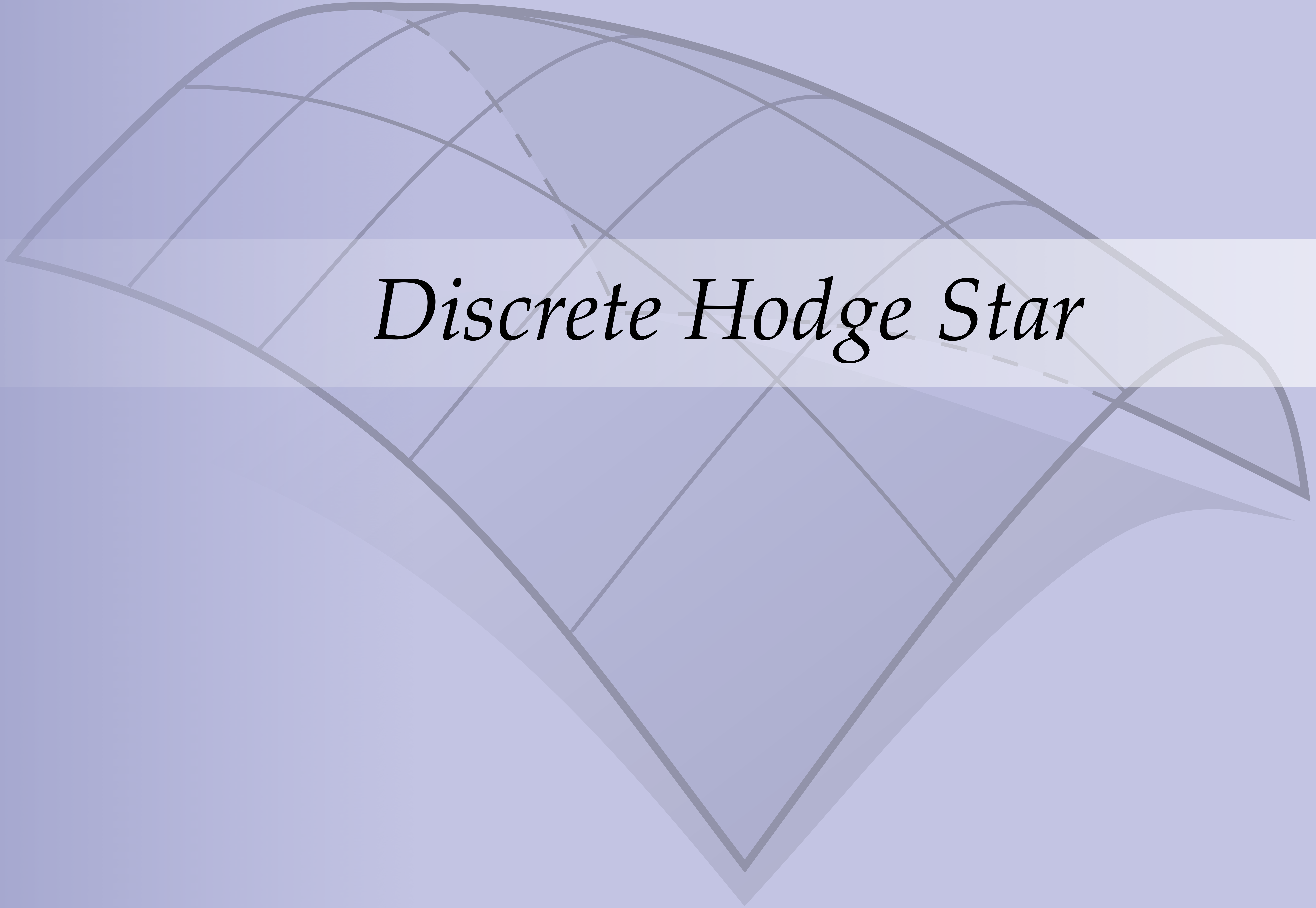
Dual Forms: Interpolation & Discretization

- Easy to interpolate primal *k*-forms:
  - *k*-simplices have clear geometry: *convex hull of vertices*
  - *k*-forms have straightforward basis: *Whitney forms*
- Not so clear cut for dual forms!
  - *e.g.*, can't interpolate dual 0-form with linear function
    - nonconvex cells even more challenging…
    - leads to *generalized barycentric coordinates* (no free lunch)
  - k-cells may not sit in a k-dimensional linear subspace
    - e.g., 2-cells in 3D can be non-planar
- Nonetheless, still easy to work with dual forms
  - e.g., discrete *d* still gives exact result, via Stokes' theorem



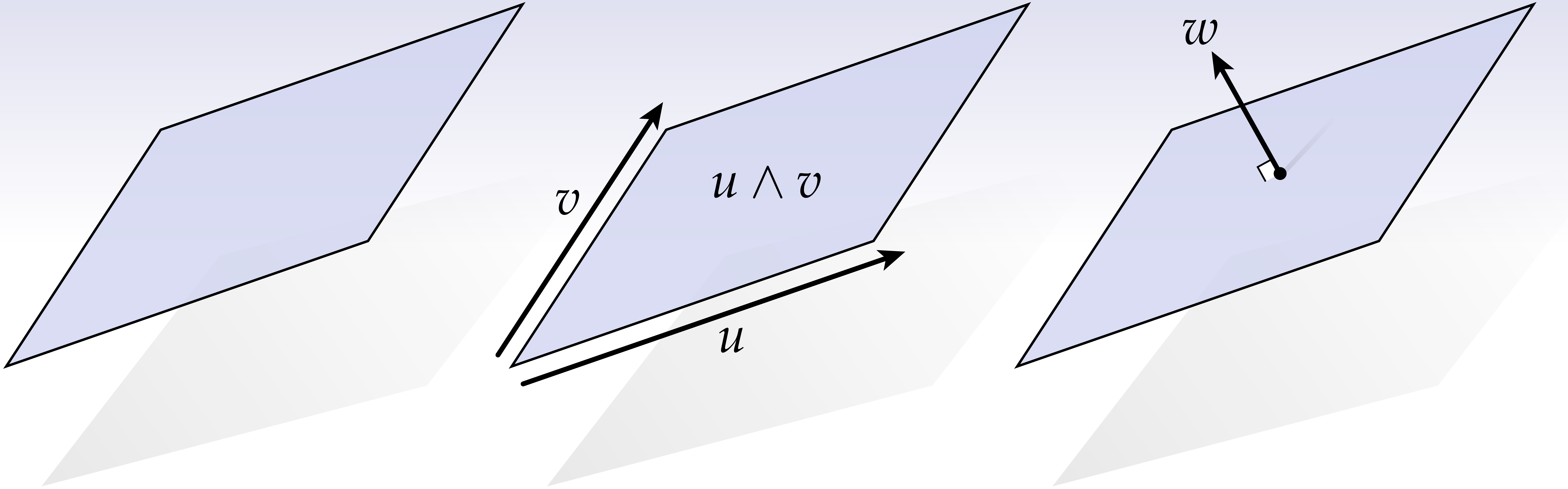


Discrete Hodge Star



Reminder: Hodge Star ($\star$)

