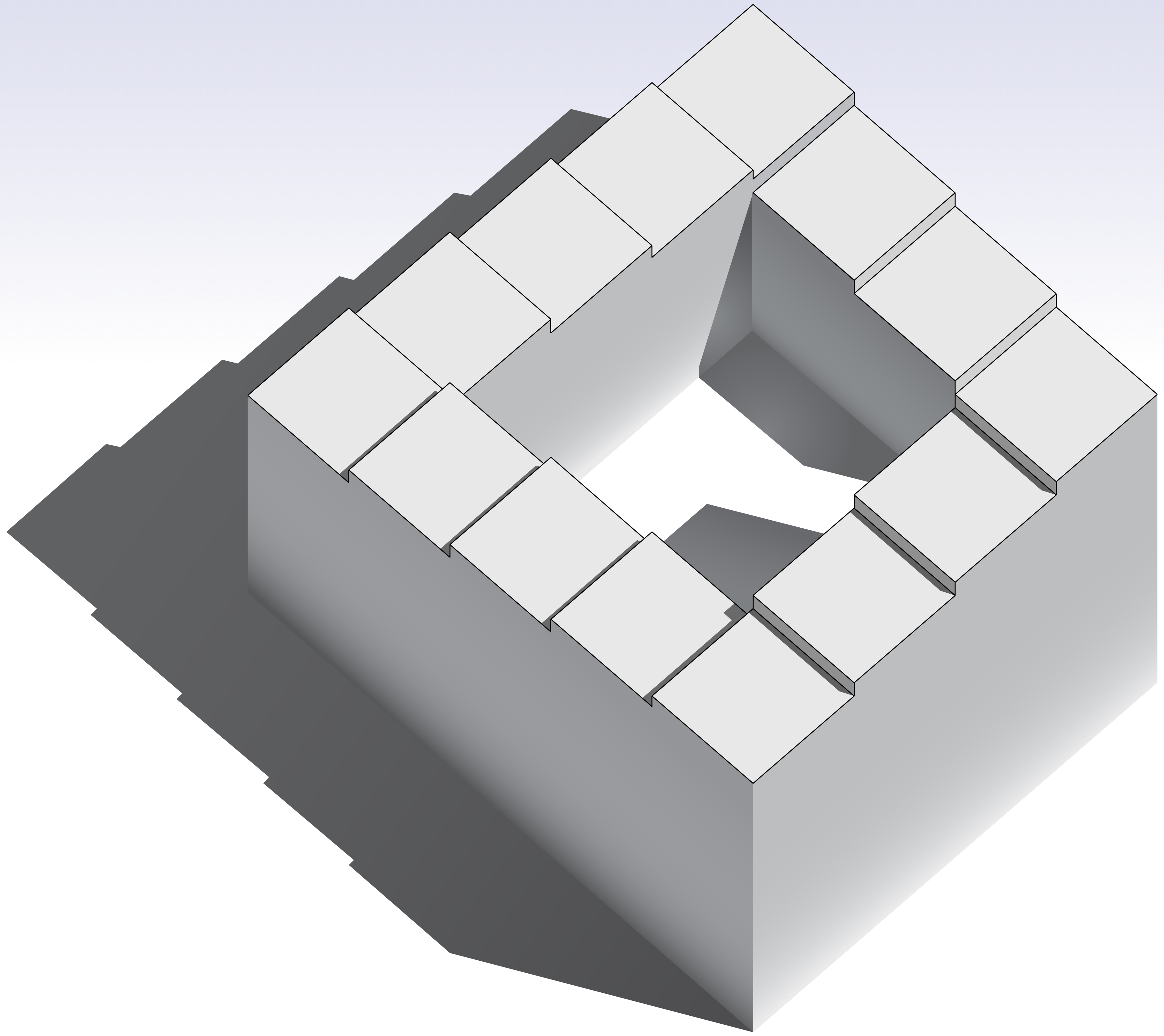

Discrete Differential Geometry: An Applied Introduction

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LECTURE 19:
CONFORMAL GEOMETRY I

DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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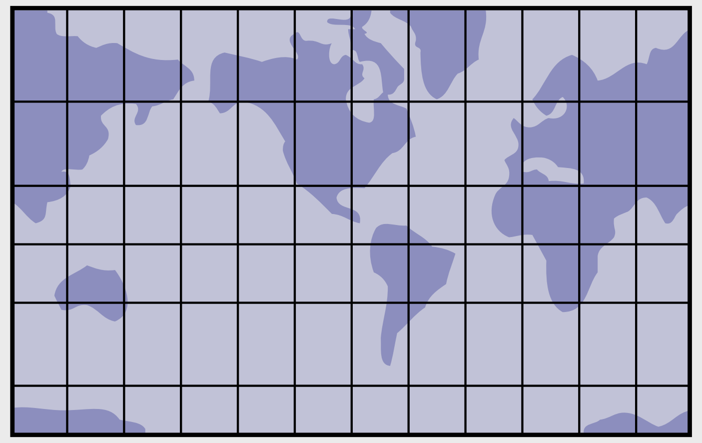
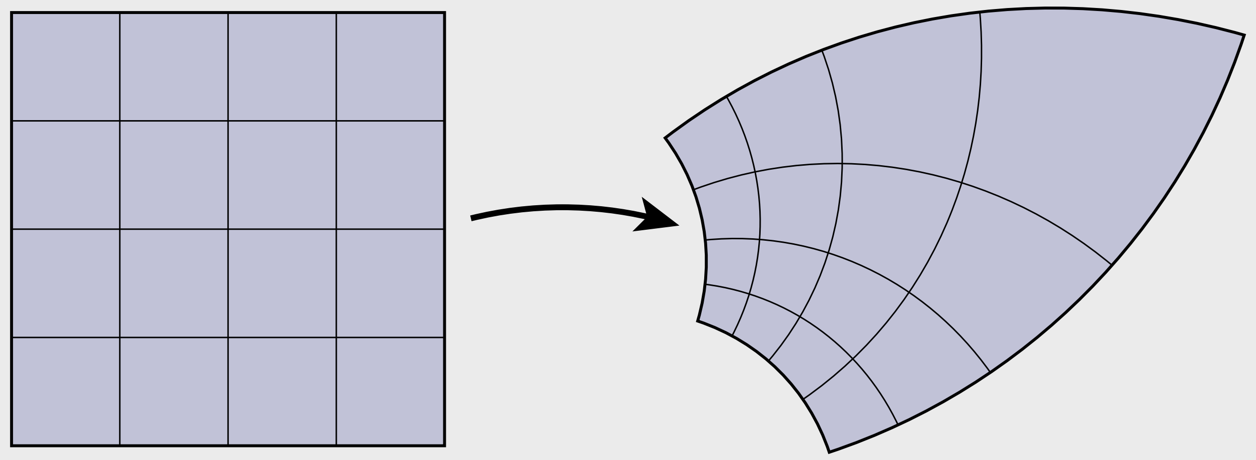
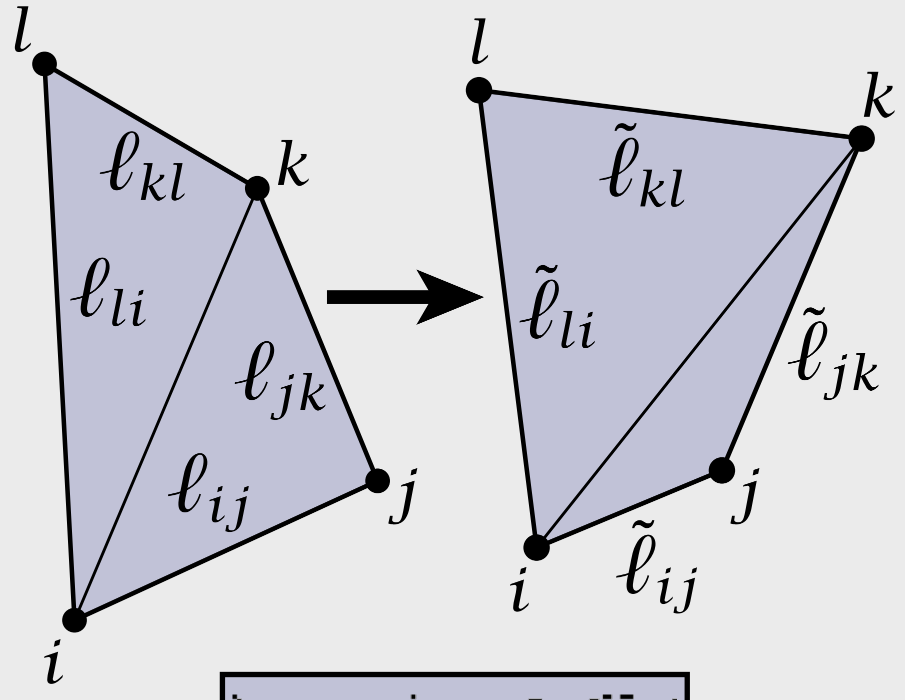
Outline



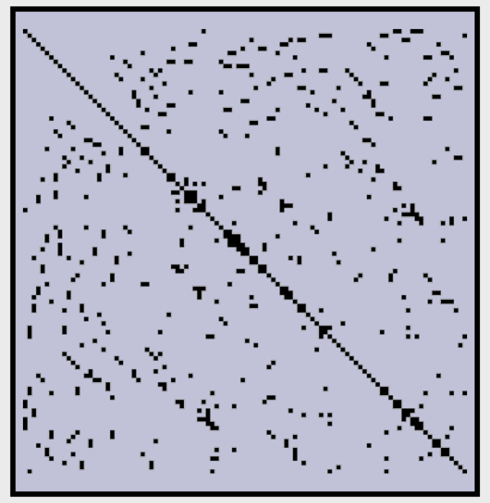
Part I: Overview



Part II: Smooth Theory

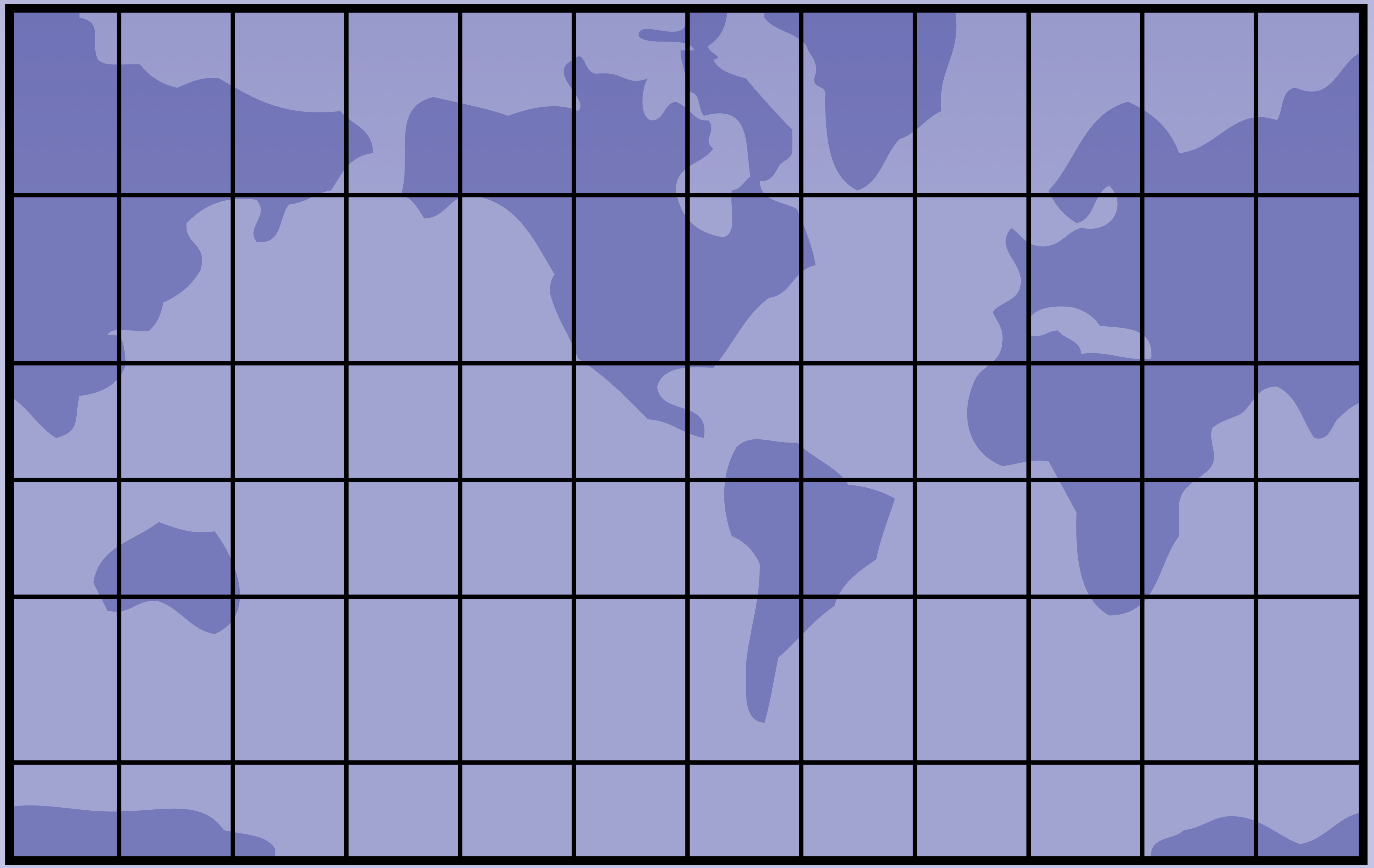


Part III: Discretization



Part IV: Algorithms

Part I:
Overview



Discrete Conformal Geometry

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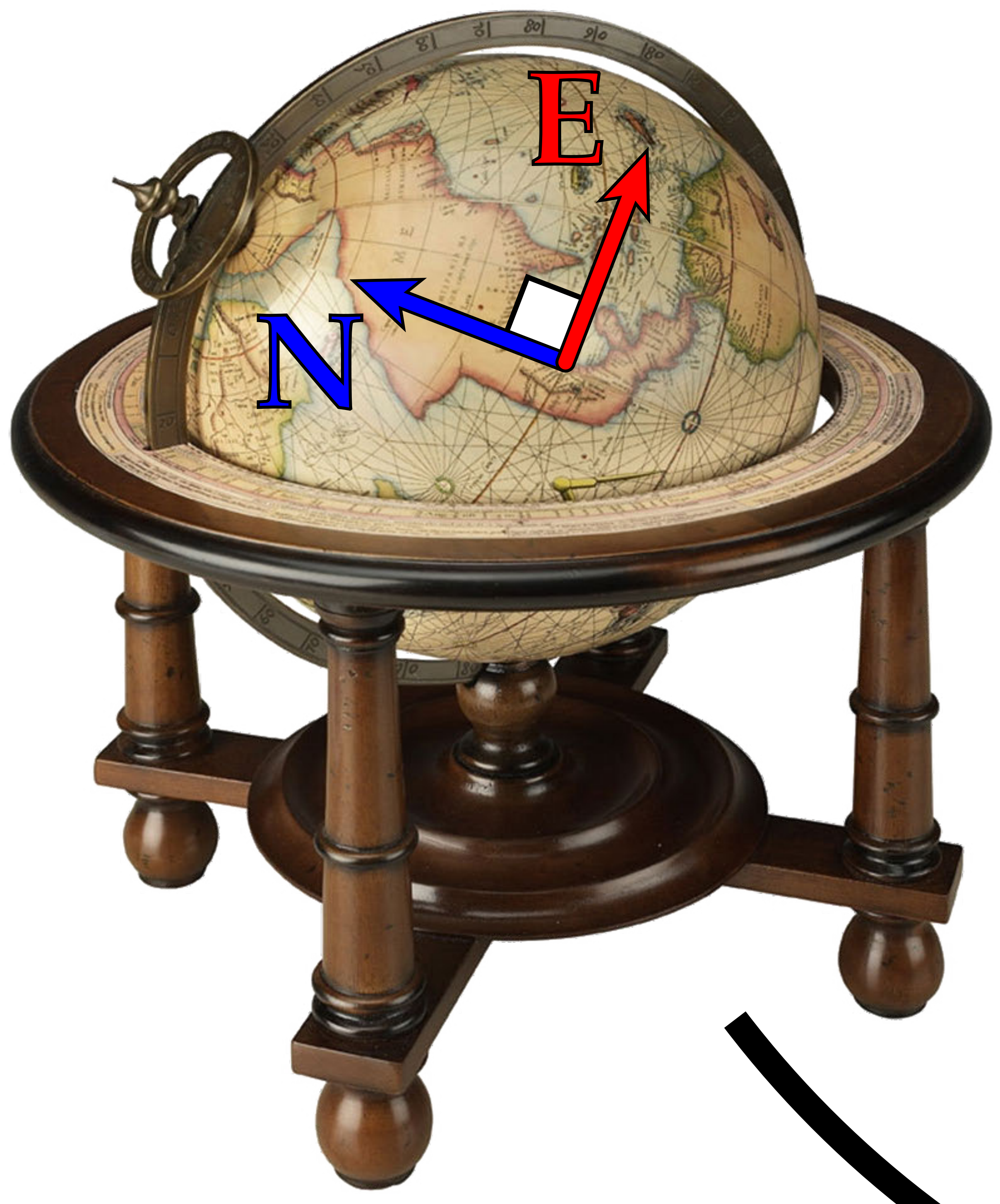
Motivation: Mapmaking Problem

- How do you make a flat map of the round globe?
- Hard to do! Like trying to flatten an orange peel…

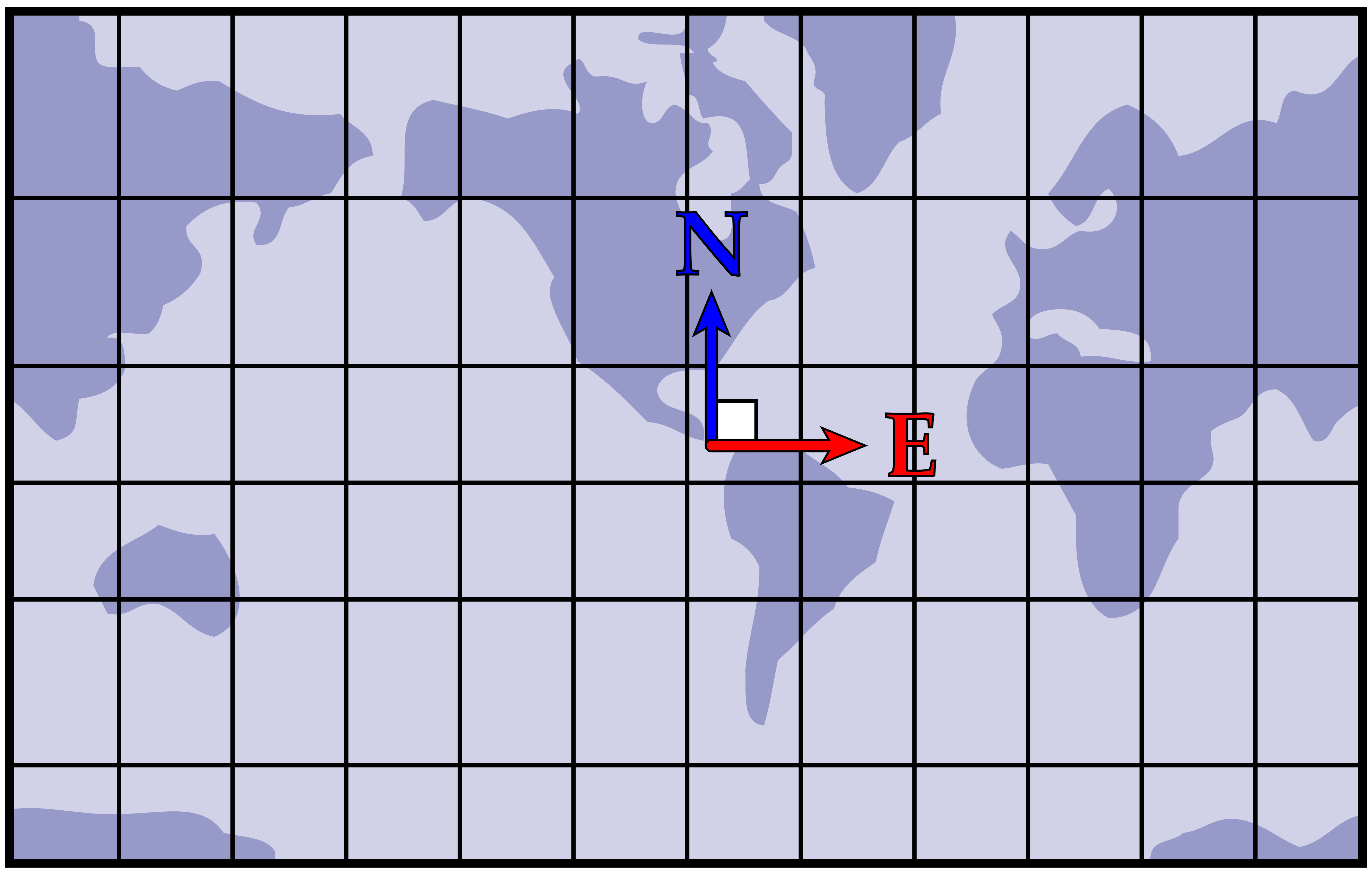
Impossible without some kind of distortion and/or cutting.

Conformal Mapmaking

- Amazing fact: can always make a map that exactly preserves **angles**.



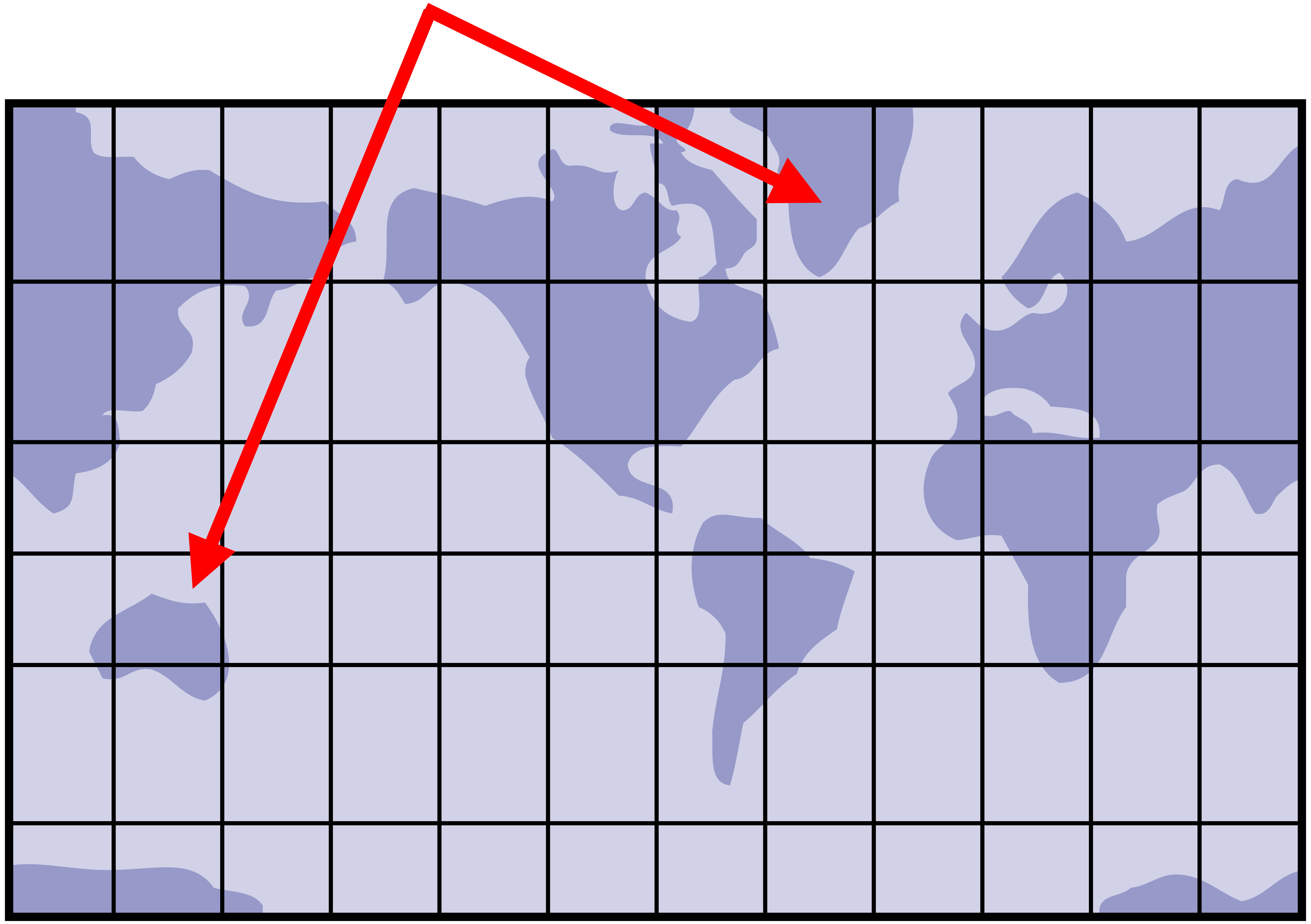
(Very useful for navigation!)



Conformal Mapmaking

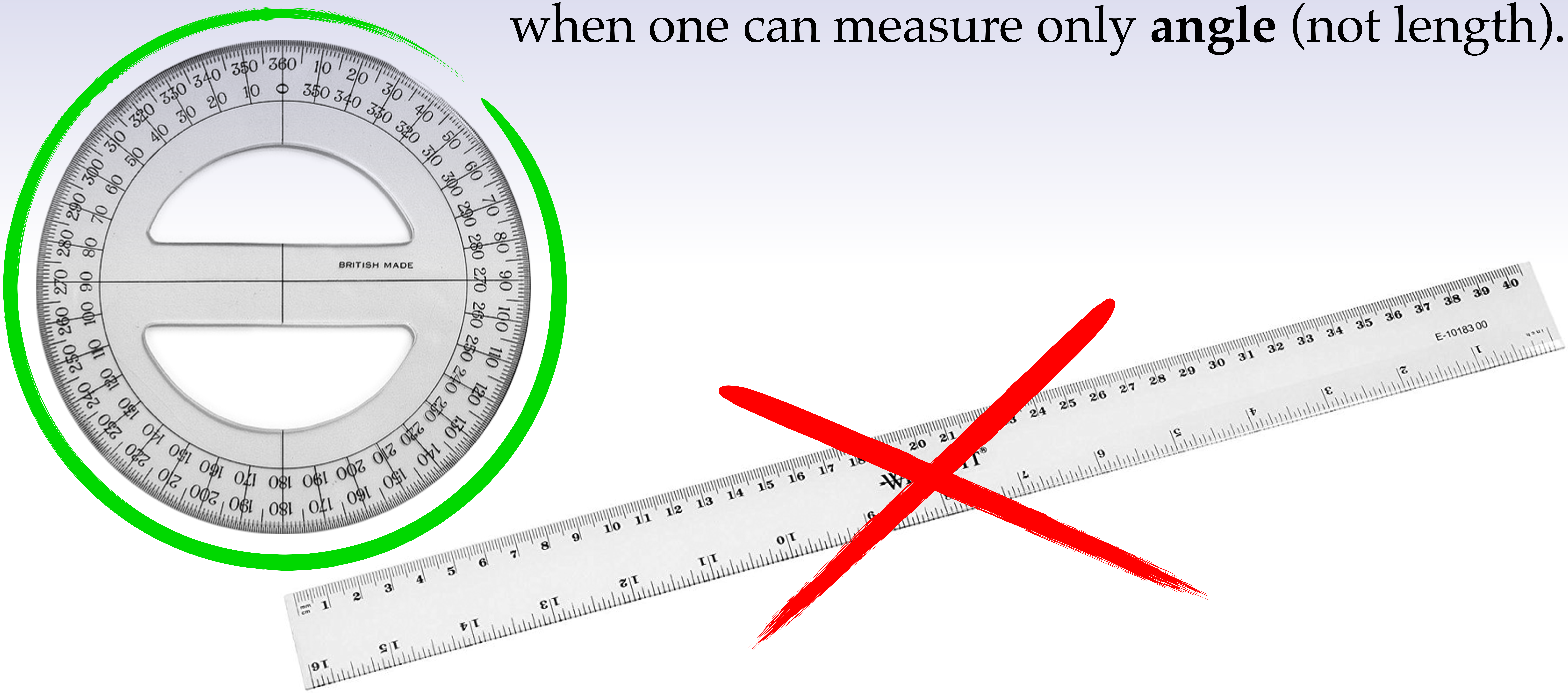
- However, **areas** may be badly distorted…

(Greenland is not bigger than Australia!)

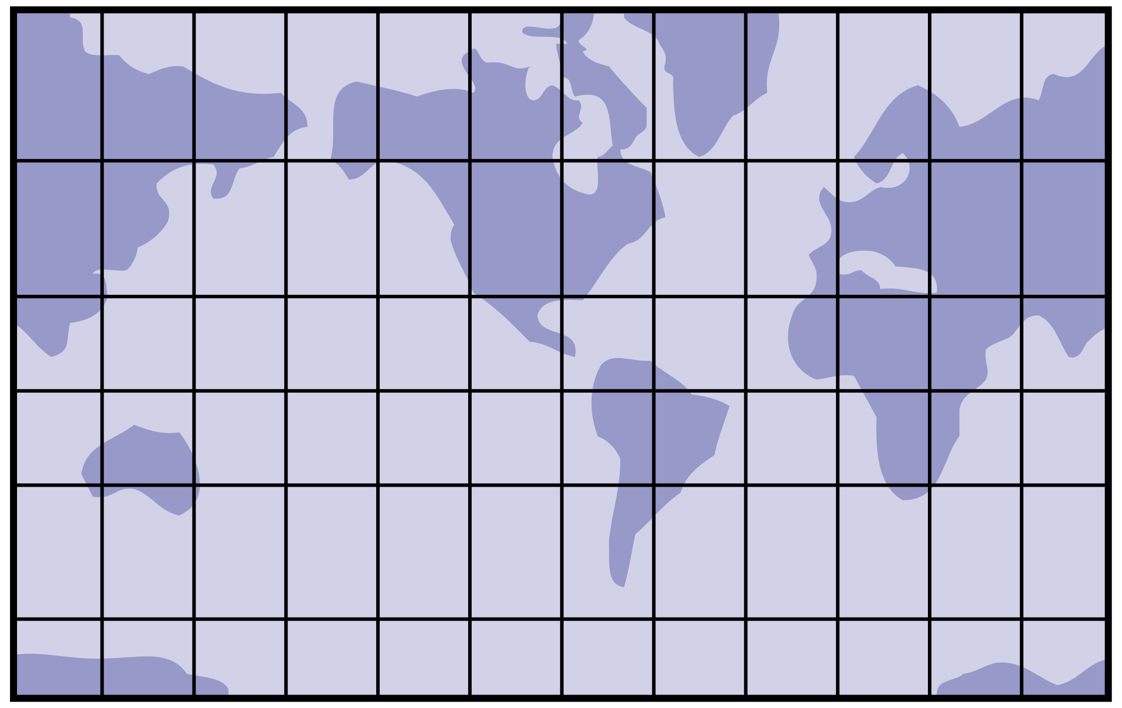


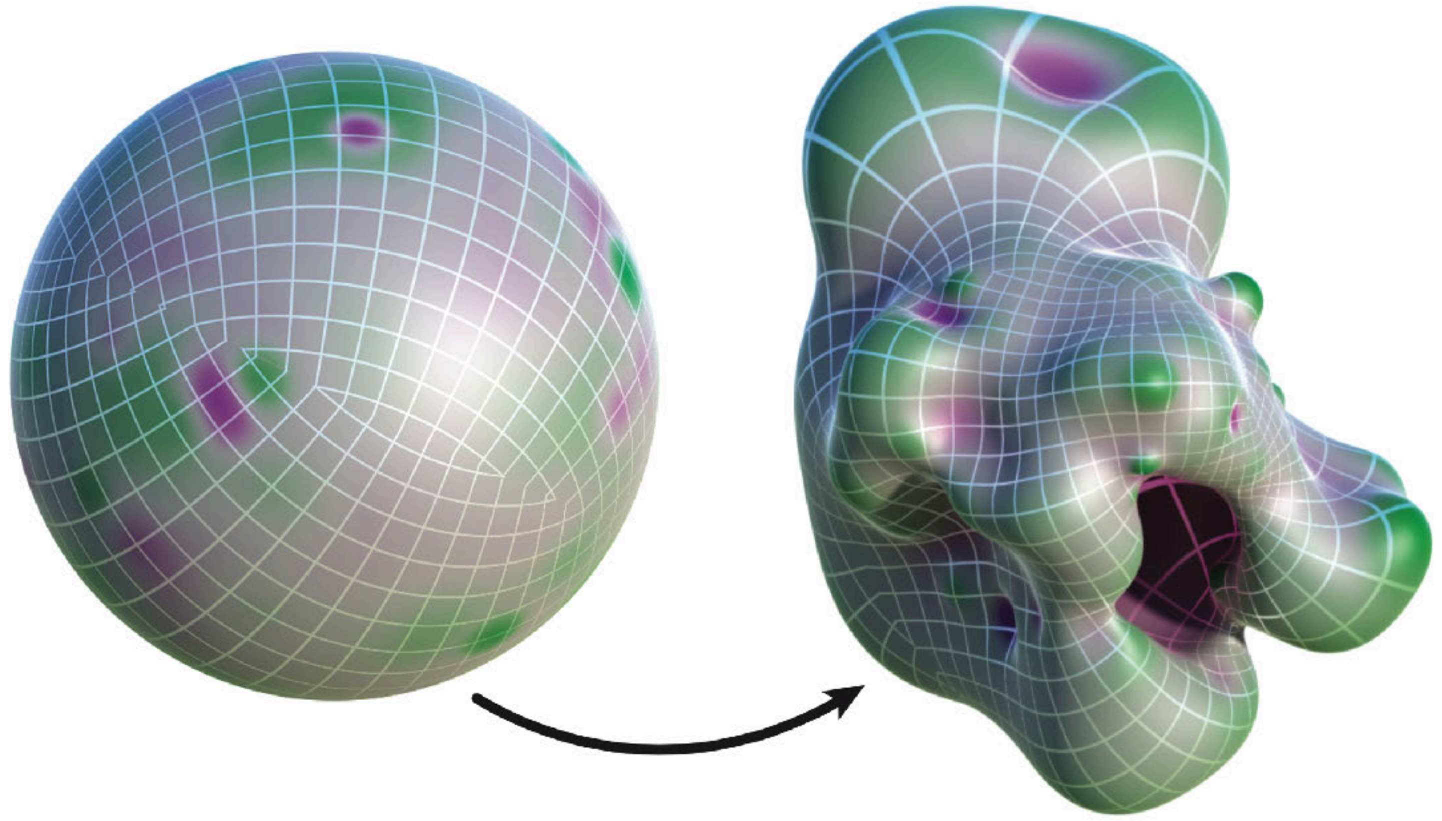
Conformal Geometry

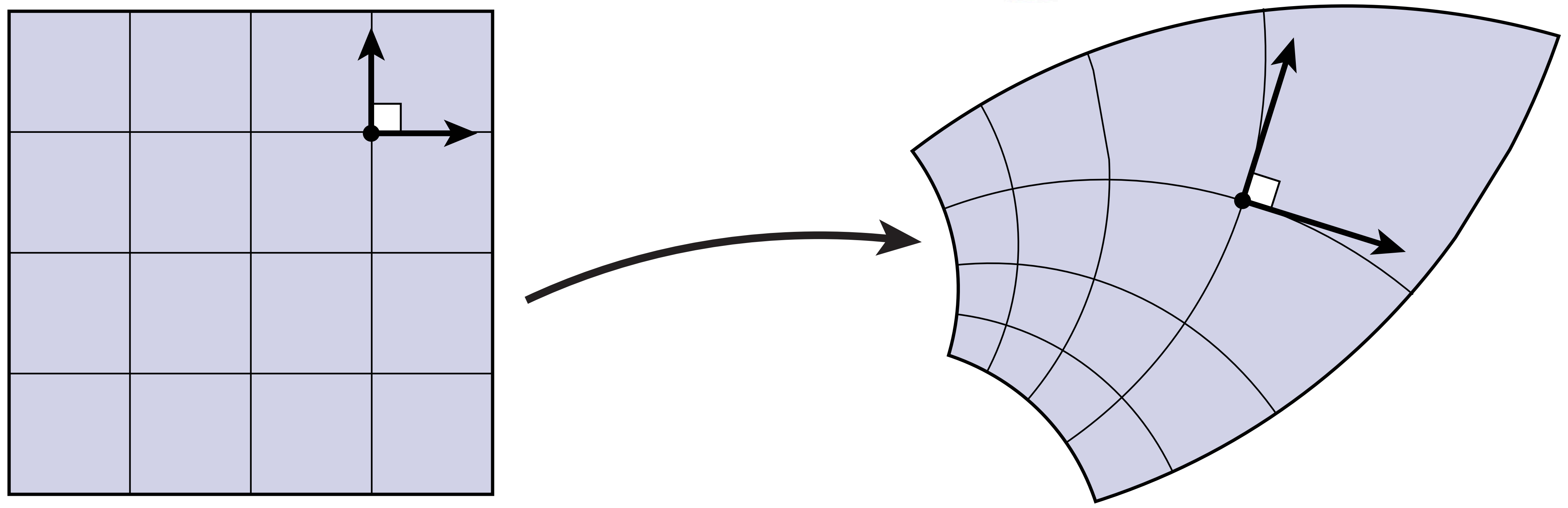
More broadly, *conformal geometry* is the study of shape when one can measure only **angle** (not length).

