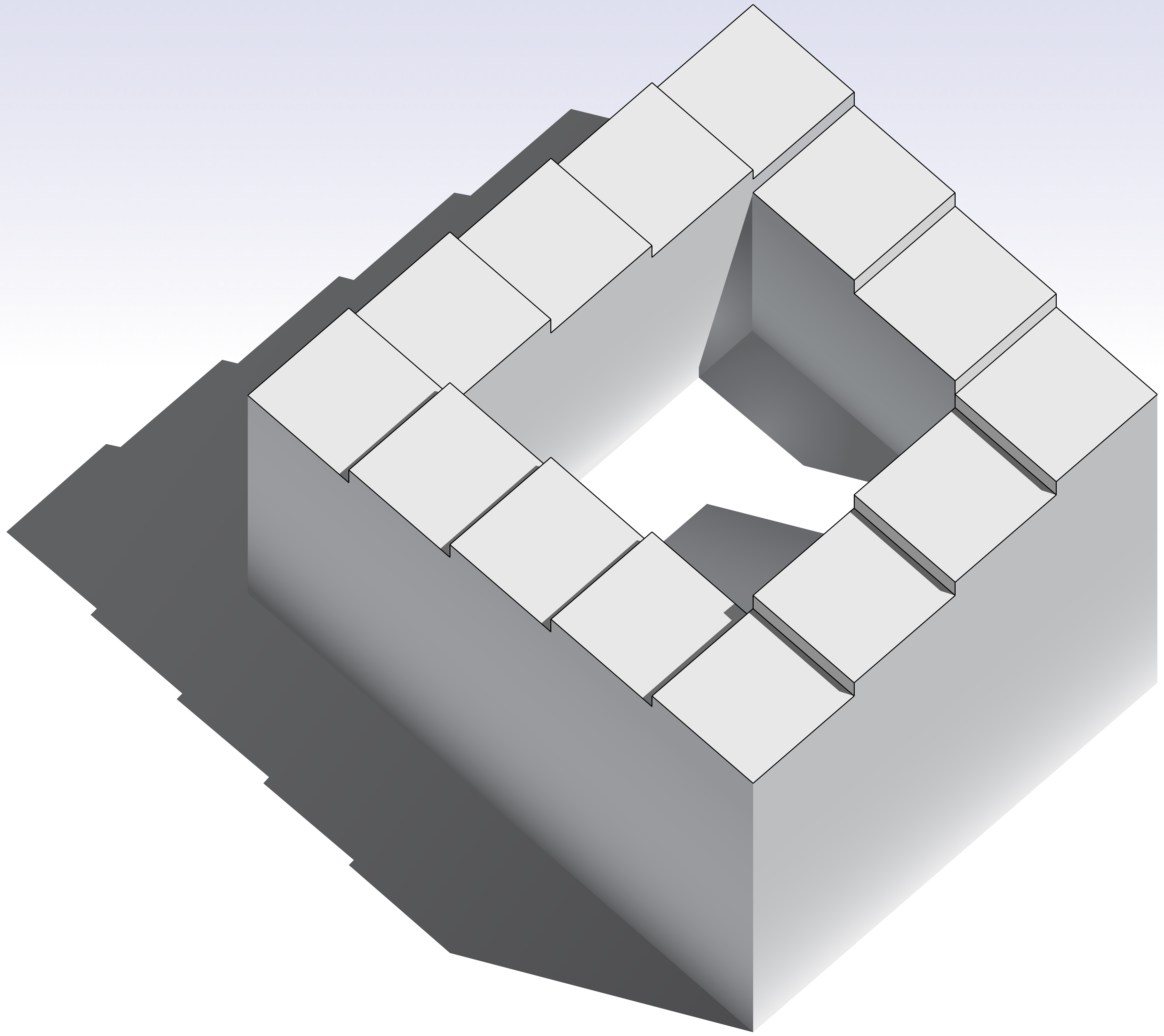
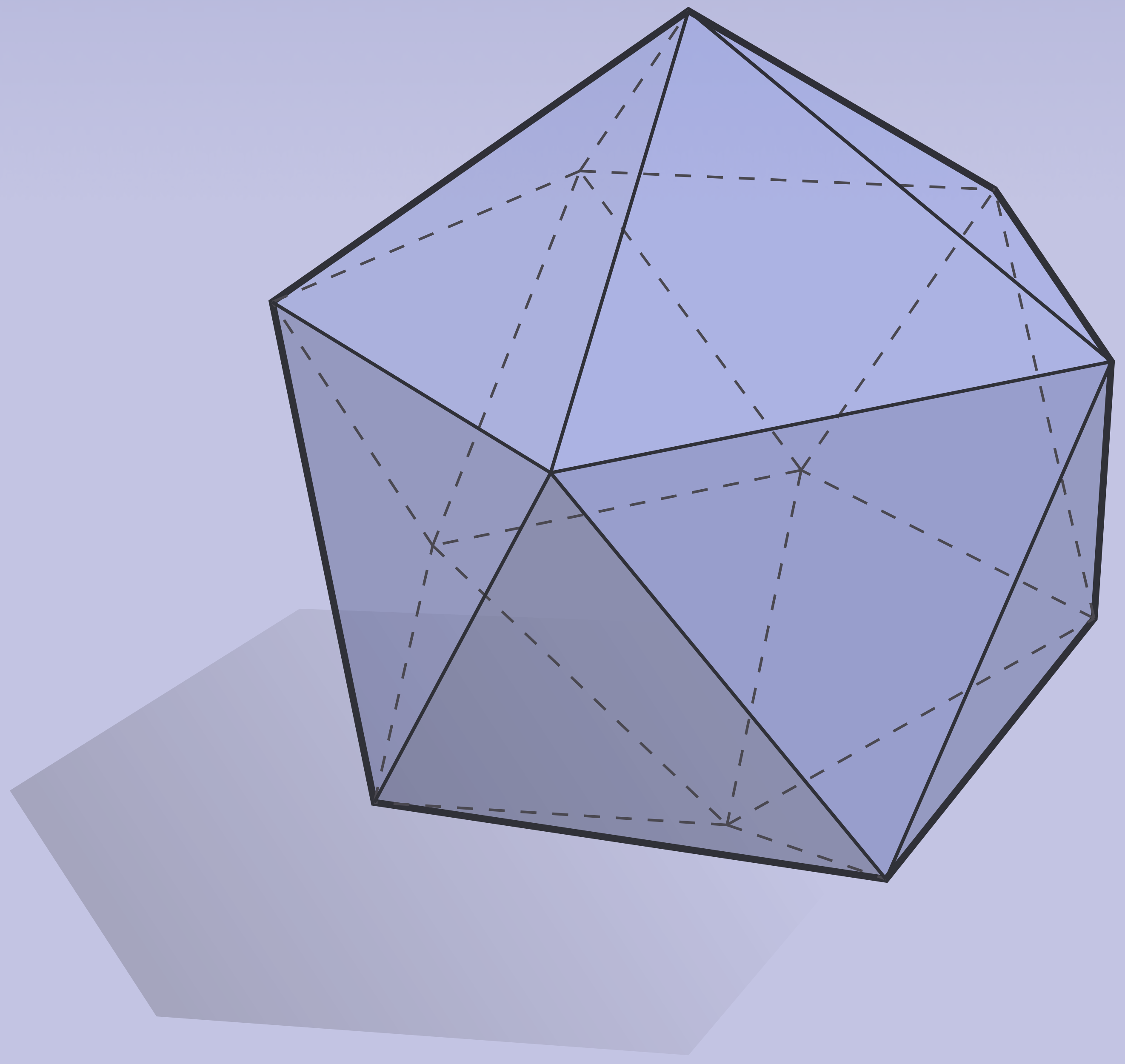