$$\star(u \wedge v) = w$$

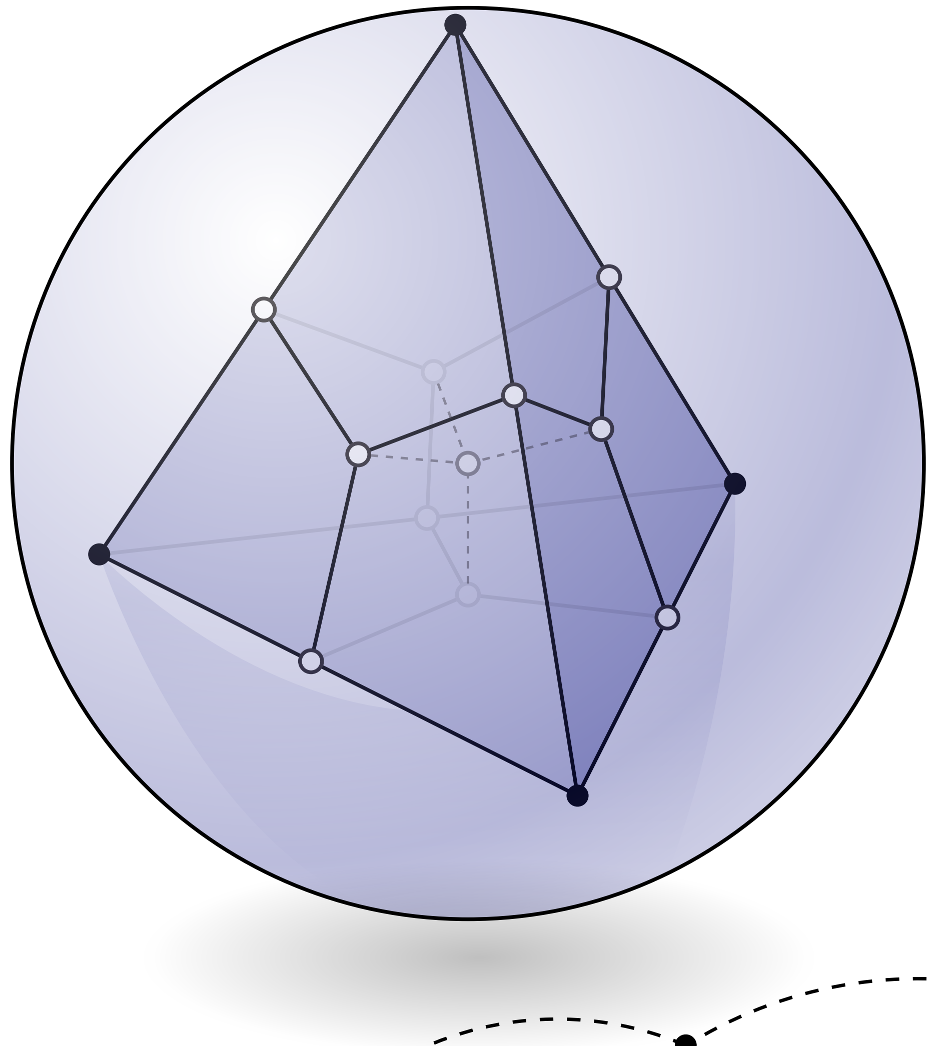


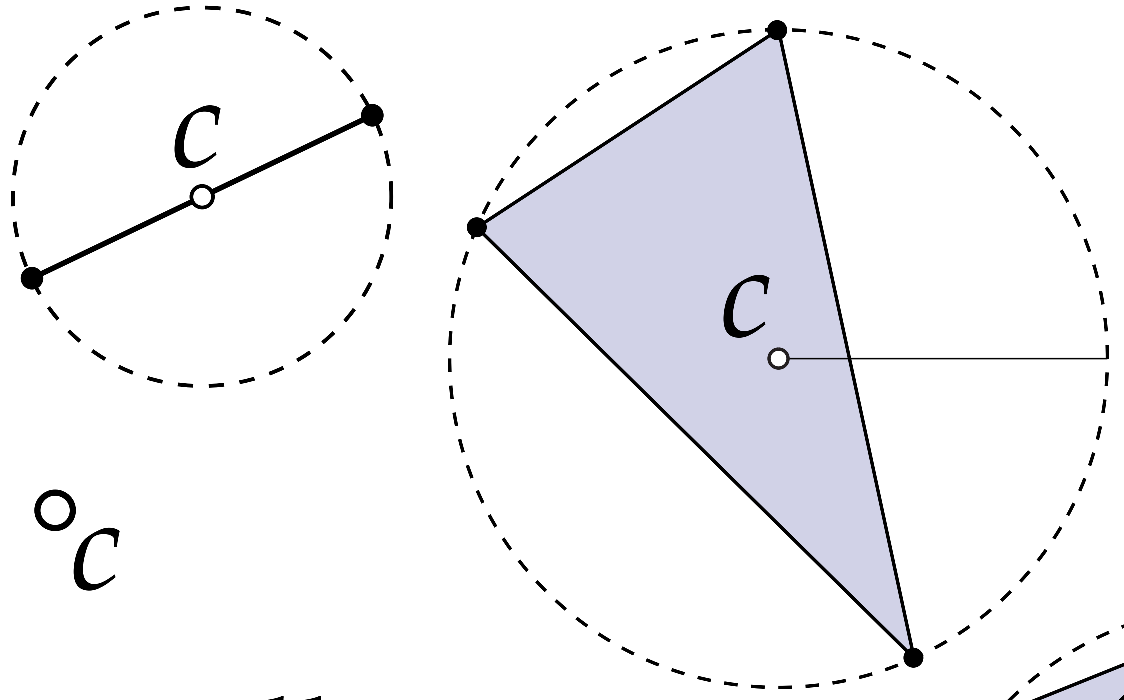
Analogy: *orthogonal complement*

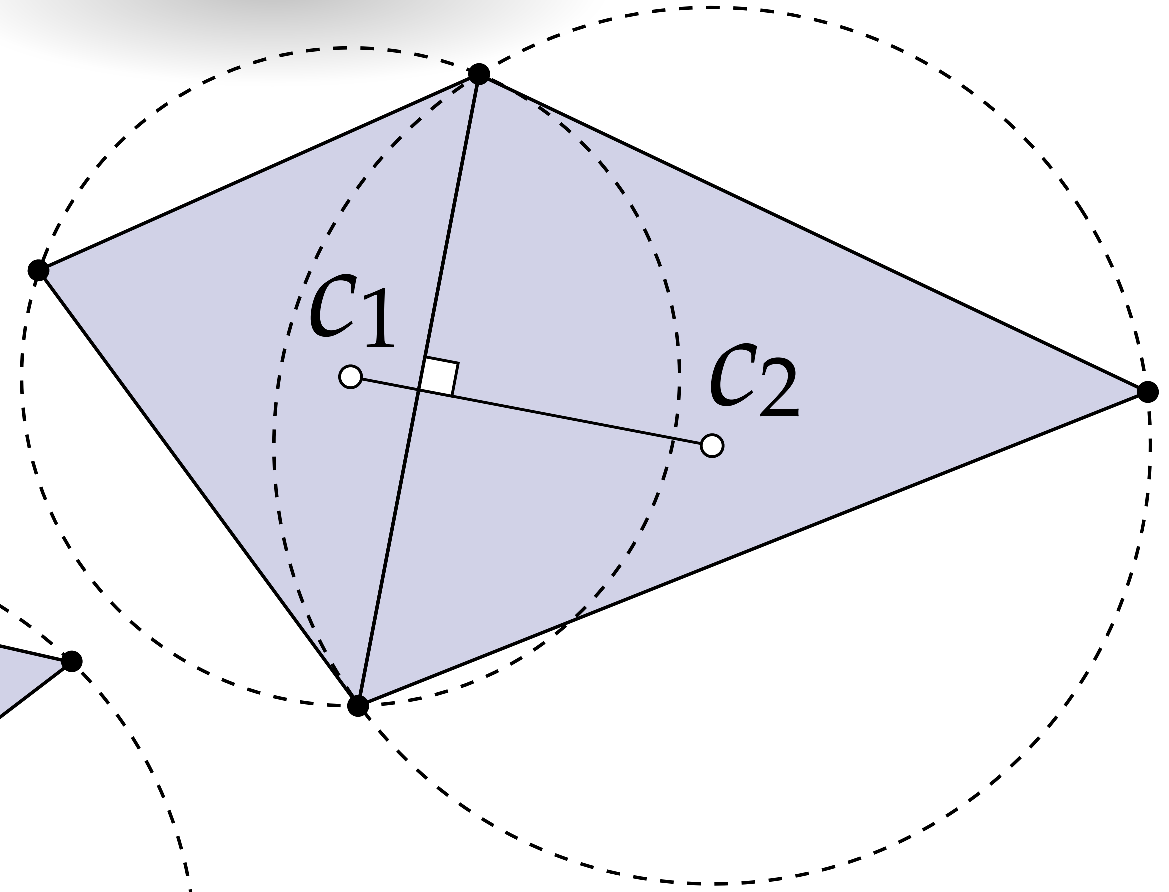
$$k \mapsto (n-k)$$

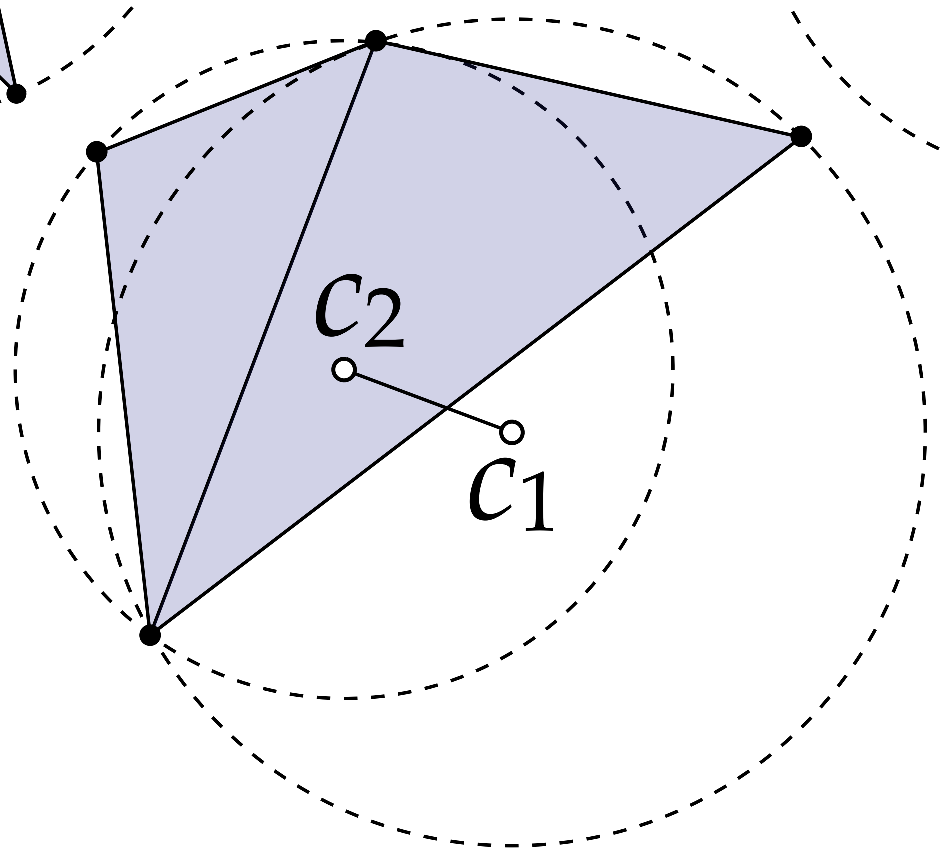
Geometry of Dual Complex

- For exterior derivative, needed only *connectivity* of the mesh
- For Hodge star, will also need a specific *geometry*
- Many possibilities, but typically use **circumcentric dual**
- **circumcenter** — center of smallest sphere through all vertices
  - 2-simplex: triangle circumcenter
  - 1-simplex: edge midpoint
  - 0-simplex: vertex itself
- **Fact:** primal & dual cells meet orthogonally
- Can yield negative signed lengths/areas/volumes



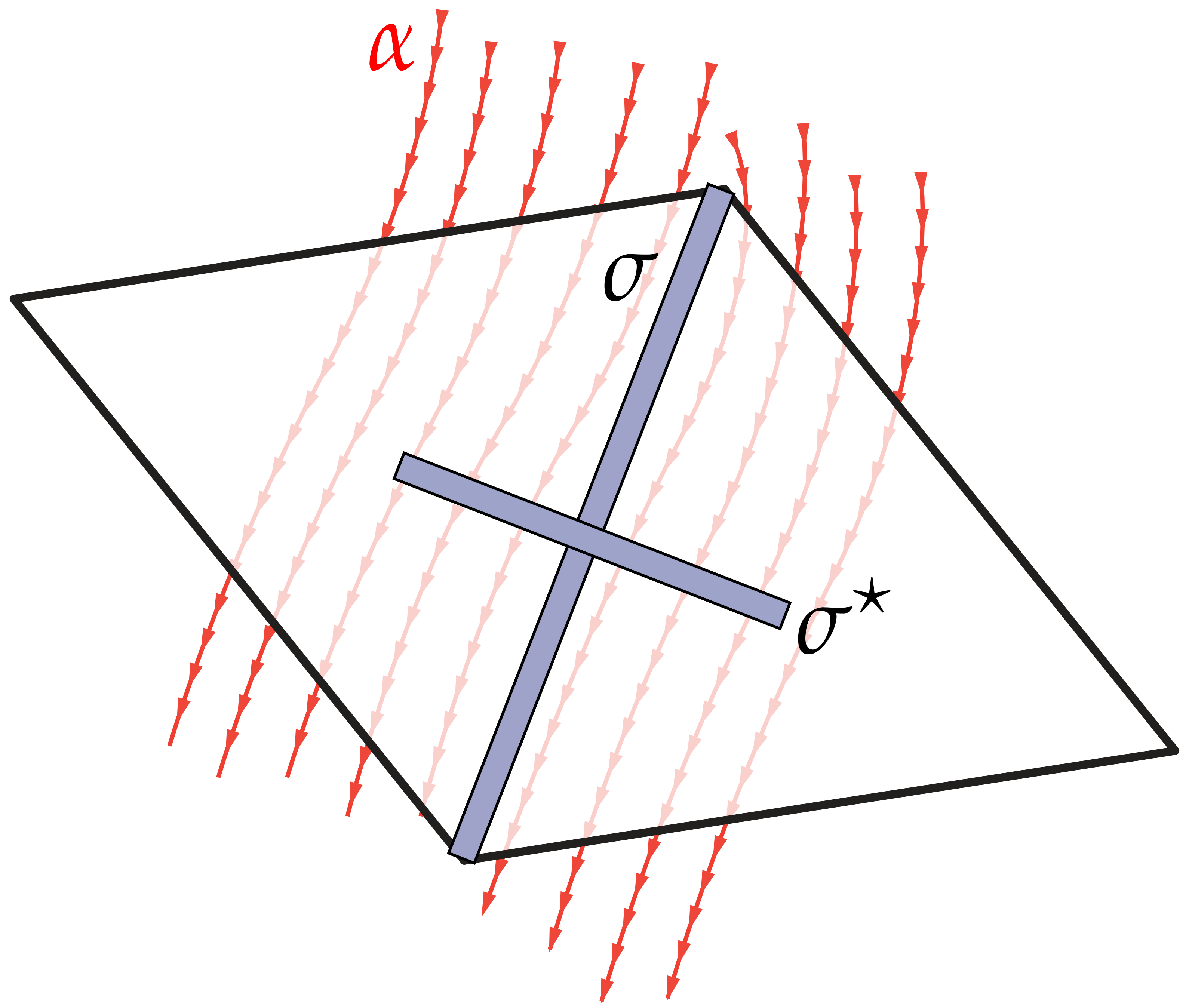






Discrete Hodge Star — Basic Idea

- Consider a k-simplex σ and dual $(n\text{-}k)$-cell $\sigma^\star$
- Integrating a k-form α over σ yields a value $\hat{\alpha}$
- Integrating $\star\alpha$ over $\sigma^\star$ yields a value $\widehat{\star\alpha}$
- **Q:** What, if anything is the relationship between these two values?
- **A:** Well, if α is constant, then they are the same up to a volume ratio
- If α is very smooth (or mesh elements small), this approximation will be reasonably good
- Hence, if we know integrals of α, we can get a good approximation of integrals of $\alpha^\star$