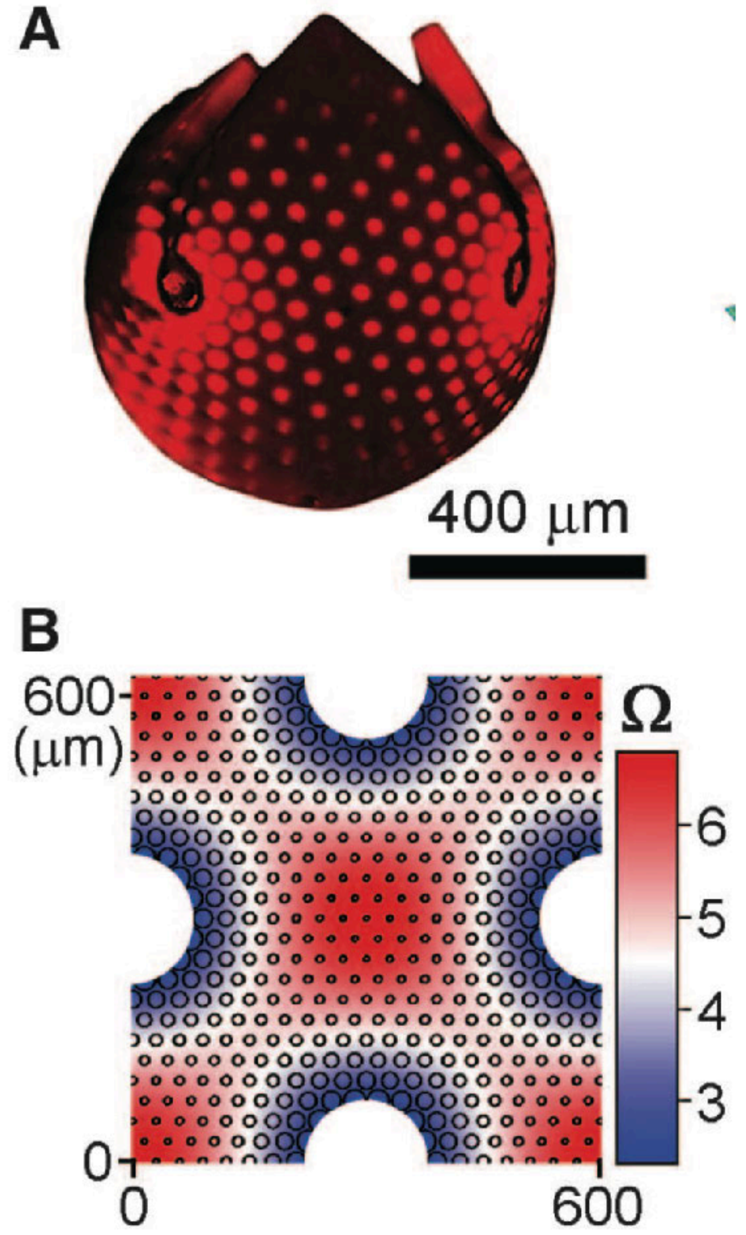


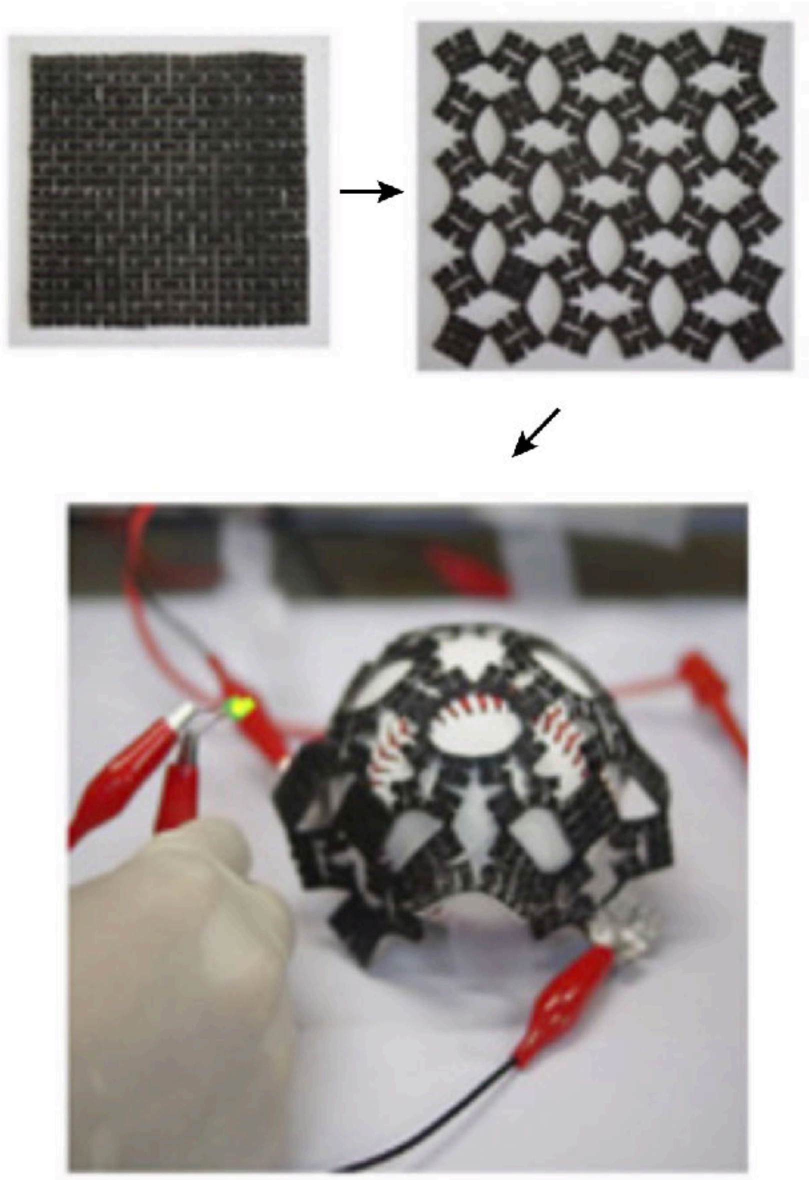
Conformal Geometry — Visualized

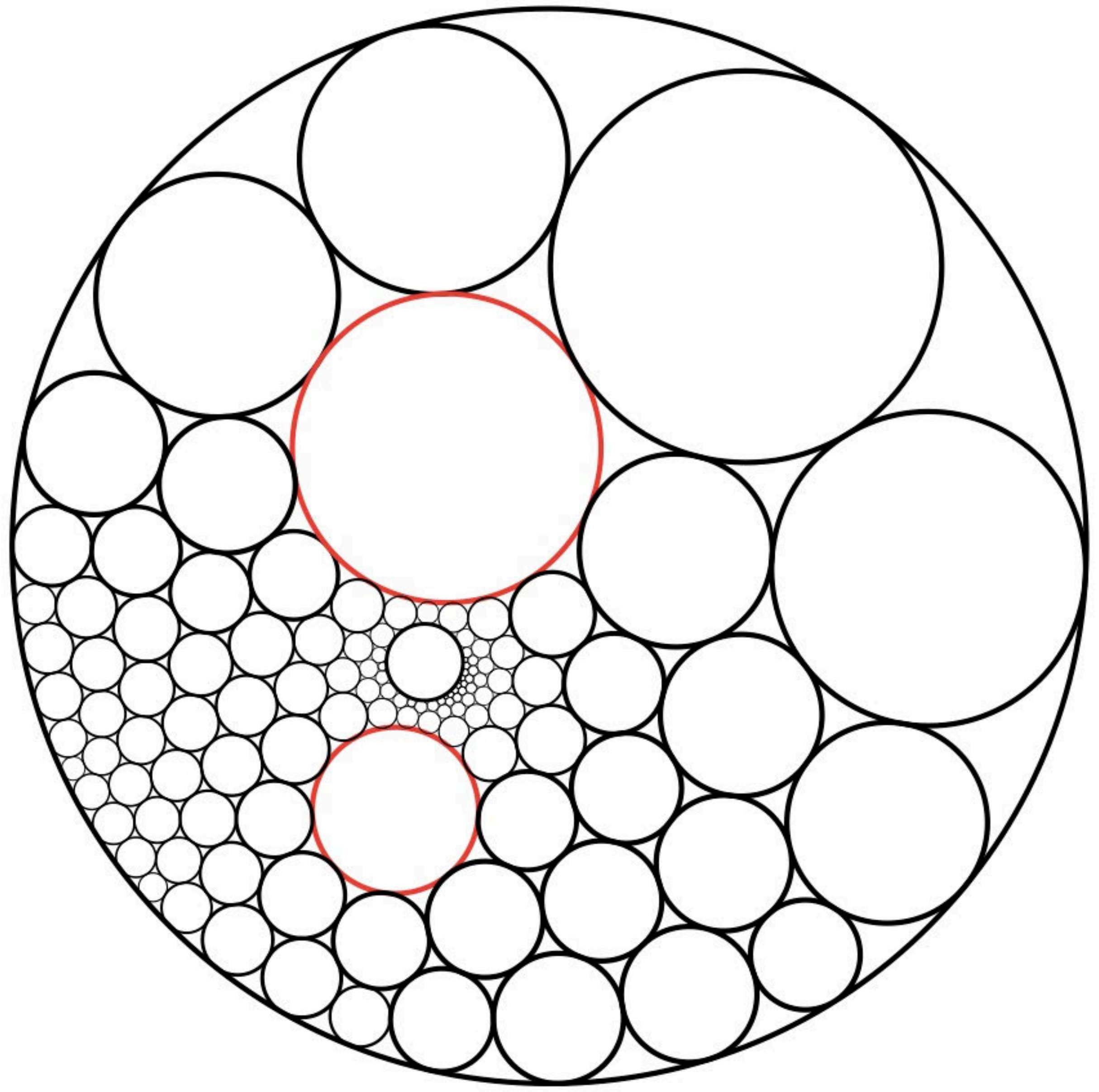


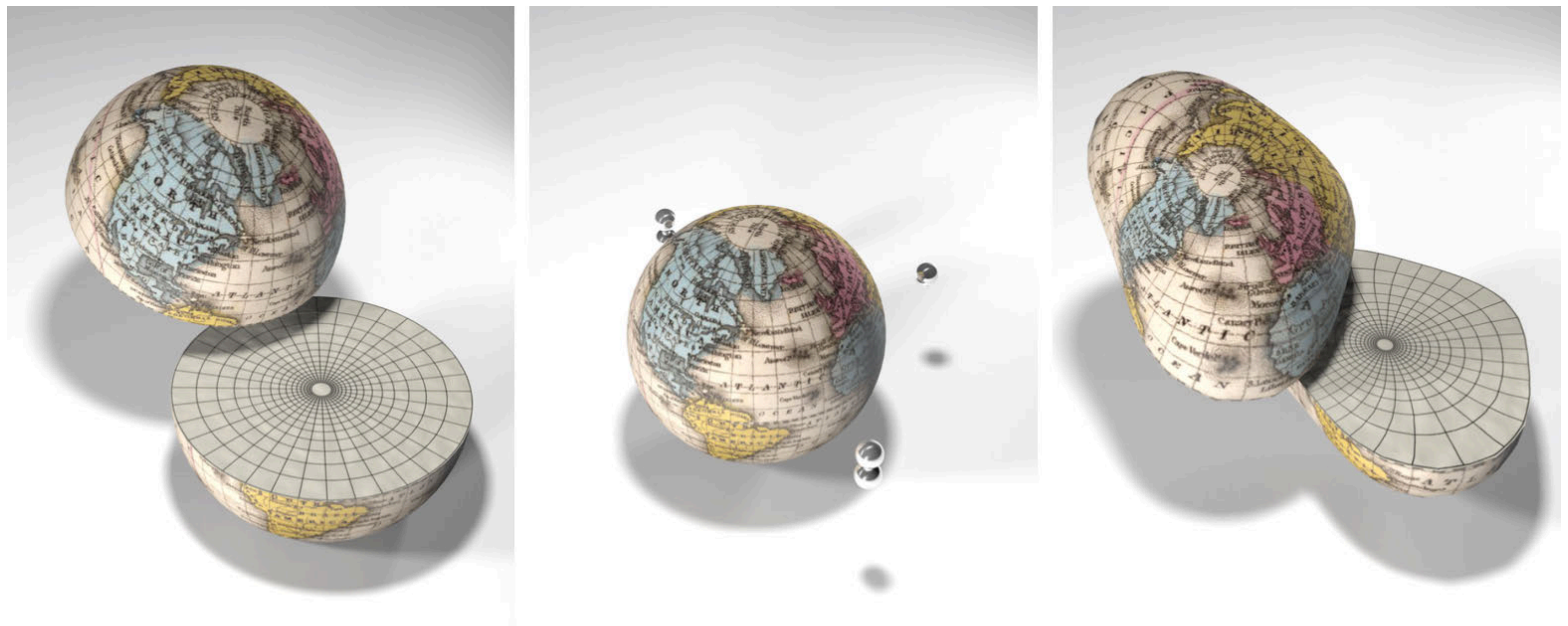






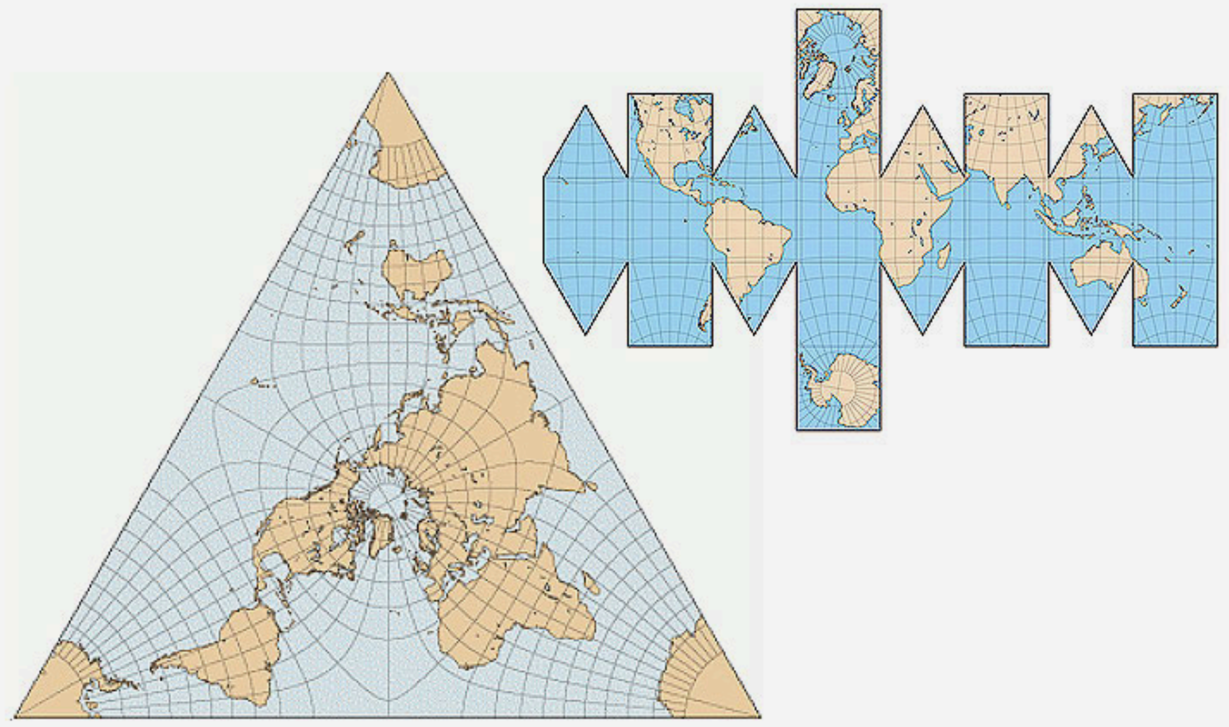




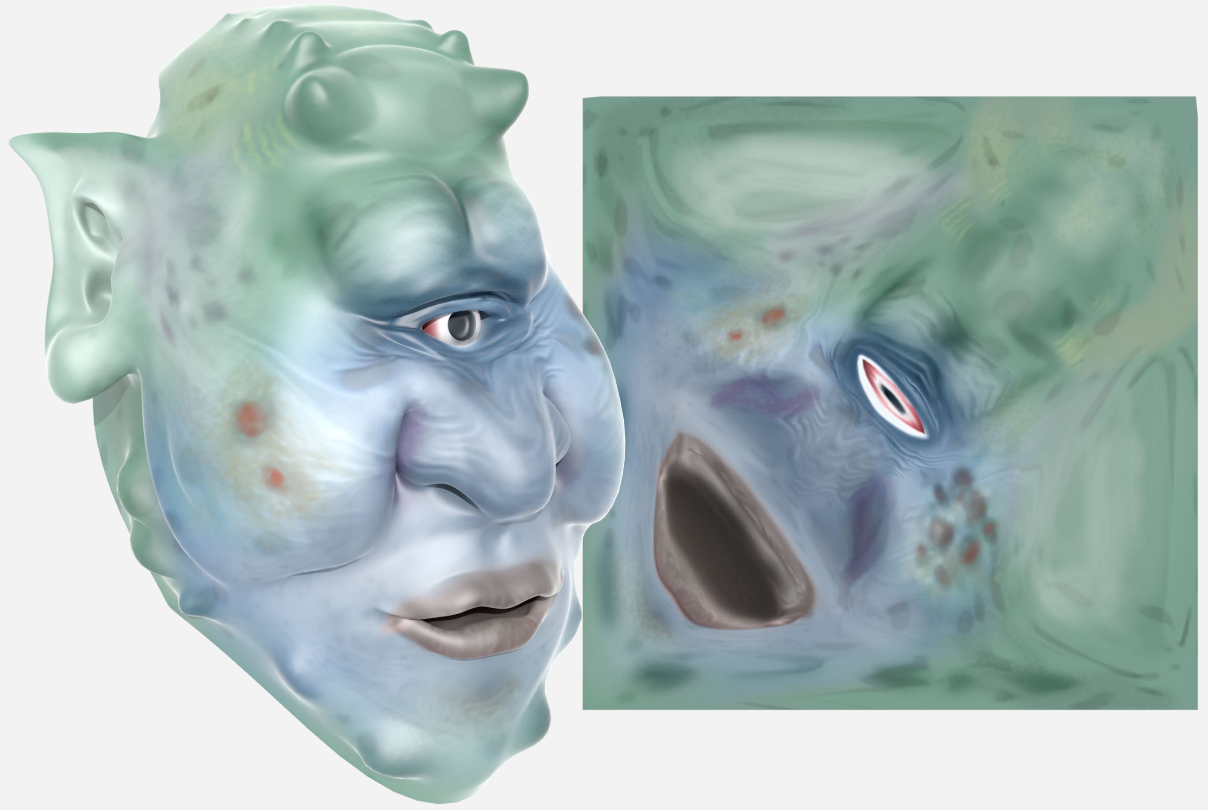


Applications of Conformal Geometry Processing

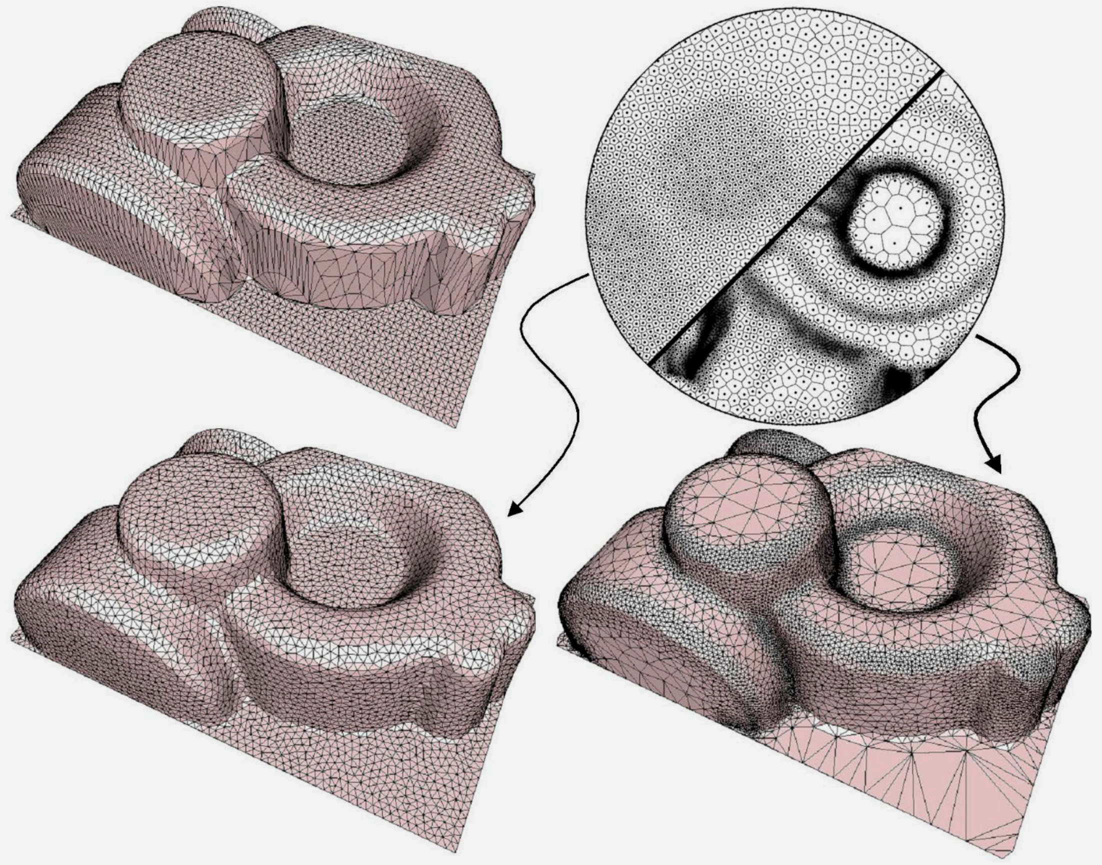
Basic building block for *many* applications…



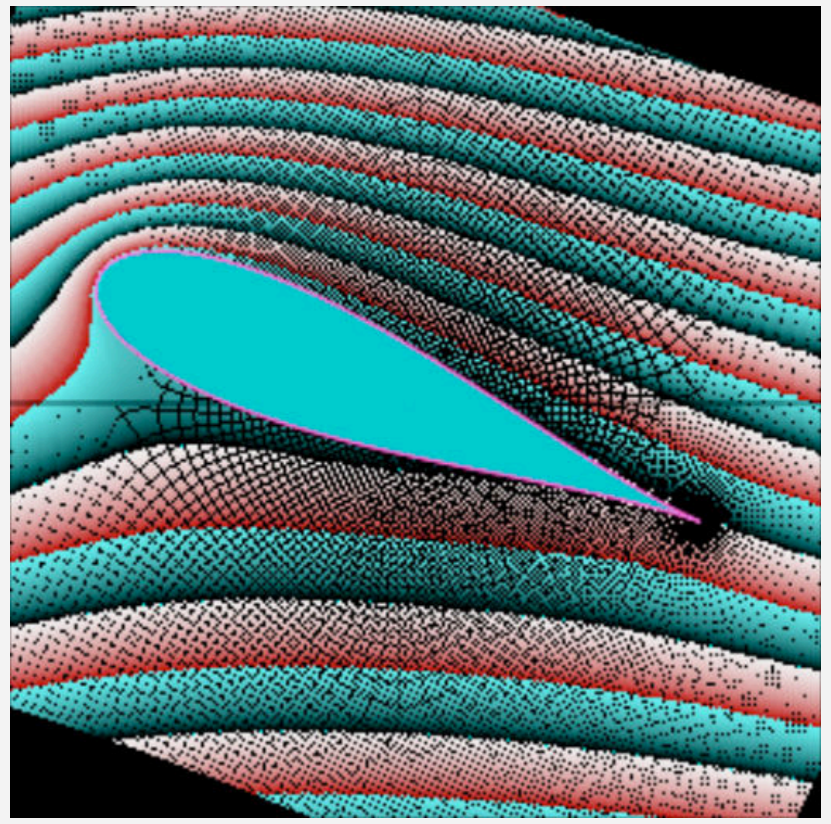
CARTOGRAPHY



TEXTURE MAPPING

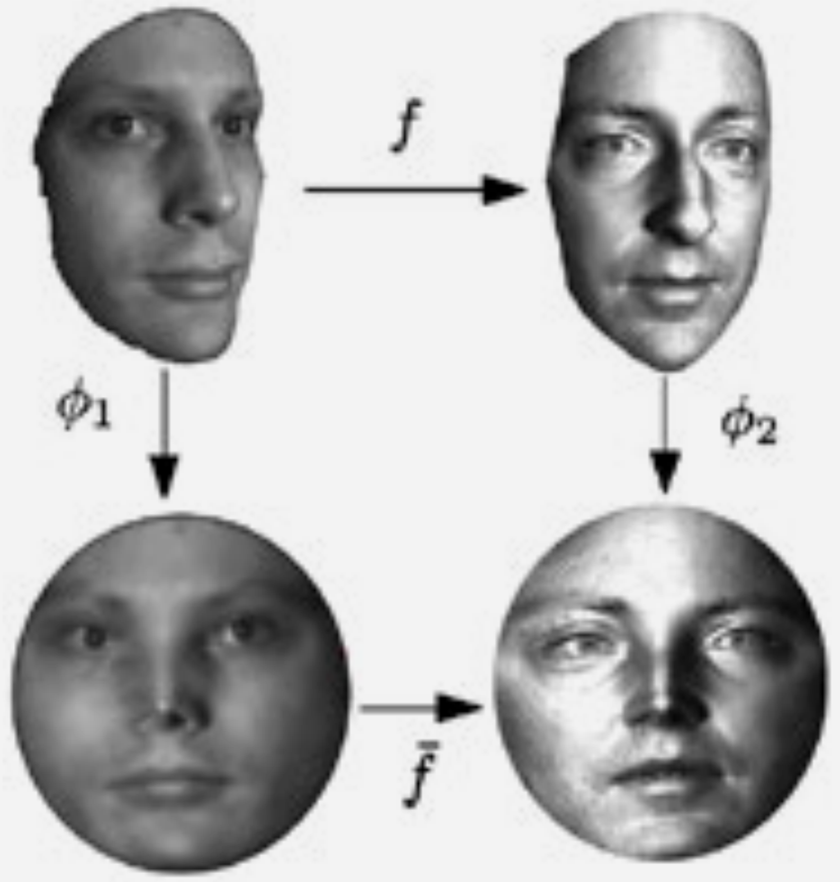


REMESHING

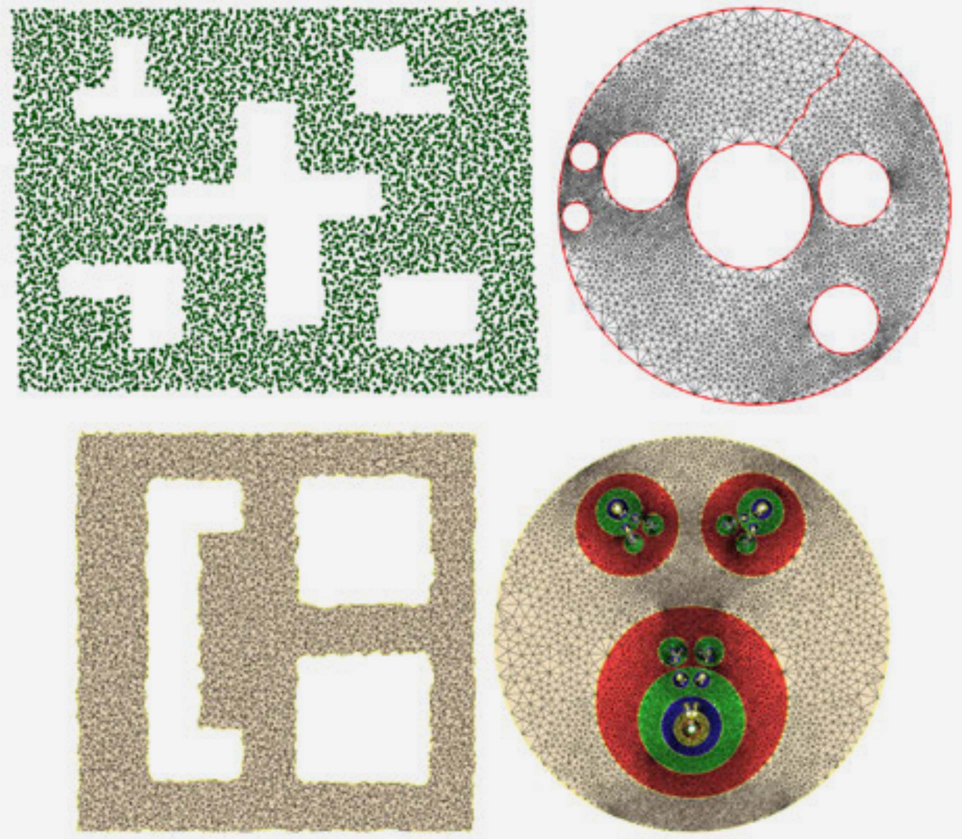


SIMULATION

3D FABRICATION



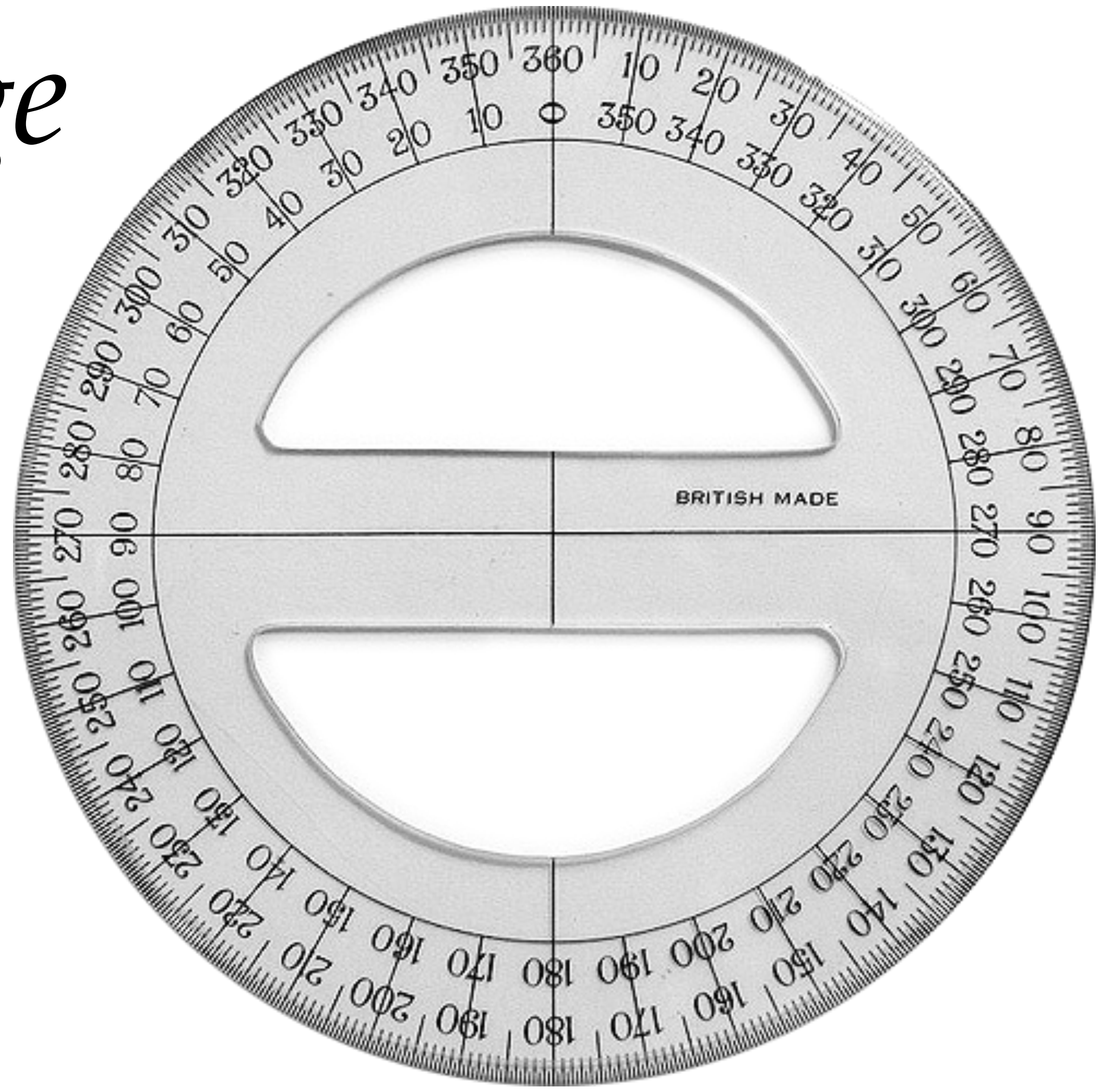
SHAPE ANALYSIS



SENSOR NETWORKS

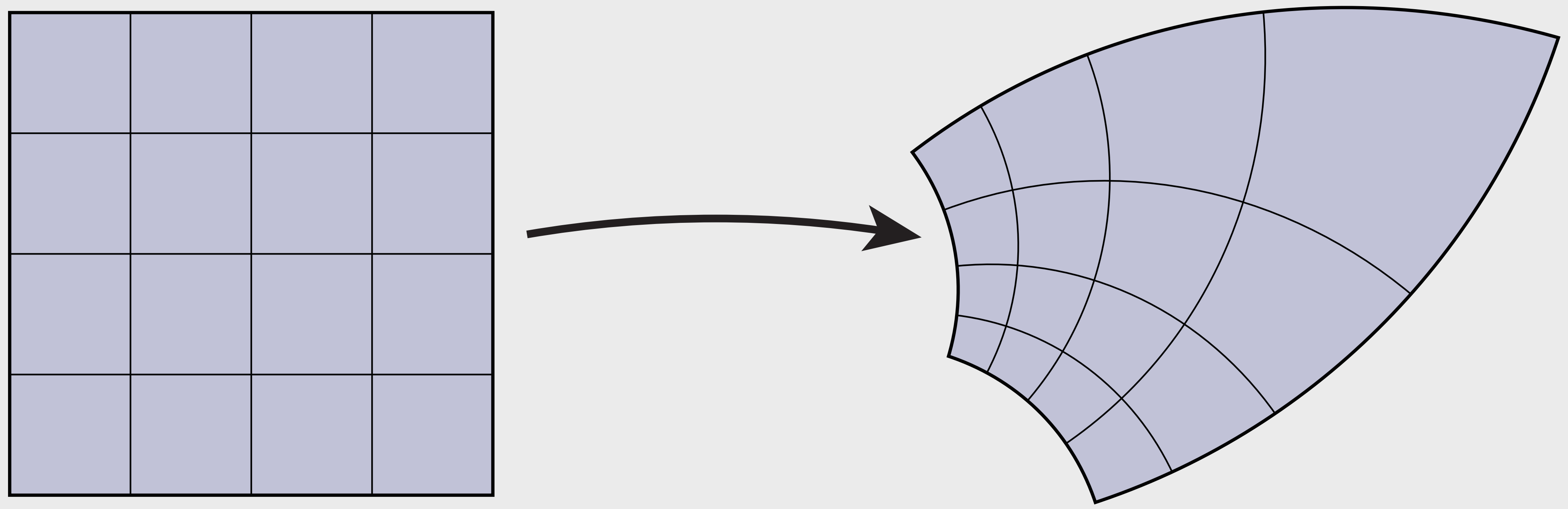
*Why **Conformal**?*

- Why so much interest in maps that preserve *angle*?
  - **QUALITY**: *Every conformal map is already "really nice"*
  - **SIMPLICITY**: *Makes "pen and paper" analysis easier*
  - **EFFICIENCY**: *Often yields computationally easy problems*
  - **GUARANTEES**: *Well understood, lots of theorems/knowledge*

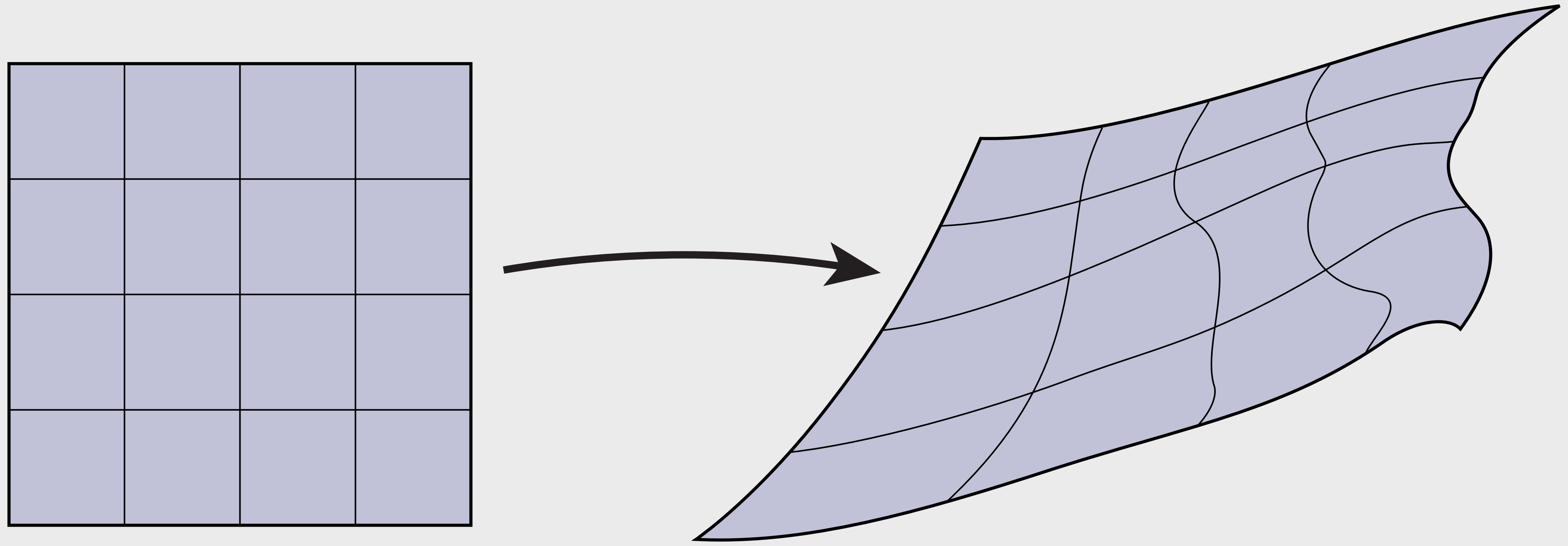


Conformal Maps are "Really Nice"

- Angle preservation already provides a lot of regularity
- E.g., every conformal map has infinitely many derivatives (C^∞)
- Scale distortion is smoothly distributed (harmonic)



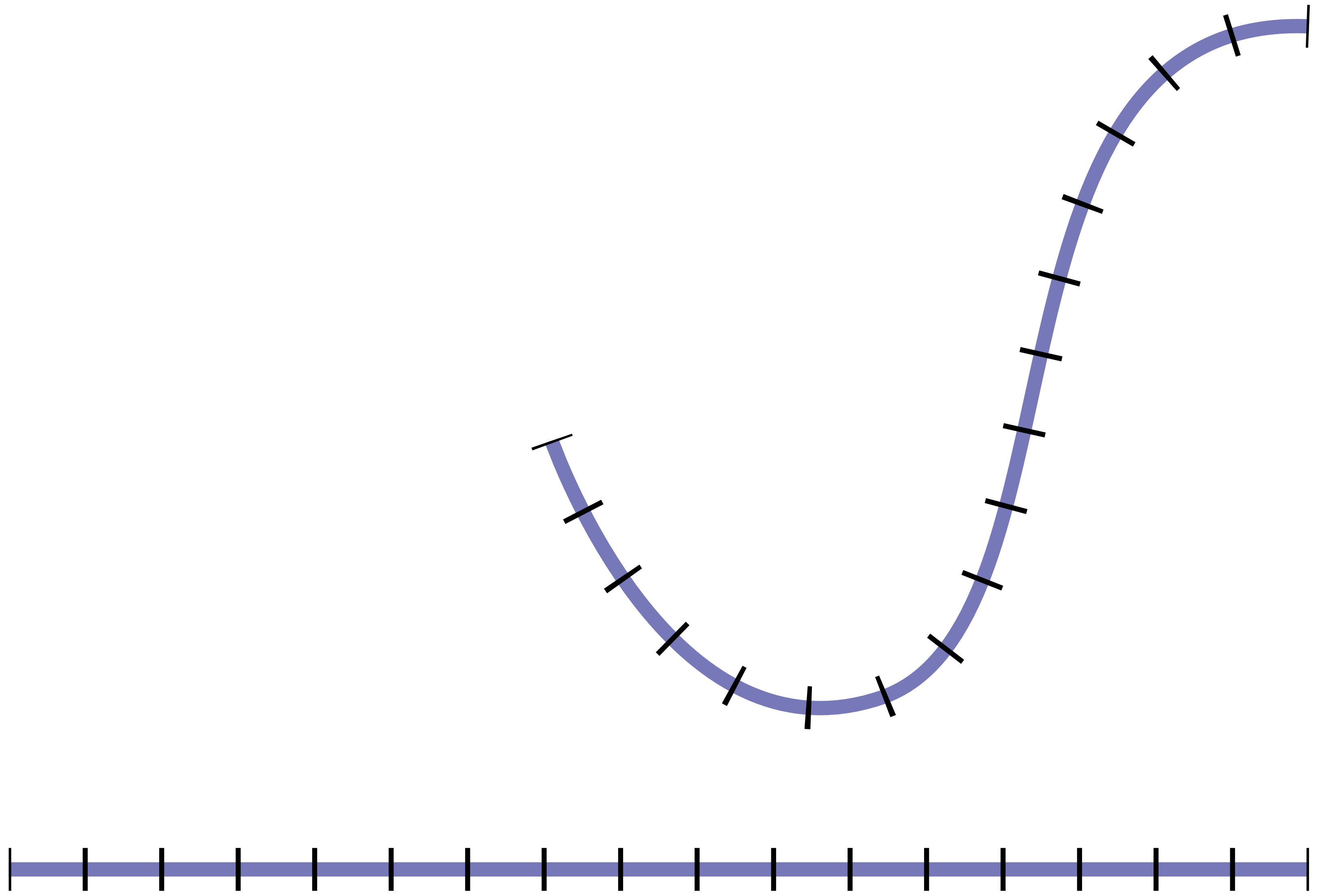
CONFORMAL

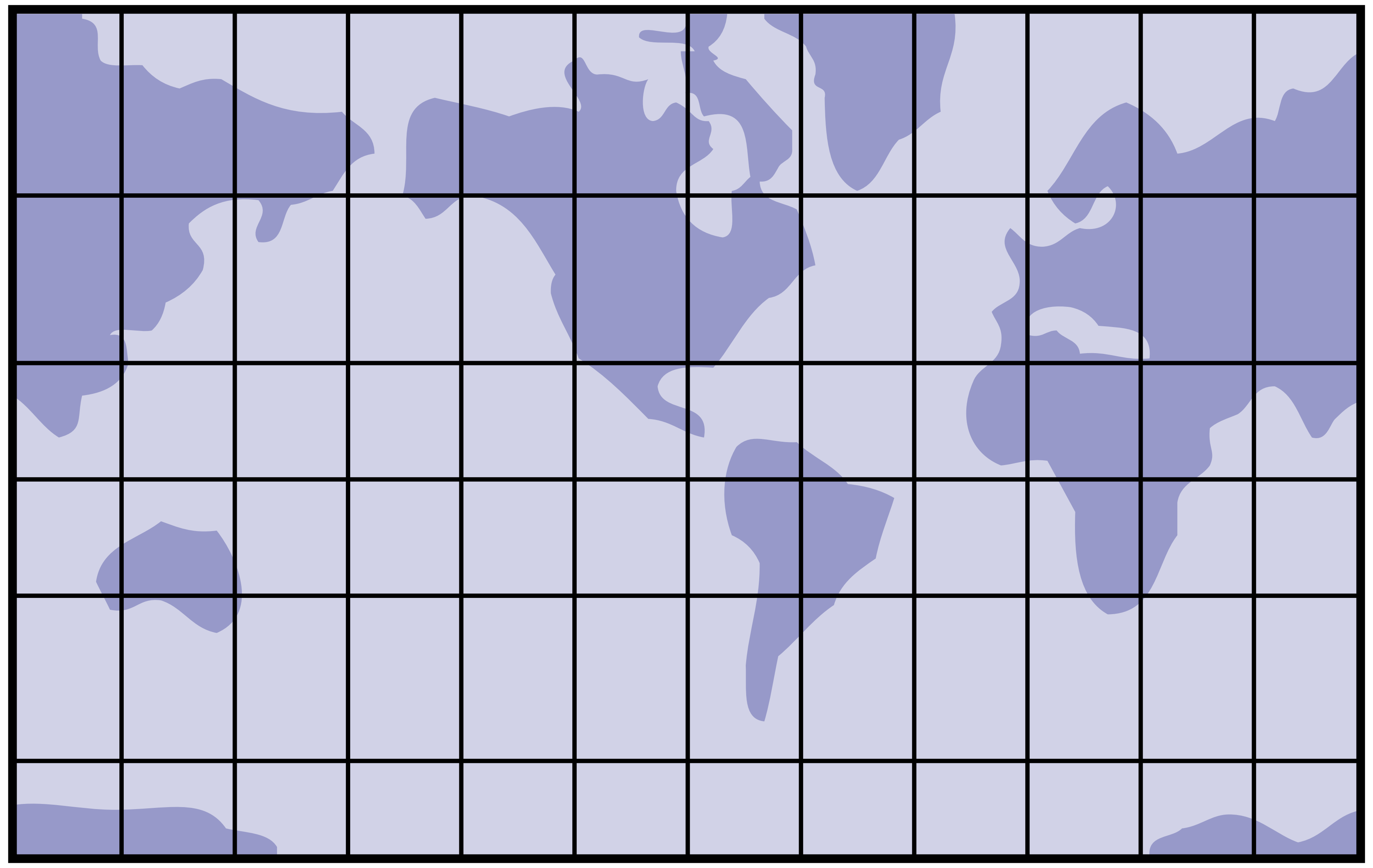


NOT CONFORMAL

Conformal Coordinates Make Life Easy

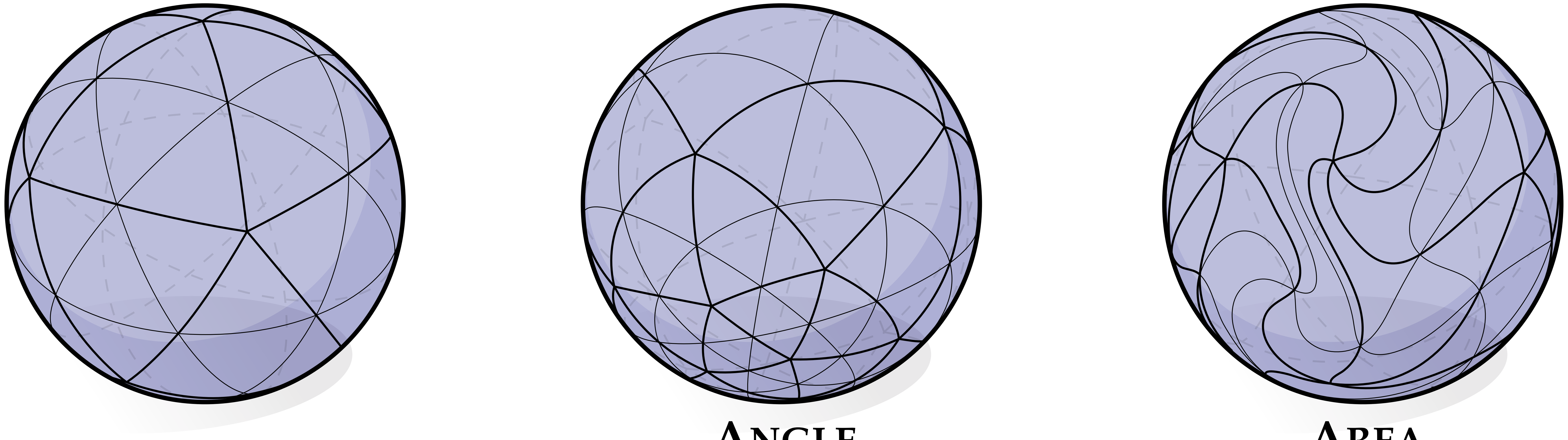
- Makes life easy "on pen and paper"
  - **Curves:** life greatly simplified by assuming *arc-length* parameterization
  - **Surfaces:** "arc-length" (isometric) not usually possible
    - conformal coordinates are "next best thing" (and always possible!)
    - only have to keep track of scale (rather than arbitrary Jacobian)





Aside: Isn't Area-Preservation "Just as Good?"

