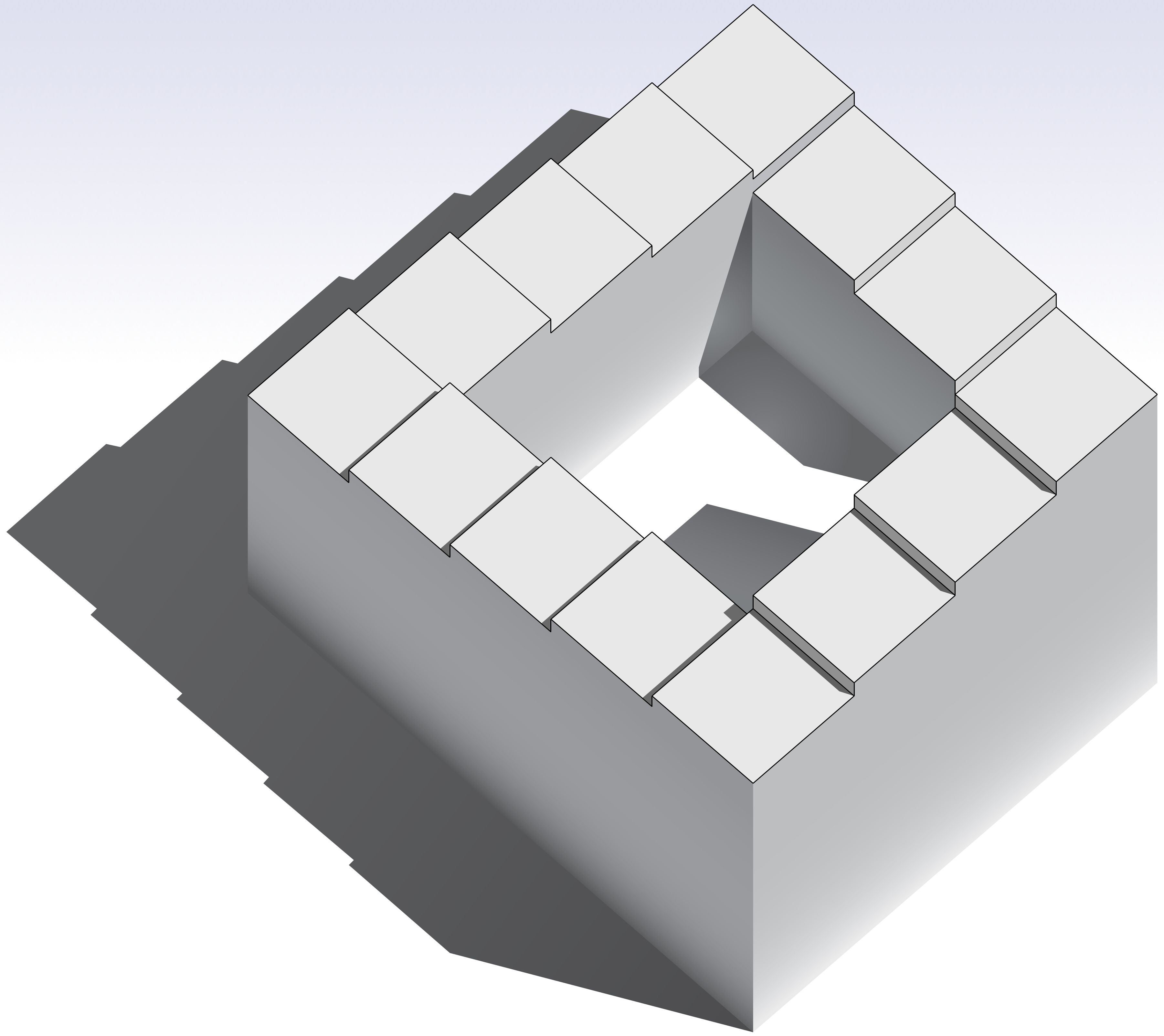

Discrete Differential Geometry: An Applied Introduction

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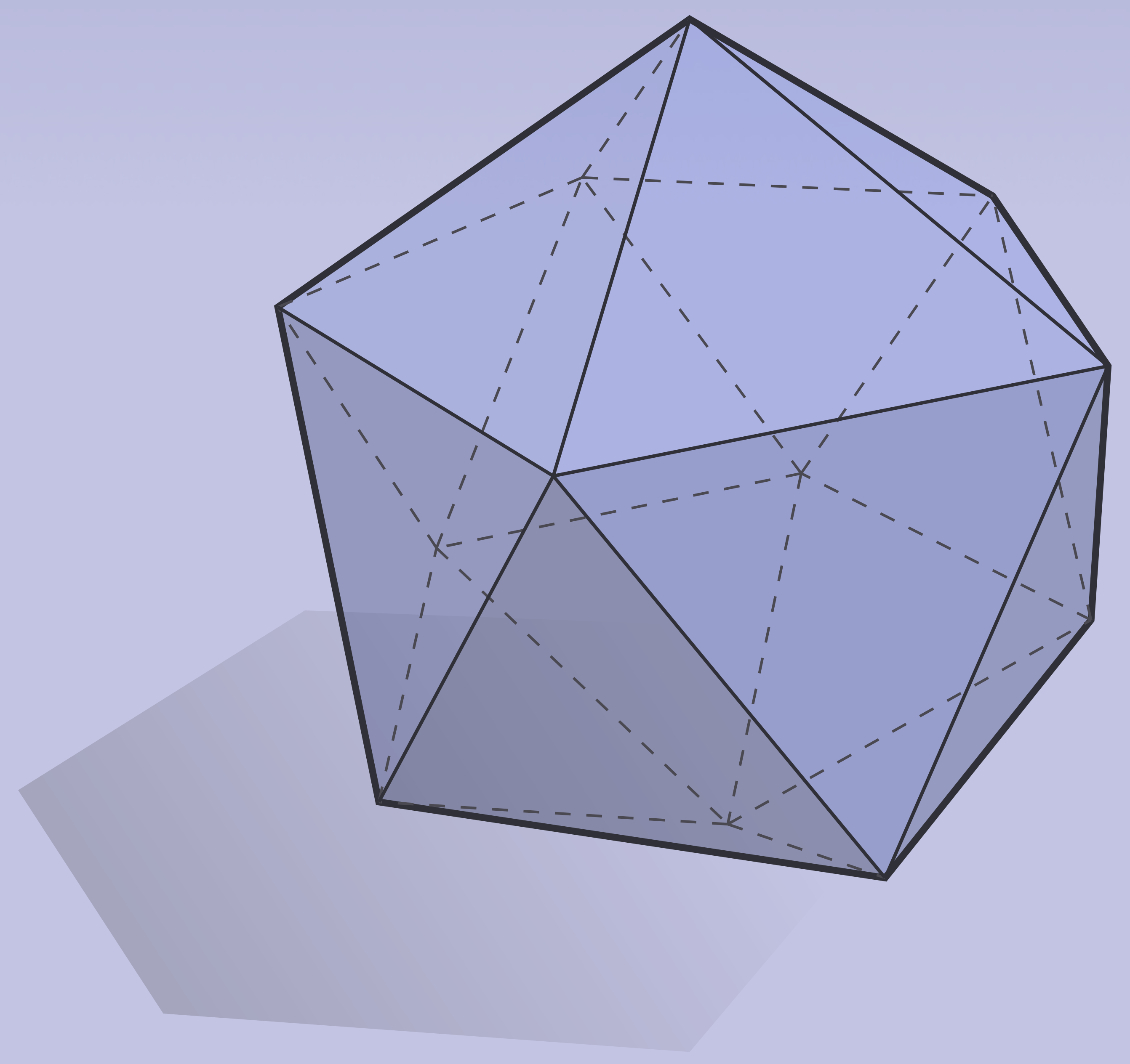
Lecture 14: Curvature



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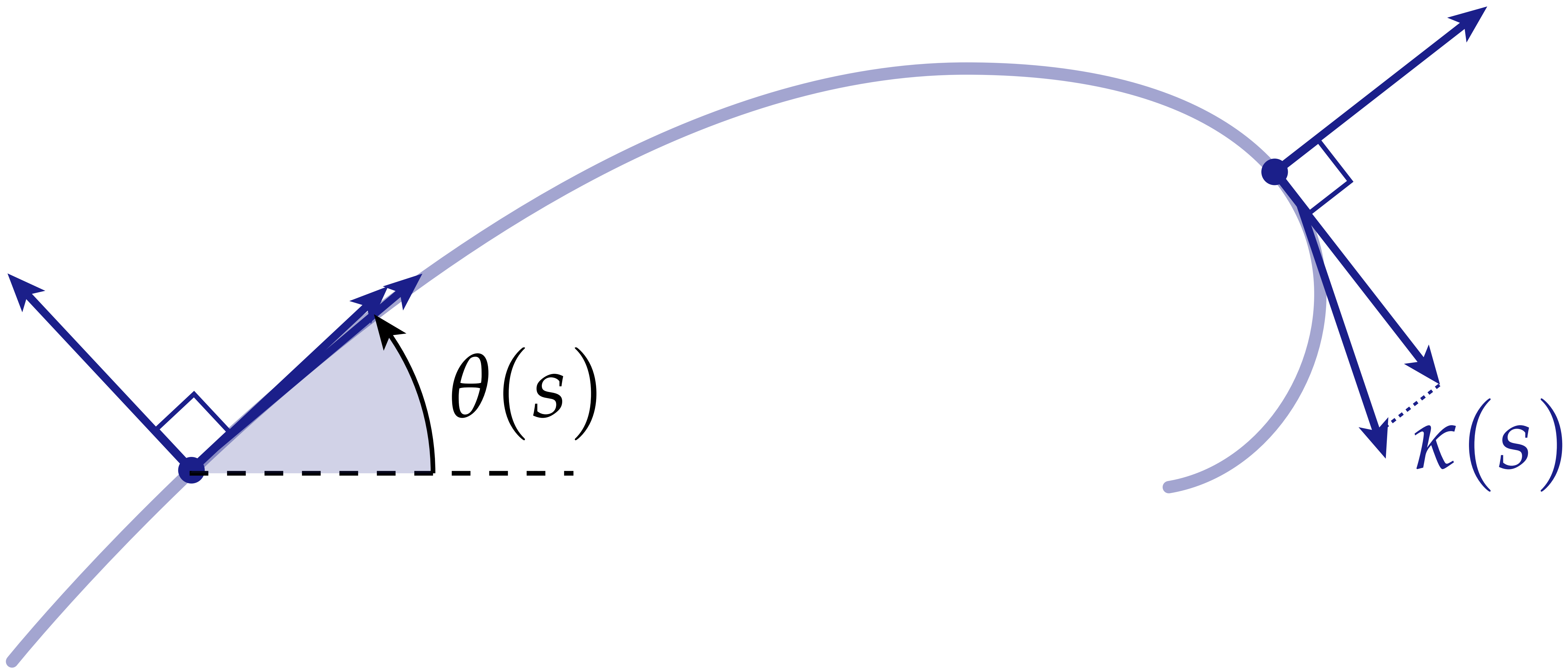
Curvature of Curves

Review: Curvature of a Plane Curve

- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent

$$\kappa(s) := \langle N(s), \tfrac{d}{ds} T(s) \rangle$$
$$= \langle N(s), \tfrac{d^2}{ds^2} \gamma(s) \rangle$$



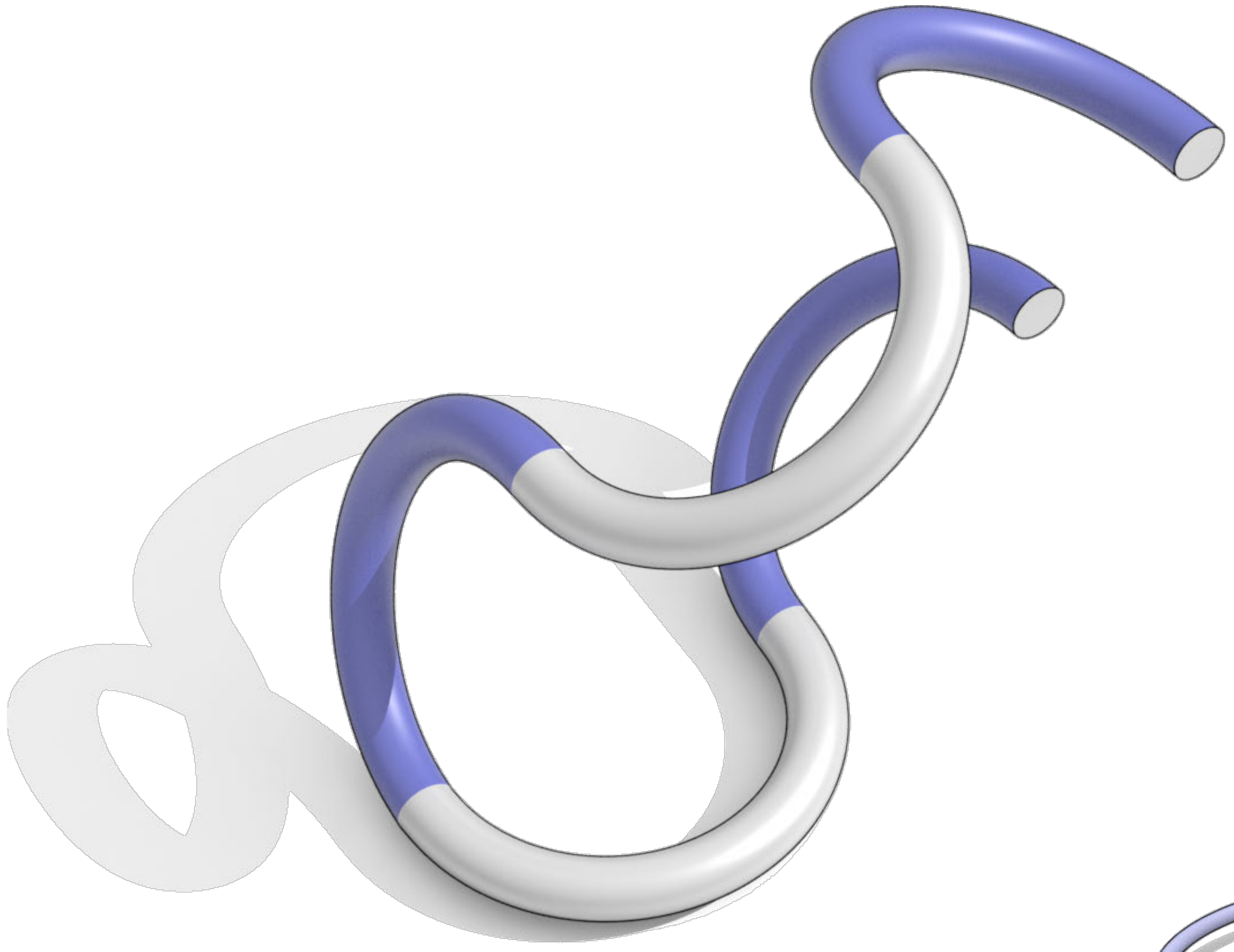
__**Equivalently:**__

$$\kappa(s) = \frac{d}{ds}\theta(s)$$

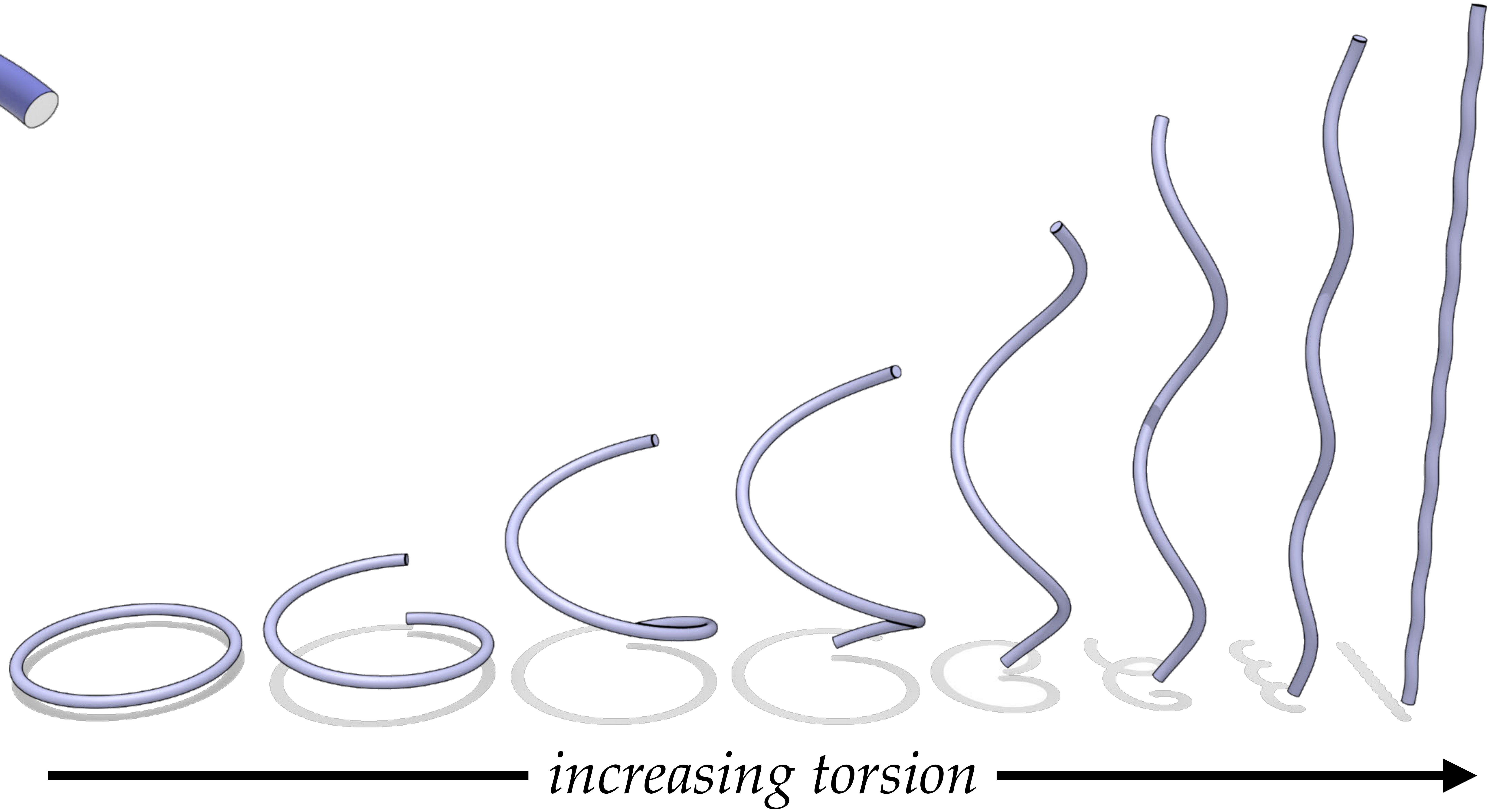
Here the angle brackets denote the usual dot product, i.e., $\langle (a,b),(x,y) \rangle := ax + by$.