circulation *along* σ

$$\hat{\alpha} = \int_\sigma \alpha$$

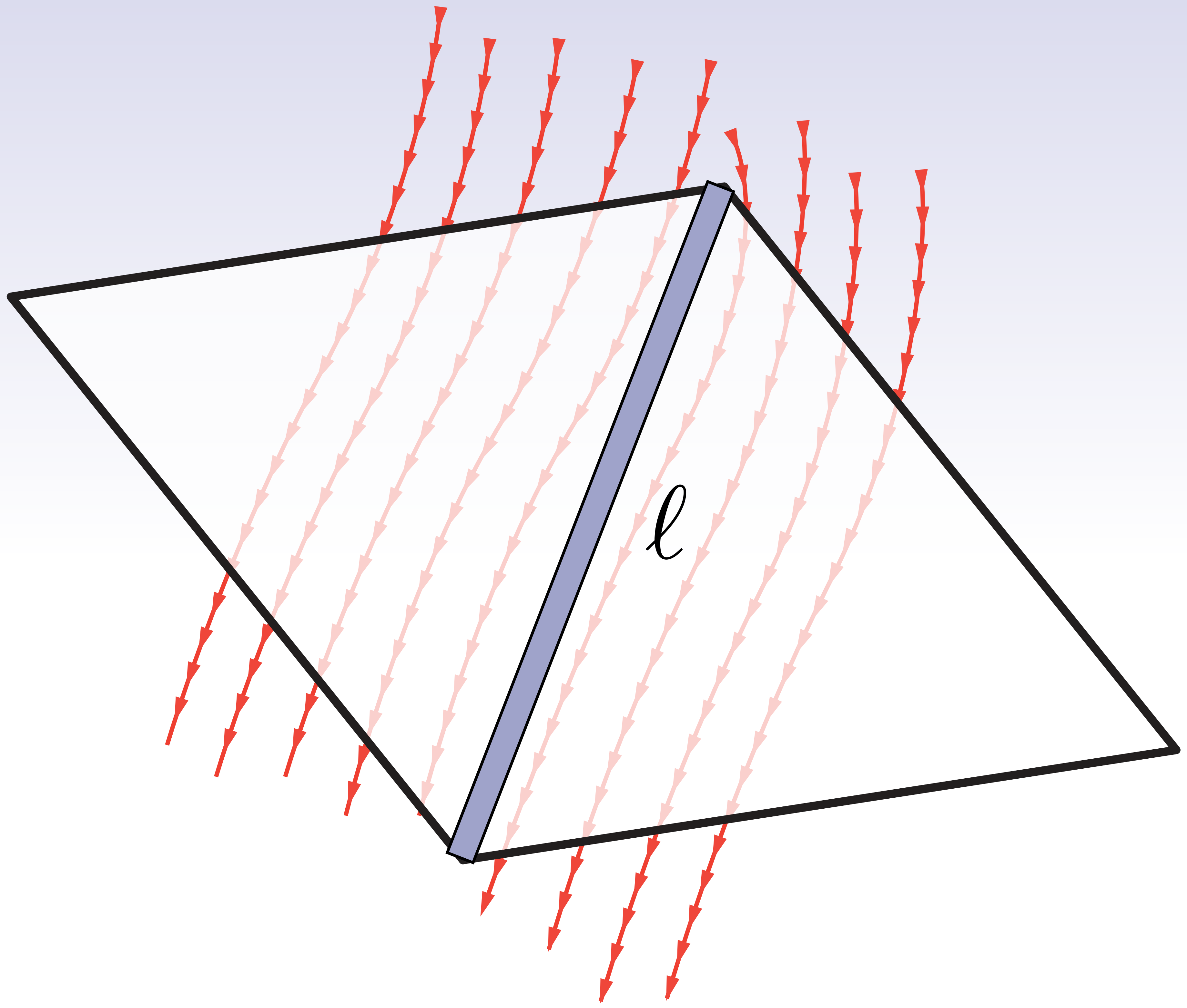
flux *through* $\sigma^\star$

$$\widehat{\star\alpha} = \int_{\sigma^\star} \star\alpha$$

If α is constant:

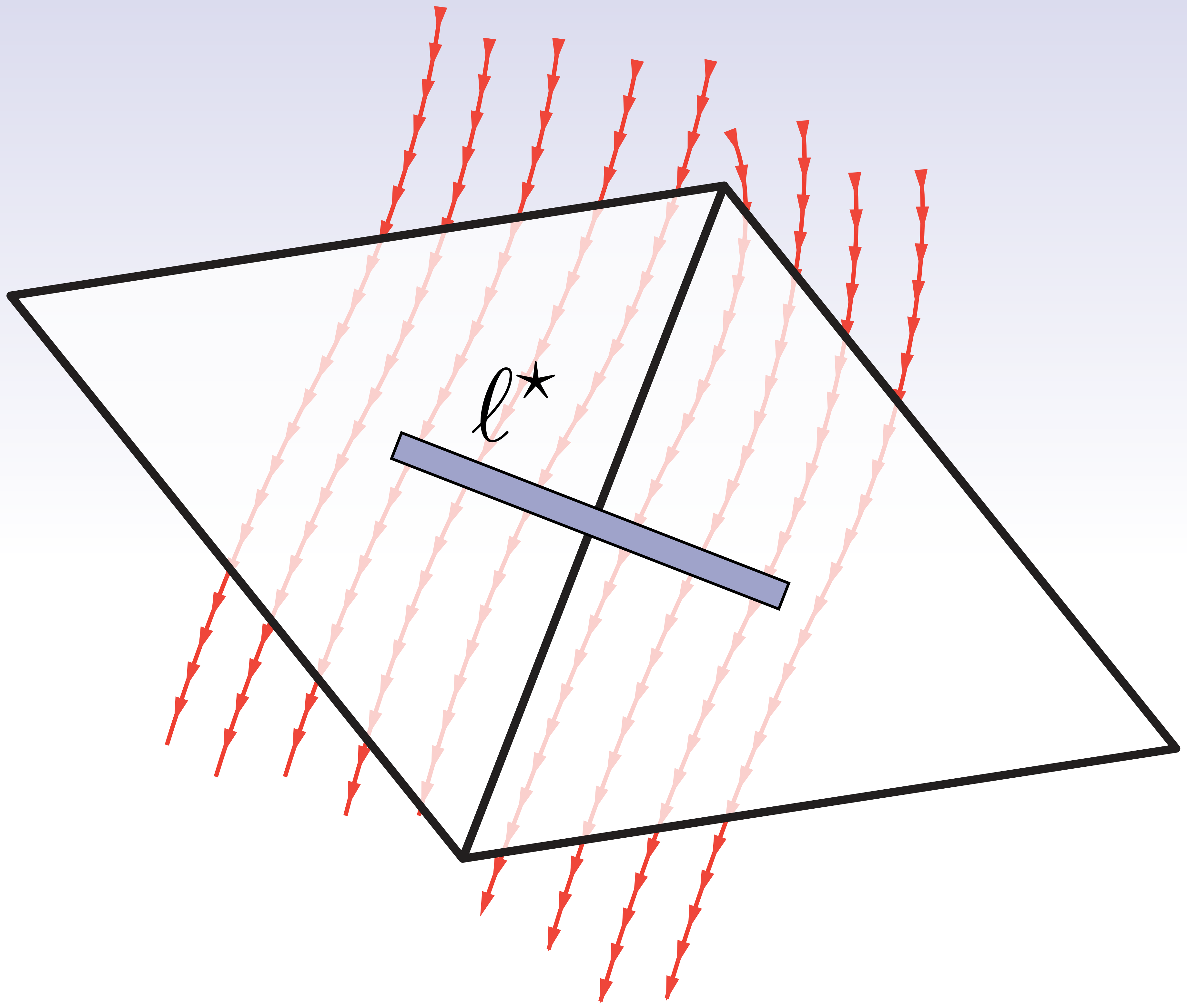
$$\frac{\widehat{\star\alpha}}{\hat{\alpha}} = \frac{|\sigma^\star|}{|\sigma|}$$

Discrete Hodge Star—1-forms in 2D



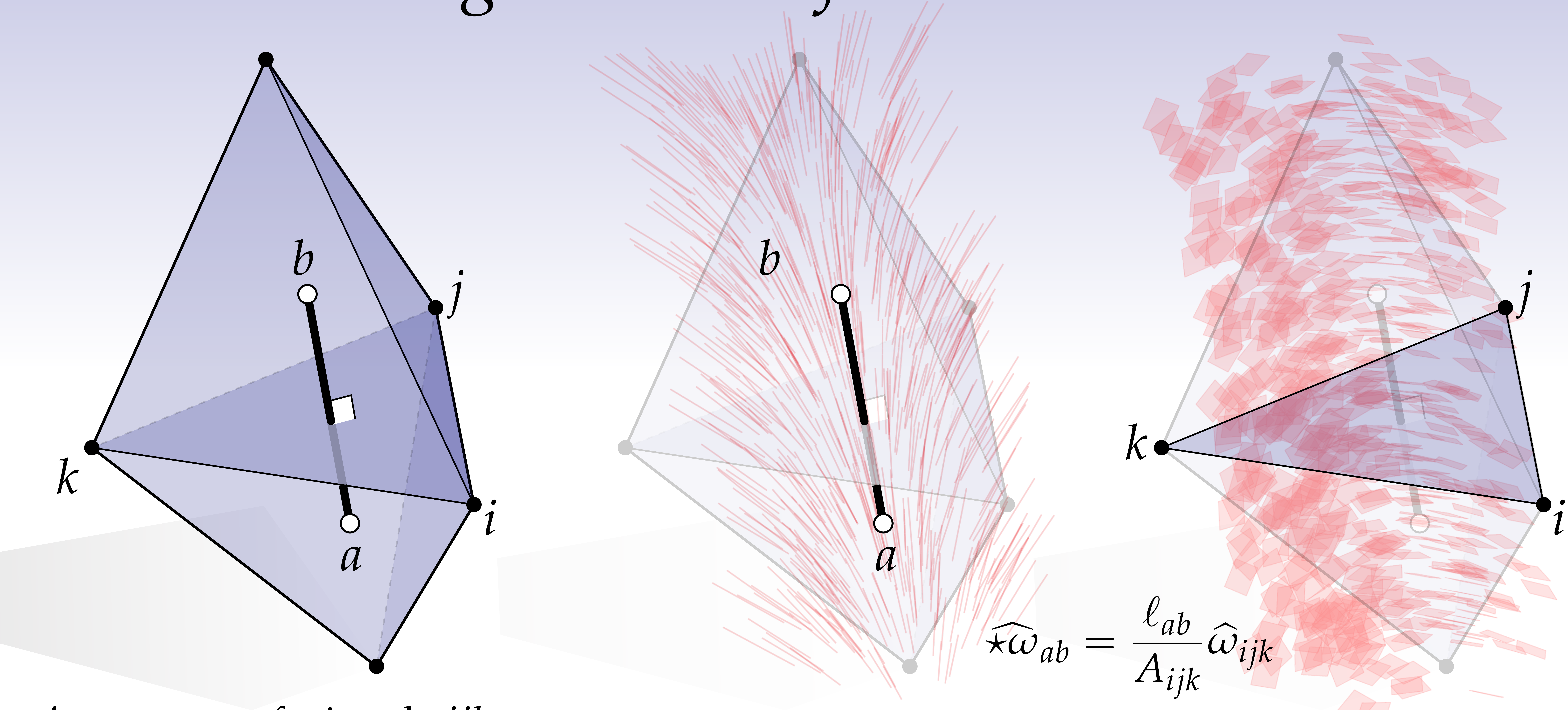
primal 1-form
(circulation)

$$\widehat{\star\alpha} := \frac{\ell^\star}{\ell}\hat{\alpha}$$



dual 1-form
(flux)