- **Q:** What's so special about *angle*? Why not preserve, say, *area* instead?
- **A:** Area-preservation alone can produce maps that are ***nasty!***
  - Don't even have to be smooth; *huge* space of possibilities.
- E.g., any motion of an incompressible fluid (e.g., swirling water):



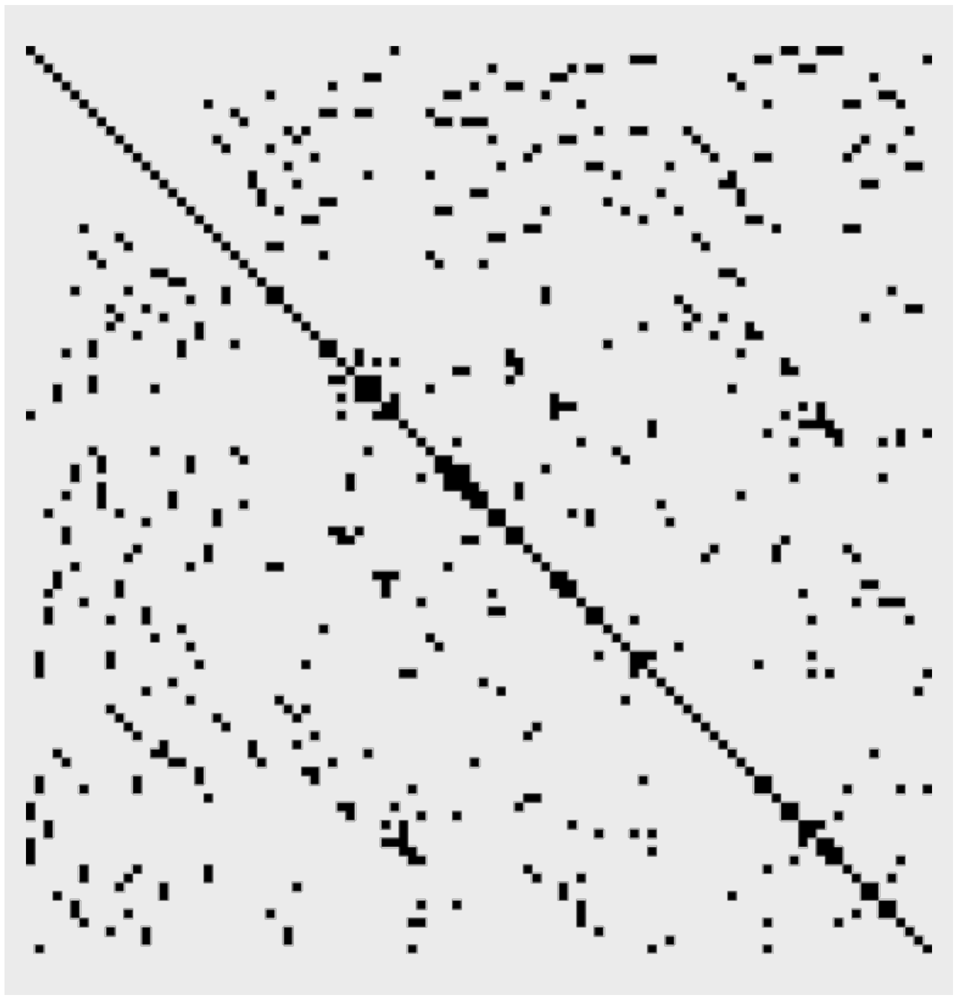
ORIGINAL

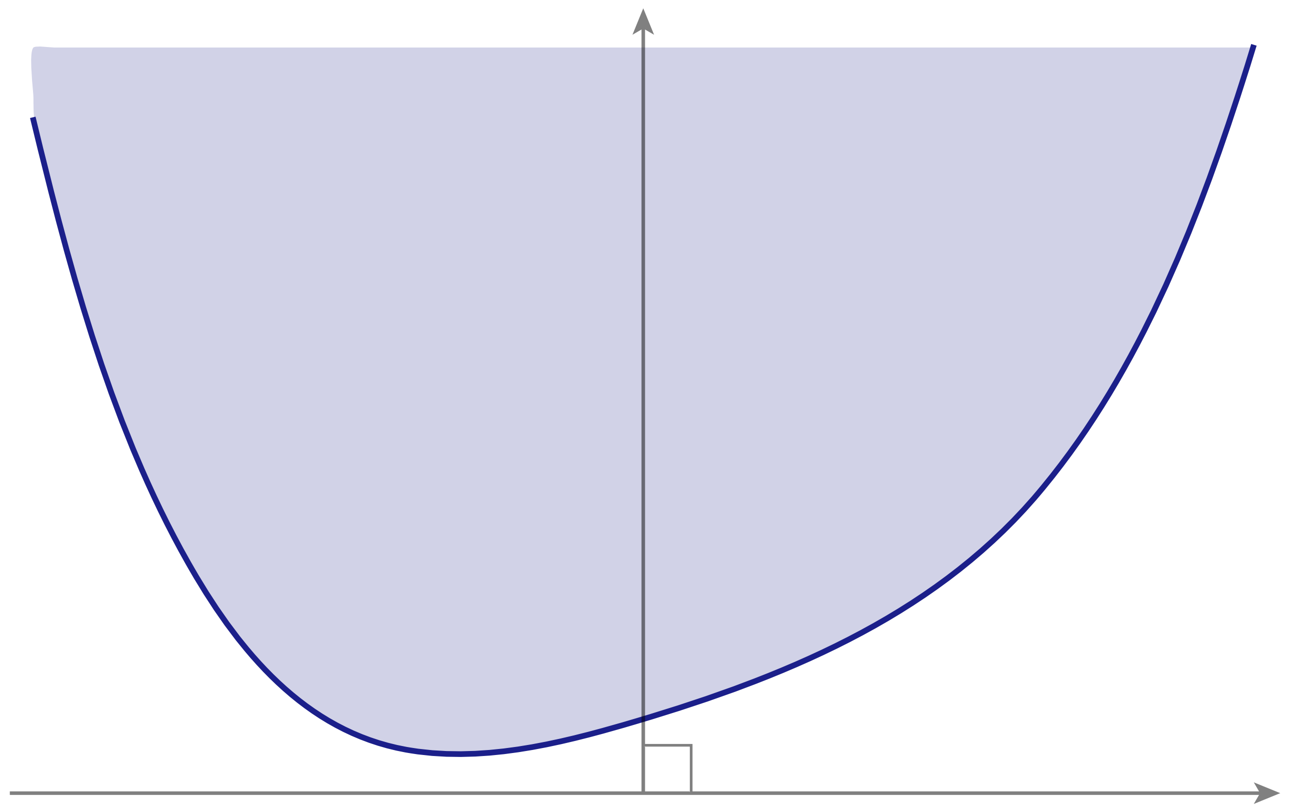
ANGLE PRESERVING

AREA PRESERVING

Computing Conformal Maps is Efficient

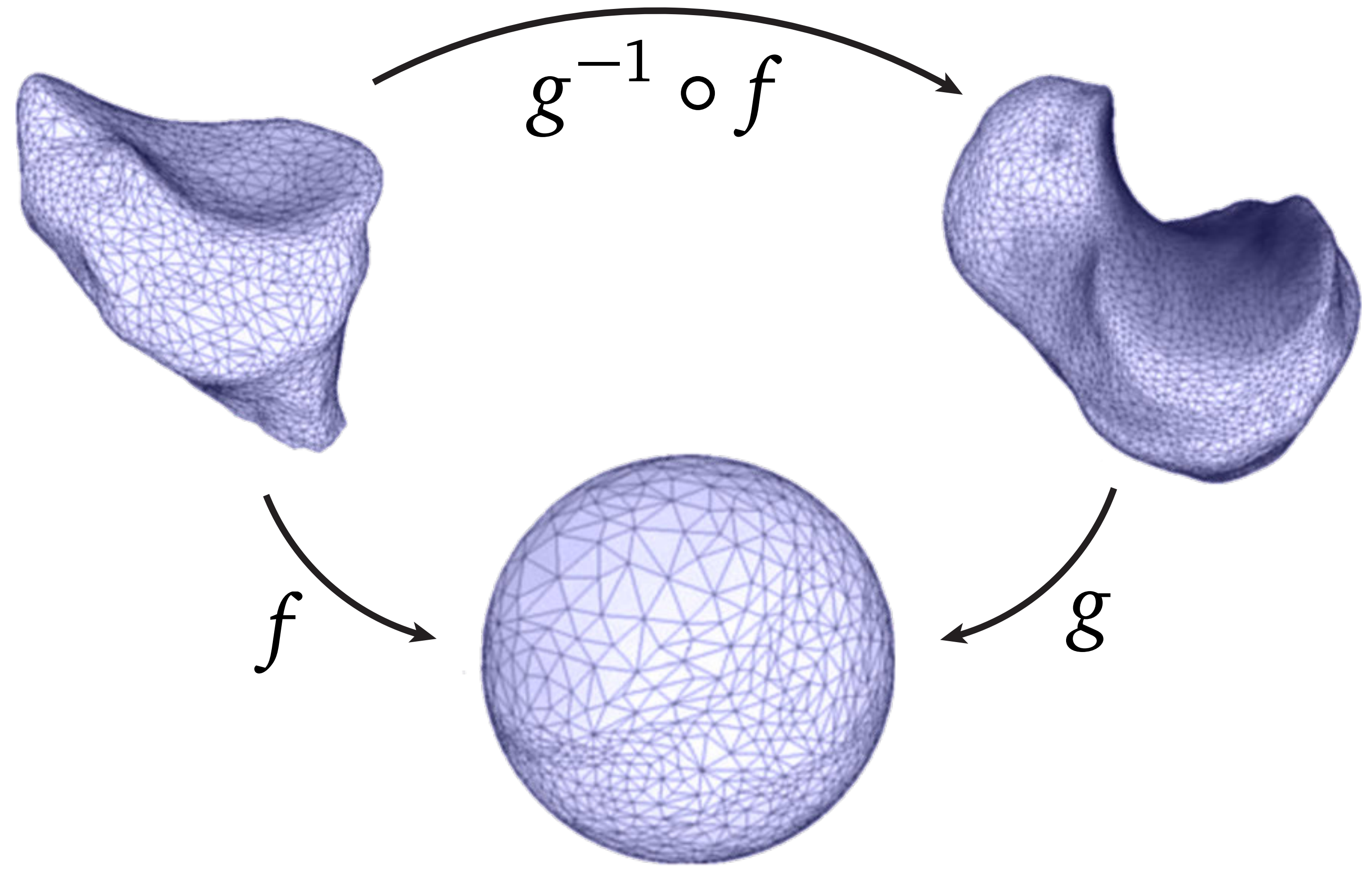
- Algorithms boil down to efficient, scalable computation
  - sparse linear systems / sparse eigenvalue problems
  - convex optimization problems
- Compare to more elaborate mapping problems
  - *bounded distortion*, *locally injective*, etc.
  - entail more difficult problems (e.g., *SOCP*)
- Much broader domain of applicability
  - real time vs. "just once"





Conformal Maps Help Provide Guarantees

- Established topic*
  - lots of existing theorems, analysis
  - connects to standard problems (e.g., Laplace)
  - makes it easier to provide guarantees (max principle, Delaunay, *etc.*)
- *Uniformization theorem* provides (nearly) canonical maps

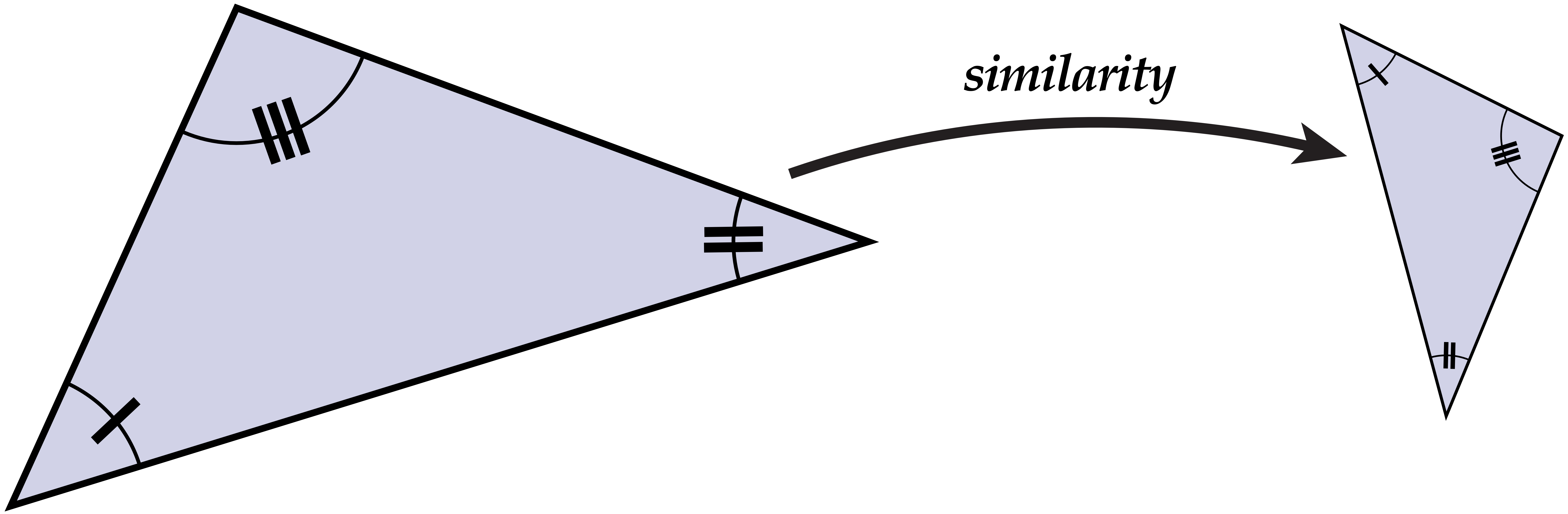


**Also makes it harder to do something truly new in conformal geometry processing…!*

Discrete Conformal Maps?

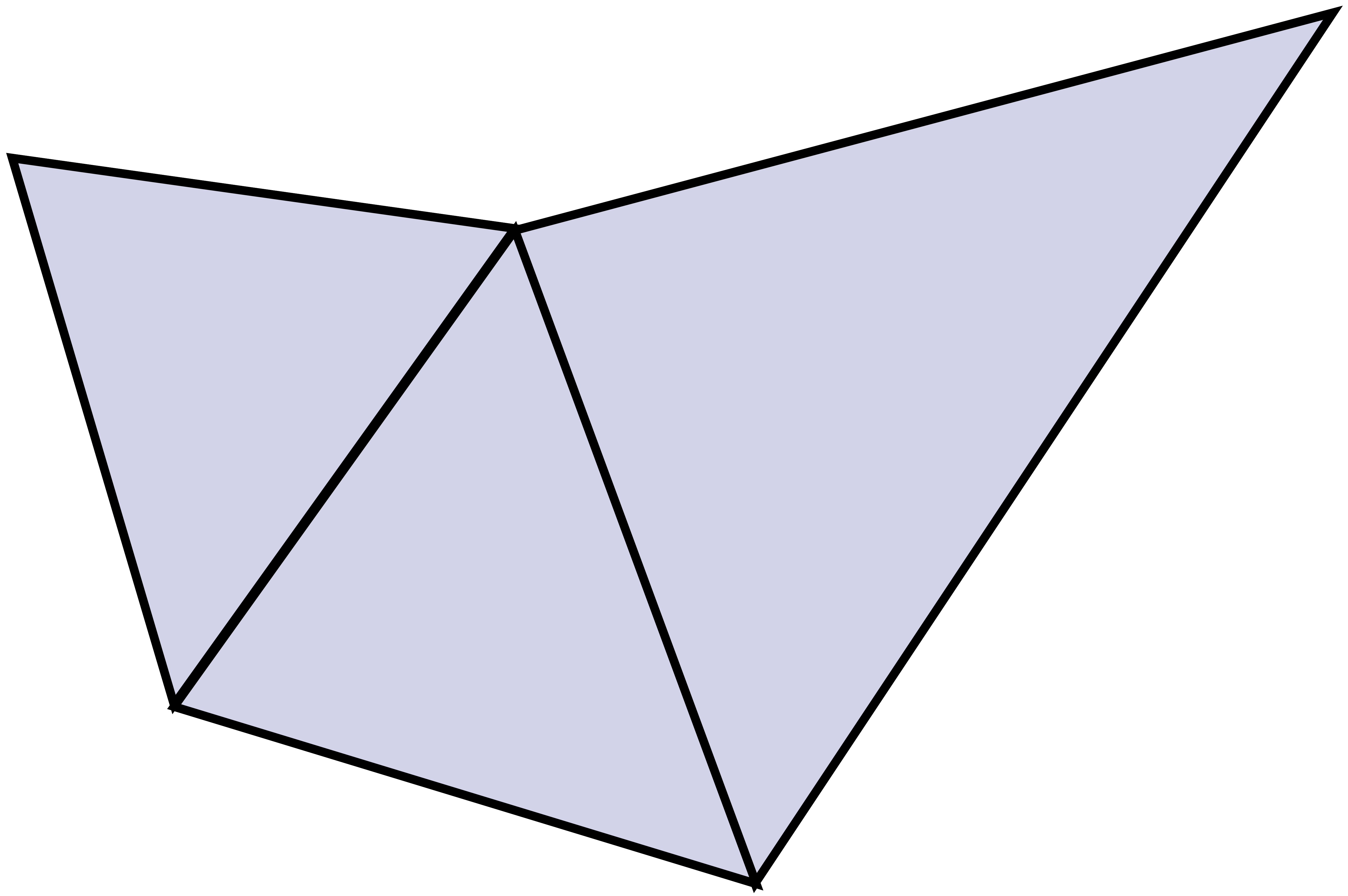
To compute conformal maps, we need some finite "discretization."

First attempt: preserve corner angles in a triangle mesh:



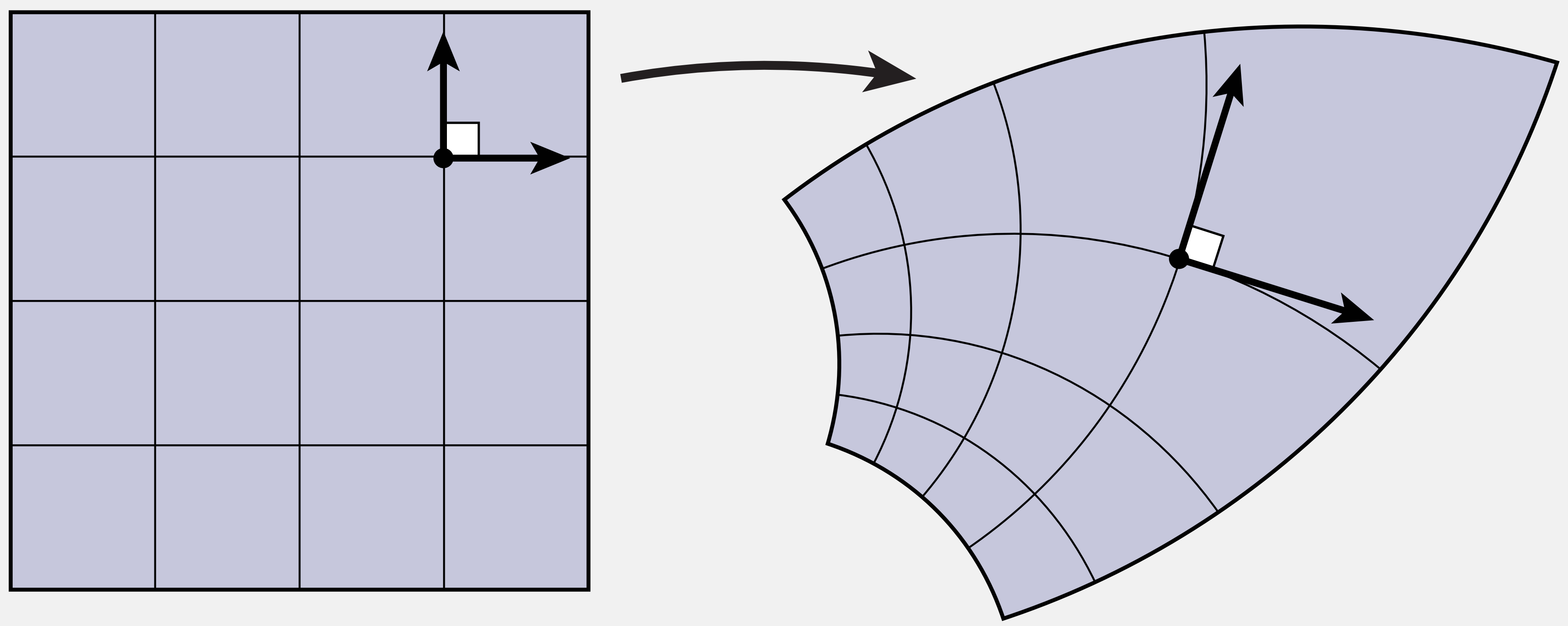
Rigidity of Angle Preservation

Problem: one triangle determines the entire map! (Too “rigid”)

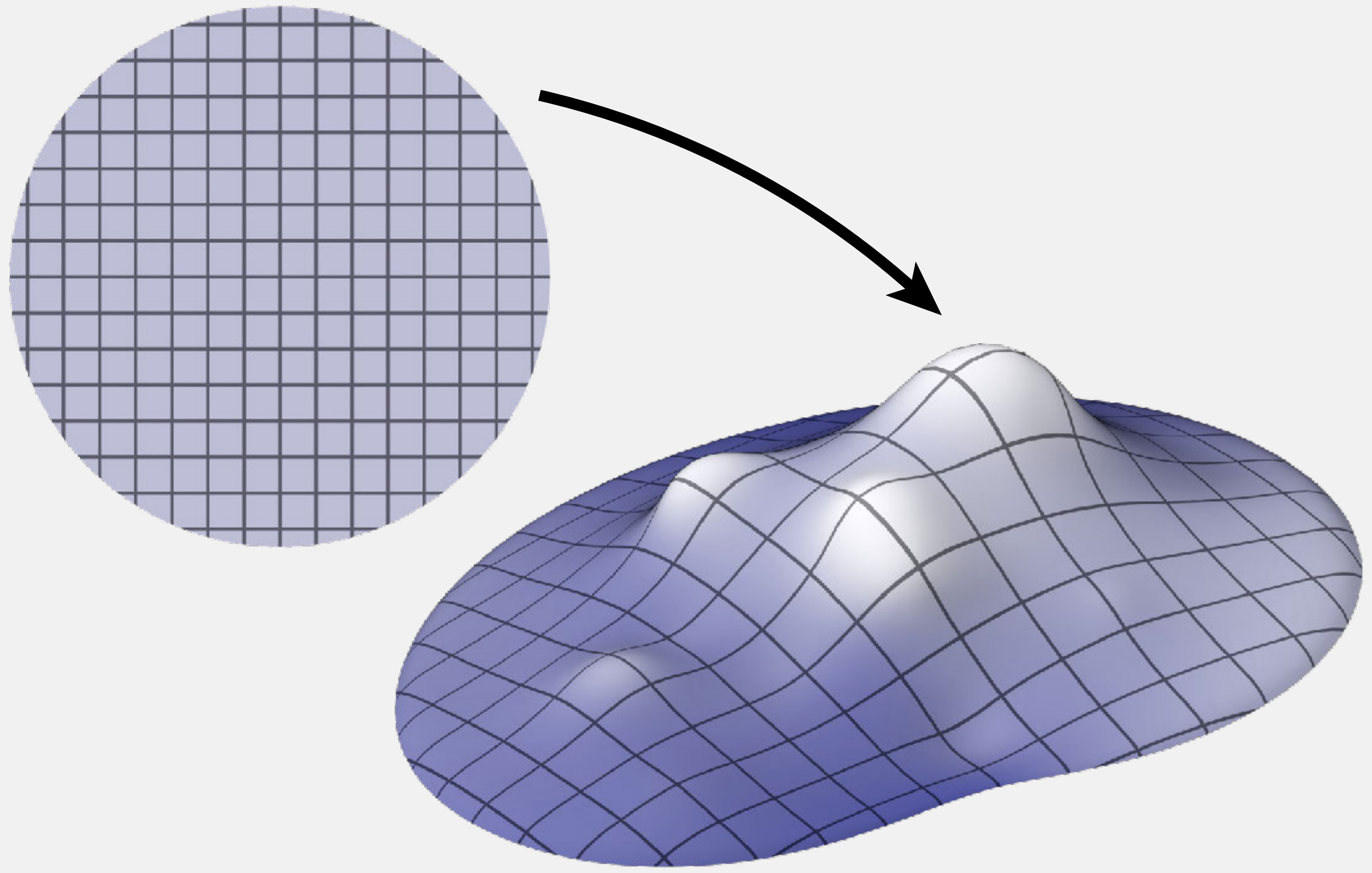


Need a different way of thinking…

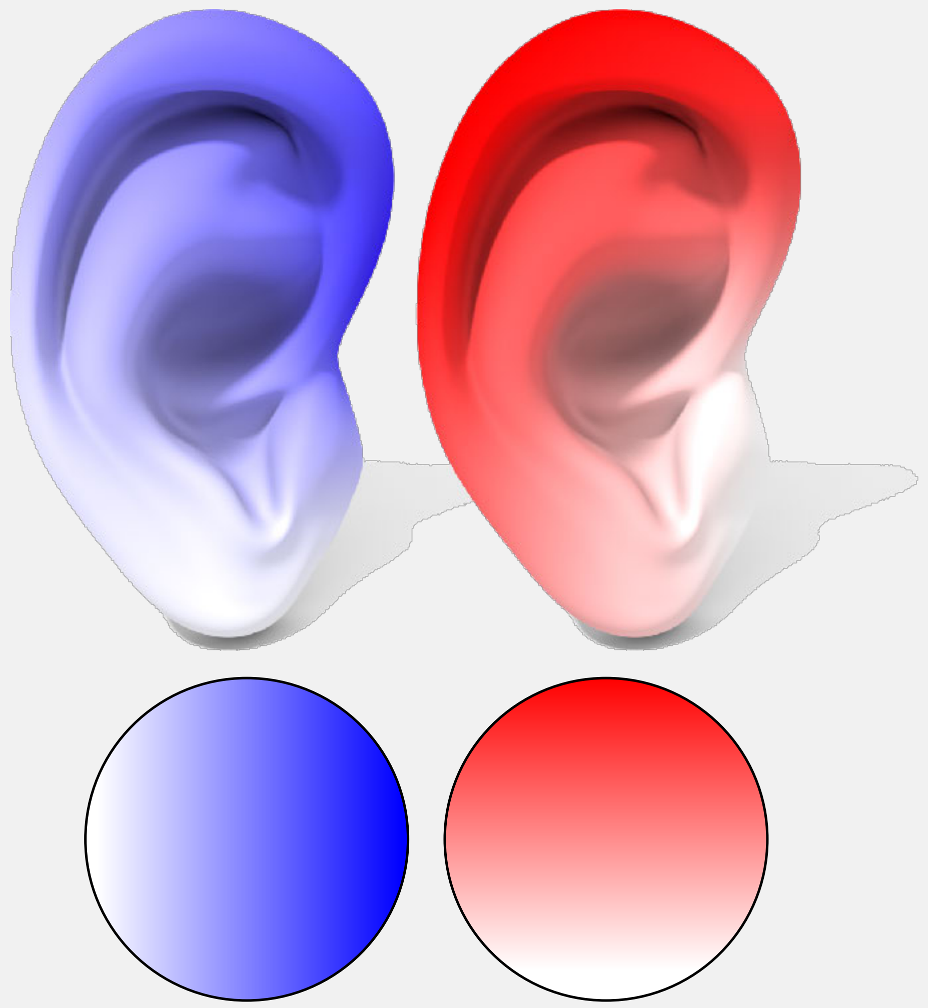
(Some) Characterizations of Conformal Maps



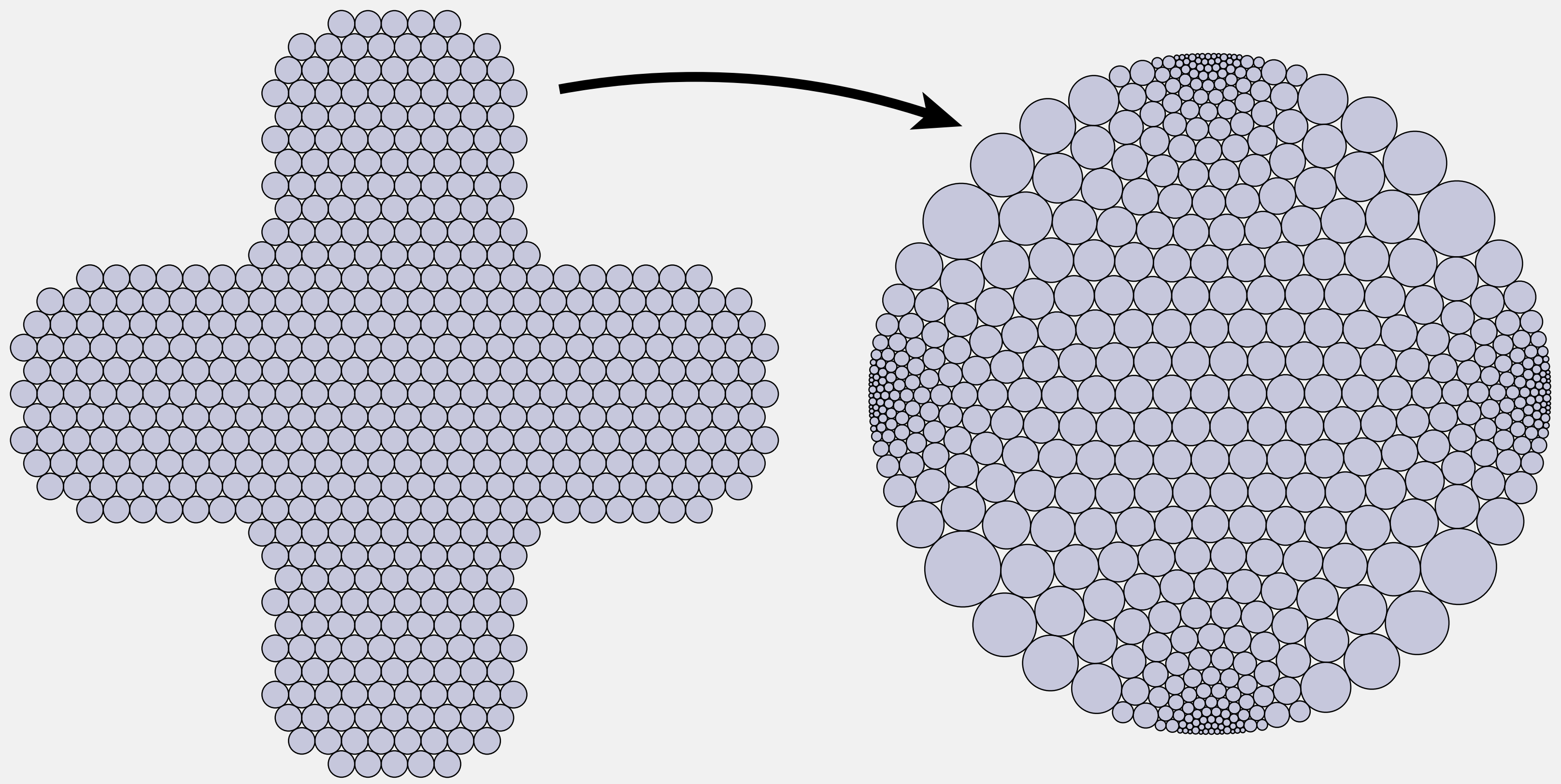
angle preservation



metric rescaling



conjugate harmonic functions



preservation of circles

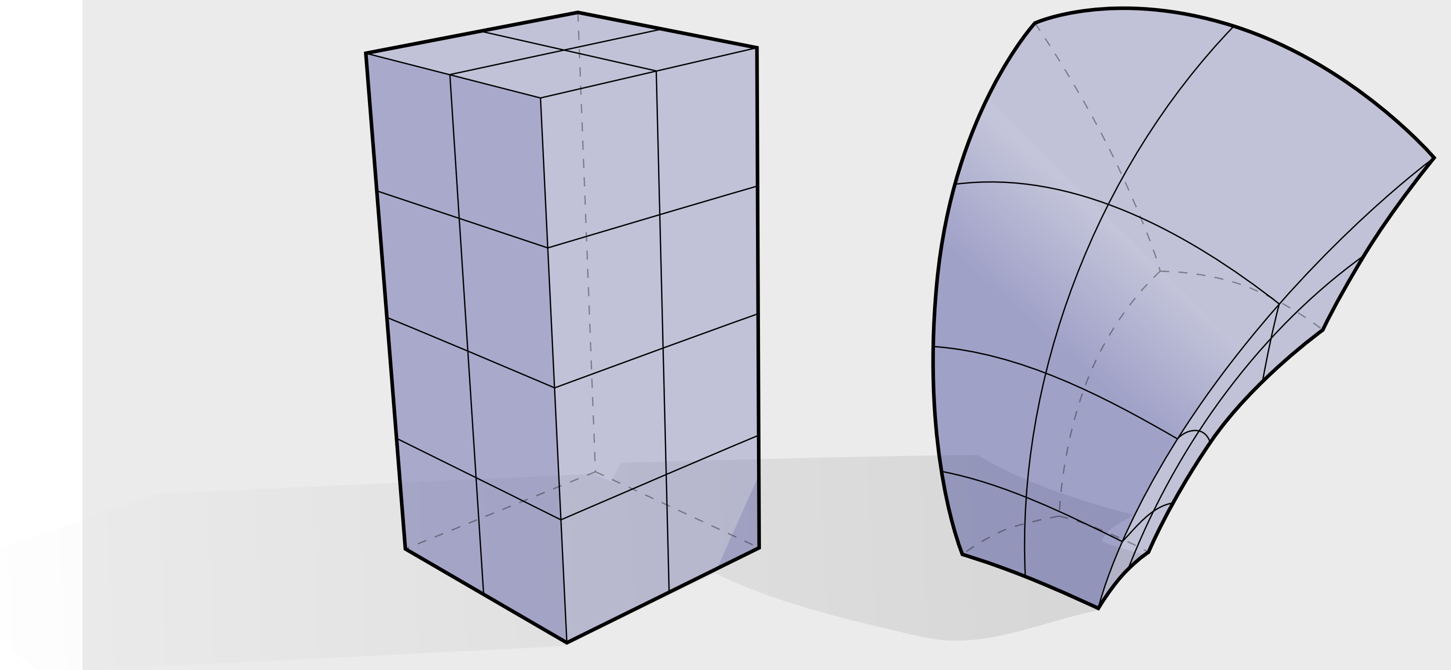
critical points of Dirichlet energy

(Some) Conformal Geometry Algorithms

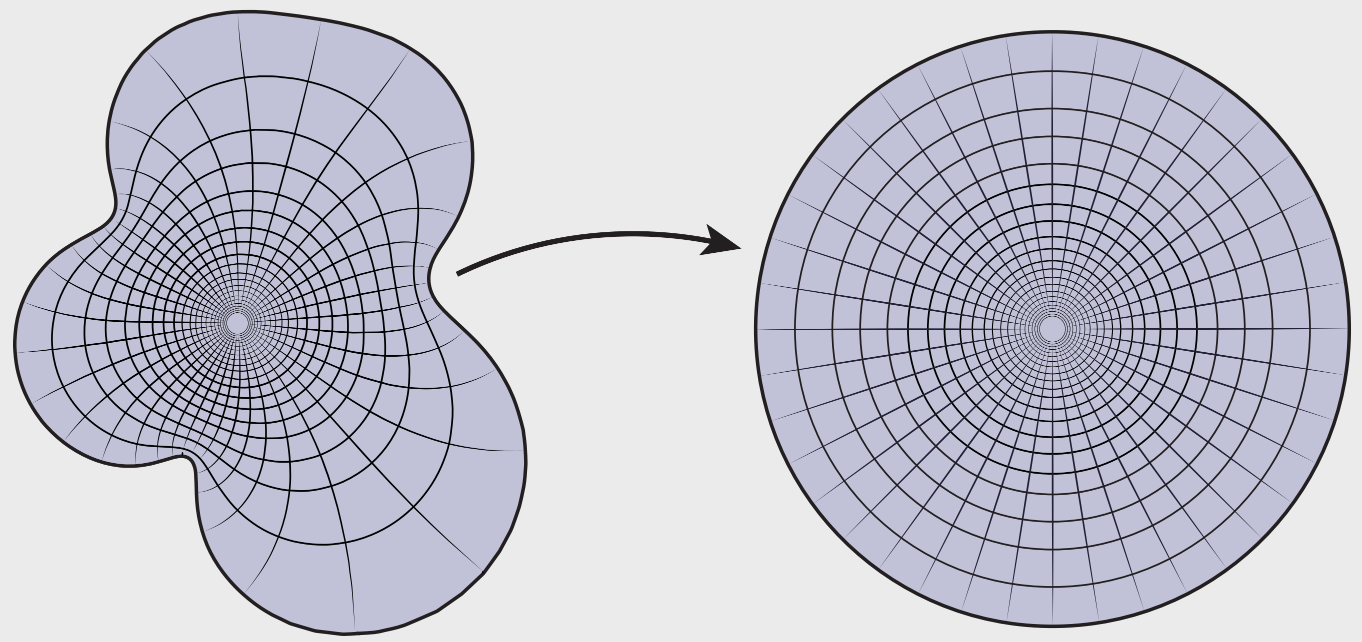
Characterization	Algorithms
Cauchy-Riemann	*least square conformal maps (LSCM)*
Dirichlet energy	*discrete conformal parameterization (DCP)* *genus zero surface conformal mapping (GZ)*
angle preservation	*angle based flattening (ABF)*
circle preservation	*circle packing* *circle patterns (CP)*
metric rescaling	*conformal prescription with metric scaling (CPMS)* *conformal equivalence of triangle meshes (CETM)*
conjugate harmonic	*boundary first flattening (BFF)*

Some Key Ideas in Conformal Surface Geometry

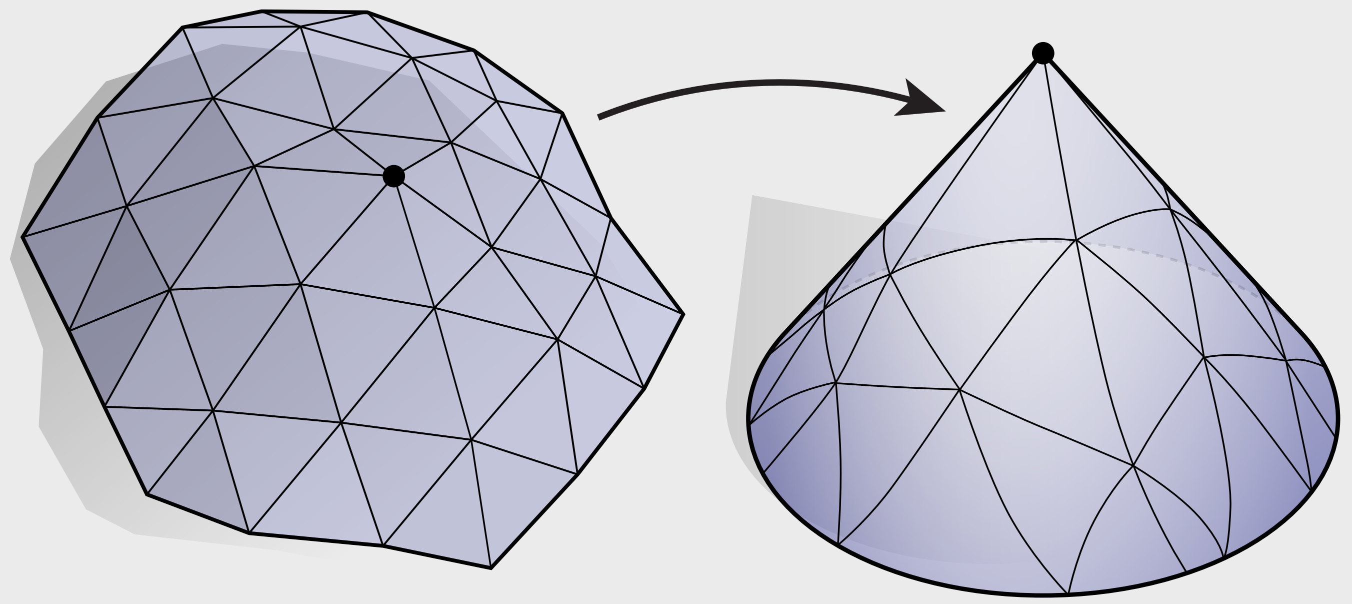
Möbius Transformations / Stereographic Projection