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

SUPPLEMENTAL:

VECTOR-VALUED DIFFERENTIAL FORMS



DISCRETE DIFFERENTIAL GEOMETRY:

AN APPLIED INTRODUCTION

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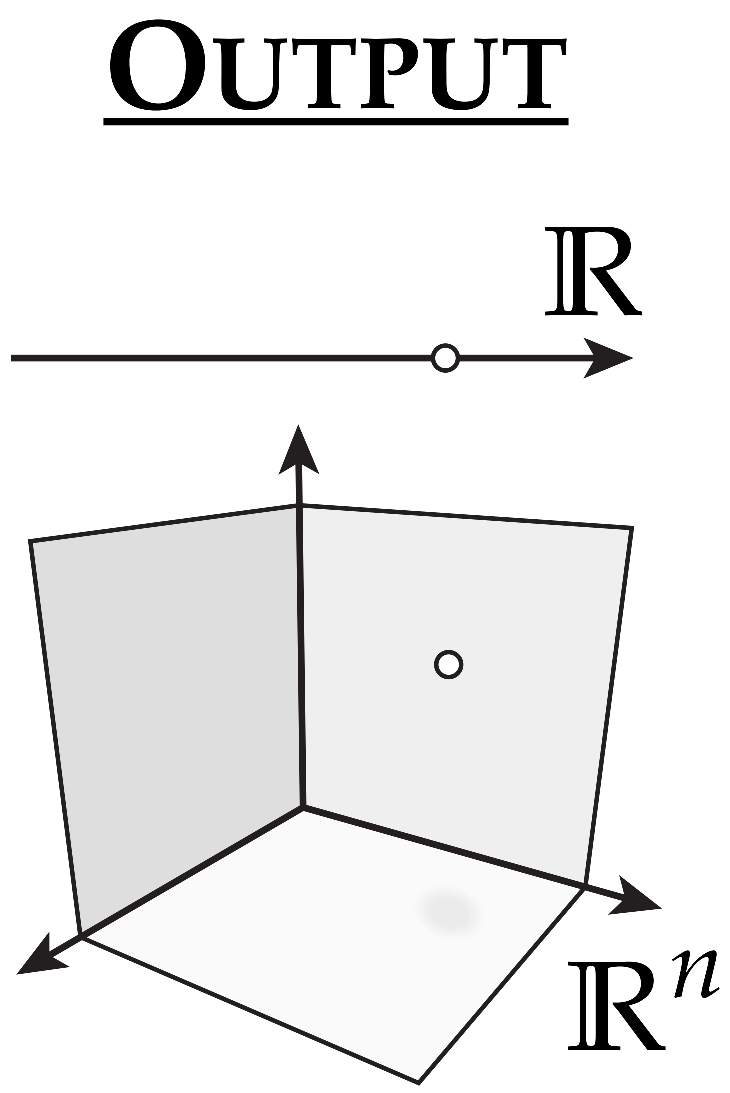
Vector Valued k-Forms

- Originally defined k-form as linear map from k vectors to real numbers
  - To encode geometry, need functions that describe points in space
  - Will therefore generalize to vector-valued k-forms