Discrete Hodge Star — 2-forms in 3D



dual 1-form

primal 2-form

$$\widehat{\star\omega}_{ab} = \frac{\ell_{ab}}{A_{ijk}}\widehat{\omega}_{ijk}$$

A_{ijk} — area of triangle ijk

ℓ_{ab} — length of dual edge ab

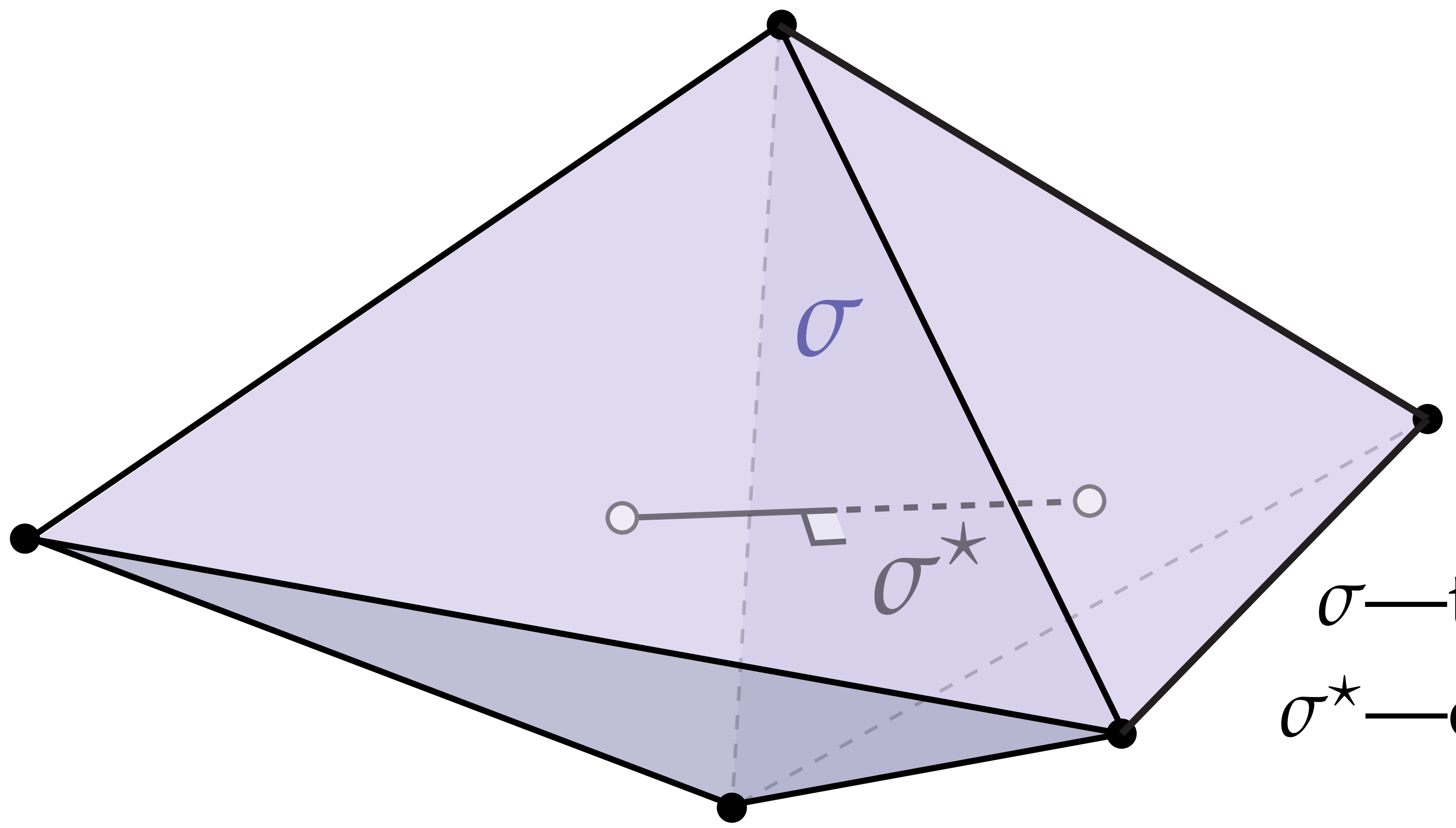
Diagonal Hodge Star

Definition. Let Ω_k and $\Omega^\star_{n-k}$ denote the primal k-forms and dual $(n-k)$ forms (respectively on an n-dimensional simplicial manifold M. The *diagonal Hodge star* is a map $\star : \Omega_k \to \Omega^\star_{n-k}$ determined by

$$\widehat{\star\alpha}(\sigma^\star) = \frac{|\sigma^\star|}{|\sigma|}\hat{\alpha}(\sigma)$$

for each k-simplex σ in M, where $\sigma^\star$ is the corresponding dual cell, and $|\cdot|$ denotes the volume of a simplex or cell.

Key idea: divide by primal area, multiply by dual area. (Why?)



σ—triangle
$\sigma^\star$—dual edge

Matrix Representation of Diagonal Hodge Star

Since the diagonal Hodge star on *k*-forms just multiplies each discrete *k*-form value by a constant (the volume ratio), it can be encoded via a *diagonal* matrix

$$\star_k := \begin{bmatrix} \frac{|\sigma_1^\star|}{|\sigma_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{|\sigma_N^\star|}{|\sigma_N|} \end{bmatrix} \in \mathbb{R}^{N\times N}$$

$\sigma_1, \ldots, \sigma_N$ — *k*-simplices in the primal mesh

$\sigma_1^\star, \ldots, \sigma_N^\star$ — $(n-k)$-cells in the dual mesh

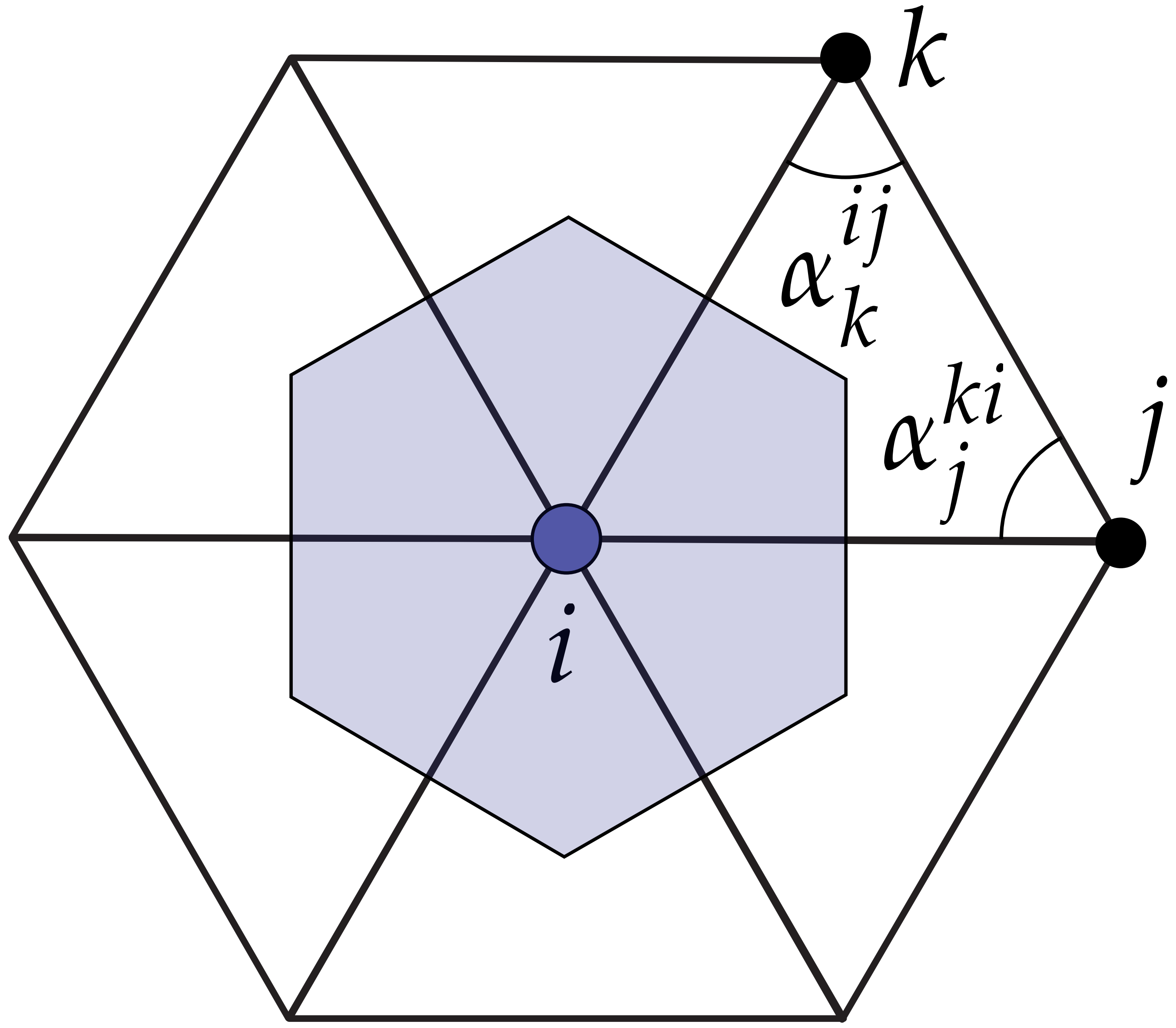
$|\cdot|$ — volume of a simplex or cell

$\star_k \in \mathbb{R}^{N\times N}$ — matrix for Hodge star on primal *k*-forms

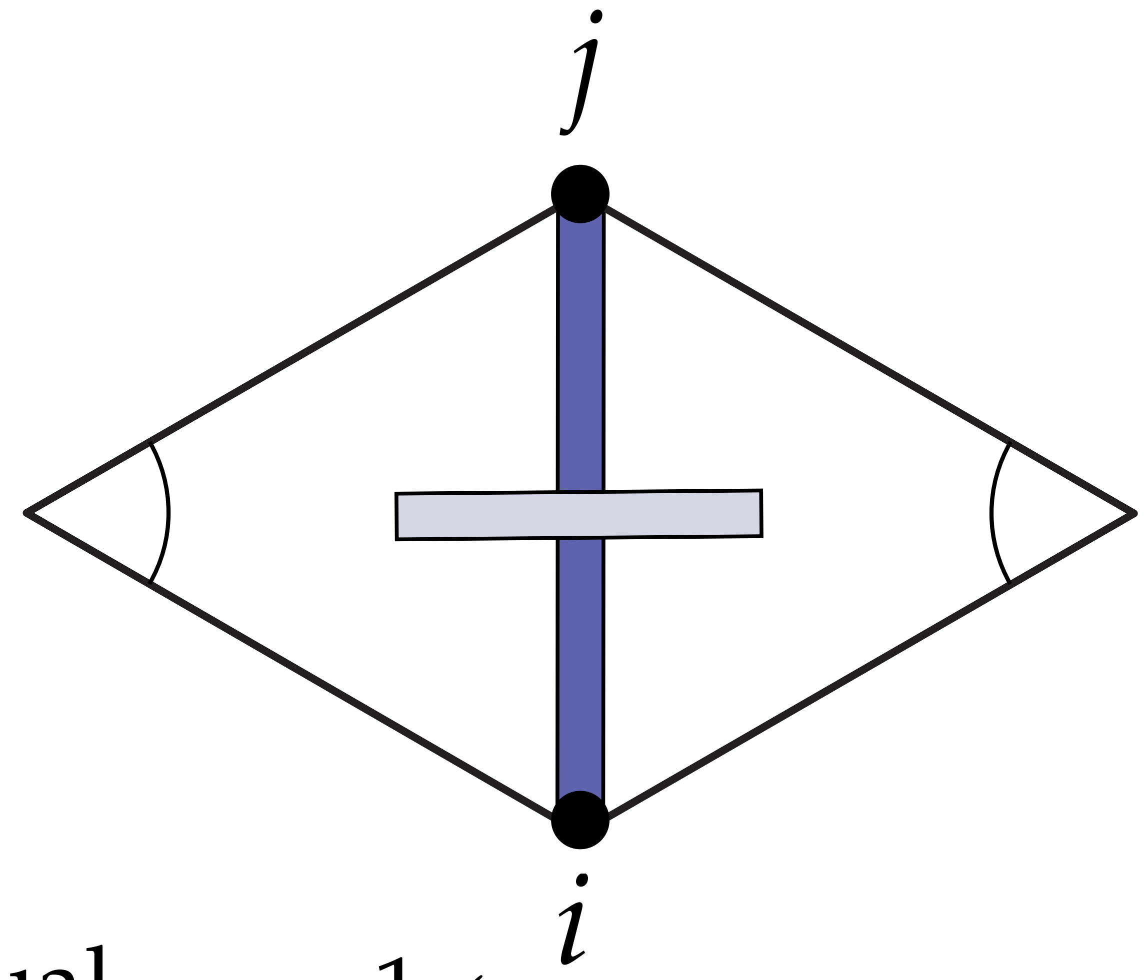
Computing Volumes

- Building Hodge star boils down to computing dual/primal volume ratios
- Often have simple expressions in terms of lengths & angles (*<u>don't</u> compute circumcenters!*)