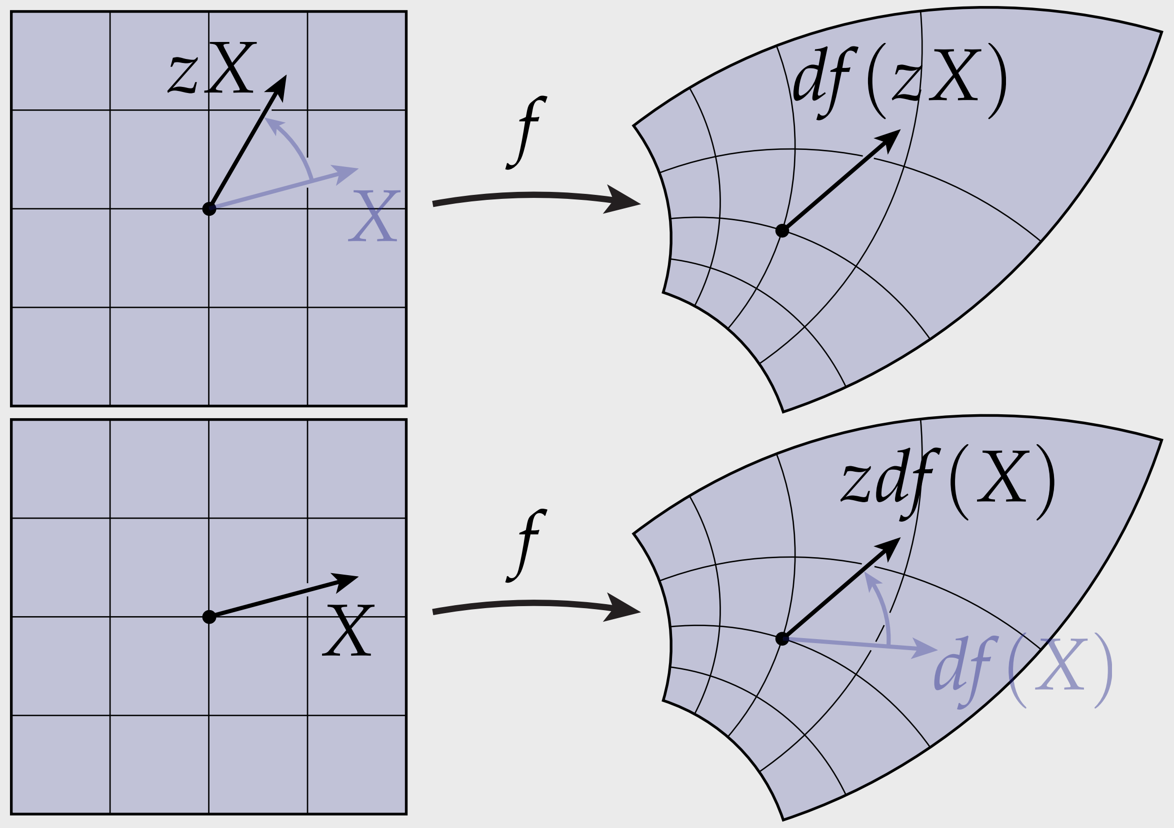
Riemann Mapping / Uniformization



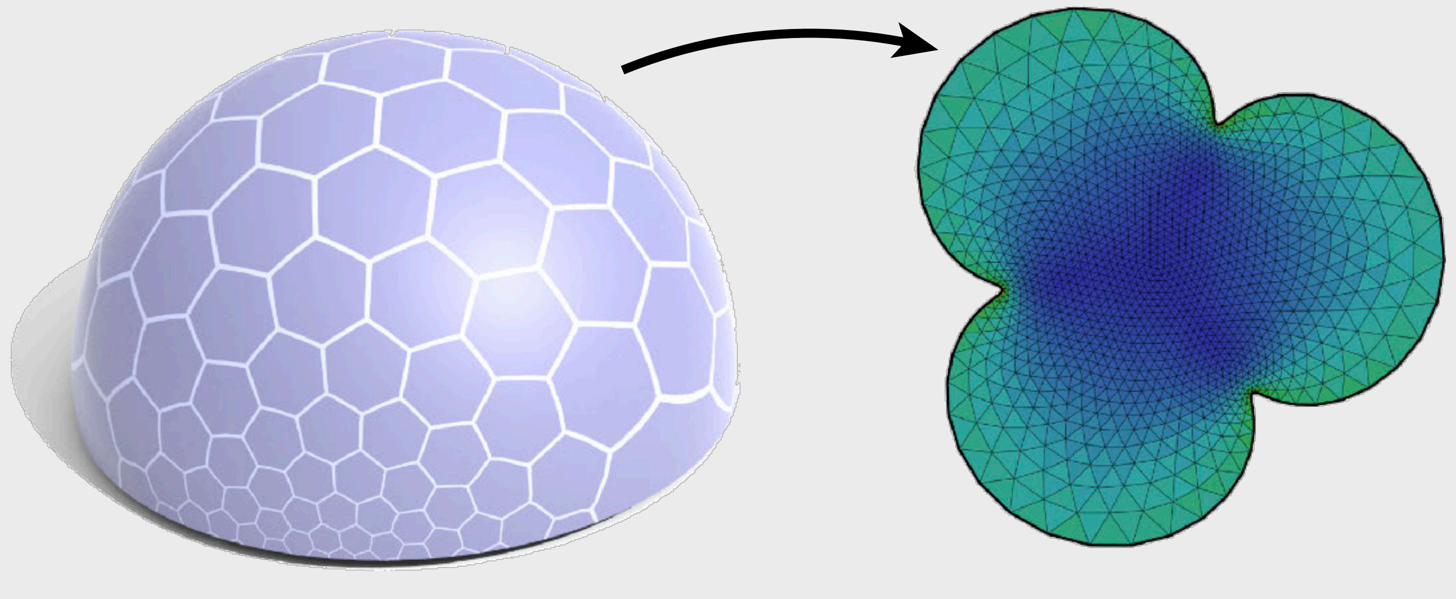
Cone Singularities



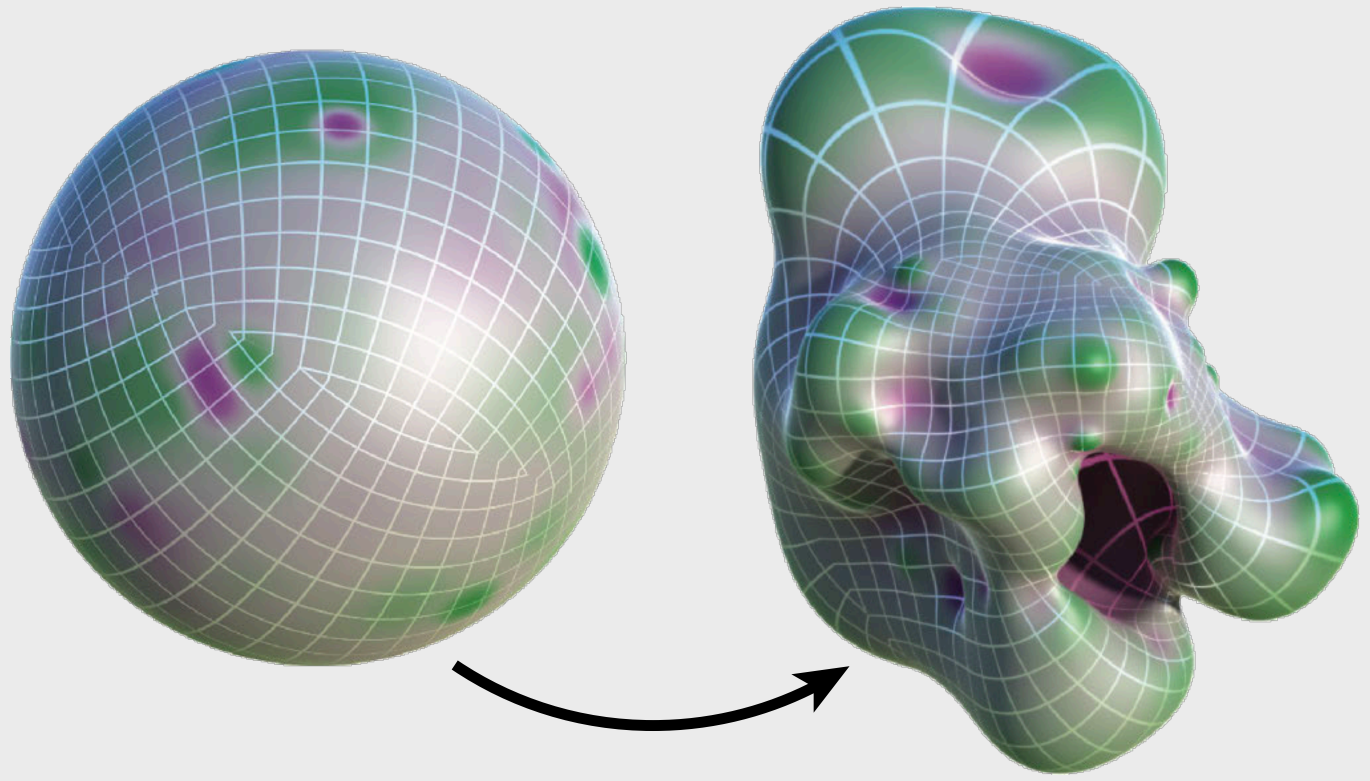
Cauchy-Riemann Equation



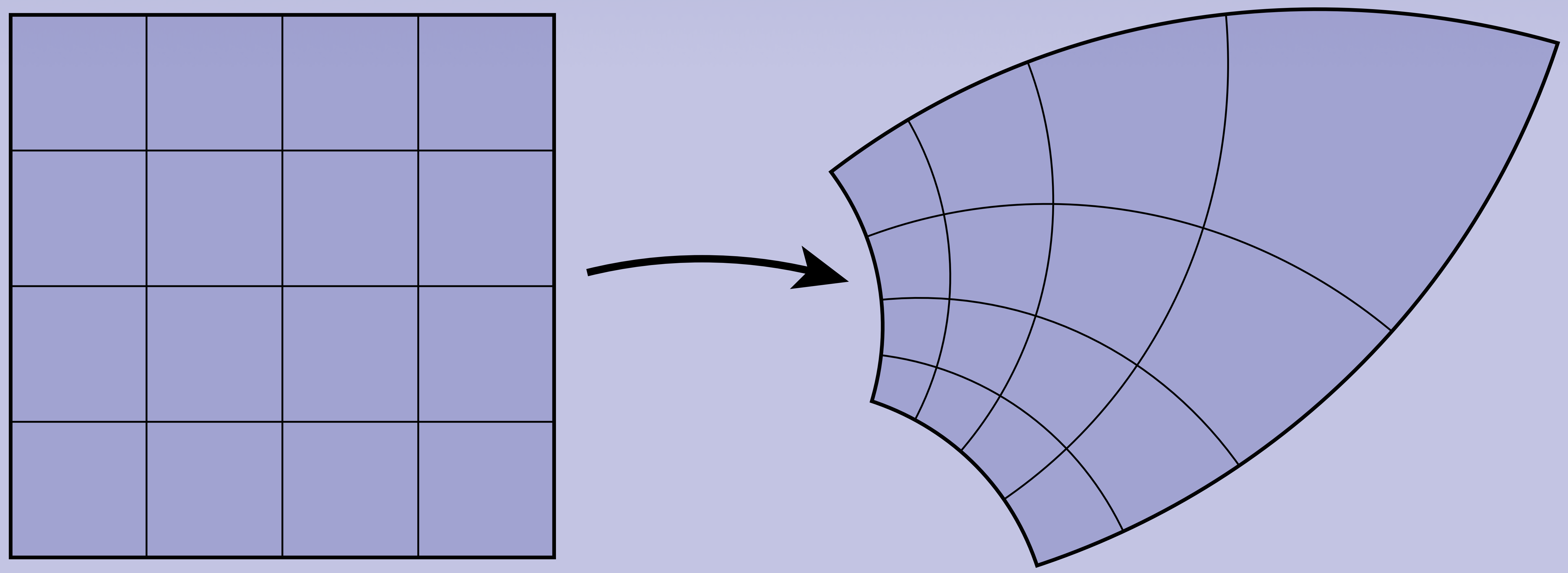
Ricci Flow / Cherrier Formula



Dirac Equation



Part II:
Smooth Theory



Discrete Conformal Geometry

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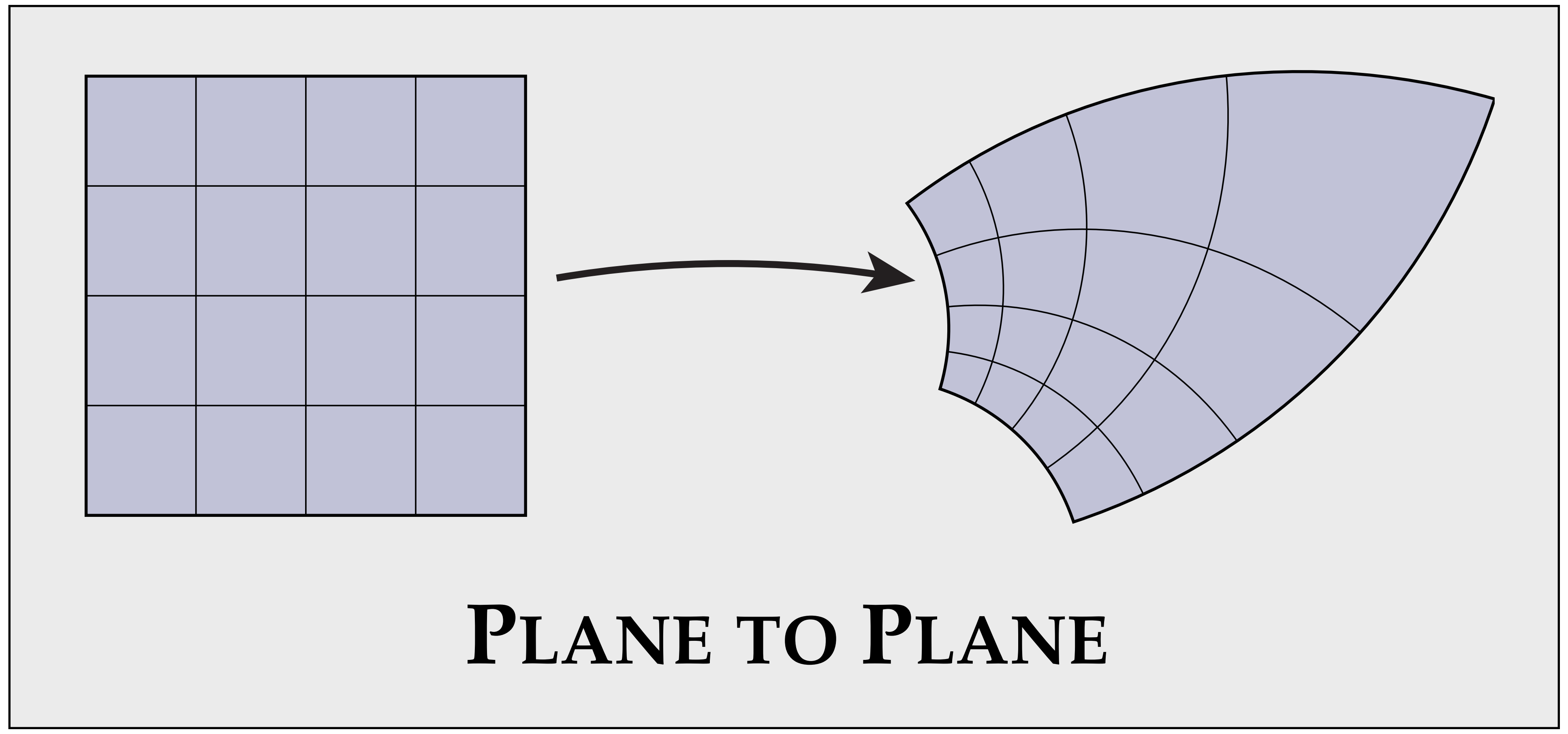
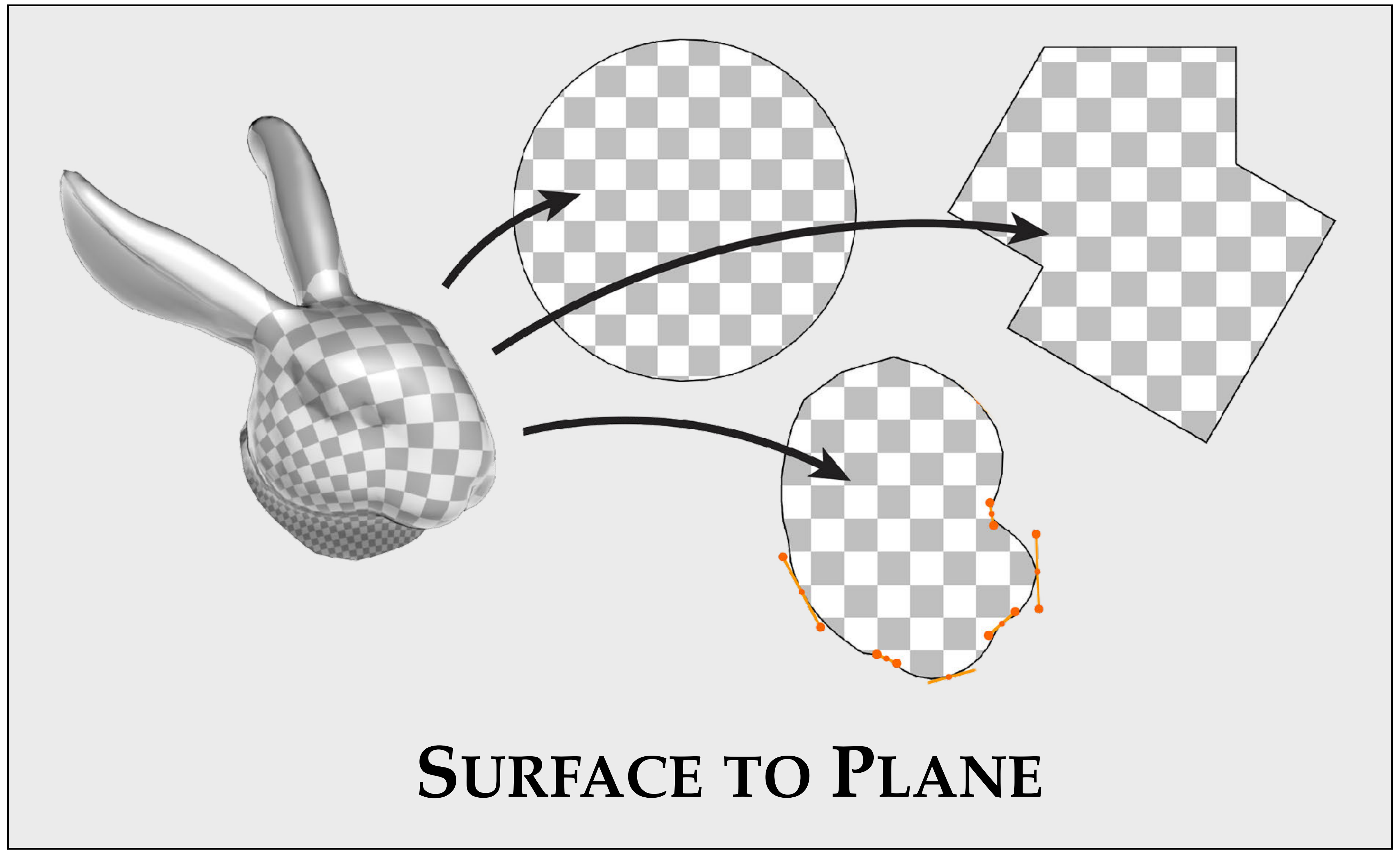
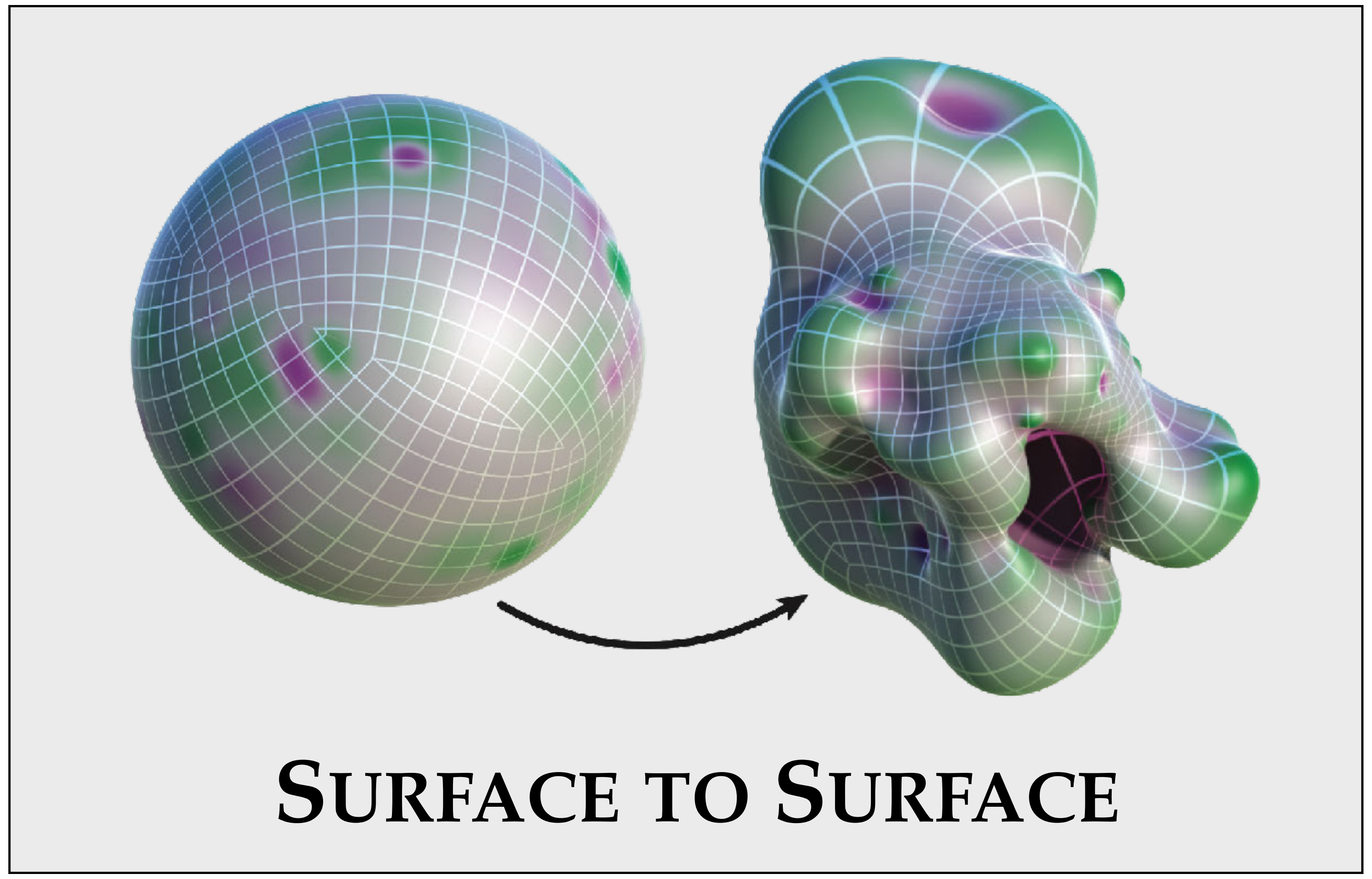
Conformal Maps of Surfaces

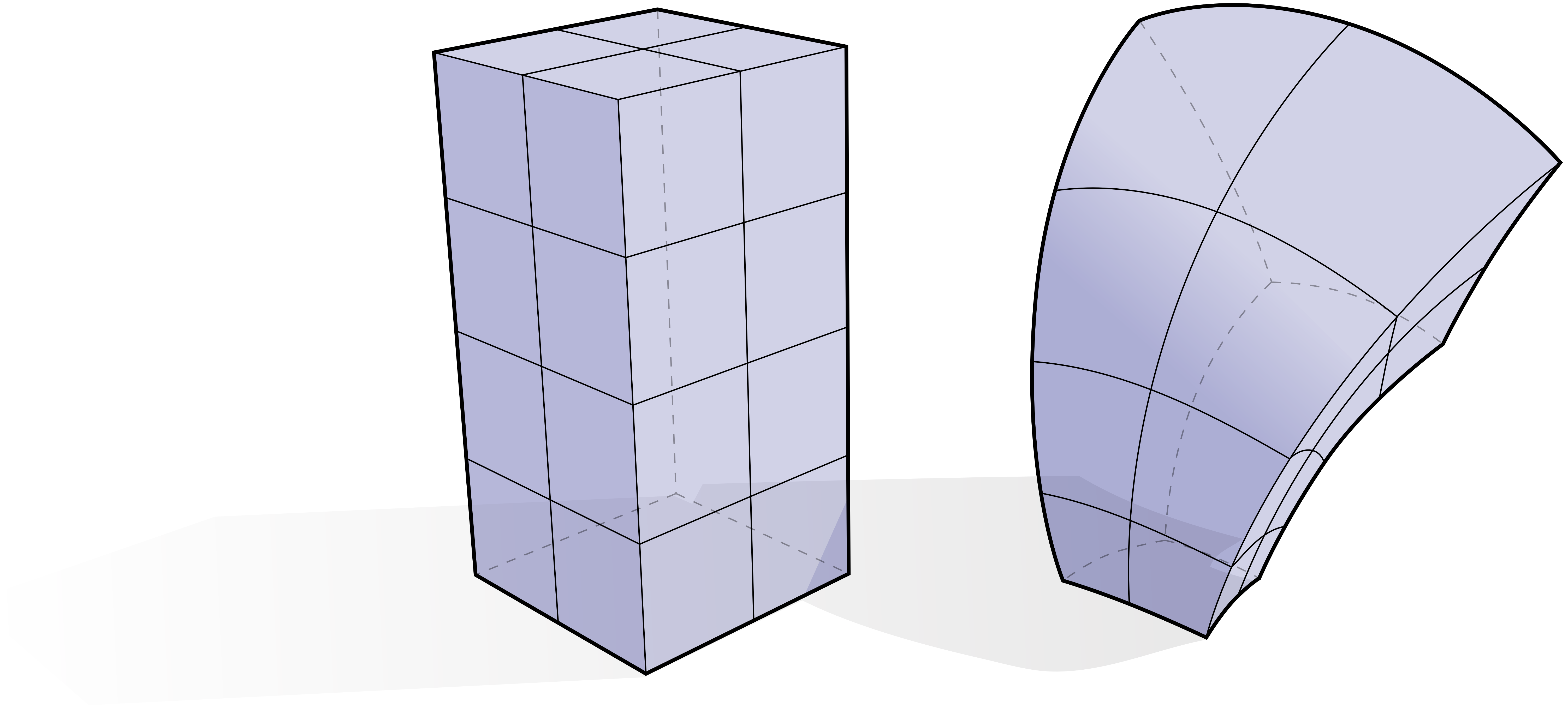






Why Not Higher Dimensions?

Theorem (Liouville)**.** For $n \geq 3$, the only angle-preserving maps from $\mathbb{R}^n$ to itself (or from a region of $\mathbb{R}^n$ to $\mathbb{R}^n$) are Möbius transformations.

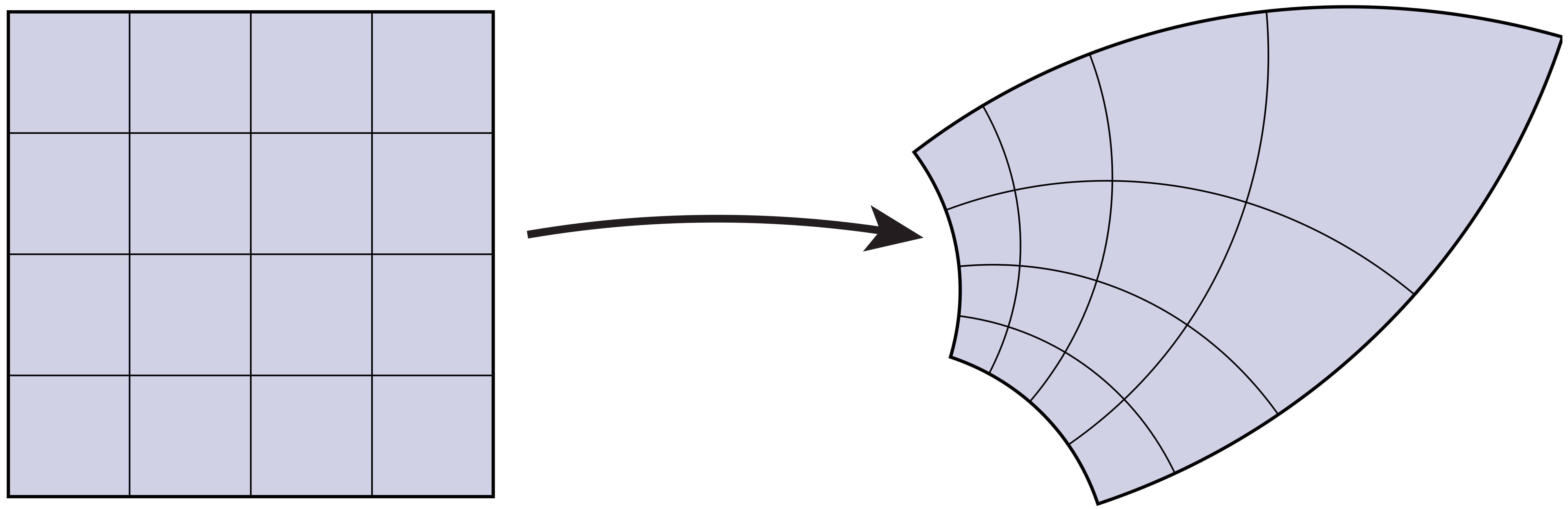


Key idea: conformal maps of volumes are *very* rigid.

Plane to Plane

Plane to Plane

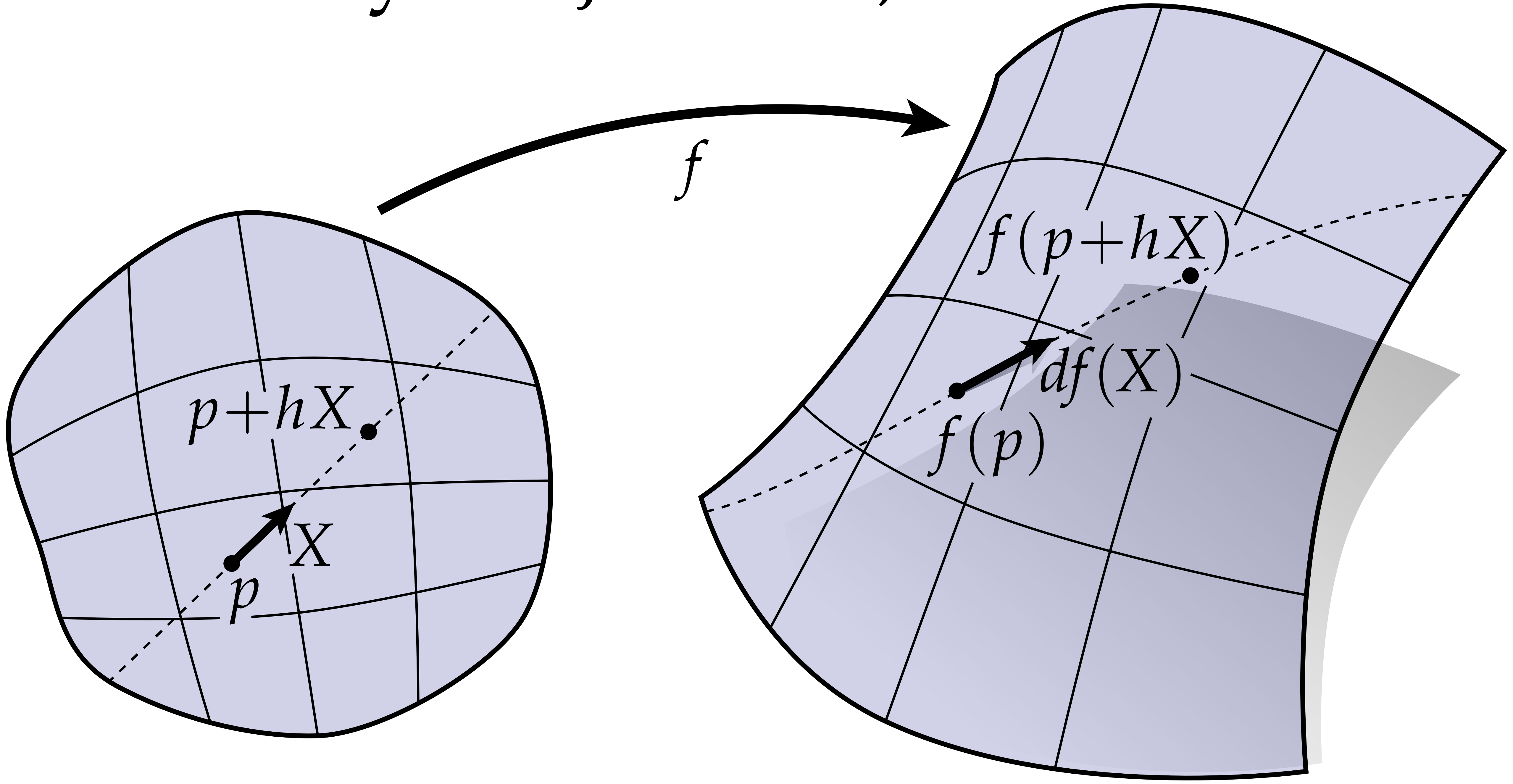
- Most basic case: conformal maps from region of 2D plane to 2D plane.
- Basic topic of complex analysis
- Fundamental equation: *Cauchy-Riemann*
- *Many* ideas we will omit (e.g., power series/analytic point of view)



Differential of a Map

- Basic idea we'll need to understand: *differential* of a map
- Describes how to *"push forward"* vectors under a differentiable map
- (In coordinates, differential is represented by the *Jacobian*)

$$df(X) = \lim_{h \to 0} \frac{f(p + hX) - f(p)}{h}$$



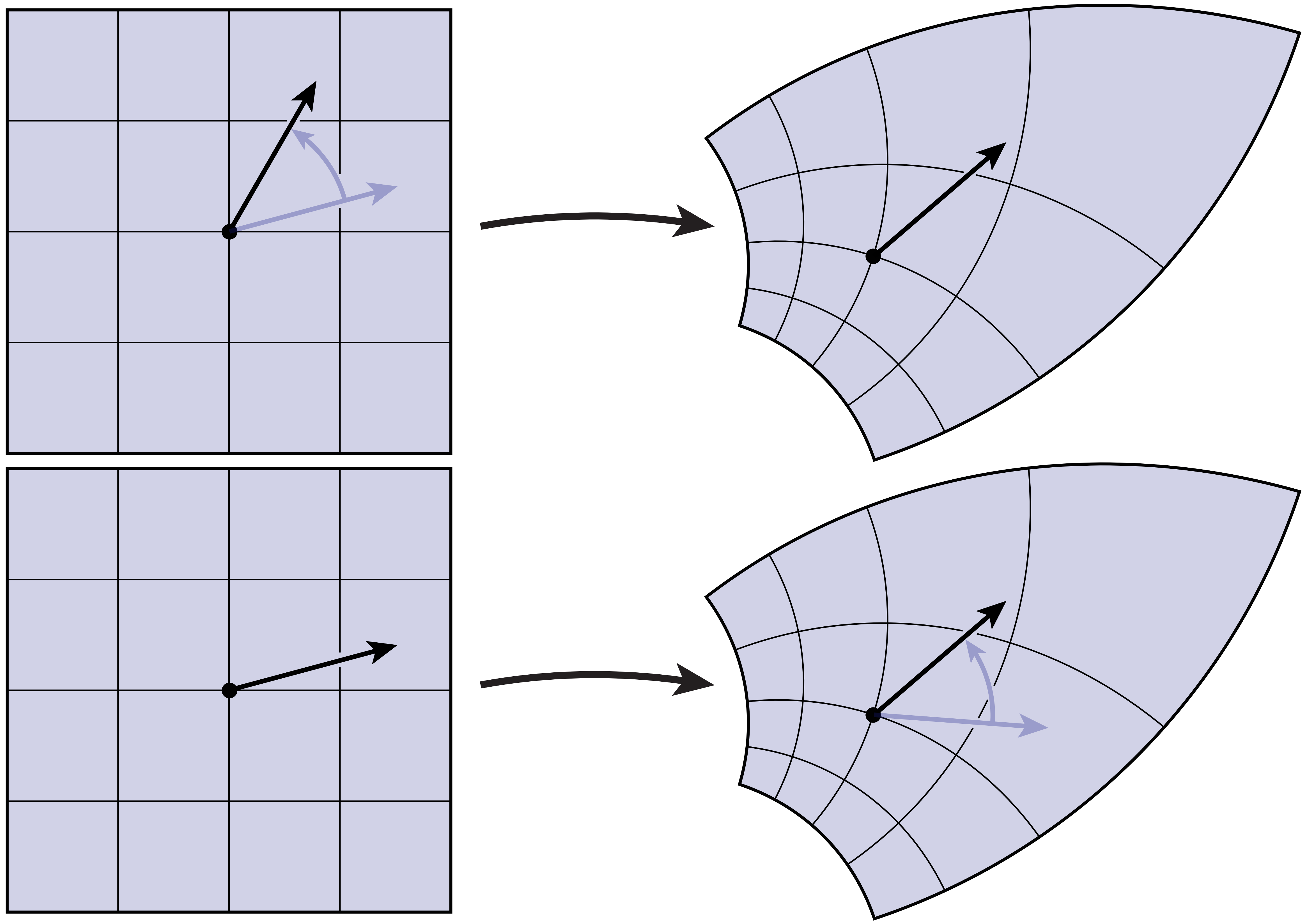
Intuition: "how do vectors get stretched out?"

Conformal Map

- A map is conformal if two operations are equivalent:

1. rotate, then push forward vector

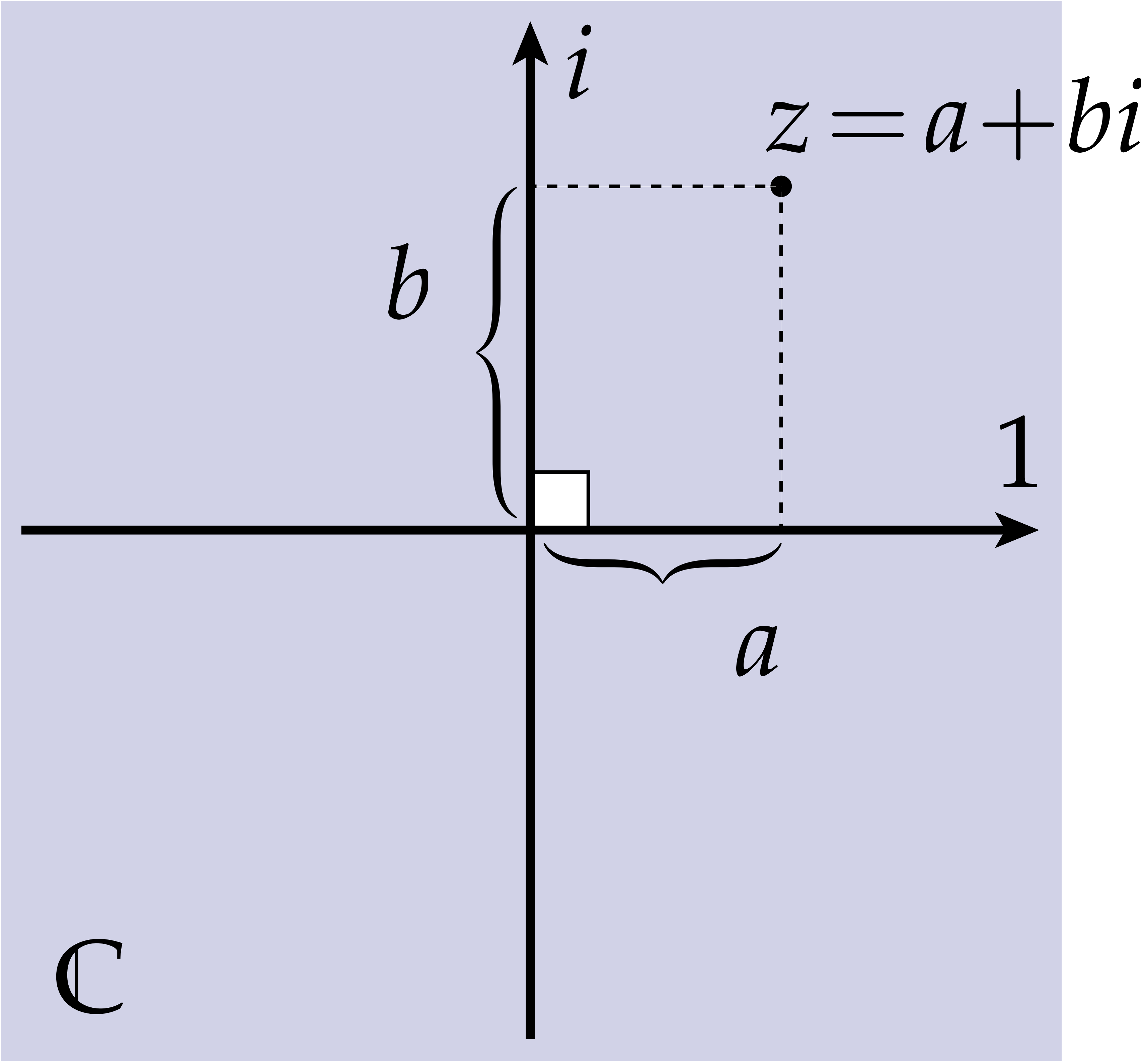
2. push forward vector, then rotate



(How can we write this condition more explicitly?)

Complex Numbers

- Not much different from the usual Euclidean plane
- Additional operations make it easy to express **scaling & rotation**
- Extremely natural for conformal geometry



- Two basis directions: 1 and i
- Points expressed as $z = a+bi$

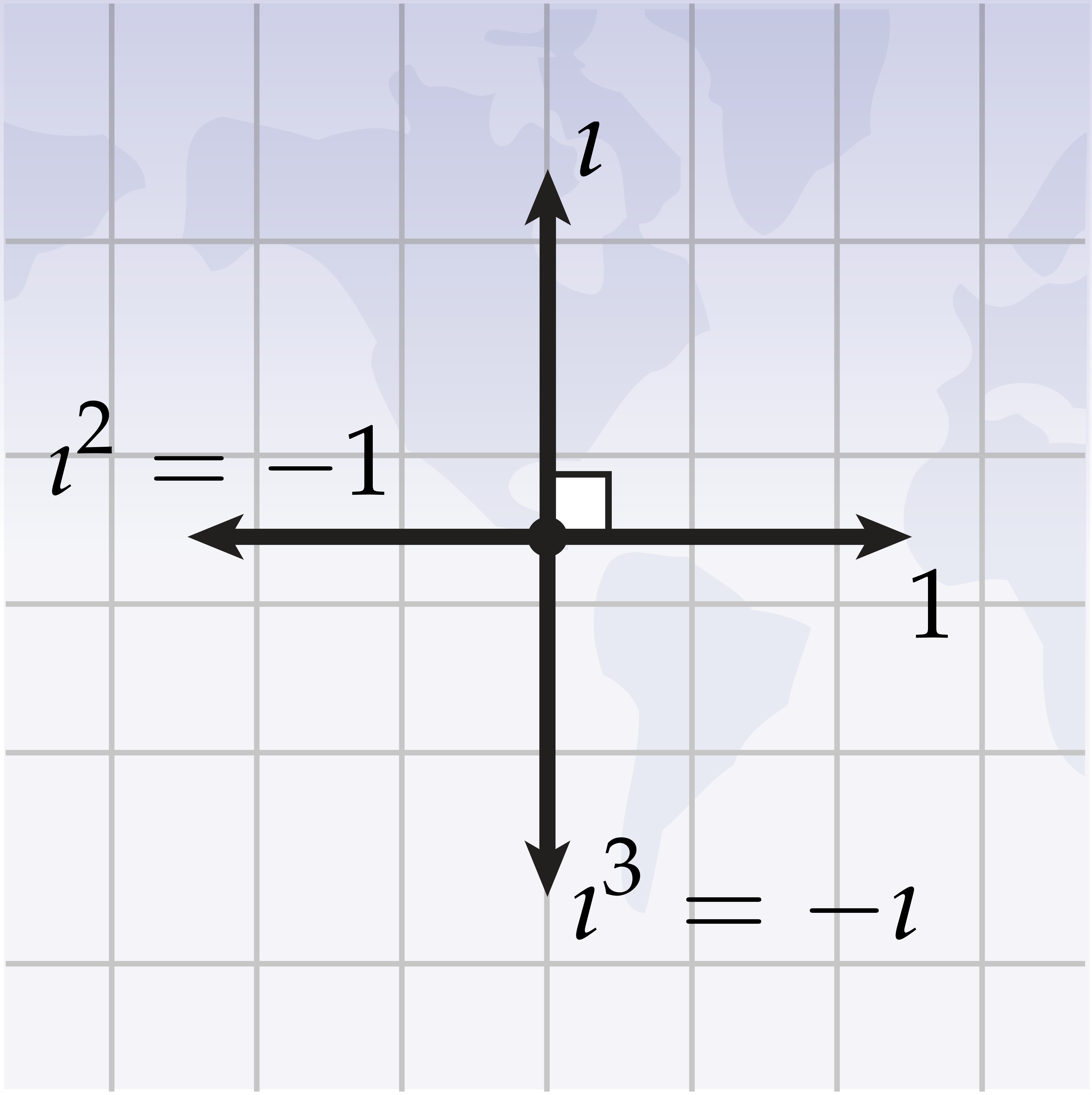
Complex Numbers

$$\cancel{\imath := \sqrt{-1}}$$

nonsense!

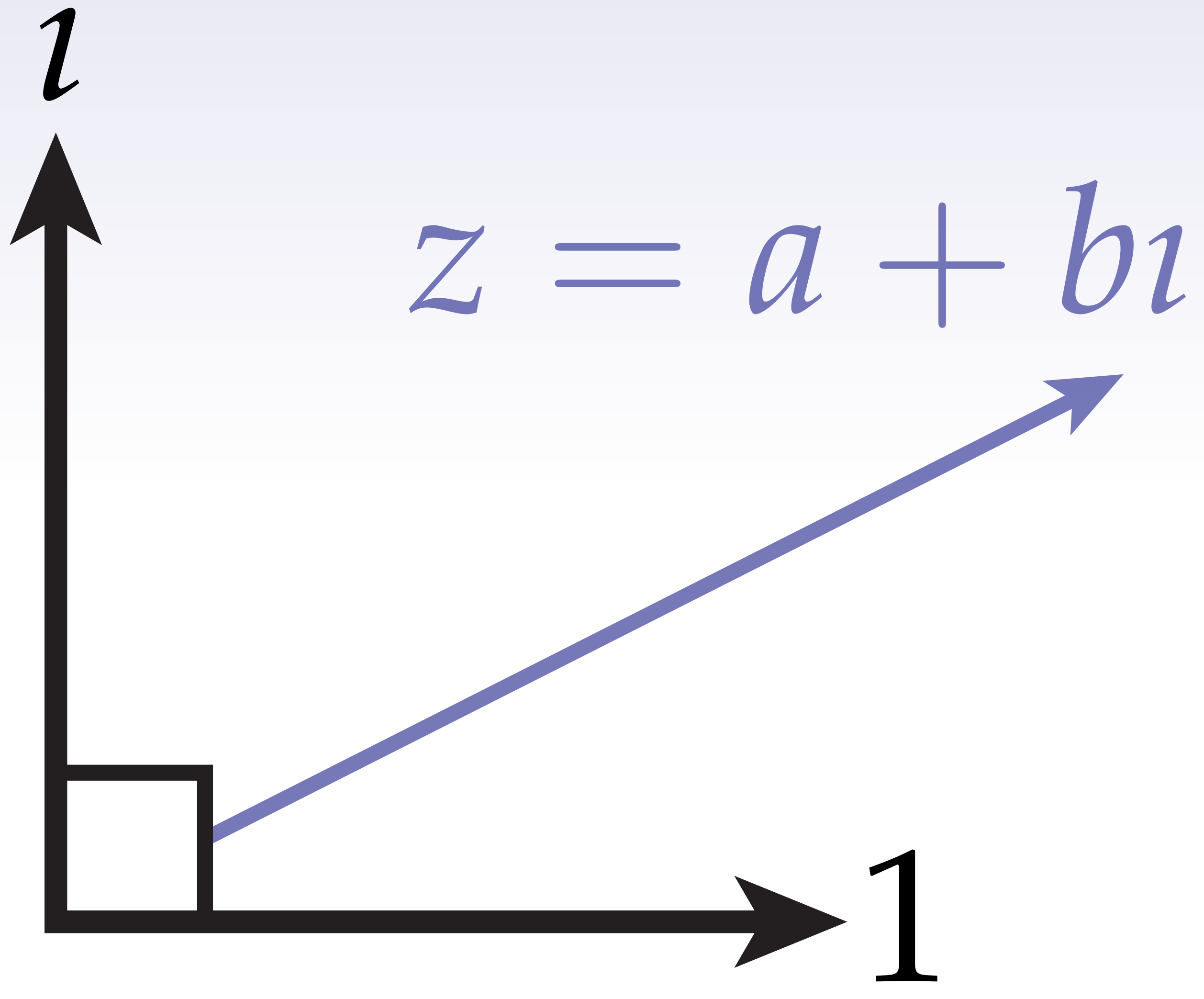
More importantly: obscures geometric meaning.

Imaginary Unit—Geometric Description

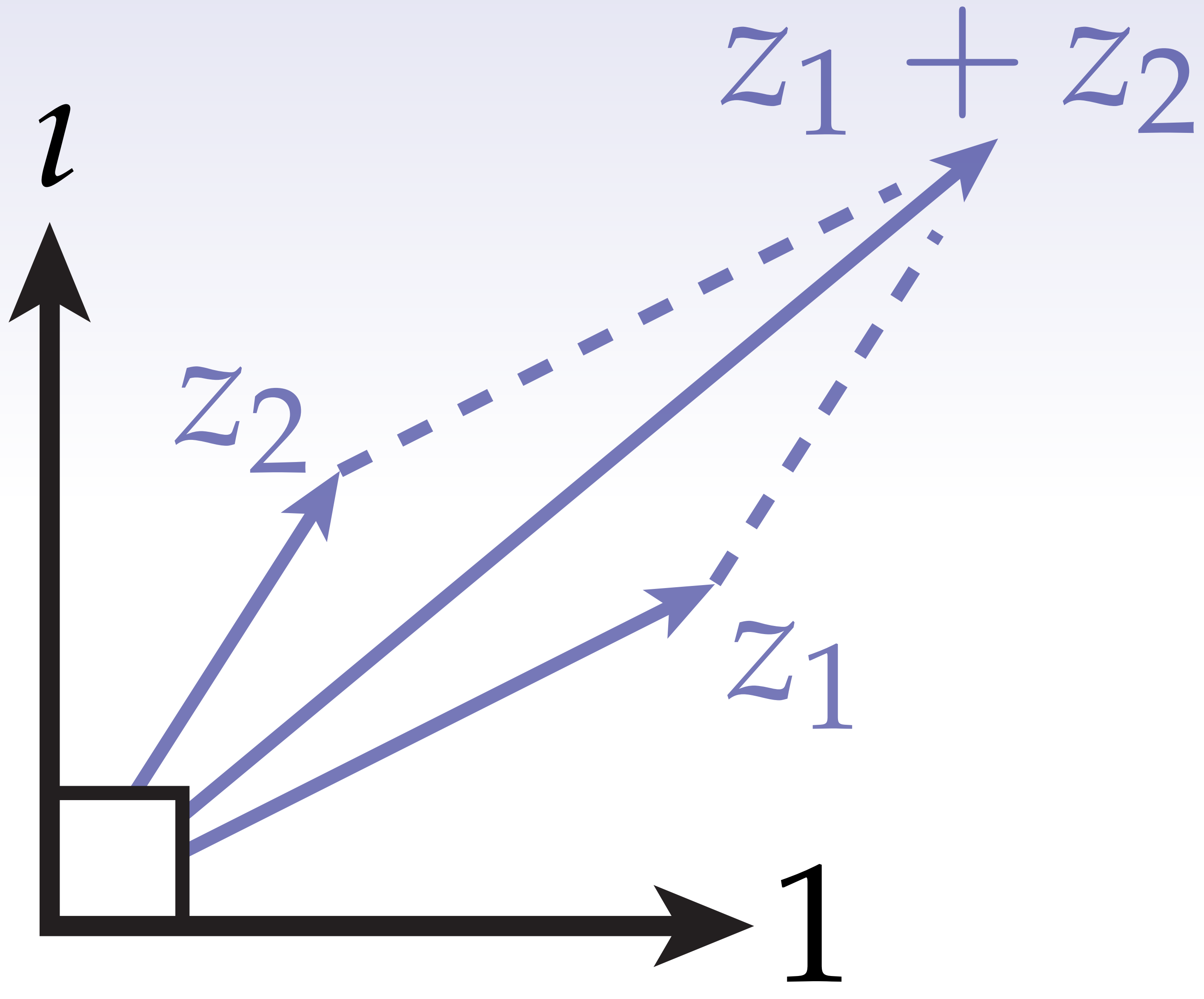


Symbol $\imath$ denotes *quarter-turn* in the *counter-clockwise* direction.

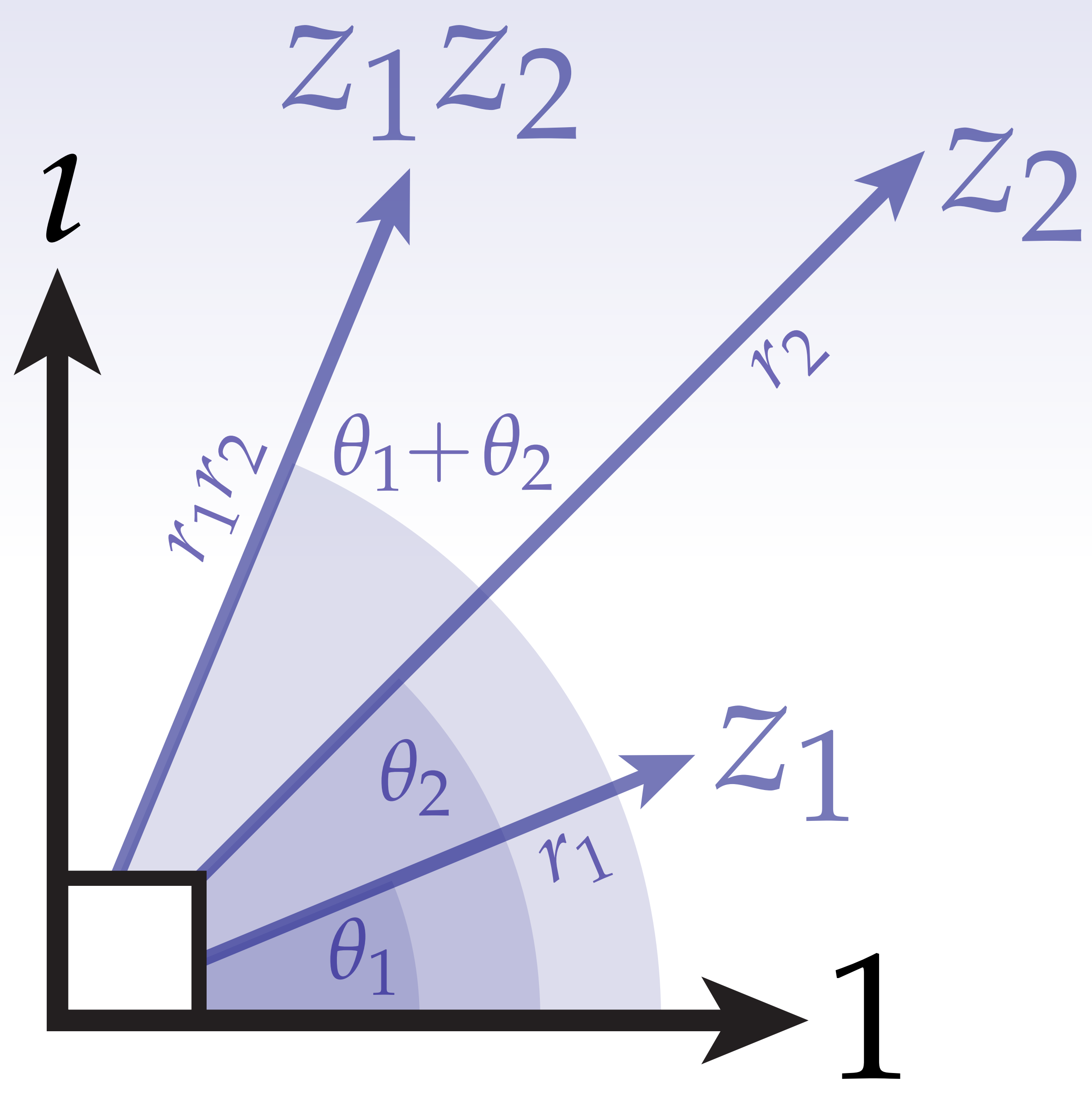
Complex Arithmetic — Visualized



rectangular coordinates



addition



multiplication

Complex Product

- Usual definition:

- Complex product distributes over addition. Hence,

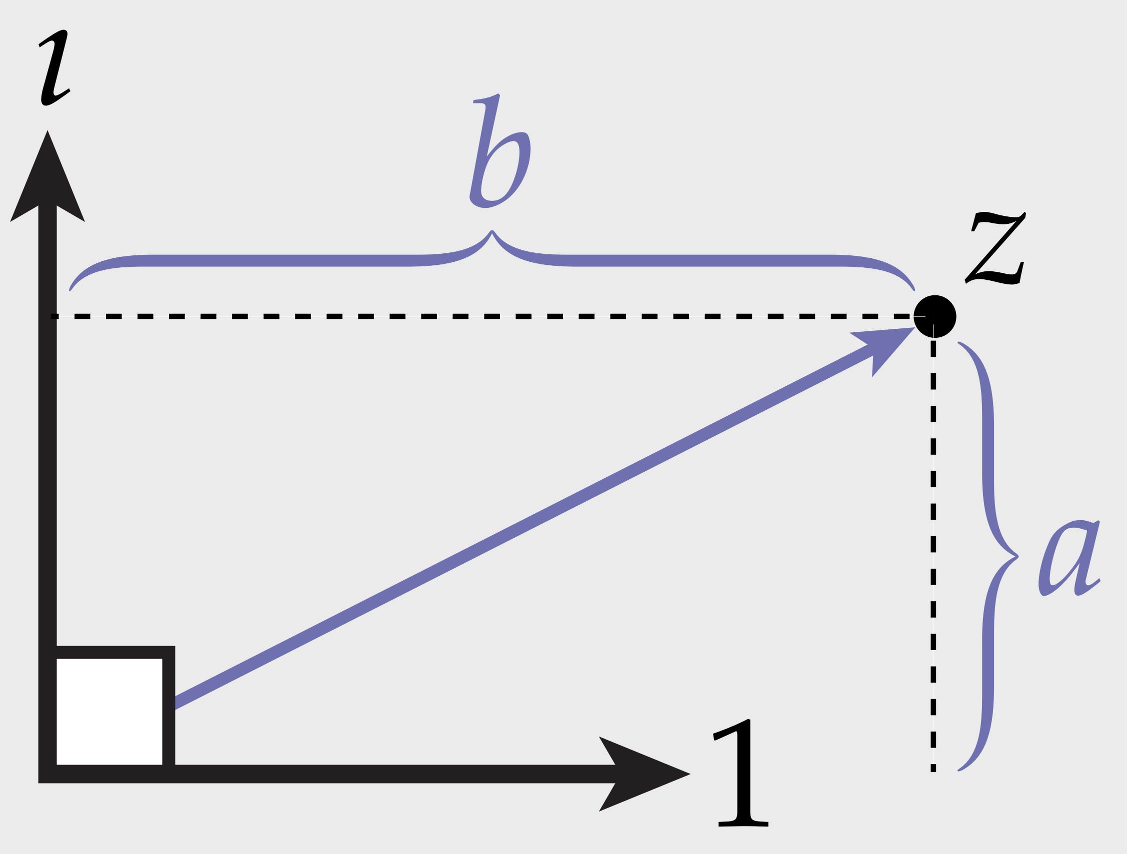
$$
\begin{aligned}
z_1 &:= a + b\imath \\
z_2 &:= c + d\imath
\end{aligned}
$$

$$
\begin{aligned}
z_1 z_2 &= (a + b\imath)(c + d\imath) \\
&= (a + b\imath)c + (a + b\imath)d\imath \\
&= ac + bc\imath + ad\imath + bd\imath^2 \\
&= \boxed{(ac - bd) + (ad + bc)\imath}
\end{aligned}
$$

Ok, terrific… but what does it mean *geometrically?*

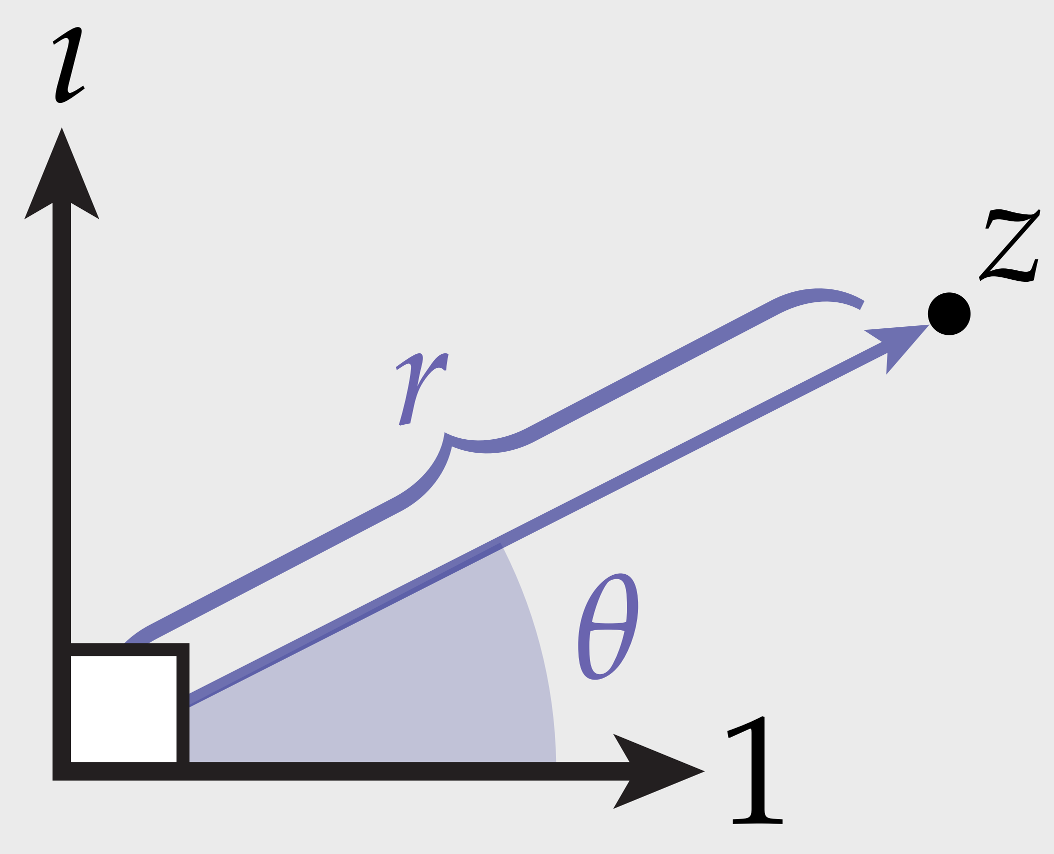
Rectangular vs. Polar Coordinates

RECTANGULAR



$$z = a + b\imath$$

POLAR



$$z = r(\cos\theta + \imath\sin\theta) = re^{\imath\theta}$$

EULER'S IDENTITY

$$\boxed{e^{\imath\theta} = \cos\theta + \imath\sin\theta}$$

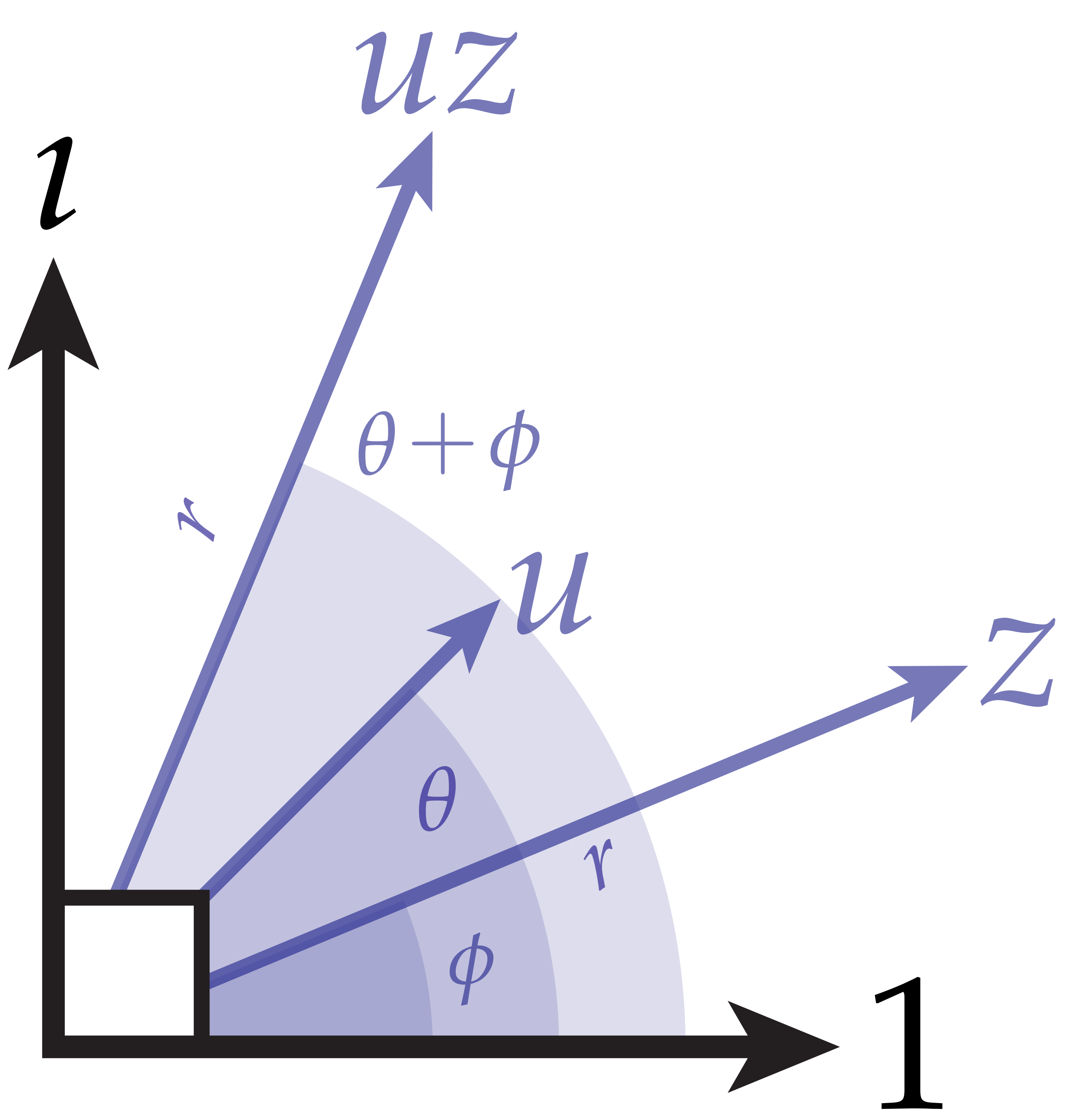
(In practice: just convenient shorthand!)

Rotations with Complex Numbers

- How can we express rotation?
- Let u be any *unit* complex number: $u = e^{\imath\theta}$
- Then for any point $z = re^{\imath\phi}$ we have

$$uz = (e^{\imath\theta})(re^{\imath\phi}) = re^{\imath(\theta+\phi)}$$

(same radius, new angle)

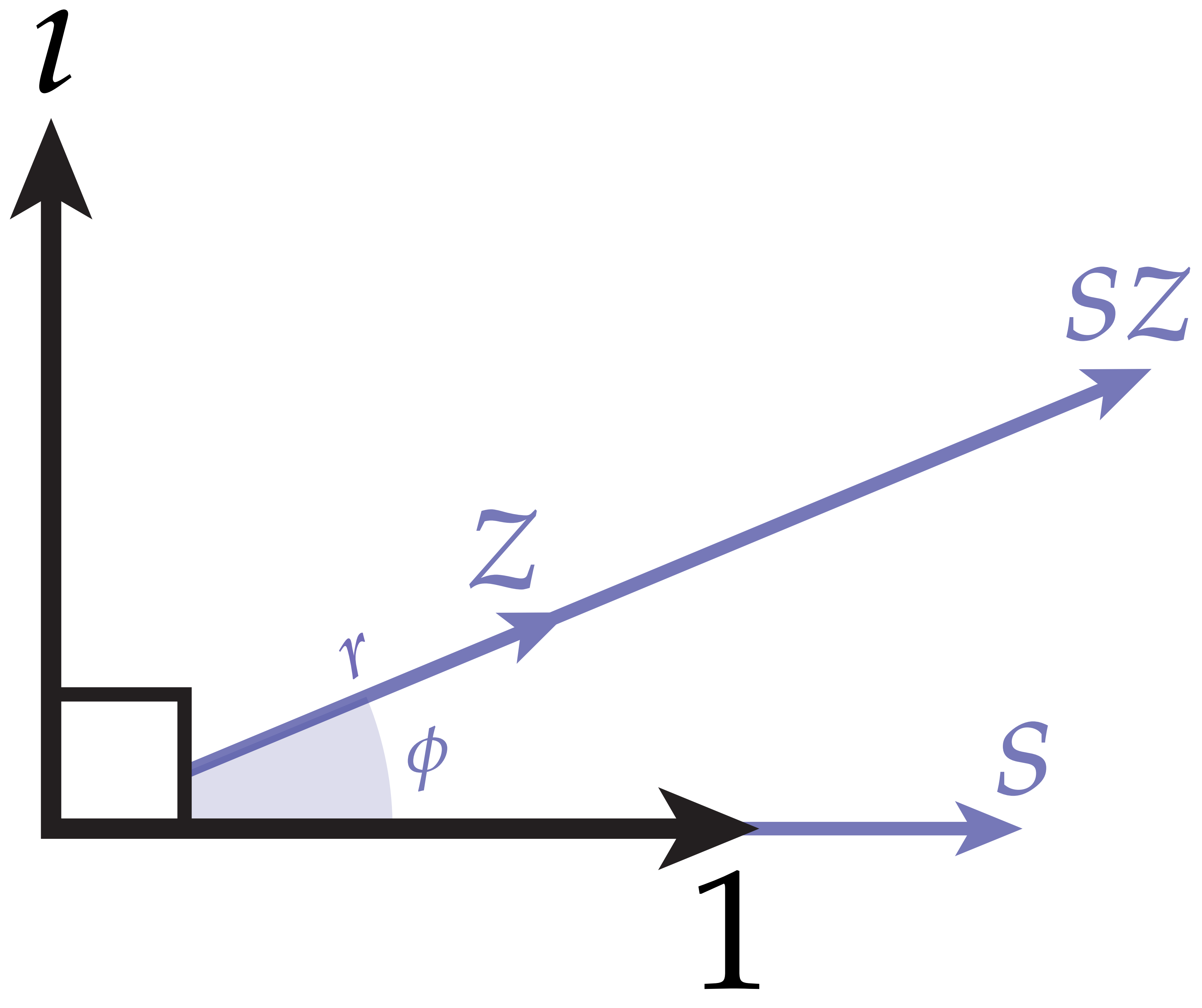


Scaling with Complex Numbers

- How can we express scaling?
- Let s be any *real* complex number: $s = a + 0\imath$
- Then for any point $z = re^{\imath\phi}$ we have

$$sz = (a + 0\imath)(re^{\imath\phi}) = are^{\imath\phi}$$

(same angle, new radius)



Complex Product—Polar Form

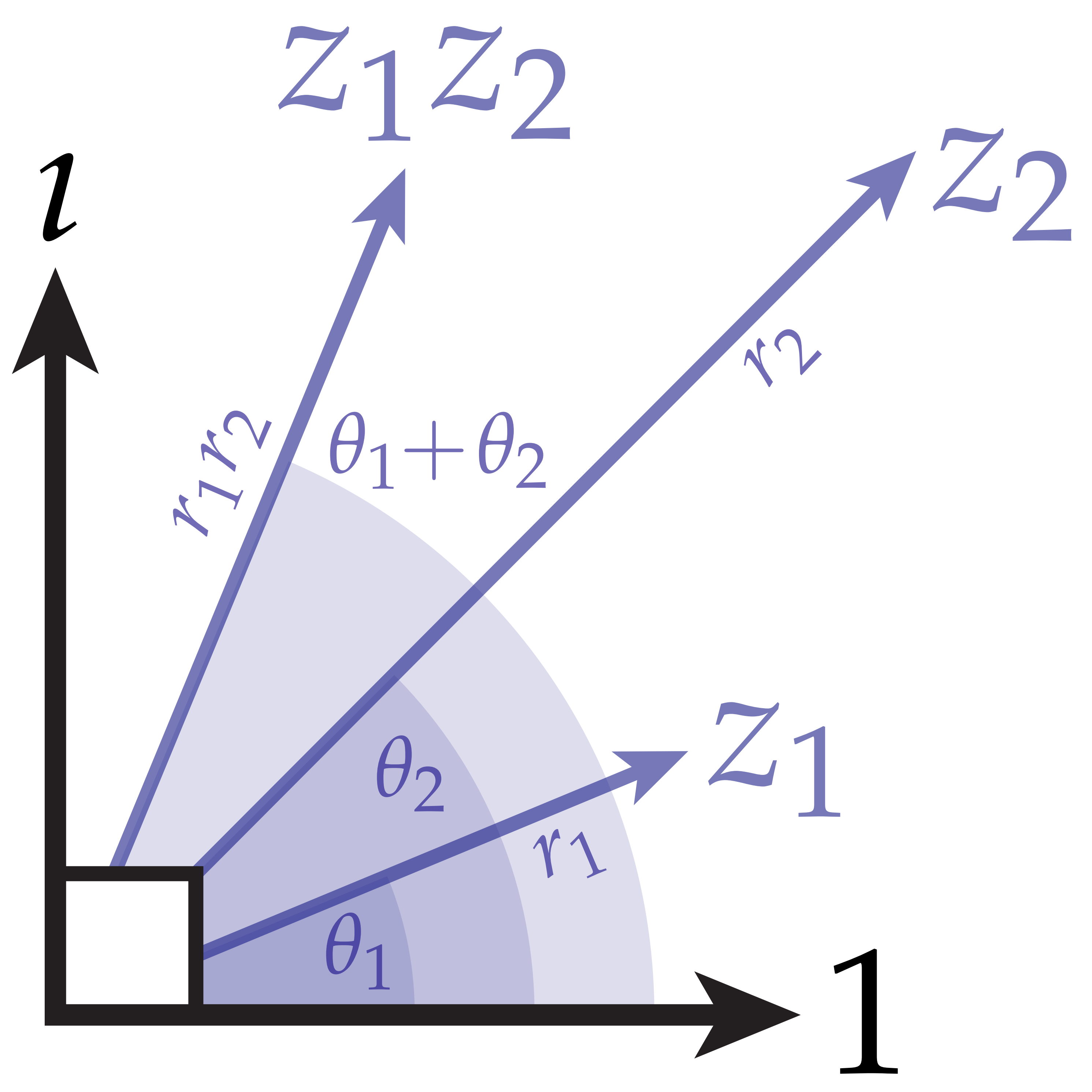
More generally, consider *any* two complex numbers:

$$\begin{aligned} z_1 &:= r_1 e^{\imath\theta_1} \\ z_2 &:= r_2 e^{\imath\theta_2} \end{aligned}$$

We can express their product as

$$z_1 z_2 = (r_1 e^{\imath\theta_1})(r_2 e^{\imath\theta_2}) = r_1 r_2 e^{\imath(\theta_1+\theta_2)}$$

- New angle is *sum* of angles
- New radius is *product* of radii

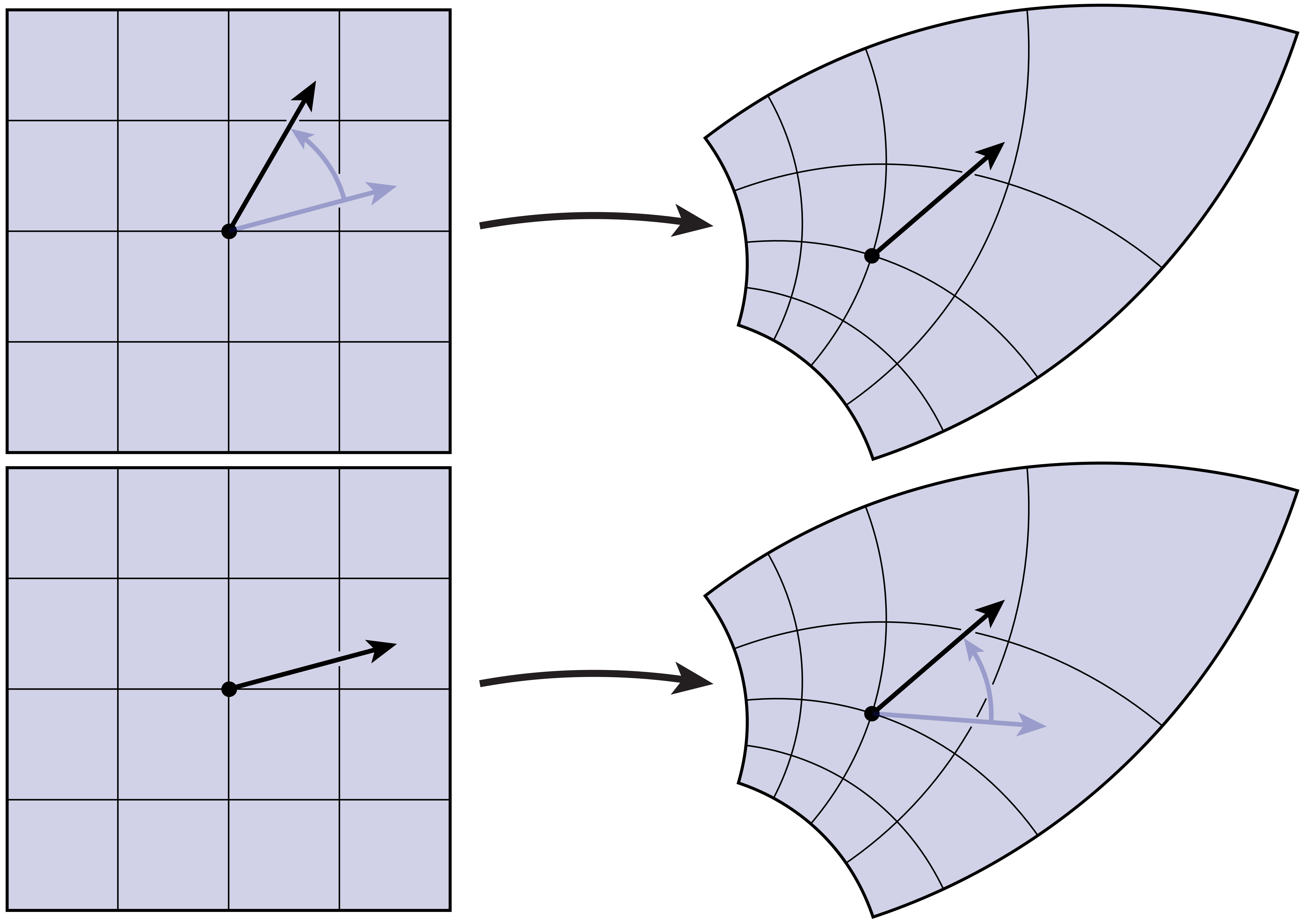


*(Now forget the algebra and remember the **geometry!**)*

Conformal Map, Revisited

- A map is conformal if two operations are equivalent:

1. rotate, then push forward vector
2. push forward vector, then rotate



(How can we write this condition more explicitly?)

Conformal Map, Revisited

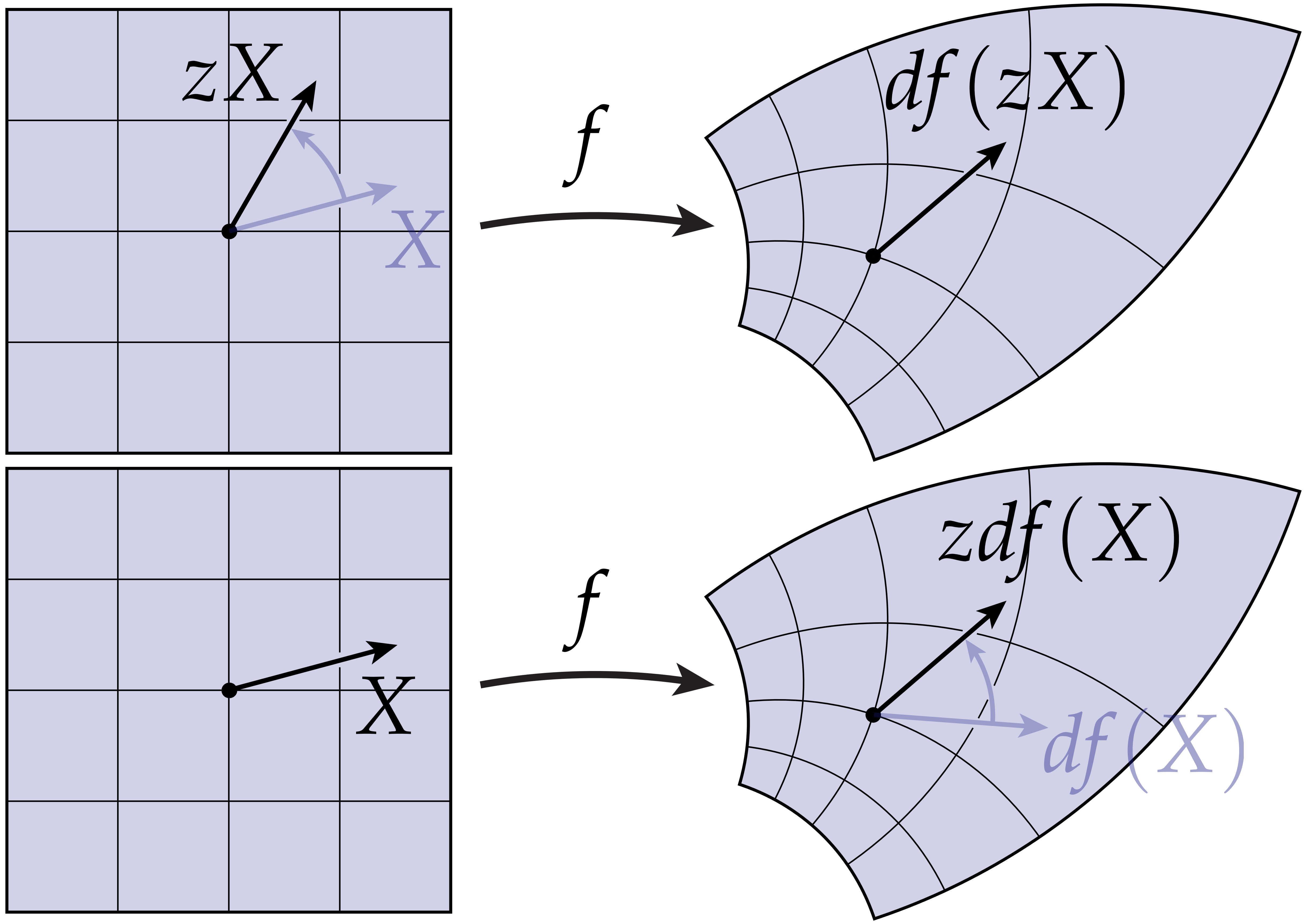
Consider a map $f : \mathbb{C} \to \mathbb{C}$

Then f is conformal as long as

$$df(zX) = zdf(X)$$

for all tangent vectors X and all complex numbers z.

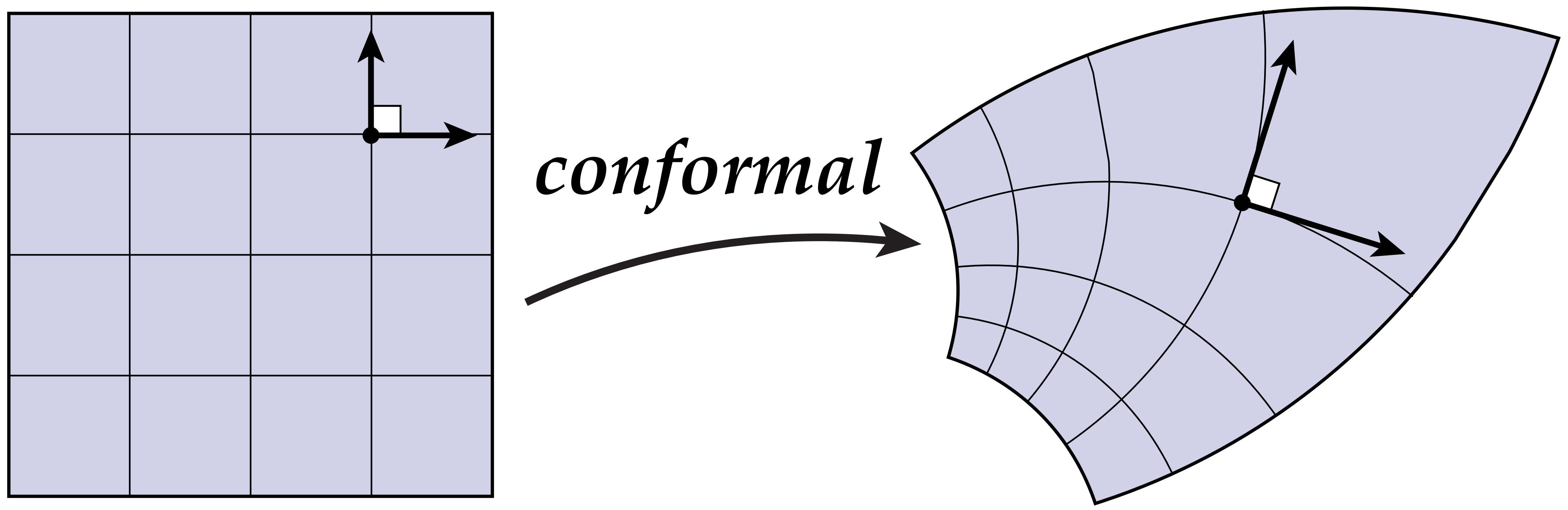
I.e., if it doesn't matter whether you rotate/scale *before* or *after* applying the map.

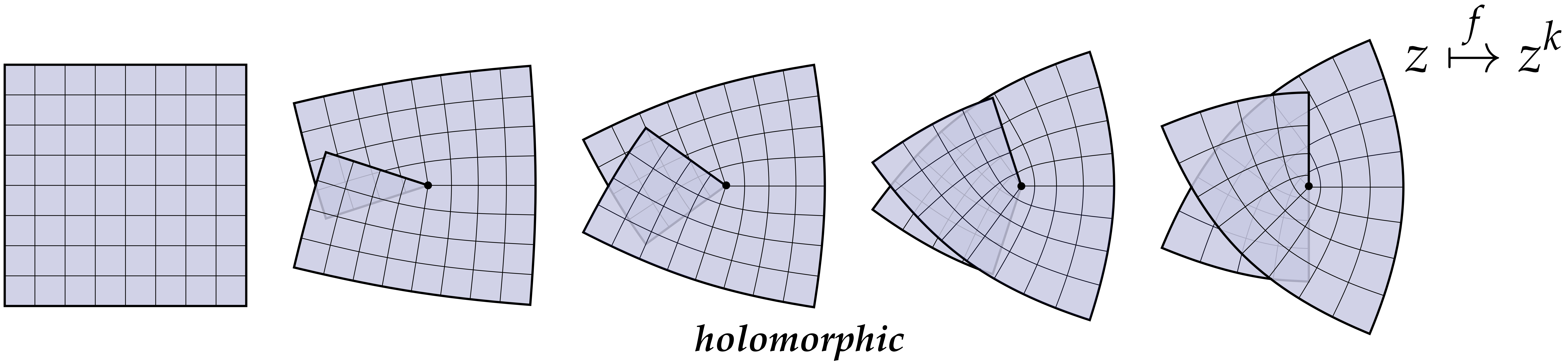


(df is "complex linear")

Holomorphic vs. Conformal

- Important linguistic distinction: a *conformal map* is a holomorphic map that is "nondegenerate", i.e., the differential is never zero.





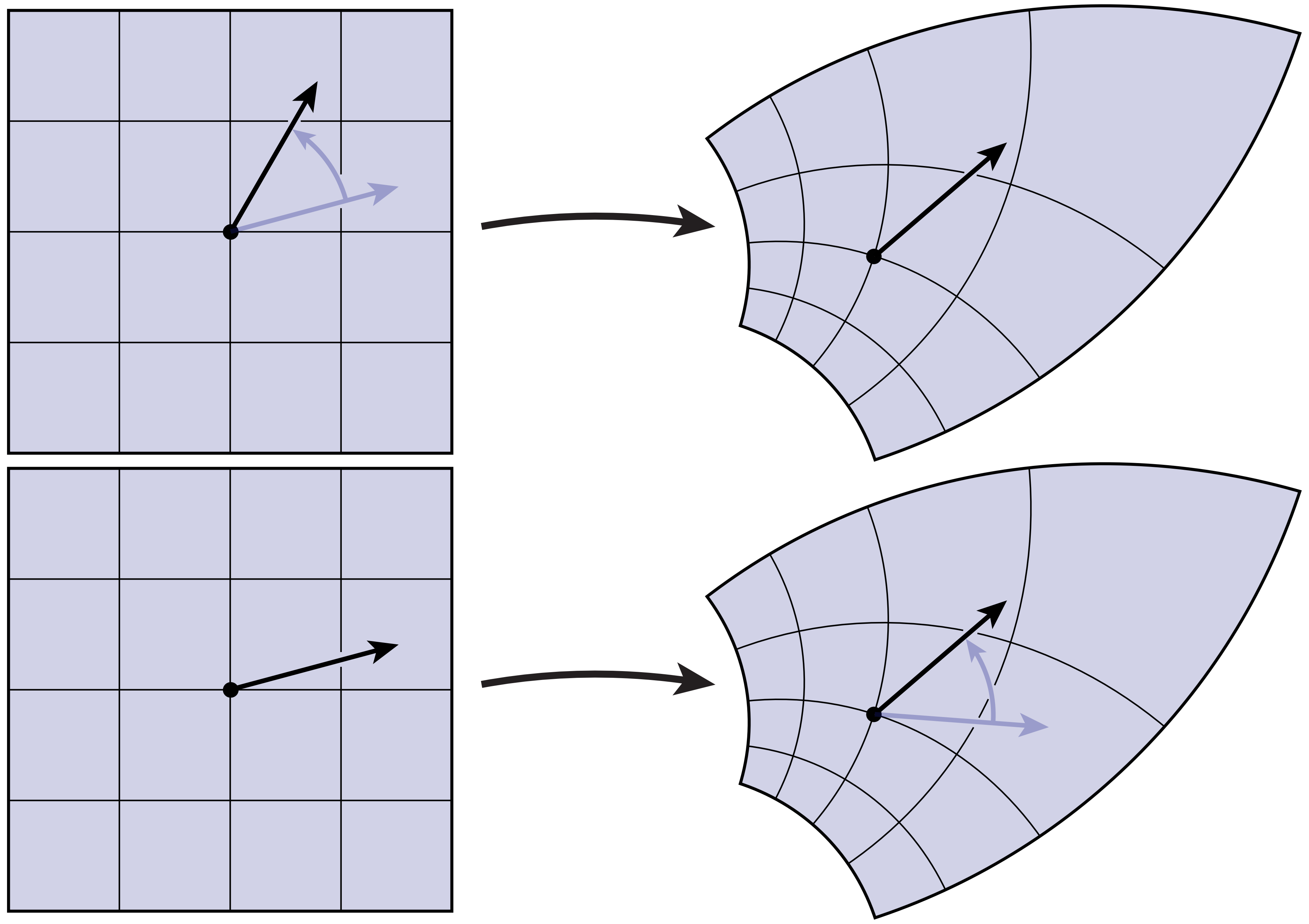
Cauchy-Riemann Equation

Several equivalent ways of writing *Cauchy-Riemann equation*:

$$df(zX) = zdf(X)$$

$$df(\imath X) = \imath df(X)$$

$$\star df = \imath df$$



$$\left.\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{\partial f_2}{\partial y} \\ \frac{\partial f_1}{\partial y} &= -\frac{\partial f_2}{\partial x} \end{aligned}\right\}$$

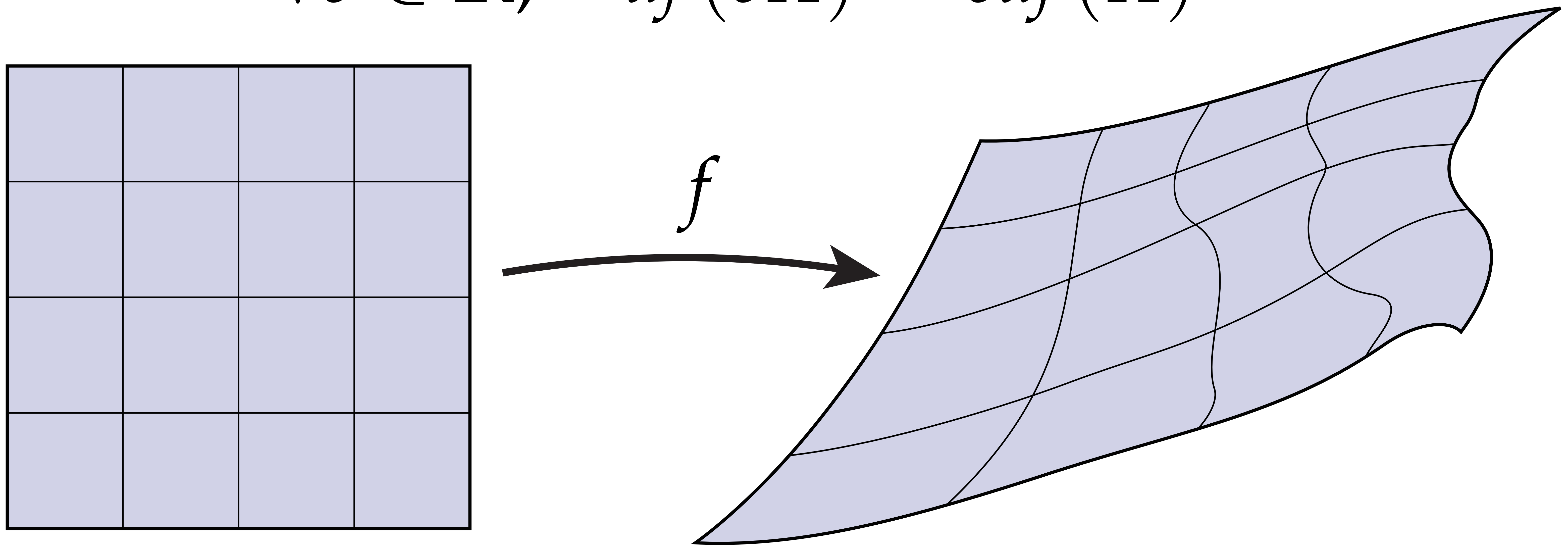
$$\bar{\partial} f = 0$$

All express the same **geometric** idea!

Aside: Real vs. Complex Linearity

What if we just ask for *real* linearity?

$$\forall c \in \mathbb{R}, \quad df(cX) = c\,df(X)$$



No angle preservation.

In fact, maps can be arbitrarily "ugly". Why?

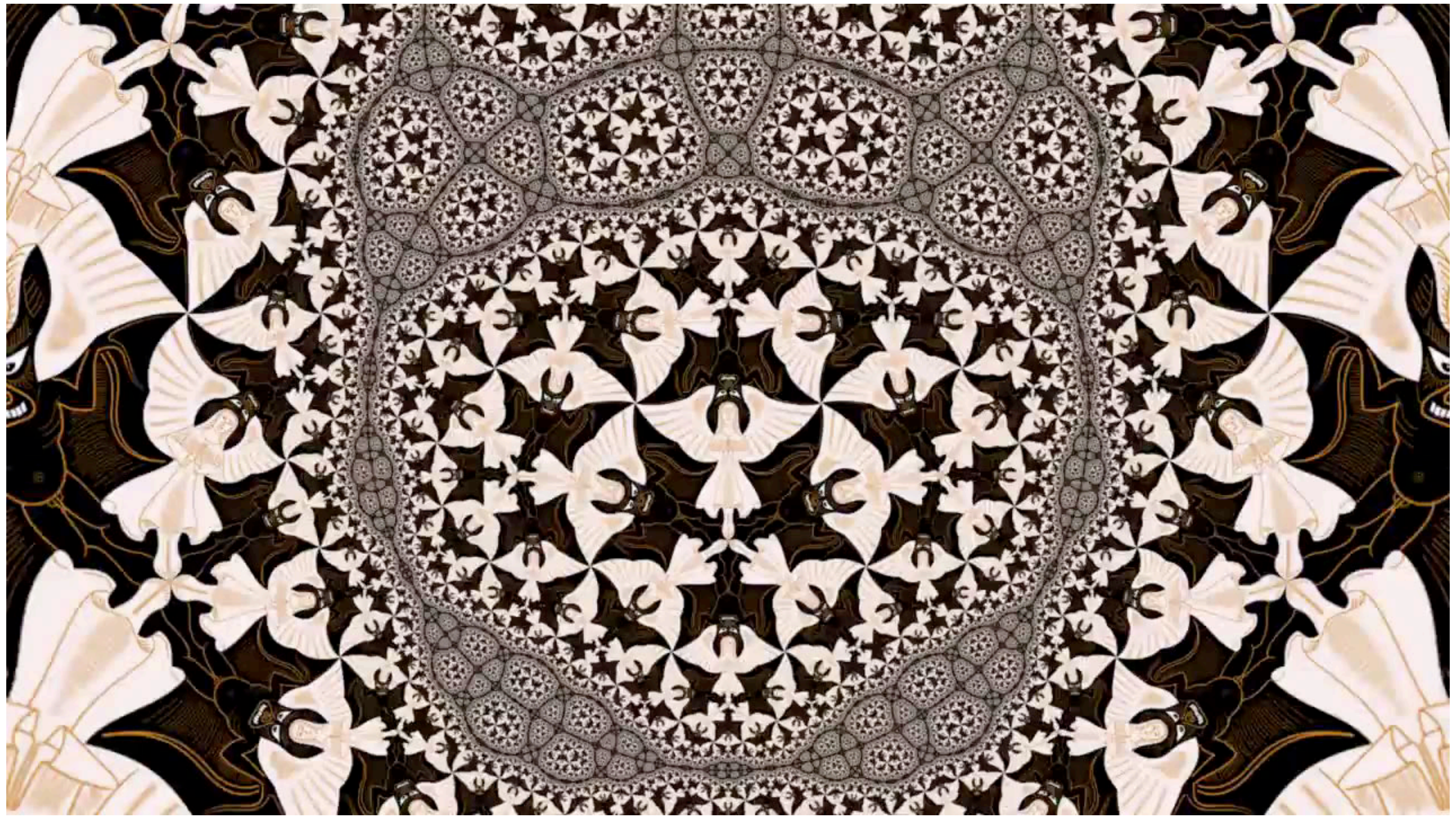
Because *any* differentiable f trivially satisfies this property!

Example—Möbius Transformations (2D)

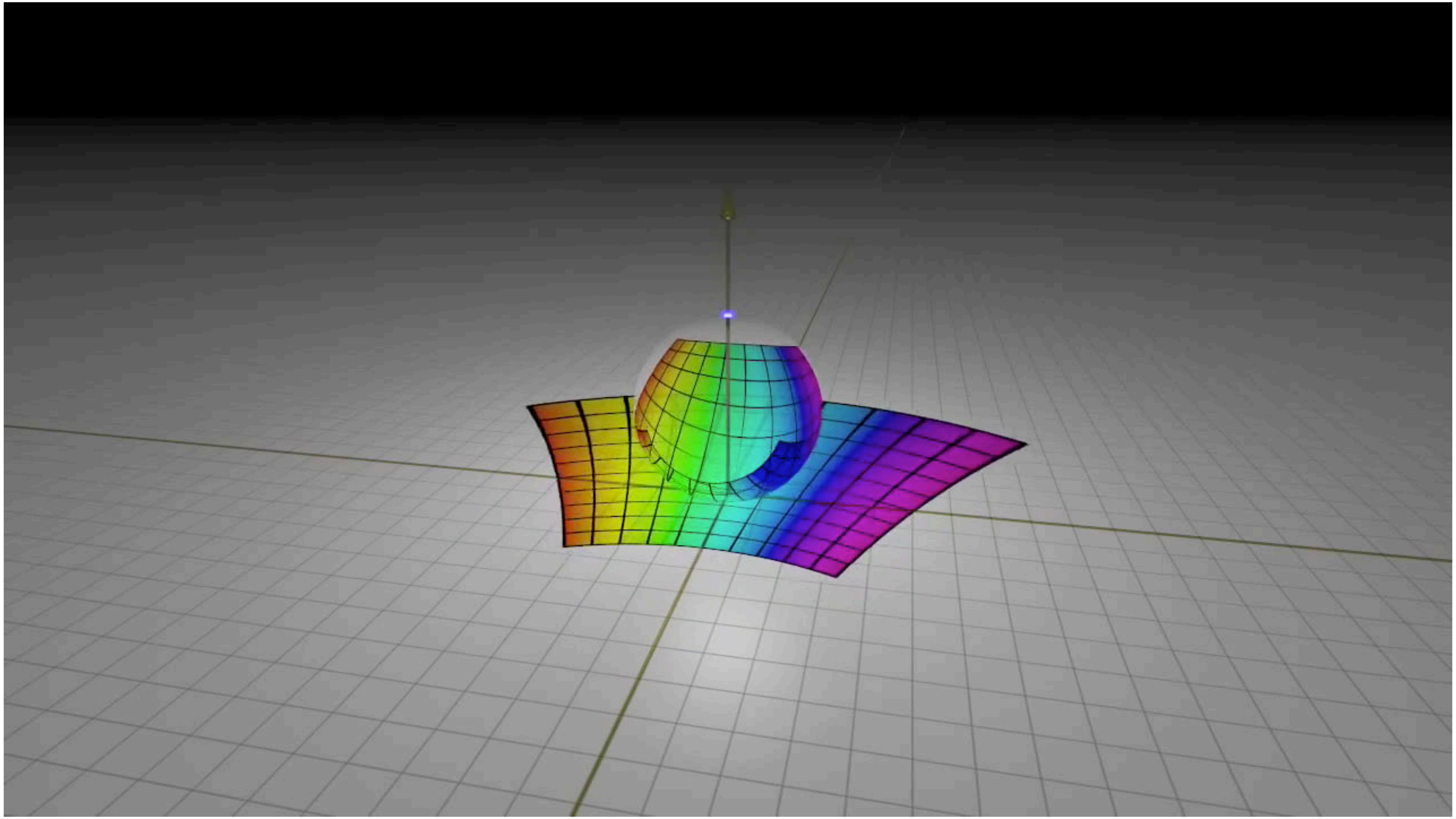
Definition. In 2D, a Möbius transformation is an orientation-preserving map taking circles to circles or lines, and lines to lines or circles. Algebraically, any Möbius transformation can be expressed as a map of the form

$$z \mapsto \frac{az+b}{cz+d}$$

for complex constants $ad \neq bc$.



Möbius Transformations "Revealed"

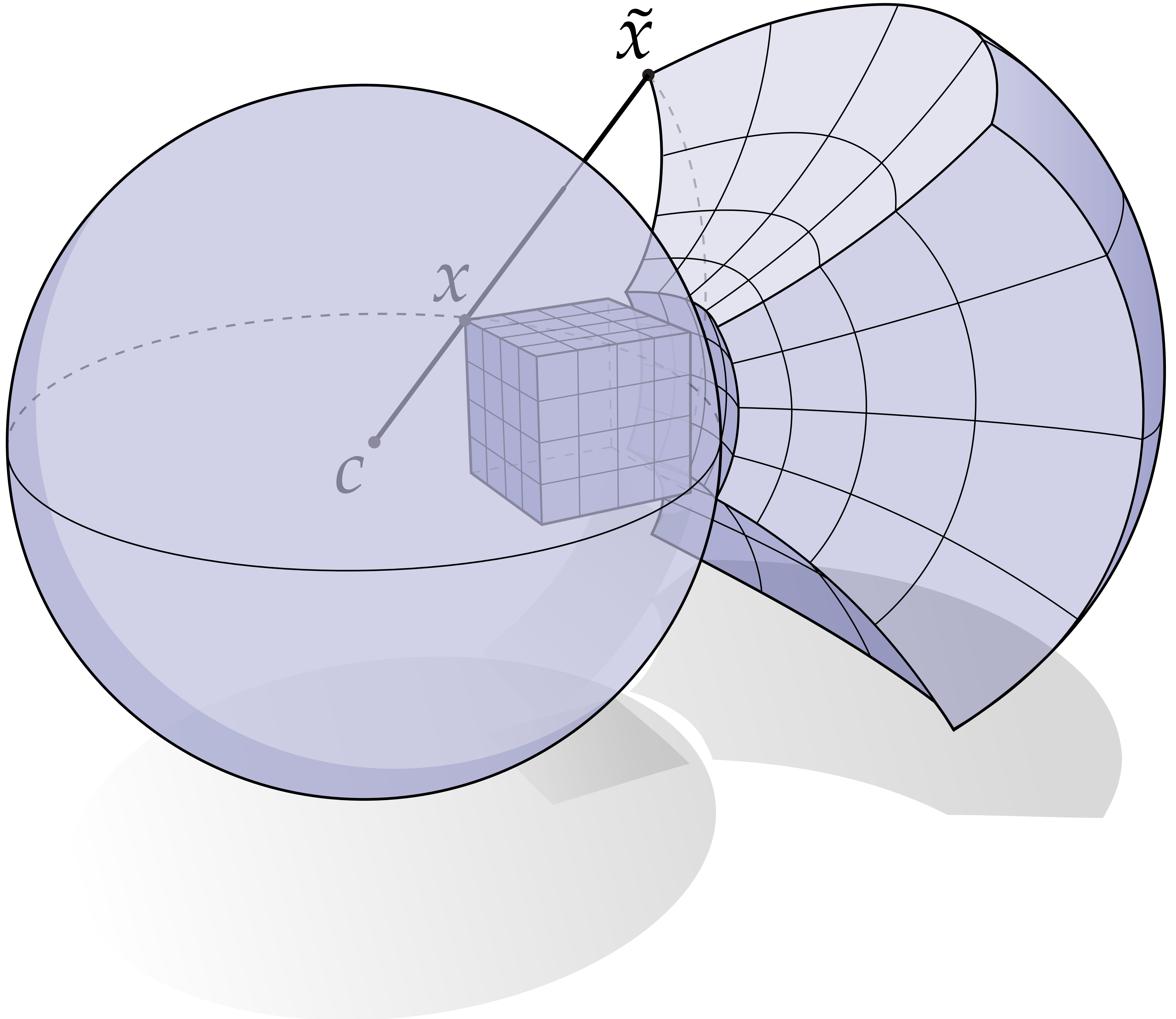


(Douglas Arnold and Jonathan Rogness)

https://www.ima.umn.edu/~arnold/moebius/

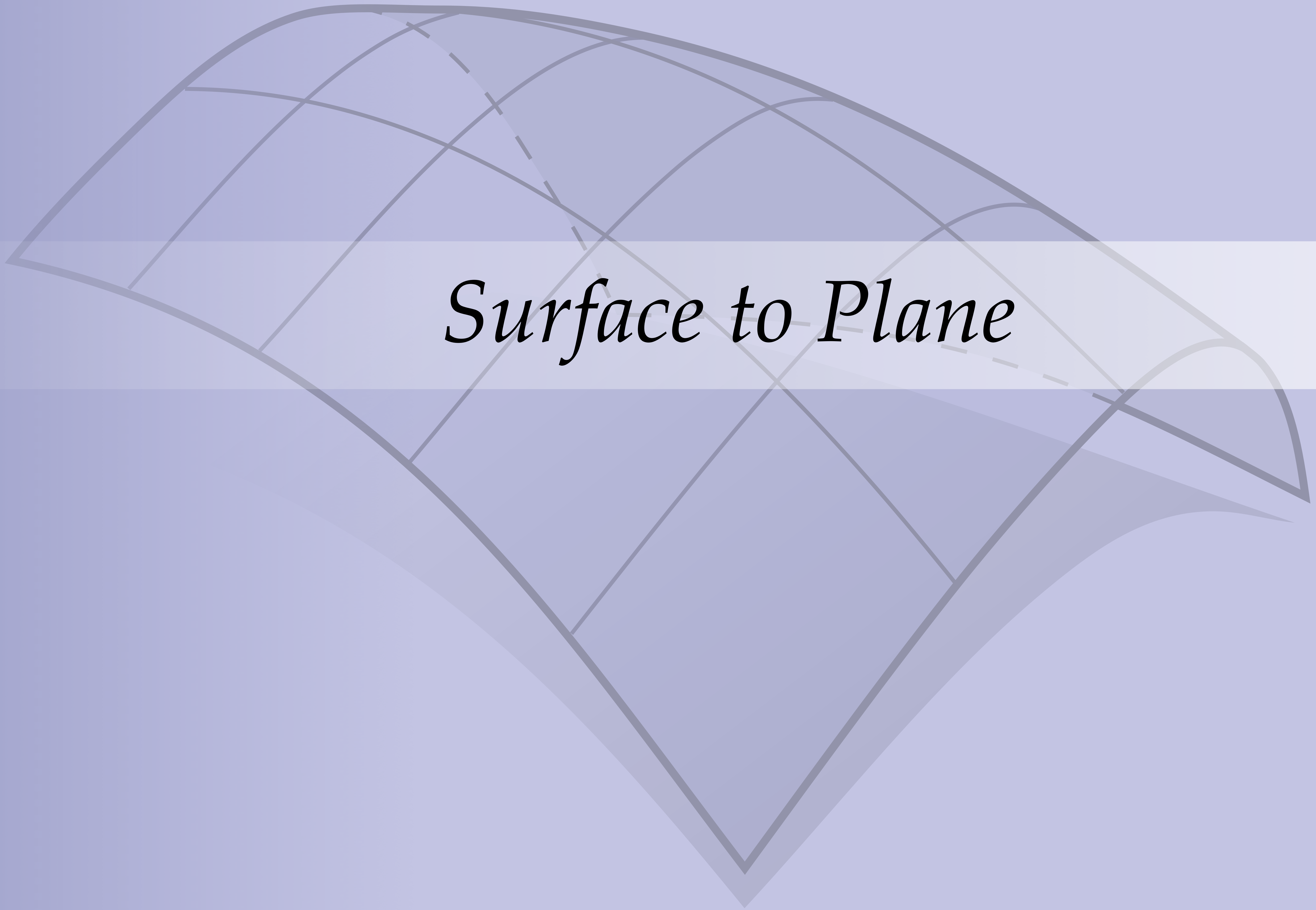
Sphere Inversion (nD)

$$x \mapsto \frac{x - c}{|x - c|^2}$$



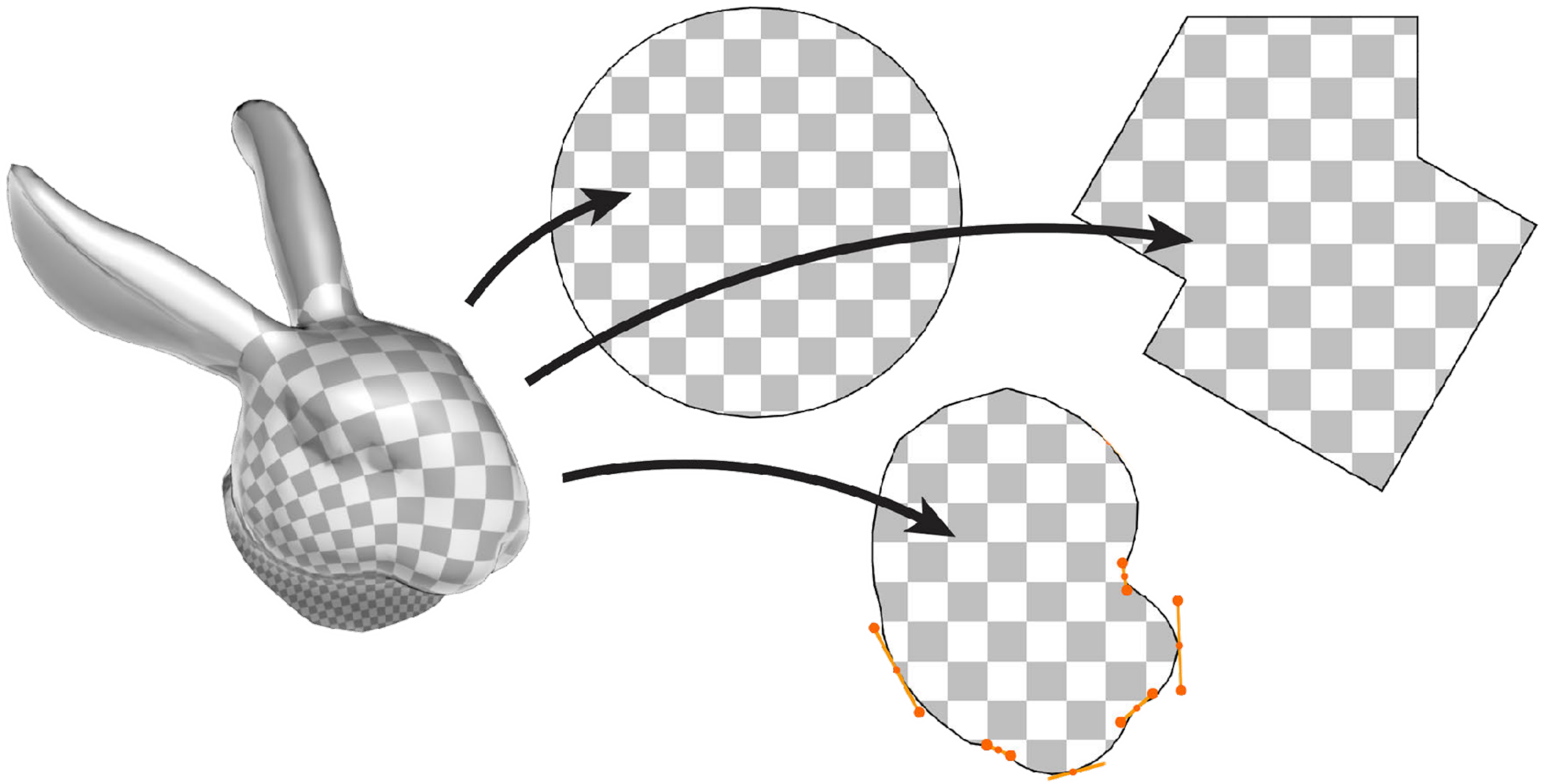
(**Note:** Reverses orientation—*anticonformal* rather than conformal)

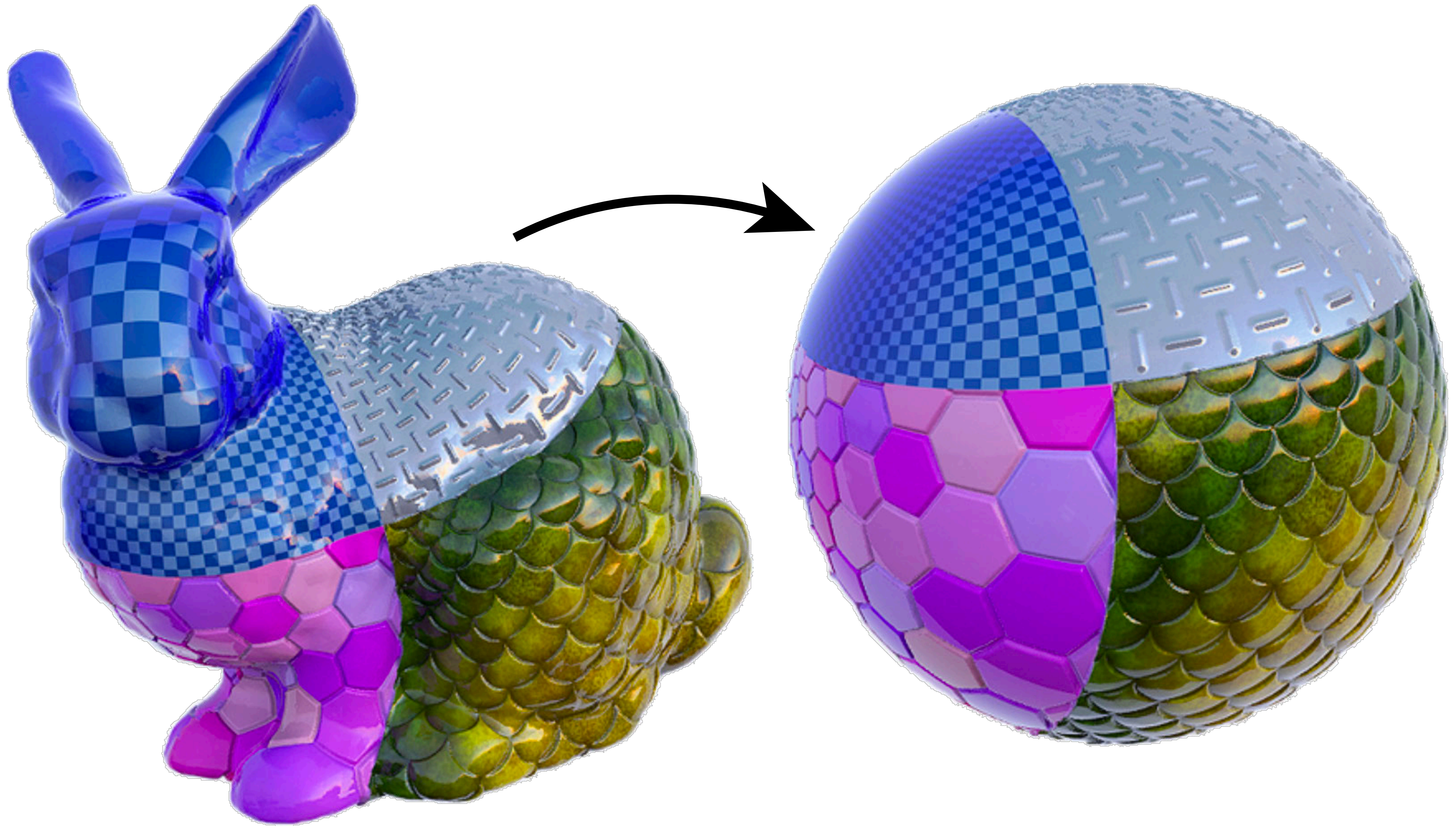
Surface to Plane



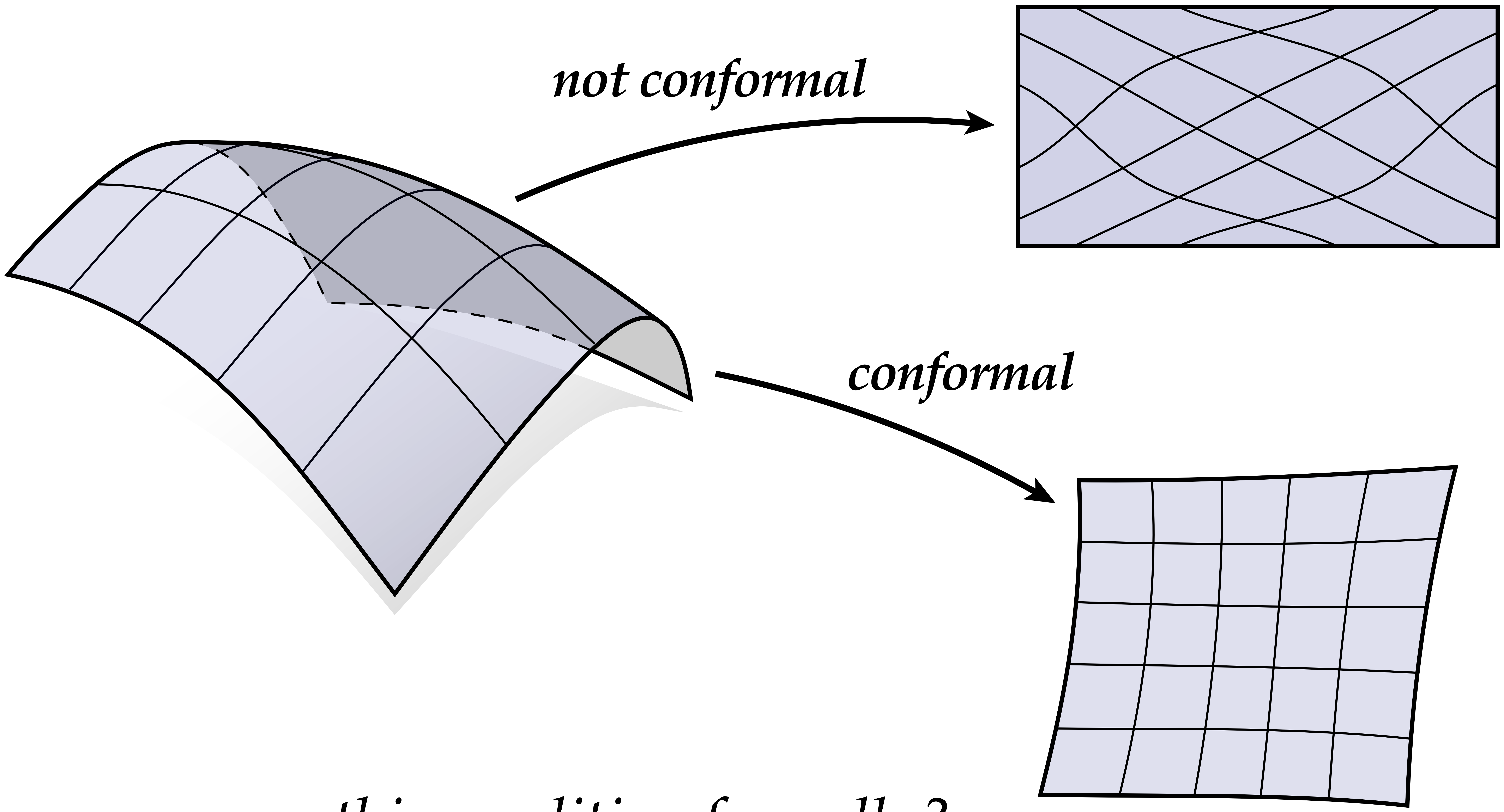
Surface to Plane

- Map curved surface to 2D plane ("conformal flattening")
- Surface does not necessarily sit in 3D
- Slight generalization: target curvature is constant but *nonzero* (*e.g.*, sphere)
- Many different equations: Cauchy-Riemann, Yamabe, …





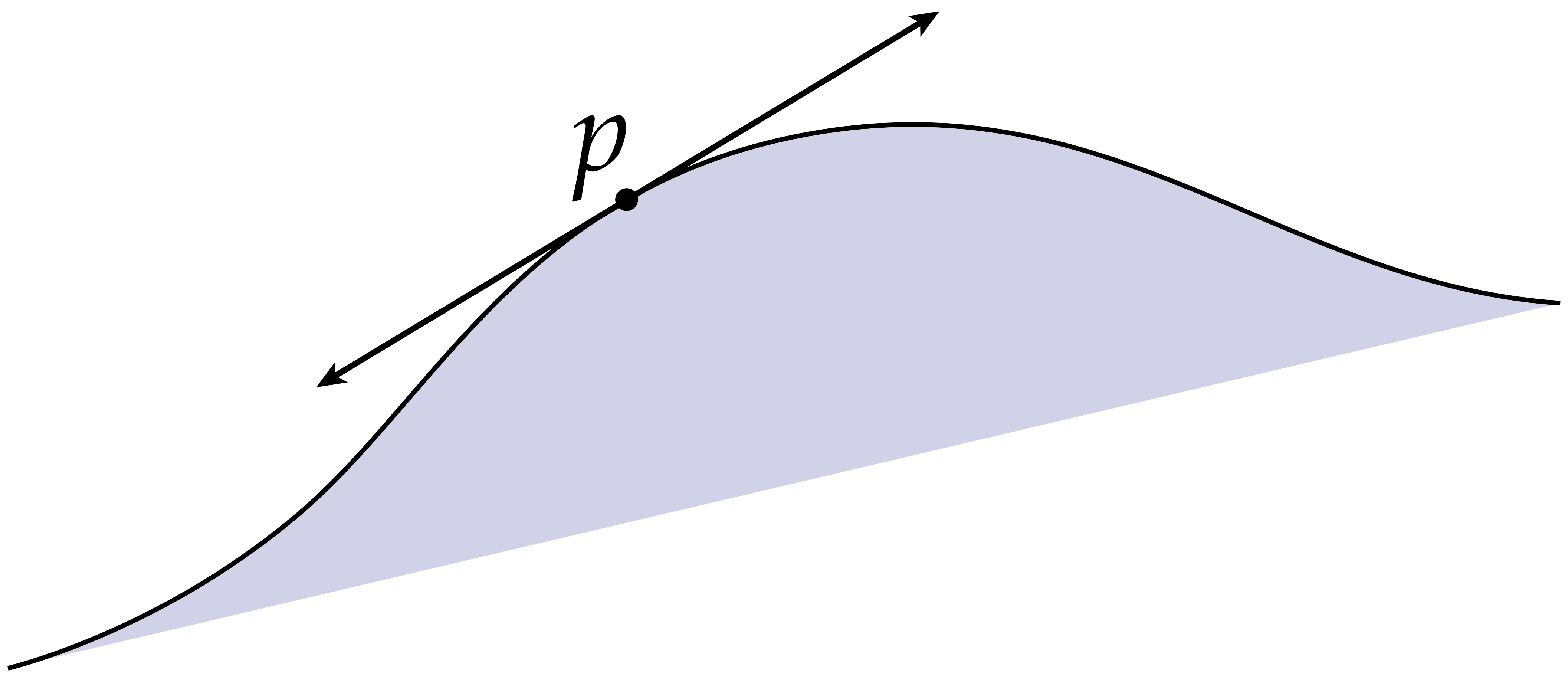
Conformal Maps on Surfaces—Visualized

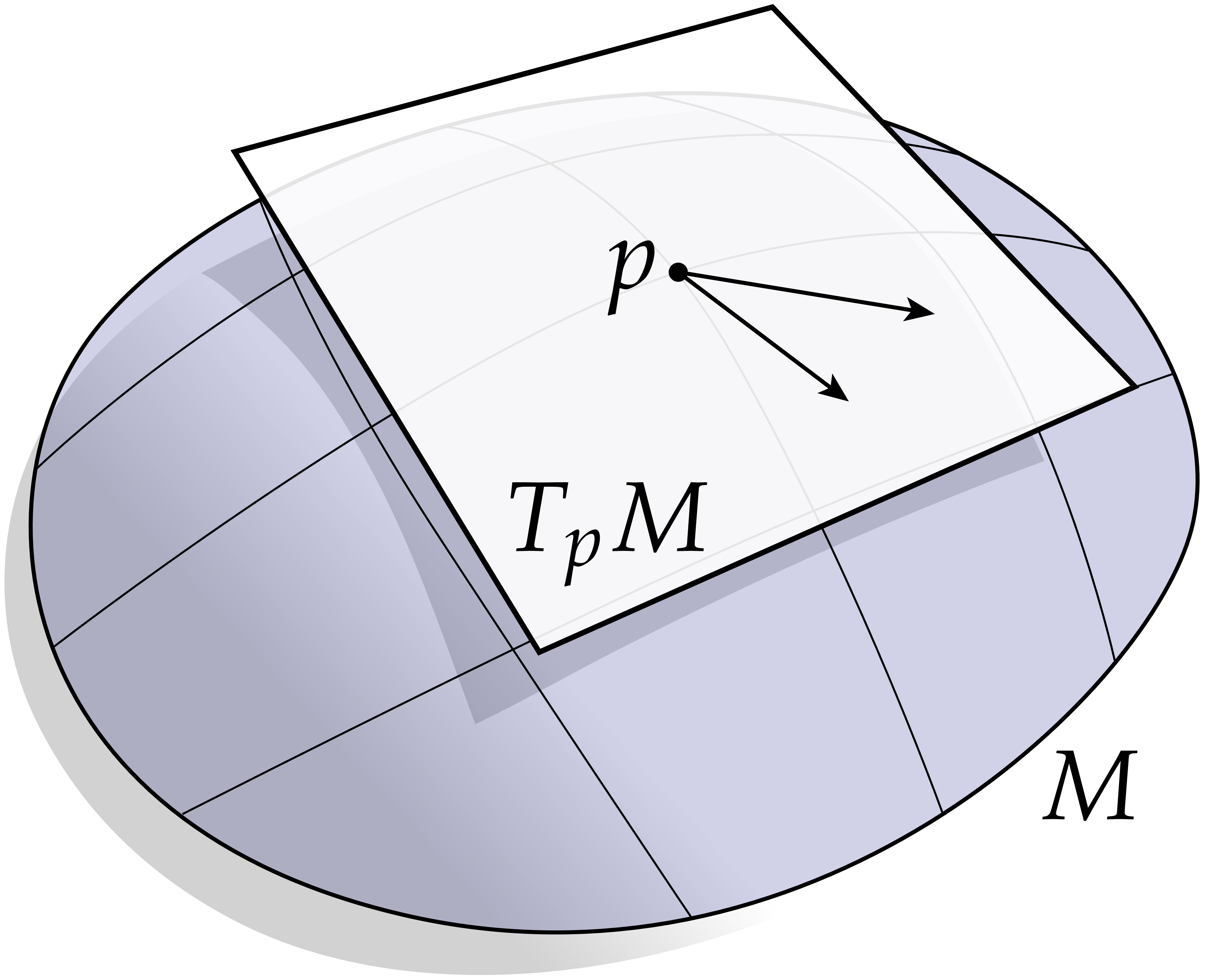


How do we express this condition formally?

Tangent Plane

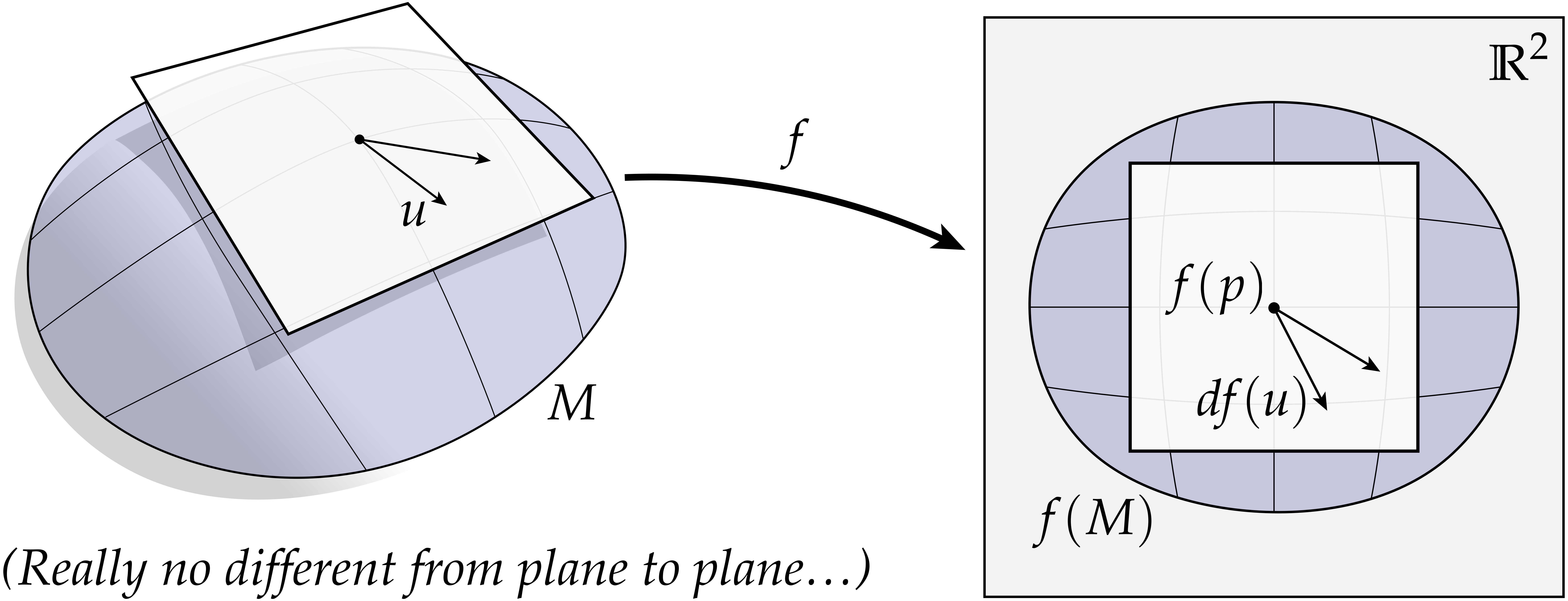
- *Tangent vectors* are those that "graze" the surface
- *Tangent plane* is all the tangent vectors at a given point





Differential of a Map from Surface to Plane

- Consider a map taking each point of a surface to a point in the plane
- *Differential* says how tangent vectors get "stretched out" under this map



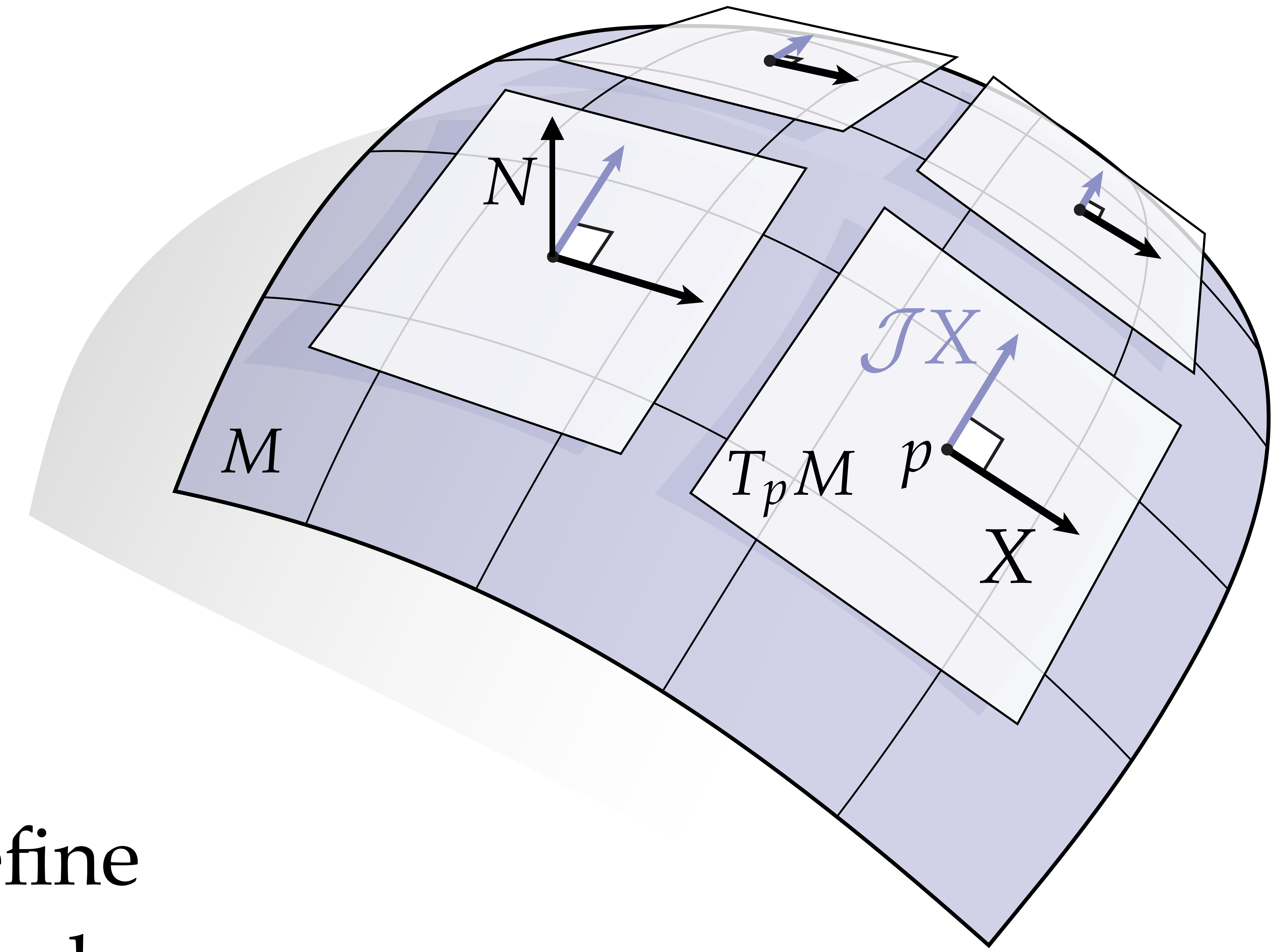
(Really no different from plane to plane…)

Complex Structure

- *Complex structure J* rotates vectors in each tangent plane by 90 degrees
- Analogous to complex unit *i*
- E.g., $\mathcal{J} \circ \mathcal{J} = -\mathrm{id}$
- For a surface in $\mathbb{R}^3$:

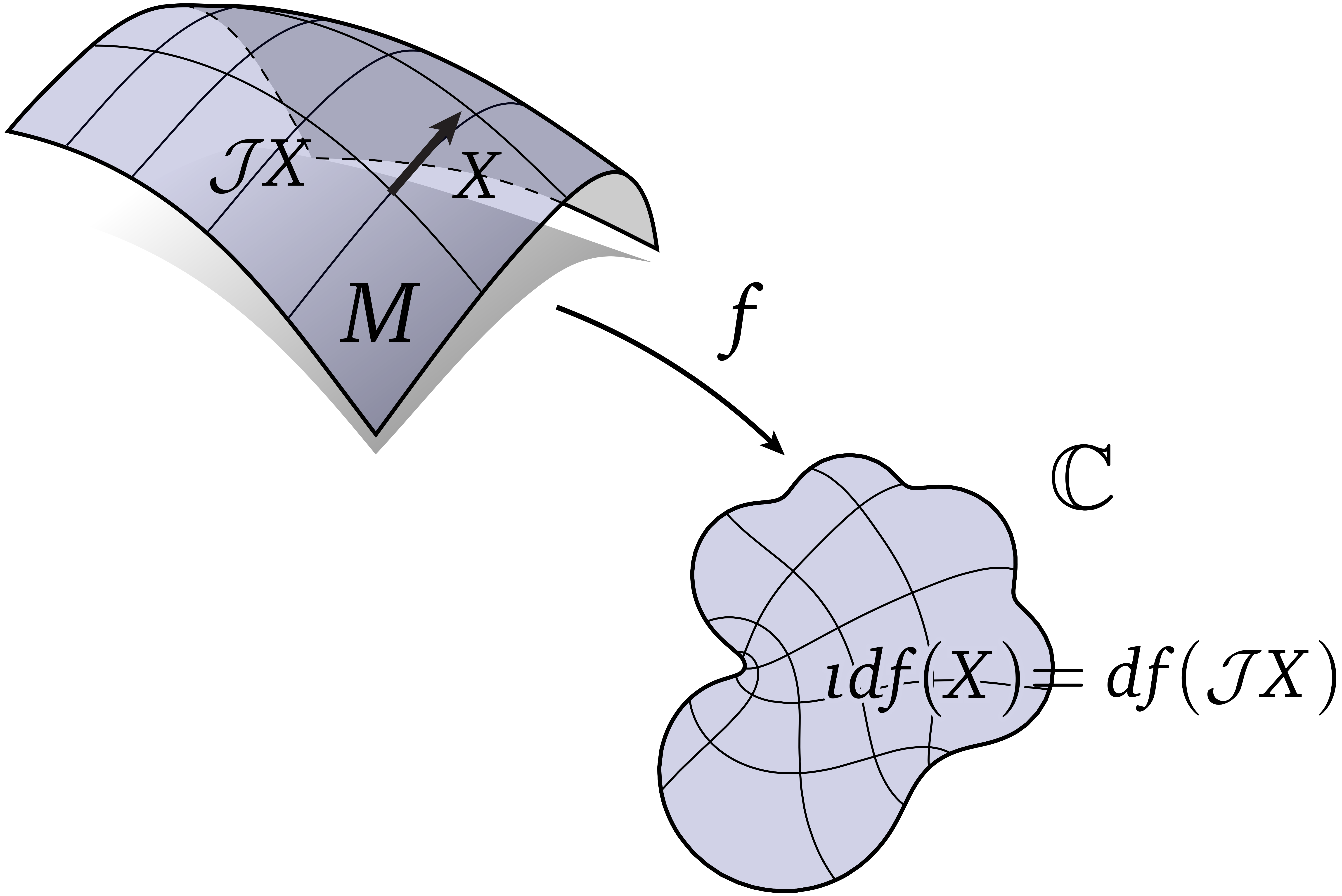
$$\mathcal{J} X = N \times X$$

(where N is unit normal)



Motivation: will enable us to define conformal maps from surface to plane.

Holomorphic Maps from a Surface to the Plane



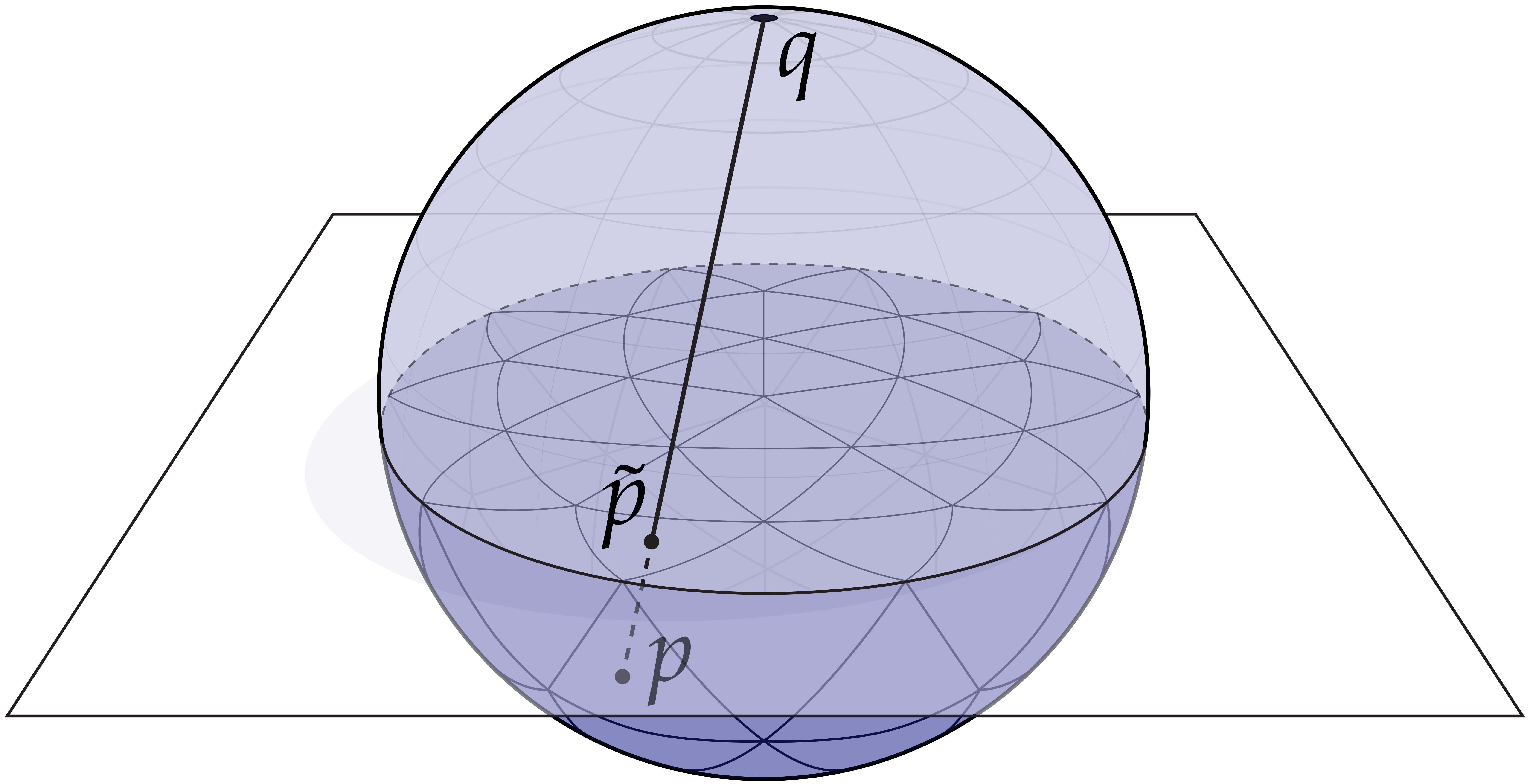
Plane to plane:

$$df(\imath X) = \imath df(X)$$

Surface to plane:

$$df(\mathcal{J}X) = \imath df(X)$$

Example—Stereographic Projection

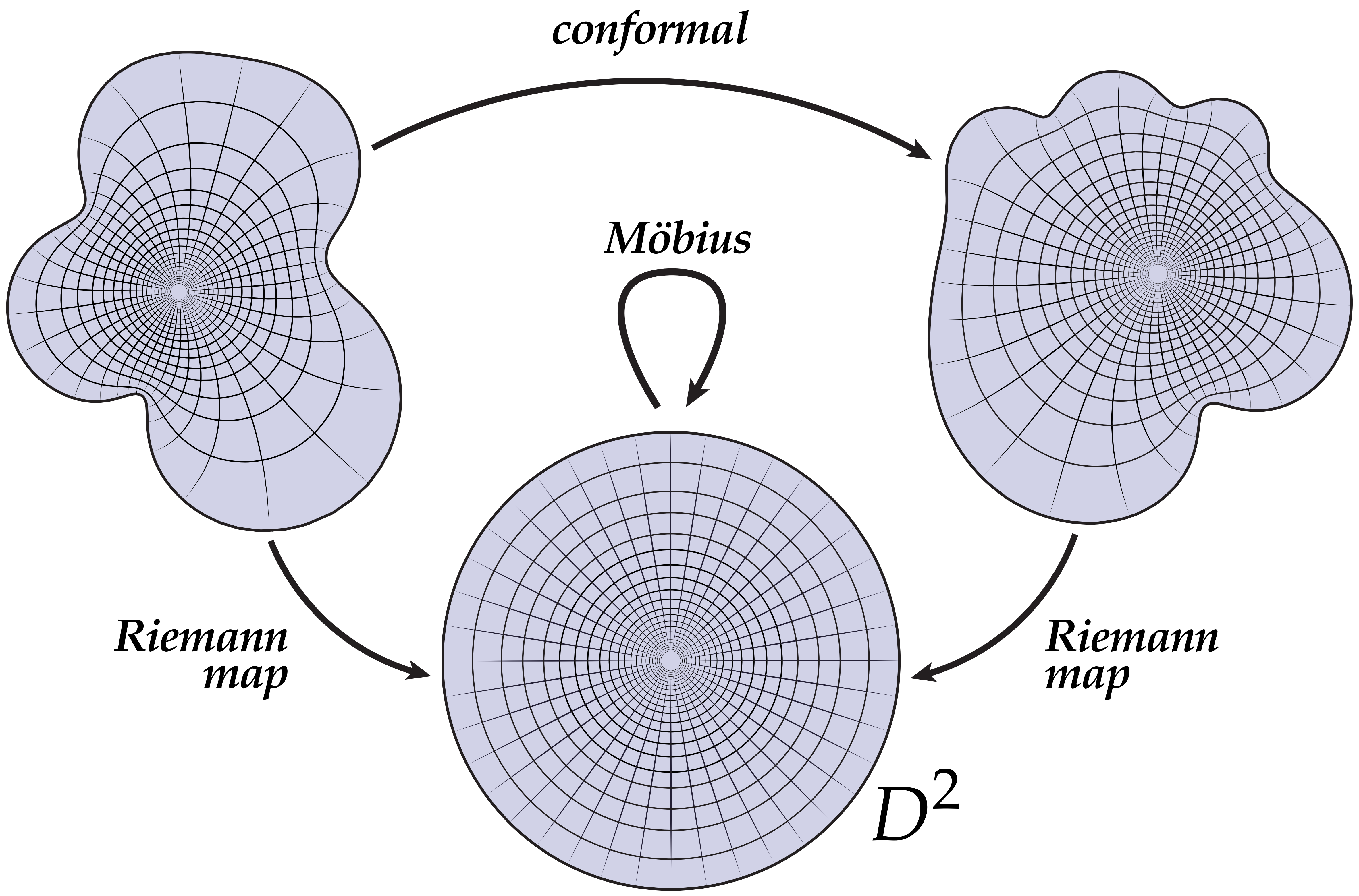


How? Don't memorize some formula—*derive it yourself!*

E.g., What's the equation for a sphere? What's the equation for a ray?

Riemann Mapping Theorem

Theorem (Riemann)**.** Any nonempty simply-connected open proper subset of $\mathbb{C}$ can be conformally mapped to the unit open disk $D^2 := \{z \in \mathbb{C} : |z| < 1\}$.



Fact. The only conformal maps from D^2 to D^2 are Möbius transformations of the form

$$z \mapsto e^{\imath\theta}\frac{z-a}{1-\bar{a}z}$$

where $a \in D^2$ and $\theta \in S^1$ (three degrees of freedom: inversion center and rotation).

Riemannian Metric

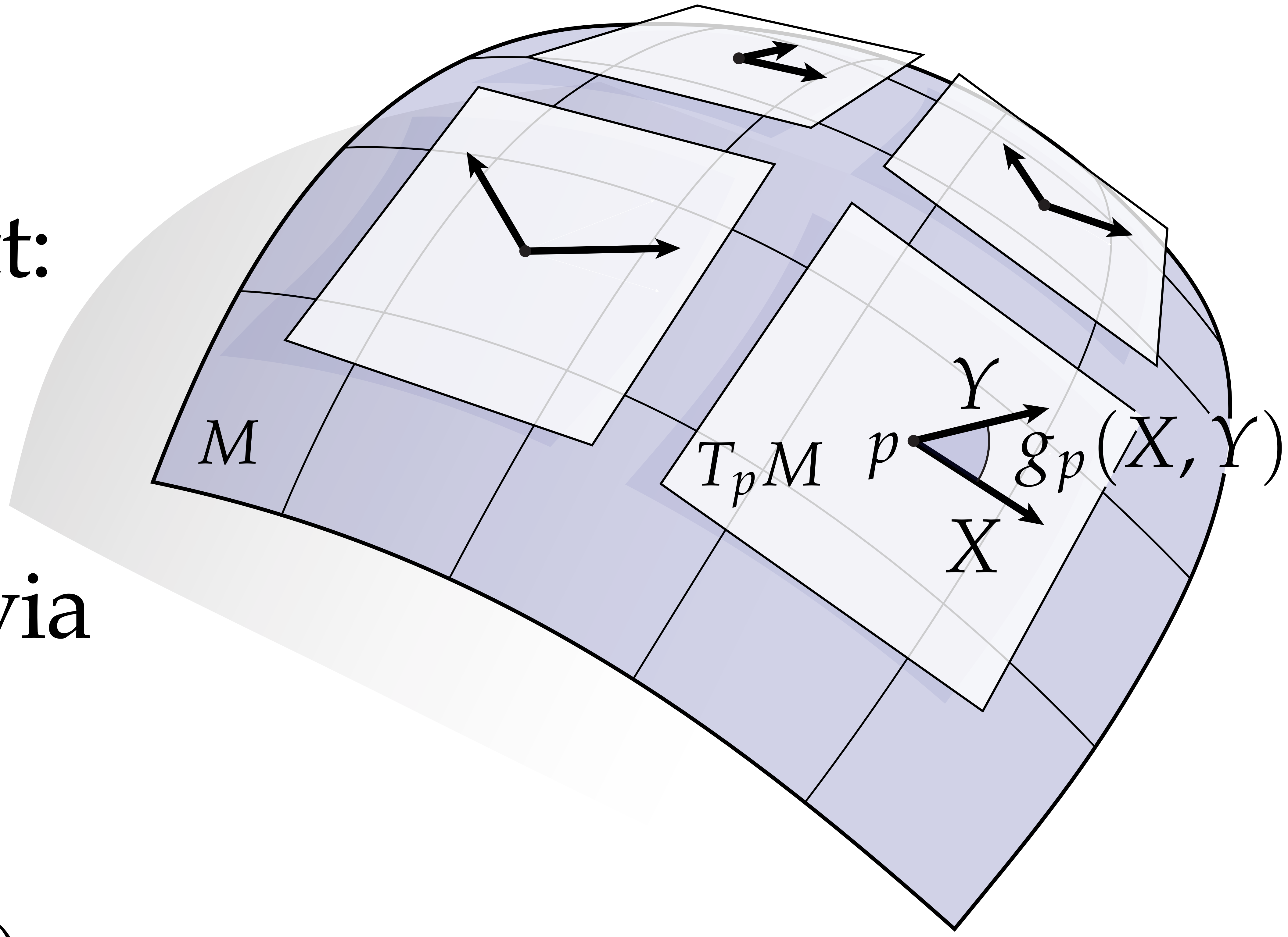
- Can also understand conformal maps in terms of *Riemannian metric*
- Riemannian metric g is simply inner product in each tangent space
- Allows us to measure length, angle, etc.
- E.g., Euclidean metric is just dot product:

$$g_{\mathbb{R}^n}(X,Y) := \sum_i X_i Y_i$$

- In general, length and angle recovered via

$$|X| := \sqrt{g(X,X)}$$

$$\angle(X,Y) := \arccos(g(X,Y)/|X||Y|)$$

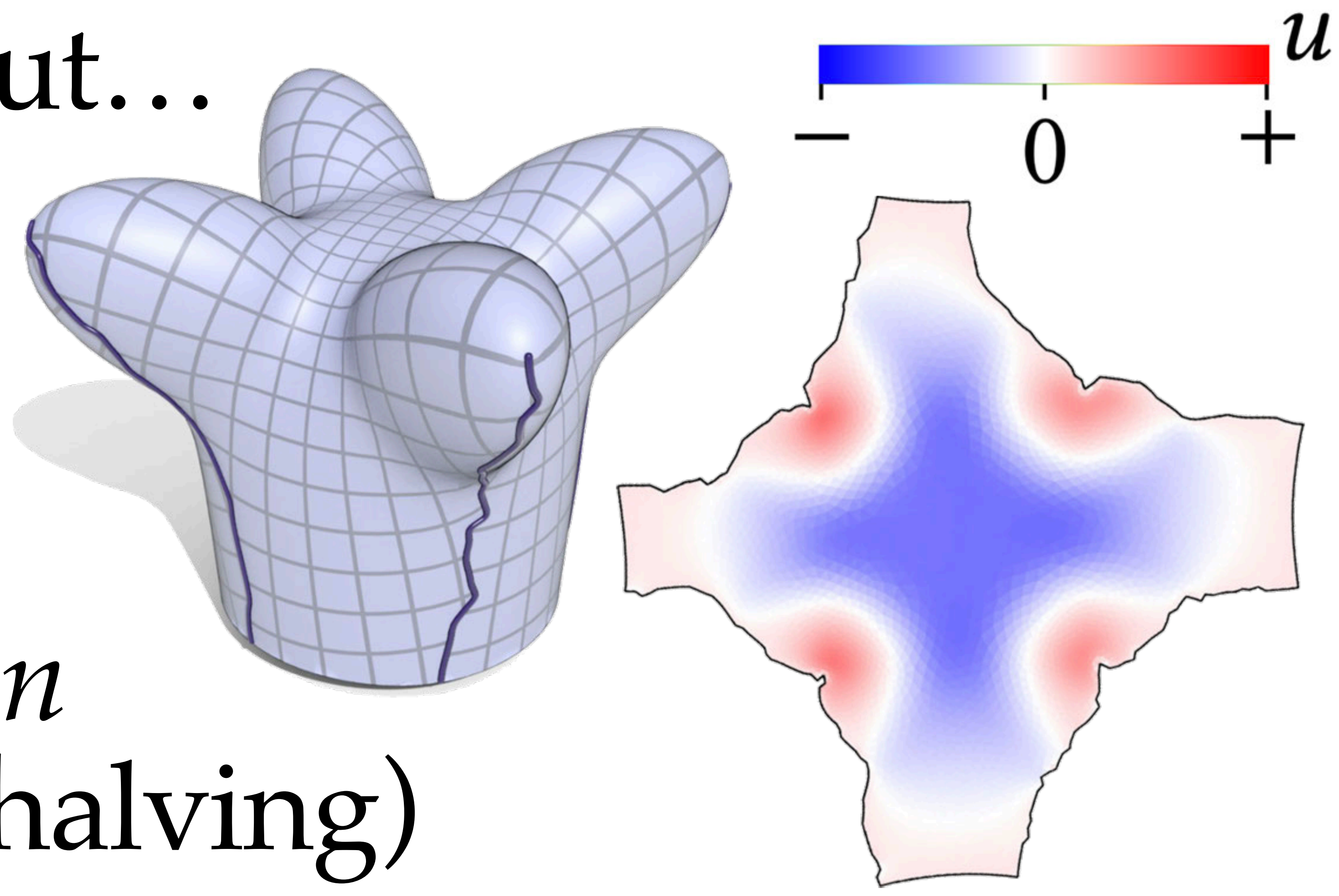


Conformally Equivalent Metrics

- Two metrics are *conformally equivalent* if they are related by a positive **conformal scale factor** at each point p:

$$\boxed{\tilde{g}_p = e^{2u(p)} g_p} \qquad u : M \to \mathbb{R}$$

- Why write scaling as e^{2u}? Initially mysterious, but…
  - ensures scaling is always *positive*
  - factor e^u gives length scaling
  - more natural way of talking about *area distortion* (e.g., doubling in scale "costs" just as much as halving)

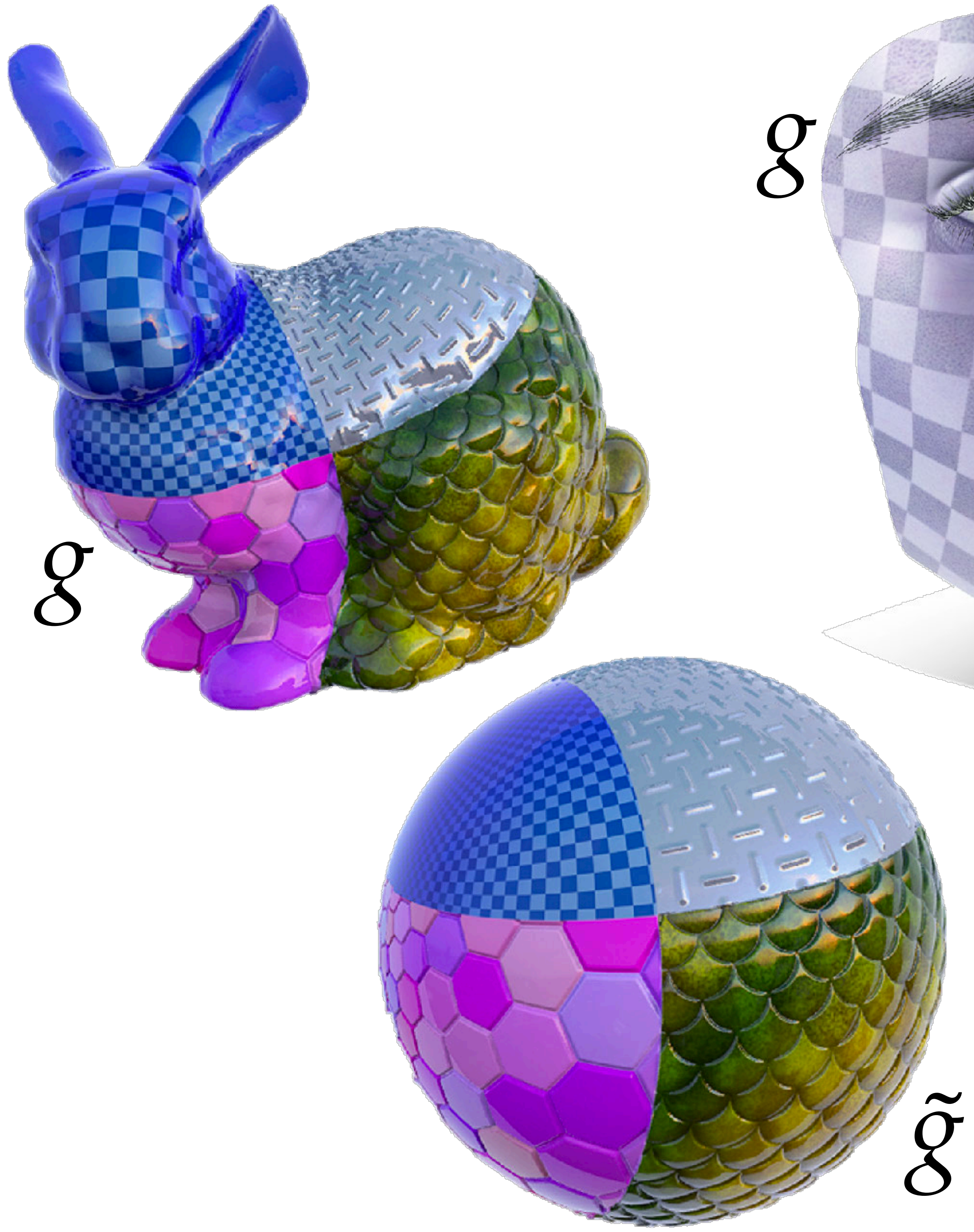


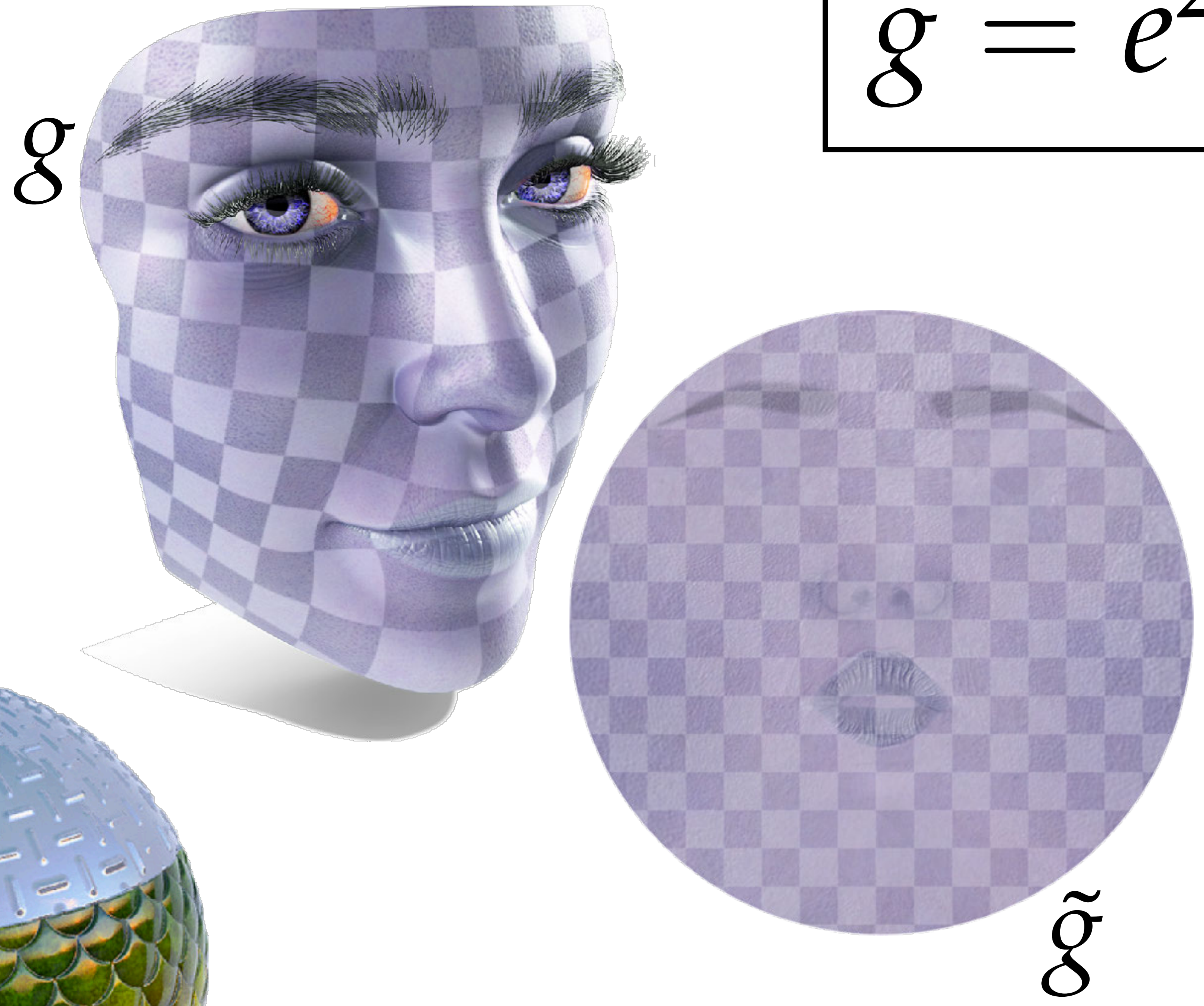
Q: Does this transformation preserve *angles*?

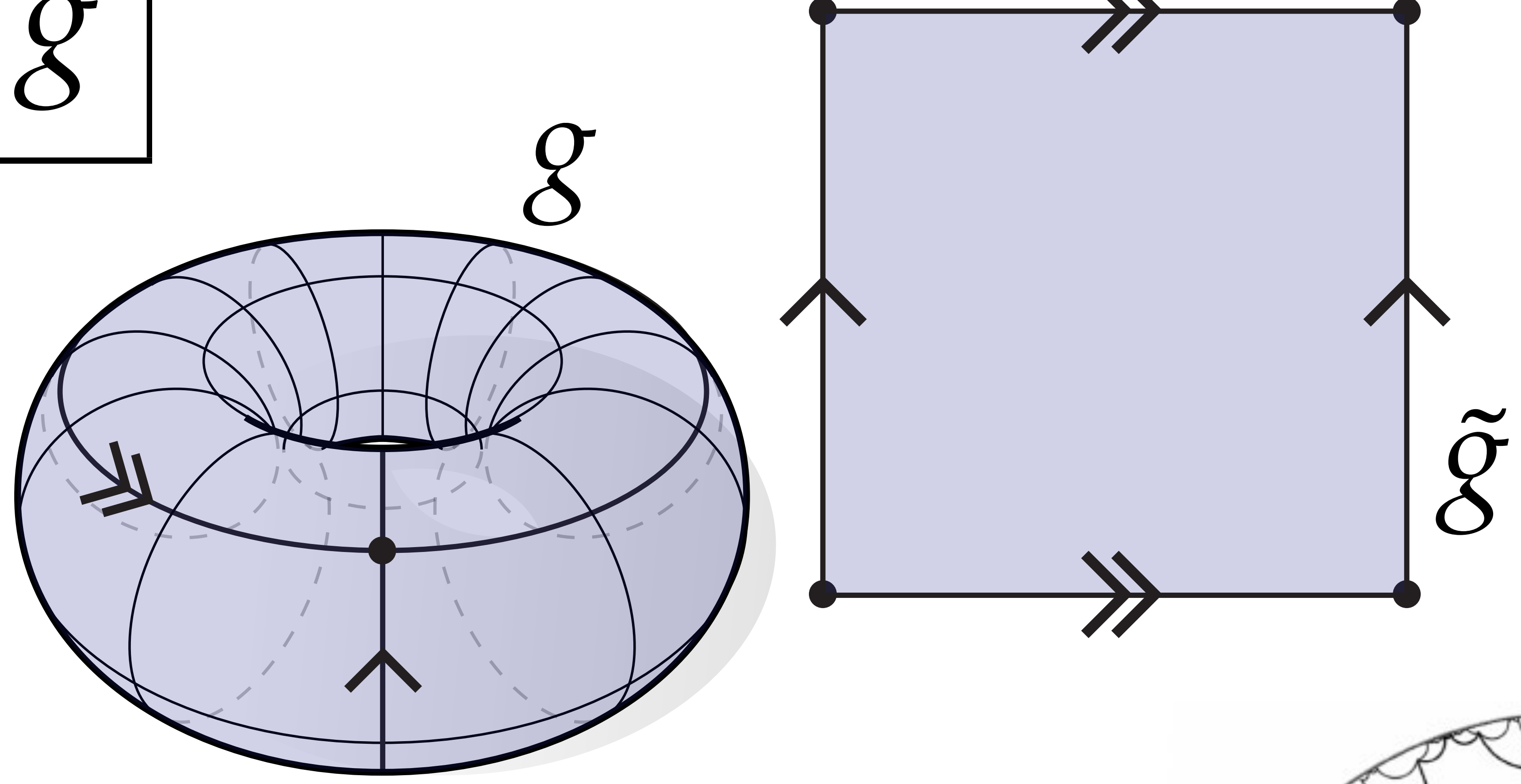
Uniformization Theorem

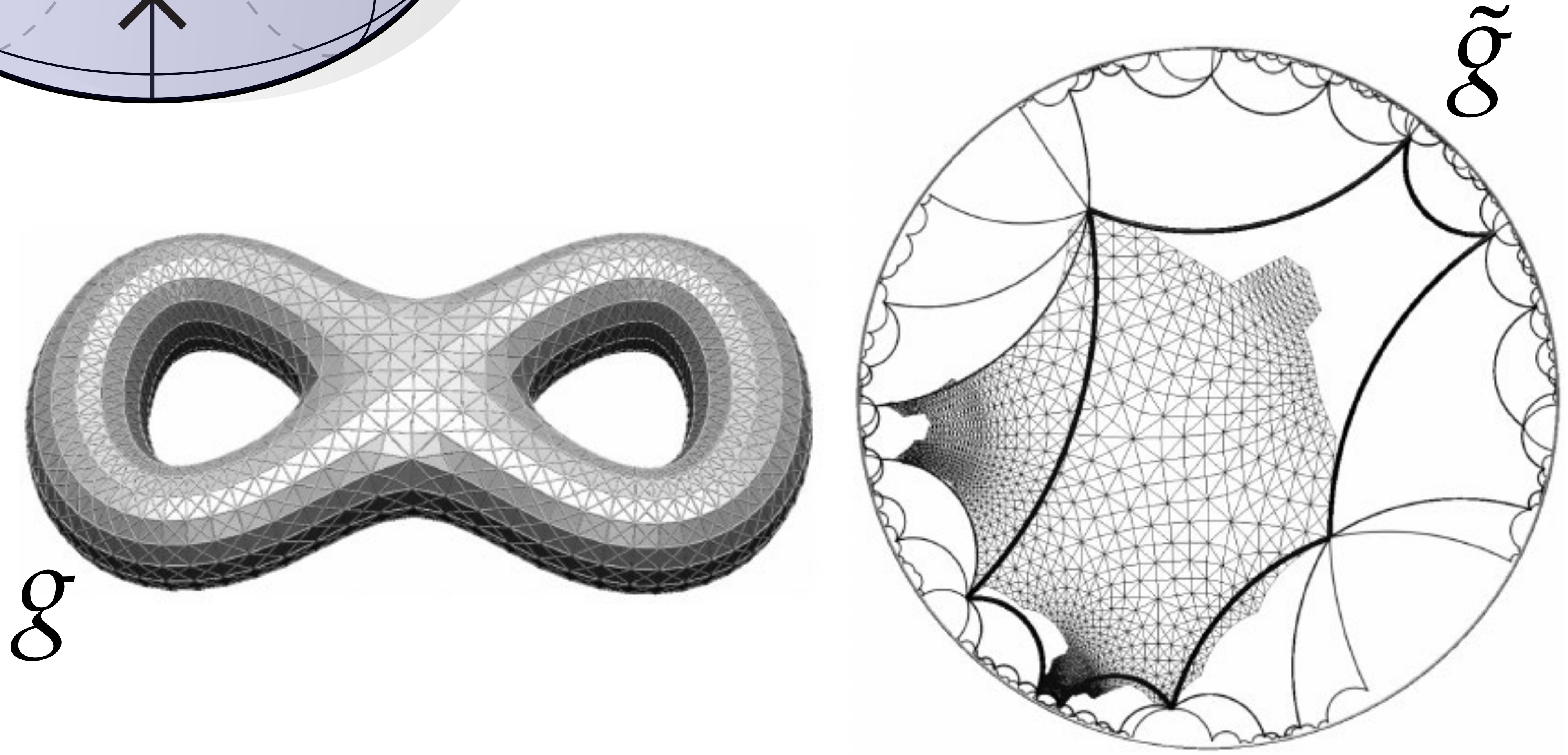
- Roughly speaking, Riemannian metric on any surface is conformally equivalent to one with *constant curvature* (flat, spherical, hyperbolic).

$$\boxed{\tilde{g} = e^{2u} g}$$



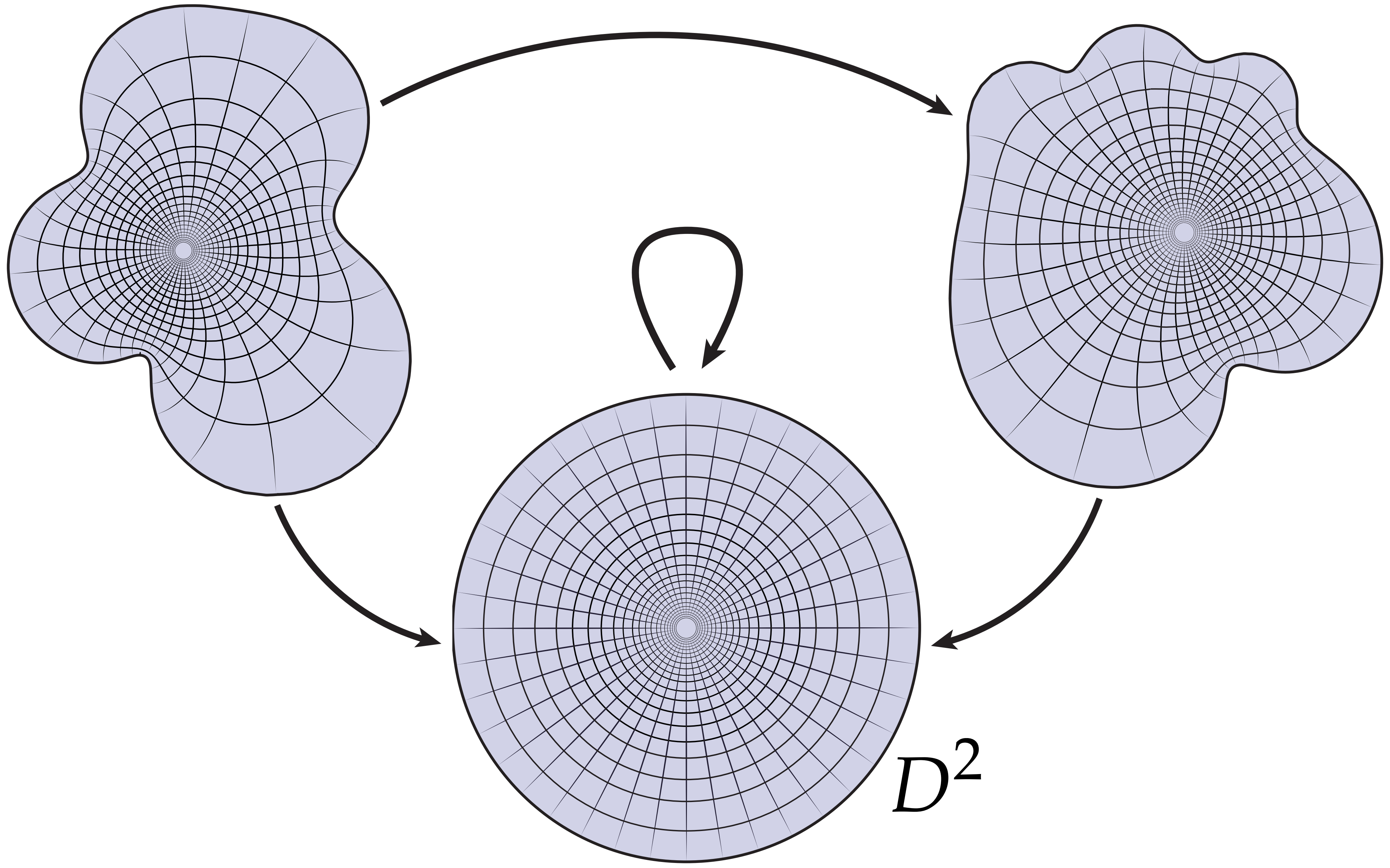


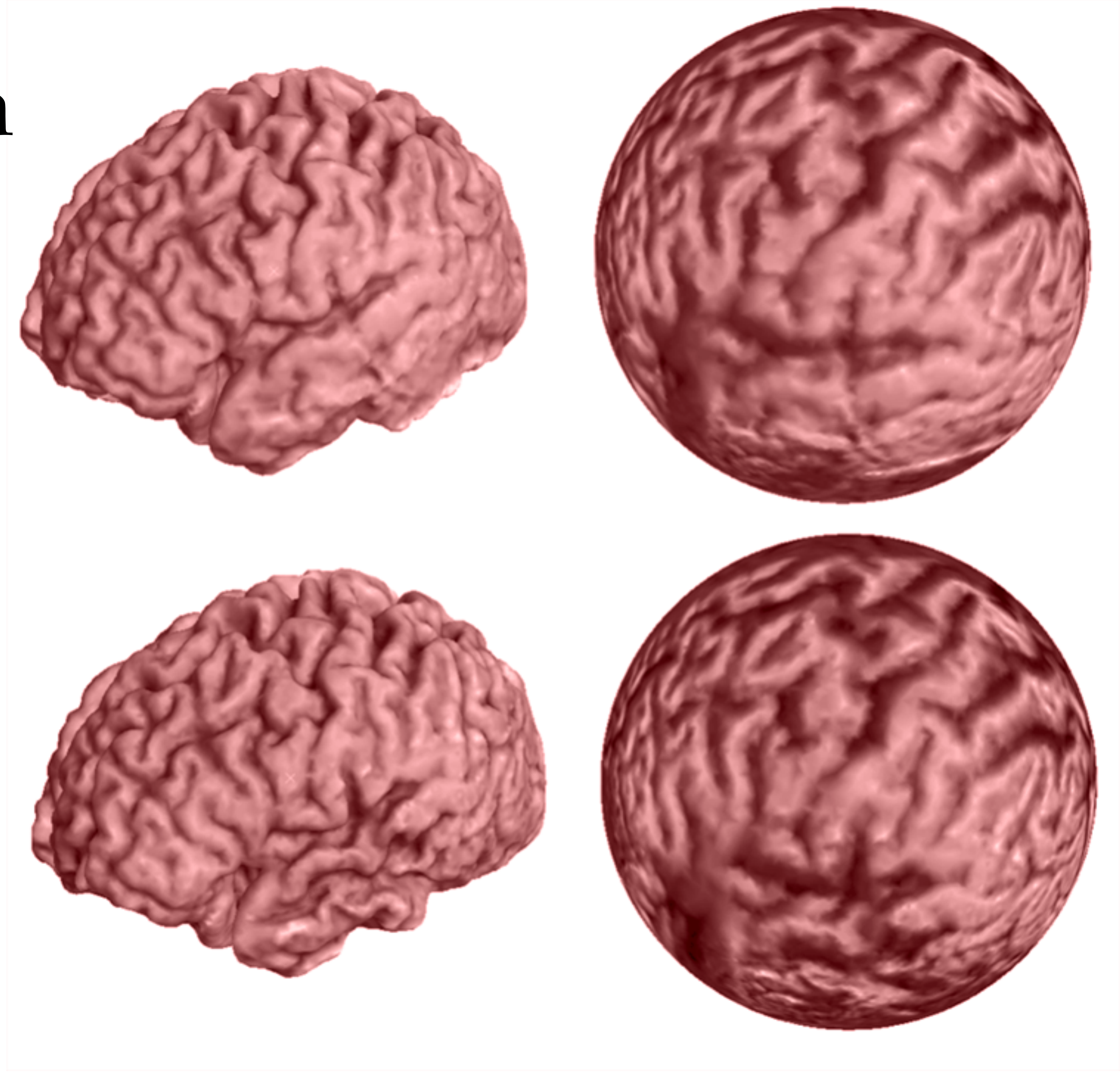




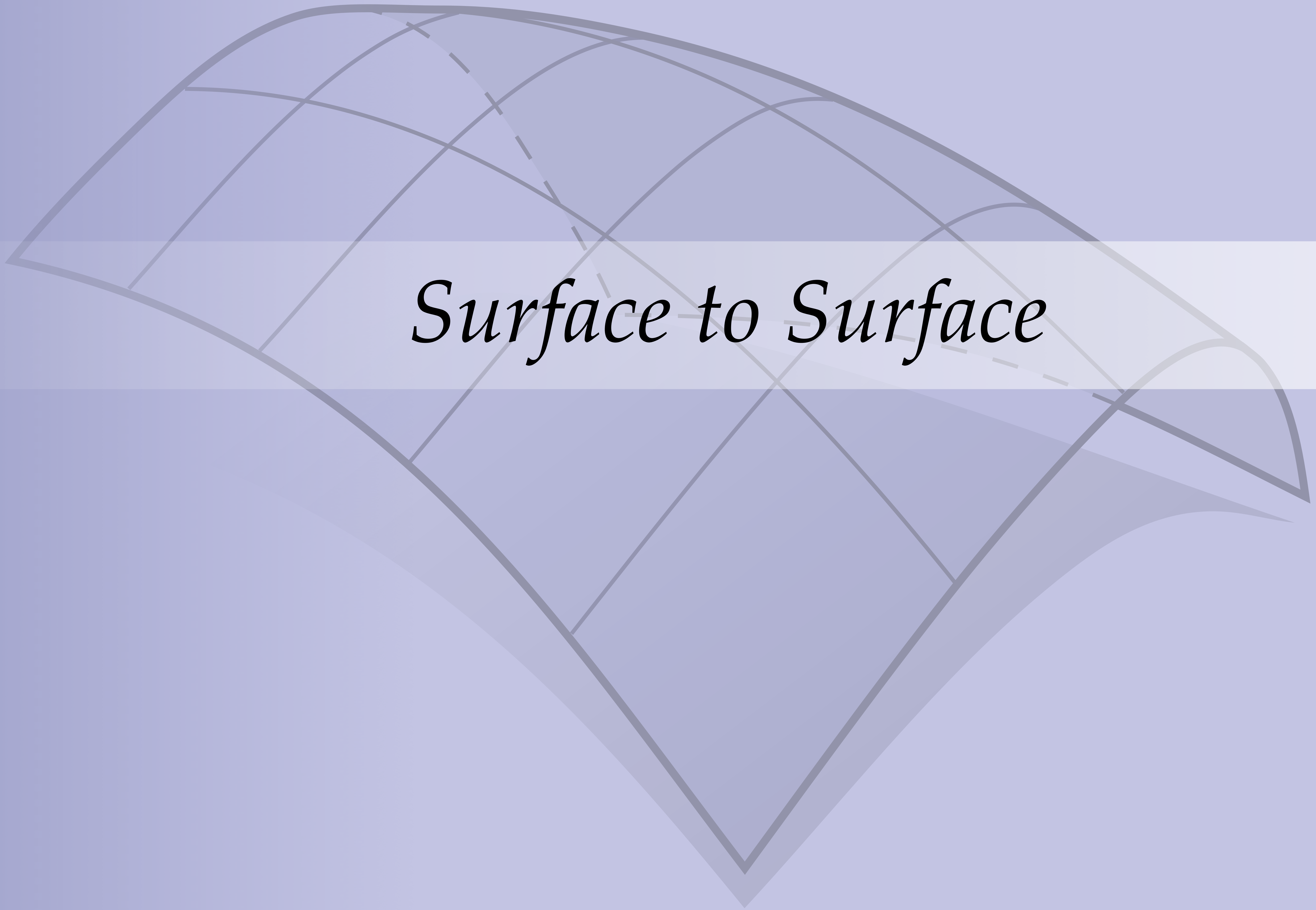
Why is Uniformization Useful?

- Provides canonical domain for solving equations, comparing data, cross-parameterization, etc.
- *Careful:* still have a few degrees of freedom (e.g., Möbius transformations)





Surface to Surface



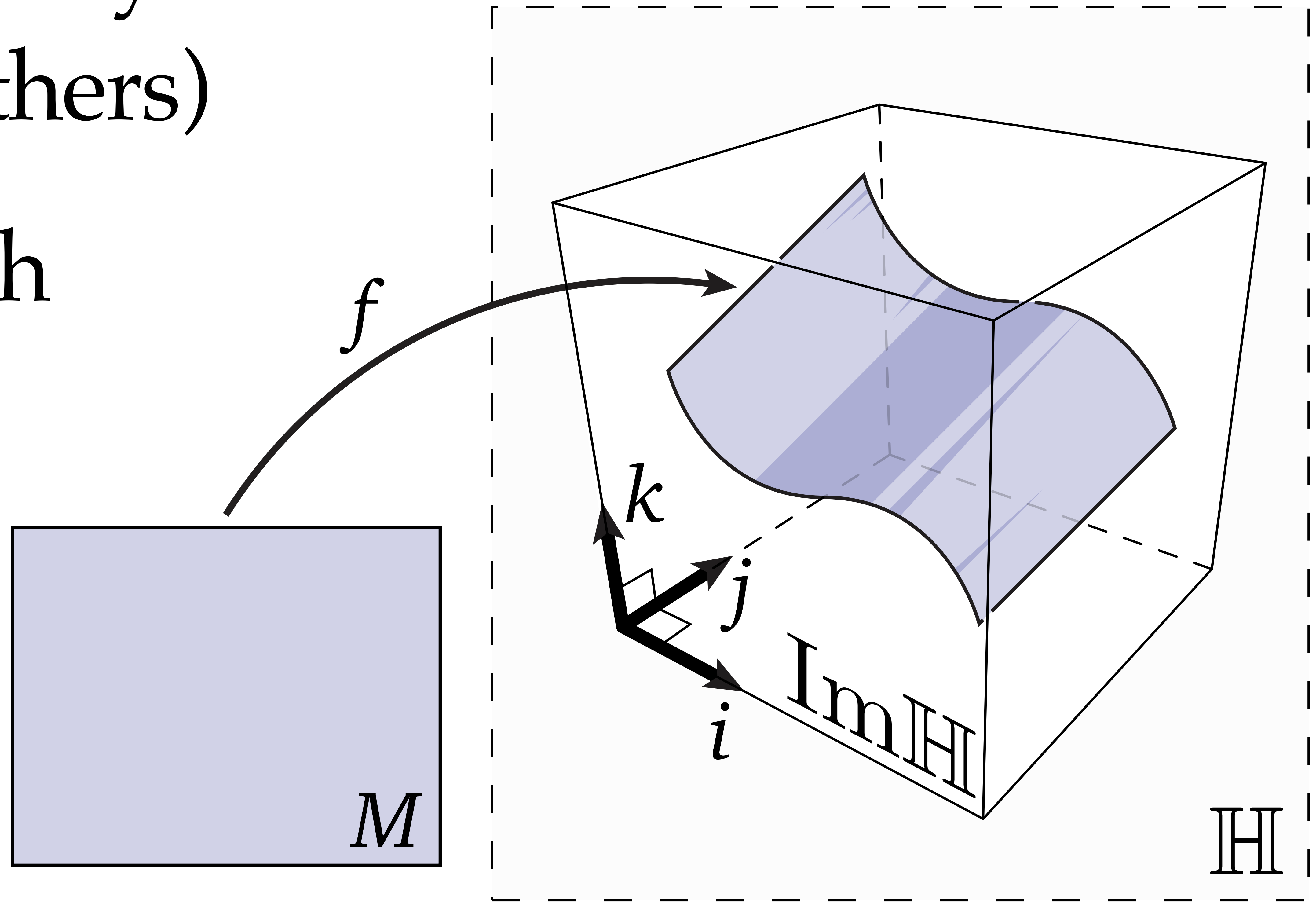
Surface to Surface

- Conformal deformations of surfaces embedded in space
- Both surfaces can have arbitrary curvature (not just sphere, disk, *etc.*)
- Opens door to much broader geometry processing applications
- Very recent theory & algorithms (~1996/2011)
- Key equation: *time-independent Dirac equation*

Won't say too much today… see https://youtu.be/UQC_emOPVK8

Geometry in the Quaternions

- Just as complex numbers helped with 2D transformations, *quaternions* provide natural language for 3D transformations
- Recent use of quaternions as alternative way of analyzing surfaces (Pedit, Pinkall, and others)
- Basic idea: points (a,b,c) get replaced with *imaginary* quaternions ai + bj + ck
- Surface is likewise an imaginary map f

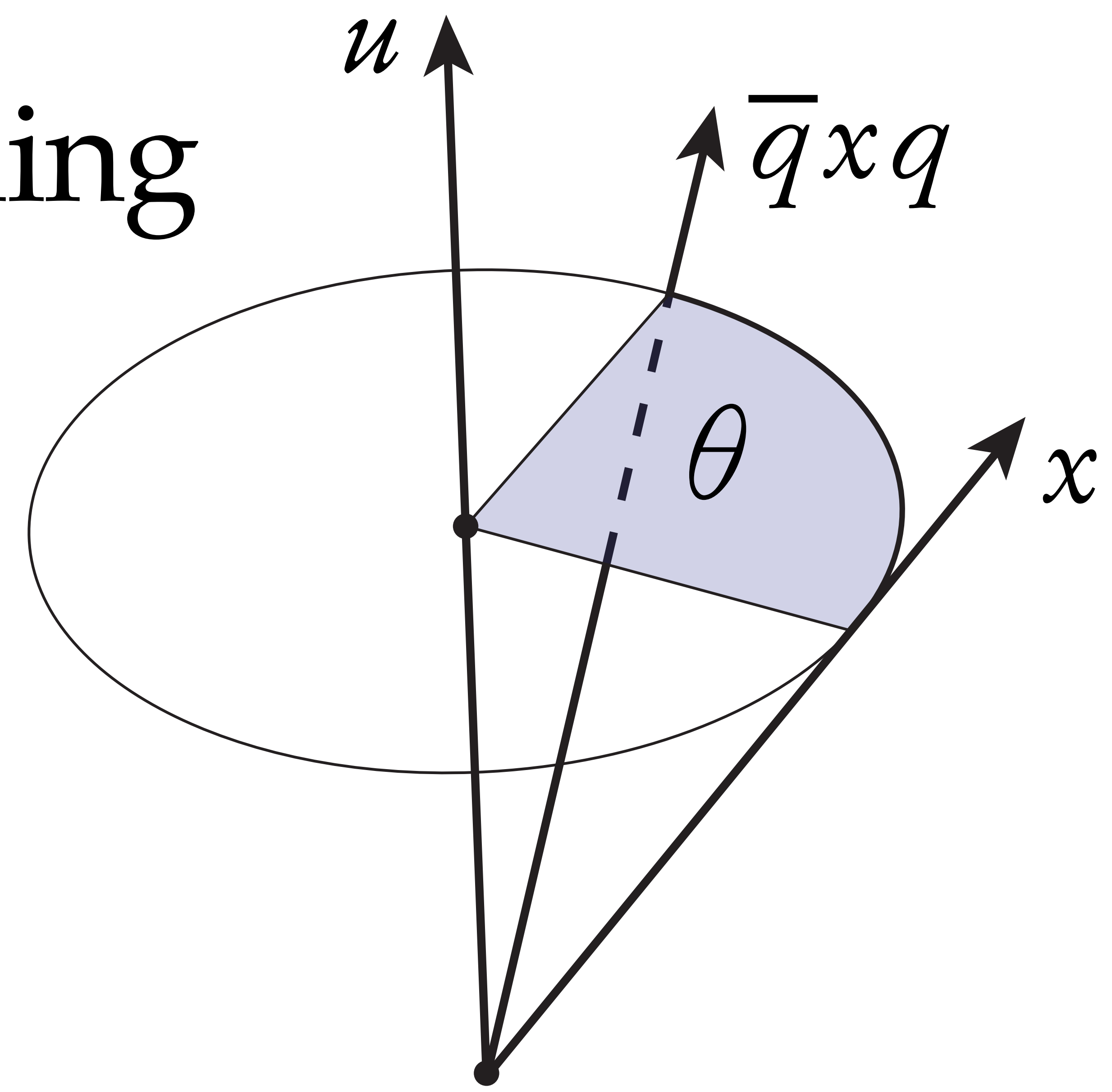


Stretch Rotations

- How do we express rotation using quaternions?
- Similar to complex case, can rotate a vector x using a unit quaternion q:

$$\underbrace{\tilde{x}}_{\textbf{rotated}} = \bar{q}\underbrace{x}_{\textbf{original}}q$$

- If q has non-unit magnitude, we get a rotation and scaling
- Should remind you of conformal map:
  scaling & *rotation* (but no shear)

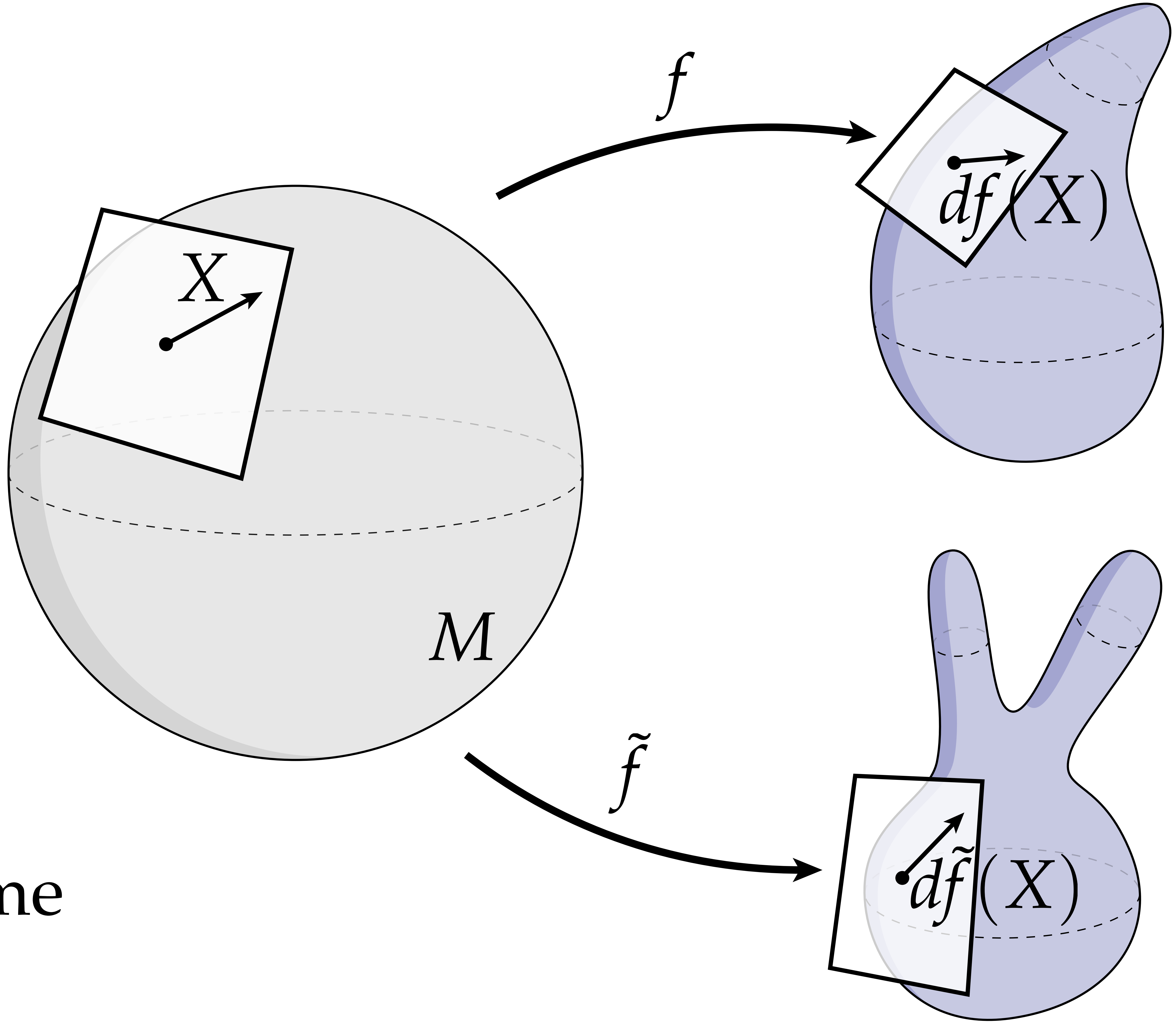


Spin Equivalence

- From here, not hard to express conformal deformation of surfaces
- Two surfaces f_0, f are *spin equivalent* if their tangent planes are related by a pure scaling and rotation at each point:

$$d\tilde{f}(X) = \bar{\psi}\, df(X)\psi$$

for all tangent vectors X and some stretch rotation $\psi : M \to \mathbb{H}$



Dirac Equation

- From here, one can derive the fundamental equation for conformal surface deformations, a *time-independent Dirac equation*

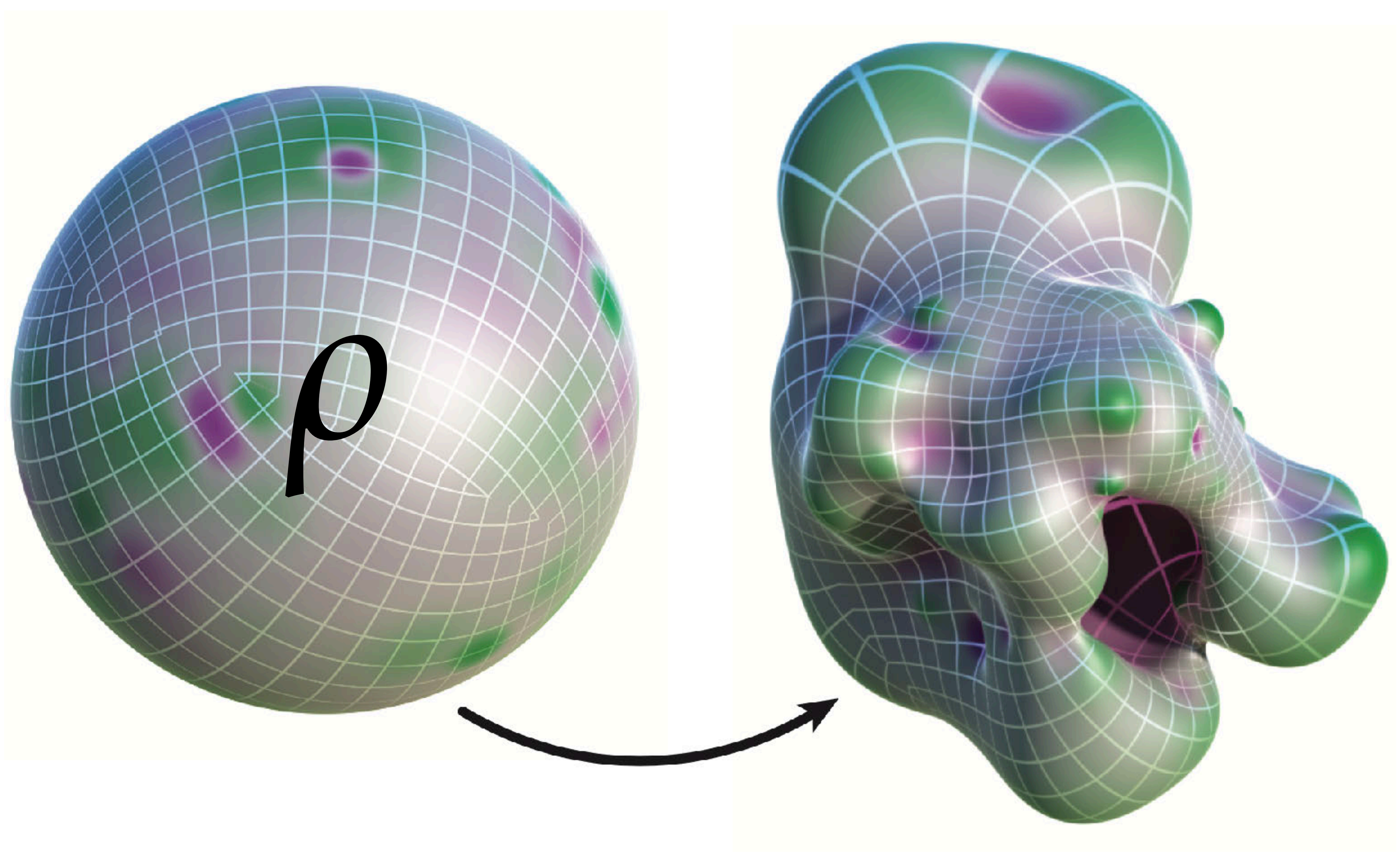
change in curvature

$$\boxed{(D - \rho)\psi = 0}$$

quaternionic Dirac operator

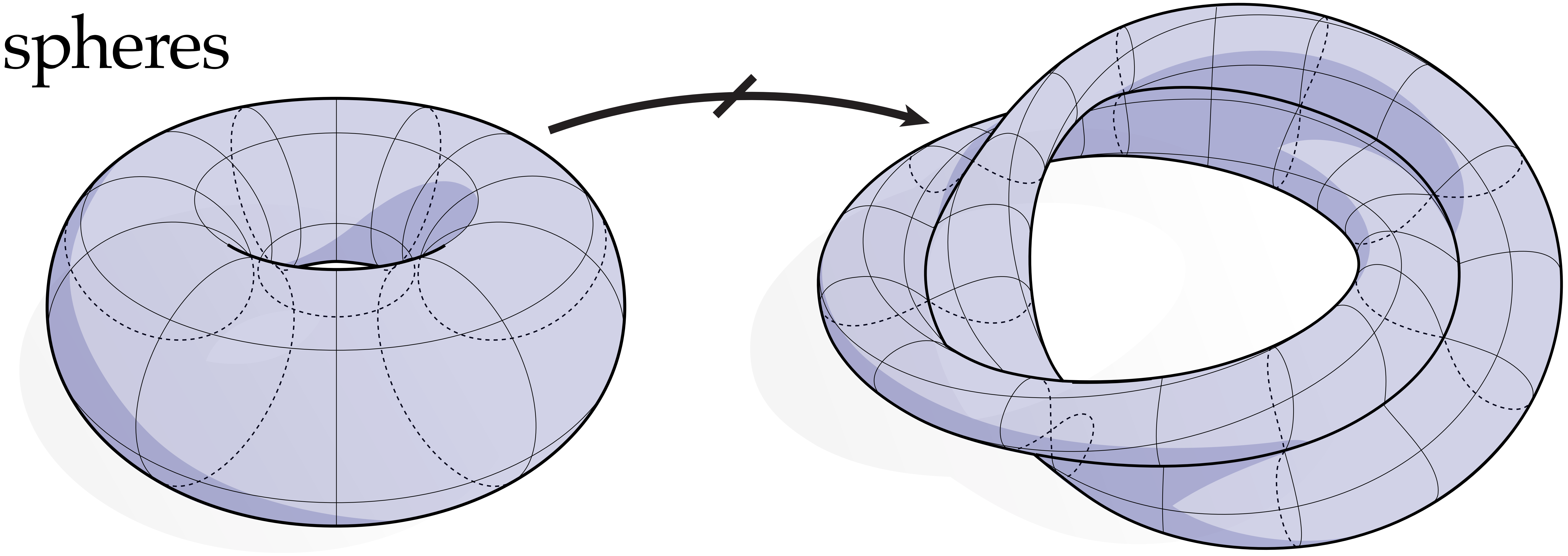
stretch rotation

$$D\psi := -\frac{df \wedge d\psi}{|df|^2}$$



Spin vs. Conformal Equivalence

- Two surfaces that are spin equivalent are also conformally equivalent: tangent vectors just get *rotated* and *scaled*! (no shearing)
- Are conformally equivalent surfaces always spin equivalent?
  - **No** in general, *e.g.*, tori that are not *regularly homotopic* (below)
  - **Yes** for topological spheres

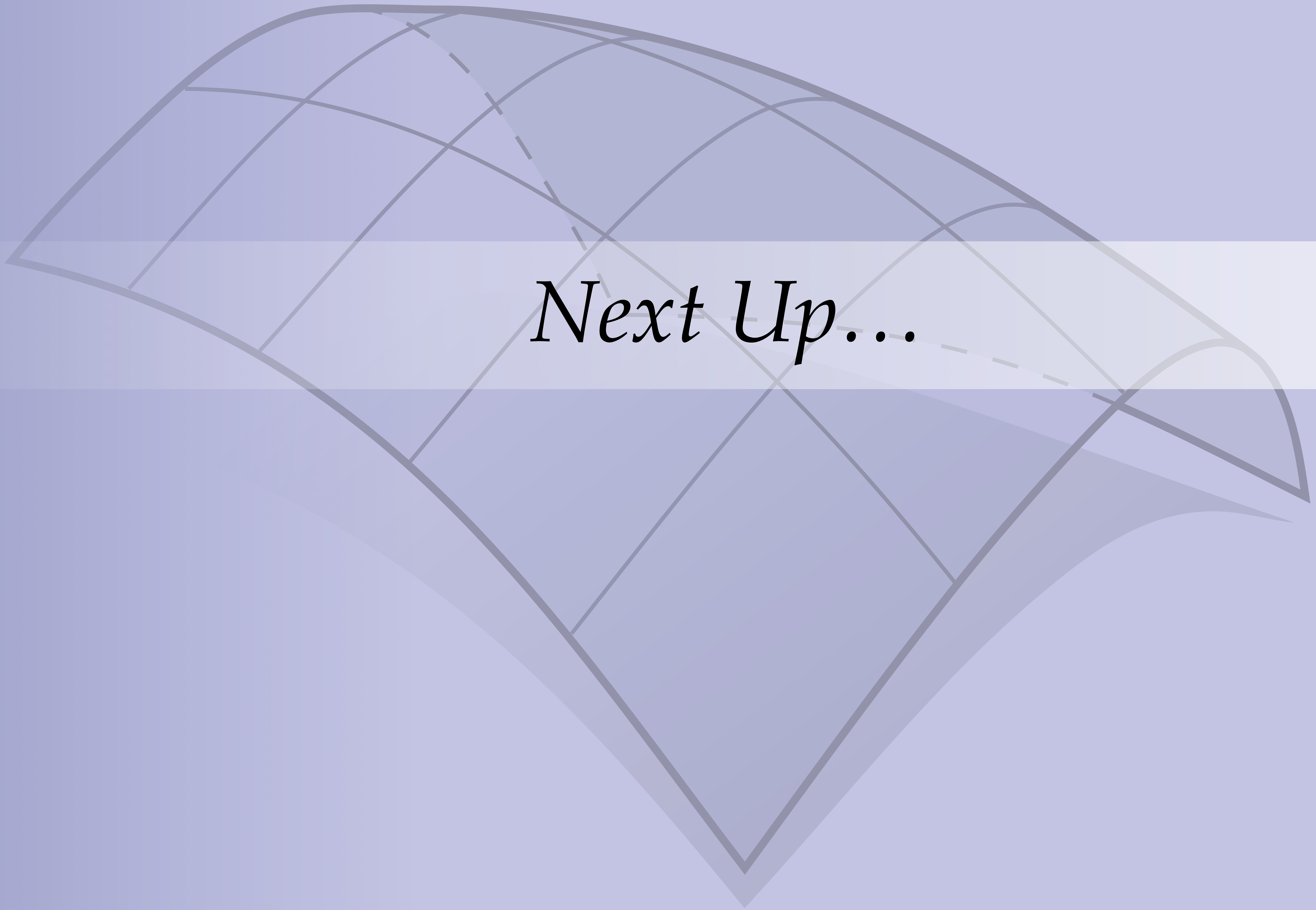


Why Not Just Optimize Angles?

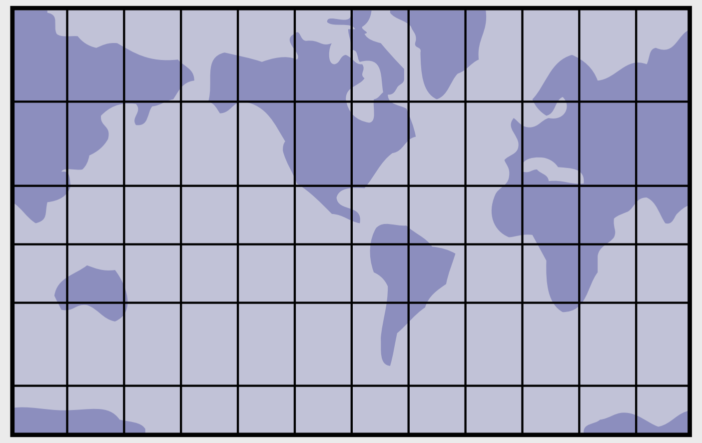
- Forget the mathematics—why not just optimize mesh to preserve angles?
- As discussed before, angle preservation is *too rigid!*
- E.g., convex surface *uniquely* determined by angles (up to rigid motion)

original **deformed** **optimize angles**

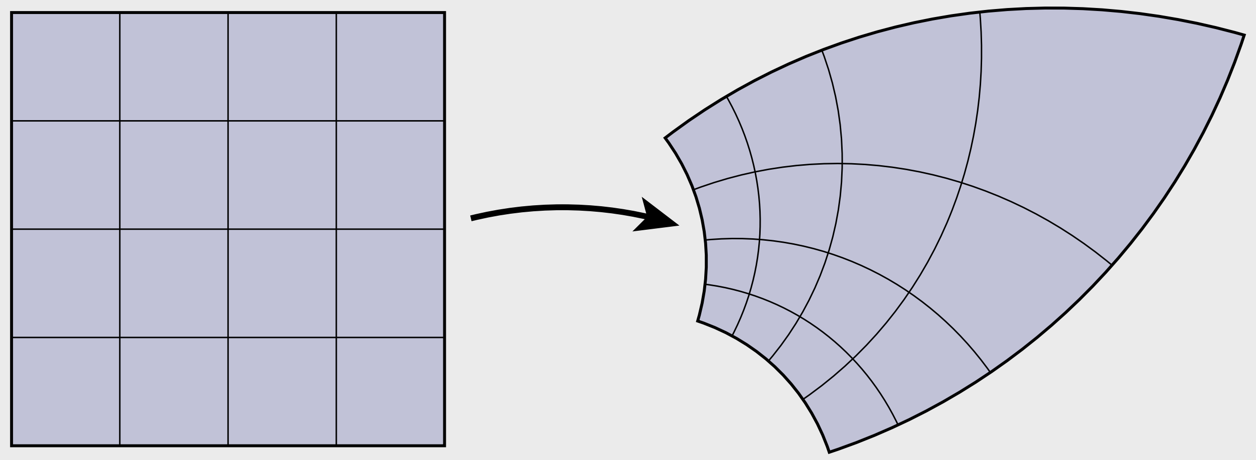
Next Up…



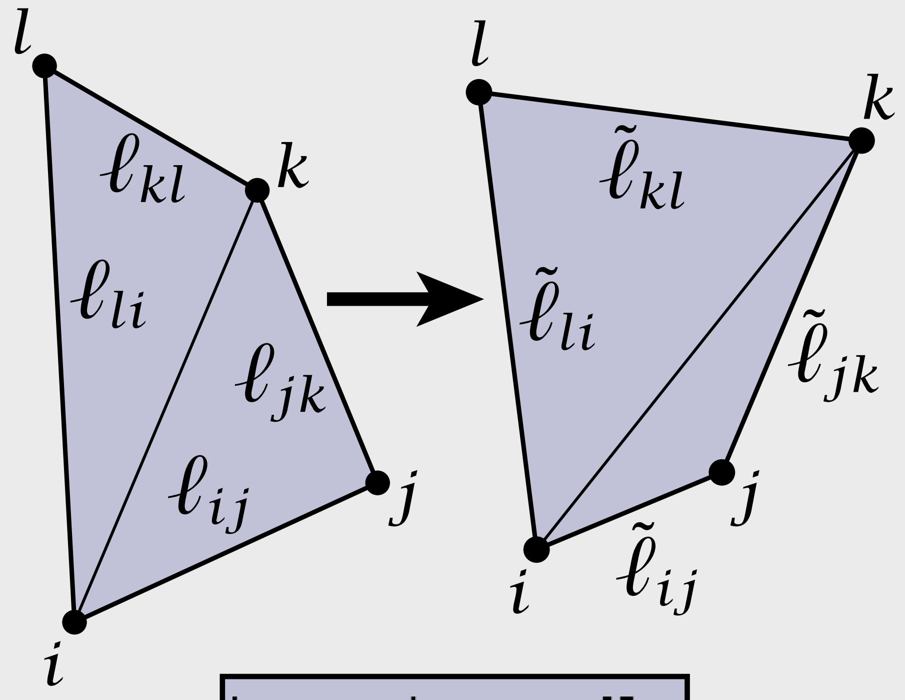
Next up… Discretization & Algorithms



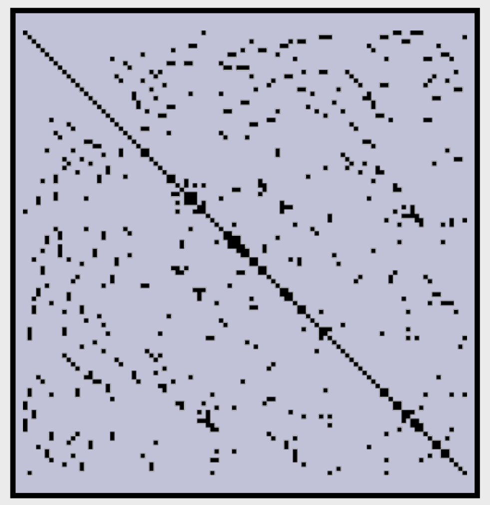
PART I: OVERVIEW



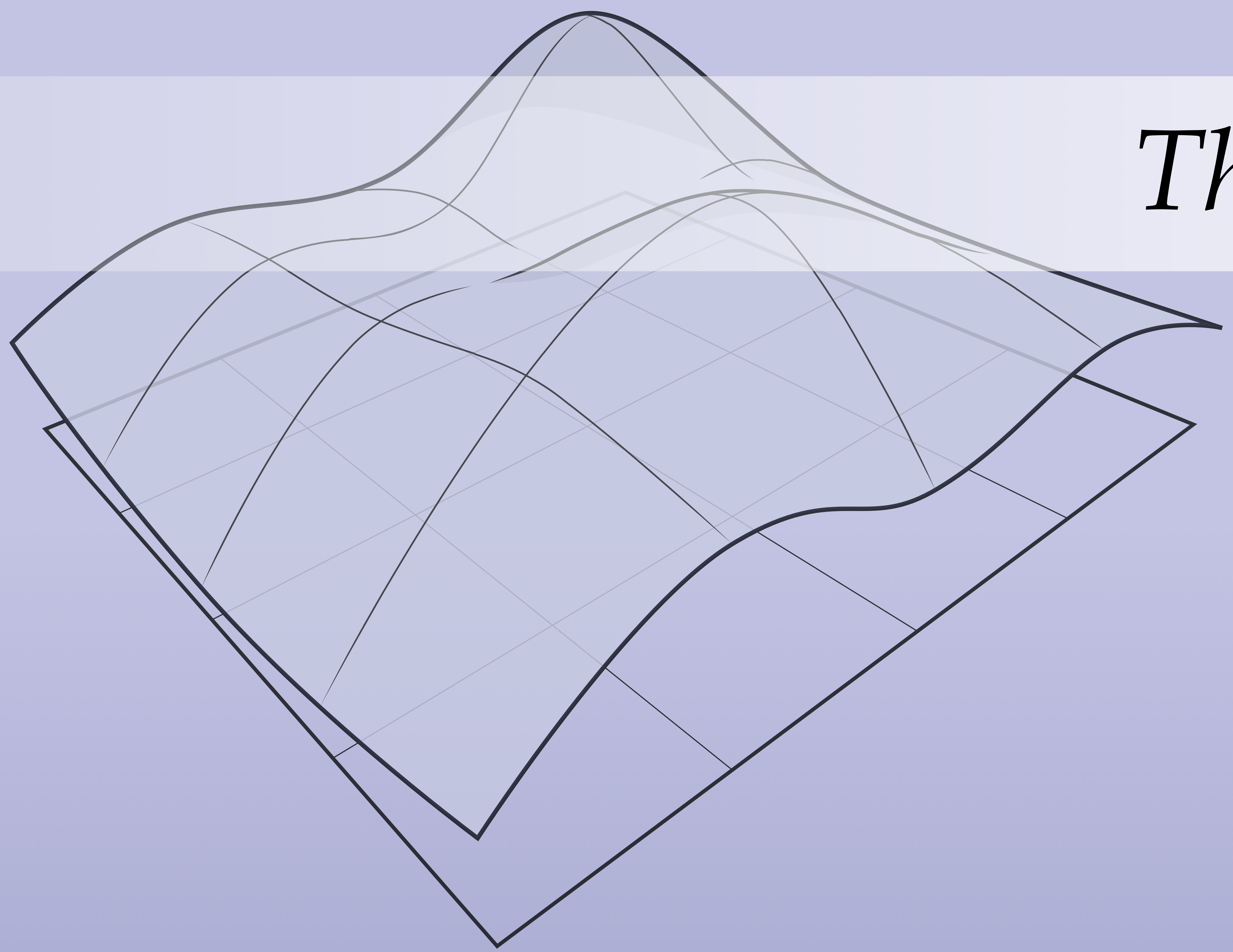
PART II: SMOOTH THEORY



PART III: DISCRETIZATION



PART IV: ALGORITHMS



Thanks!

Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-869(J) • Spring 2016