Review: Curvature and Torsion of a Space Curve

- For a plane curve, *curvature* captured the notion of “bending”
- For a space curve we also have *torsion*, which captures “twisting”



Intuition: torsion is “out of plane bending”



Review: Fundamental Theorem of Plane Curves

Fact. Up to rigid motions, an arc-length parameterized plane curve is uniquely determined by its curvature.

Q: Given only the curvature function, how can we recover the curve?

A: Just "invert" the two relationships $\frac{d}{ds}\theta = \kappa$, $\frac{d}{ds}\gamma = T$

First integrate curvature to get angle: $\theta(s) := \int_0^s \kappa(t)\, dt$

Then evaluate unit tangents: $T(s) := (\cos(\theta), \sin(\theta))$

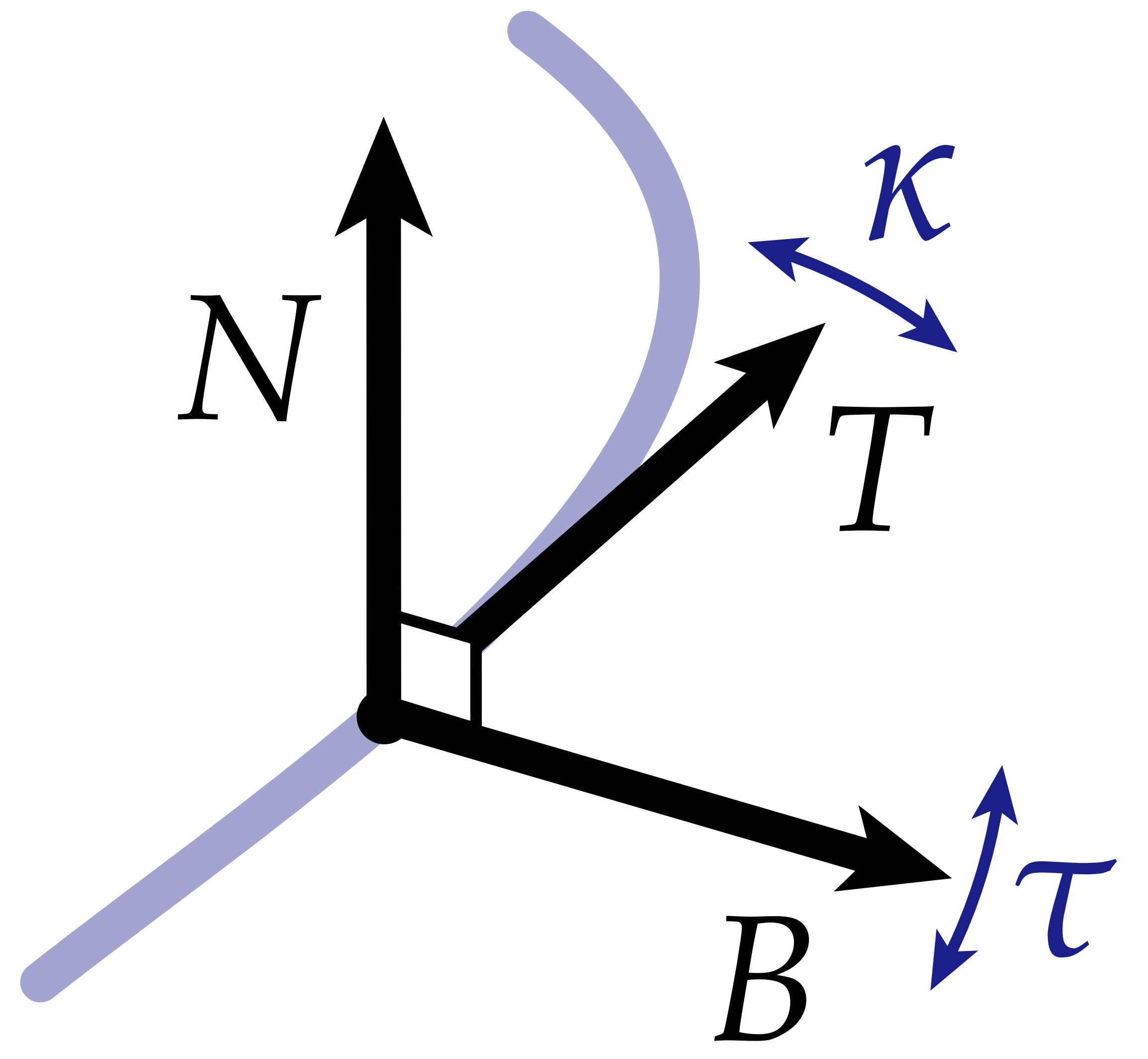
Finally, integrate tangents to get curve: $\gamma(s) := \int_0^s T(t)\, dt$

Q: What about the rigid motion? Will this work for *closed* curves?

Review: Fundamental Theorem of Space Curves

- The *fundamental theorem of space curves* tells us we can also go the other way: given the curvature and torsion of an arc-length parameterized space curve, we can recover the curve itself
- In 2D we just had to integrate a single ODE; here we integrate a system of three ODEs—namely, Frenet-Serret!

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

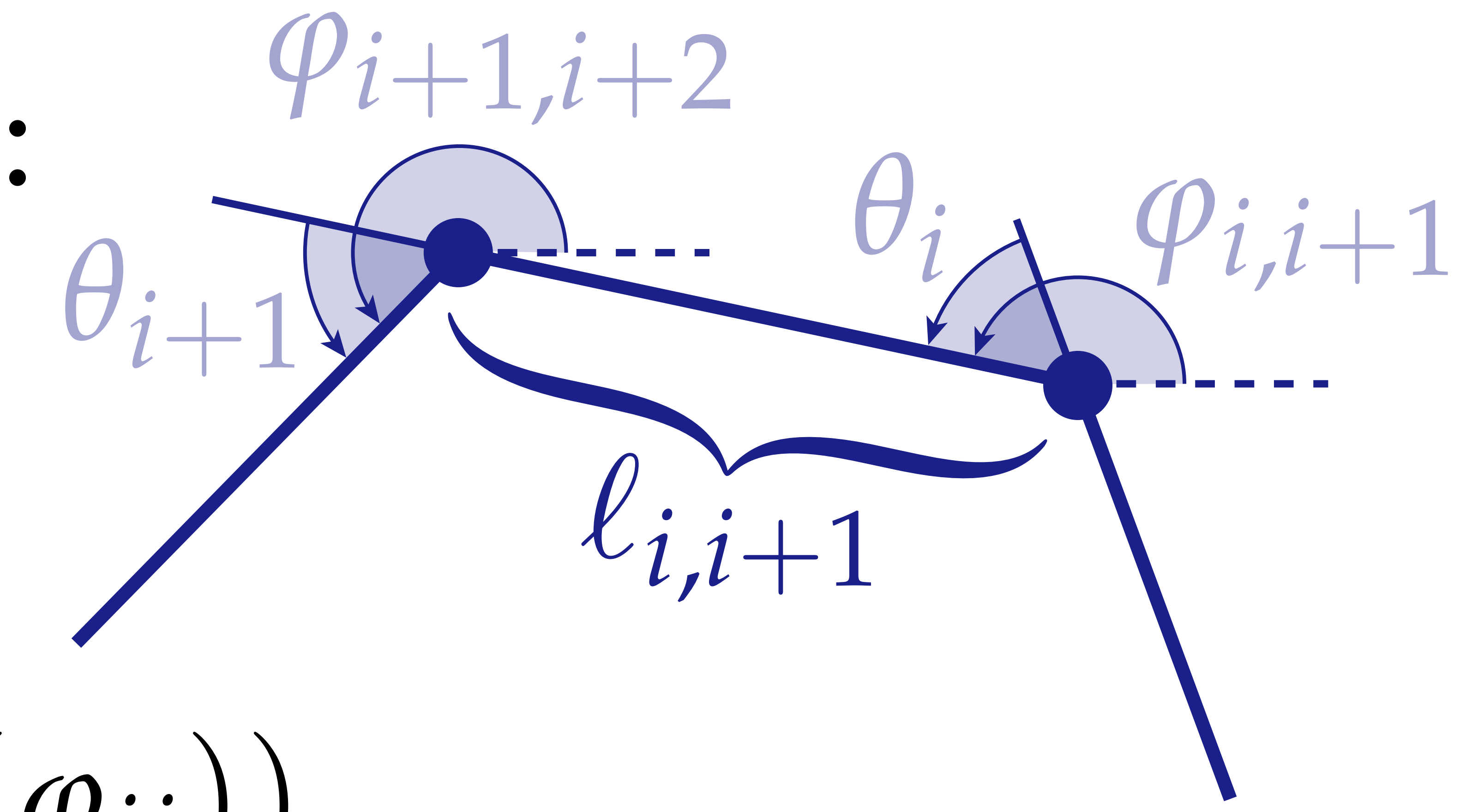


Algorithm: Recover Plane Curve from Curvature

Fact. Up to rigid motions, a regular discrete plane curve is uniquely determined by its edge lengths and turning angles.

Q: Given only this data, how can we recover the curve?

A: Mimic the procedure from the smooth setting:



Sum curvatures to get angles: $\varphi_{i,i+1} := \sum_{k=1}^{i} \theta_k$

Evaluate unit tangents: $T_{ij} := (\cos(\varphi_{ij}), \sin(\varphi_{ij}))$

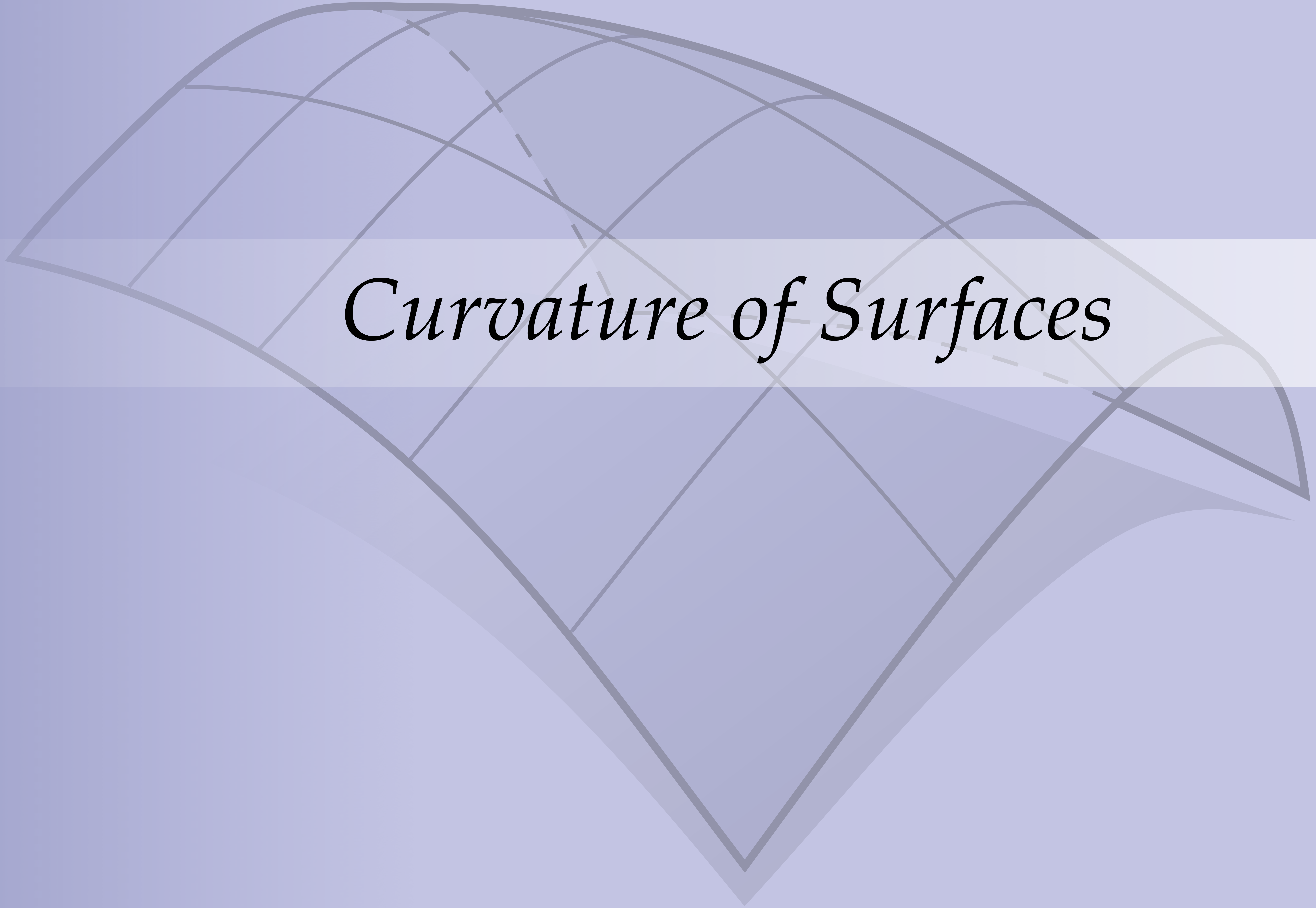
Sum tangents to get curve: $\gamma_i := \sum_{k=1}^{i} \ell_{k,k+1} T_{k,k+1}$

Q: Rigid motions?