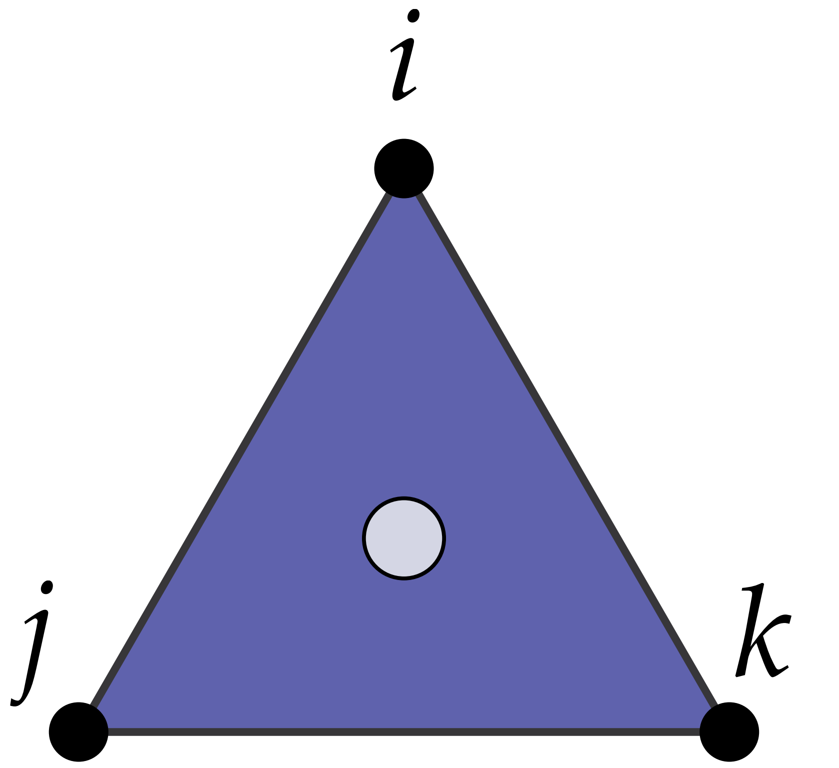
<u>Example: 2D circumcentric dual</u>



$$\frac{A_{\text{dual}}}{1} = \frac{1}{8} \sum_{ijk \in F} \left(\ell_{ij}^2 \cot \alpha_k^{jk} + \ell_{ik}^2 \cot \alpha_j^{ki} \right)$$



$$\frac{\ell_{\text{dual}}}{\ell_{\text{primal}}} = \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})$$



$$\frac{1}{A_{ijk}} = \frac{1}{\sqrt{s(s-\ell_{ij})(s-\ell_{jk})(s-\ell_{ki})}}$$

$$s = \frac{1}{2}(\ell_{ij} + \ell_{jk} + \ell_{ki})$$

Possible Choices for Discrete Hodge Star

- Many choices—*none* give exact results!
- **Volume ratio**
  - diagonal matrix; most typical choice in DEC (Hirani, Desbrun et al)
    - typical choice: circumcentric dual (Delaunay / Voronoi)
    - more general orthogonal dual (weighted triangulation / power diagram)
    - can also use barycentric dual (*e.g.*, Auchmann & Kurz, Alexa & Wardetzky)
      - easy, dual volumes are always positive, but no orthogonality (less accurate)
- **Galerkin Hodge star**
  - L_2 norm on Whitney forms
    - non-diagonal, but still sparse; standard in, e.g., FEEC (Arnold et al).
    - appropriate "mass lumping" again yields circumcentric Hodge star

(Thanks: Fernando de Goes)

Summary

Discrete Exterior Calculus — Basic Operators

Basic operators can be summarized in a very useful diagram (here in 2D):

$$
\begin{array}{ccccc}
\Omega_0 & \xrightarrow{d_0} & \Omega_1 & \xrightarrow{d_1} & \Omega_2 \\
\downarrow \star_0 & & \downarrow \star_1 & & \downarrow \star_2 \\
\Omega_2^\star & \xleftarrow[d_0^\mathsf{T}]{} & \Omega_1^\star & \xleftarrow[d_1^\mathsf{T}]{} & \Omega_0^\star
\end{array}
$$

Ω_k — primal k-forms

$\Omega_k^\star$ — dual k-forms

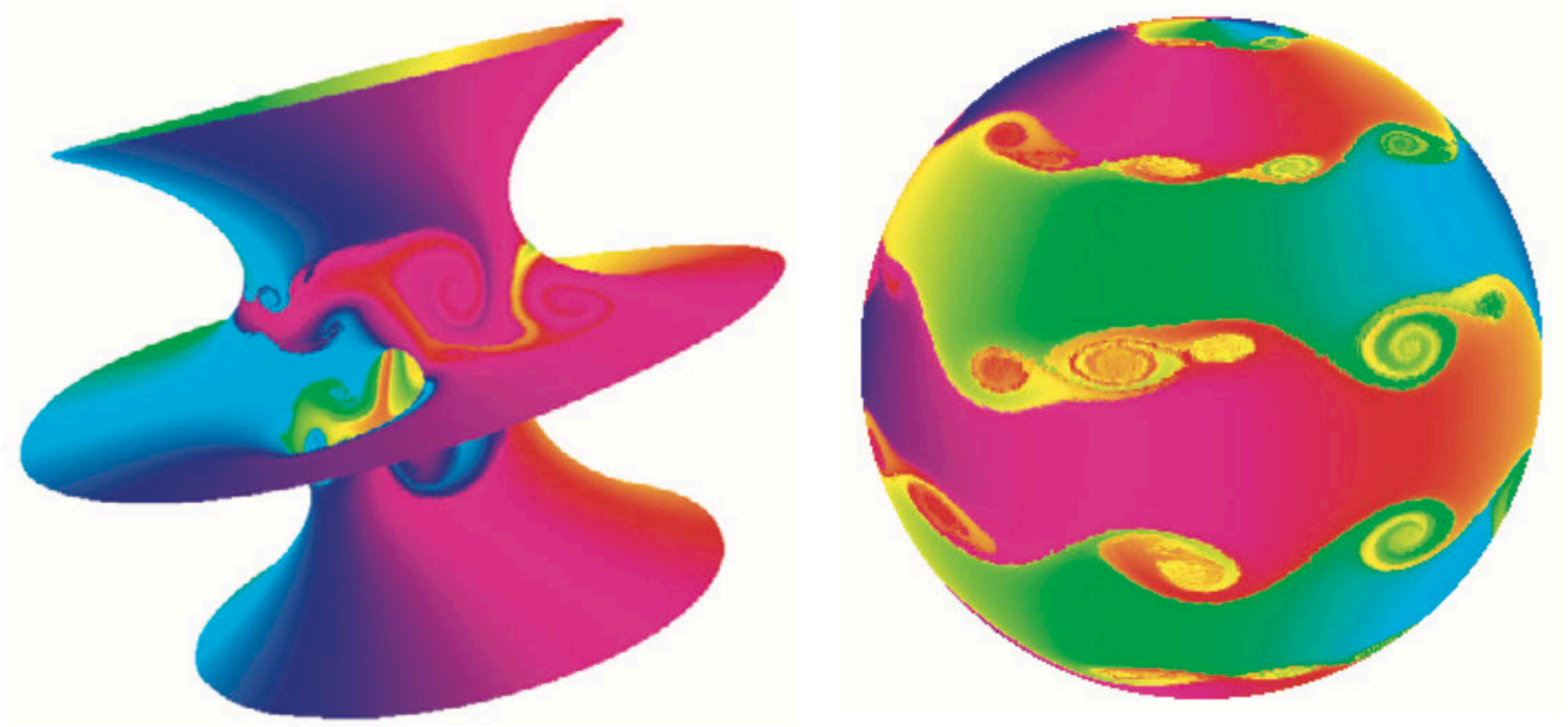
Composition of Operators

By composing matrices, we can easily solve equations involving operators like those from vector calculus (grad, curl, div, Laplacian…) but in much greater generality (*e.g.*, curved surfaces, k-forms…) and on complicated domains (meshes)

$$\text{grad} \longrightarrow d_0 \qquad \text{curl} \longrightarrow \star_2 d_1$$

