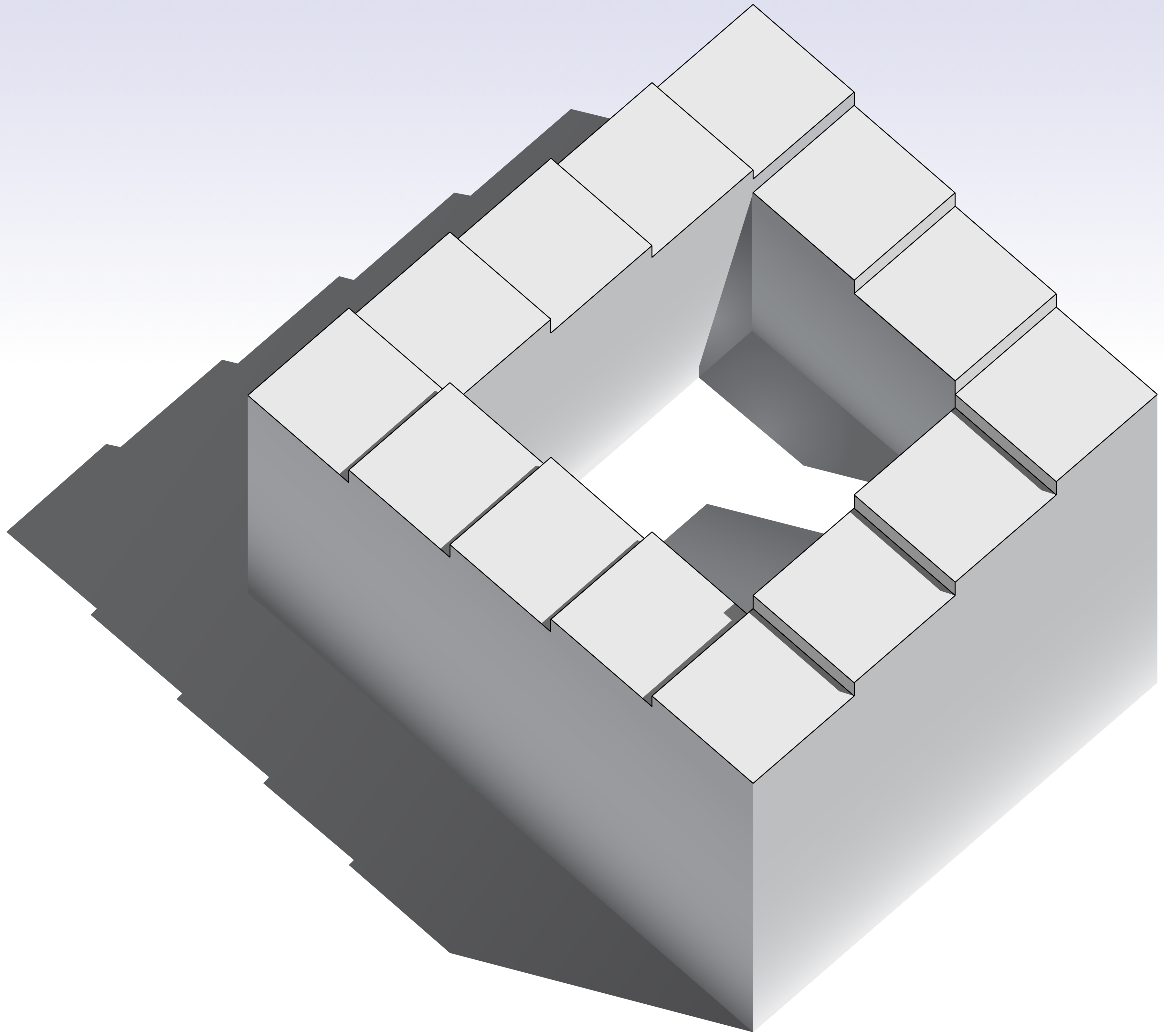
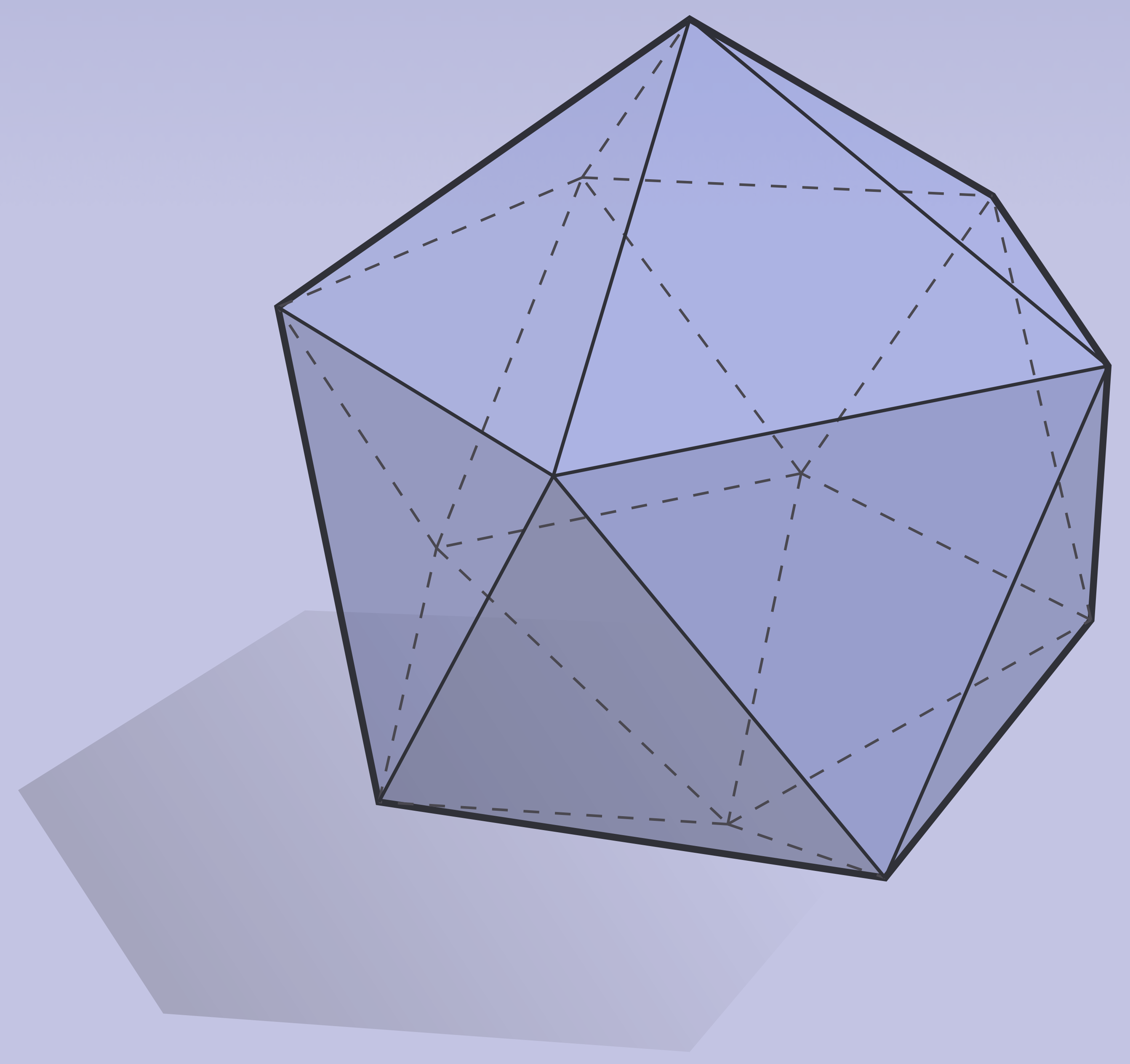


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

LECTURE 15:
DISCRETE CURVATURE I (INTEGRAL)



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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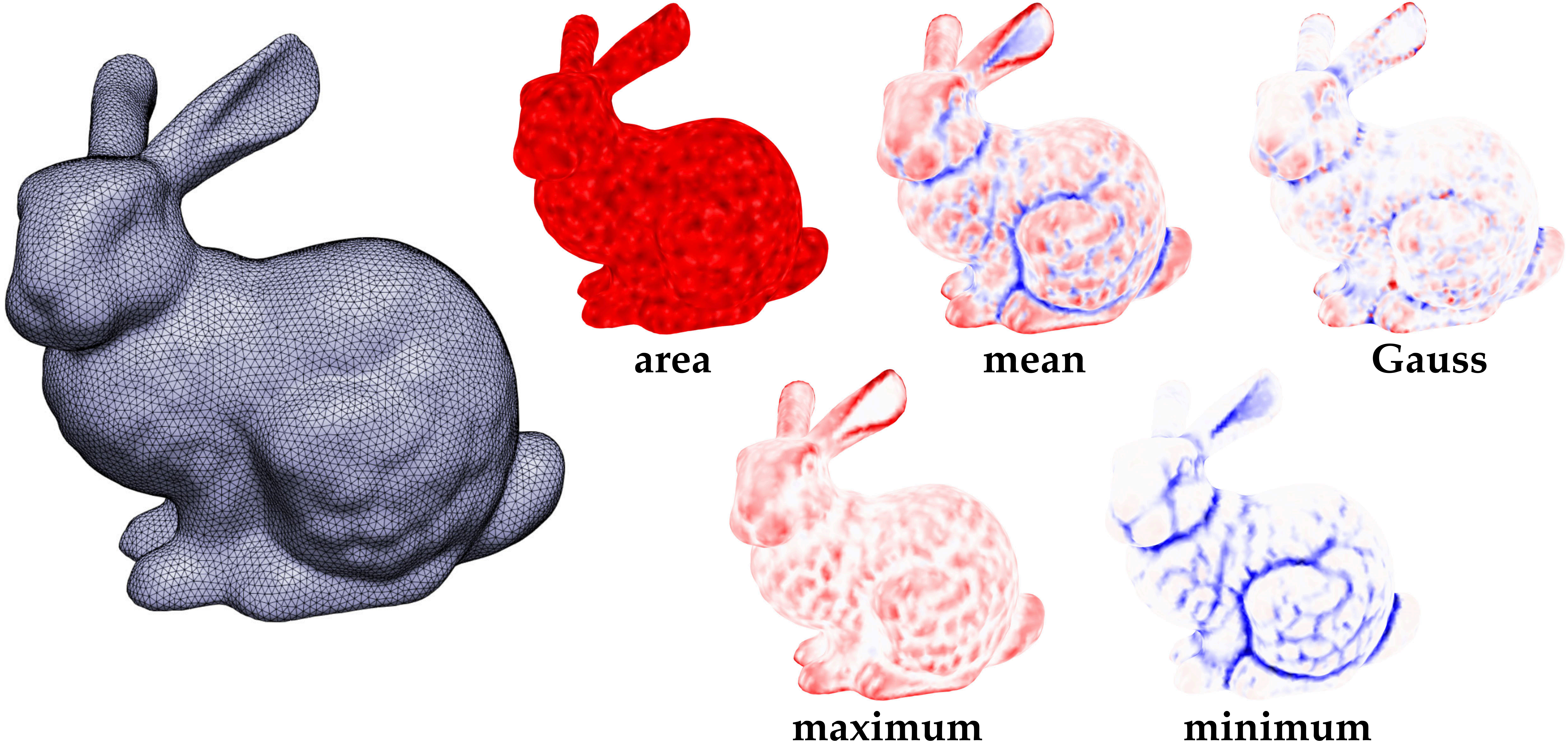
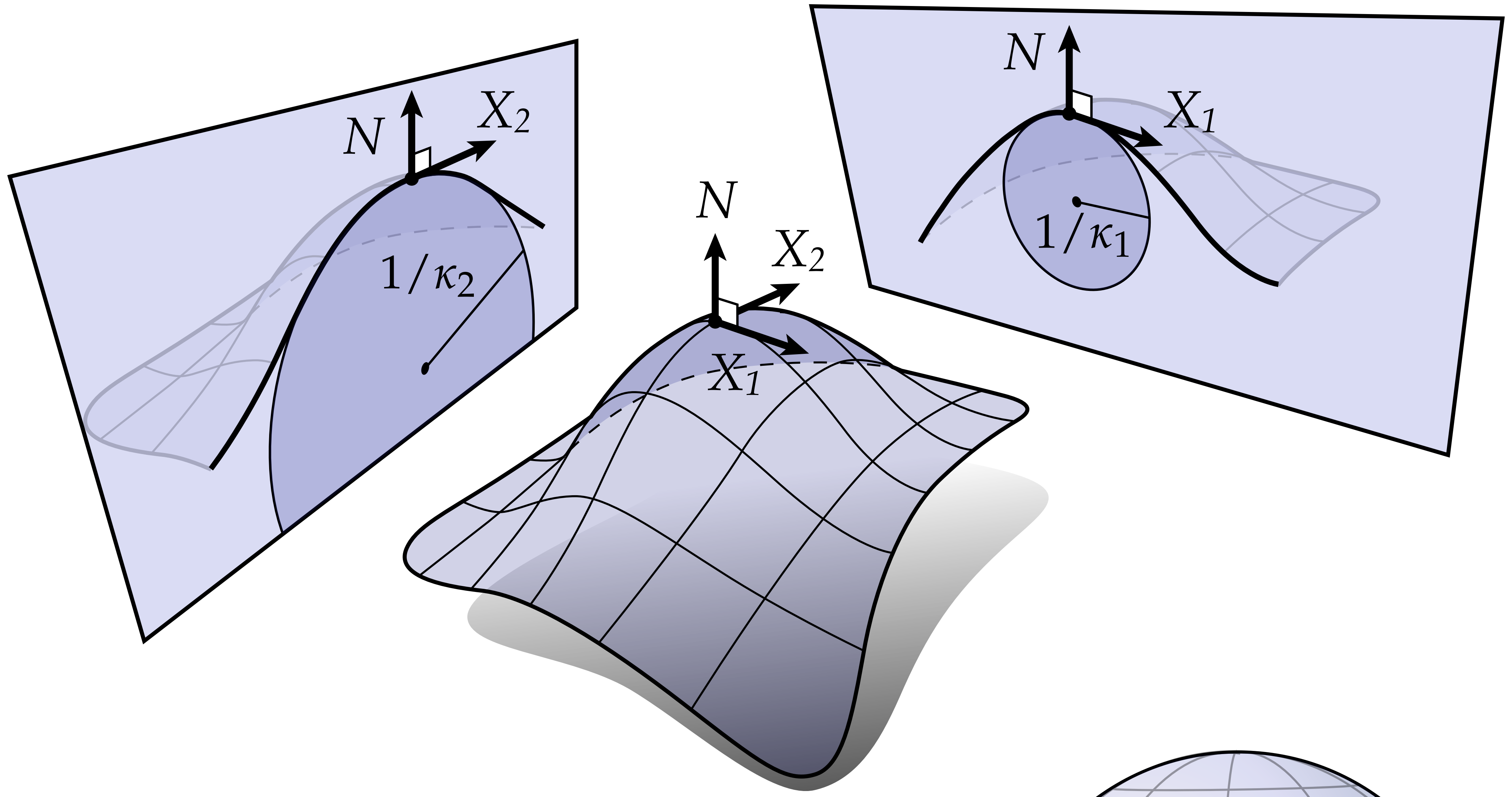
Discrete Curvature

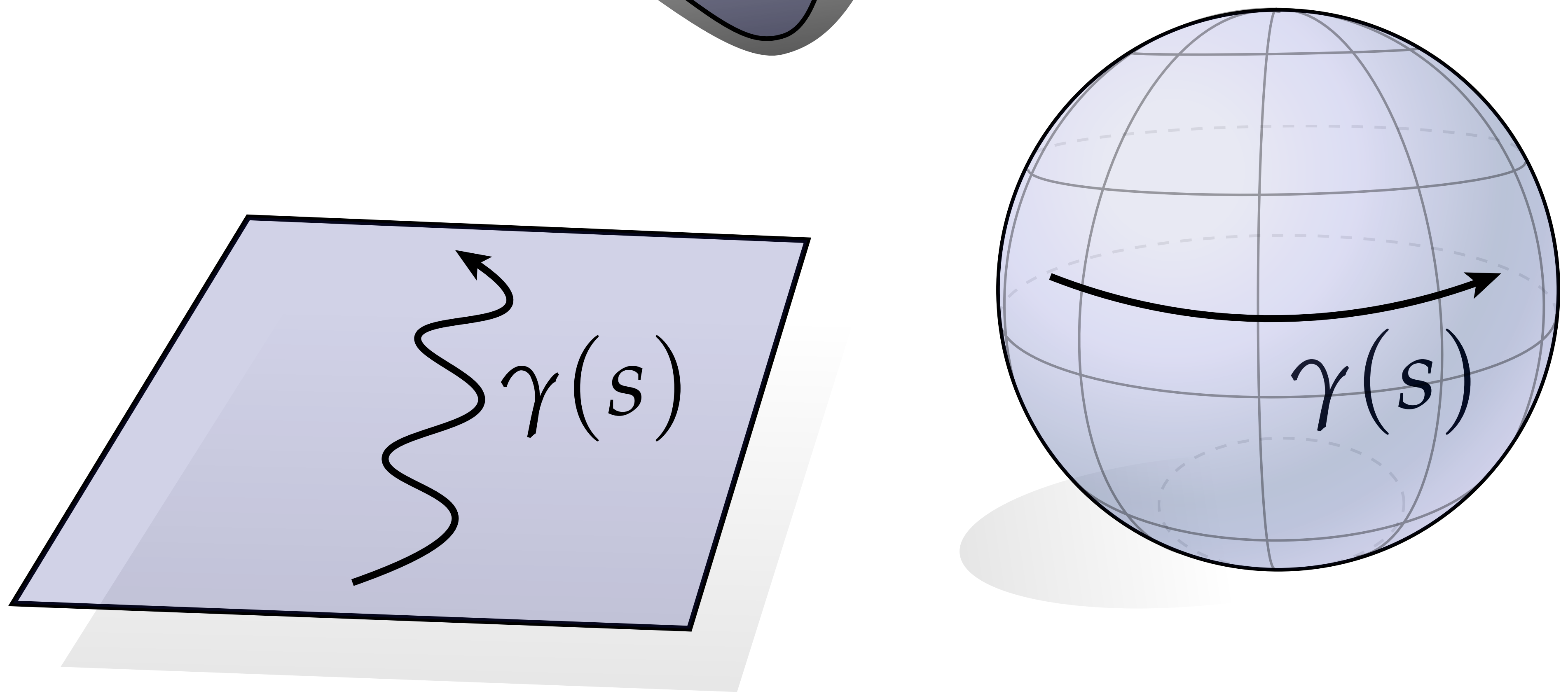
Discrete Curvature—Visualized



Curvature of Surfaces

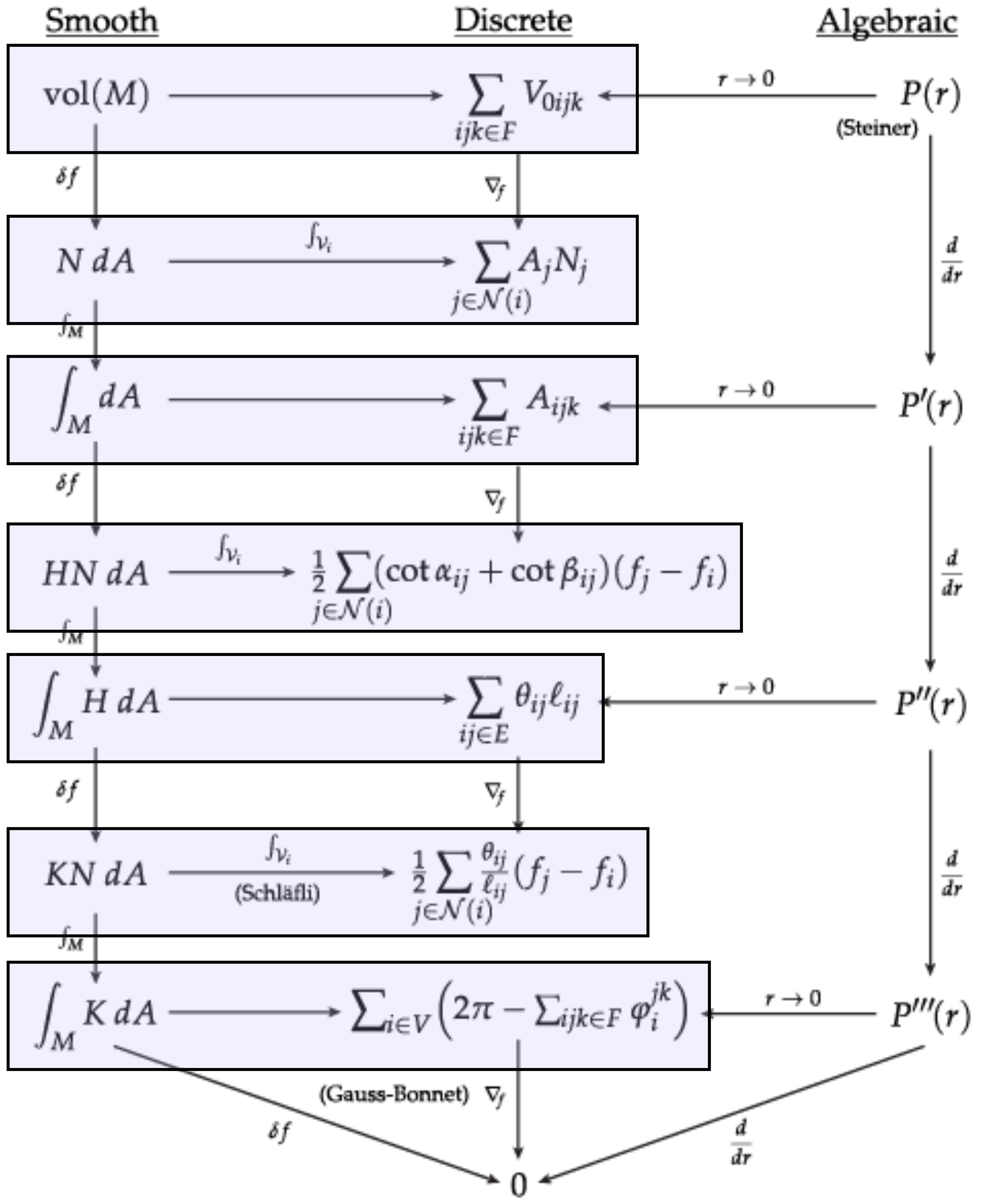
- In smooth setting, had many different curvatures (normal, principal, Gauss, mean, geodesic, …)
- In discrete setting, appear to be many different choices for discretization
- Actually, there is a unified viewpoint that helps explain many common choices…