Algorithm: Recover Space Curve from Curvature

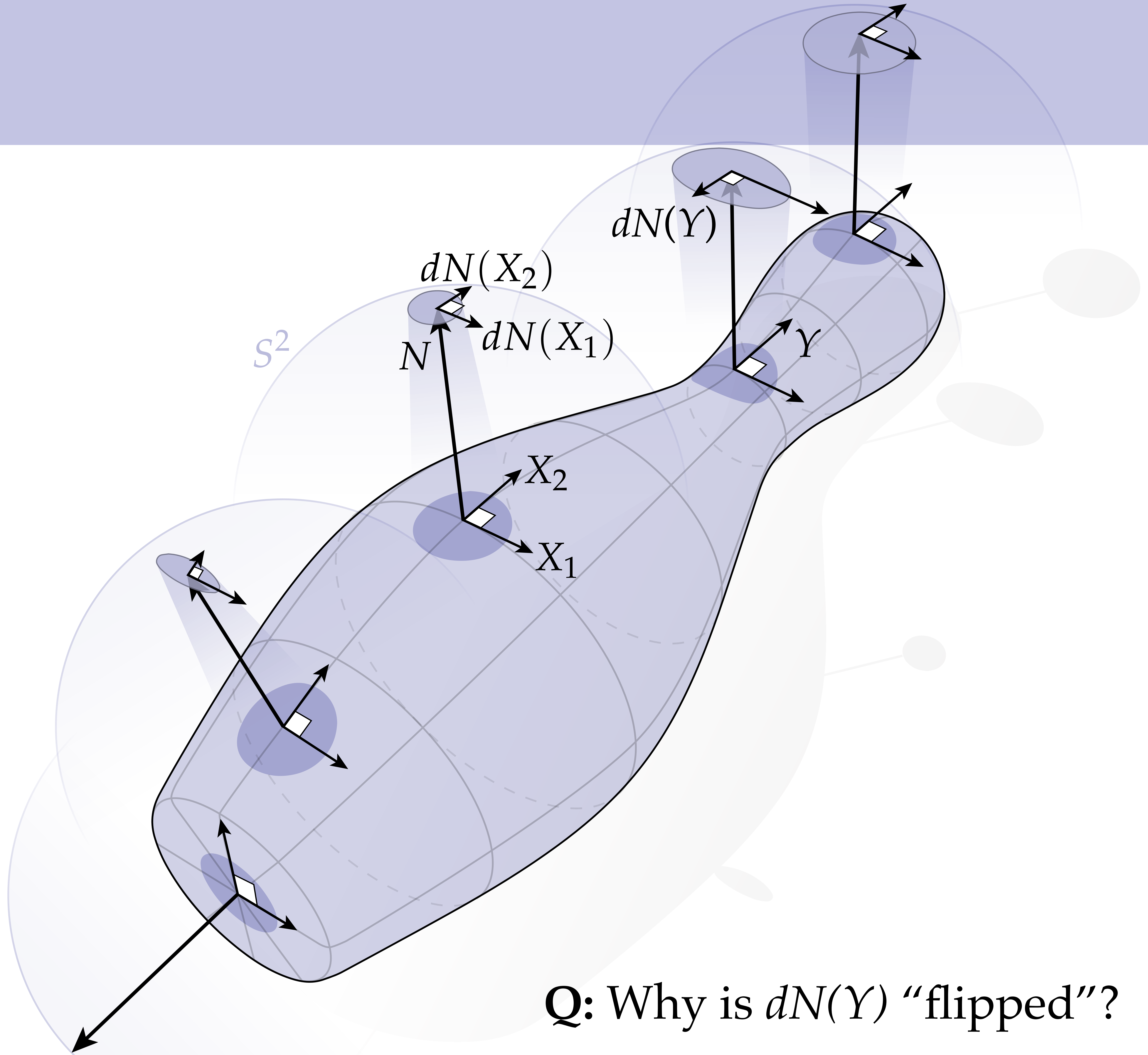
TODO. Define discrete torsion, give algorithm.

Curvature of Surfaces



Weingarten Map

- The **Weingarten map** dN is the differential of the Gauss map N
- At each point, tells us the change in the normal vector along any given direction X
- Since change in *unit* normal cannot have any component in the normal direction, $dN(X)$ is always tangent to the surface
- Can also think of it as a vector tangent to the unit sphere S^2



Q: Why is $dN(Y)$ "flipped"?

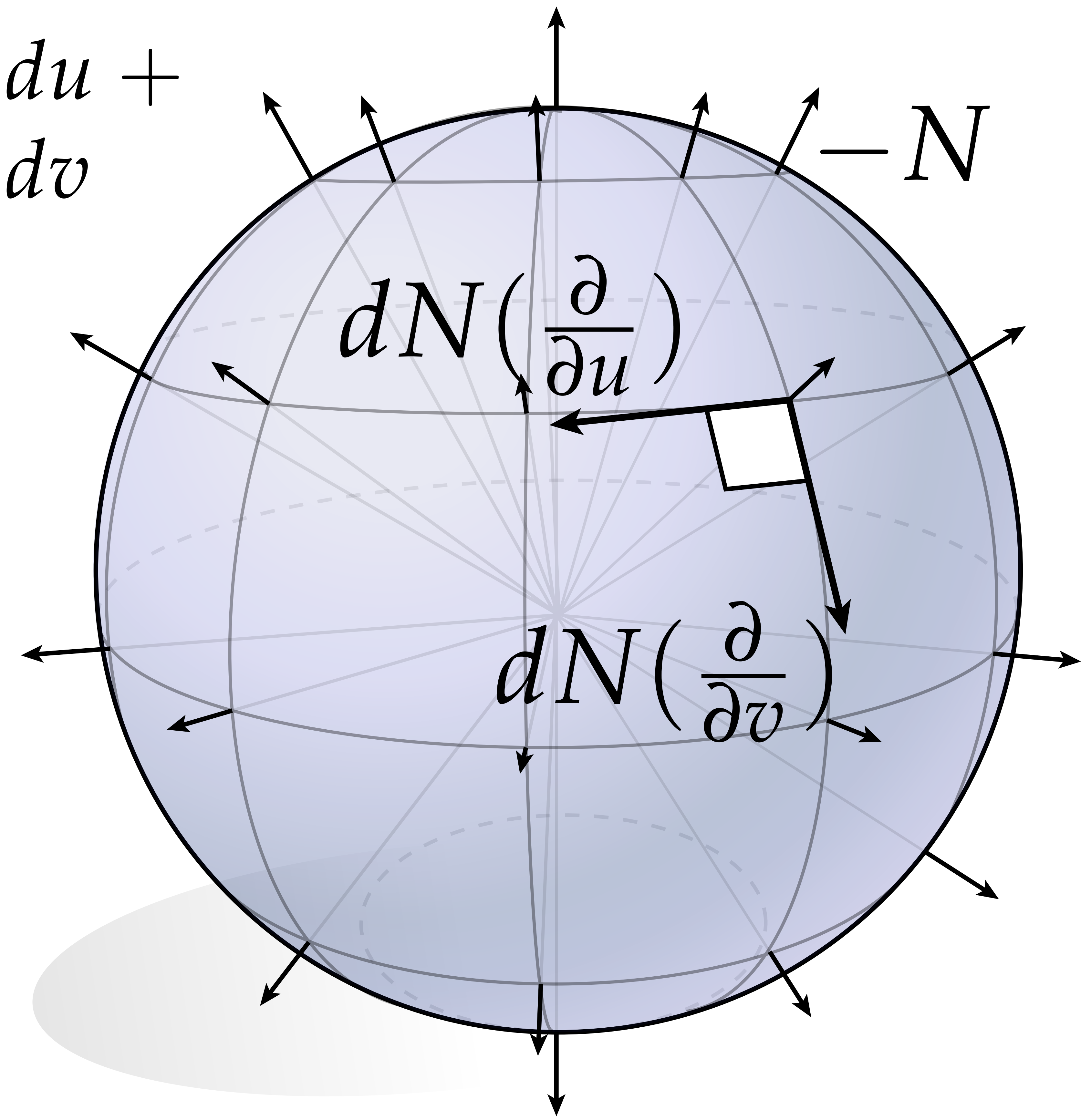
Weingarten Map—Example

- Recall that for the sphere, $N = -f$. Hence, Weingarten map dN is just $-df$:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{array}{llll} (-\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &)\ du\ + \\ (\cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &)\ dv \end{array}$$

$$dN = \begin{array}{llll} (\sin(u)\sin(v), & -\cos(u)\sin(v), & 0 &)\ du \\ (-\cos(u)\cos(v), & -\cos(v)\sin(u), & \sin(v) &)\ dv \end{array}$$



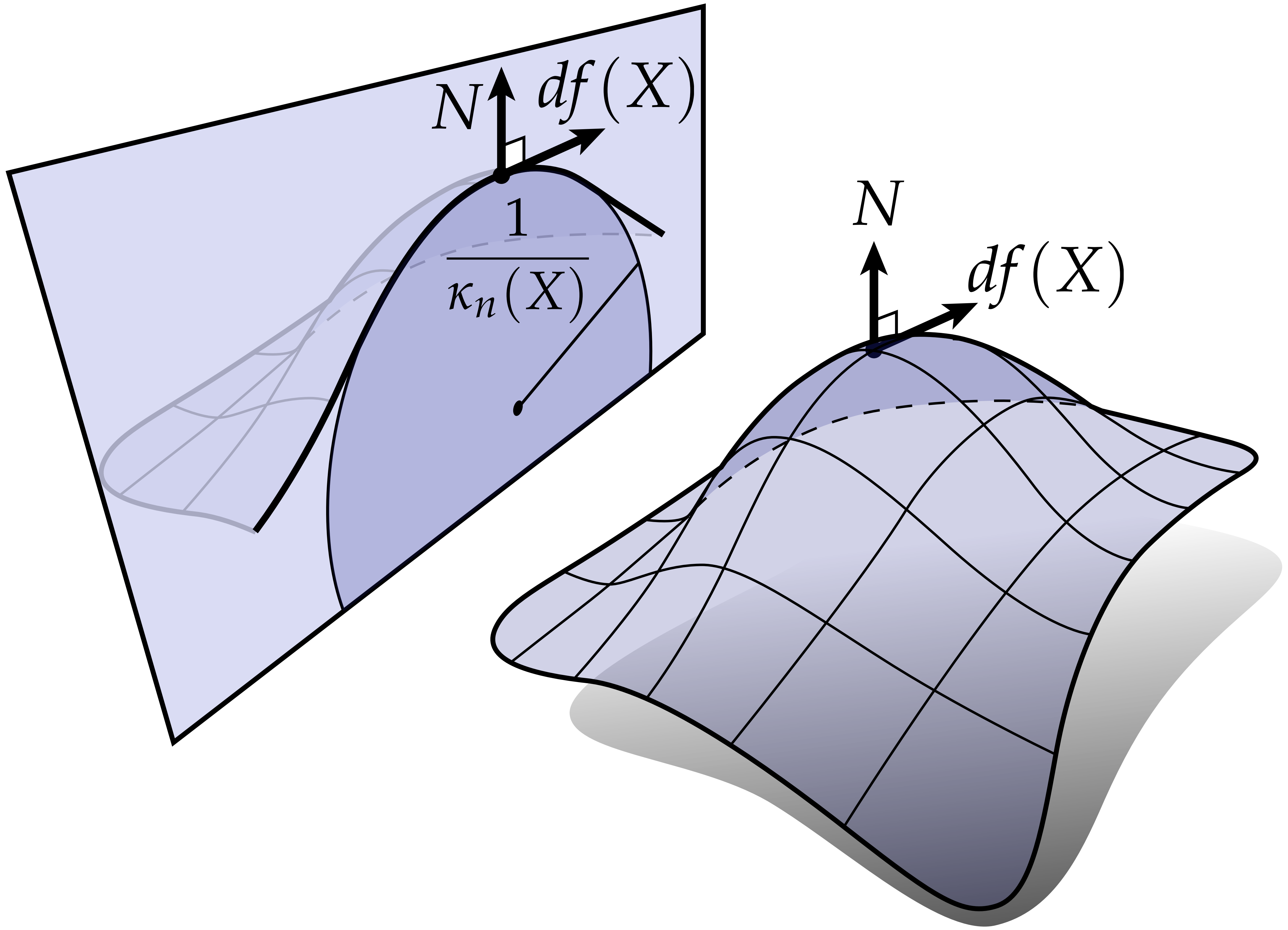
Key idea: computing the Weingarten map is no different from computing the differential of a surface.

Normal Curvature

- For curves, curvature was the rate of change of the *tangent*; for immersed surfaces, we'll instead consider how quickly the *normal* is changing.*
- In particular, **normal curvature** is rate at which normal is bending along a given tangent direction:

$$\kappa_N(X) := \frac{\langle df(X), dN(X)\rangle}{|df(X)|^2}$$

- Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve



*For plane curves, what would happen if we instead considered change in *N*?

Normal Curvature—Example

Consider a parameterized cylinder:

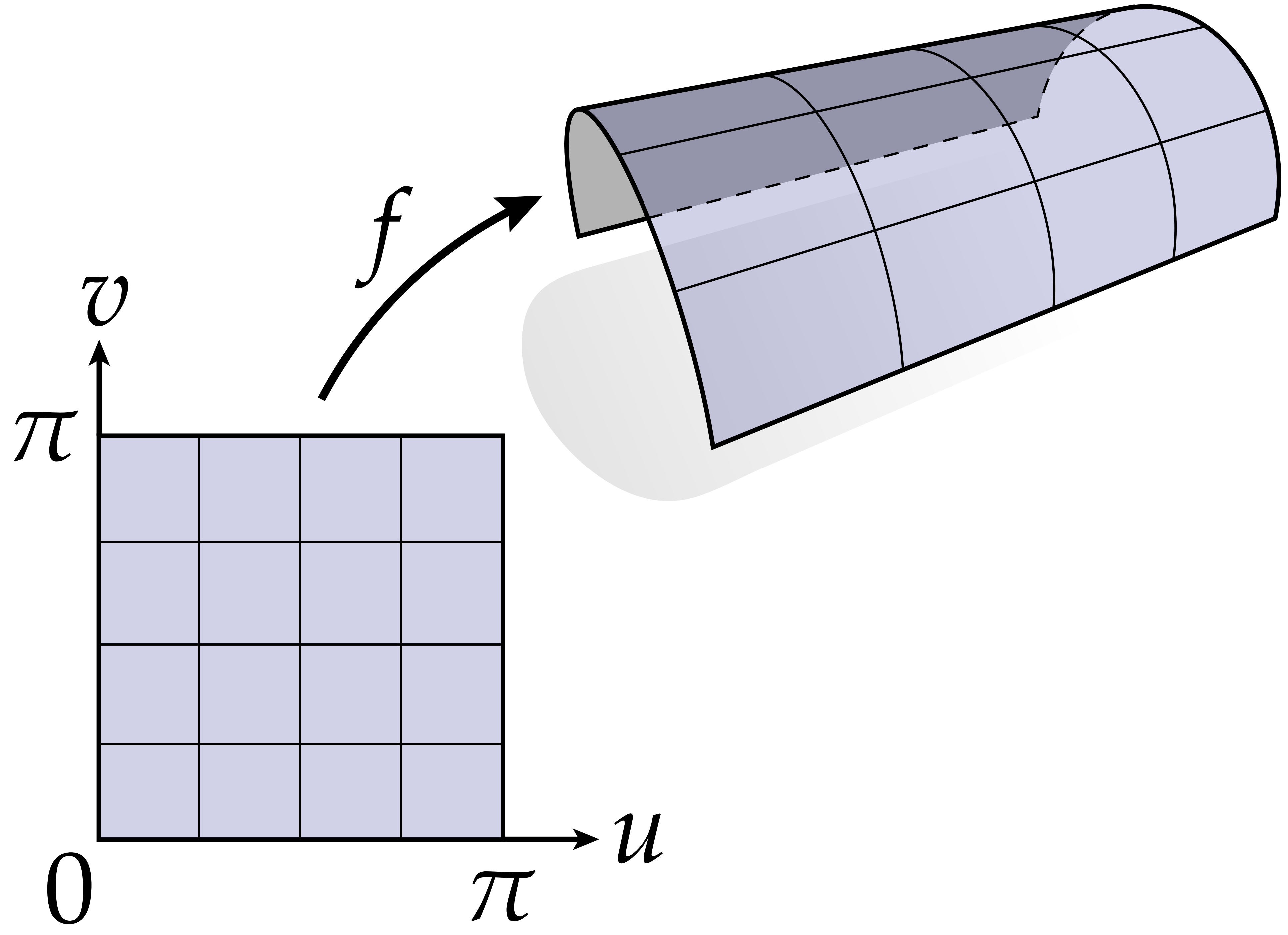
$$f(u,v) := (\cos(u), \sin(u), v)$$

$$df = (-\sin(u), \cos(u), 0)du + (0,0,1)dv$$

$$N = (-\sin(u), \cos(u), 0) \times (0,0,1)$$

$$= (\cos(u), \sin(u), 0)$$

$$dN = (-\sin(u), \cos(u), 0)du$$



$$\kappa_N(\tfrac{\partial}{\partial u}) = \frac{\langle df(\frac{\partial}{\partial u}), dN(\frac{\partial}{\partial u})\rangle}{|df(\frac{\partial}{\partial u})|^2} = \frac{(-\sin(u),\cos(u),0)\cdot(-\sin(u),\cos(u),0)}{|(-\sin(u),\cos(u),0)|^2} = 1$$

$$\kappa_N(\tfrac{\partial}{\partial v}) = \cdots = 0$$

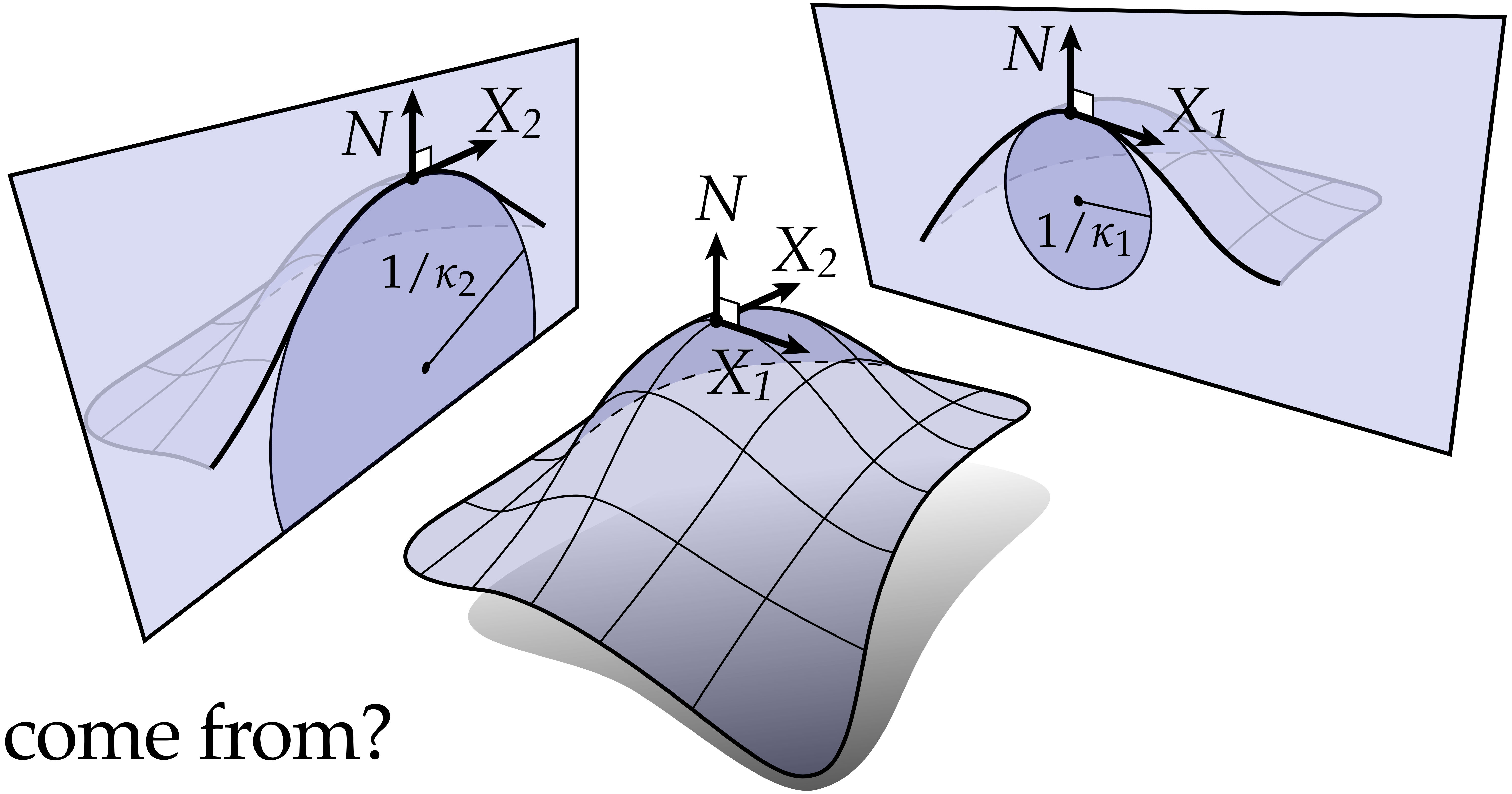
Q: Does this result make sense geometrically?

Principal Curvature

- Among all directions X, there are two **principal directions** X_1, X_2 where normal curvature has minimum/maximum value (respectively)
- Corresponding normal curvatures are the **principal curvatures**
- Two critical facts*:

1. $g(X_1, X_2) = 0$

2. $dN(X_i) = \kappa_i df(X_i)$



Where do these relationships come from?

Shape Operator

- The change in the normal N is always *tangent* to the surface
- Must therefore be some linear map S from tangent vectors to tangent vectors, called the **shape operator**, such that

$$df(SX) = dN(X)$$

- Principal directions are the *eigenvectors* of S
- Principal curvatures are *eigenvalues* of S
- **Note:** S is not a symmetric matrix! Hence, eigenvectors are not orthogonal in R^2; only orthogonal with respect to induced metric g.

Shape Operator — Example

Consider a nonstandard parameterization of the cylinder (*sheared* along z):

$$f(u,v) := (\cos(u), \sin(u), u+v) \qquad df = (-\sin(u), \cos(u), 1)du + (0,0,1)dv$$

$$N = (\cos(u), \sin(u), 0) \qquad dN = (-\sin(u), \cos(u), 0)du$$

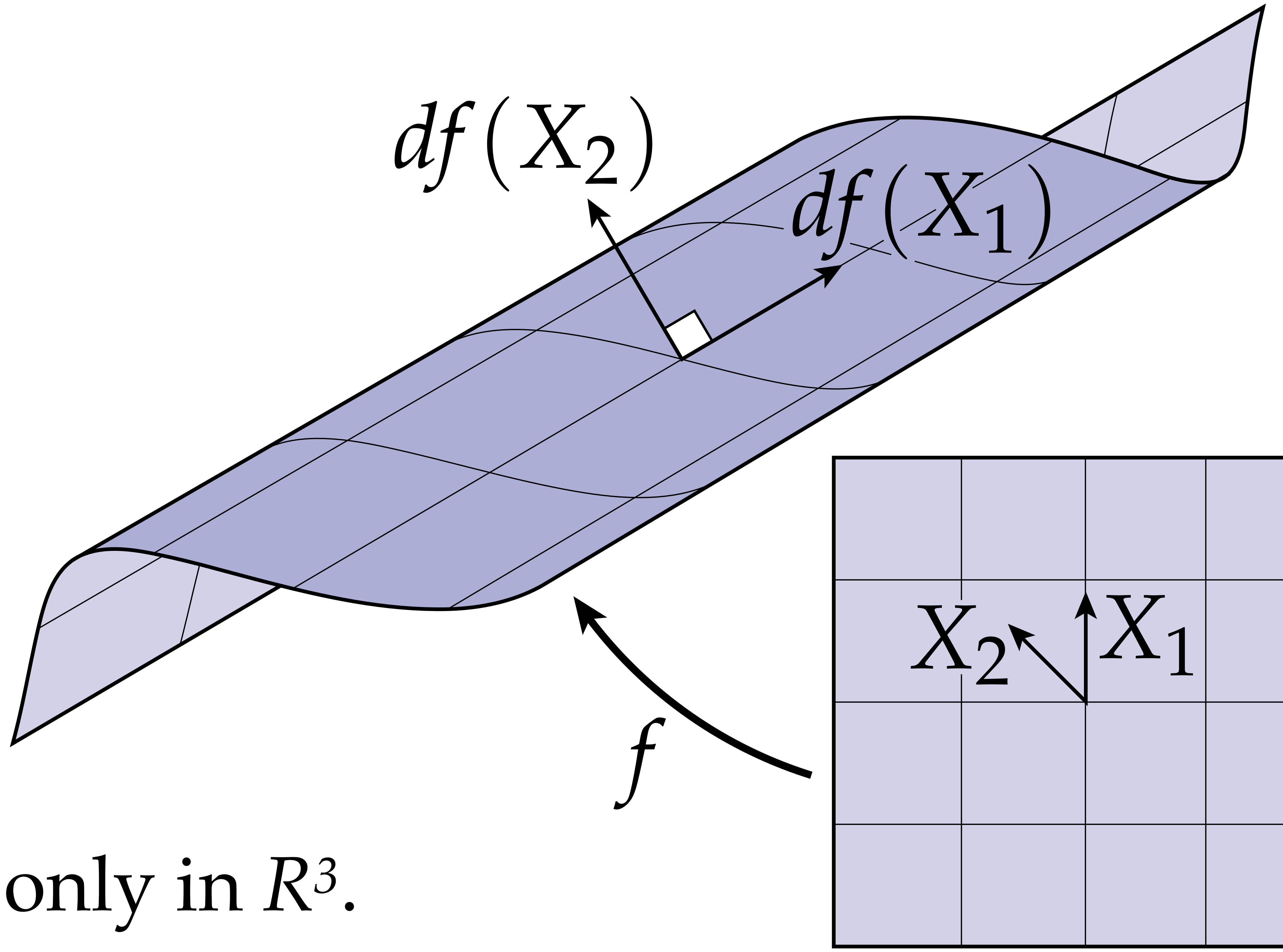
$$\boxed{df \circ S = dN}$$

$$\begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$df(X_1) = (0,0,1) \qquad \kappa_1 = 0$$

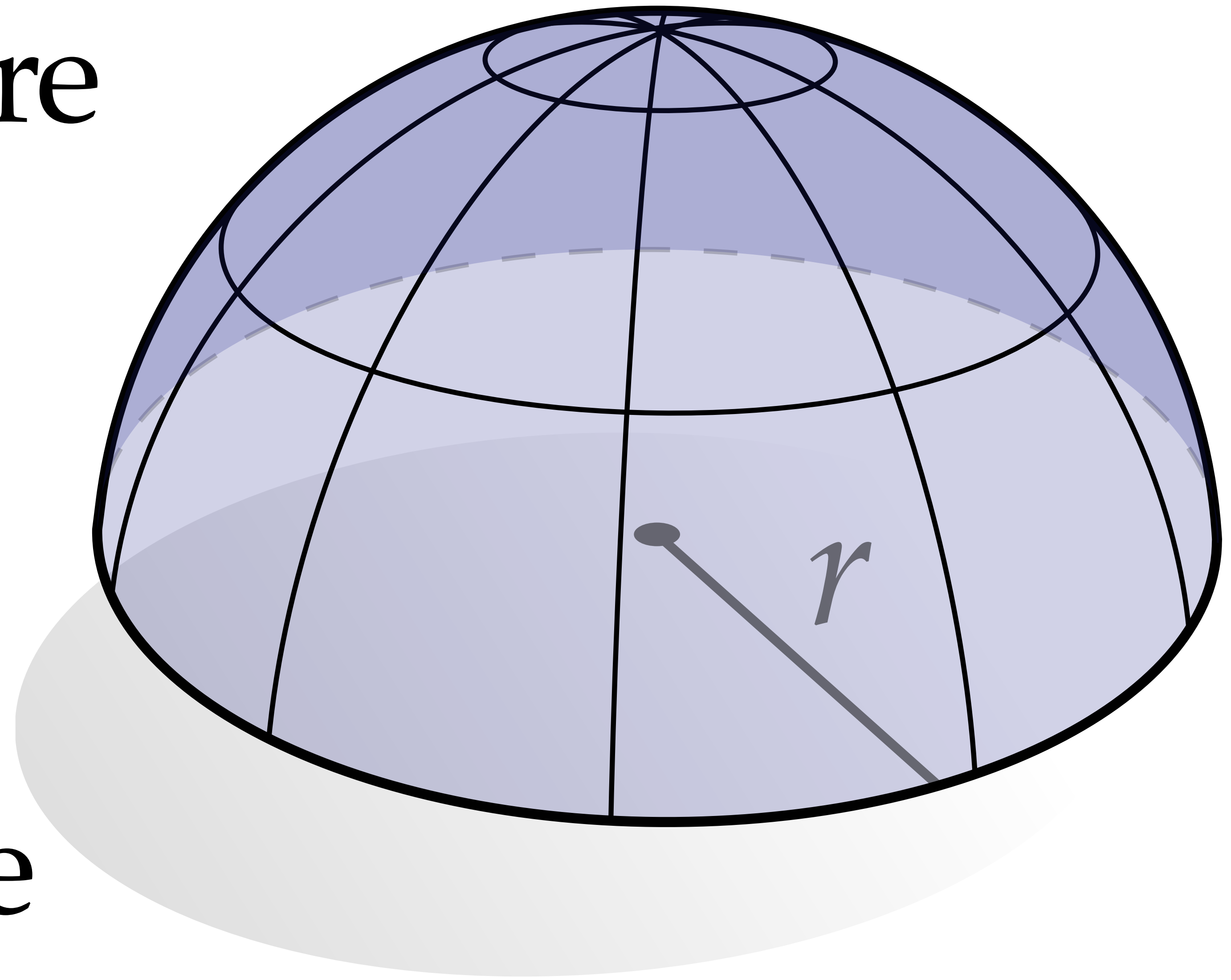
$$df(X_2) = (\sin(u), -\cos(u), 0) \qquad \kappa_2 = 1$$



Key observation: principal directions orthogonal only in R^3.

Umbilic Points

- Points where principal curvatures are equal are called **umbilic points**
- Principal *directions* are not uniquely determined here
- What happens to the shape operator S?
  - May still have full rank!
  - Just have repeated eigenvalues, 2-dim. eigenspace

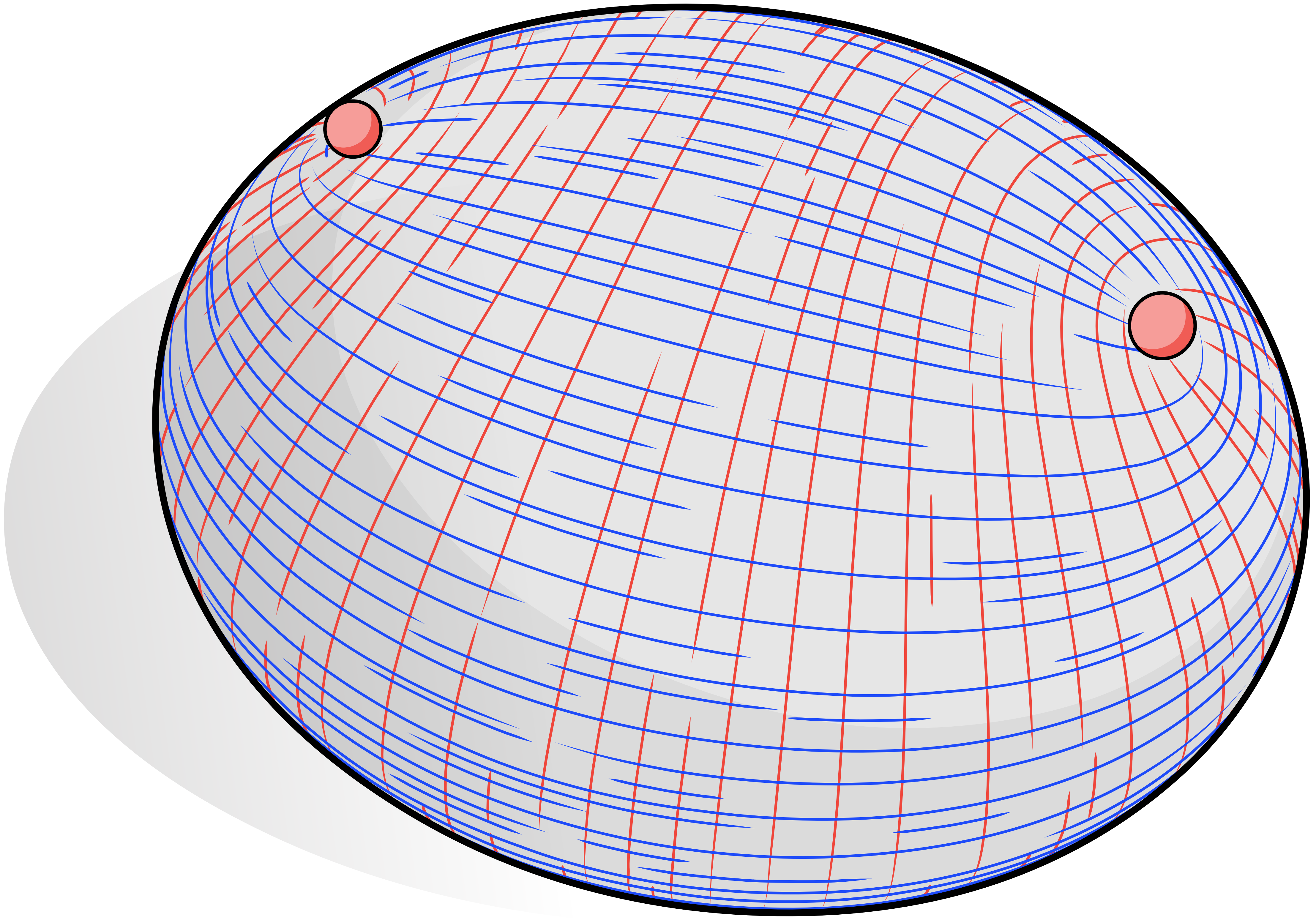


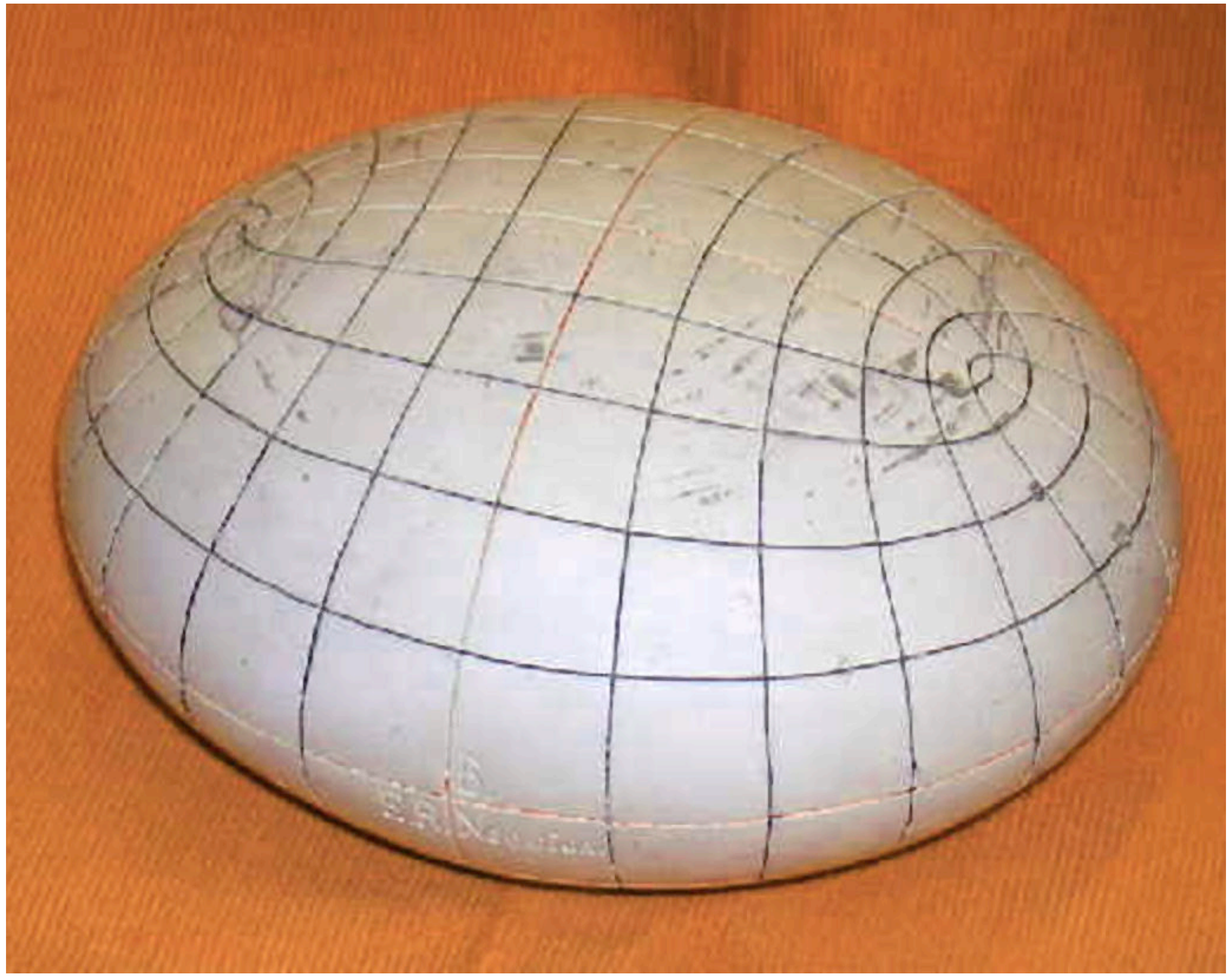
$$S = \begin{bmatrix} 1/r & 0 \\ 0 & 1/r \end{bmatrix} \qquad \kappa_1 = \kappa_2 = \frac{1}{r} \qquad \forall X,\ SX = \tfrac{1}{r}X$$

Could still of course choose (arbitrarily) an orthonormal pair X_1, X_2…

Principal Curvature Nets

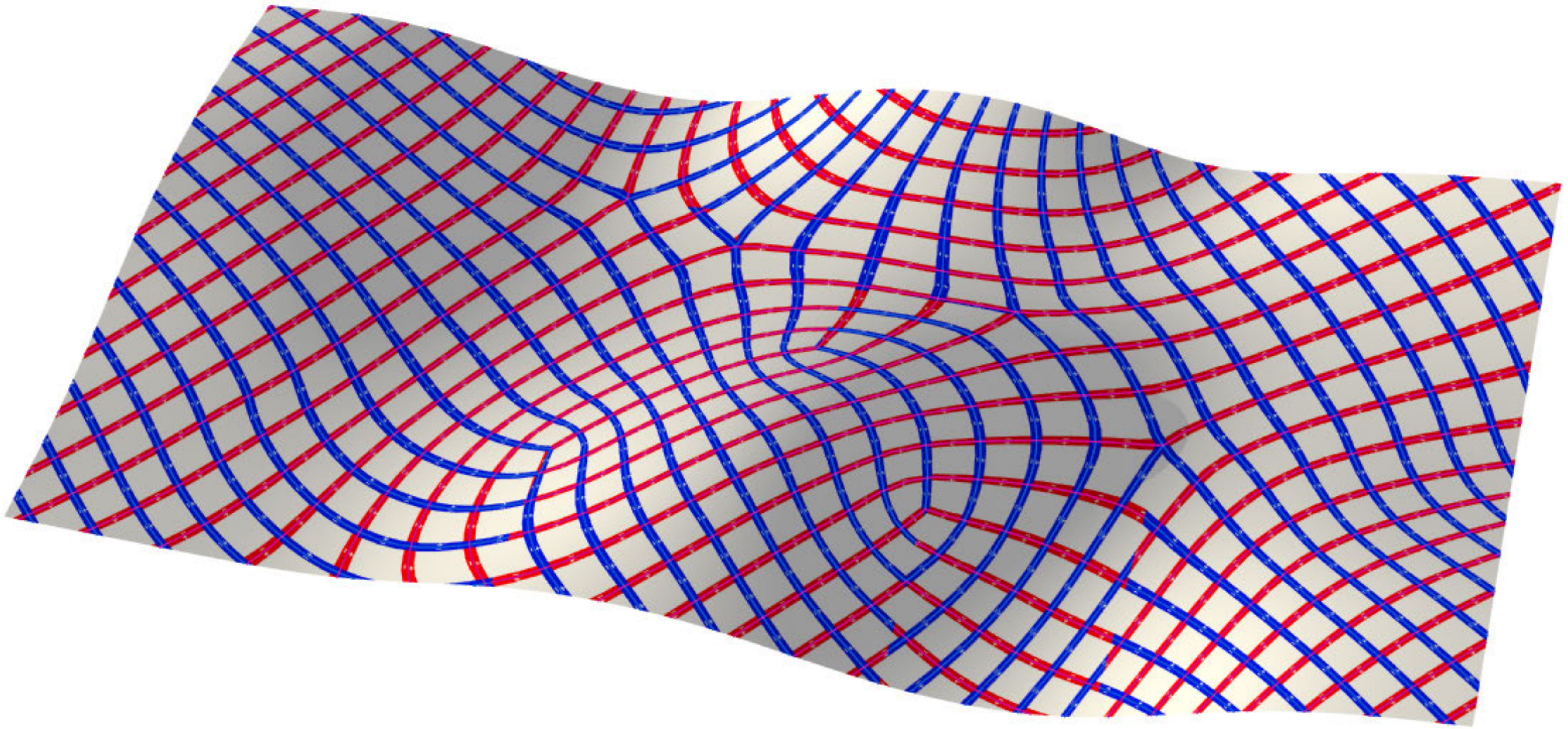
- Walking along principal direction field yields **principal curvature lines**
- Collection of all such lines is called the **principal curvature network**

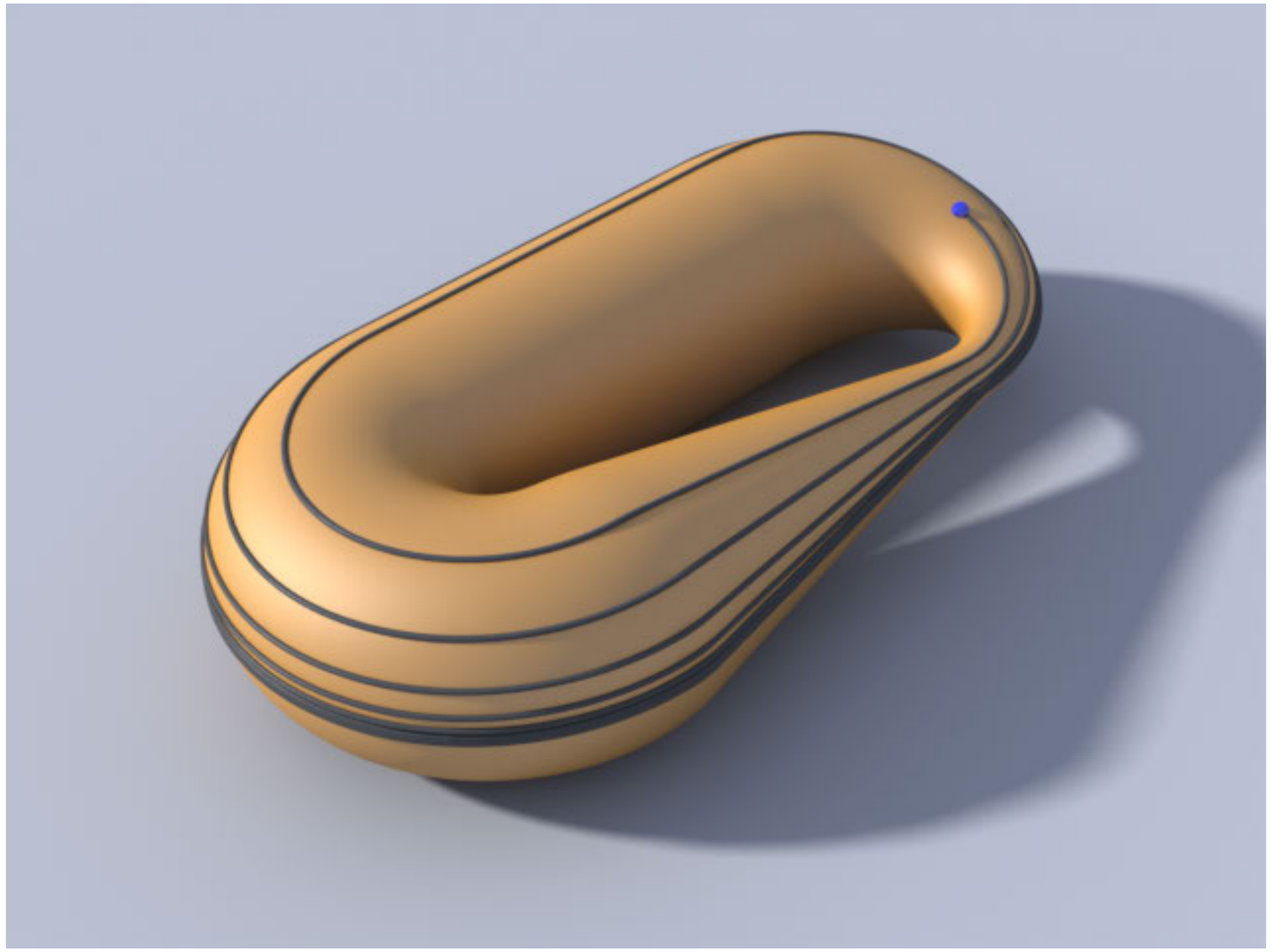




Separatrices and Spirals

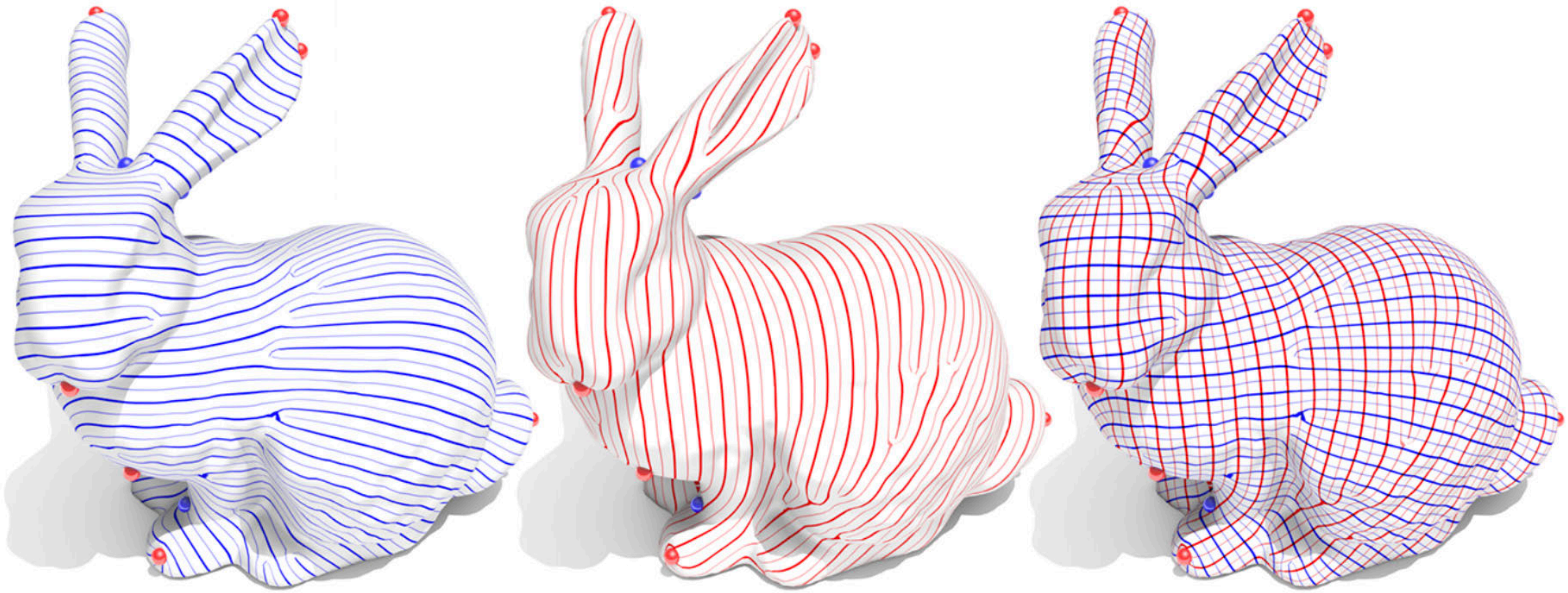
- If we walk along a principal curvature line, where do we end up?
- Sometimes, a curvature line terminates at an umbilic point in both directions; these so-called **separatrices** (can) split network into regular patches.
- Other times, we make a closed loop. More often, however, behavior is *not* so nice!





Application—Quad Remeshing

- Recent approach to meshing: construct net *roughly* aligned with principal curvature—but with separatrices & loops, not spirals.

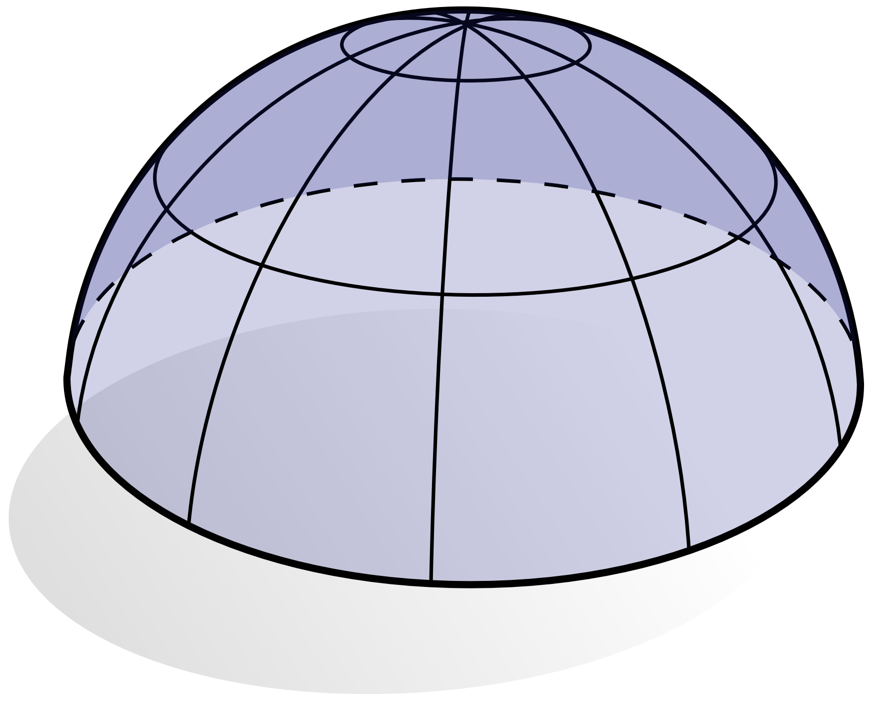


from Knöppel, Crane, Pinkall, Schröder, *"Stripe Patterns on Surfaces"*

Gaussian and Mean Curvature

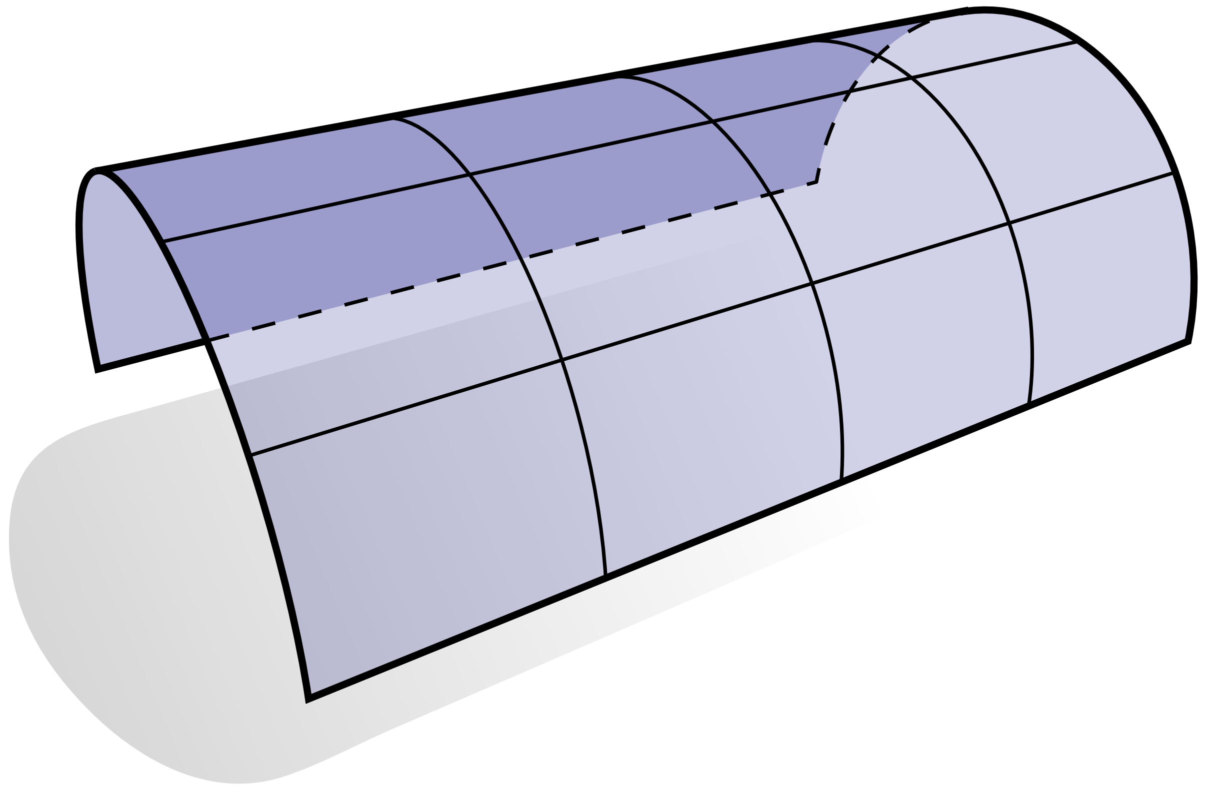
Gaussian and *mean* curvature also fully describe local bending:

$$\textbf{Gaussian} \quad K := \kappa_1 \kappa_2$$
$$\textbf{mean*} \quad H := \tfrac{1}{2}(\kappa_1 + \kappa_2)$$



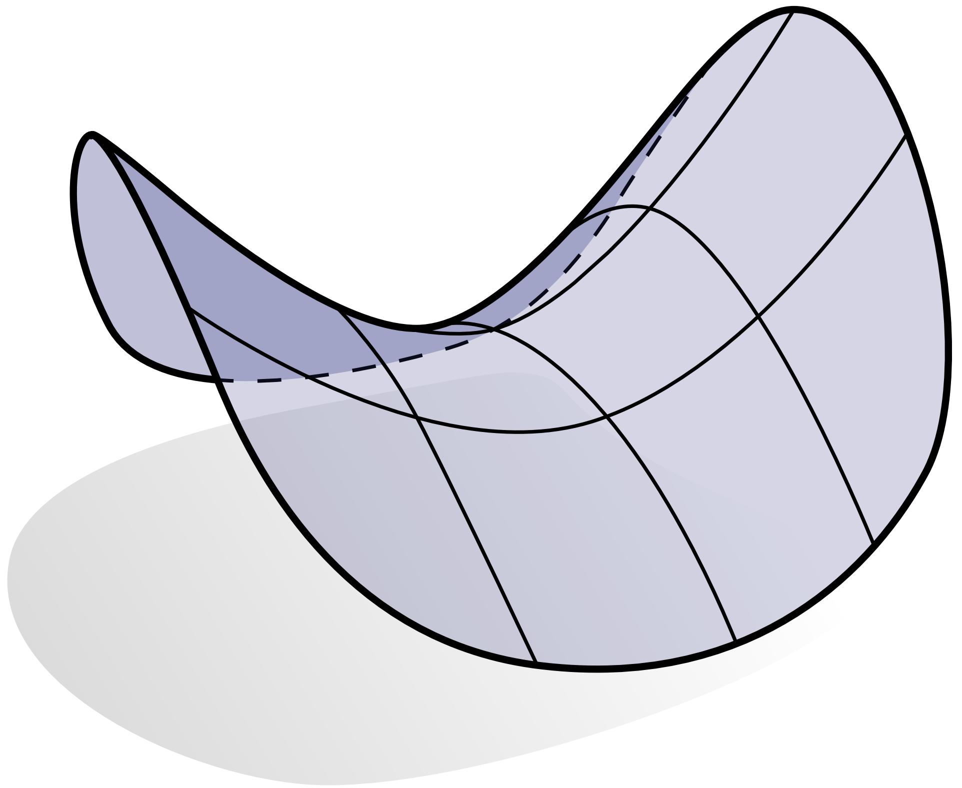
$K > 0$

$H \neq 0$



"developable" $K = 0$

$H \neq 0$



$K < 0$

"minimal" $H = 0$

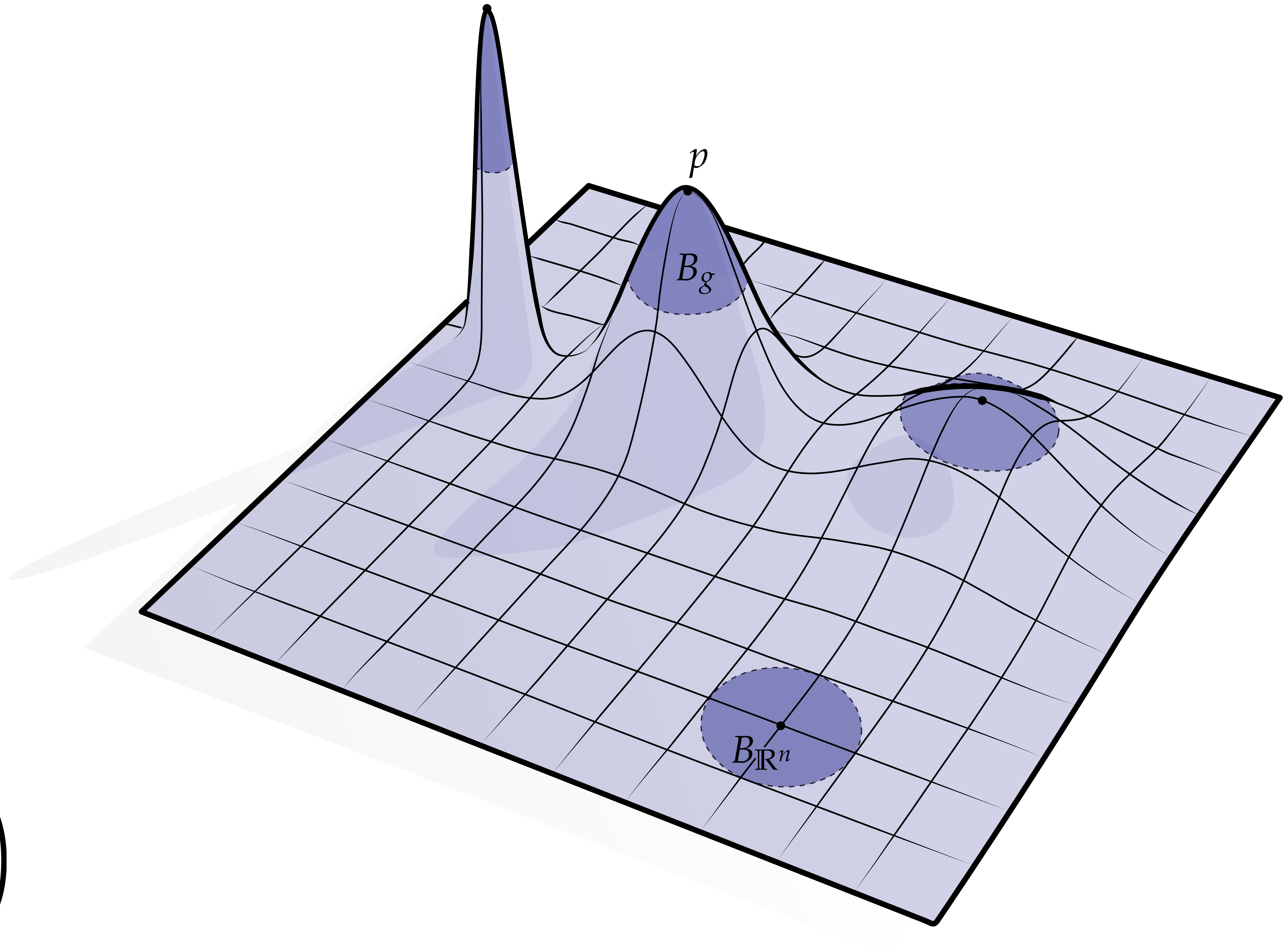
*$\textbf{Warning:}$ another common convention is to omit the factor of $1/2$

Gaussian Curvature as Ratio of Ball Areas

- Originally defined Gaussian curvature as product of principal curvatures
- Can also view it as "failure" of balls to behave like Euclidean balls

Roughly speaking,

$$K \propto 1 - \frac{|B_g|}{|B_{\mathbb{R}^2}|}$$



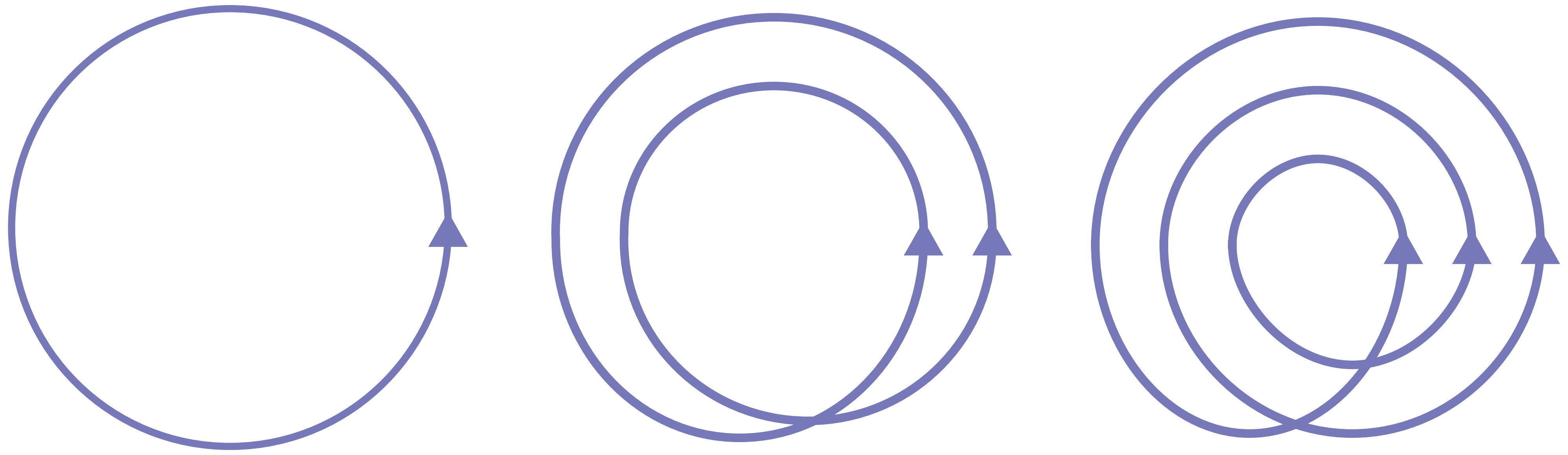
More precisely:

$$|B_g(p,\varepsilon)| = |B_{\mathbb{R}^2}(p,\varepsilon)| \left(1 - \frac{K}{12}\varepsilon^2 + O(\varepsilon^3)\right)$$

Gauss-Bonnet Theorem

- Recall that the total curvature of a closed plane curve was always equal to 2π times *turning number k*
- **Q:** Can we make an analogous statement about surfaces?
- **A:** Yes! Gauss-Bonnet theorem says total Gaussian curvature is always 2π times *Euler characteristic* χ
- For tori, Euler characteristic expressed in terms of the *genus* (number of "handles")

$$\chi := 2 - 2g$$



k=1 k=2 k=3

g=0 g=1 g=2 g=3

Curves

$$\int_0^L \kappa\, ds = 2\pi k$$

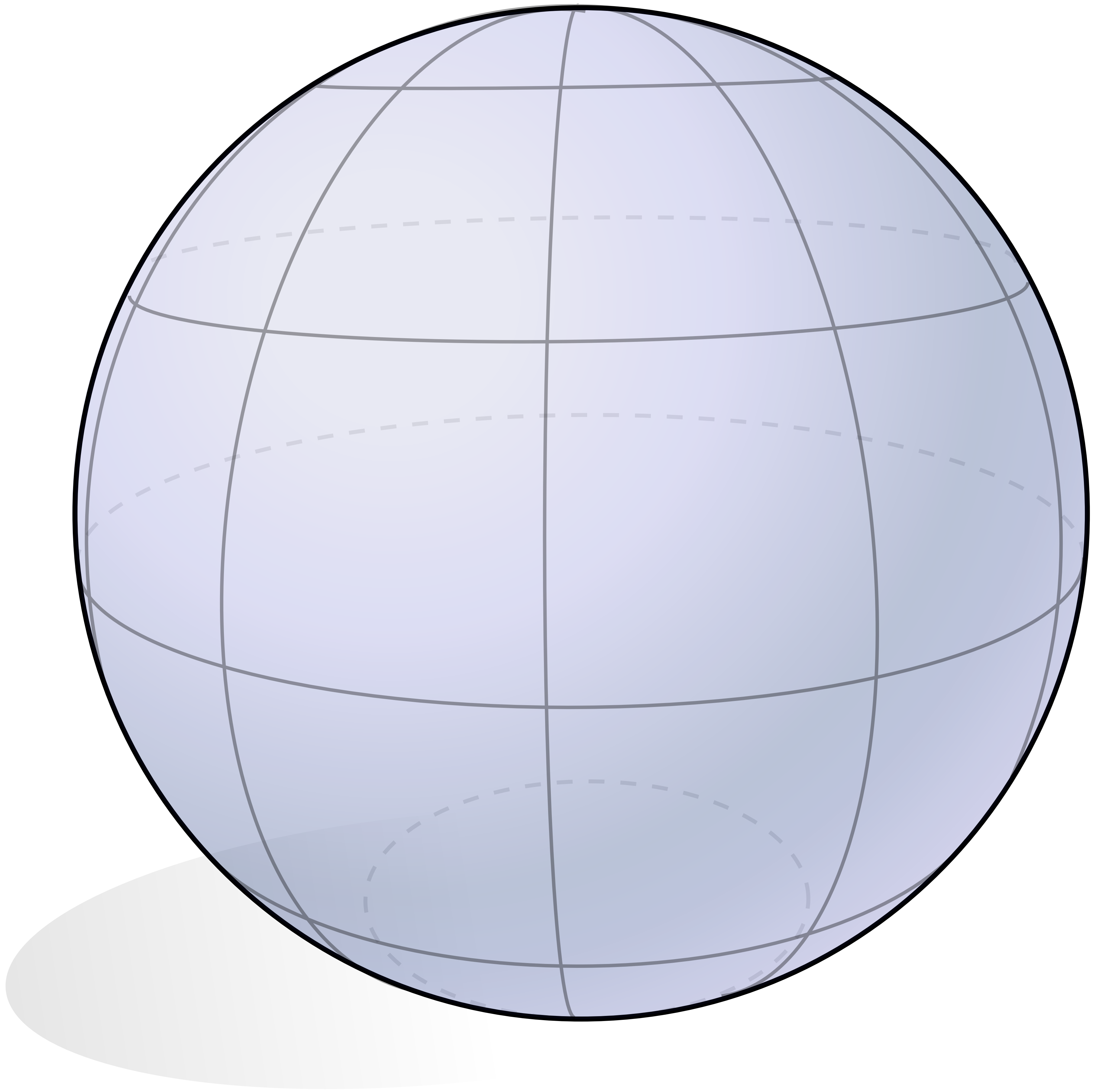
Surfaces

$$\int_M K\, dA = 2\pi\chi$$

Total Mean Curvature?

Theorem (Minkowski): for a regular closed embedded surface,

$$\int_M H\, dA \geq \sqrt{4\pi A}$$

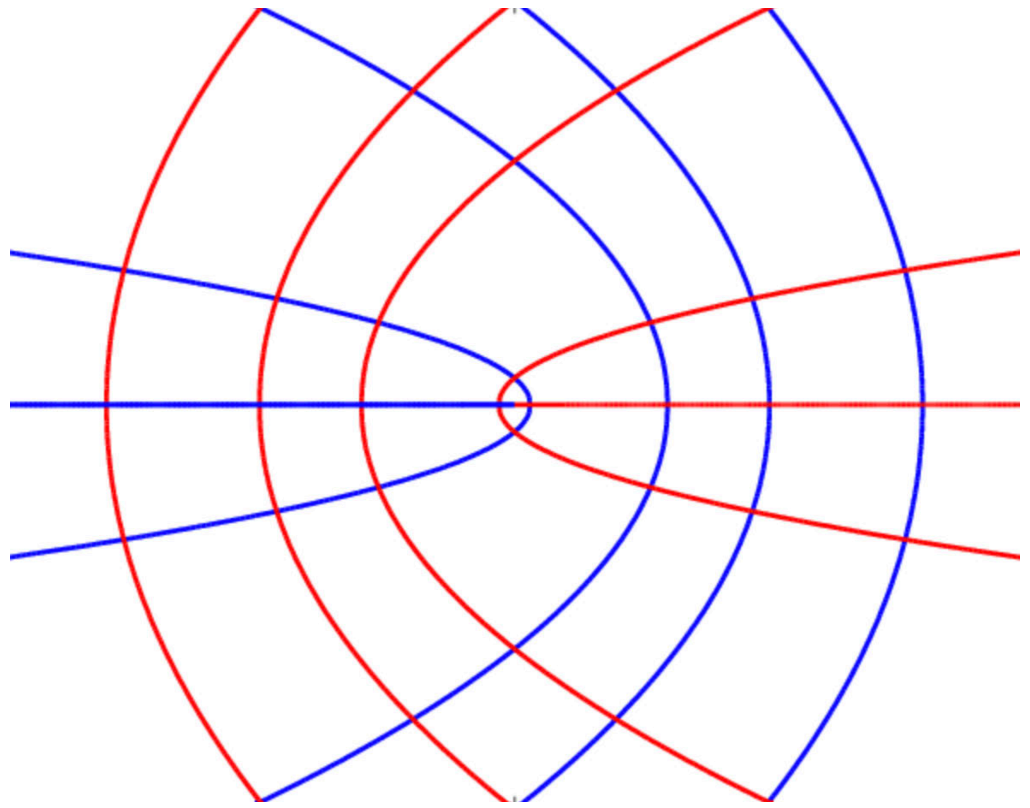


Q: When do we get equality?

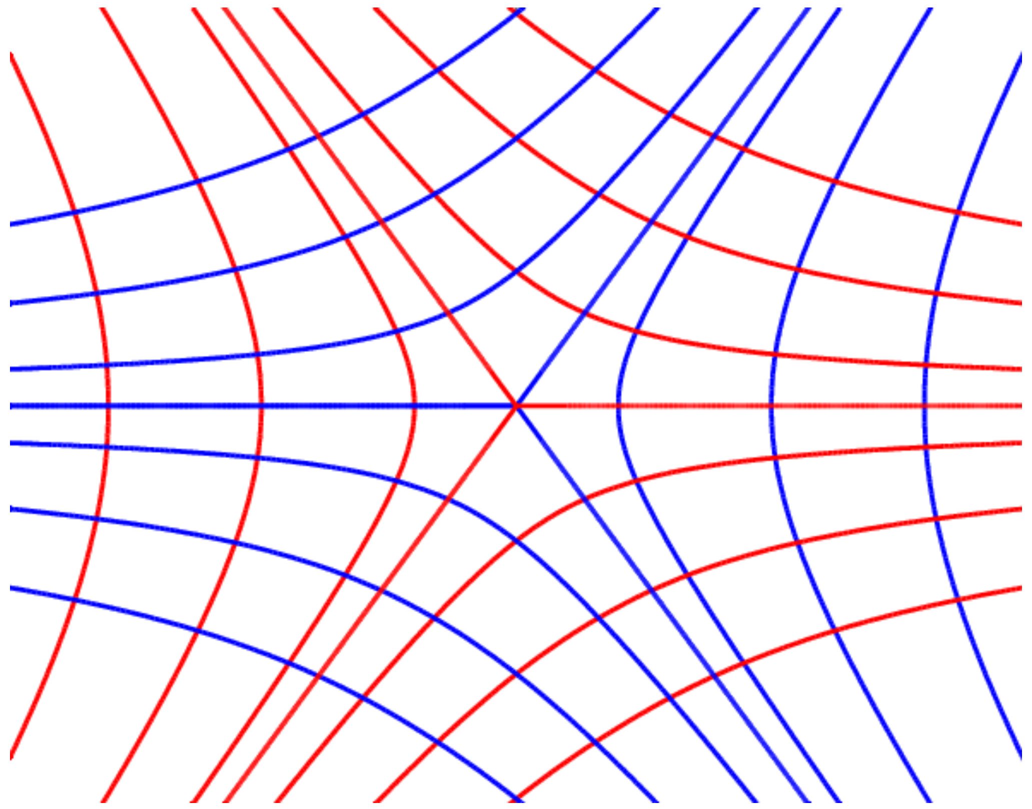
A: For a sphere.

Topological Invariance of Umbilic Count

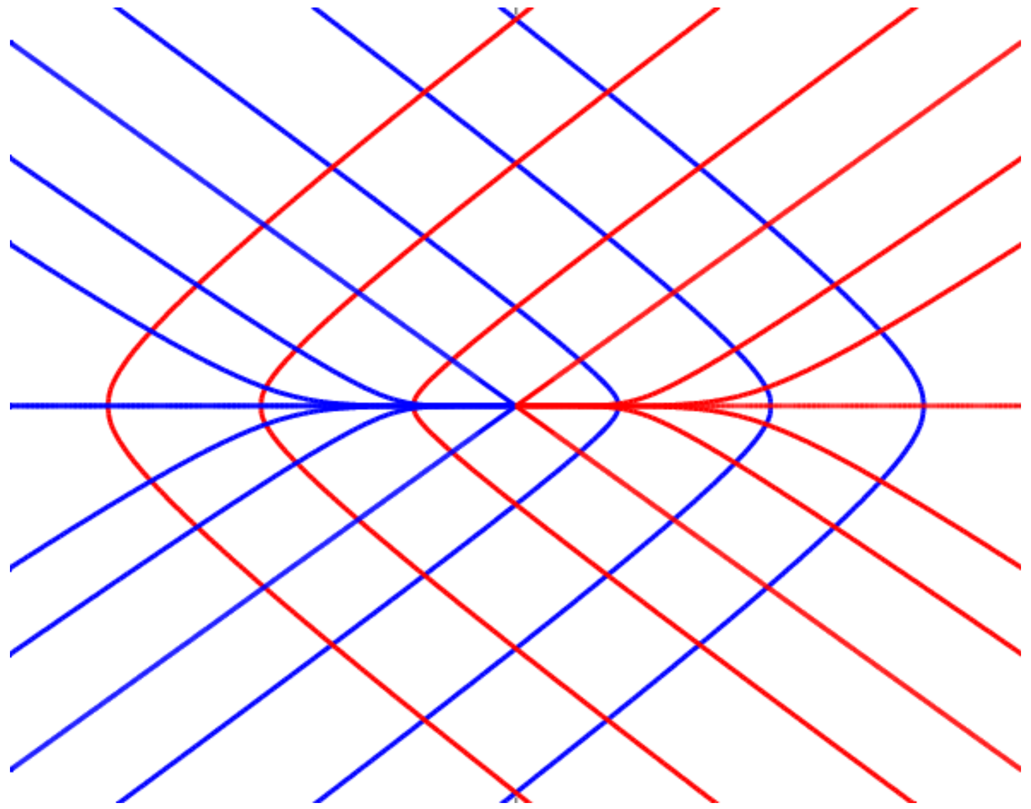
Can classify regions around **isolated** umbilics into three types based on behavior of principal network: *lemon*, *star*, and *monstar*



lemon (k_1)



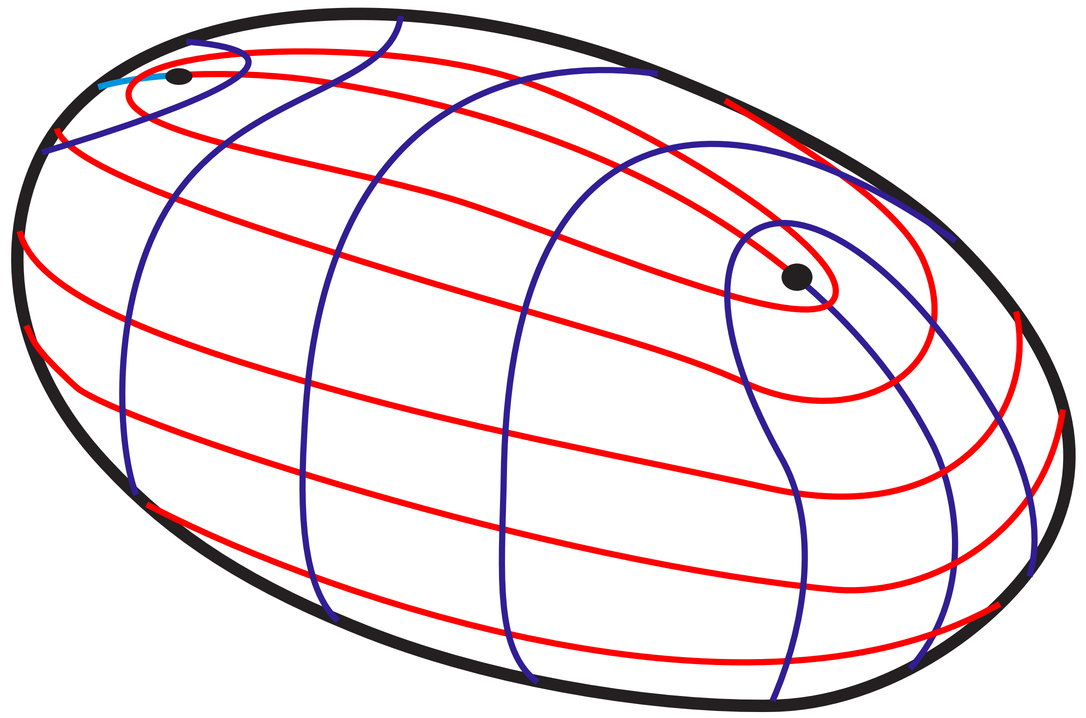
star (k_2)



monstar (k_3)

Fact. If k_1, k_2, k_3 are number of umbilics of each type, then

$$\kappa_1 - \kappa_2 + \kappa_3 = 2\chi$$



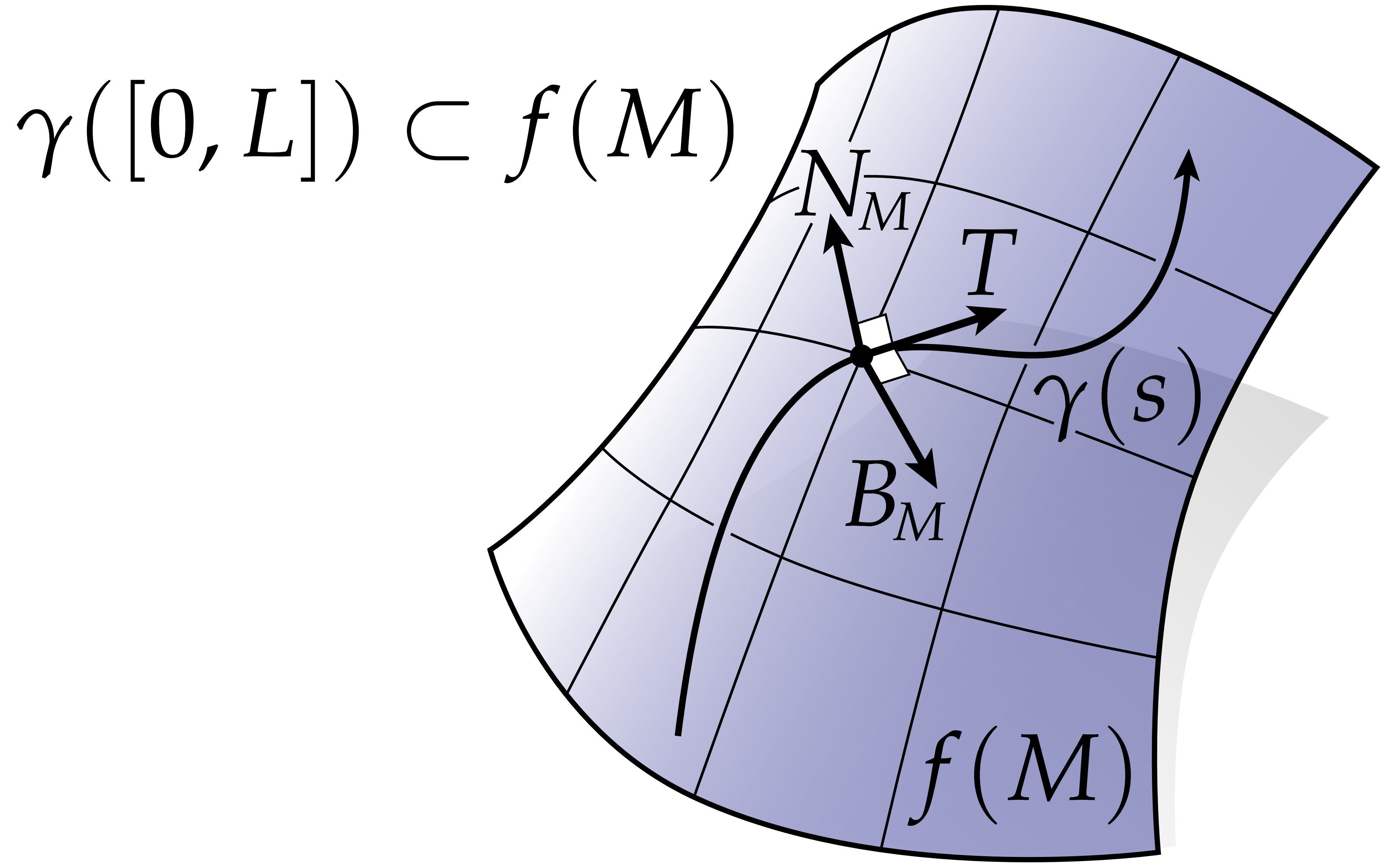
Curvature of a Curve in a Surface

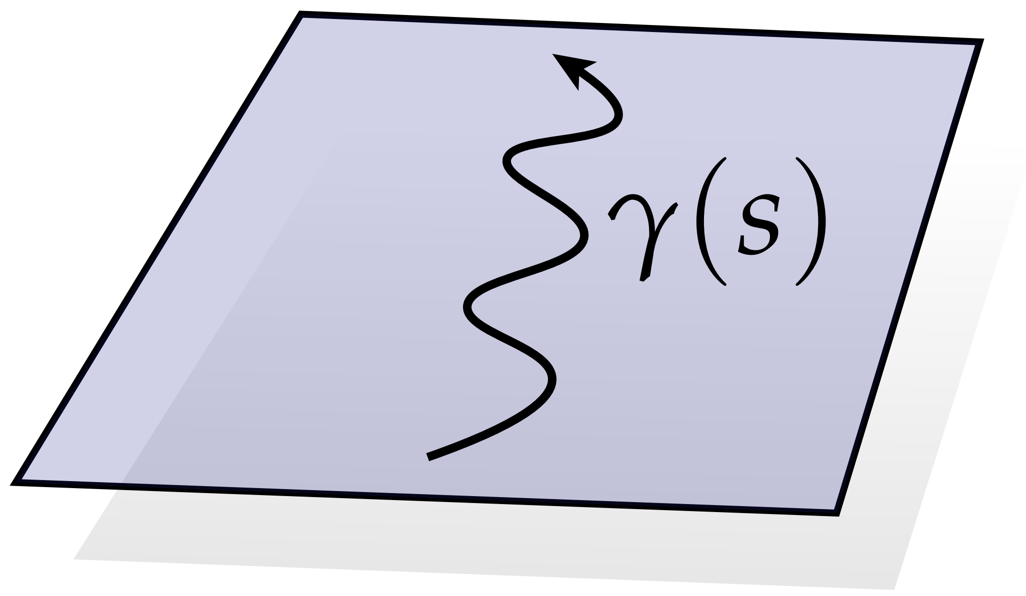
- Earlier, broke the "bending" of a space curve into curvature (κ) and torsion (τ)
- For a curve *in a surface*, can instead break into *normal* and *geodesic* curvature:

$$\kappa_n := \langle N_M, \tfrac{d}{ds} T \rangle$$
$$\kappa_g := \langle B_M, \tfrac{d}{ds} T \rangle$$

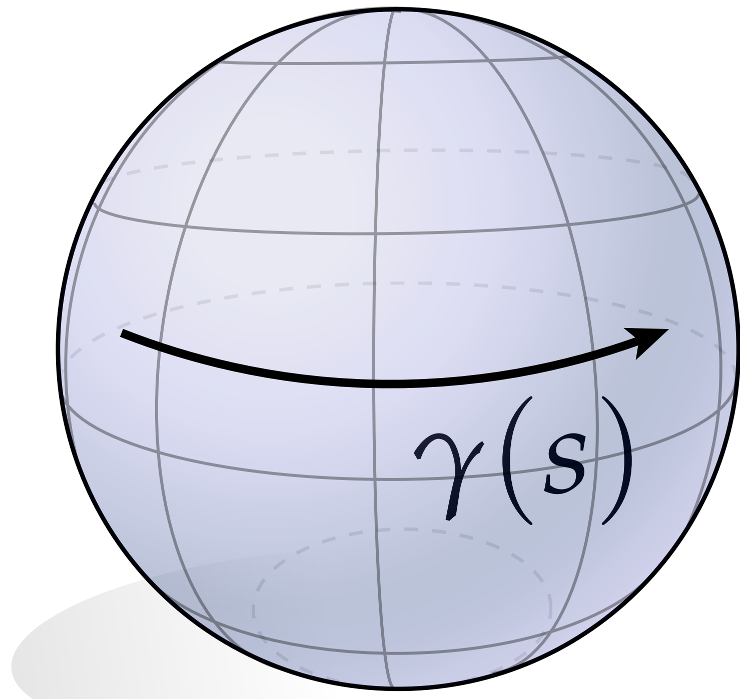
- T is still tangent of the curve; but unlike the Frenet frame, N_M is the normal of the surface and $B_M := T \times N_M$

Q: Why no third curvature $\langle T_M, \frac{d}{ds} T \rangle$?





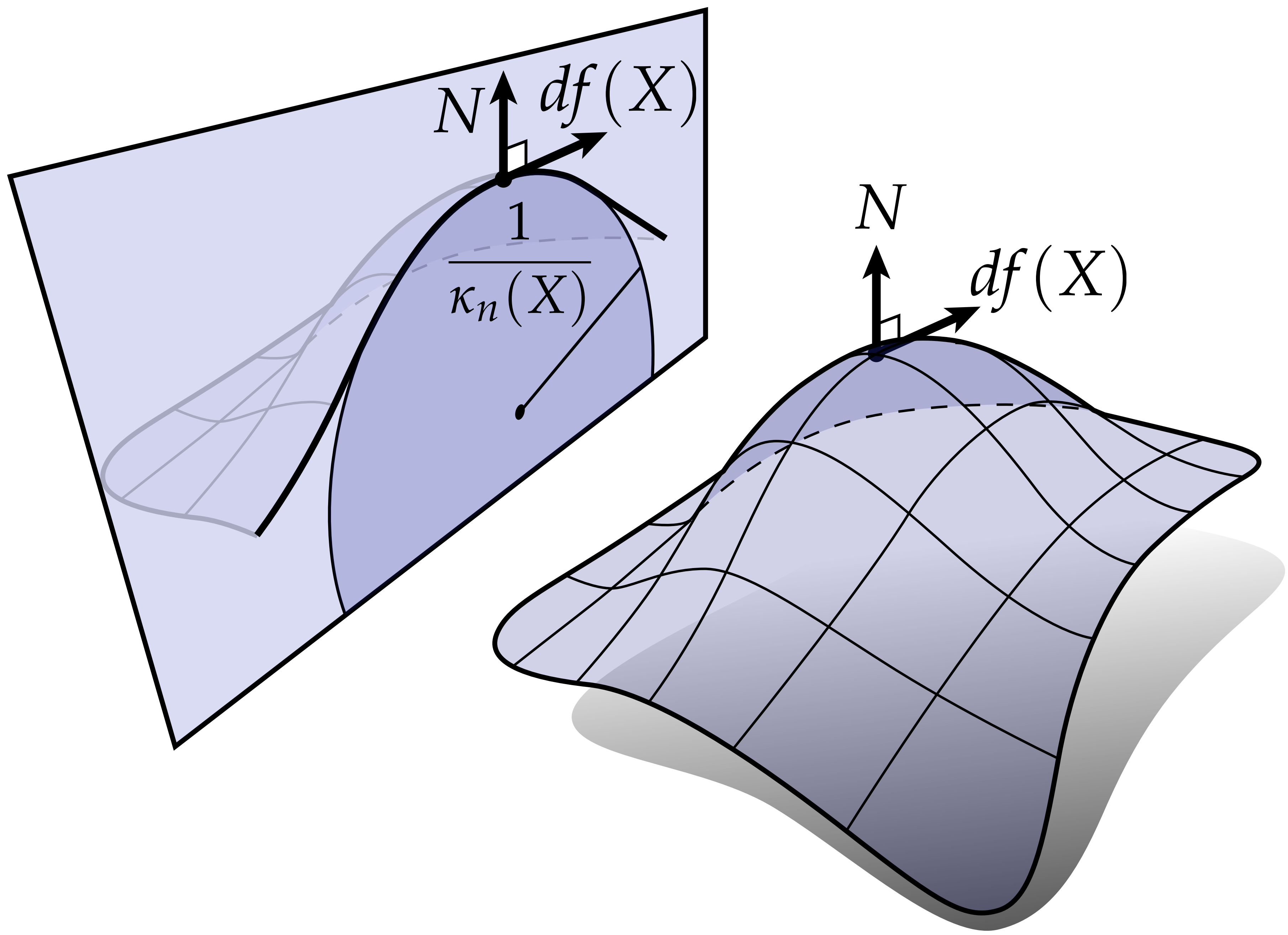
large κ_g;
small κ_n



large κ_n;
small κ_g

Second Fundamental Form

- Second fundamental form is closely related to principal curvature
- Can also be viewed as change in *first* fundamental form under motion in normal direction
- Why "fundamental?"First & second fundamental forms play role in important theorem…



$$\mathbf{II}(X,Y) := \langle dN(X), df(Y)\rangle$$

$$\kappa_N(X) := \frac{df(X), dN(X)}{|df(X)|^2} = \frac{\mathbf{II}(X,X)}{\mathbf{I}(X,X)}$$

Fundamental Theorem of Surfaces

- **Fact.** Two surfaces in R^3 are congruent if and only if they have the same first and second fundamental forms
  - …However, not every pair of bilinear forms **I**, **II** on a domain U describes a valid surface—must satisfy the **Gauss Codazzi** equations
- Analogous to fundamental theorem of plane curves: determined up to rigid motion by curvature
  - …However, for *closed* curves not every curvature function is valid (*e.g.*, must integrate to $2k\pi$)

Fundamental Theorem of Discrete Surfaces

- **Fact.** Up to rigid motions, can recover a discrete surface from its *dihedral angles* and *edge lengths*.
- Fairly natural analogue of Gauss-Codazzi; data is split into edge lengths (encoding **I**) and dihedral angles (encoding **II**)
- Basic idea: construct each triangle from edge lengths; use dihedral angles to globally glue together

from Wang, Liu, and Tong,
"Linear Surface Reconstruction from Discrete Fundamental Forms on Triangle Meshes"

Other Descriptions of Surfaces?

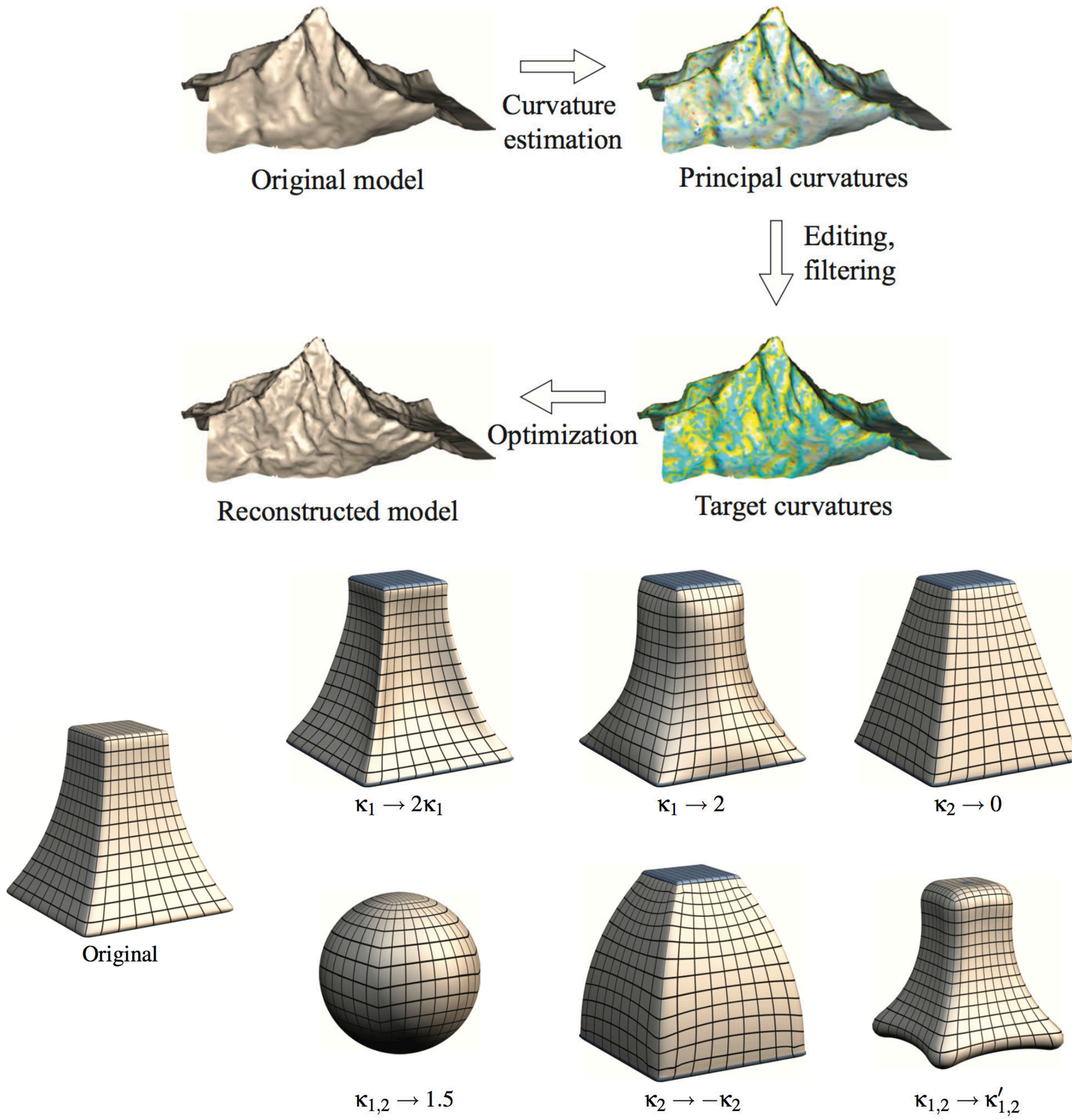
- Classic question in differential geometry:

> *"What data is sufficient to completely determine a surface in space?"*

- Many possibilities…
  - First & second fundamental form (Gauss-Codazzi)
  - Mean curvature and metric (up to "Bonnet pairs")
  - Convex surfaces: metric alone is enough (Alexandrov/Pogorolev)
  - Gauss curvature essentially determines metric (Kazdan-Warner)
- …in general, still a surprisingly murky question!

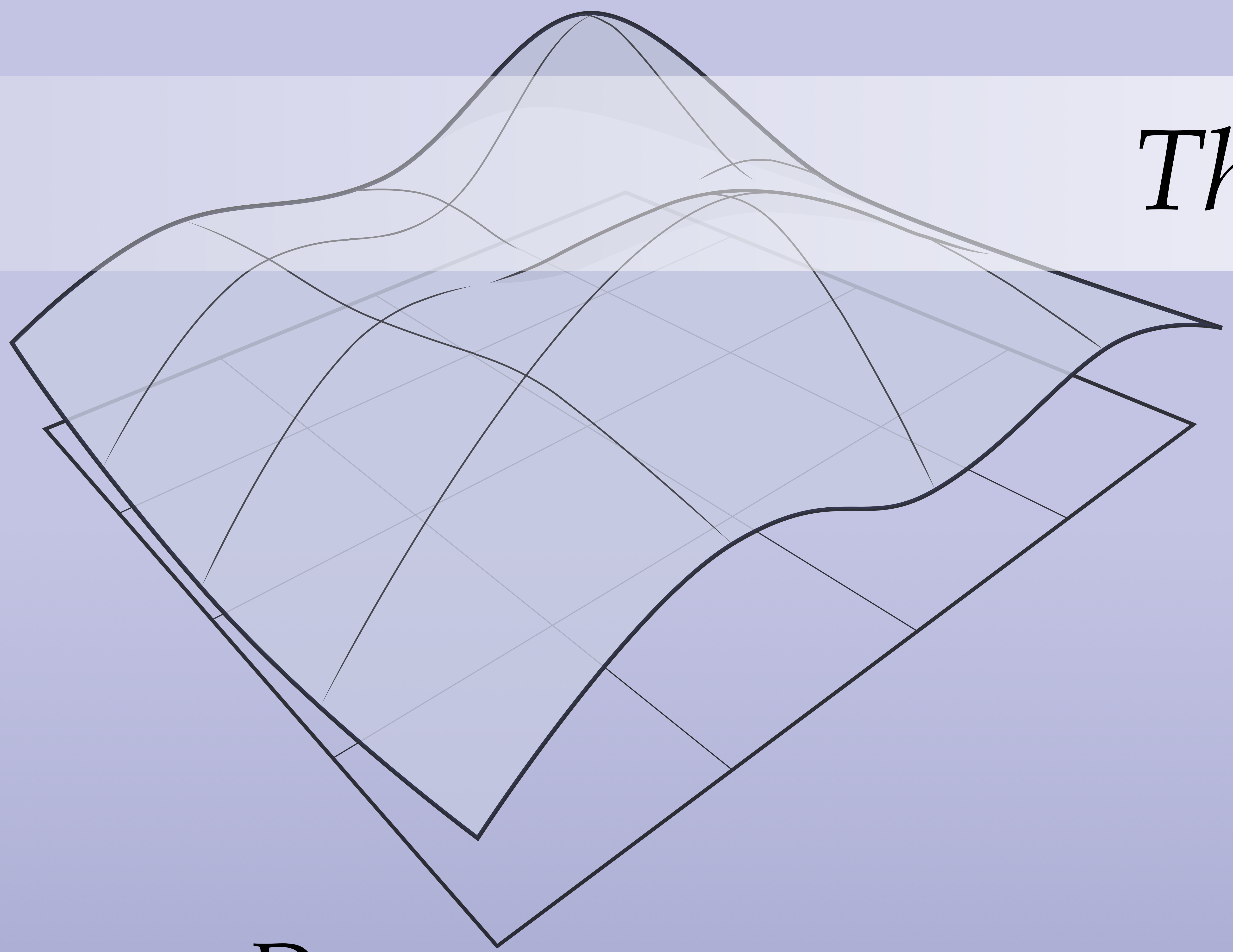
Open Challenges in Shape Recovery

- What other **discrete** quantities determine a surface?
- …and how can we (efficiently) recover a surface from this data?
  - Lengths + dihedral angles work in general (*fundamental theorem of discrete surfaces*); lengths alone are sufficient for convex surfaces. What about just dihedral angles?
  - Have a variety of discrete curvatures. Which are sufficient, for which classes of surfaces?
- Why bother? Offers new & different ways to analyze, process, edit, transmit, … curved surfaces digitally.



from Eigensatz & Pauly, *"Curvature Domain Shape Processing"*

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858