$$\text{div} \longrightarrow \star_0^{-1} d_0^T \star_1$$

$$\Delta \longrightarrow \star_0^{-1} d_0^T \star_1 d_0$$



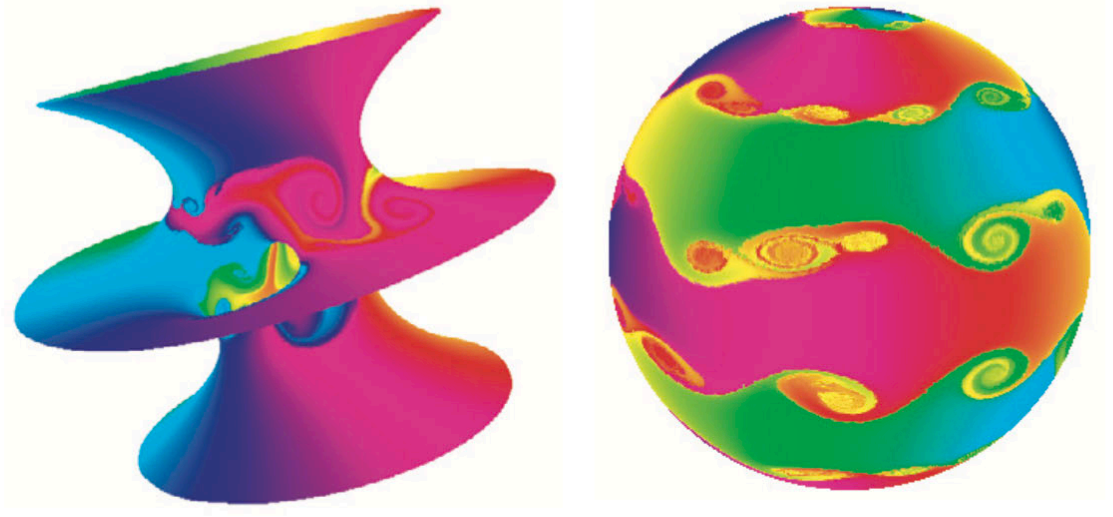
images: Elcott et al 2007

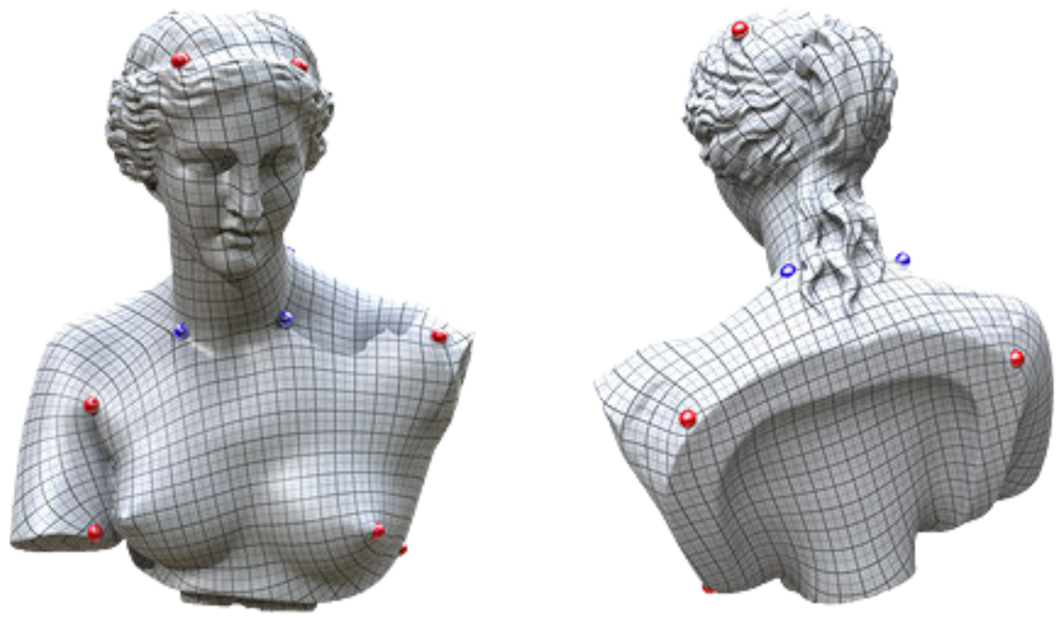
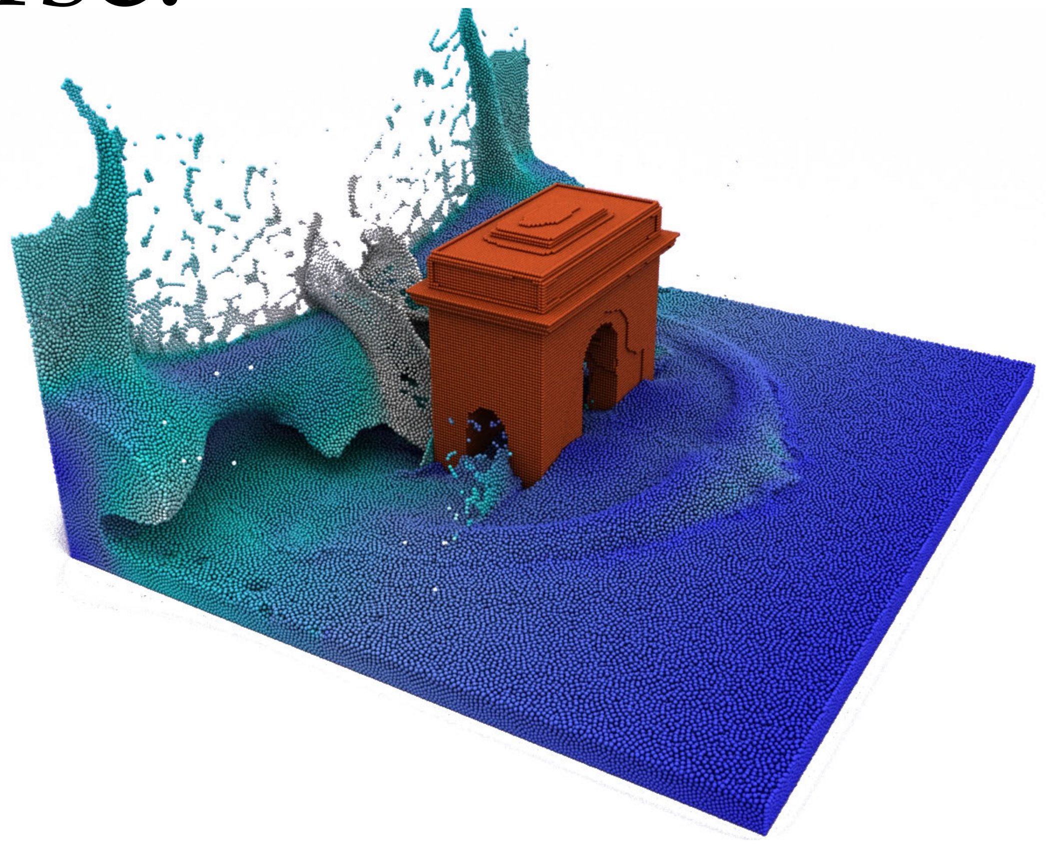
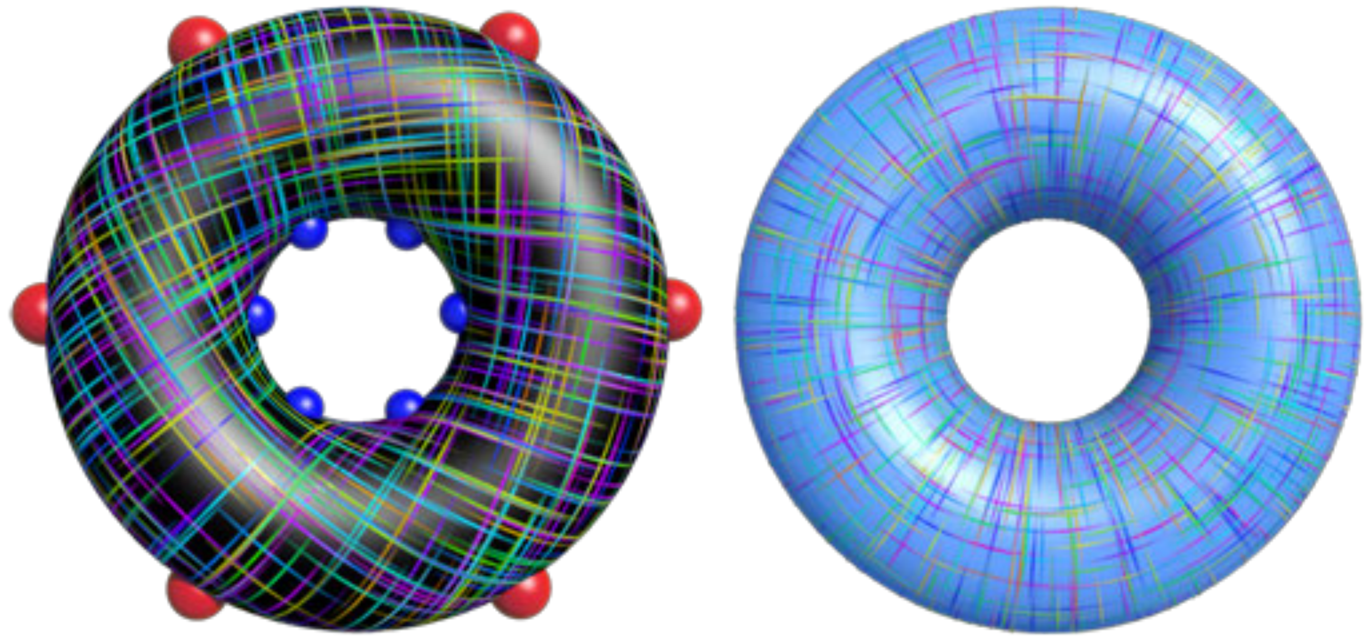
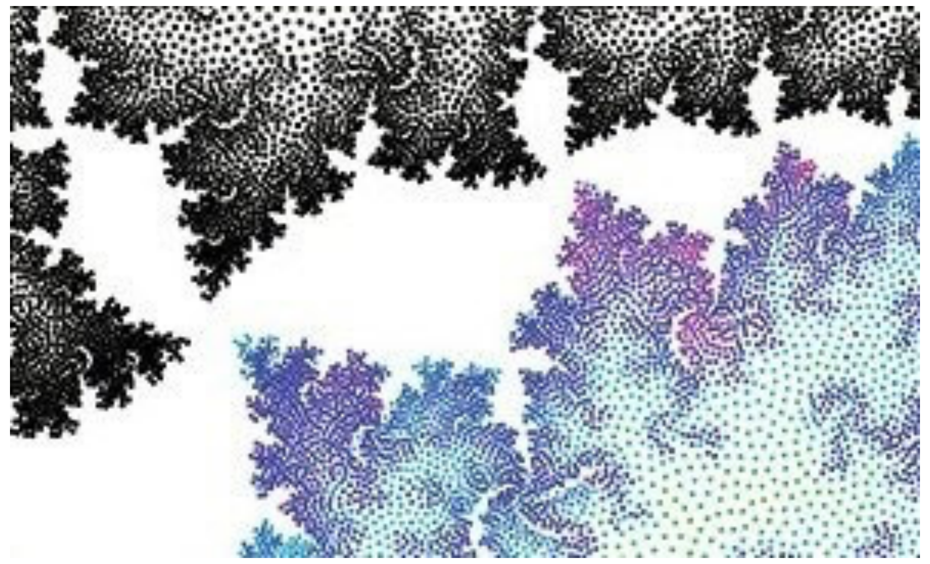
$$\Delta_k \longrightarrow d_{k-1} \star_{k-1}^{-1} d_{k-1}^T \star_k + \star_k^{-1} d_k^T \star_{k+1} d_k$$

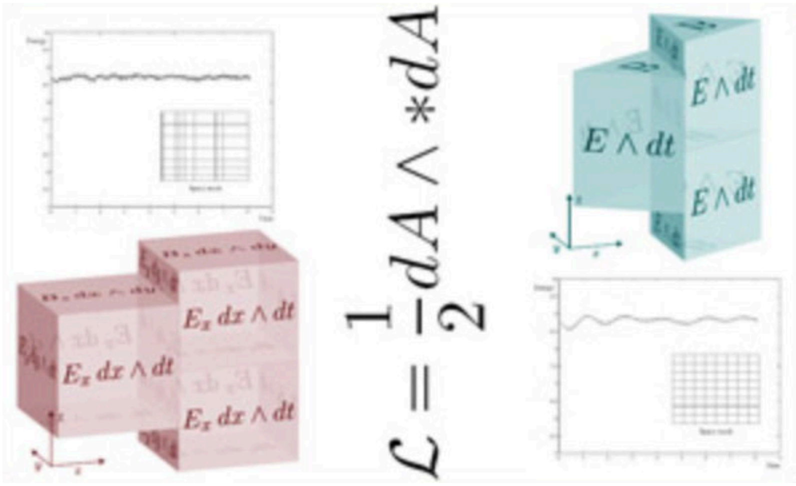
Basic recipe: load a mesh, build a few basic matrices, solve a linear system.

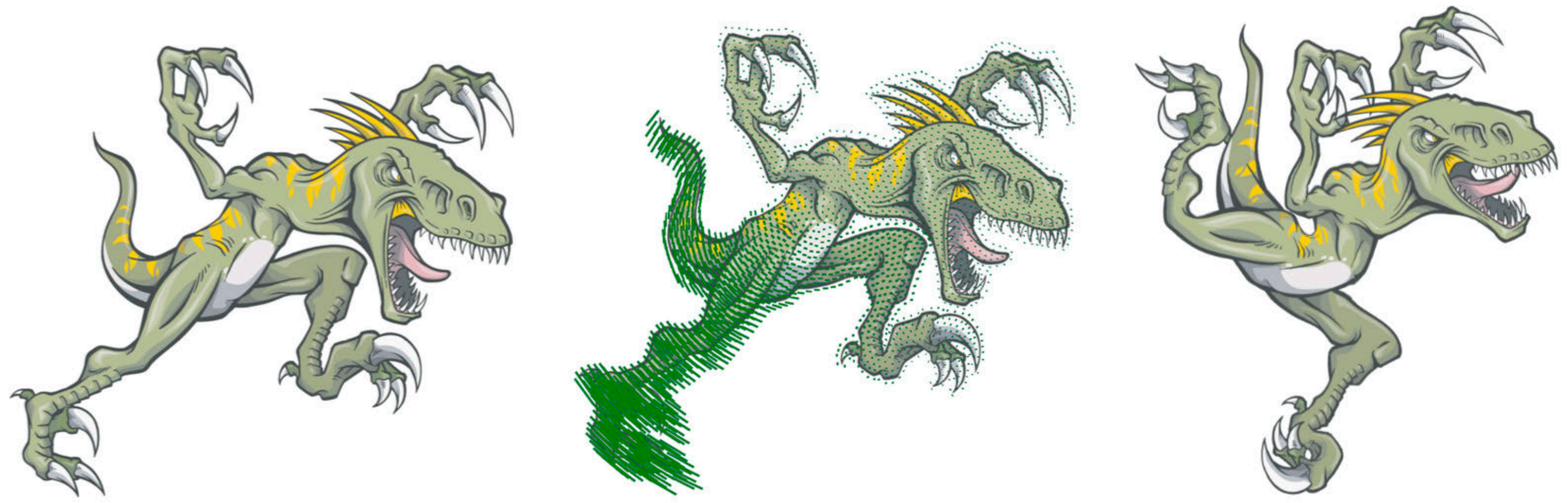
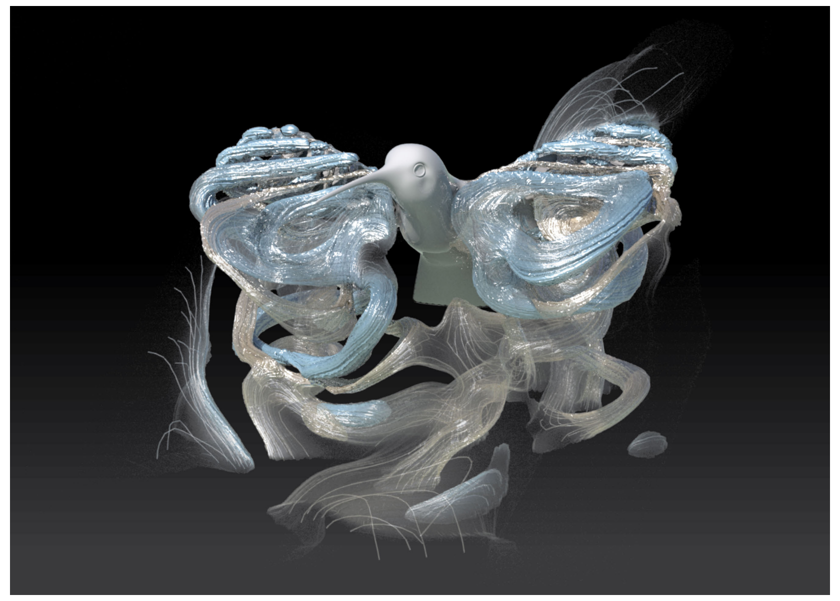
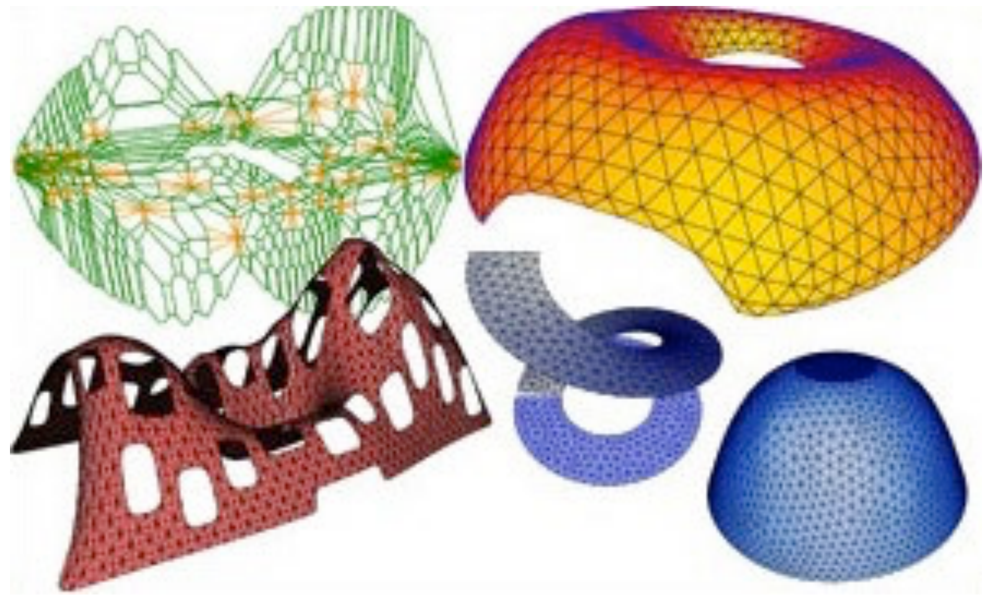
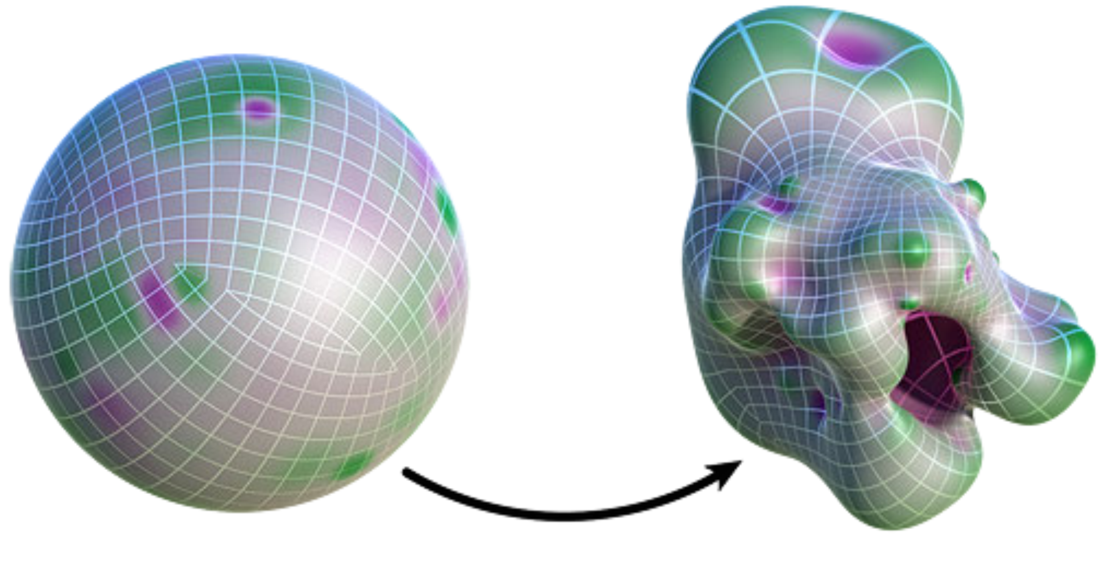
Applications