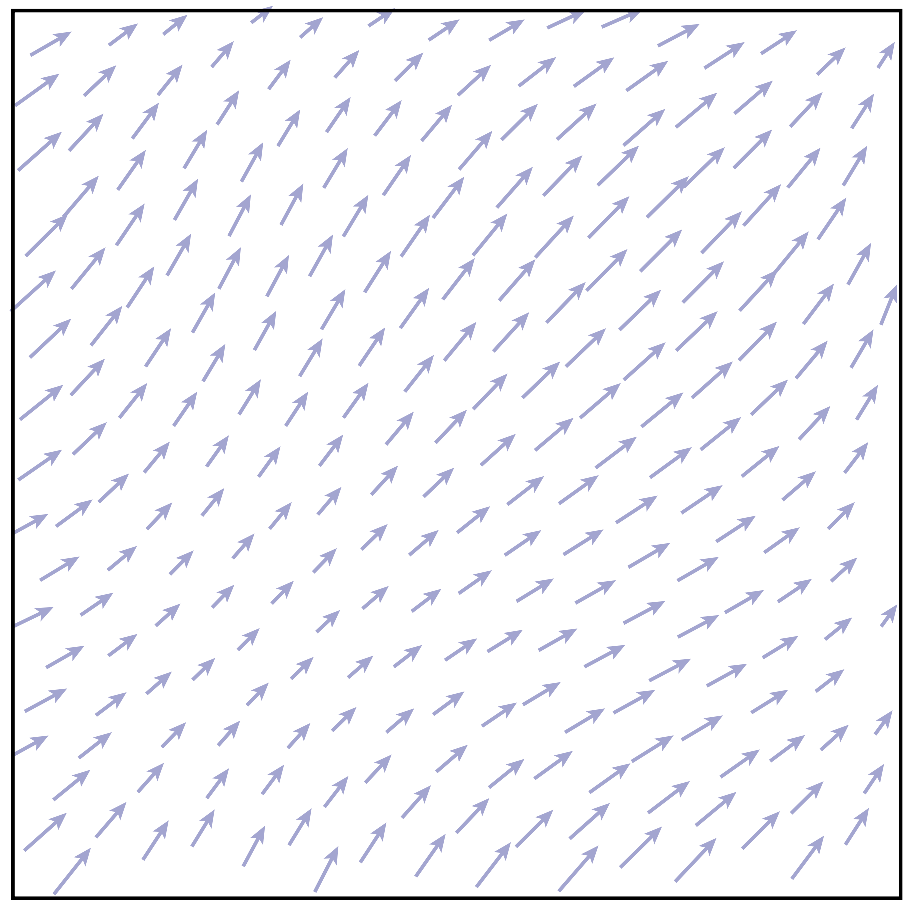
Definition. A *vector-valued k-form* is a fully antisymmetric multi-linear map from k vectors in a vector space V to another vector space U.

- Have already seen many $\mathbb{R}$-valued k-forms on $\mathbb{R}^n$ ($V = \mathbb{R}^n$, $U = \mathbb{R}$)
- A $\mathbb{R}^3$-valued 2-form on $\mathbb{R}^2$ would instead be a multilinear, fully-antisymmetric map from a pair of vectors u,v in $\mathbb{R}^2$ to a single vector in $\mathbb{R}^3$:

$$\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \qquad \alpha(u,v) = -\alpha(v,u)$$

$$\alpha(au + bv, w) = a\alpha(u,w) + b\alpha(v,w), \quad \forall u,v,w \in \mathbb{R}^2,\ a,b \in \mathbb{R}$$

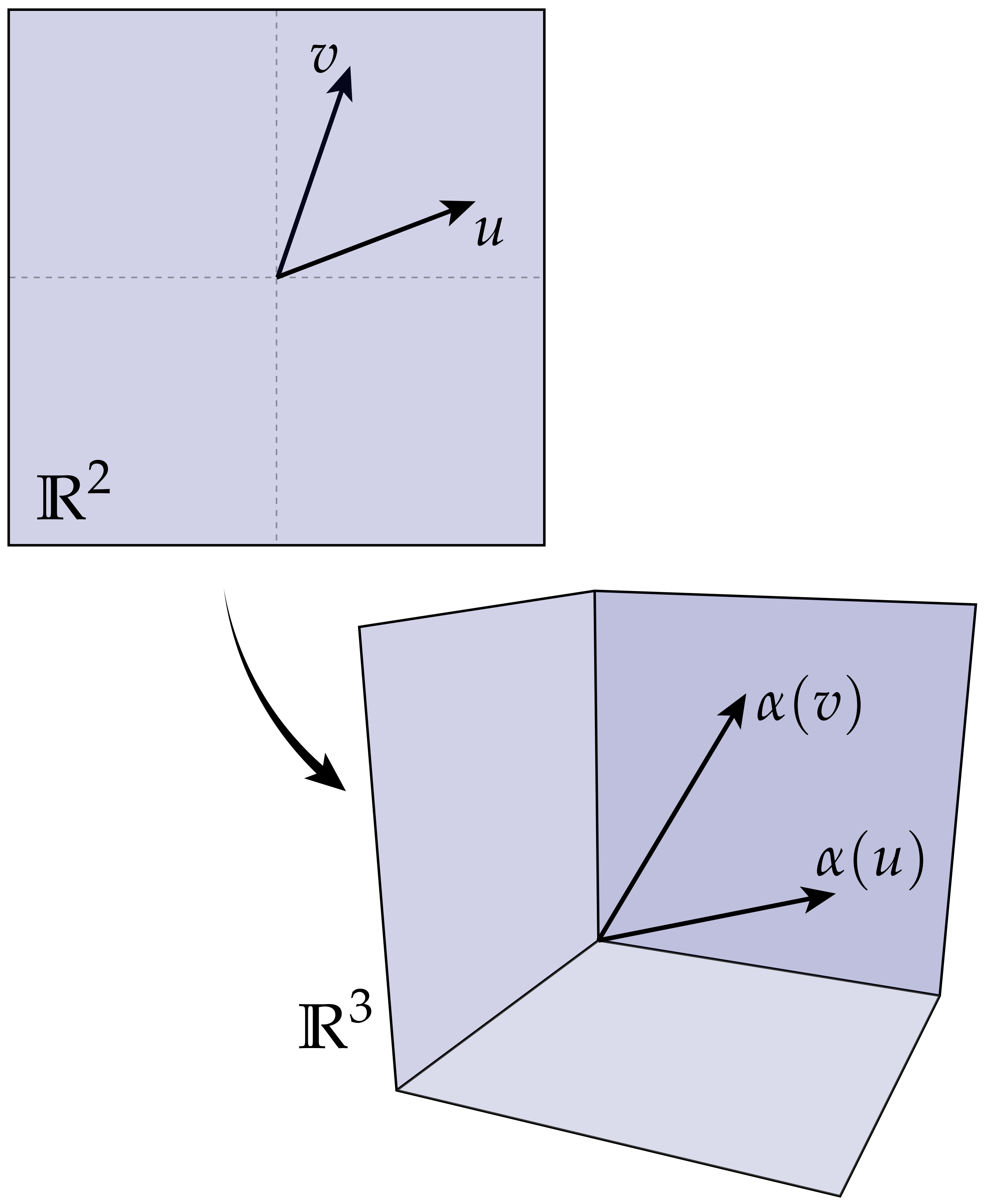
Q: What kind of object is a $\mathbb{R}^2$-valued 0-form on $\mathbb{R}^2$?



Vector-Valued k-forms—Example

Consider for instance the following $\mathbb{R}^3$-valued 1-form on $\mathbb{R}^2$:

$$\alpha := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^1 + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} e^2$$



Q: What do we get if we evaluate this 1-form on the vector

$$u := e_1 - e_2$$

A: Evaluation is not much different from real-valued forms:

$$\alpha(u) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cancelto{1}{e^1(e_1 - e_2)} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cancelto{-1}{e^2(e_1 - e_2)} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Key idea: most operations just look like scalar case, applied to each component

Wedge Product of Vector-Valued k-Forms

- Most important change is how we evaluate wedge product for vector-valued forms.
- Consider for instance a pair of $\mathbb{R}^3$-valued 1-forms:

$$\alpha, \beta : V \to \mathbb{R}^3$$

- To evaluate their wedge product on a pair of vectors u,v we would normally write:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

- If α and β were *real*-valued, then $\alpha(u)$, $\beta(v)$, $\alpha(v)$, $\beta(u)$, would just be *real numbers*, so we could just multiply the two pairs and take their difference.
- But what does it mean to take the "product" of two vectors from $\mathbb{R}^3$?
- Many possibilities (*e.g.*, dot product), but if we want result to be an $\mathbb{R}^3$-valued 2-form, the product we choose must produce another vector in $\mathbb{R}^3$!

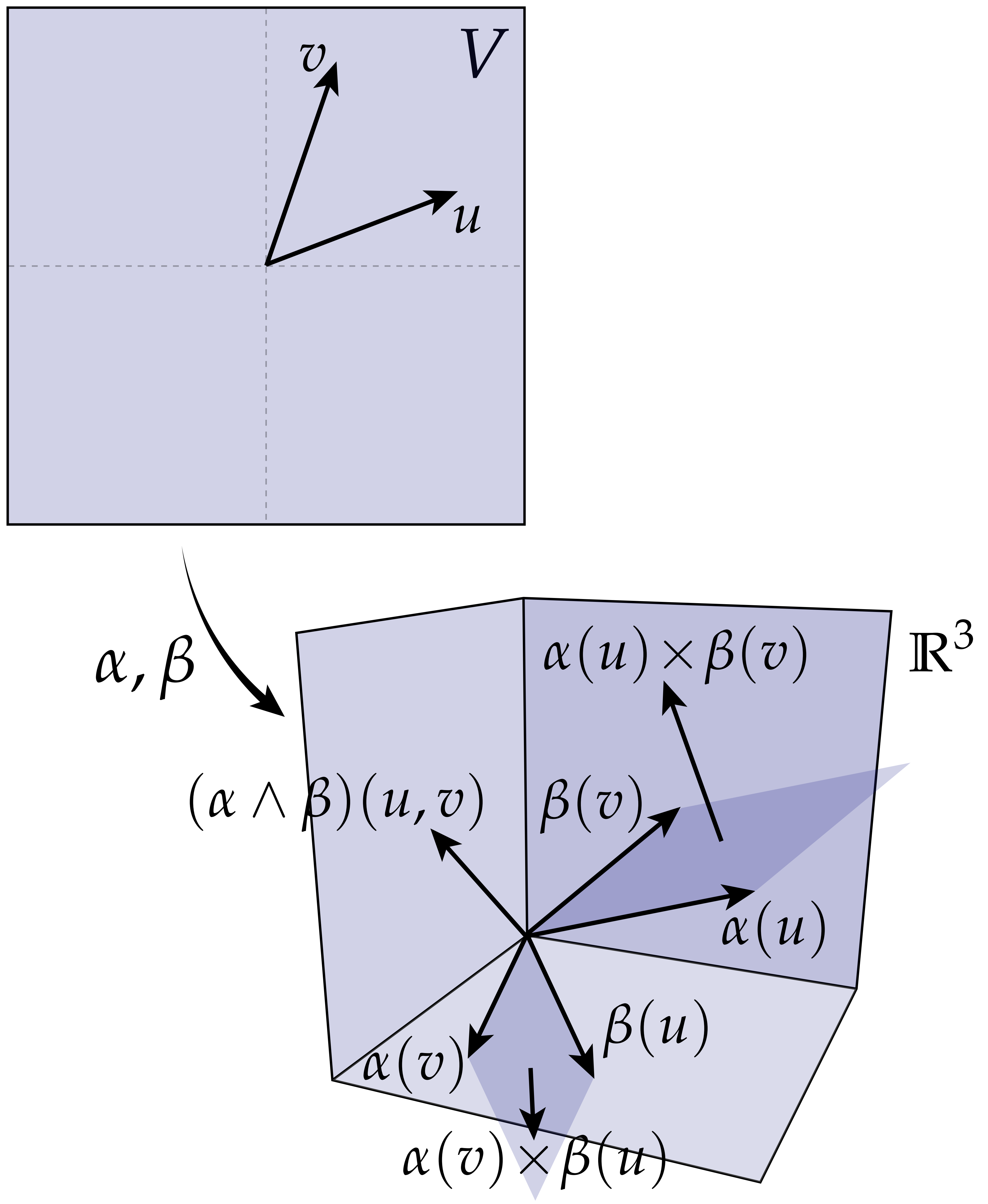
Wedge Product of $\mathbb{R}^3$-Valued k-Forms

- When working with 3D geometry:
  - k-forms are $\mathbb{R}^3$-valued
  - use **cross product** to multiply vectors in $\mathbb{R}^3$

$$\alpha, \beta : V \to \mathbb{R}^3$$

$$\alpha \wedge \beta : V \times V \to \mathbb{R}^3$$

$$\boxed{(\alpha \wedge \beta)(u, v) := \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u)}$$



$\mathbb{R}^3$-valued 1-forms: Antisymmetry & Symmetry

With real-valued forms, we observed antisymmetry in both the wedge product of 1-forms as well as the application of the 2-form to a pair of vectors, *i.e.*,

$$(\alpha \wedge \beta)(u, v) = -(\alpha \wedge \beta)(v, u)$$
$$(\beta \wedge \alpha)(u, v) = -(\alpha \wedge \beta)(u, v)$$

What happens w/ $\mathbb{R}^3$-valued 1-forms? Since cross product is antisymmetric, we get

$$\begin{aligned}(\alpha \wedge \beta)(v, u) &= \alpha(v) \times \beta(u) - \alpha(u) \times \beta(v) \\ &= -(\alpha(u) \times \beta(v) - \alpha(v) \times \beta(u))\end{aligned}$$

$$\Rightarrow \boxed{(\alpha \wedge \beta)(u, v) = -(\alpha \wedge \beta)(v, u)}$$

(same as with real-valued forms)

$$\begin{aligned}(\beta \wedge \alpha)(u, v) &= \beta(u) \times \alpha(v) - \beta(v) \times \alpha(u) \\ &= \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u) \\ &= (\alpha \wedge \beta)(u, v)\end{aligned}$$

$$\Rightarrow \boxed{\alpha \wedge \beta = \beta \wedge \alpha}$$

(no sign change)

Key idea: "antisymmetries cancel"

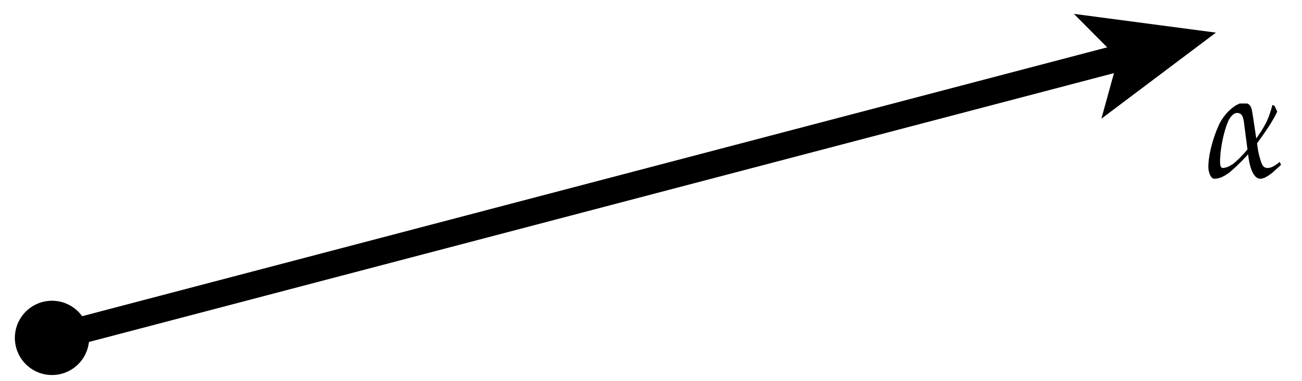
$\mathbb{R}^3$-*valued 1-forms: Self-Wedge*

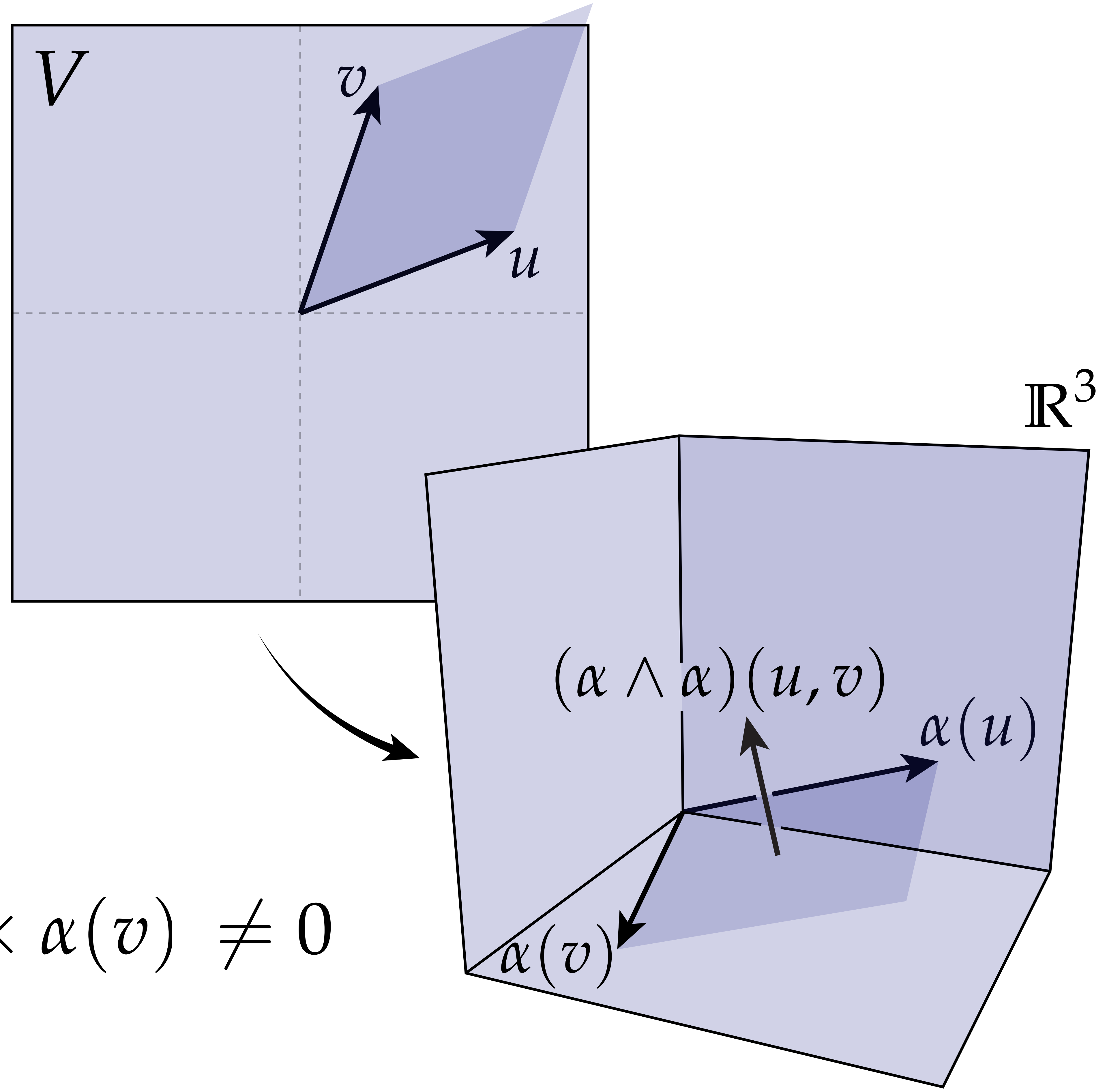
Likewise, we saw that wedging a real-valued 1-form with itself yields zero:

$$\alpha \wedge \alpha = 0$$

Q: What was the *geometric* interpretation?

A: Parallelogram made from two copies of the same vector has zero area!





No longer true with $(\mathbb{R}^3, \times)$-valued 1-forms:

$$(\alpha \wedge \alpha)(u,v) = \alpha(u) \times \alpha(v) - \alpha(v) \times \alpha(u) = 2\alpha(u) \times \alpha(v) \neq 0$$

Q: Geometric meaning?

A: Vector with (twice) area of "stretched out" parallelogram.

Vector-Valued Differential k-Forms

- Just as we distinguished between a *k-form* (value at a single point) and a *differential k-form* (value at each point), will say that a *vector-valued differential k-form* is a vector-valued *k*-form at each point.

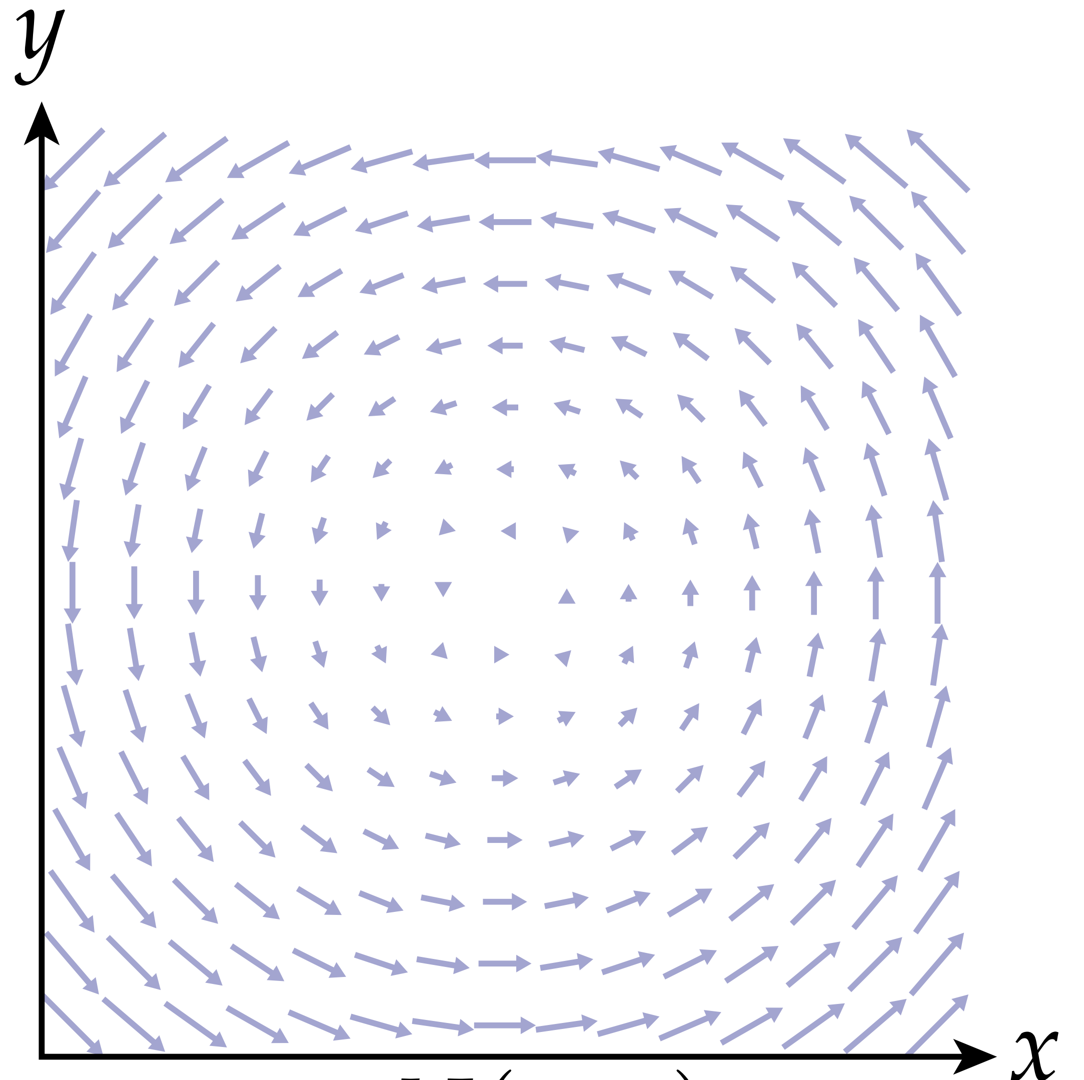
- Just like any differential form, a vector-valued differential *k*-form gets evaluated on k vector fields $X_1, \ldots, X_k$.

- **<u>Example:</u>** an $\mathbb{R}^3$-valued differential 1-form on $\mathbb{R}^2$:

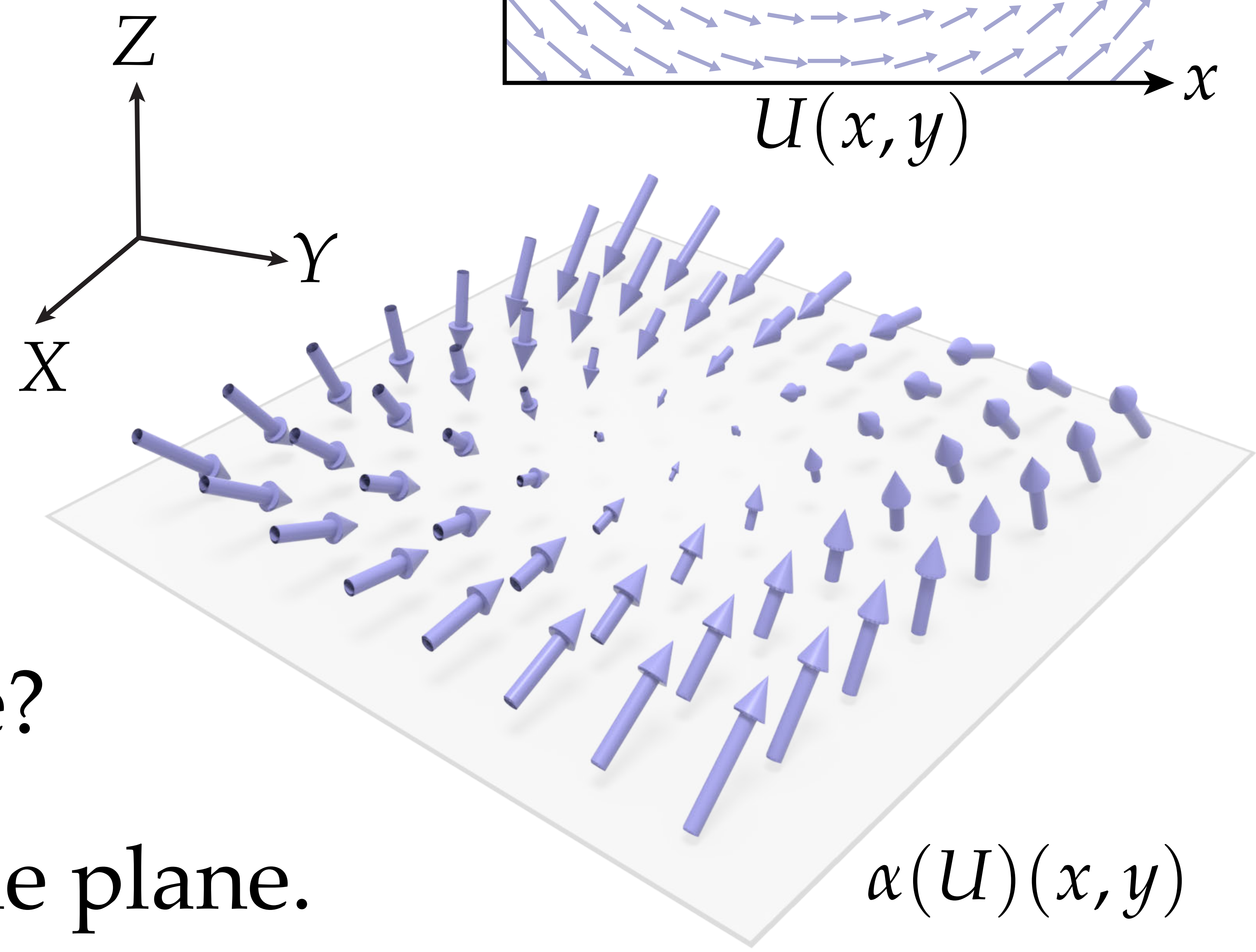
$$\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dx + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} dy$$

Q: What does α do to a given vector field U in the plane?

A: It turns it into a 3D vector field that "sticks out" of the plane.



$U(x,y)$



$\alpha(U)(x,y)$

Exterior Derivative on Vector-Valued Forms

Unlike the wedge product, not much changes with the exterior derivative.
For instance, if we have an $\mathbb{R}^n$-valued k-form we can simply imagine we have n real-valued k-forms and differentiate as usual.

Example.

Consider an $\mathbb{R}^2$-valued differential 0-form $\phi_{(x,y)} := \begin{bmatrix} x^2 \\ xy \end{bmatrix}$

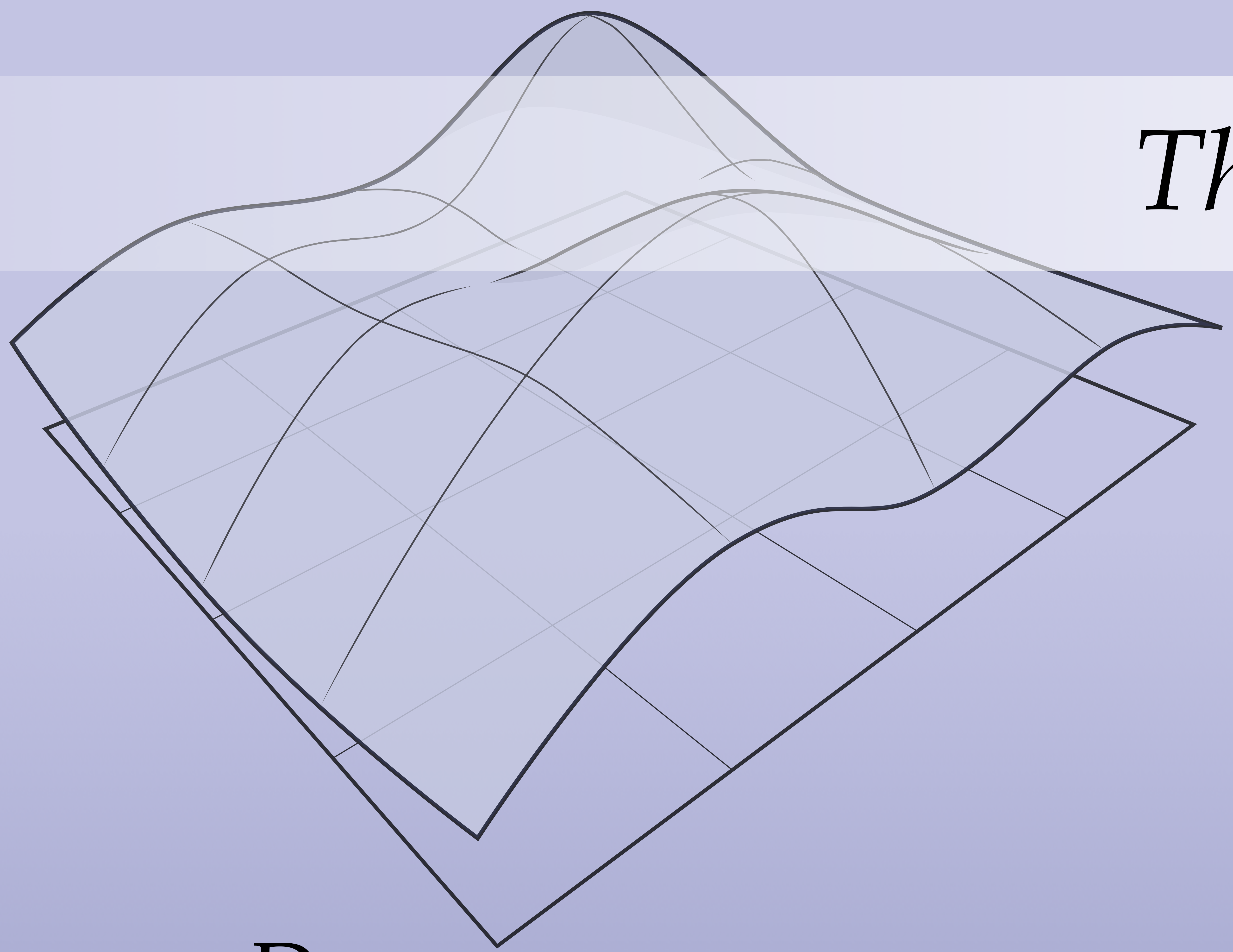
Then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = \begin{bmatrix} 2x \\ y \end{bmatrix} dx + \begin{bmatrix} 0 \\ x \end{bmatrix} dy$

Example.

Consider an $\mathbb{R}^2$-valued differential 1-form $\alpha_{(x,y)} := \begin{bmatrix} x^2 \\ xy \end{bmatrix} dx + \begin{bmatrix} xy \\ y^2 \end{bmatrix} dy$

Then $d\alpha = \left(\begin{bmatrix} 2x \\ y \end{bmatrix} dx + \begin{bmatrix} 0 \\ x \end{bmatrix} dy \right) \wedge dx + \left(\begin{bmatrix} y \\ 0 \end{bmatrix} dx + \begin{bmatrix} x \\ 2y \end{bmatrix} dy \right) \wedge dy = \begin{bmatrix} y \\ -x \end{bmatrix} dx \wedge dy$

Thanks!



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