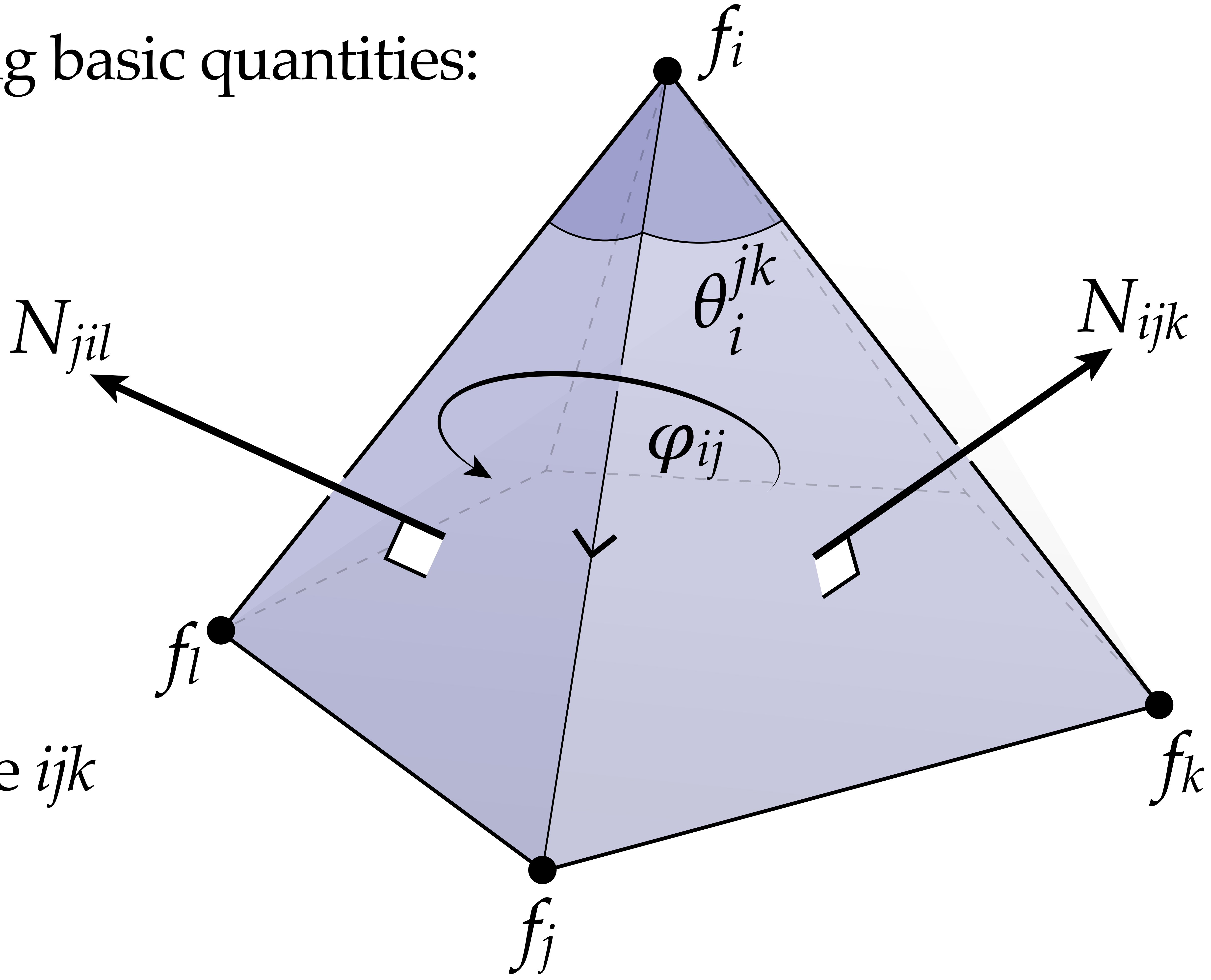
A Unified Picture of Discrete Curvature

- By making some connections between smooth and discrete surfaces, we get a unified picture of many different discrete curvatures scattered throughout the literature
- To tell the full story we'll need a few pieces:
  - **geometric derivatives**
  - **Steiner polynomials**
  - **sequence of curvature variations**
  - **assorted theorems** (Gauss-Bonnet, Schläfli, $\Delta f = 2HN$)
- Start with *integral* viewpoint (1st lecture), then cover *variational* viewpoint (2nd lecture).



Quantities & Conventions

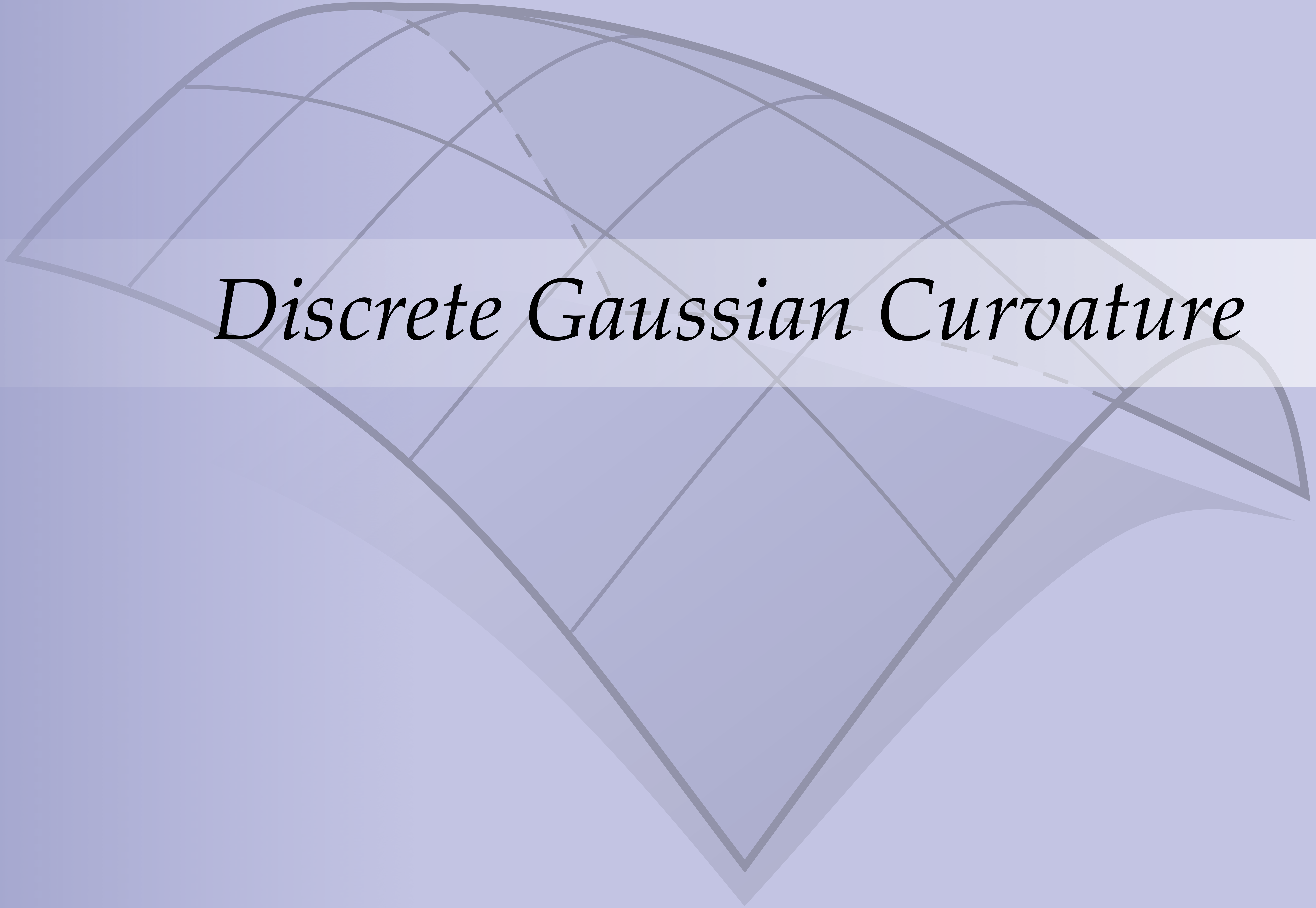
- Throughout we will consider the following basic quantities:
  - f_i — position of vertex i
  - e_{ij} — vector from i to j
  - ℓ_{ij} — length of edge ij
  - A_{ijk} — area of triangle ijk
  - N_{ijk} — unit normal of triangle ijk
  - θ_i^{jk} — interior angle at vertex i of triangle ijk
  - φ_{ij} — dihedral angle at oriented edge ij



$$\varphi_{ij} := \text{atan2}(\hat{e} \cdot N_{ijk} \times N_{jil}, N_{ijk} \cdot N_{jil}), \qquad \hat{e}_{ij} := e_{ij}/\ell_{ij}$$

Q: Which of these quantities are discrete differential forms? (And what kind?)

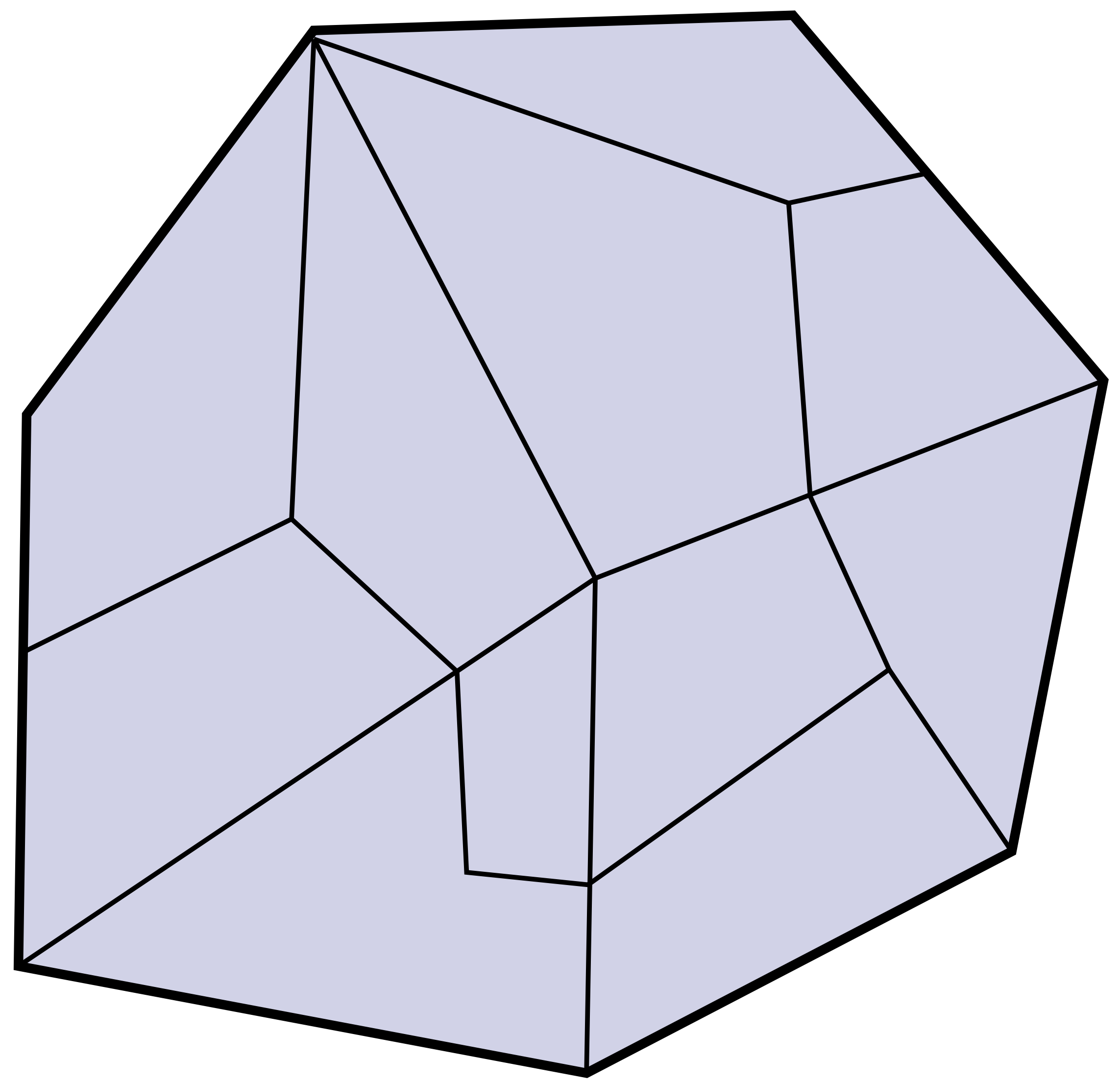
Discrete Gaussian Curvature



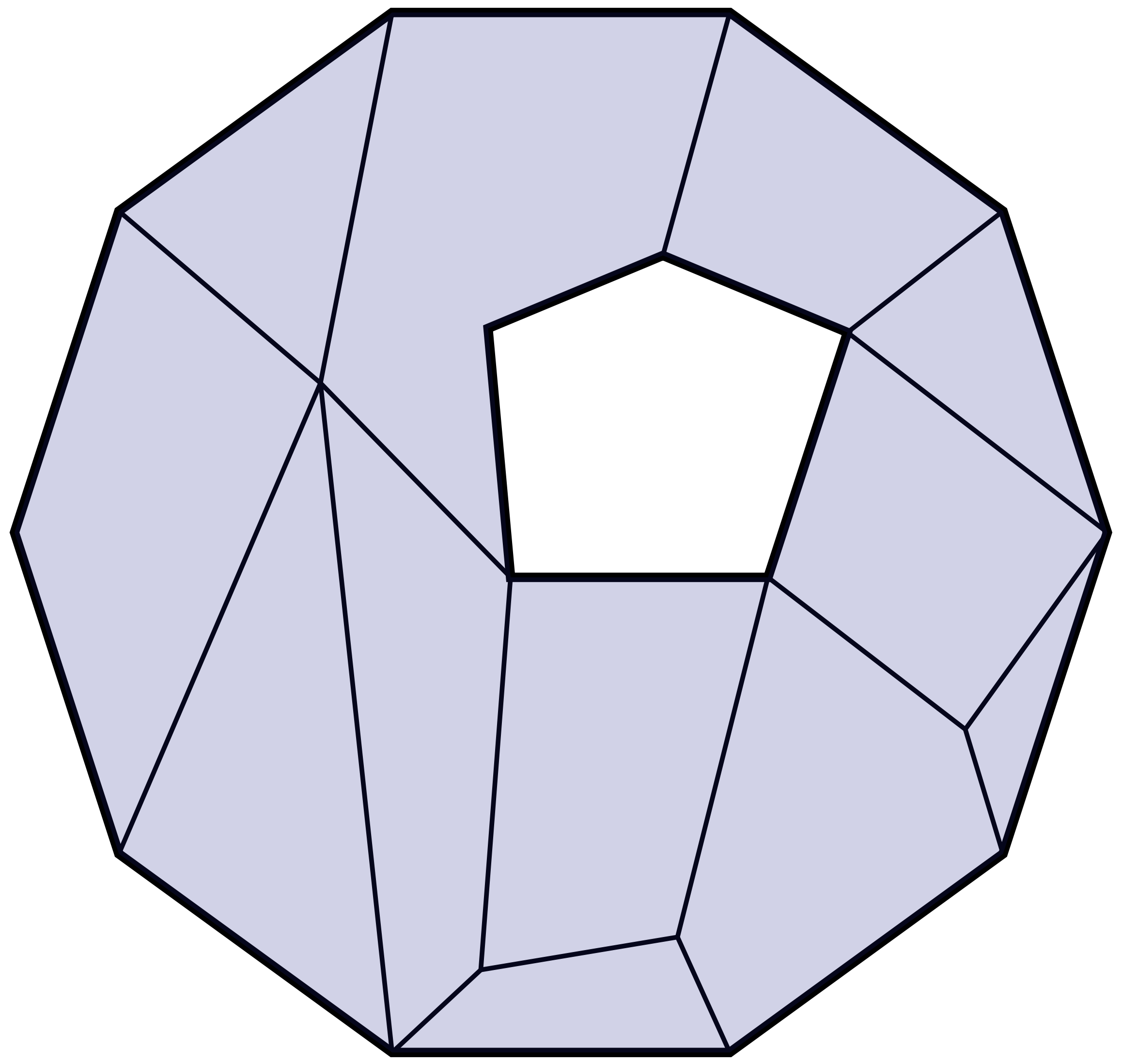
Euler Characteristic

The **Euler characteristic** of a simplicial 2-complex $K=(V,E,F)$ is the constant

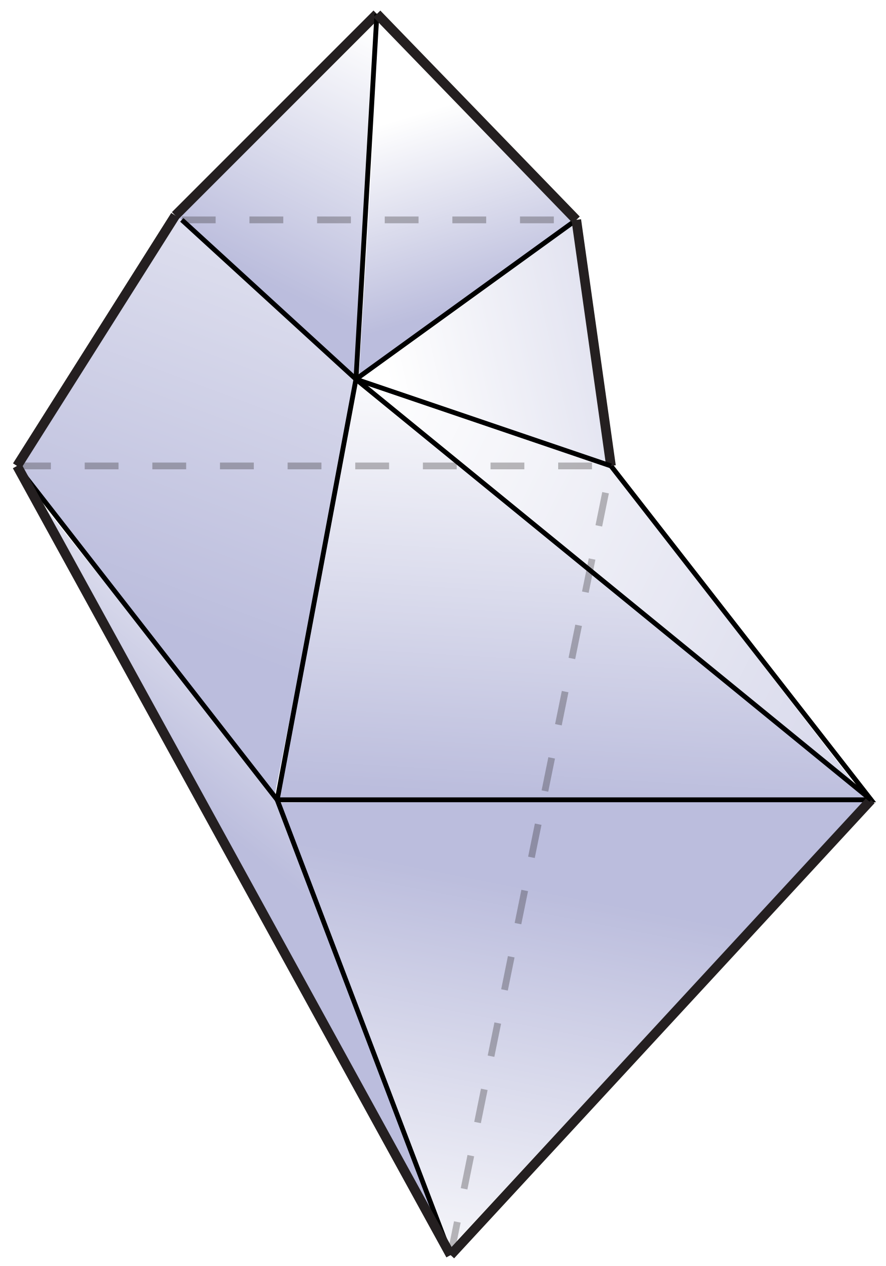
$$\boxed{\chi := V - E + F}$$



$\chi = 1$



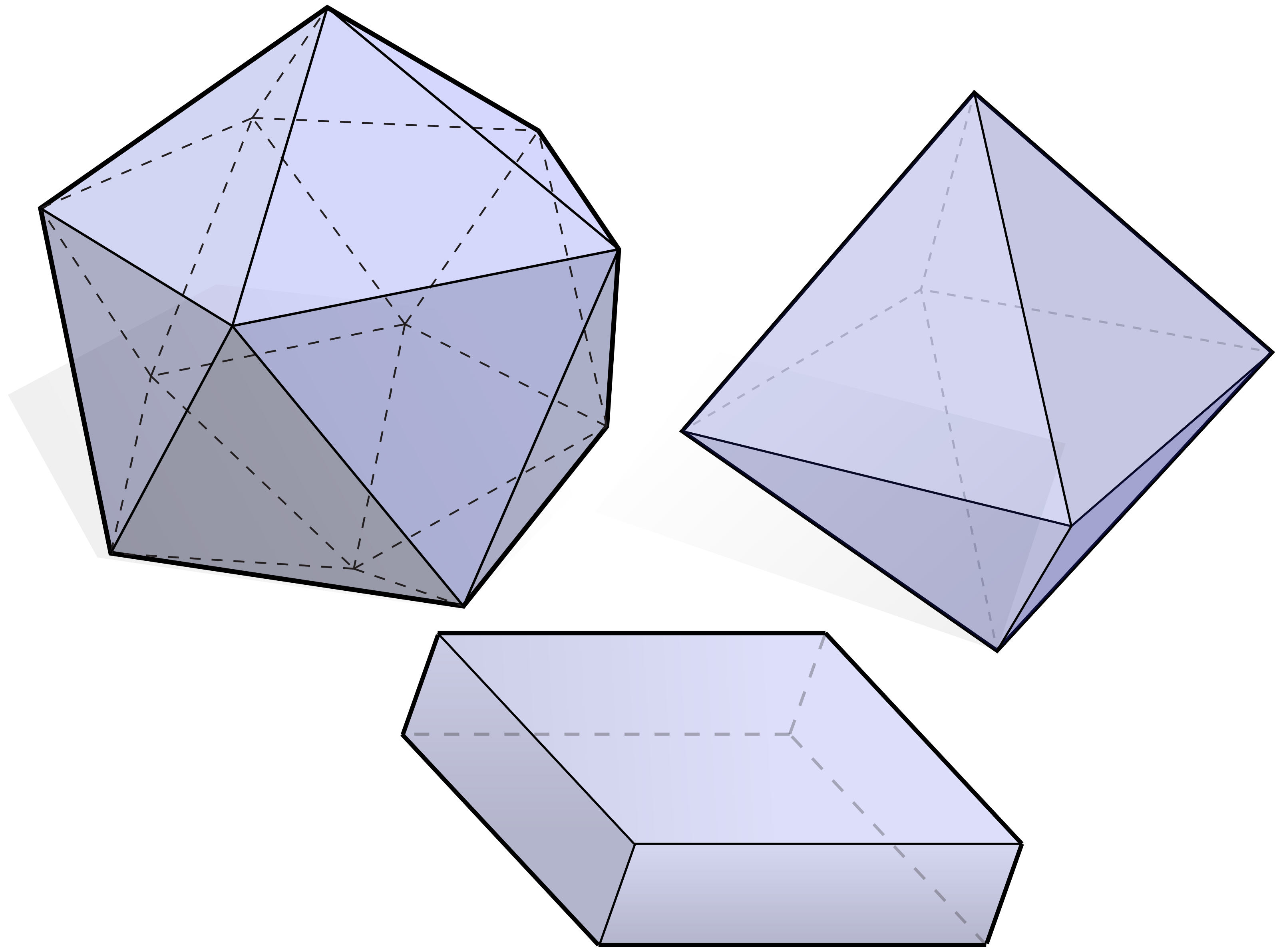
$\chi = 0$



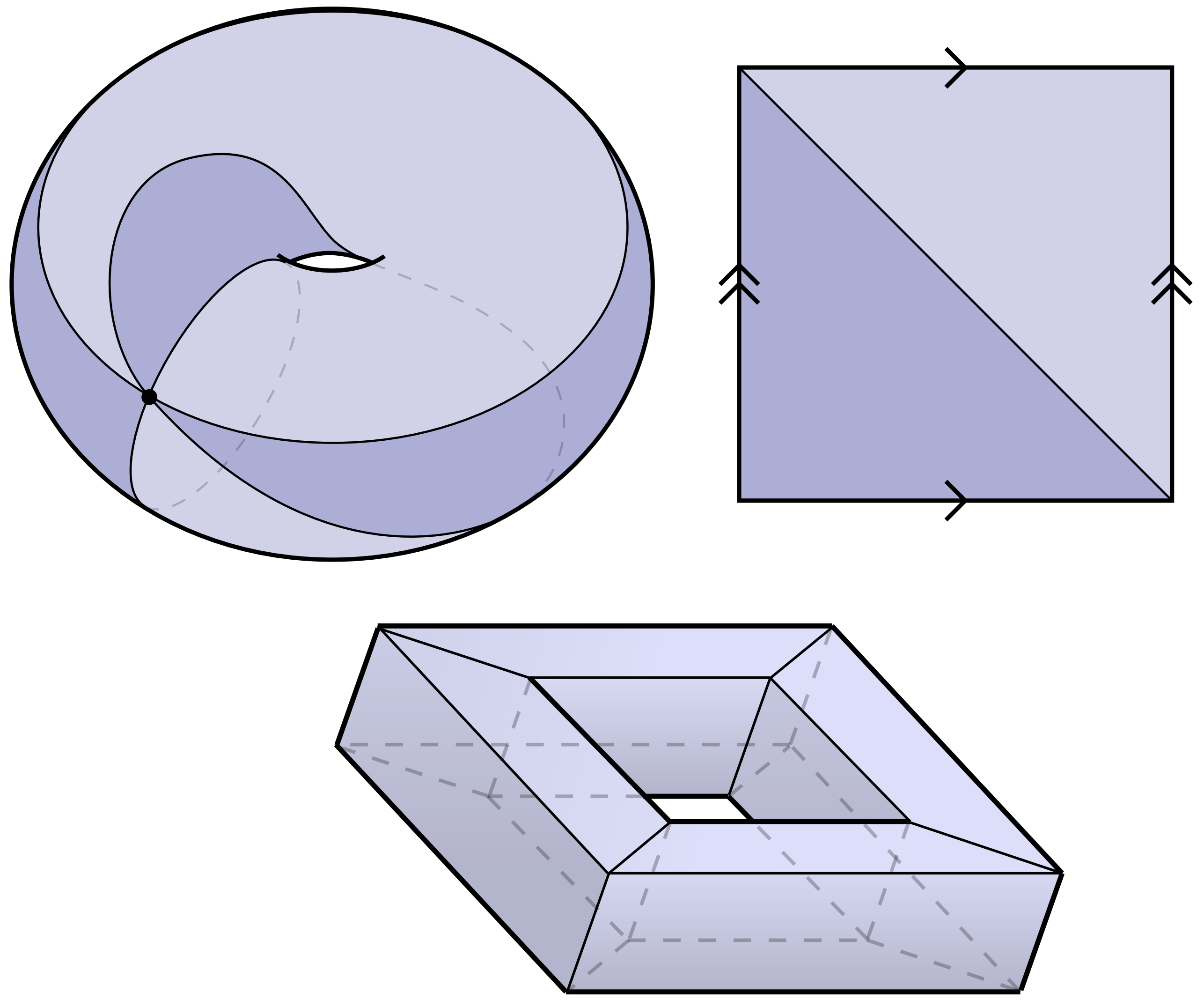
$\chi = 2$

Topological Invariance of the Euler Characteristic

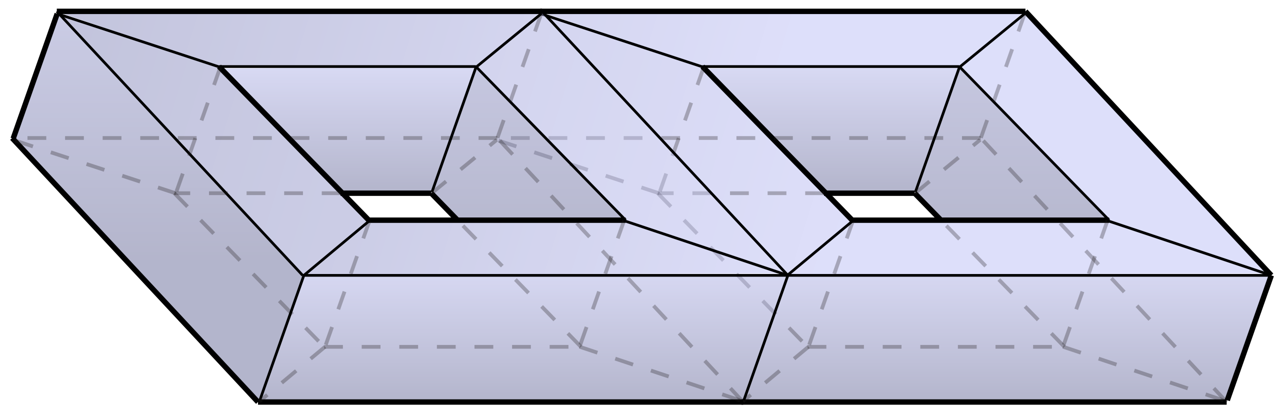
Fact. (L'Huilier) For simplicial surfaces w/out boundary, the Euler characteristic is a topological invariant. *E.g.*, for a torus of genus g, $\chi = 2\text{-}2g$ (independent of the particular tessellation).



$g = 0$



$g = 1$

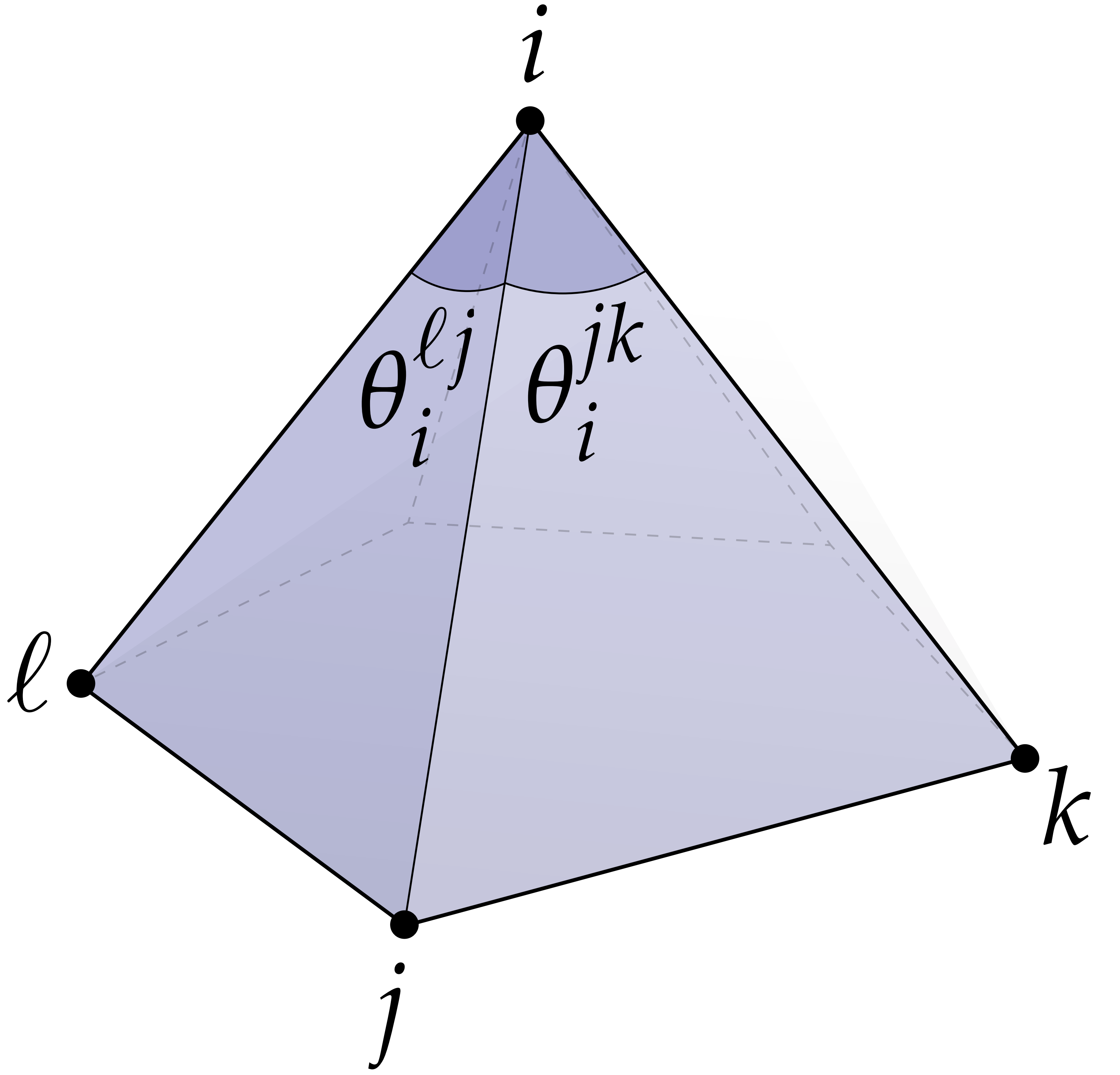


$g = 2$

Angle Defect

- The **angle defect** at a vertex i is the deviation of the sum of interior angles from the Euclidean angle sum of 2π:

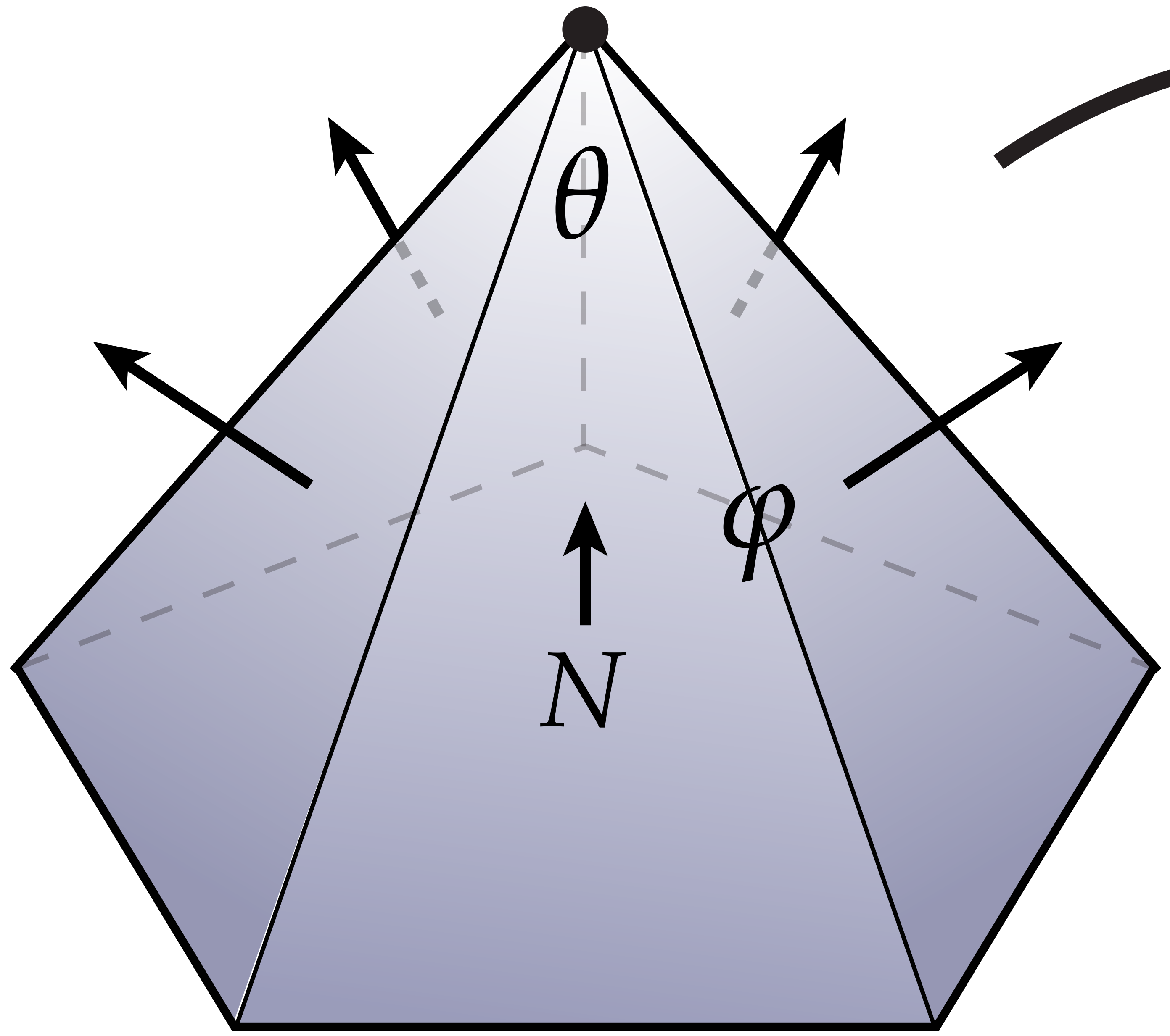
$$\Omega_i := 2\pi - \sum_{ijk} \theta_i^{jk}$$

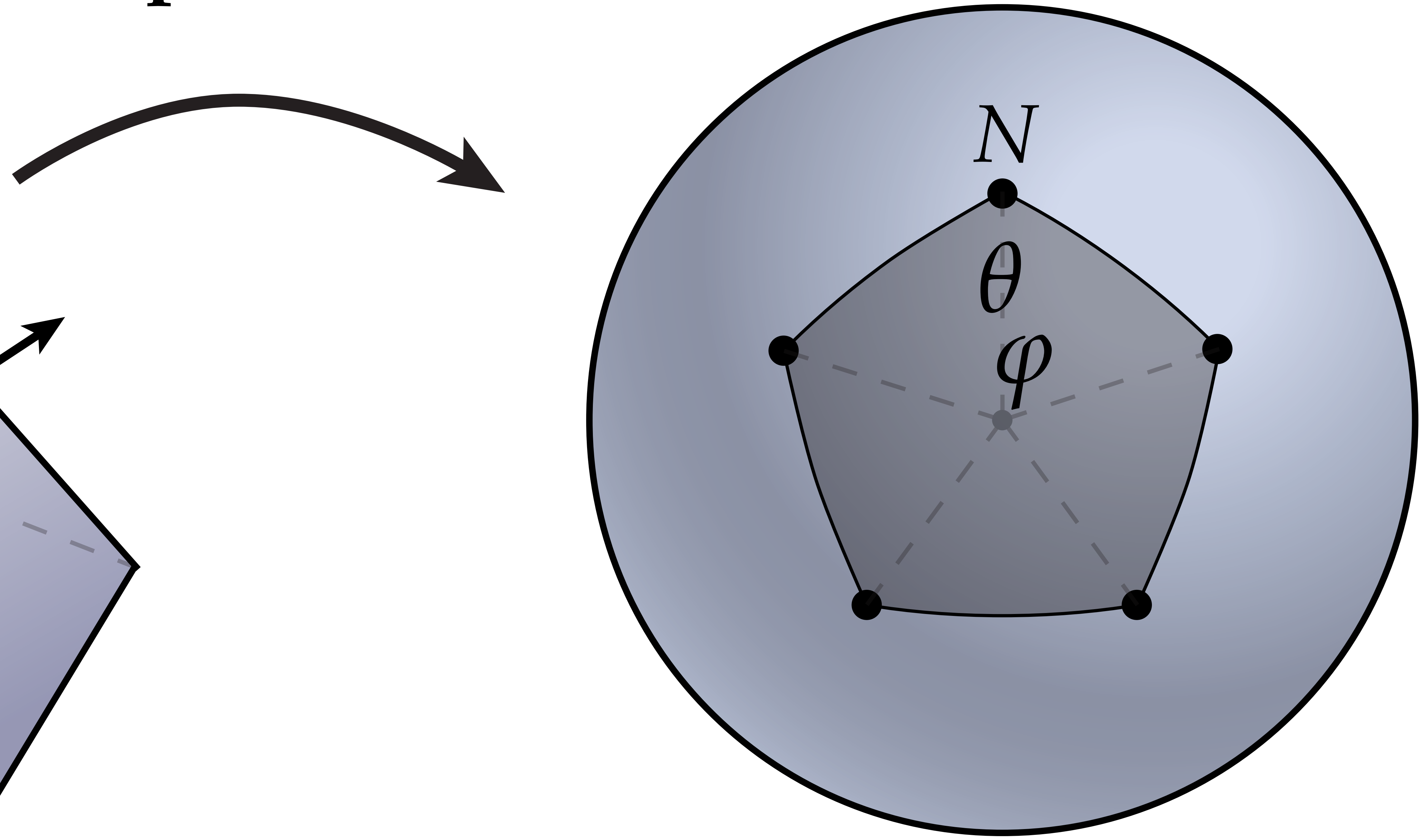


Intuition: how "flat" is the vertex?

Angle Defect and Spherical Area

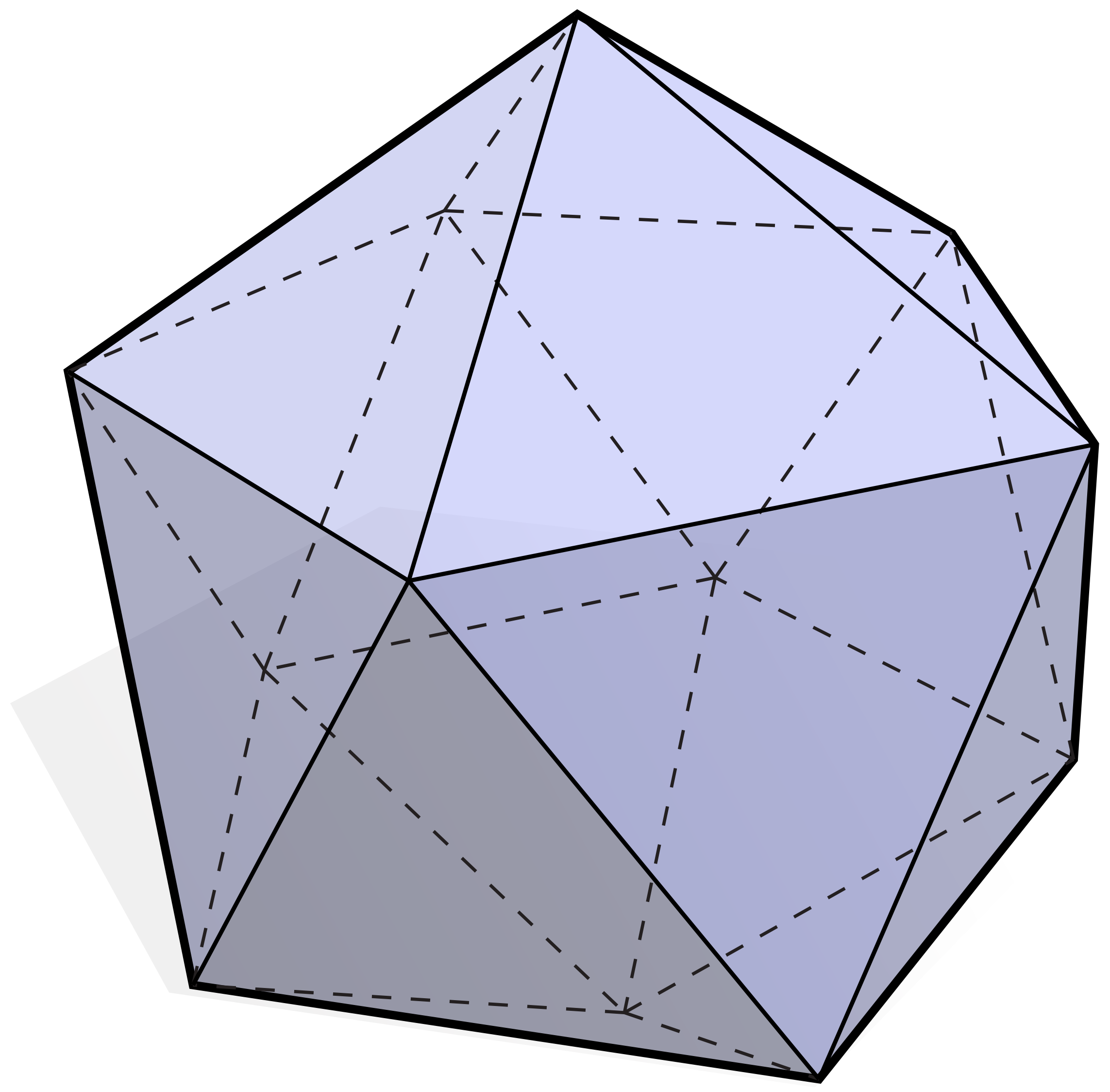
- Consider the discrete Gauss map…
  - …unit normals on surface become points on the sphere
  - …dihedral angles on surface become interior angles on sphere
  - …interior angles on surface become dihedral angles on the sphere
  - …**angle defect on surface becomes area on the sphere**

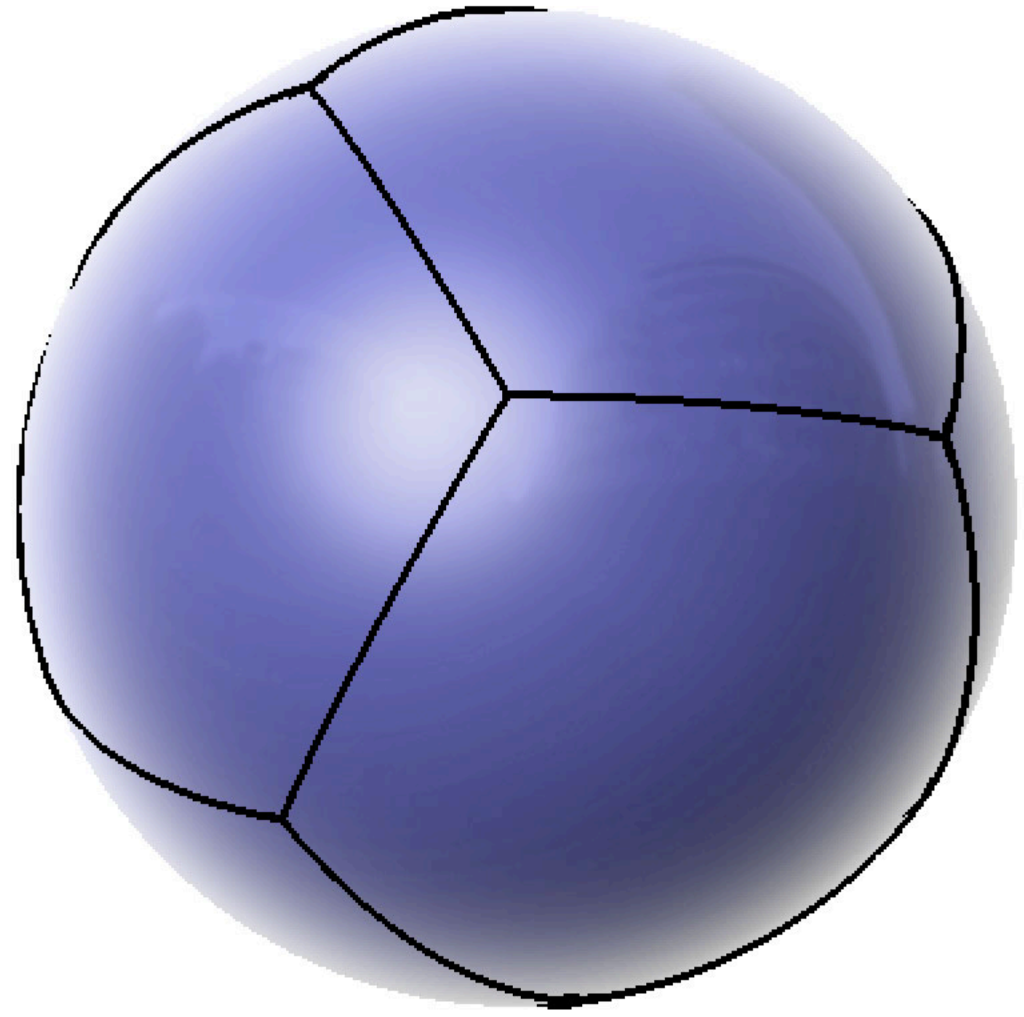




Total Angle Defect of a Convex Polyhedron

- Consider a closed convex polyhedron in R^3
- **Q:** Given that angle defect is equivalent to spherical area, what might we guess about total angle defect?
- **A:** Equal to 4π! (Area of unit sphere)
- More generally, can argue that total angle defect is equal to 4π for *any* polyhedron with spherical topology, and $2\pi(2\text{-}2g)$ for any polyhedron of genus g
- Should remind you of *Gauss-Bonnet theorem*

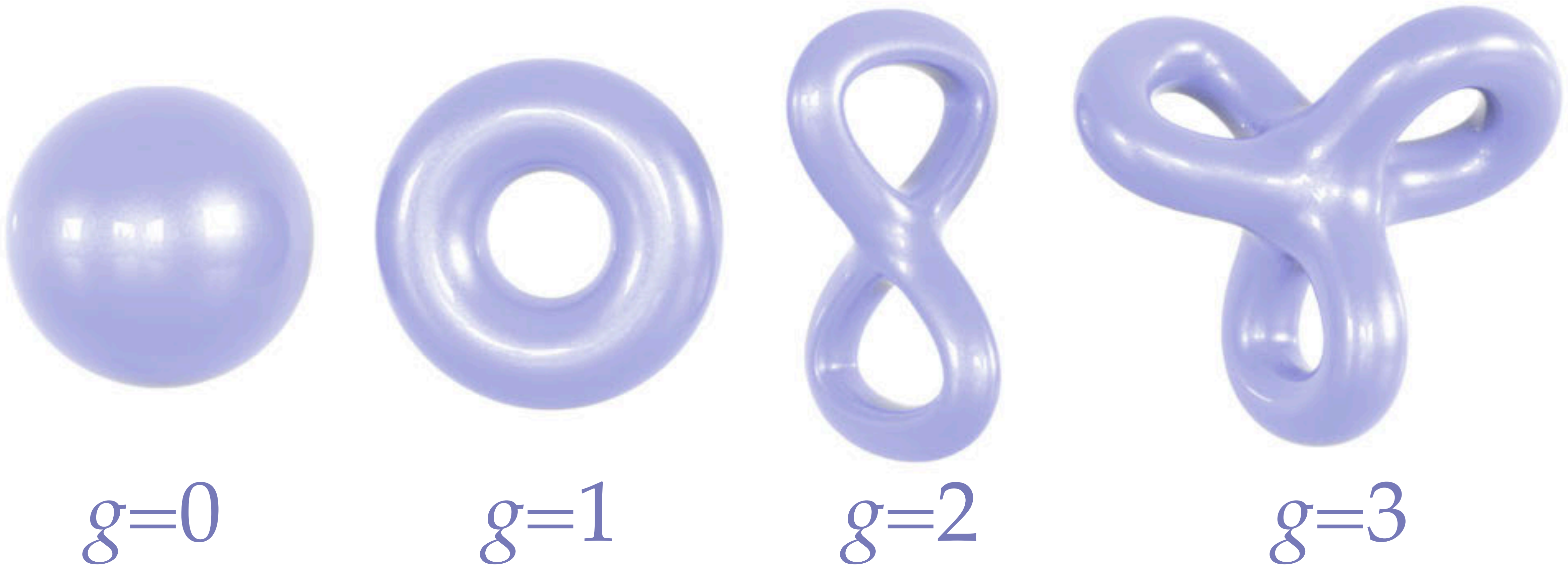




Review: Gauss-Bonnet Theorem

- Classic example of *local-global* theorems in differential geometry
- Gauss-Bonnet theorem says total Gaussian curvature is always 2π times *Euler characteristic* χ
- For tori, Euler characteristic expressed in terms of the *genus* (number of “handles”)

$$\chi := 2 - 2g$$



Gauss-Bonnet

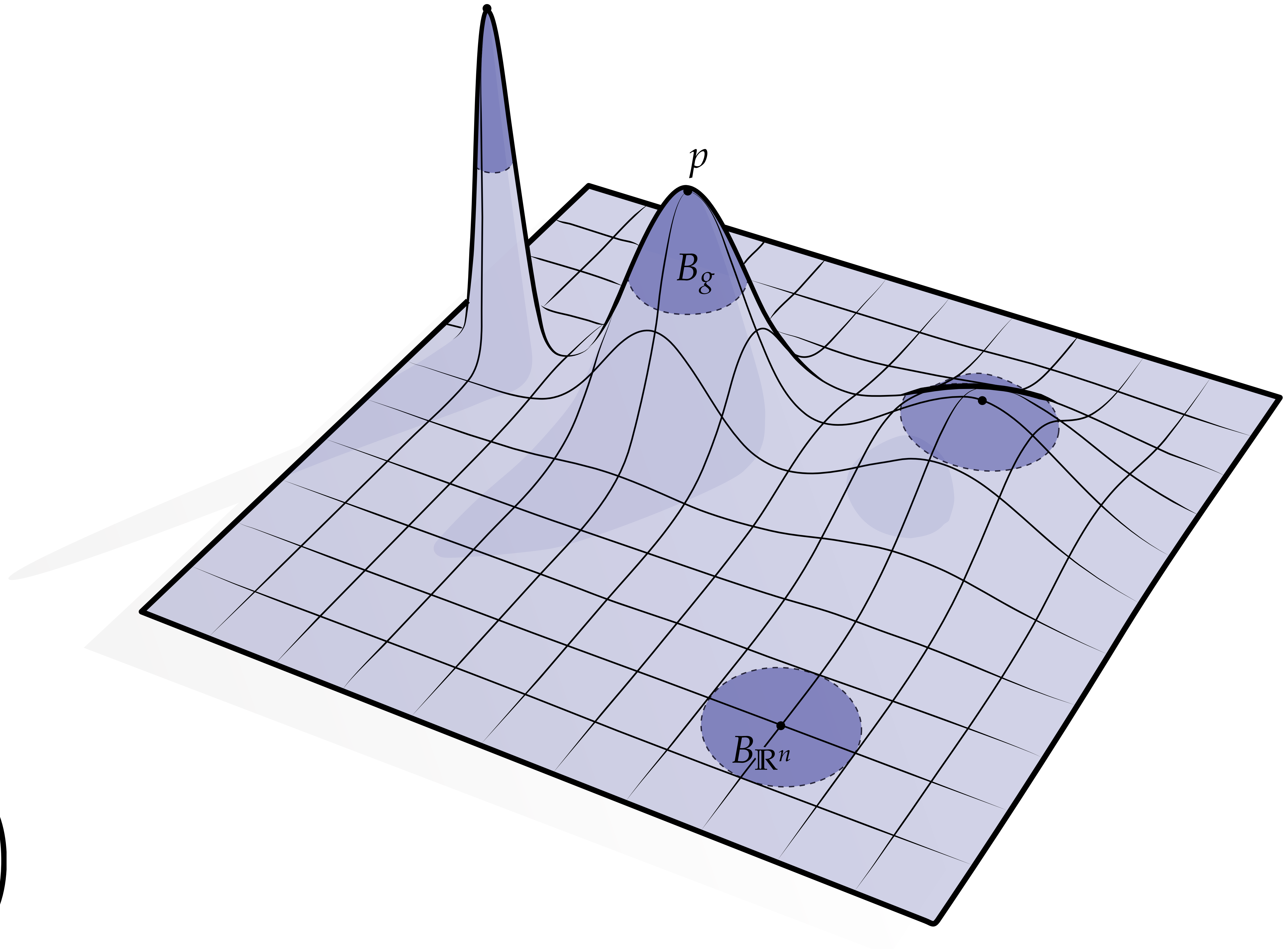
$$\int_M K\,dA = 2\pi\chi$$

Gaussian Curvature as Ratio of Ball Areas

- Originally defined Gaussian curvature as product of principal curvatures
- Can also view it as “failure” of balls to behave like Euclidean balls

Roughly speaking,

$$K \propto 1 - \frac{|B_g|}{|B_{\mathbb{R}^2}|}$$



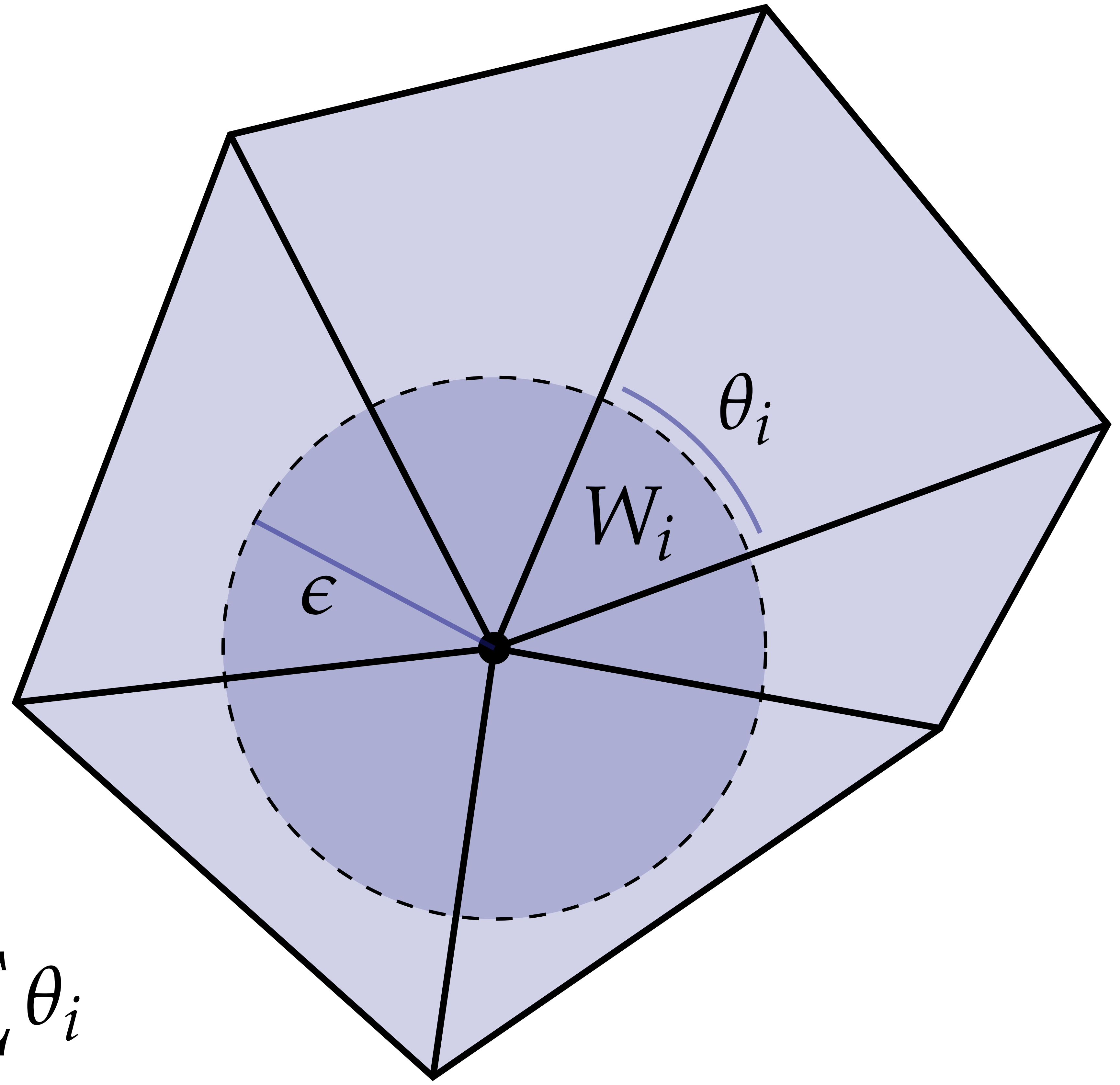
More precisely:

$$|B_g(p,\varepsilon)| = |B_{\mathbb{R}^2}(p,\varepsilon)| \left(1 - \frac{K}{12}\varepsilon^2 + O(\varepsilon^3)\right)$$

Discrete Gaussian Curvature as Ratio of Areas

- For small values of ε, we have

$$\frac{\varepsilon^2}{12}K \approx 1 - \frac{|B_g(\varepsilon)|}{|B_{\mathbb{R}^2}(\varepsilon)|}$$



Substitute

area of Euclidean ball $|B_{\mathbb{R}^2}(\varepsilon)| = \pi\varepsilon^2$

area of geodesic "wedge" $W_i(\varepsilon) = \frac{\theta_i}{2\pi}|B_{\mathbb{R}^2}| = \frac{1}{2}\varepsilon^2\theta_i$

area of geodesic ball $|B_g(\varepsilon)| = \sum_i W_i(\varepsilon) = \frac{\varepsilon^2}{2}\sum_i \theta_i$

Then

$$\frac{\varepsilon^2}{12}K = 1 - \frac{1}{2\pi}\sum_i \theta_i \iff \boxed{2\pi - \sum_i \theta_i = \frac{1}{6}\pi\varepsilon^2 K}$$

Angle defect is *integrated* curvature

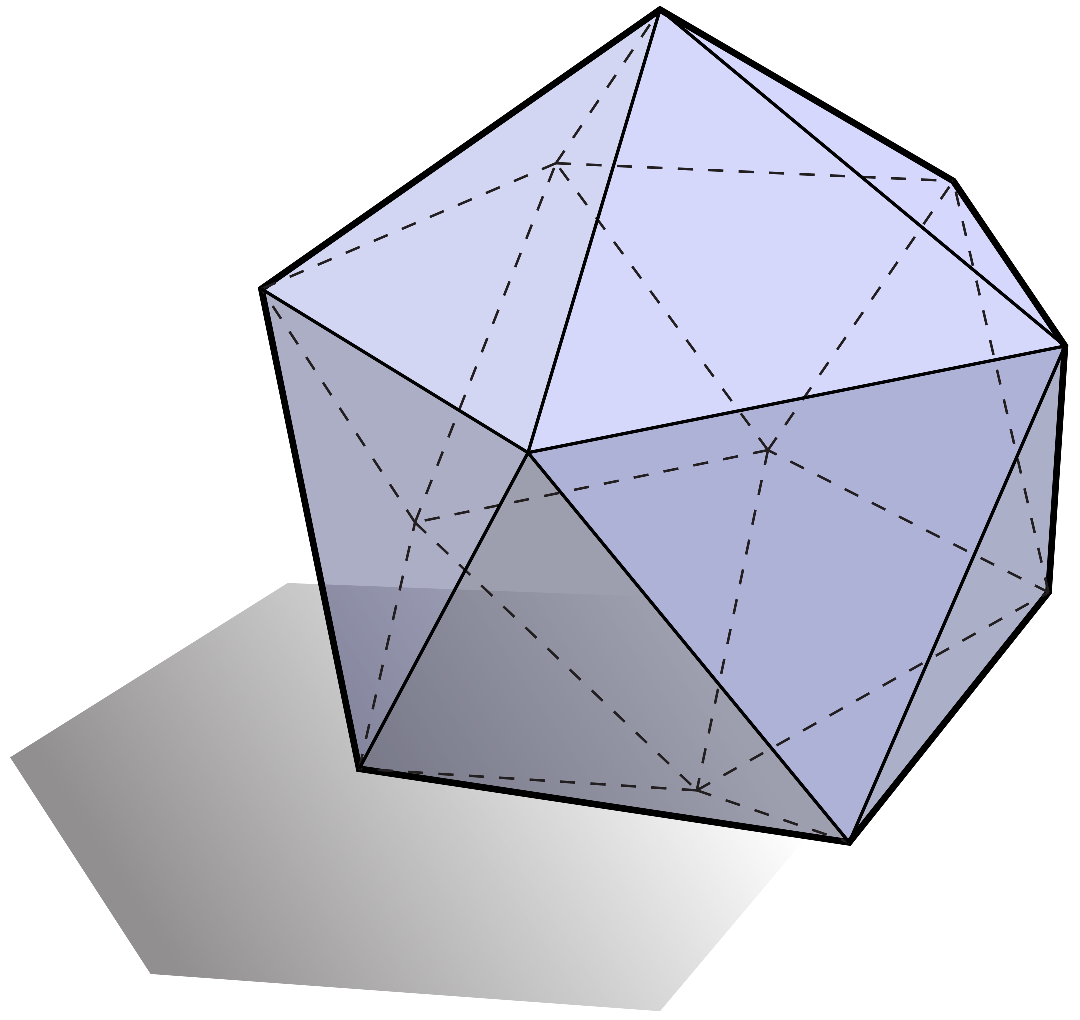
Discrete Gauss Bonnet Theorem

Theorem. For a smooth surface of genus g, the total Gauss curvature is

$$\int_M K\, dA = 2\pi\chi$$

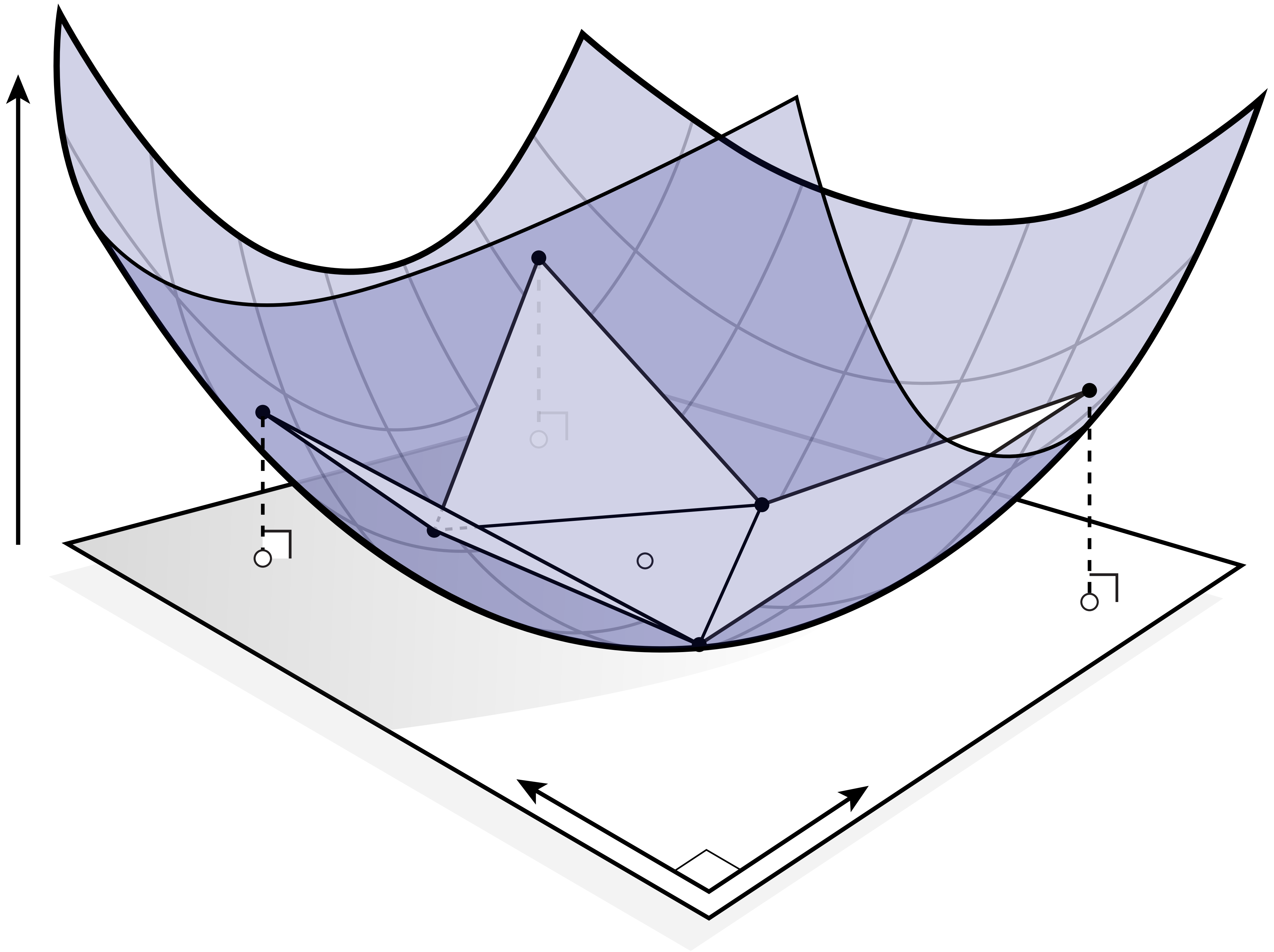
Theorem. For a simplicial surface of genus g, the total angle defect is

$$\sum_{i \in V} \Omega_i = 2\pi\chi$$



Approximating Gaussian Curvature

- Many other ways to approximate Gaussian curvature
- *E.g.*, locally fit quadratic functions, compute smooth Gaussian curvature
- Which way is "best"?
  - values from quadratic fit won't satisfy Gauss-Bonnet
  - angle defects won't converge[1] unless vertex valence is 4 or 6
- In general, no best way; each choice has its own pros & cons

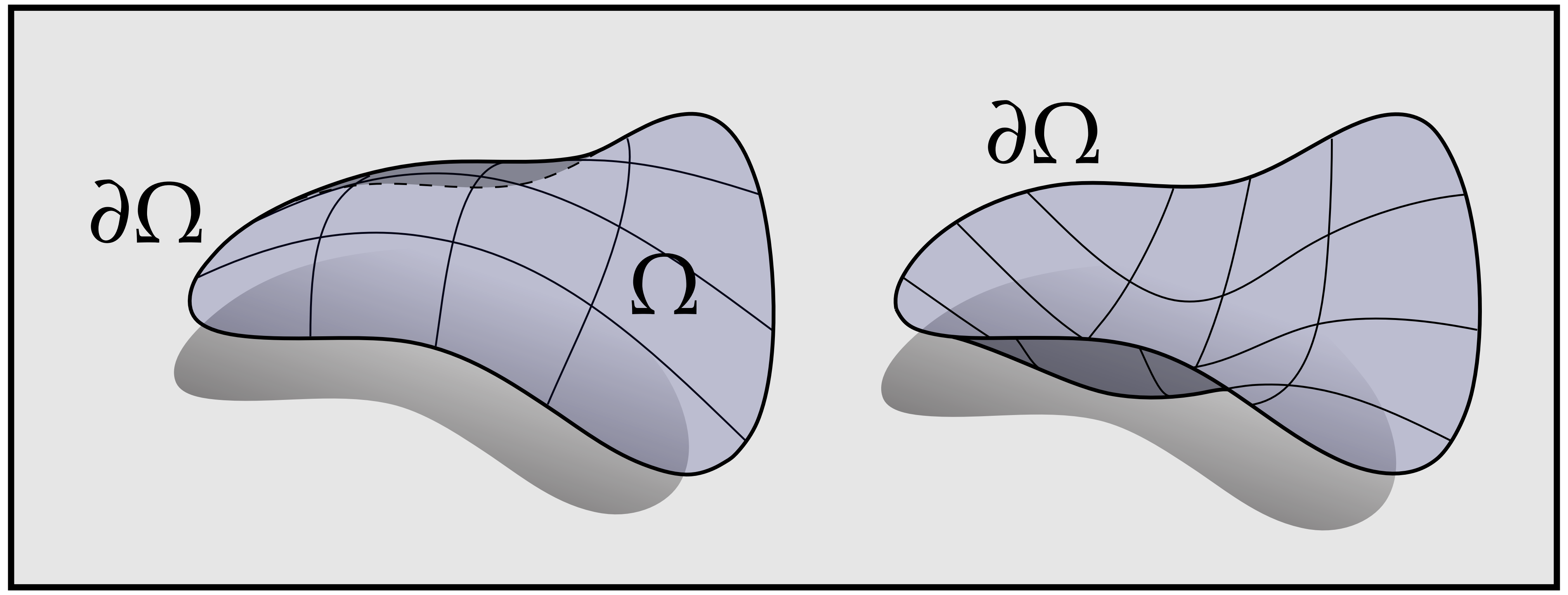


[1]Borrelli, Cazals, Morvan, *"On the angular defect of triangulations and the pointwise approximation of curvatures"*

Curvature Normals

Curvature Normals

- Earlier we saw vector area, which was the integral of the 2-form NdA



- This 2-form is one of three quantities we can naturally associate with a surface:

$$\frac{1}{2}\, df \wedge df = NdA \qquad \textbf{(area normal)}$$

$$\frac{1}{2}\, df \wedge dN = HNdA \qquad \textbf{(mean curvature normal)}$$

$$\frac{1}{2}\, dN \wedge dN = KNdA \qquad \textbf{(Gauss curvature normal)}$$

- Effectively *mixed areas* of change in position & normal (more later)

Curvature Normals—Derivation

- Let X_1, X_2 be principal curvature directions (recall that $dN(X_i) = \kappa_i \, df(X_i)$). Then

$$df \wedge df(X_1, X_2) = df(X_1) \times df(X_2) - df(X_2) \times df(X_1) =$$
$$2df(X_1) \times df(X_2) = \boxed{2NdA(X_1, X_2)}$$

$$df \wedge dN(X_1, X_2) = df(X_1) \times dN(X_2) - df(X_2) \times dN(X_1) =$$
$$\kappa_1 df(X_1) \times df(X_2) - \kappa_2 df(X_2) \times df(X_1) =$$
$$(\kappa_1 + \kappa_2) df(X_1) \times df(X_2) = \boxed{2HNdA(X_1, X_2)}$$

$$dN \wedge dN(X_1, X_2) = dN(X_1) \times dN(X_2) - dN(X_2) \times dN(X_1) =$$
$$\kappa_1 \kappa_2 df(X_1) \times df(X_2) - \kappa_2 \kappa_1 df(X_2) \times df(X_1) =$$
$$2Kdf(X_1) \times df(X_2) = \boxed{2KNdA(X_1, X_2)}$$

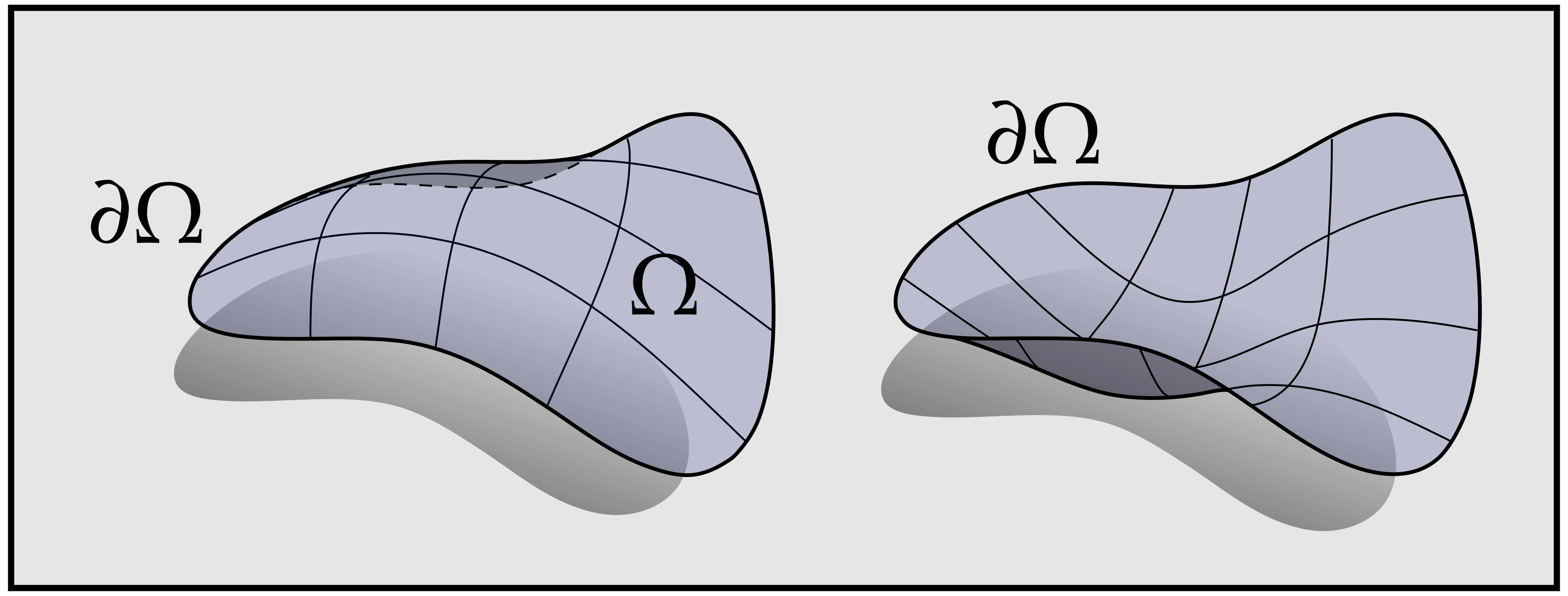
Discrete Vector Area

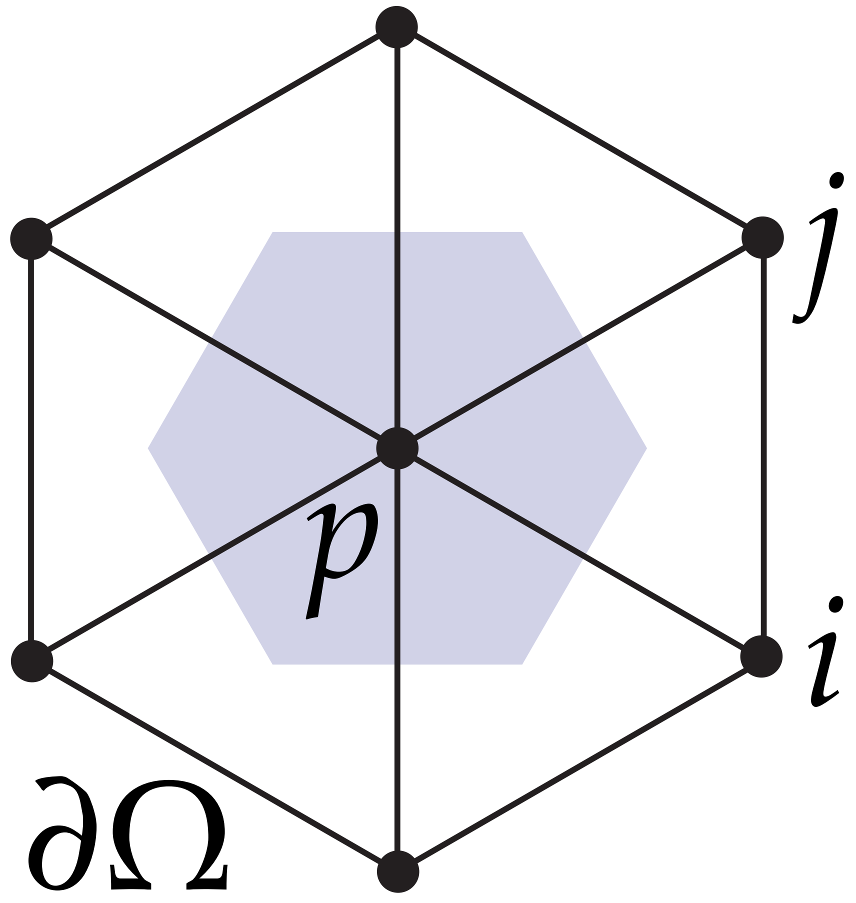
- Recall smooth vector area: $\int_\Omega N\,dA = \frac{1}{2}\int_\Omega df \wedge df = \frac{1}{2}\int_{\partial\Omega} f \times df$
- **Idea:** Integrate NdA over dual cell to get normal at vertex p

$$\frac{1}{3}\int_\Omega N\,dA = \frac{1}{6}\int_{\partial\Omega} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\int_{e_{ij}} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\frac{f_i+f_j}{2}\times(f_j - f_i) = \boxed{\frac{1}{6}\sum_{ij\in\partial\Omega} f_i \times f_j}$$





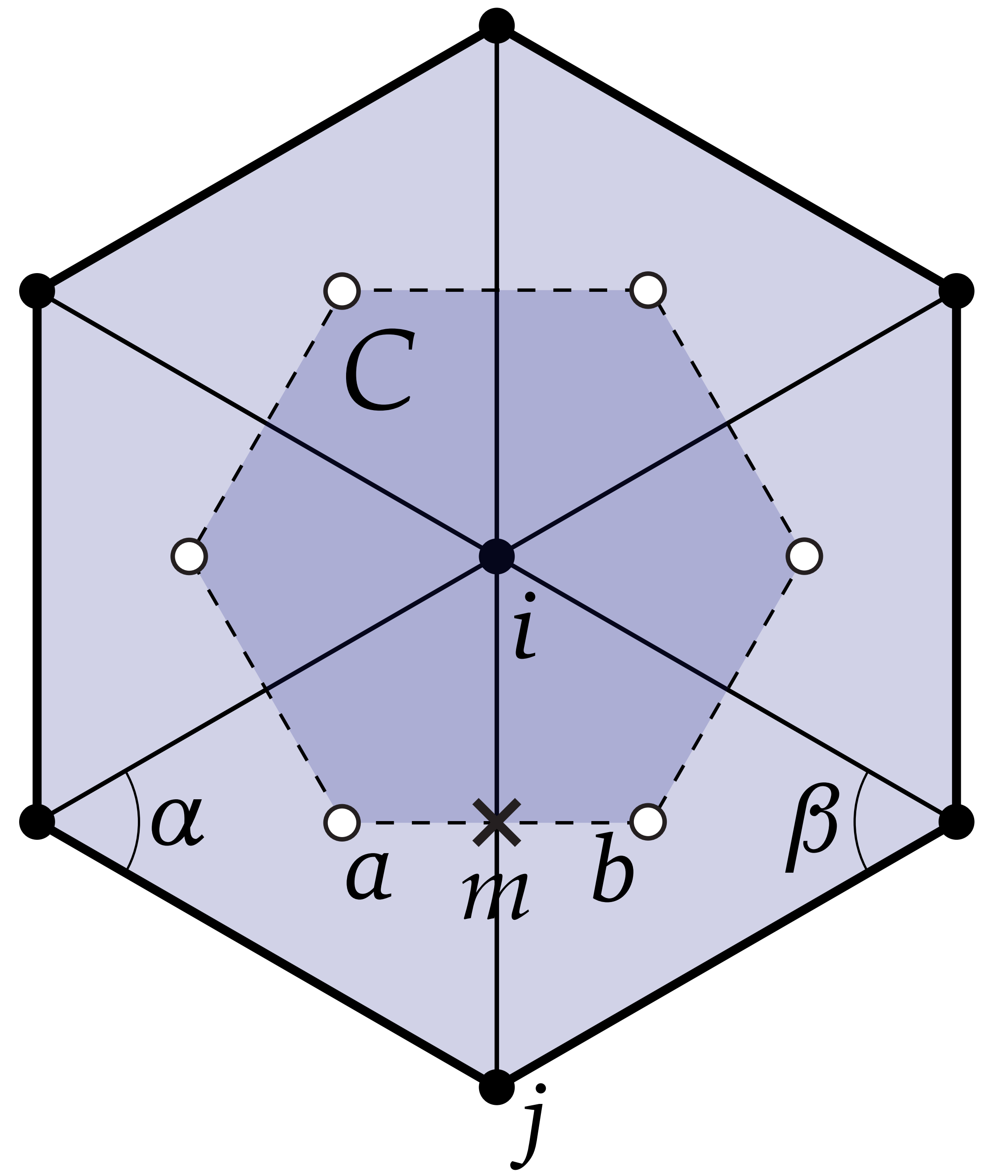
Q: What kind of quantity is the final expression? Does that matter?

Discrete Mean Curvature Normal

Similarly, integrating HN over a circumcentric dual cell C yields

$$\int_C HN\, dA = \int_C df \wedge dN = \int_C dN \wedge df = \int_C d(N \wedge df) =$$

$$\int_{\partial C} N \wedge df = \sum_j \int_{e^\star_{ij}} N \wedge df = \sum_j N_a \times (m-a) + N_b \times (b-m)$$



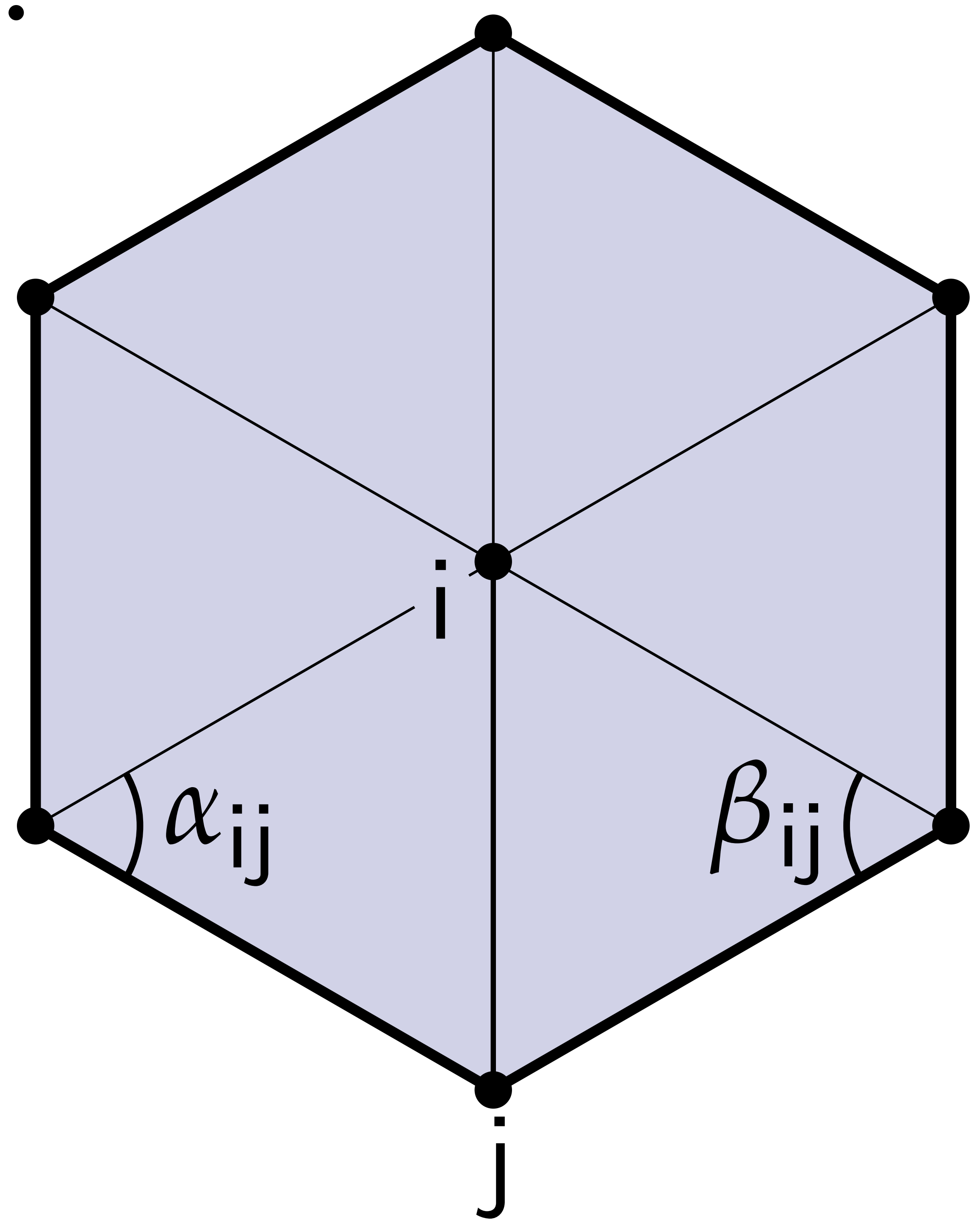
- Since N × is an in-plane 90-degree rotation, each term in the sum is parallel to the edge vector e_{ij}
- The length of the vector is the length of the dual edge
- Ratio of dual/primal length is given by cotan formula, yielding

$$(HN)_i := \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$

Mean Curvature Normal via Laplace-Beltrami

- Another well-known fact: mean curvature normal can be expressed via the *Laplace-Beltrami operator** Δ
- **Fact.** For any smooth immersed surface f, $\Delta f = 2HN$.
- Can discretize Δ via the *cotangent formula*, leading again to

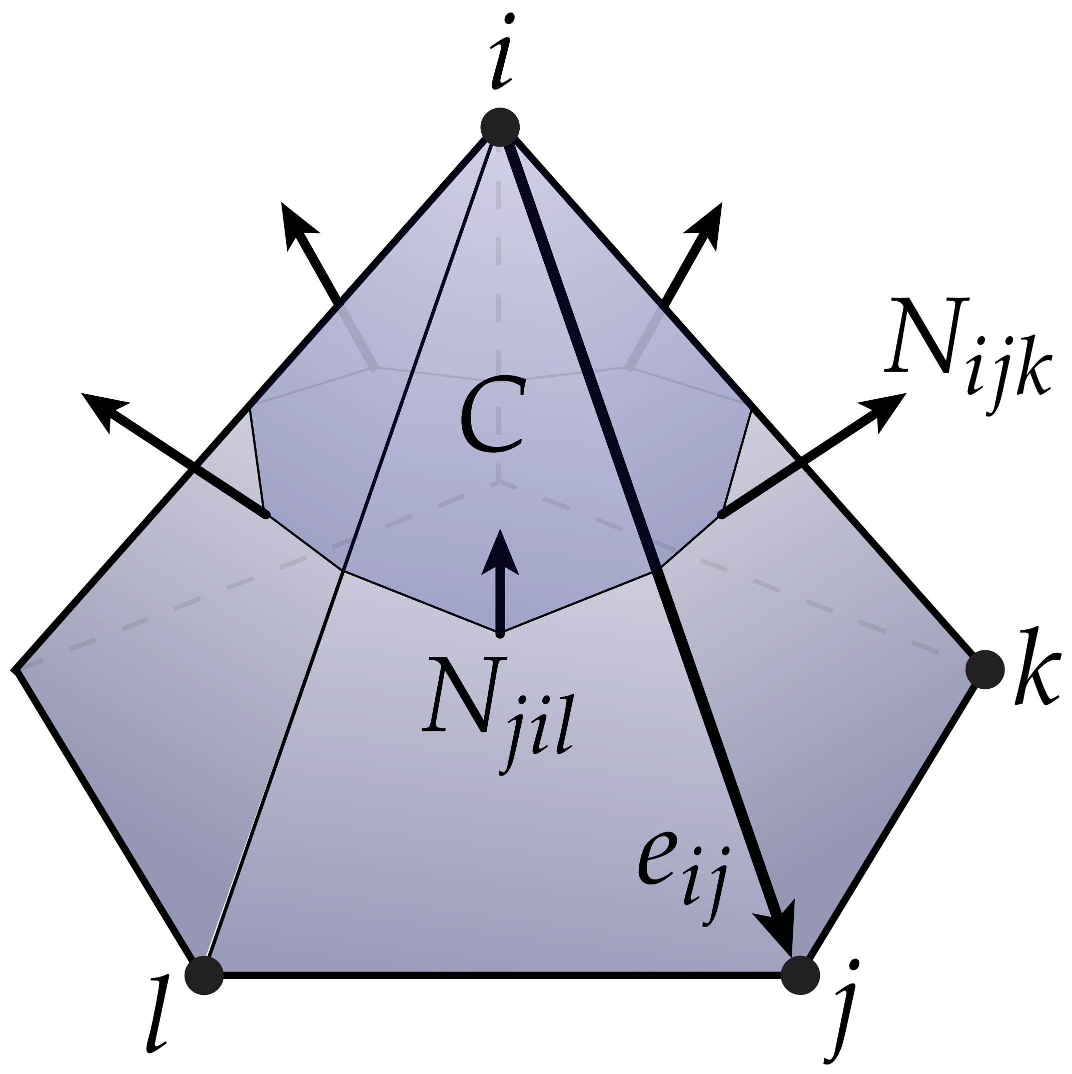
$$(\Delta f)_i = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$



*Will say *much* more in upcoming lectures!

Discrete Gauss Curvature Normal

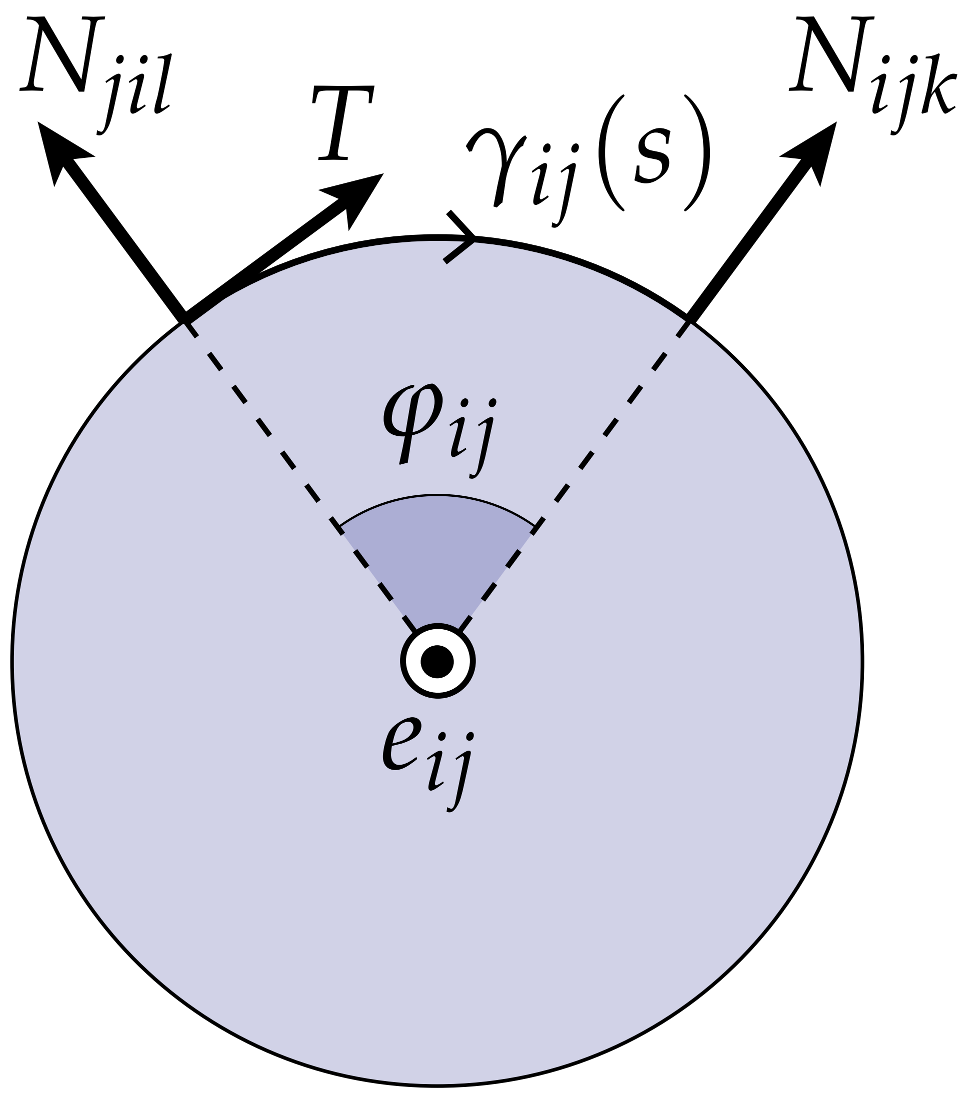
- A similar calculation leads to an expression for the (discrete) Gauss curvature normal
- One key difference: rather than viewing N as linear along edges, we imagine it makes an arc on the unit sphere

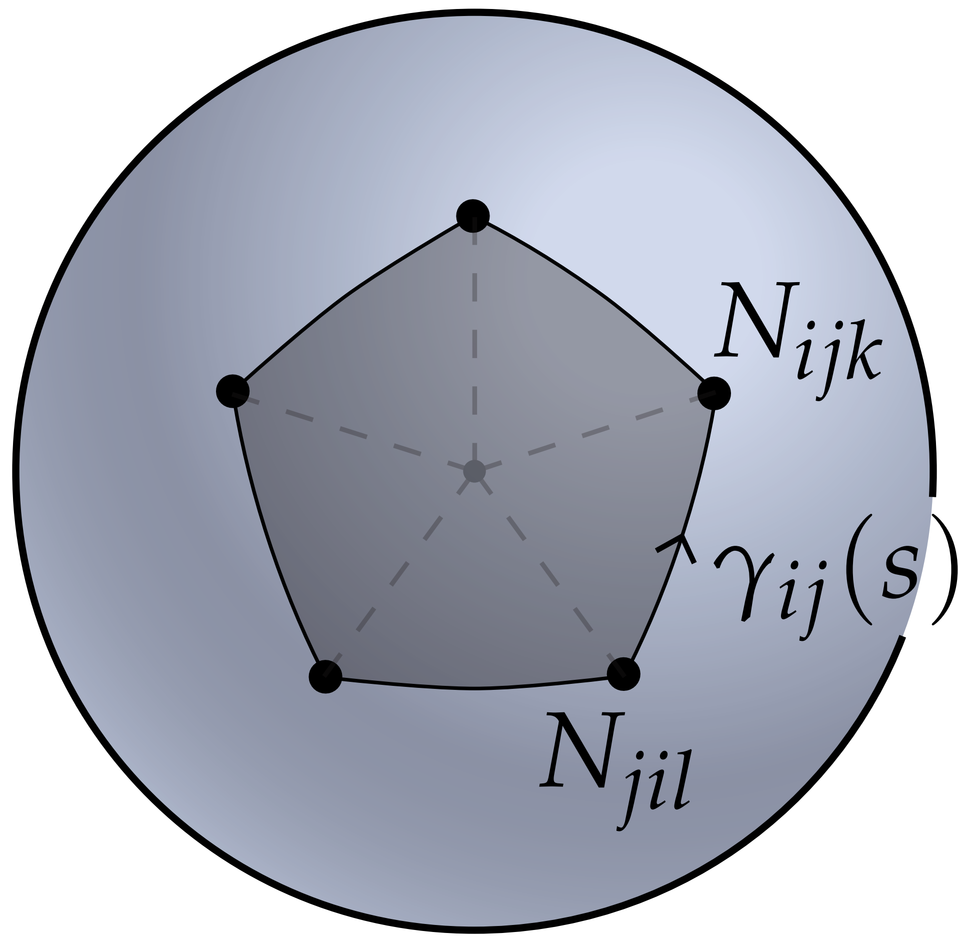


$$2\int_C KN\,dA = \int_C dN \wedge dN = \int_C d(N \wedge dN) =$$

$$\int_{\partial C} N \wedge dN = \int_{\partial C} N \times dN(\gamma')\,ds =$$

$$\int_{\partial C} N \times T\,ds = \sum_j \int_{\partial C} \frac{e_{ij}}{|e_{ij}|}\,ds = \sum_j \frac{e_{ij}}{\ell_{ij}}\varphi_{ij}$$





$$(KN)_i := \frac{1}{2}\sum_{ij\in E}\frac{\varphi_{ij}}{\ell_{ij}}(f_j - f_i)$$

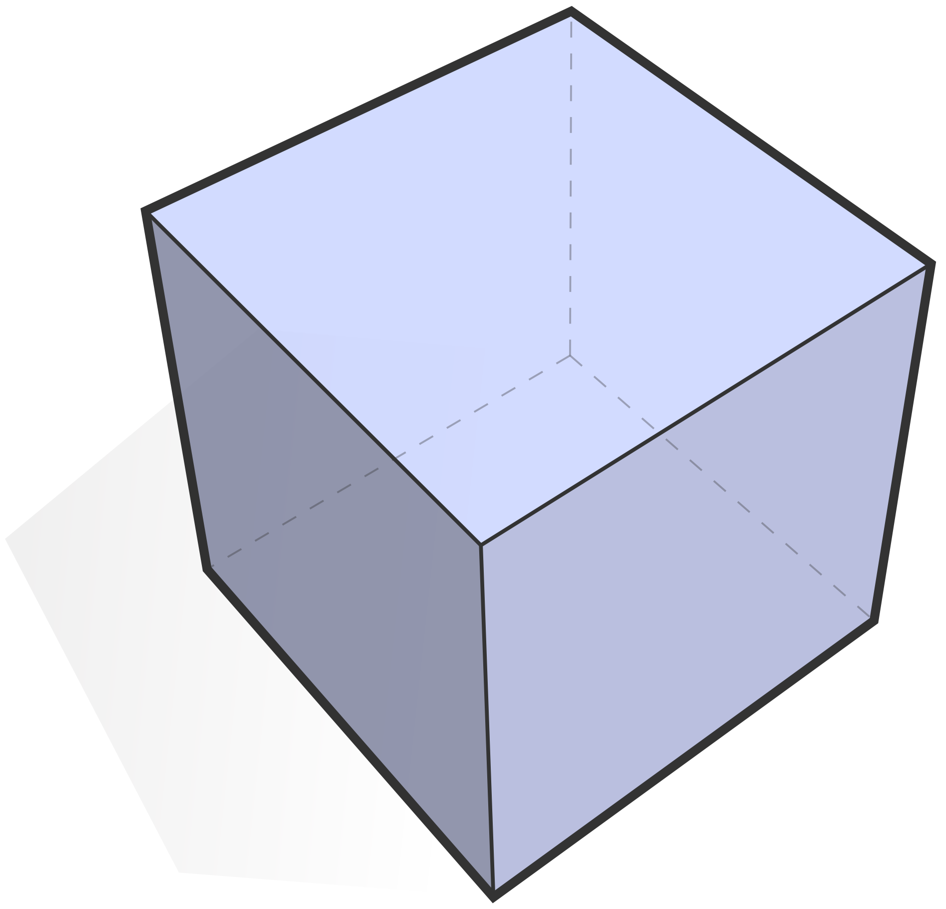
Discrete Curvature Normals — Summary

	area (NdA)	**mean** ($HNdA$)	**Gauss** ($KNdA$)
smooth	$\frac{1}{2}\, df \wedge df$	$\frac{1}{2}\, df \wedge dN$	$\frac{1}{2}\, dN \wedge dN$
discrete	$\frac{1}{6} \sum_{ijk \in \mathrm{St}(i)} f_j \times f_k$	$\frac{1}{2} \sum_{ij \in \mathrm{St}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$	$\frac{1}{2} \sum_{ij \in \mathrm{St}(i)} \frac{\varphi_{ij}}{\ell_{ij}} (f_j - f_i)$

Steiner's Formula

Steiner Approach to Curvature

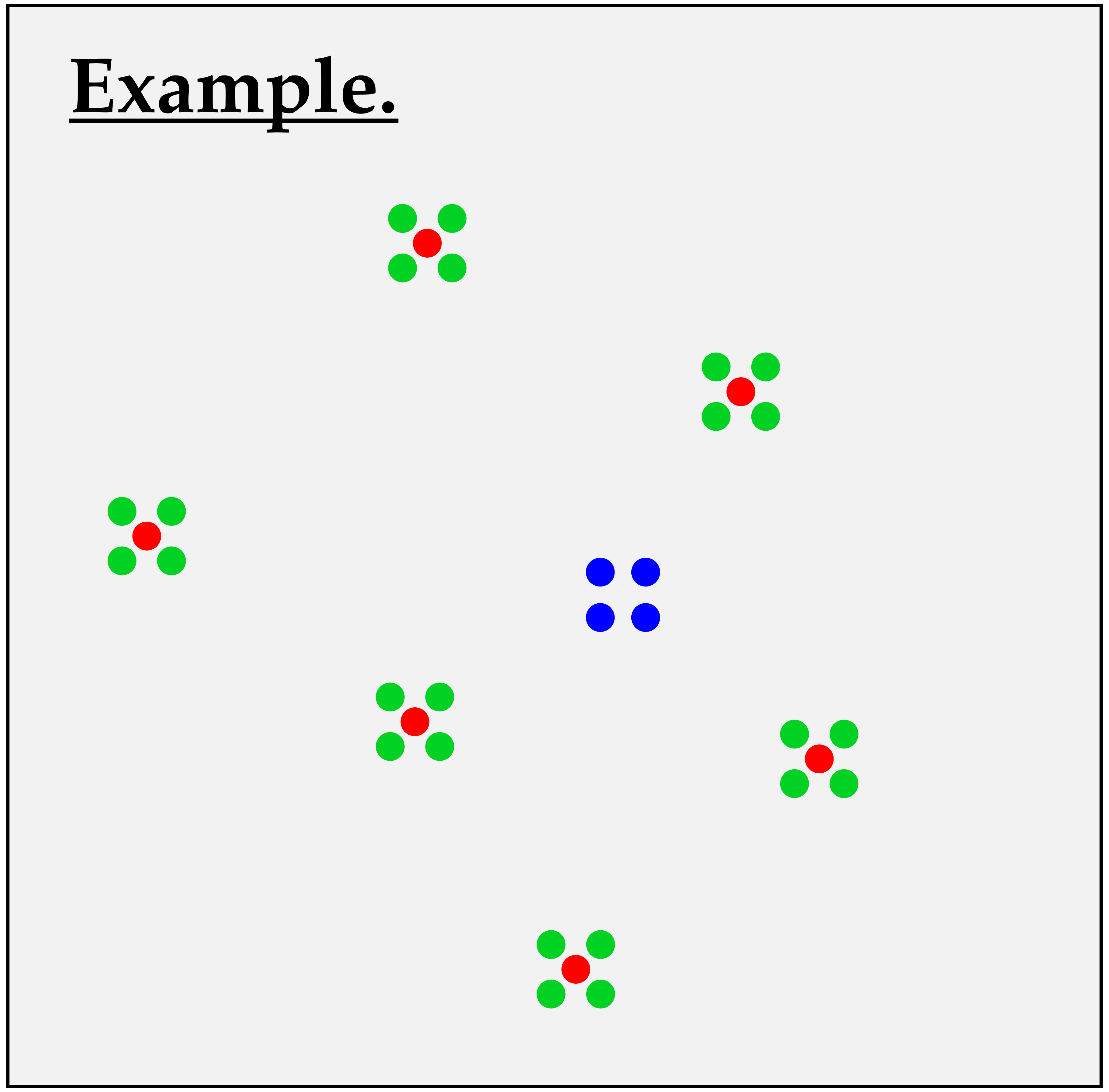
- What's the curvature of a discrete surface (polyhedron)?
- Simply taking derivatives of the normal yields a useless answer: zero except at vertices/edges, where derivative is ill-defined ("infinite")
- Steiner approach: "smooth out" the surface; define discrete curvature in terms of this *mollified* surface

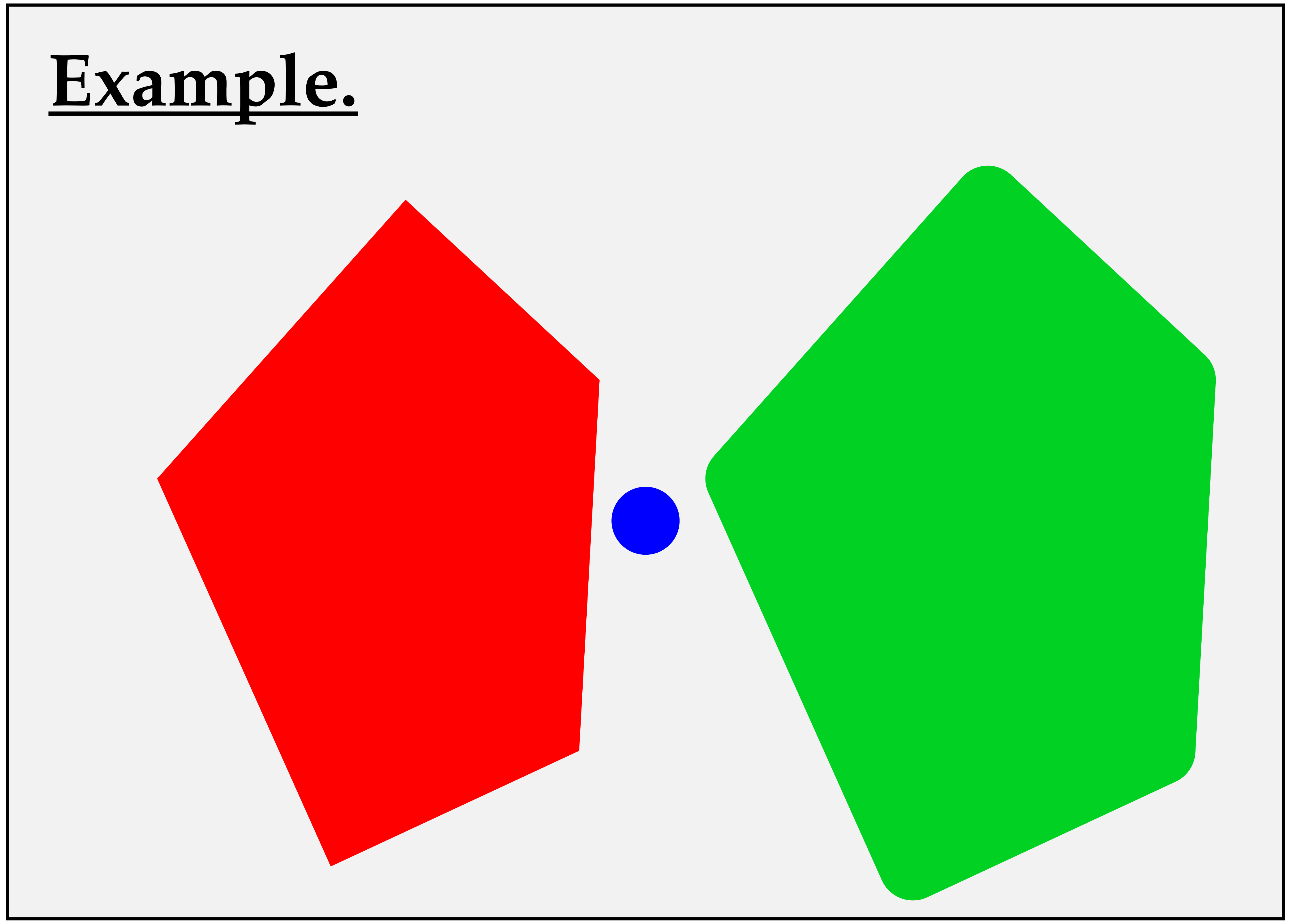


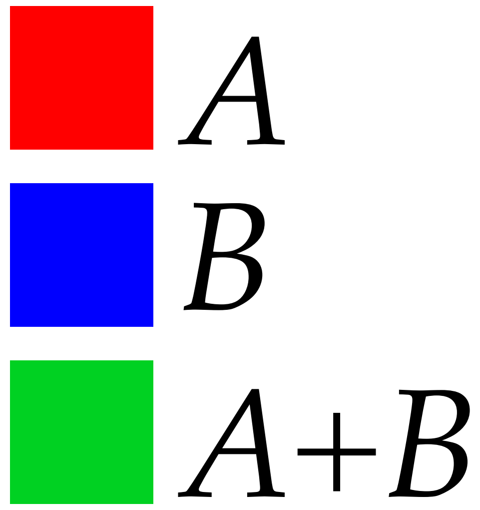
Minkowski Sum

- Given two sets A, B in R^n, their *Minkowski sum* is the set of points

$$A + B := \{a + b \mid a \in A, b \in B\}$$







Q: Does translation of A, B matter?

Mollification of Polyhedral Surfaces

- Steiner approach mollifies polyhedral surface by taking Minkowski sum with ball of radius $\varepsilon > 0$
- Measure curvature, take limit as ε goes to zero to get discrete definition
- (Have to think carefully about nonconvex polyhedra…)

Steiner Formula

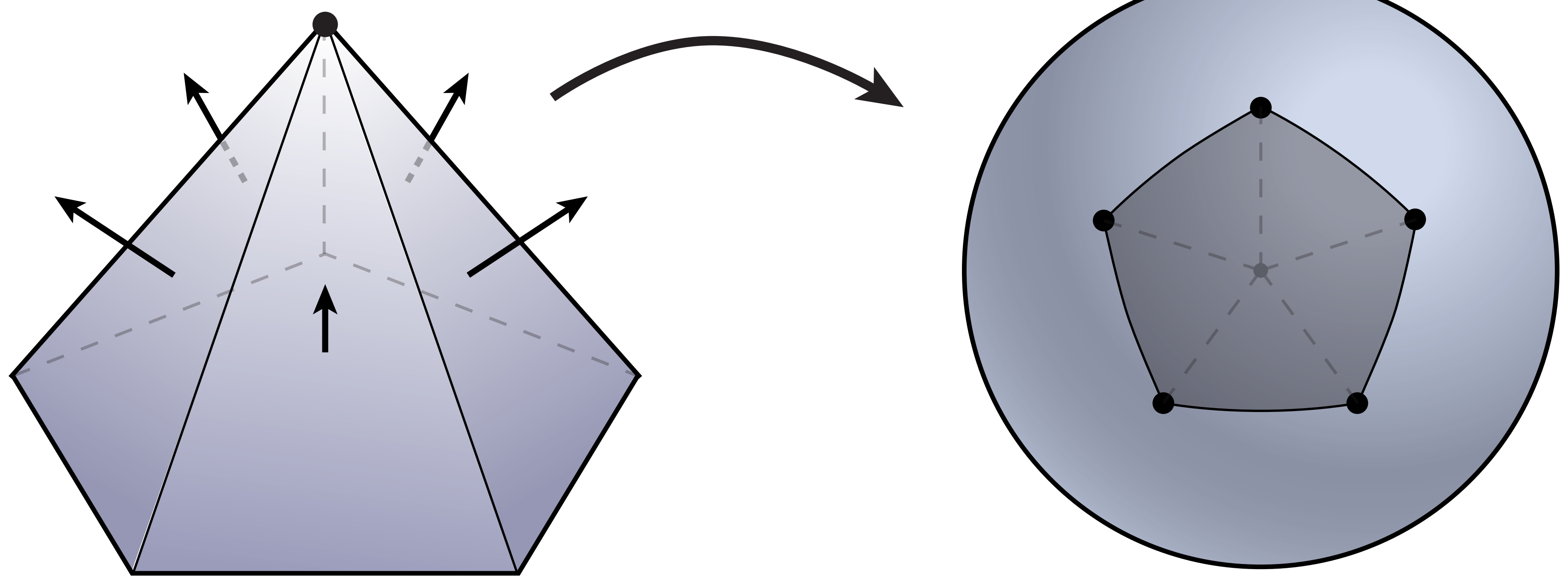
- **Theorem.** (Steiner) Let A be any convex body in R^n, and let B_ε be a ball of radius ε. Then the volume of the Minkowski sum $A+B_\varepsilon$ can be expressed as a polynomial in ε:

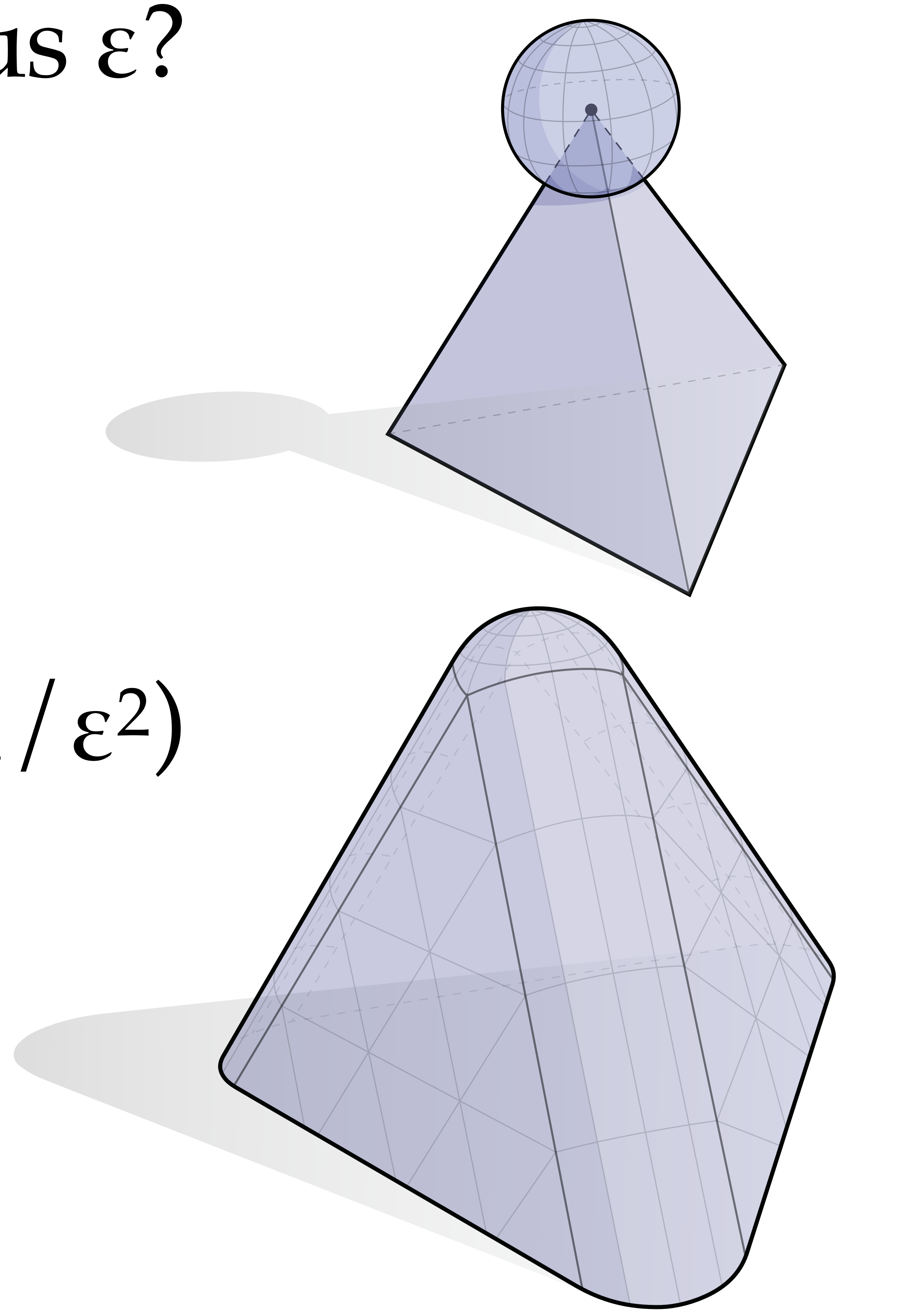
$$\text{volume}(A + B_\varepsilon) = \text{volume}(A) + \sum_{k=1}^{n} \Phi_k(A)\varepsilon^k$$

- Constant coefficients are called *quermassintegrals*, and determine how quickly the volume grows
- This volume growth in turn has to do with (discrete) curvature, as we are about to see…

Gaussian Curvature of Mollified Surface

- **Q:** Consider a *closed, convex* polyhedron in R^3; what's the Gaussian curvature K of the mollified surface for a ball of radius ε?
  - **Triangles:** $K = 0$
  - **Edges:** $K = 0$
  - **Vertices?**
    - each contributes a piece of sphere of radius ε ($K = 1/\varepsilon^2$)
    - recall (unit) spherical area given by *angle defect* Ω_i
    - *total* curvature associated with vertex i is then



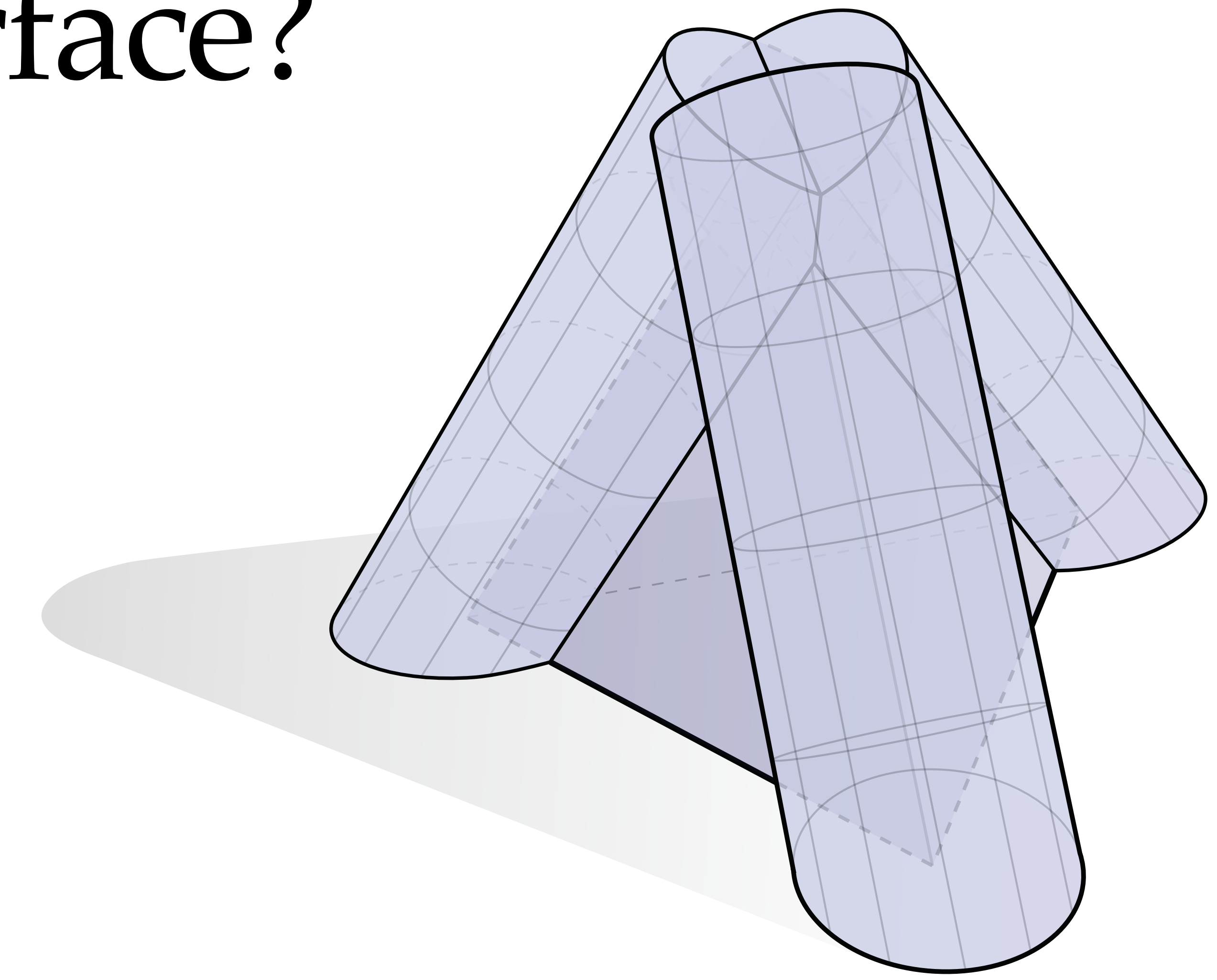


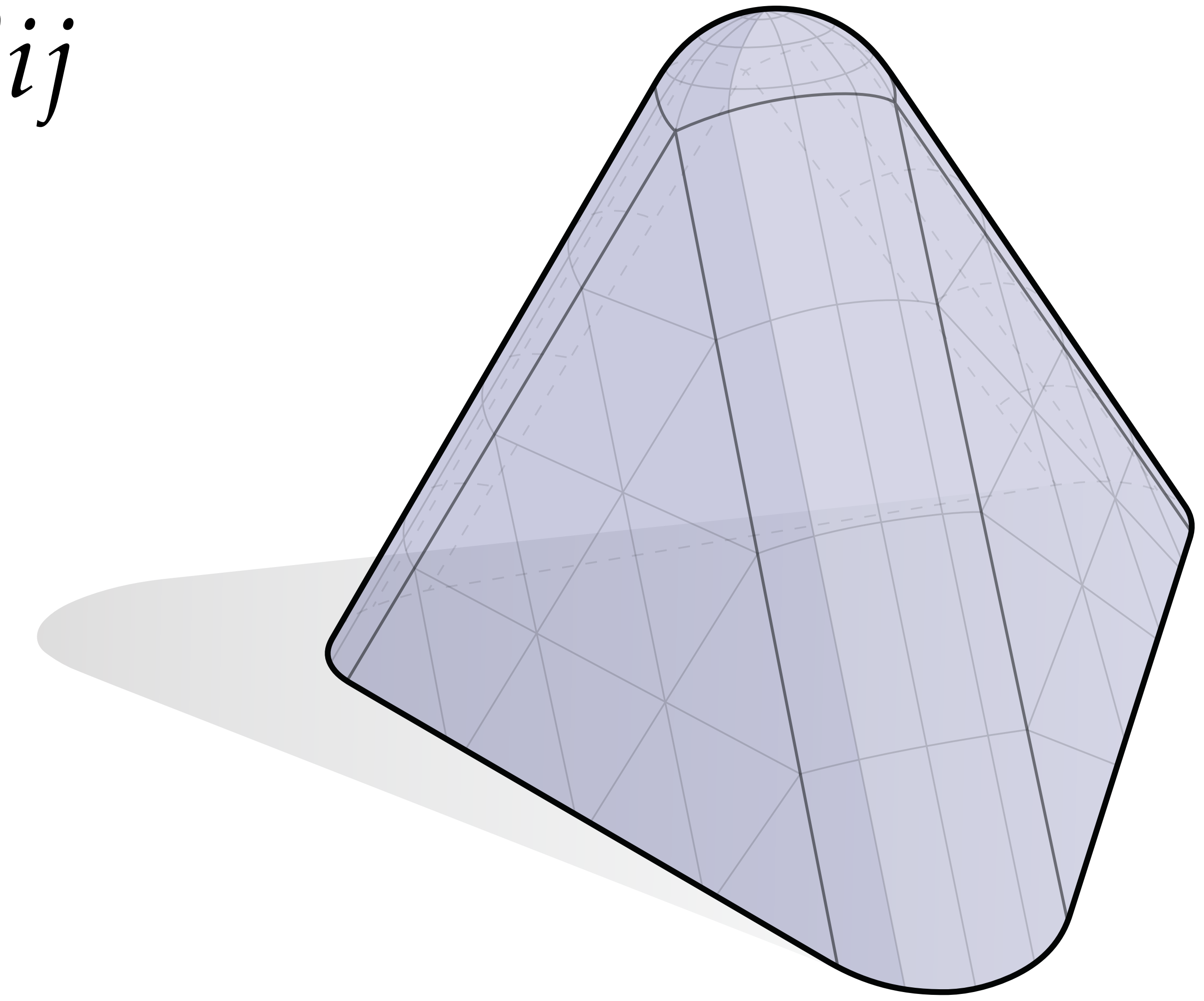
$$A_i K_i = \left(\frac{\Omega_i}{4\pi} 4\pi\varepsilon^2 \right) \frac{1}{\varepsilon^2} = \Omega_i$$

(Spherical polygon is all normals associated with vertex.)

Mean Curvature of Mollified Surface

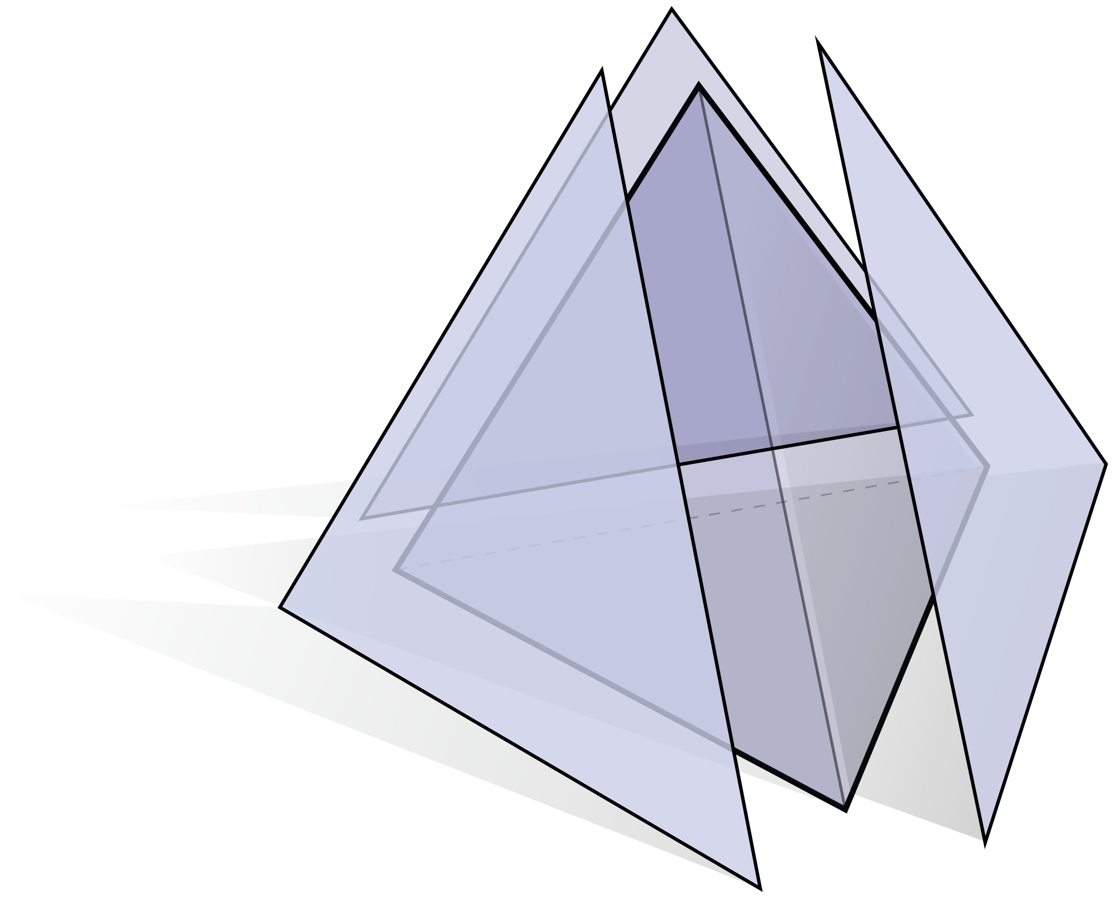
- **Q:** What's the mean curvature H of the mollified surface?
  - **Faces:** $H = 0$
  - **Edges?**
    - each contributes a piece of a cylinder (H=1/2ε)
    - area of cylindrical piece is $\ell_{ij}\varphi_{ij}\varepsilon$
    - *total* mean curvature for edge is hence $H_{ij} = \frac{1}{2}\ell_{ij}\varphi_{ij}$
  - **Vertices?**
    - each contributes a piece of the sphere (H=1/ε)
    - area is $(\Omega_i/4\pi)\,4\pi\varepsilon^2 = \Omega_i\varepsilon^2$
    - *total* mean curvature for vertex is then $H_i = \Omega_i\varepsilon$





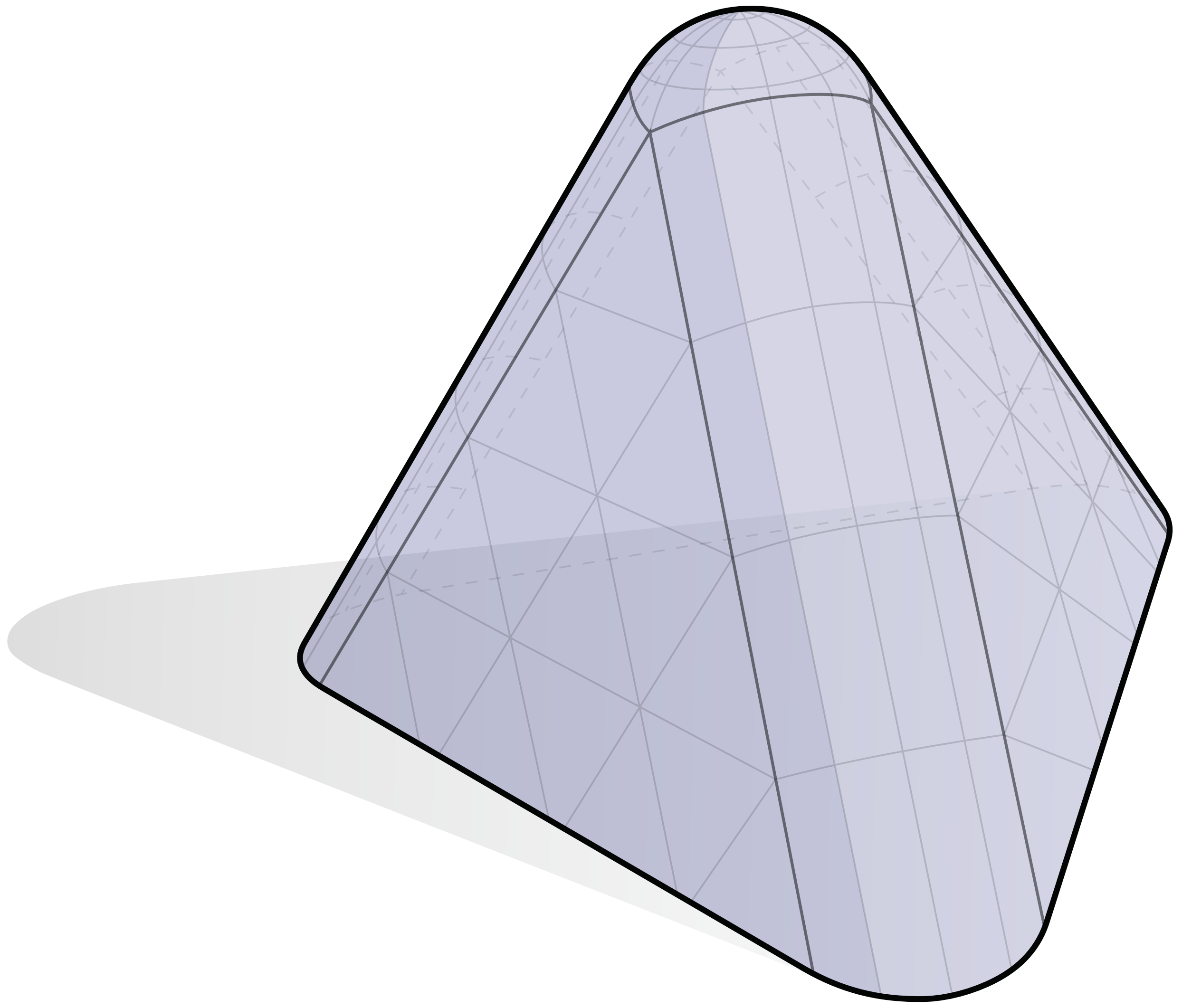
Area of a Mollified Surface

- **Q:** What's the area of the mollified surface?
  - **Faces:** just the original area A_{ijk}
  - **Edges:** $\ell_{ij}\varphi_{ij}\varepsilon$
  - **Vertices:** $\Omega_i\varepsilon^2$



- Total area of the whole surface is then

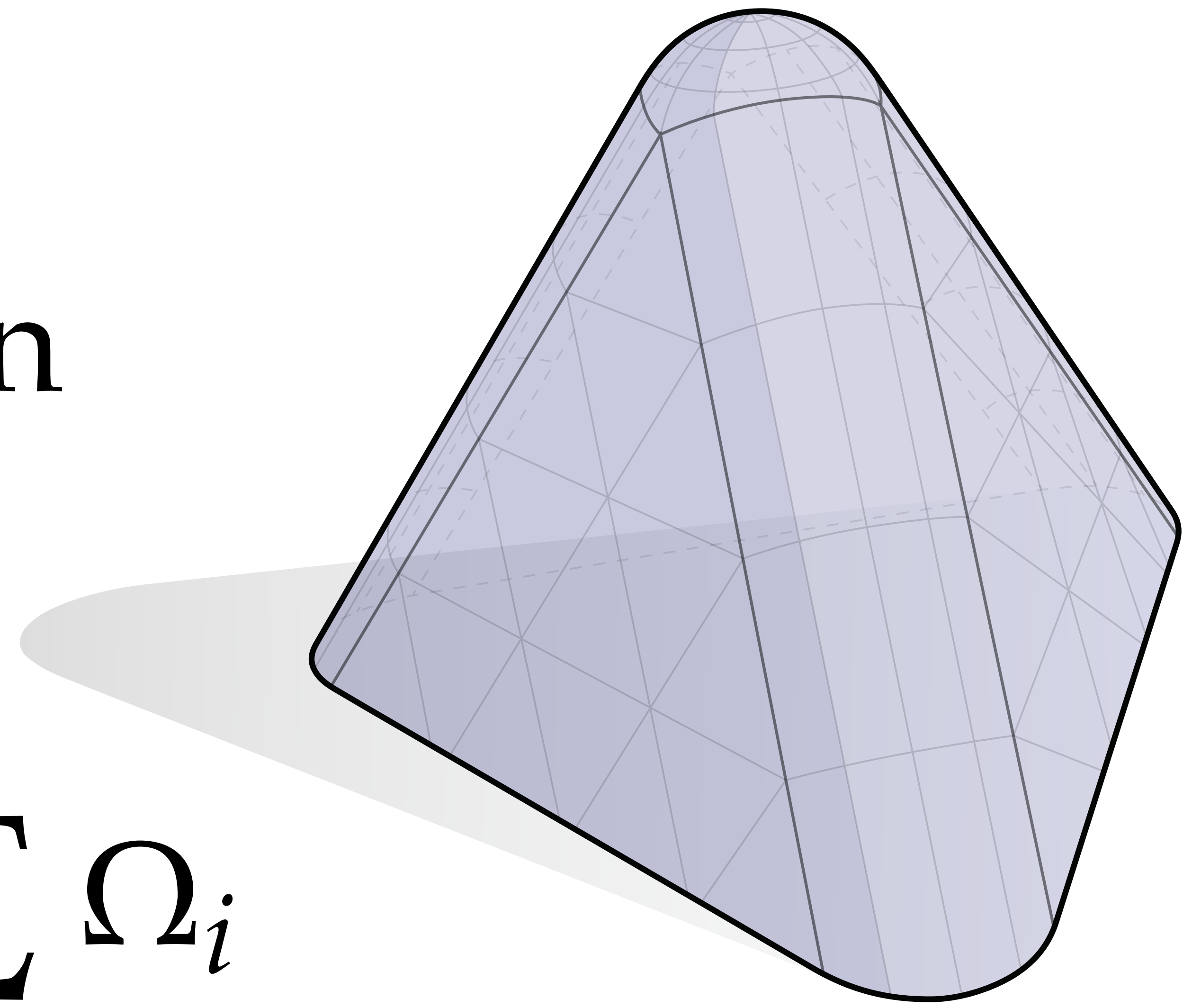
$$\text{area}_\varepsilon(f) = \sum_{ijk\in F} A_{ijk} + \varepsilon \sum_{ij\in E} \ell_{ij}\varphi_{ij} + \varepsilon^2 \sum_{i\in V} \Omega_i$$



- By (discrete) Gauss-Bonnet, last term is also $2\pi\chi$

Volume of Mollified Surface

- Q: What's the total volume of the mollified surface?
- Starting to see a pattern—if V_0 is original volume, then



$$\text{volume}_\varepsilon(f) = V_0 + \varepsilon \sum_{ijk \in F} A_{ijk} + \varepsilon^2 \overset{1/2}{\sum_{ij \in E}} \ell_{ij}\varphi_{ij} + \varepsilon^3 \overset{1/3}{\sum_{i \in V}} \Omega_i$$

- **Q:** How did we get here from our area expression?
- **A:** Increasing radius by ε increases volume proportional to area

Steiner Polynomial for Polyhedra in R³

- Volume of mollified polyhedron is a polynomial in radius ε
- Derivatives w.r.t. ε give total area, mean curvature, Gauss curvature

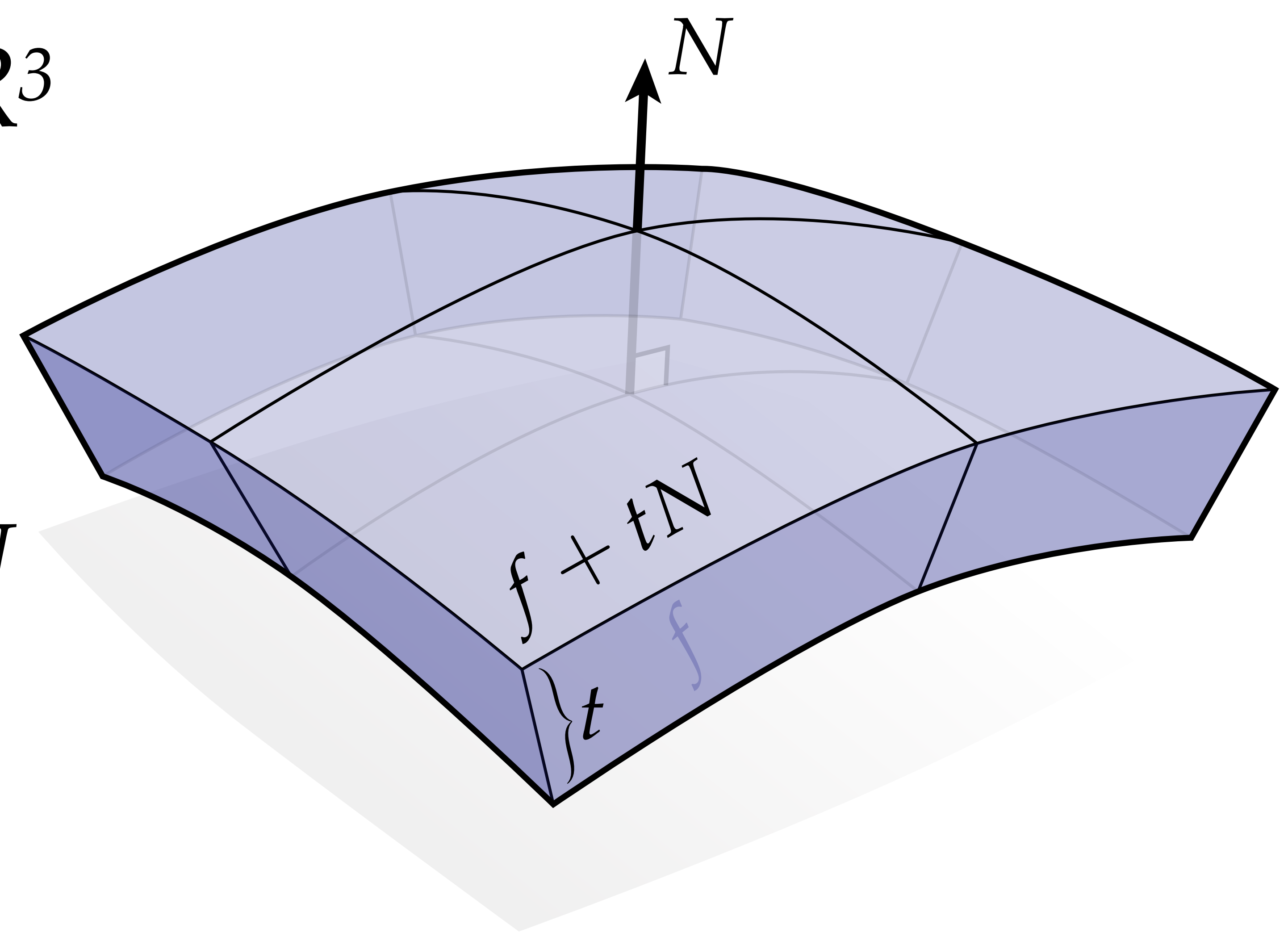
$$\text{volume}_\varepsilon(f) = V_0 + \varepsilon \sum_{ijk \in F} A_{ijk} + \varepsilon^2 \sum_{ij \in E} \ell_{ij}\varphi_{ij} + \varepsilon^3 \sum_{i \in V} \Omega_i$$

$\frac{d}{d\varepsilon}\text{volume}_\varepsilon = \text{area}_\varepsilon$ $\quad \frac{d}{d\varepsilon}\text{area}_\varepsilon = \text{mean}_\varepsilon$ $\quad \frac{d}{d\varepsilon}\text{mean}_\varepsilon = \text{Gauss}_\varepsilon$ $\quad \frac{d}{d\varepsilon}\text{Gauss}_\varepsilon = 0$

Q: Why are there only four terms?

Steiner Polynomial for Surfaces in R^3

- Not surprisingly, there is an analogous formula for surfaces in R^3
- Taking a Minkowski sum with a ball* of radius ε is the same as shifting the surface in the normal direction a distance ε
- Consider a surface $f : M \longrightarrow R^3$ with Gauss map N; let $f_t := f + tN$
- How is the area of the "smoothed" surface changing?



$$dA_t = \tfrac{1}{2}\langle N, df_t \wedge df_t \rangle$$

$$df_t \wedge df_t =$$

$$(df + tdN) \wedge (df + tdN) =$$

$$df \wedge df + 2t df \wedge dN + t^2 dN \wedge dN =$$

$$(1 + 2tH + t^2 K) df \wedge df$$

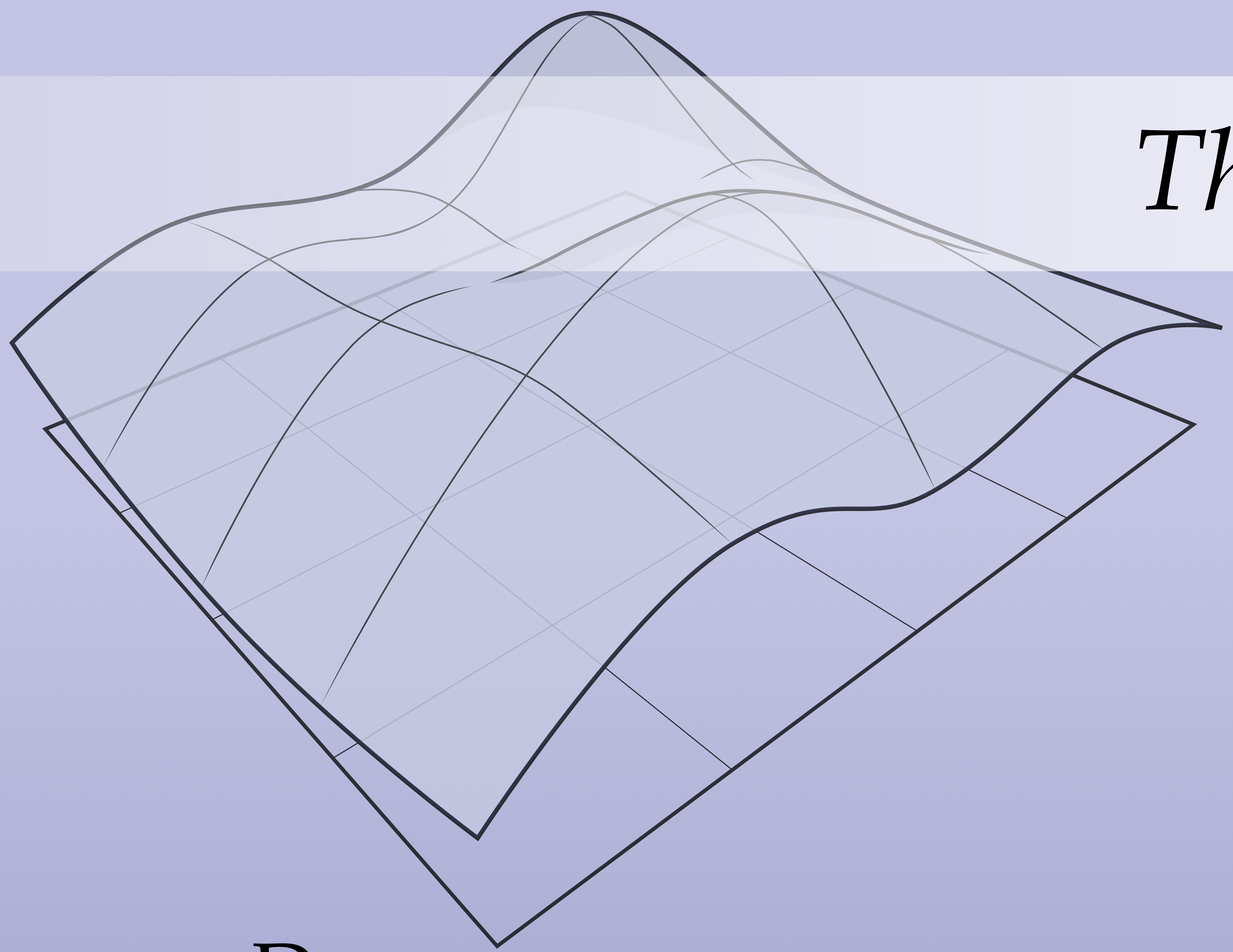
$$\boxed{\Longrightarrow dA_t = (1 + 2tH + t^2 K) dA_0}$$

Notice:

- surface area given by $df \wedge df$
- spherical area $dN \wedge dN$ gives Gauss curvature
- mixed area $df \wedge dN$ gives mean curvature

*sufficiently small

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017