- Lots! (And growing.) We'll see many as we continue with the course.



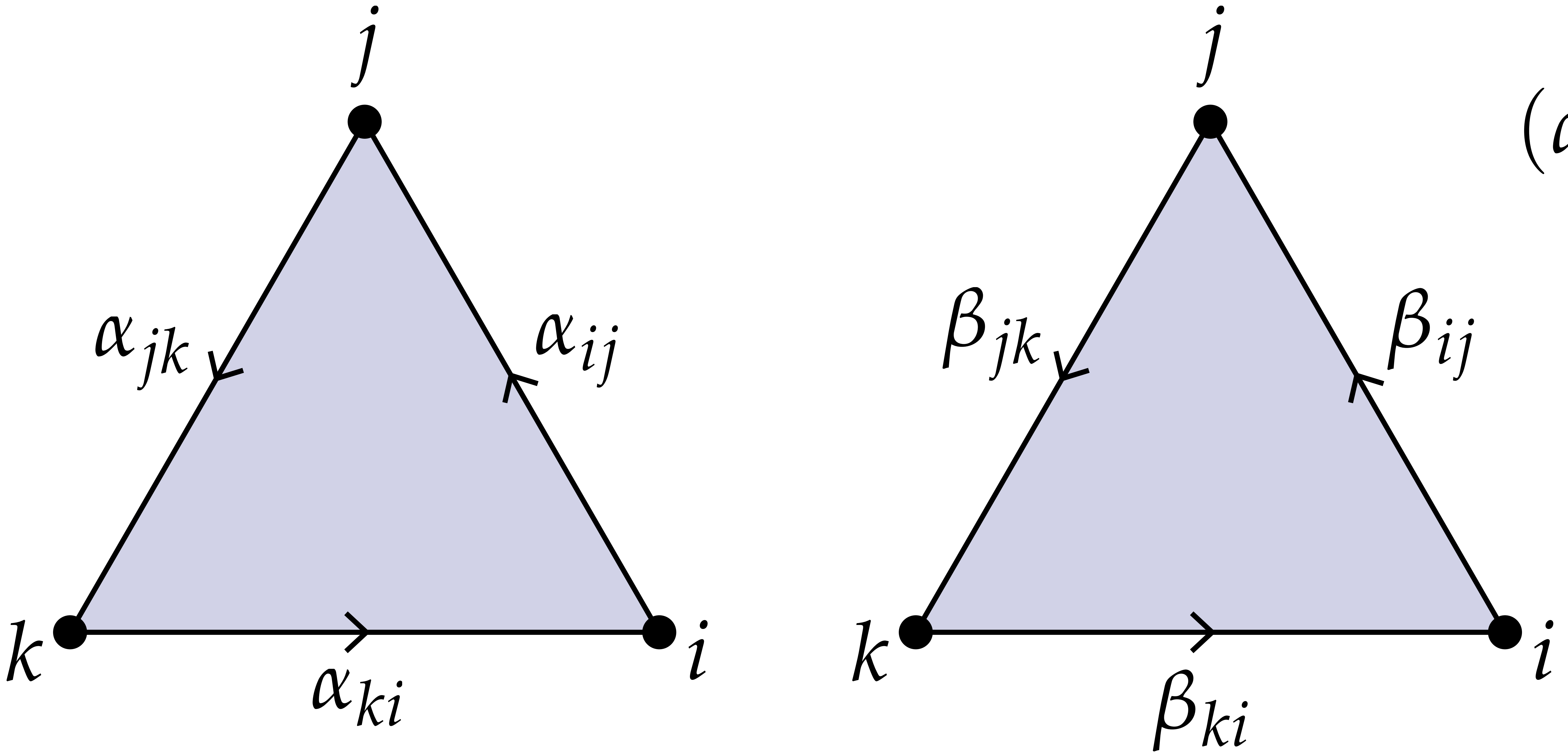







Other Discrete Operators

- Many other operators in exterior calculus (wedge, sharp, flat, Lie derivative, …)
- E.g., wedge product on two discrete 1-forms:

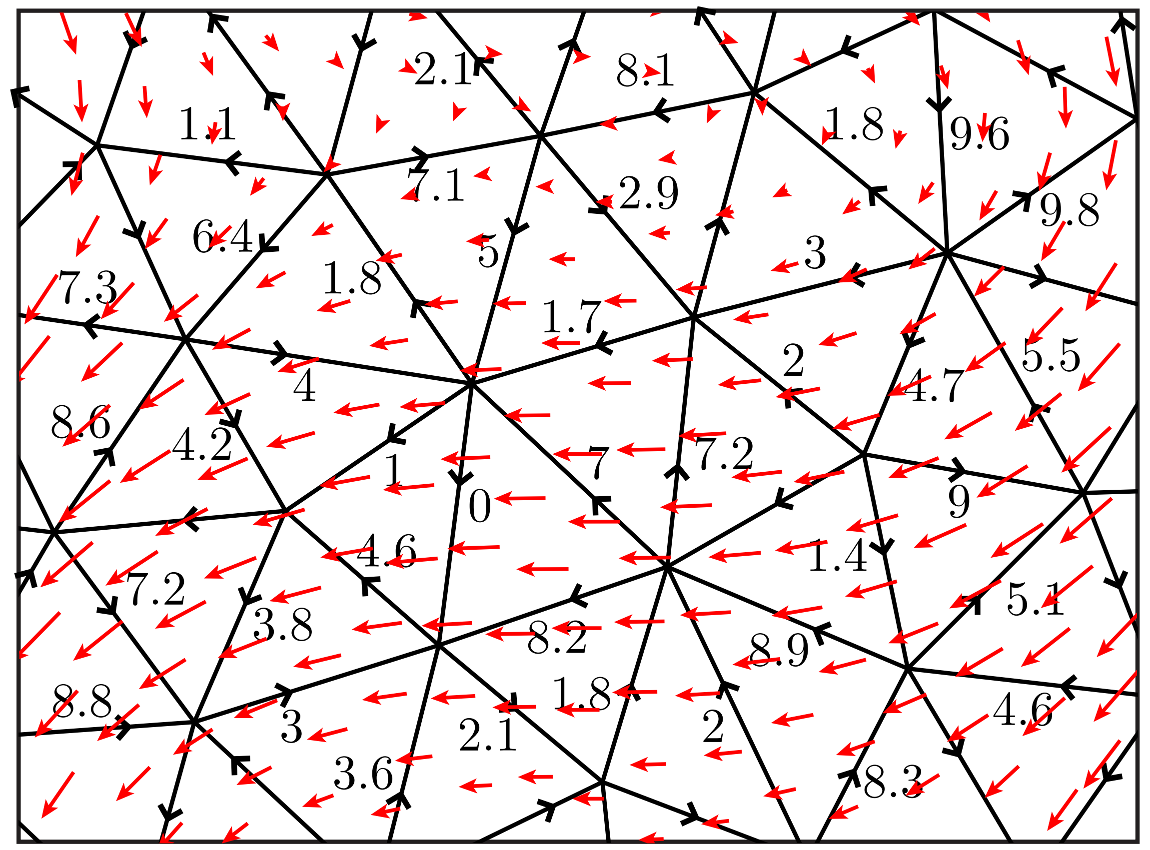


$$(\alpha \wedge \beta)_{ijk} := \frac{1}{6}\big(\alpha_{ij}\beta_{jk} - \alpha_{jk}\beta_{ij} + \alpha_{jk}\beta_{ki} - \alpha_{ki}\beta_{jk} + \alpha_{ki}\beta_{ij} - \alpha_{ij}\beta_{ki} \big)$$

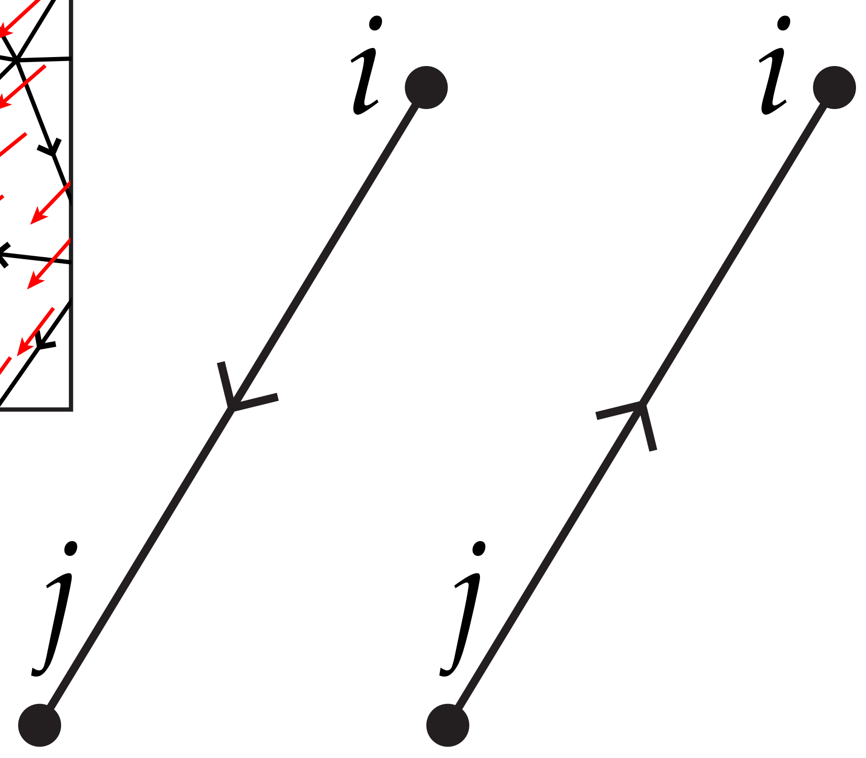
(More broadly, many open questions about how to discretize exterior calculus…)

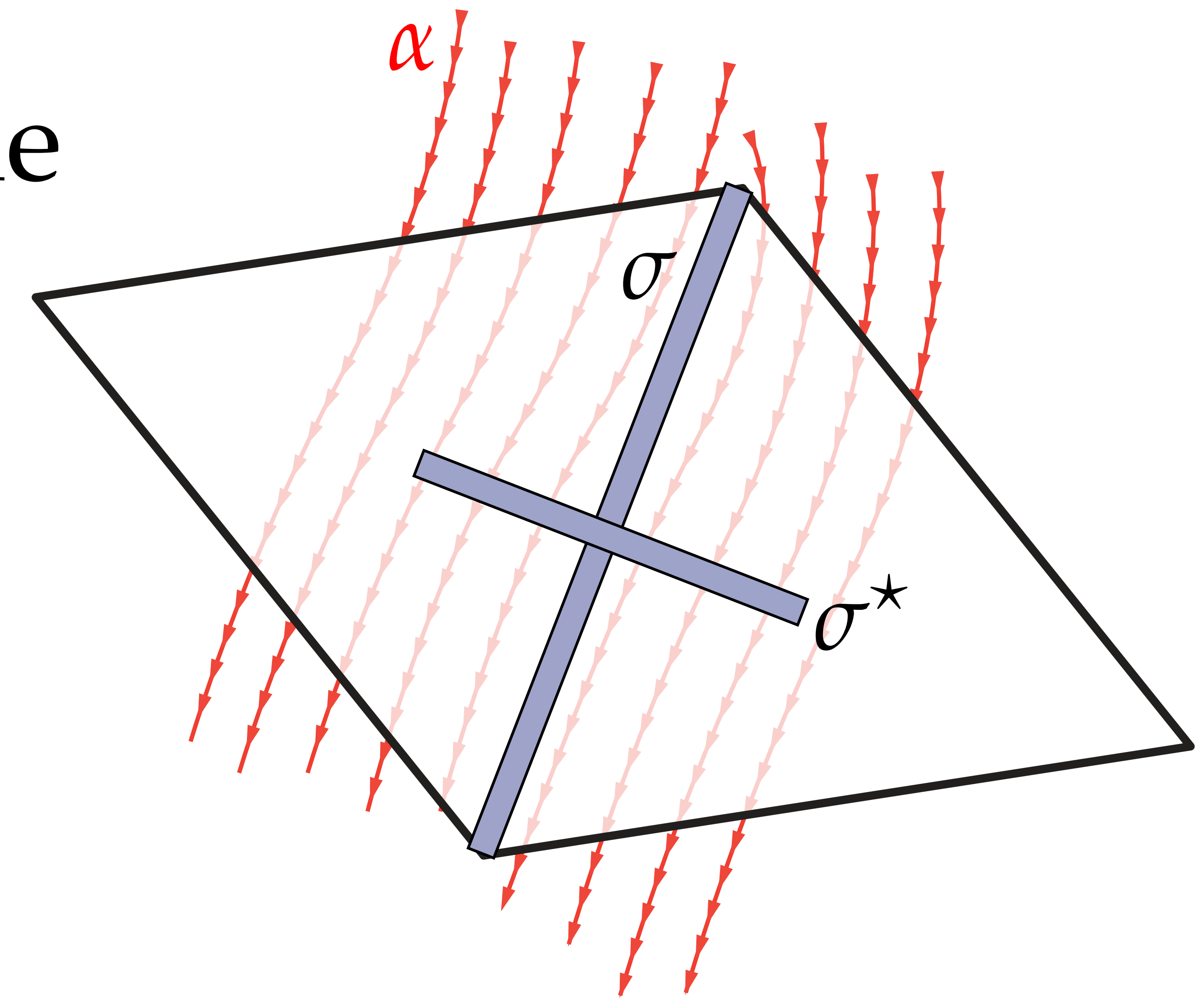
Discrete Exterior Calculus - Summary

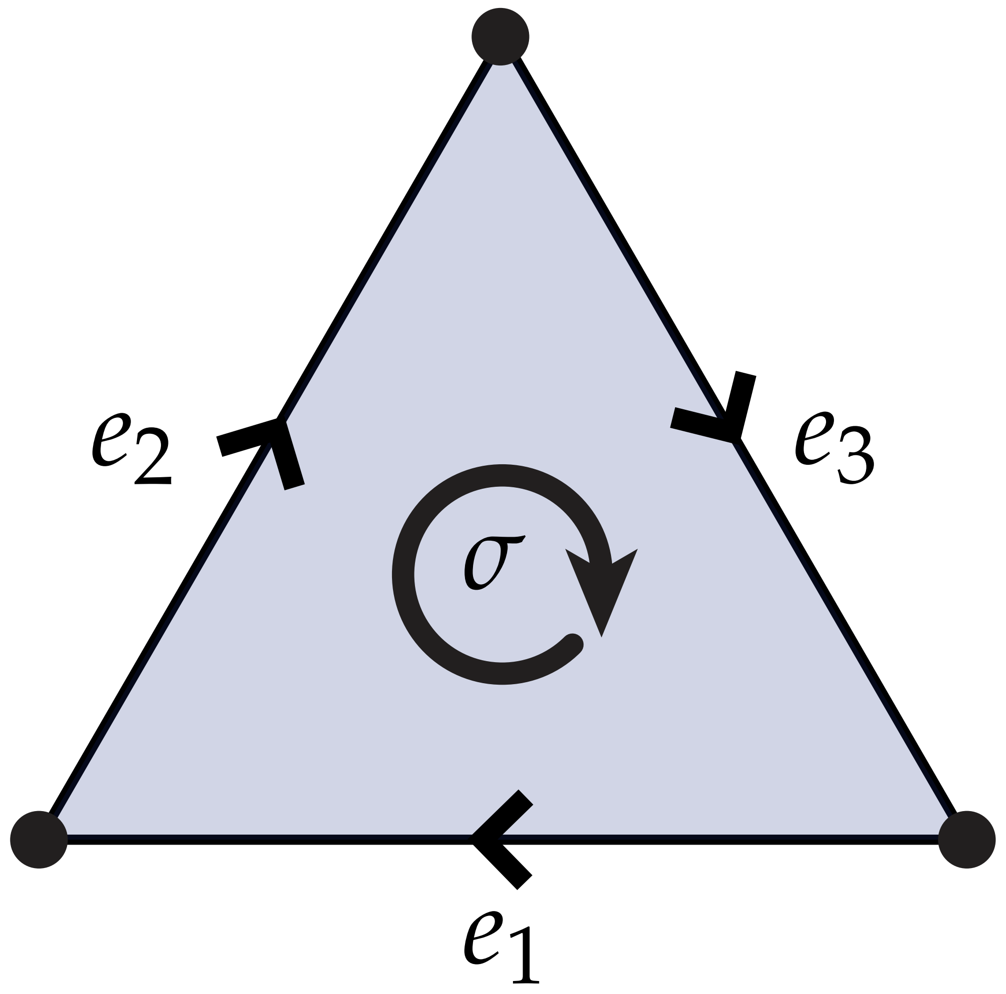
- integrate k-form over k-simplices
  - result is *discrete* k-form
  - sign changes according to orientation
- can also integrate over dual elements (*dual* forms)
- Hodge star converts between primal and dual (*approximately*!)
  - multiply by ratio of dual/primal volume
- discrete exterior derivative is just a sum
  - gives *exact* value (via Stokes' theorem)
- **Next up:** apply these tools to geometry!

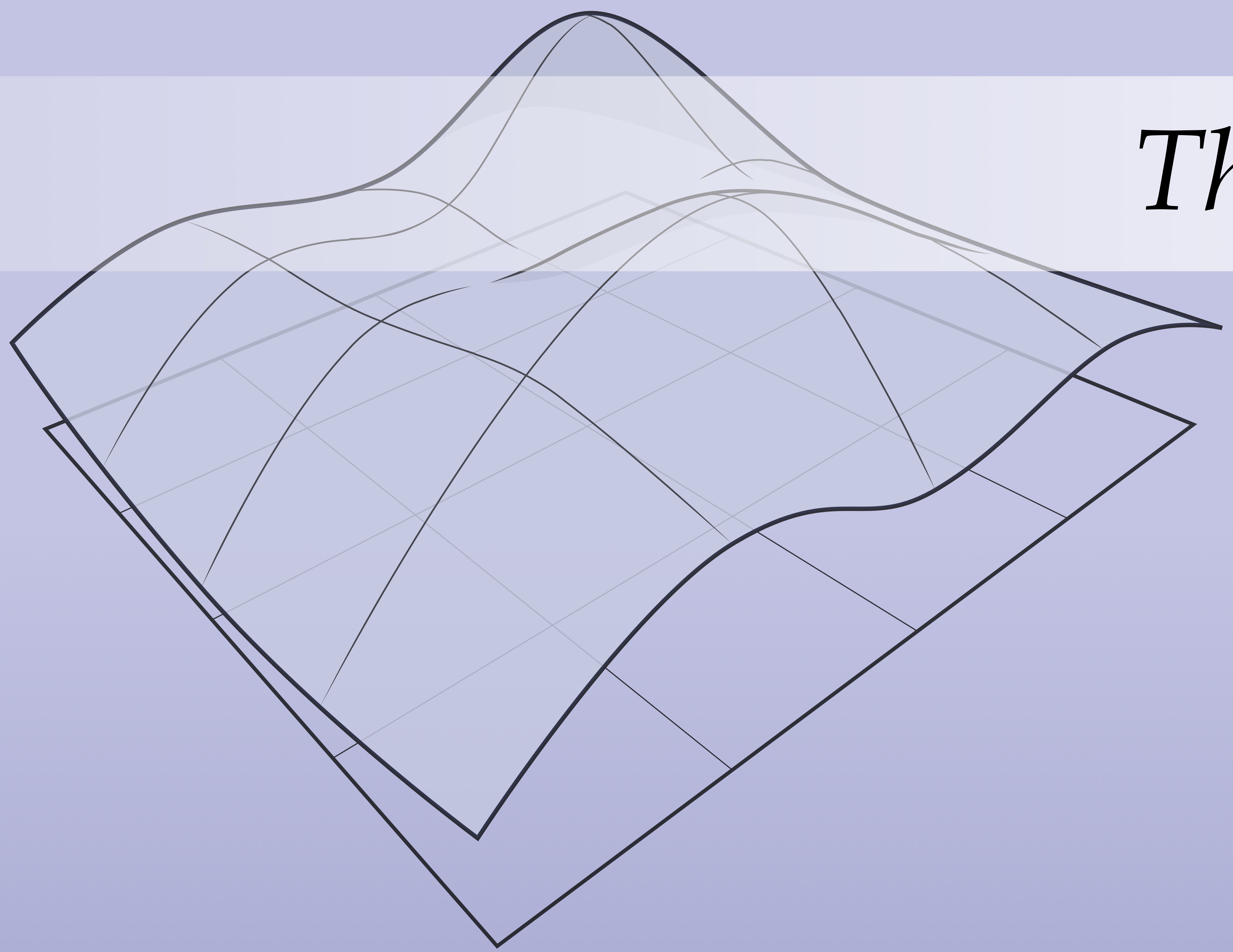


$$\hat{\alpha}_{ij} = -\hat{\alpha}_{ji}$$









Thanks!

DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION