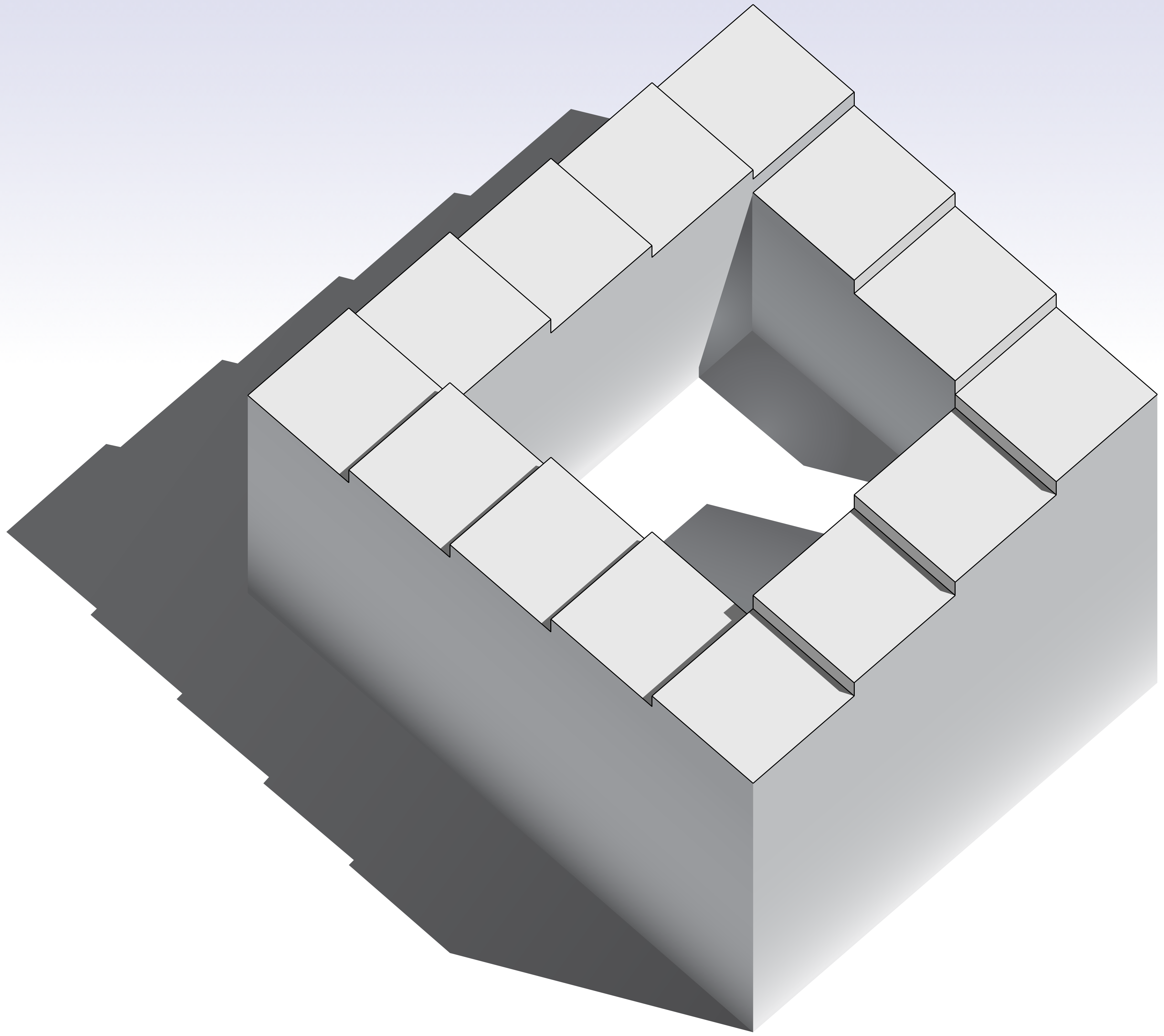
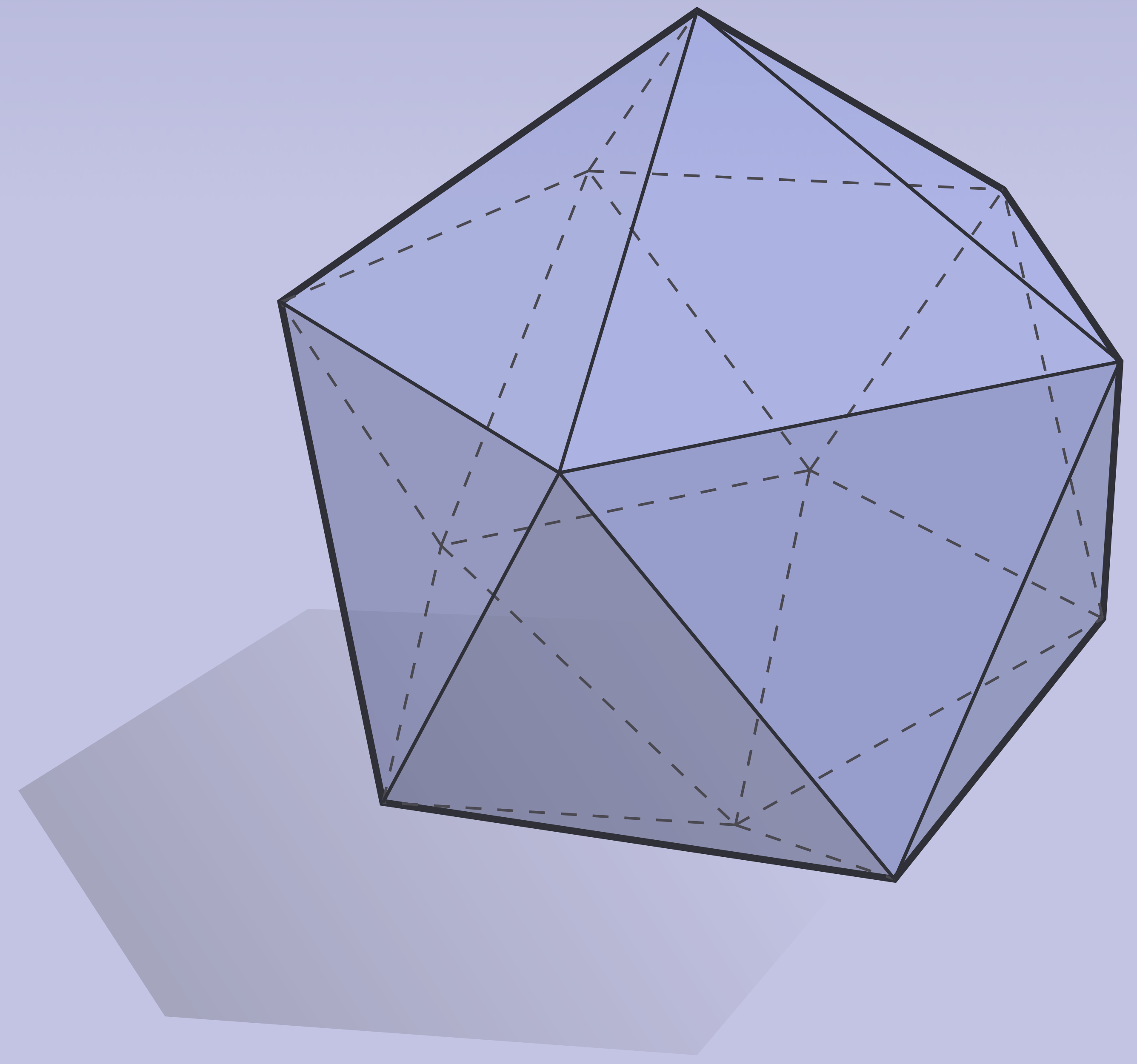


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 4:

Differential Forms in R^n



Discrete Differential Geometry: An Applied Introduction

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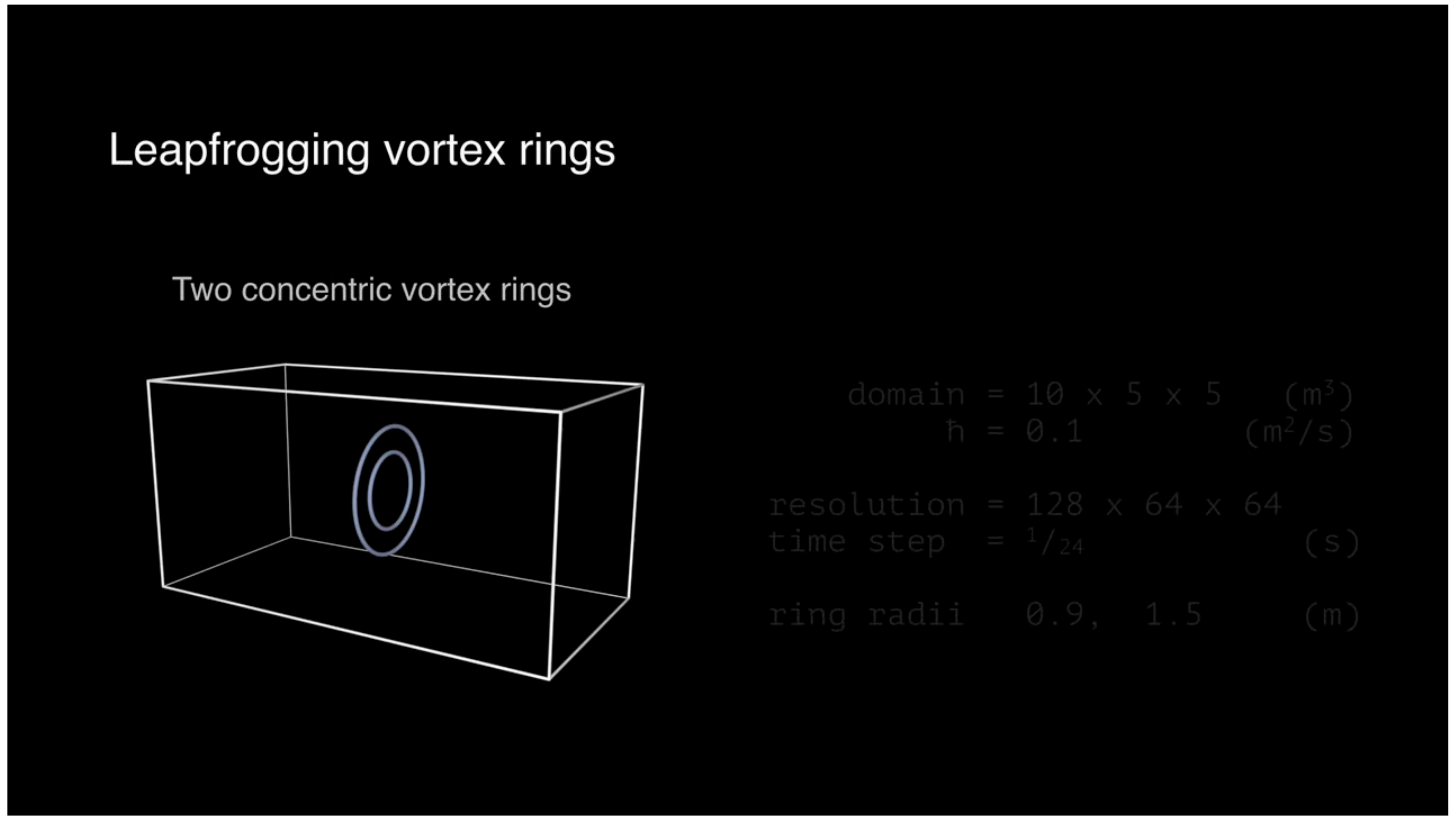
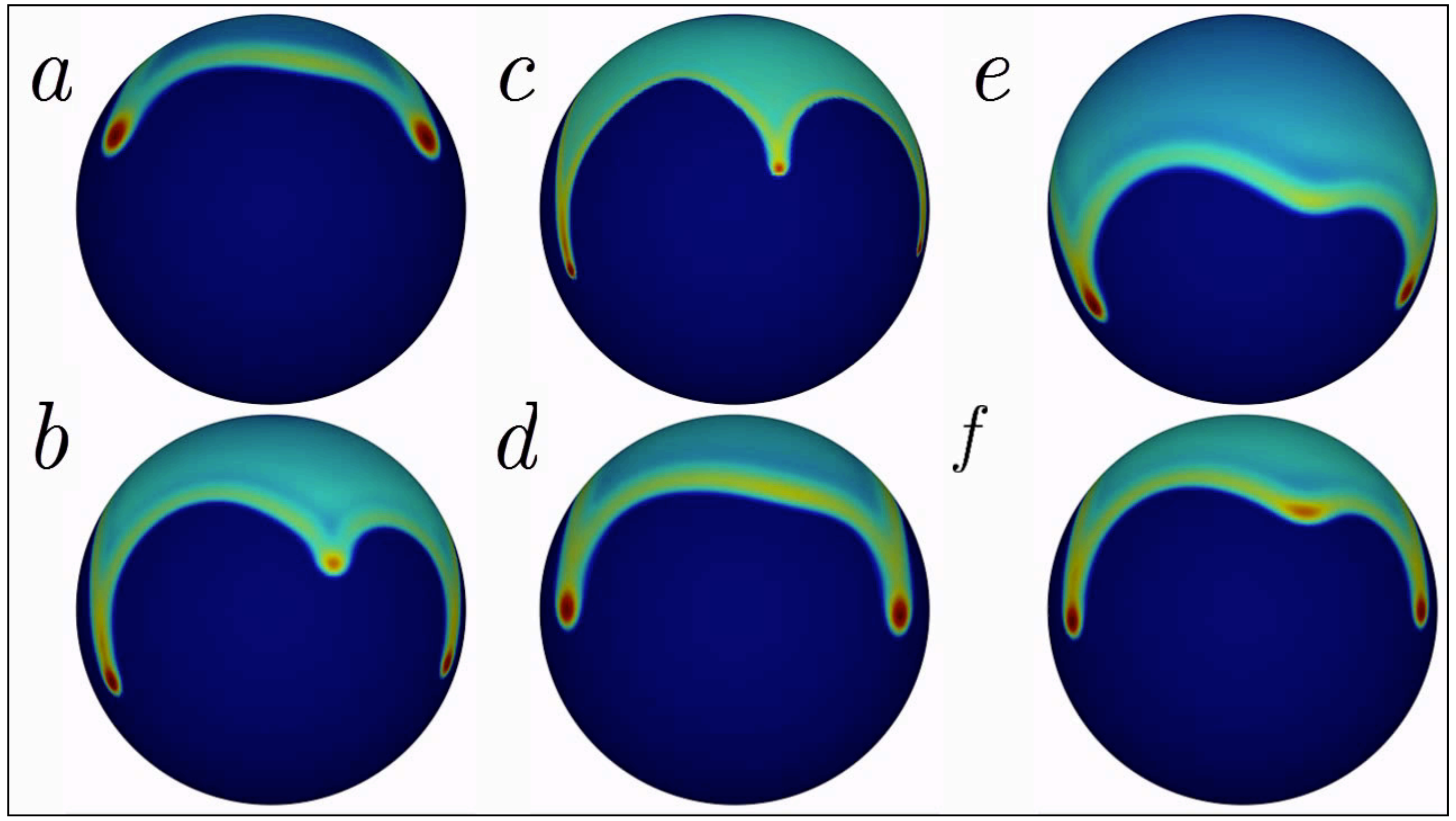
Motivation: Applications of Differential Forms





*Need to **measure** k-dimensional quantities that are changing in space & time!*

Where Are We Going Next?

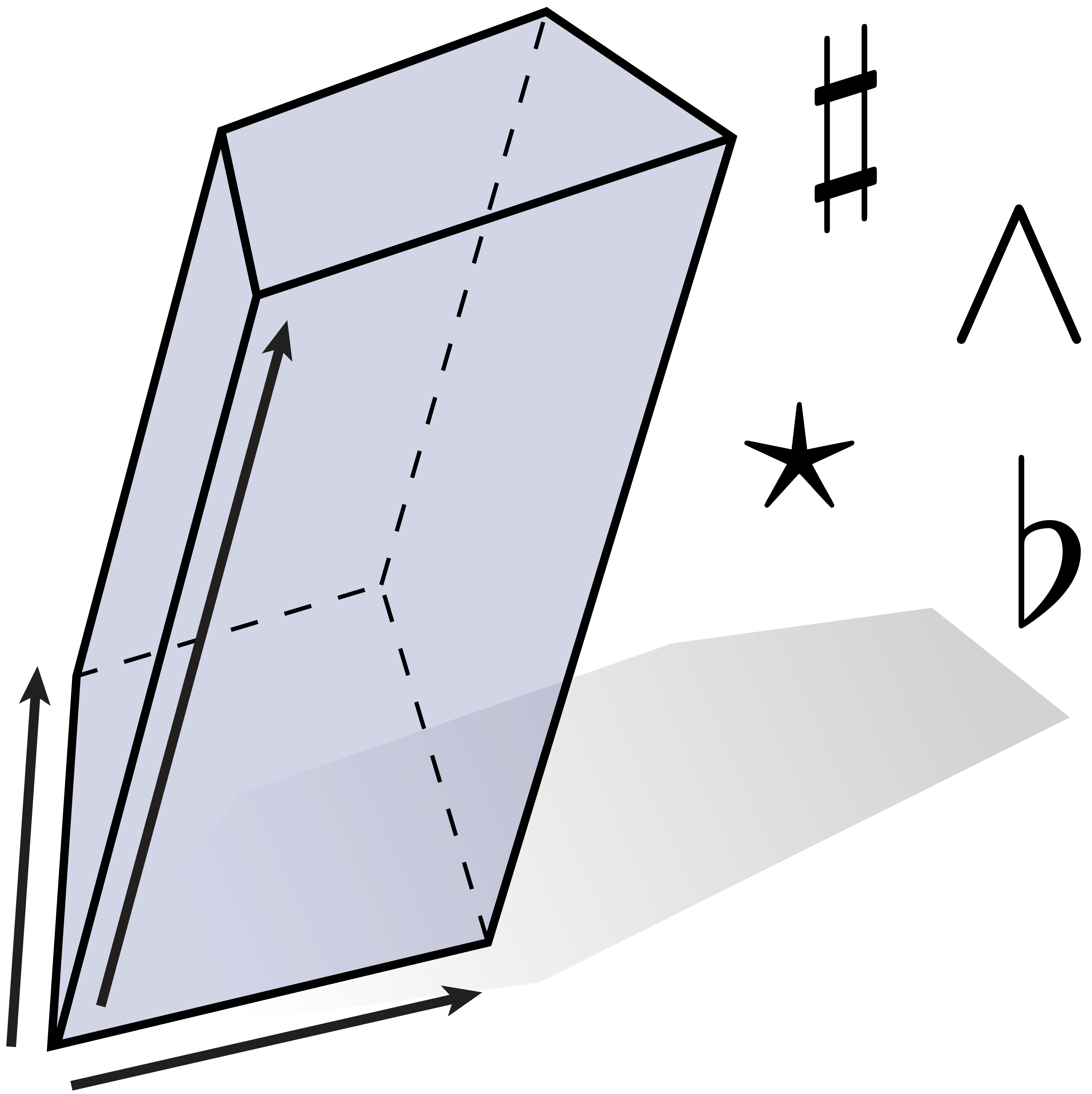
GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

- **Linear algebra:** "little arrows" (vectors)
- **Vector Calculus:** how do vectors *change?*

Next few lectures:

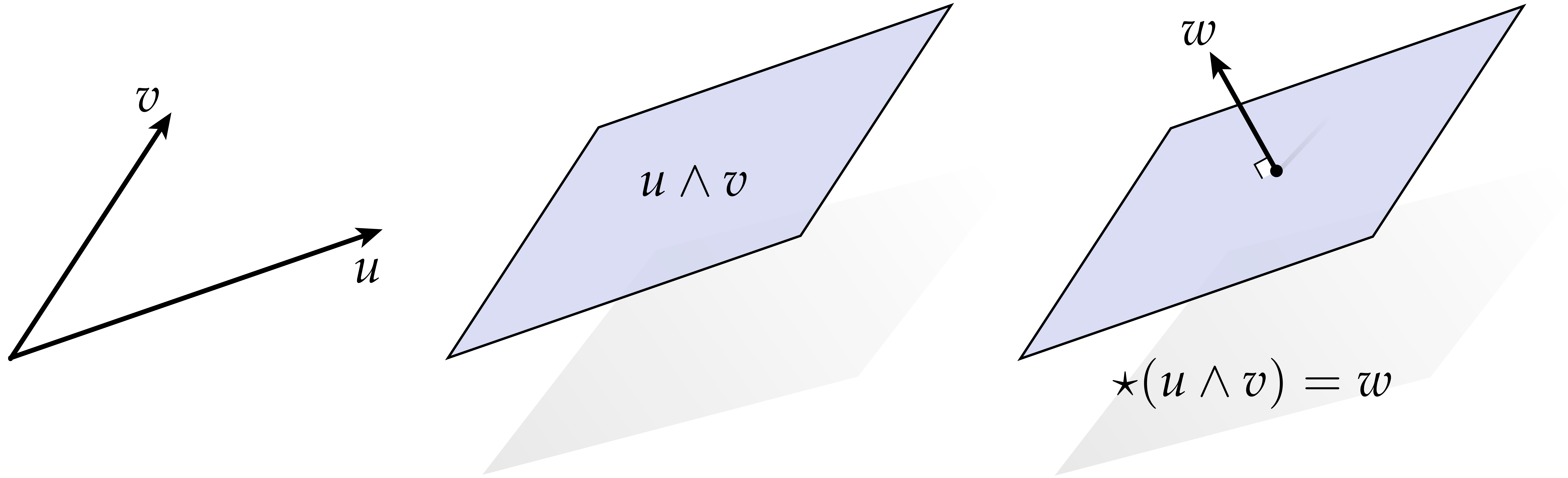
- **Exterior algebra**: "little volumes" (k-vectors)
- **Differential forms:** spatially-varying k-form
- **Exterior calculus**: how do k-vectors change?
- **DEC:** how do we do all of this on meshes?



Basic idea: replace vector calculus with computation on meshes.

Recap: Exterior Algebra

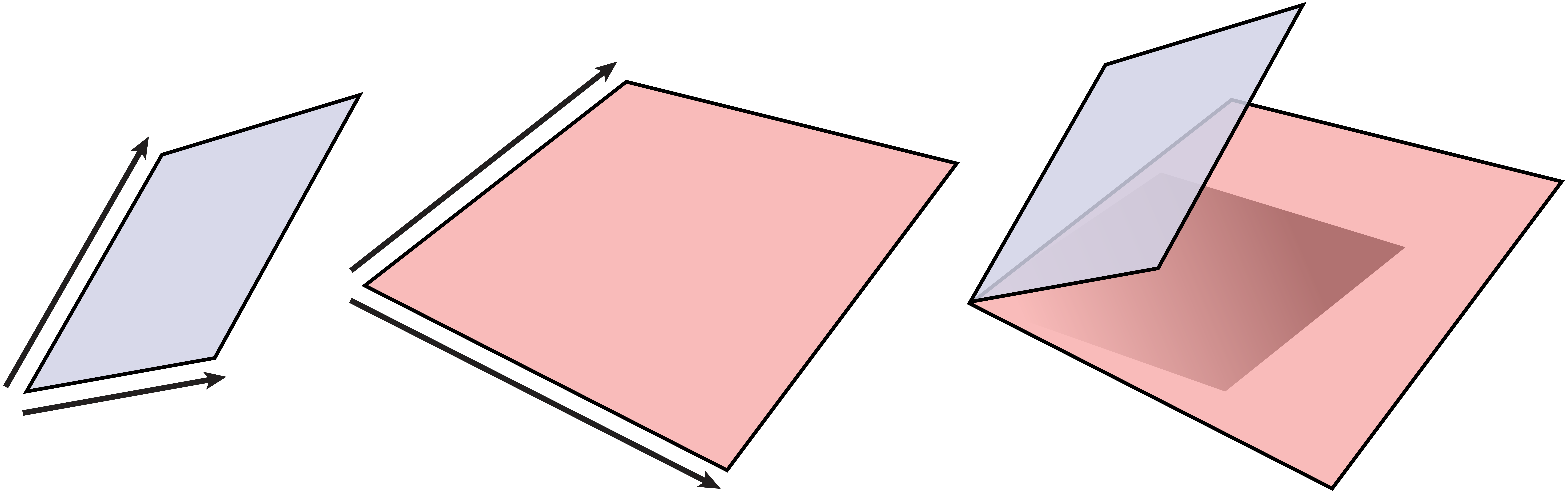
- Use wedge product to build up "little volumes" (*k-vectors*) from ordinary vectors



- Like linear subspaces, but have *magnitude* and *orientation*
- Use Hodge star to describe complementary volumes

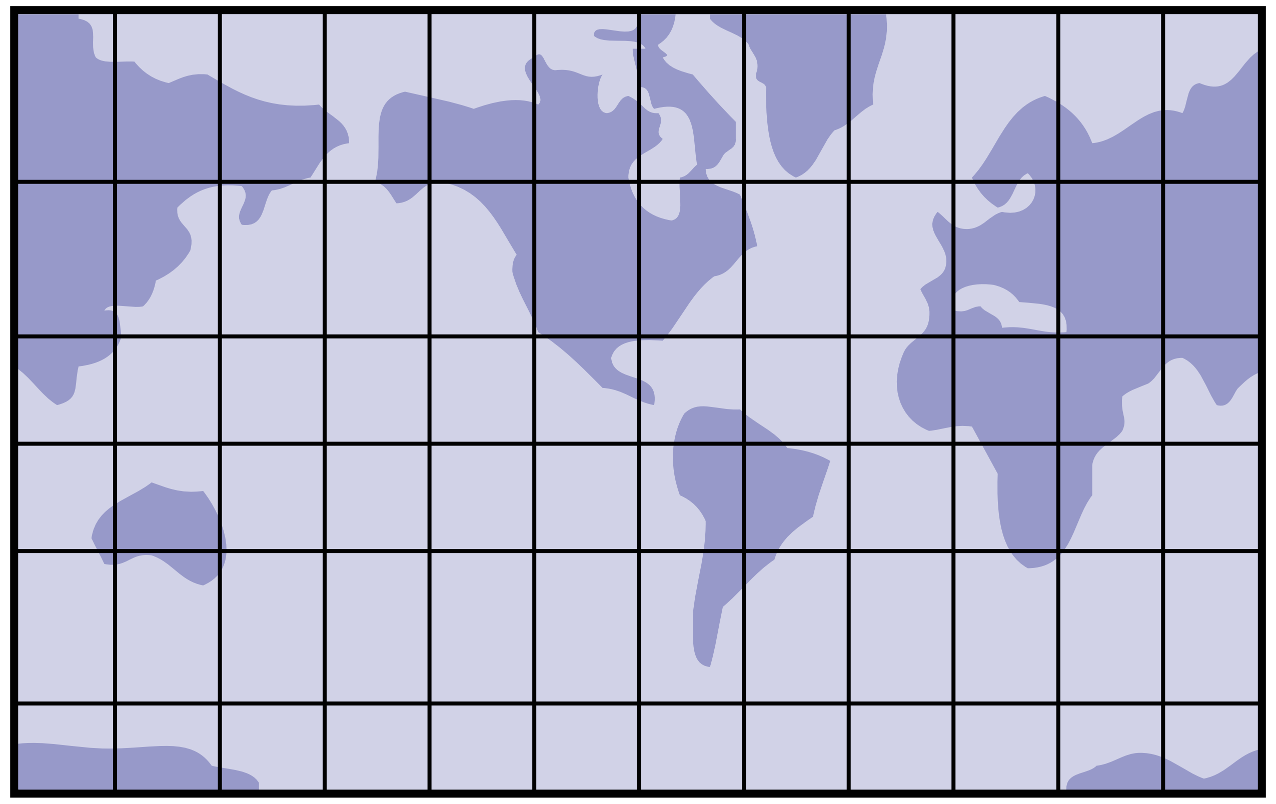
Recap: k-Forms

- Can measure a vector with a *covector;* can measure a *k*-vector with a *k-form*



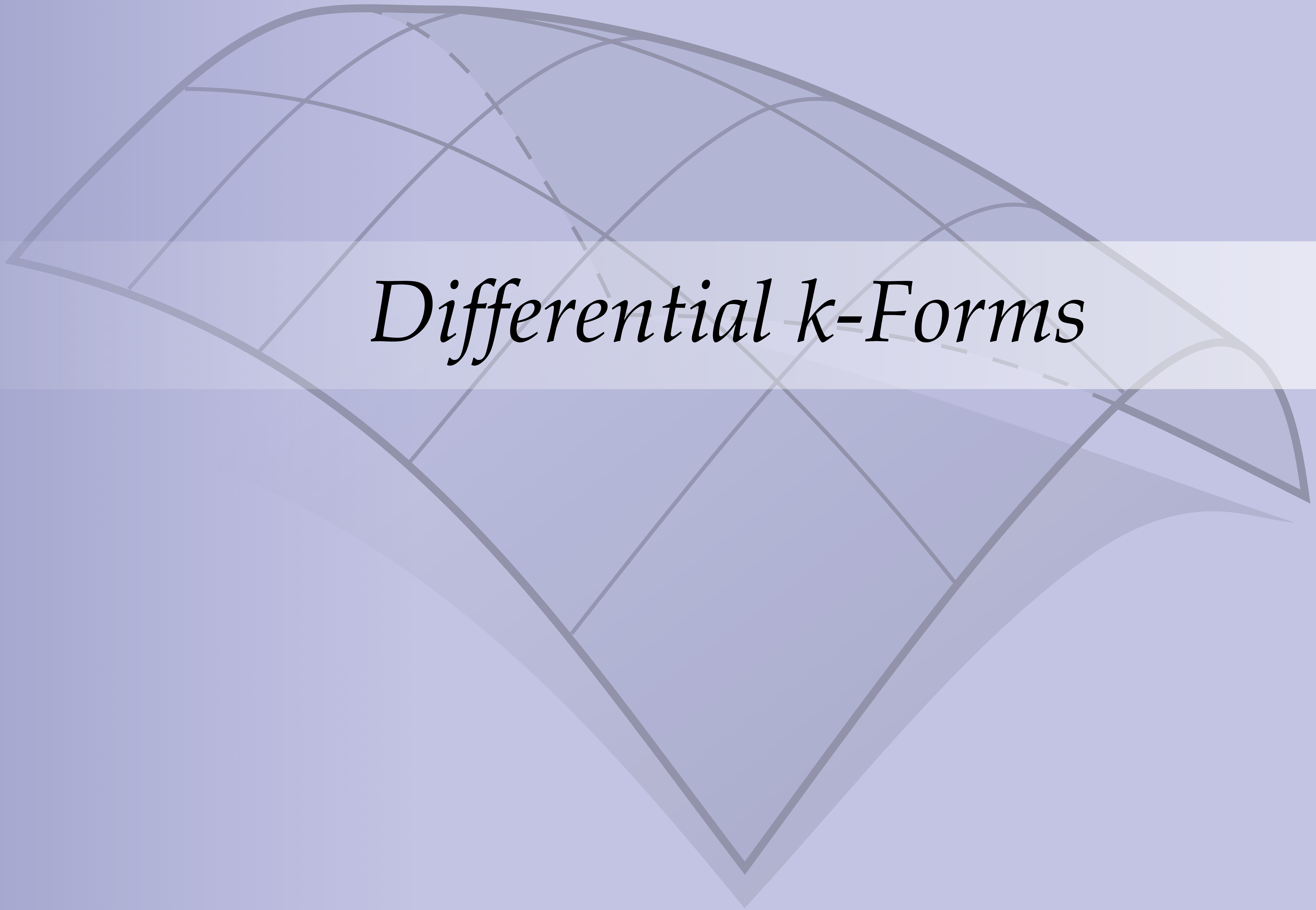
- Build up *k*-forms by wedging together covectors
- To measure, project k-vector onto k-form and take volume (e.g., via determinant)

Exterior Calculus: Flat vs. Curved Spaces



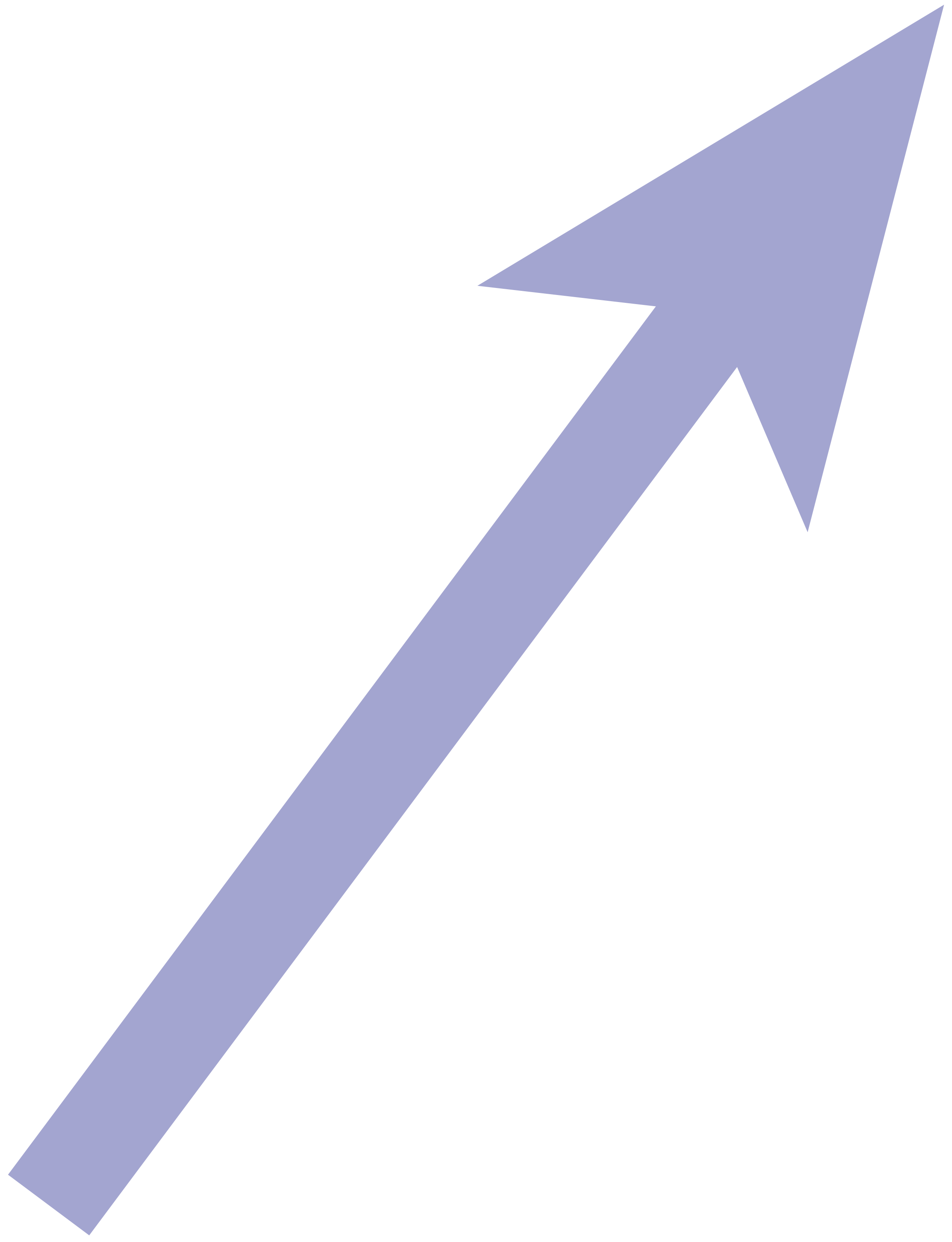
- For now, we'll only consider *flat* spaces like the 2D plane
  - Keeps all our calculations simple
  - Don't have to define *manifolds* (yet!)
- True power of exterior calculus revealed on *curved* spaces
  - In flat spaces, vectors and forms look very similar
  - Difference is less superficial on curved spaces
  - Close relationship to *curvature* (geometry)
  - Also close relationship to *mass* (physics)

Differential k-Forms

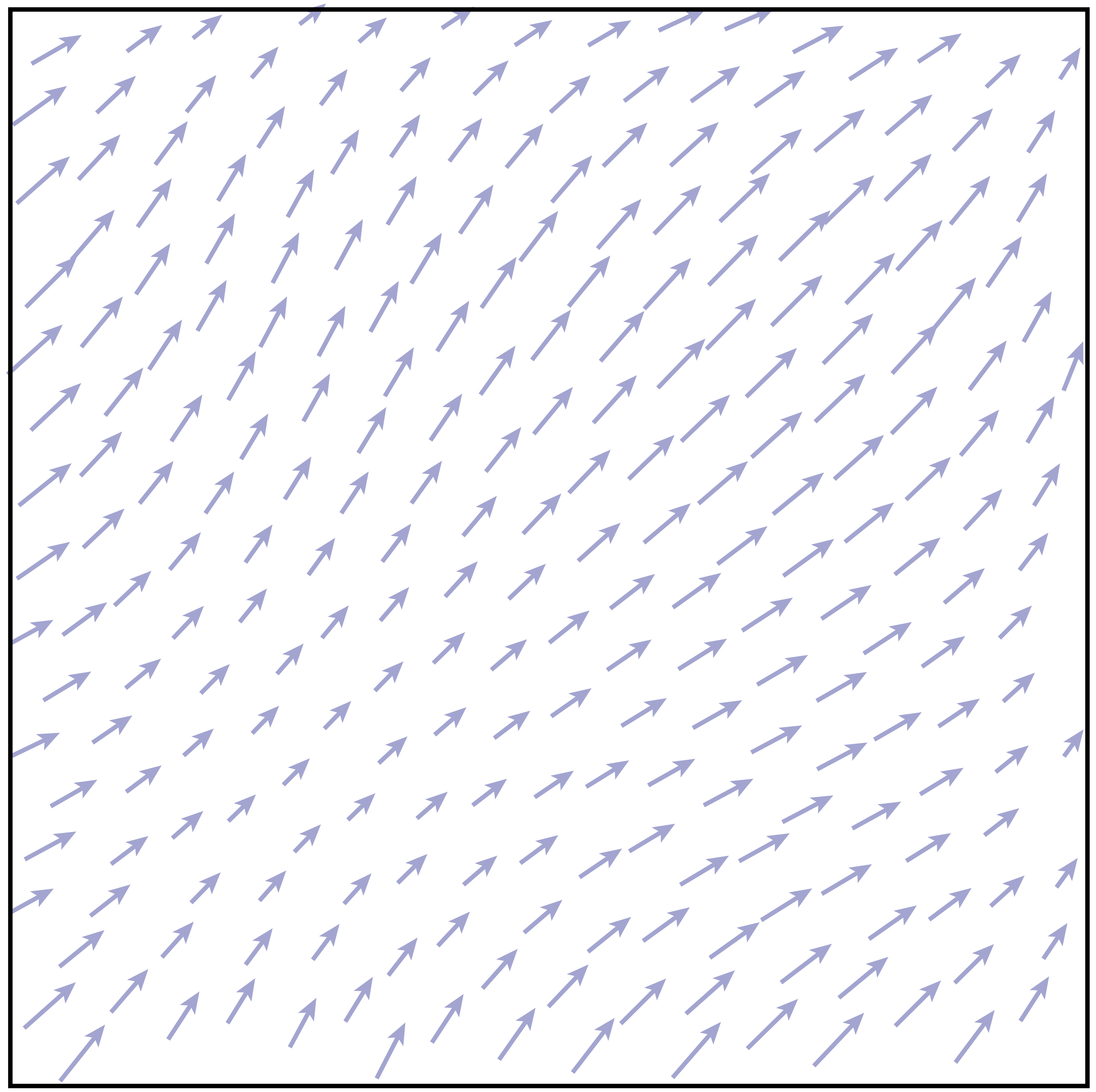


Review: Vector vs. Vector Field

- Recall that a vector *field* is an assignment of a vector to each point:



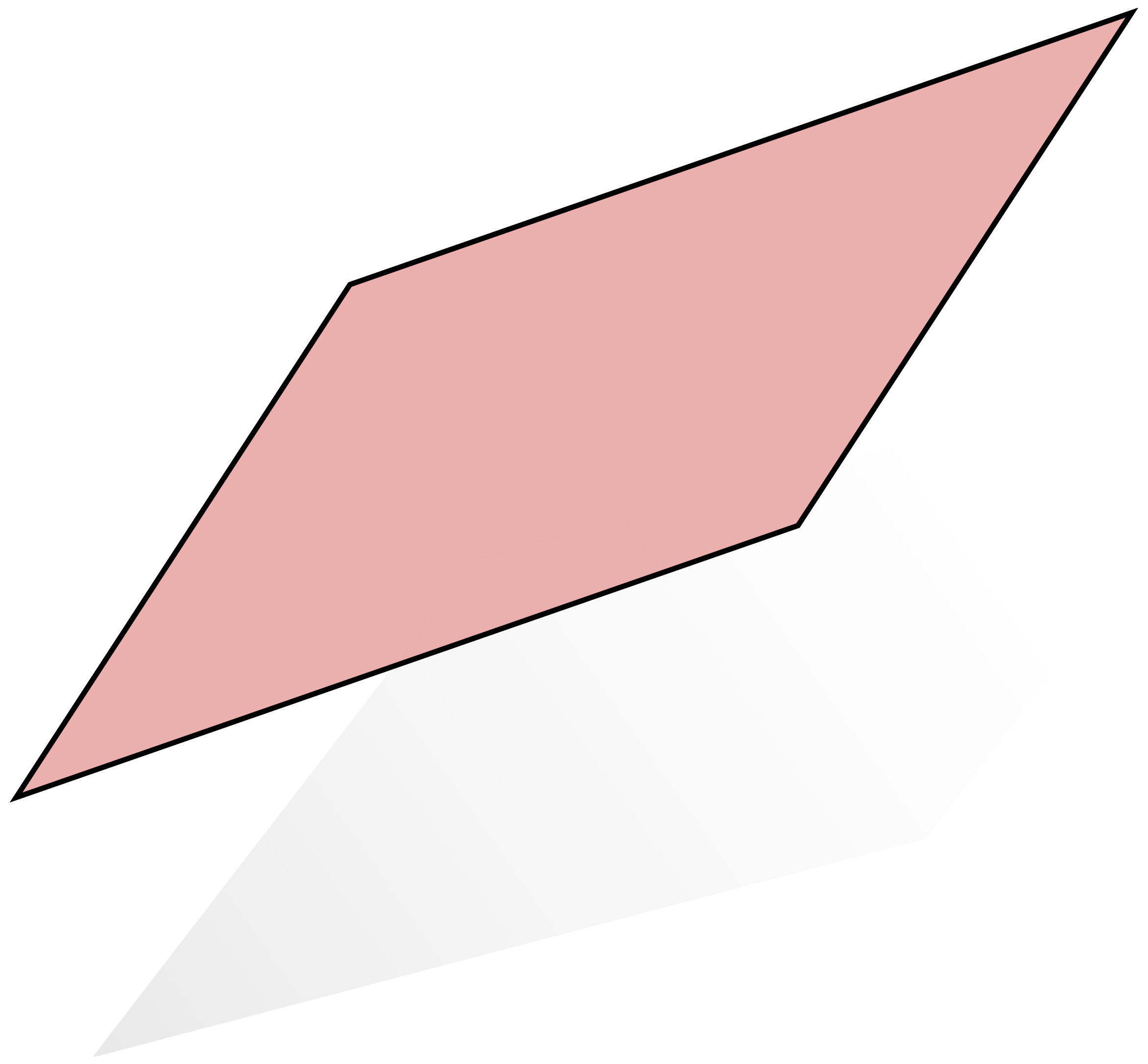
vector



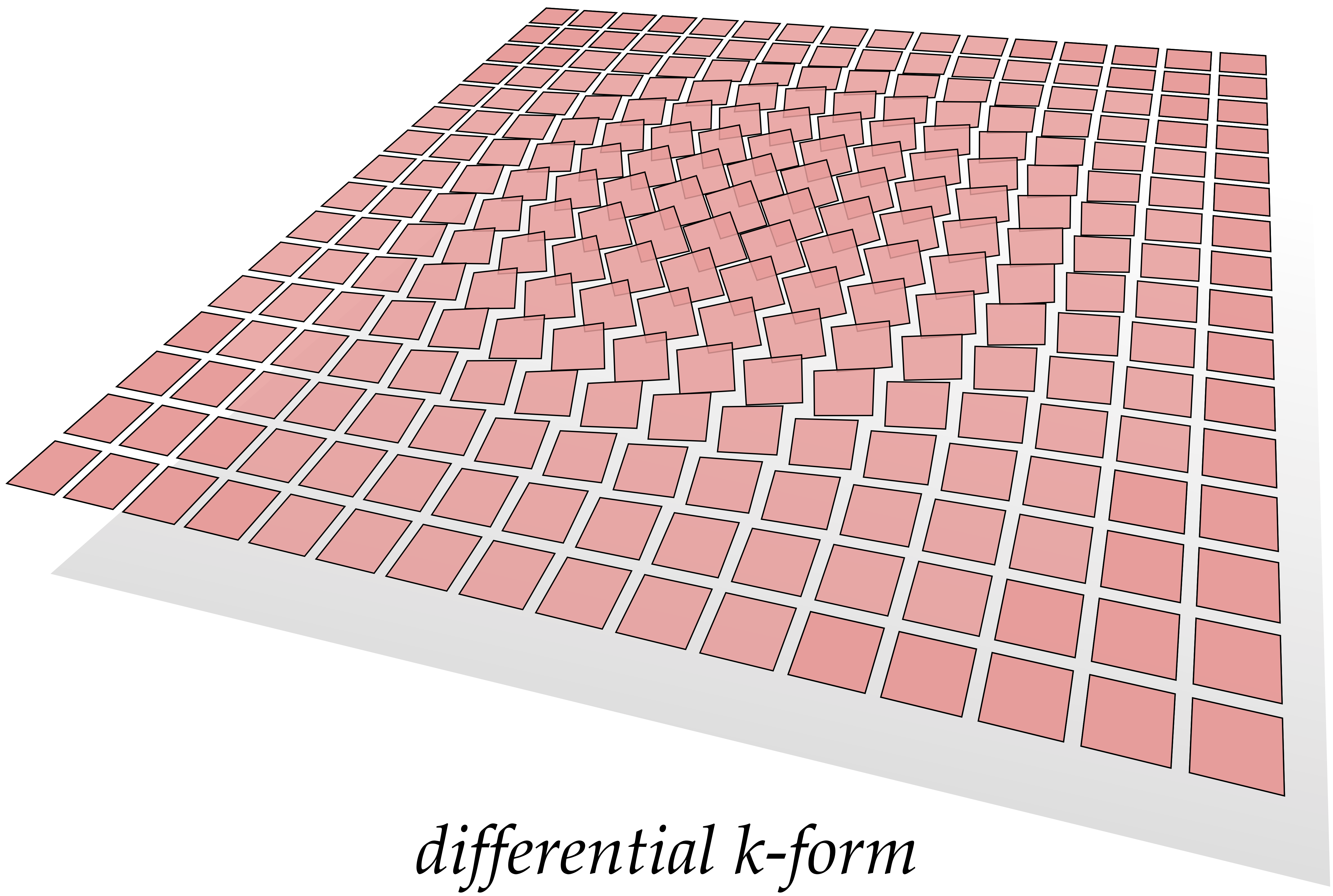
vector field

Differential Form

- A *differential k-form* is an assignment of a k-form to each point*:



k-form

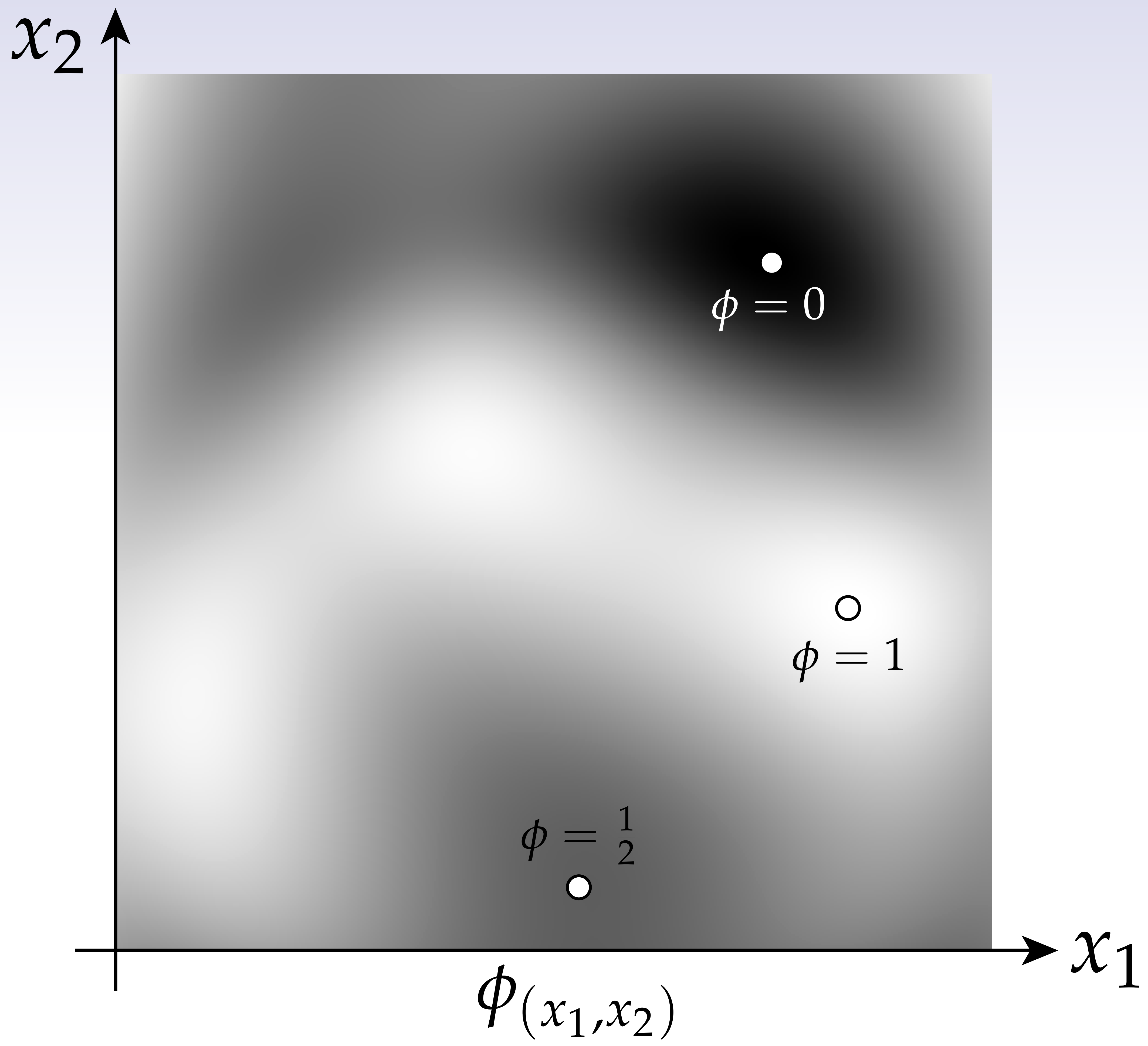


differential k-form

*Common (and confusing!) to abbreviate "differential k-form" as just "k-form"!

Differential 0-Form

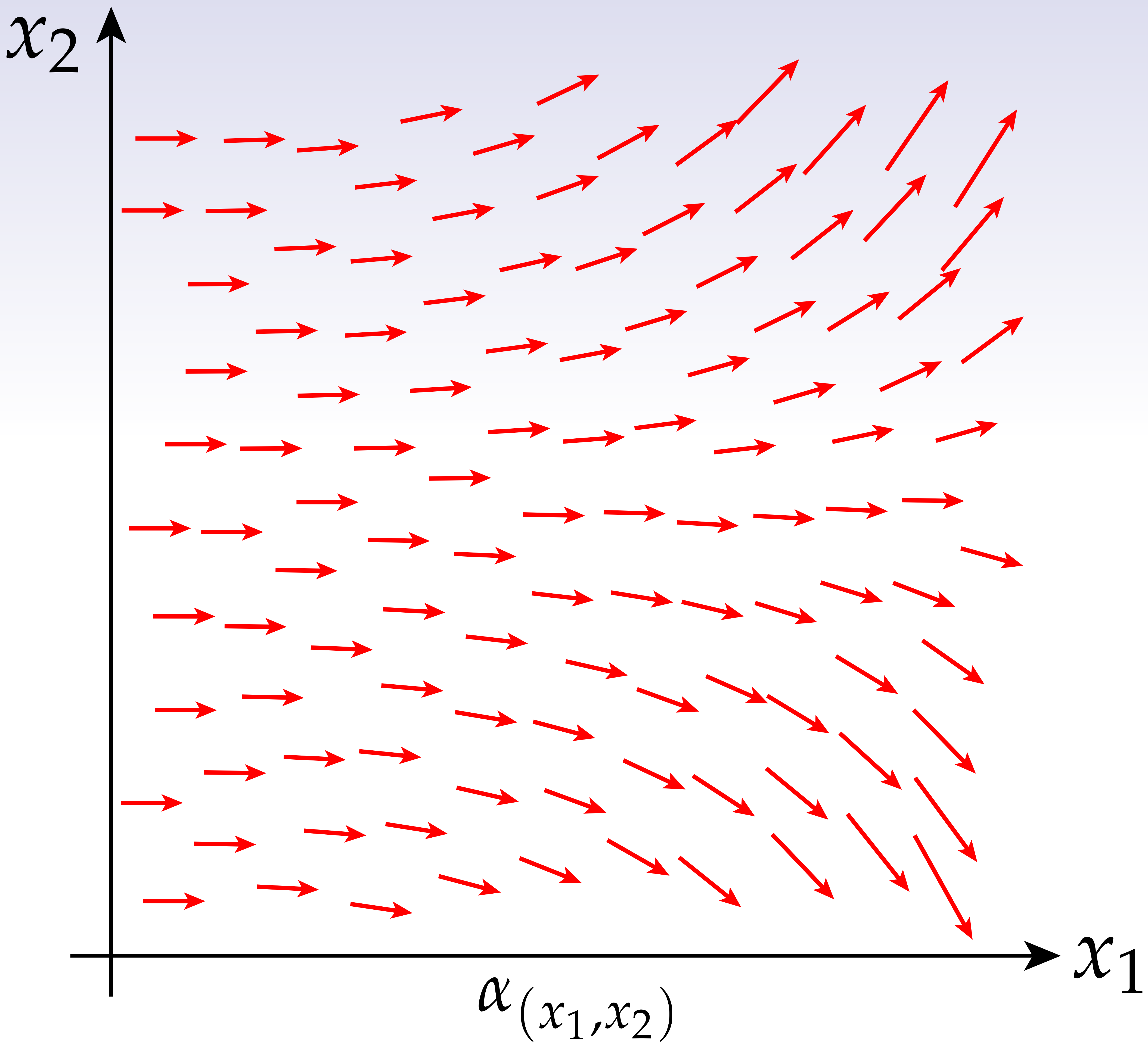
Assigns a scalar to each point. *E.g.*, in 2D we have a value at each point (x_1, x_2):



Note: exactly the same thing as a *scalar function!*

Differential 1-Form

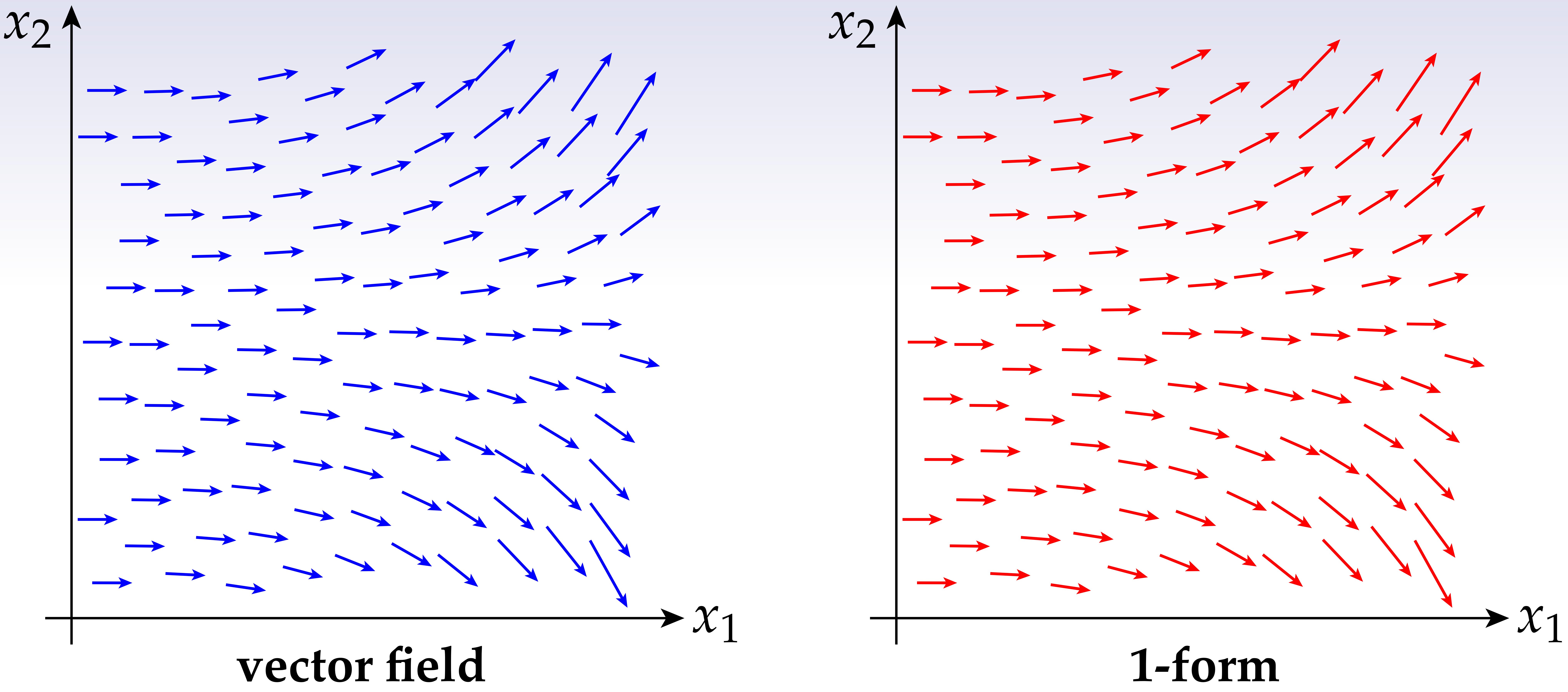
Assigns a 1-form each point. *E.g.*, in 2D we have a 1-form at each point (x_1,x_2):



Note: NOT the same thing as a vector field!

Vector Field vs. Differential 1-Form

Superficially, vector fields and differential 1-forms look the same (in R^n):



But recall that a 1-form is a *linear function* from a vector to a scalar (here, at each point.)

Applying a Differential 1-Form to a Vector Field

At each point (x_1, x_2), we can therefore use a 1-form to measure a vector field:

vector field
$X_{(x_1,x_2)}$

differential 1-form
$\alpha_{(x_1,x_2)}$

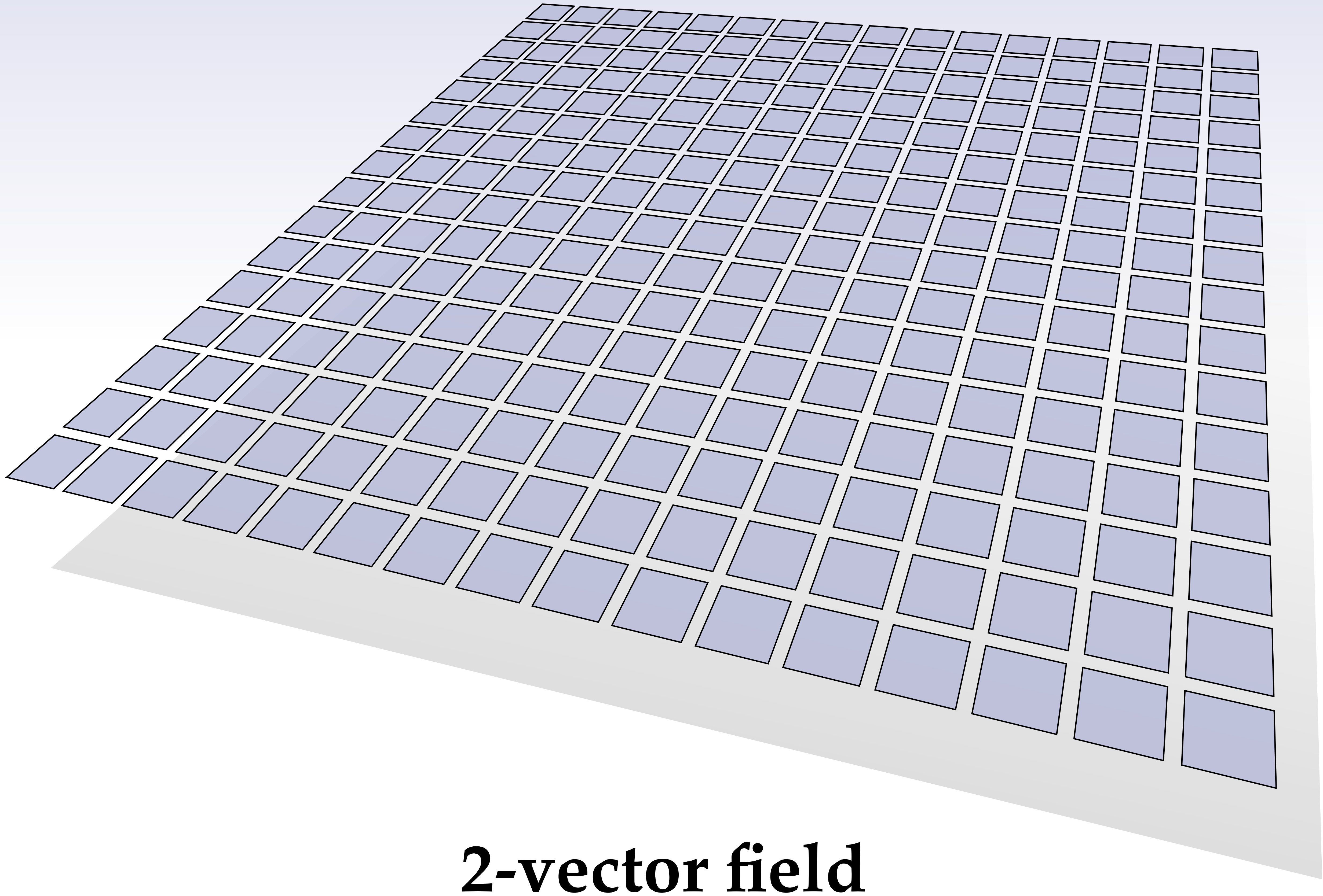
(superimposed)

scalar function
$\alpha(X)_{(x_1,x_2)}$

Intuition: resulting function indicates "how strong" X is along α.

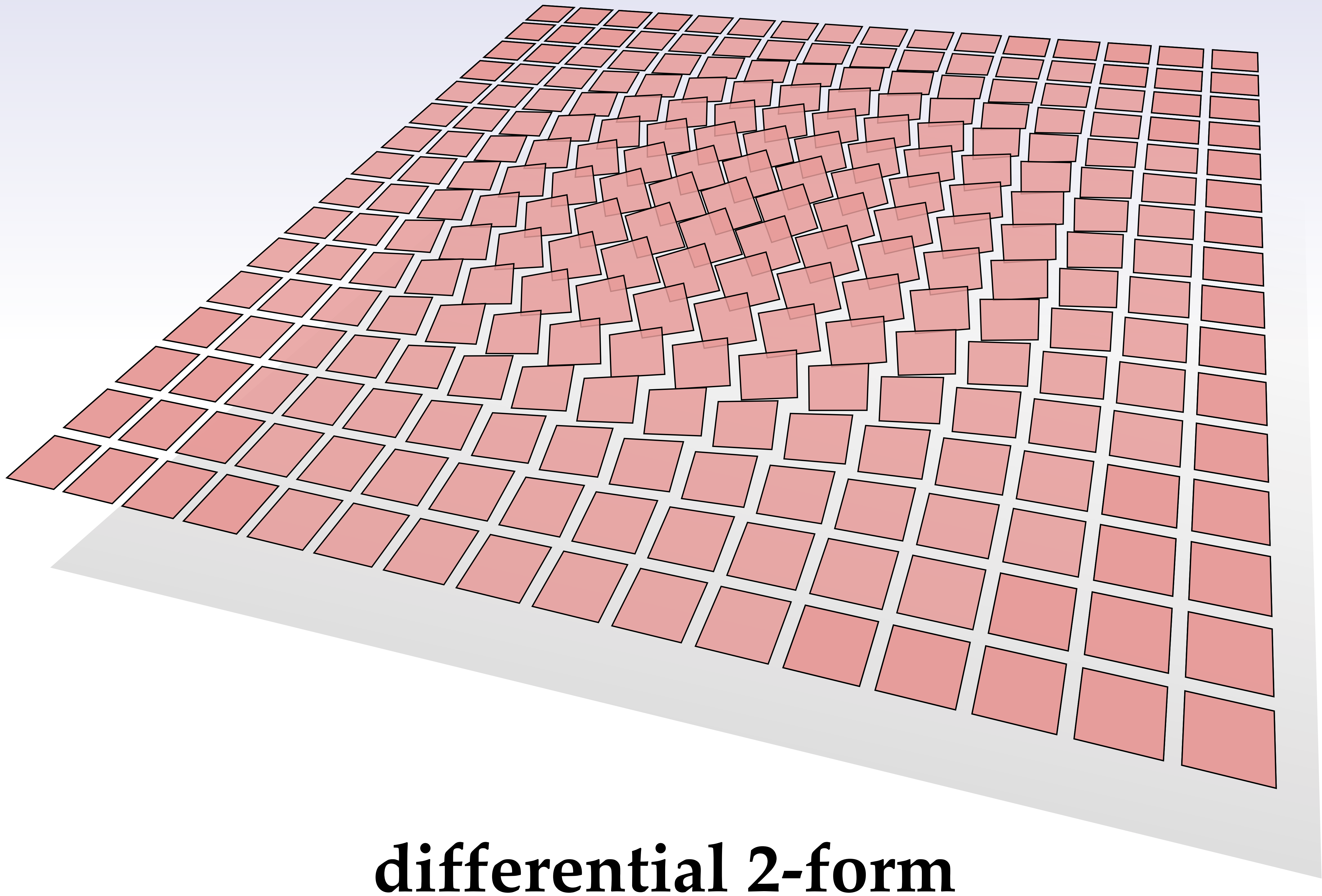
Differential 2-Forms

Likewise, a differential 2-form is an area measurement at each point (x_1, x_2, x_3):



2-vector field

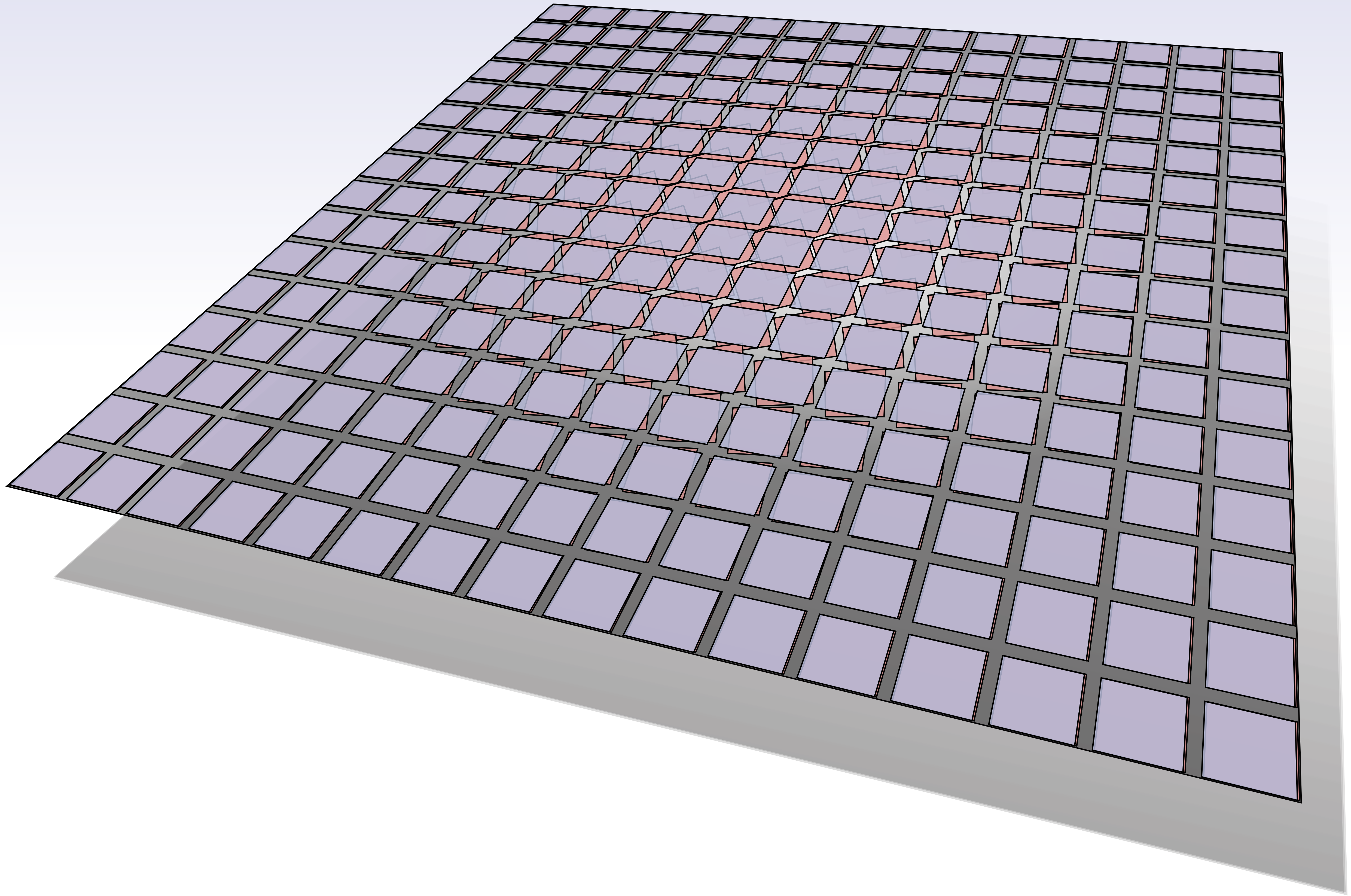
$(X \wedge Y)_{(x_1, x_2, x_3)}$



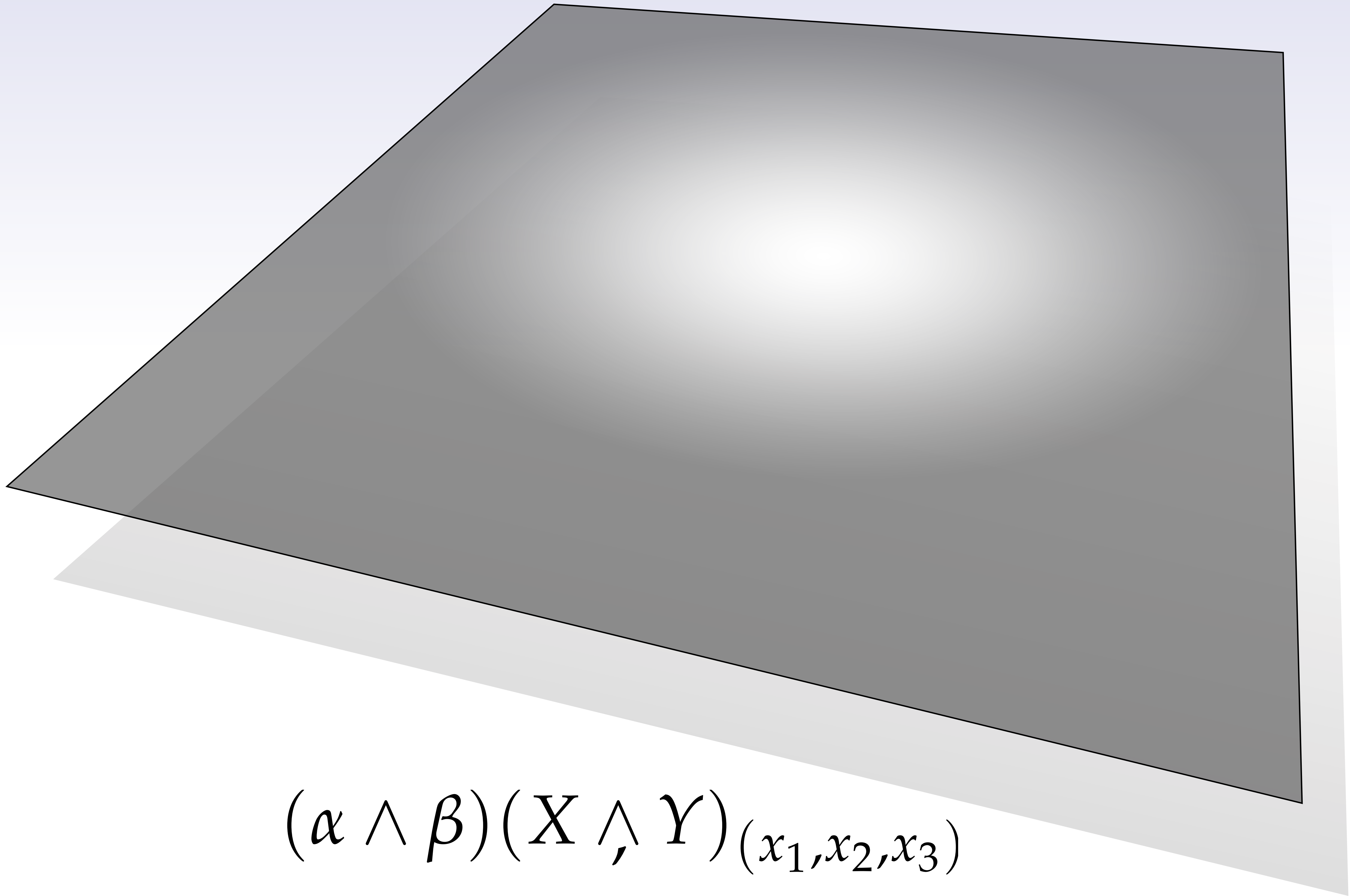
differential 2-form

$(\alpha \wedge \beta)_{(x_1, x_2, x_3)}$

Differential 2-Forms



Differential 2-Forms



$$(\alpha \wedge \beta)(X \wedge Y)_{(x_1,x_2,x_3)}$$

Resulting function says how much a 2-vector field “lines up” with a given 2-form.

Pointwise Operations on Differential k-Forms

- Most operations on differential k-forms simply apply that operation at each point.
- *E.g.*, consider two differential forms α, β on R^n. At each point $p := (x_1,\ldots,x_n)$,

$$(\star\alpha)_p := \star(\alpha_p)$$

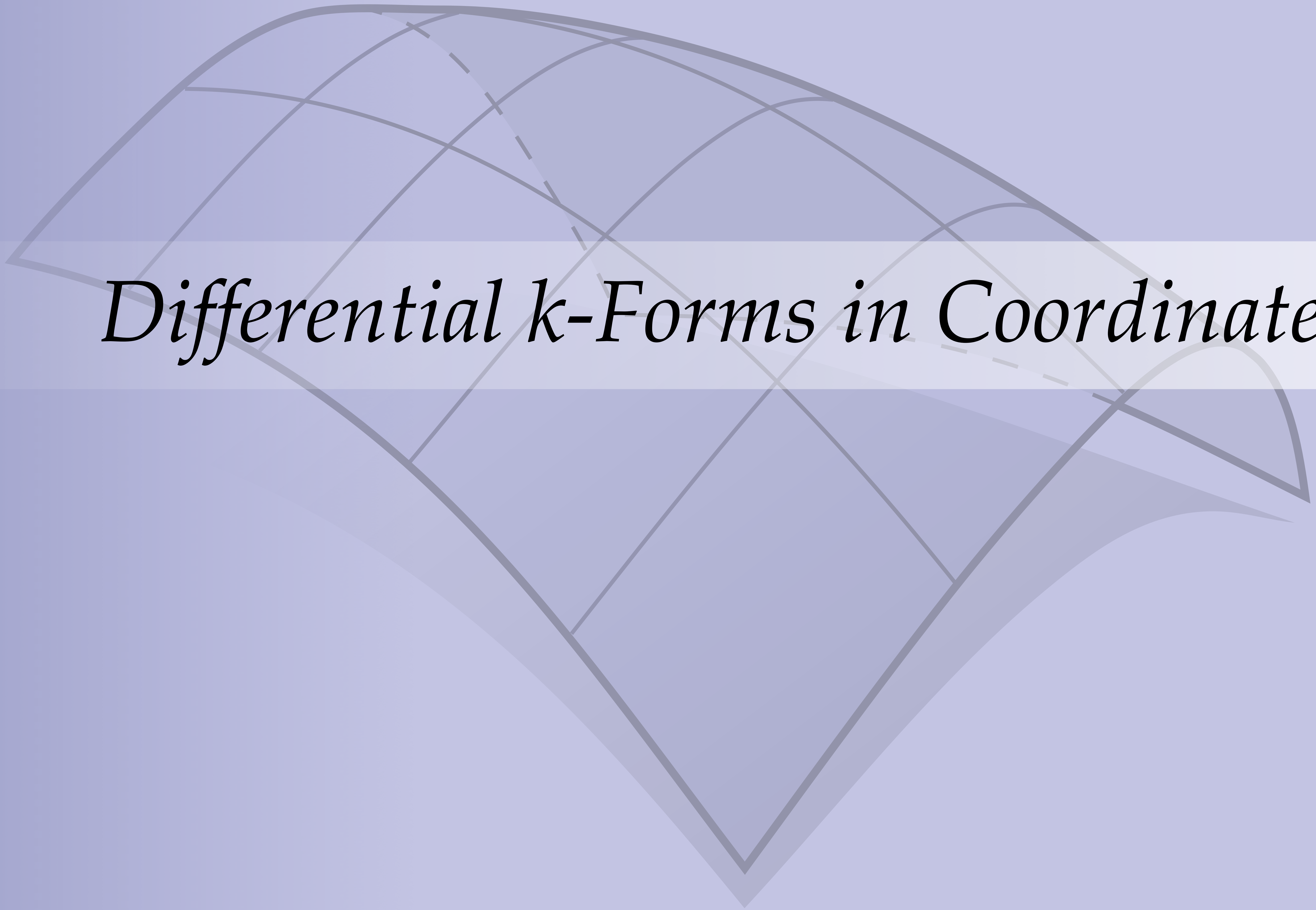
$$(\alpha \wedge \beta)_p := (\alpha_p) \wedge (\beta_p)$$

- In other words, to get the Hodge star of the *differential* k-form, we just apply the Hodge star to the individual k forms at each point p; to take the wedge of two differential k-forms we just wedge their values at each point.
- Likewise, if $X_1, \ldots, X_k$ are vector fields on all of R^n, then

$$\alpha(X_1,\ldots,X_k)_p := (\alpha_p)((X_1)_p,\ldots,(X_k)_p)$$

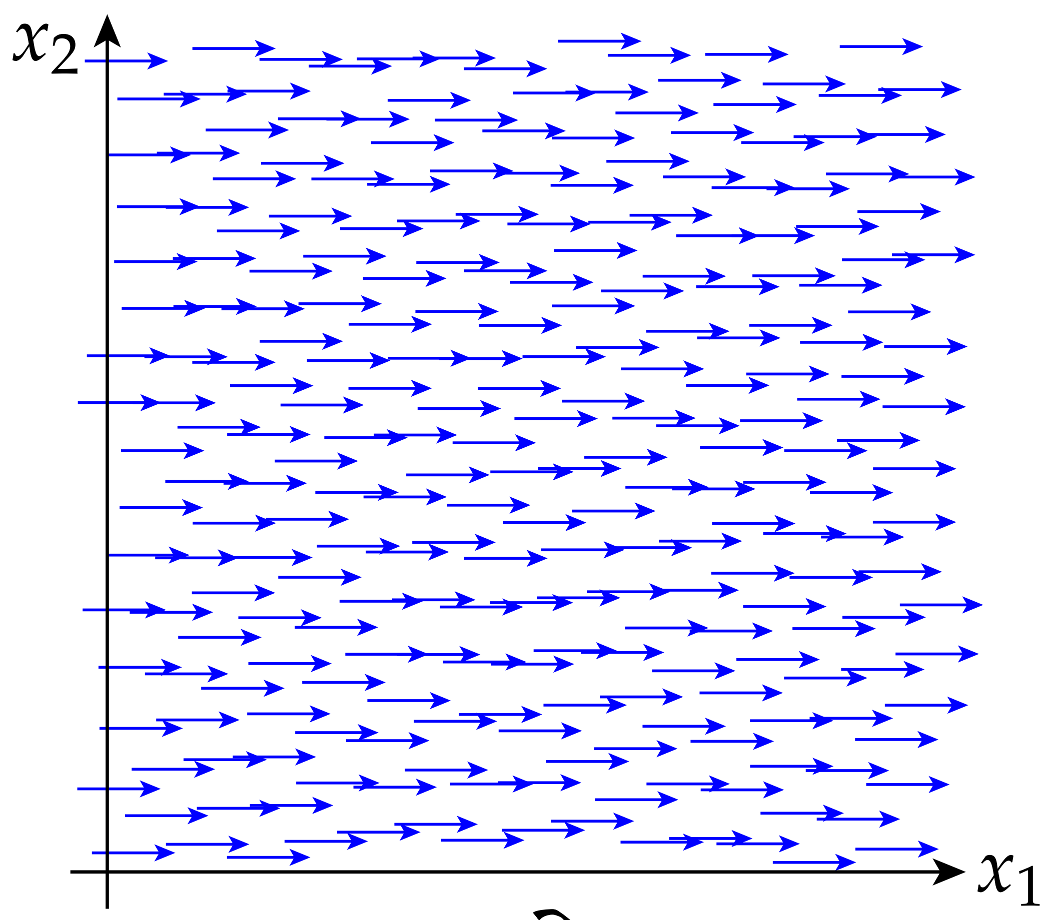
Typically we just drop the p entirely and write $\star\alpha$, $\alpha \wedge \beta$, $\alpha(X,Y)$, *etc.*

Differential k-Forms in Coordinates

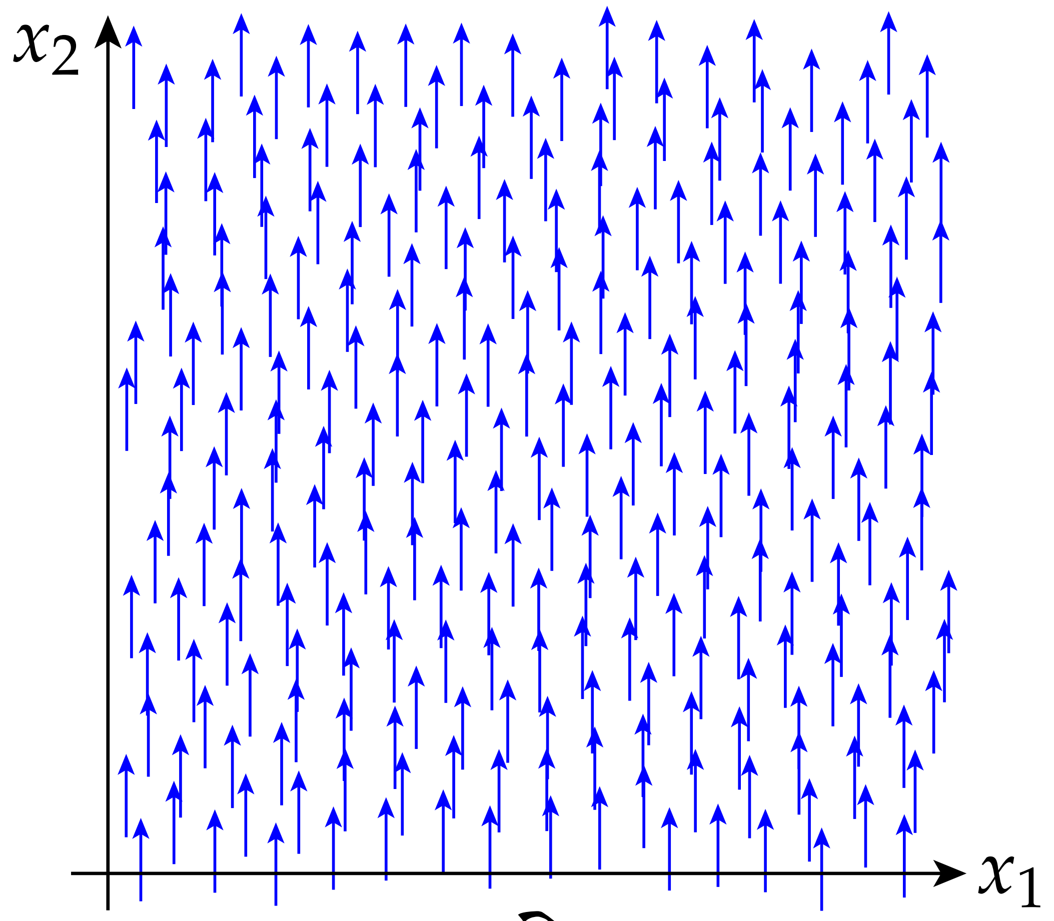


Basis Vector Fields

- Just as we can pick a basis for *vectors*, we can also pick a basis for *vector fields*
- The standard basis for vector fields on R^n are just **constant** vector fields of unit magnitude pointing along each of the coordinate axes:



$$\frac{\partial}{\partial x^1}$$

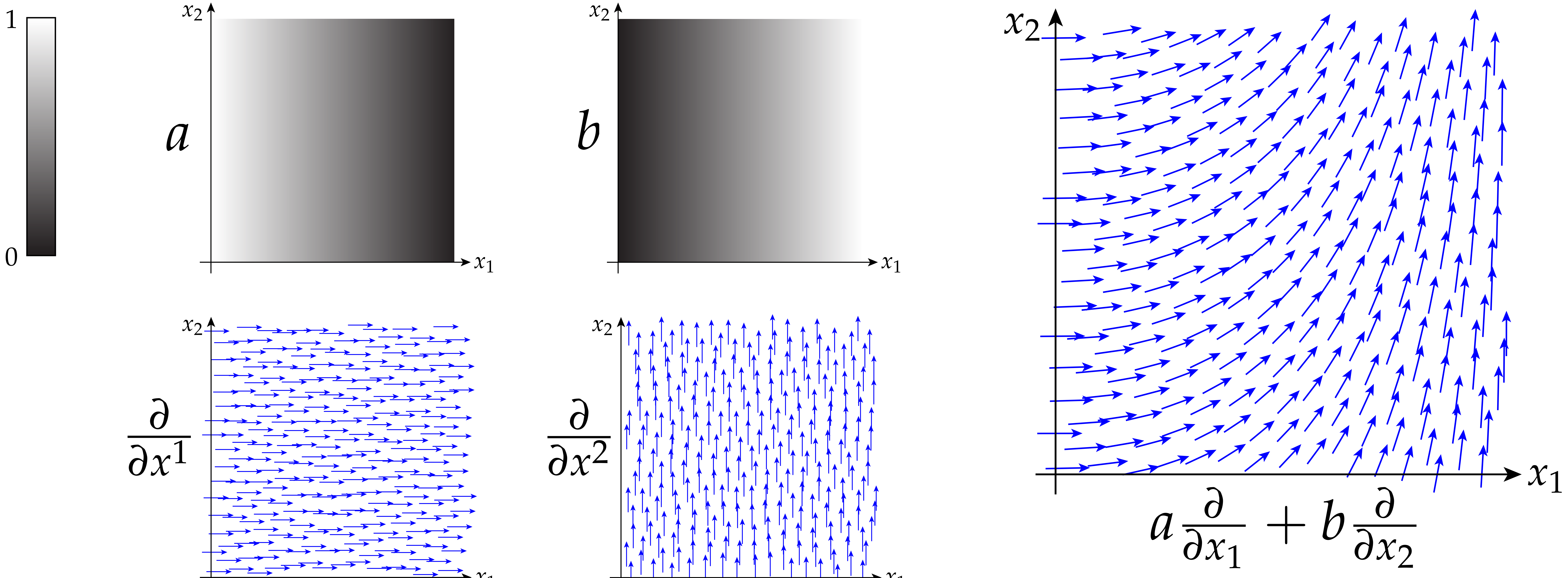


$$\frac{\partial}{\partial x^2}$$

- For historical reasons, these fields have funny-looking names that look like partial derivatives. But you will do yourself a *huge* favor by **forgetting that they have anything at all to do with derivatives!** (For now…)

Basis Expansion of Vector Fields

- Any other vector field is then a linear combination of the basis vector fields…
  - …*but*, coefficients of linear combination vary across the domain:

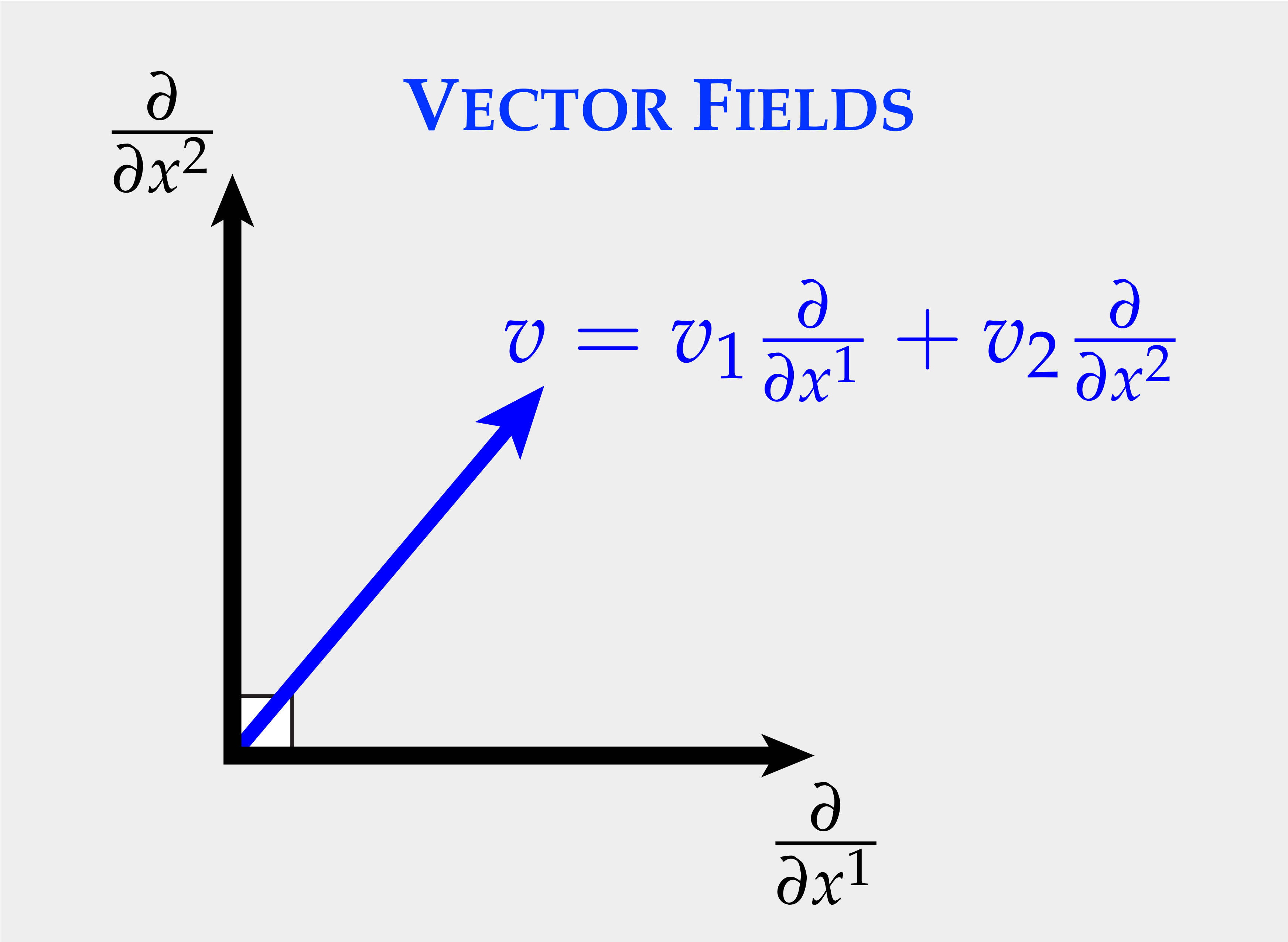


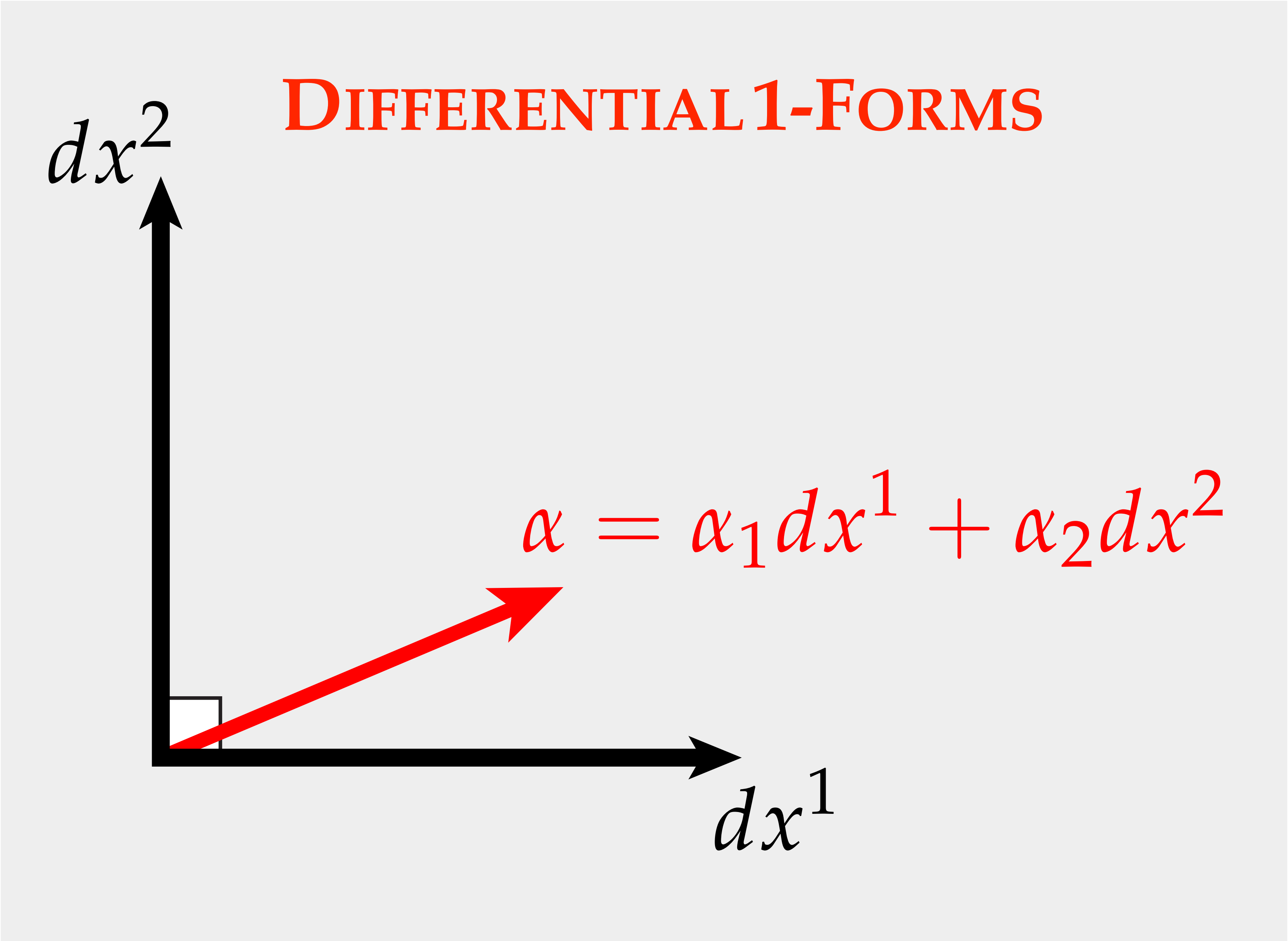
GOOF GOOF

Q: What would happen if we didn't allow coefficients to vary?

Bases for Vector Fields and Differential 1-forms

The story is nearly identical for differential 1-forms, but with different bases:



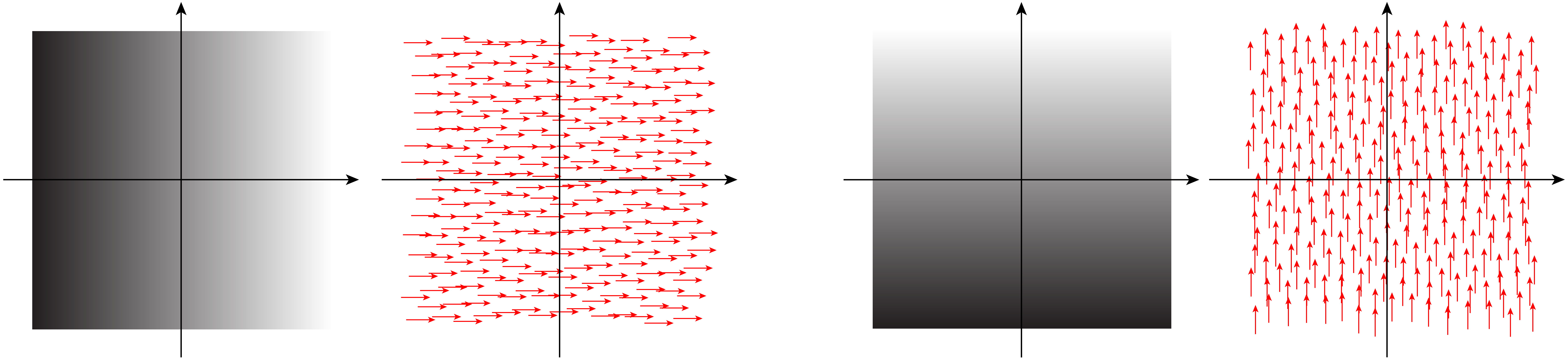


$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

Stay sane: think of these symbols as *bases*; forget they look like *derivatives!*

Coordinate Bases as Derivatives

Q: That being said, why the heck do we use symbols that look like *derivatives?*



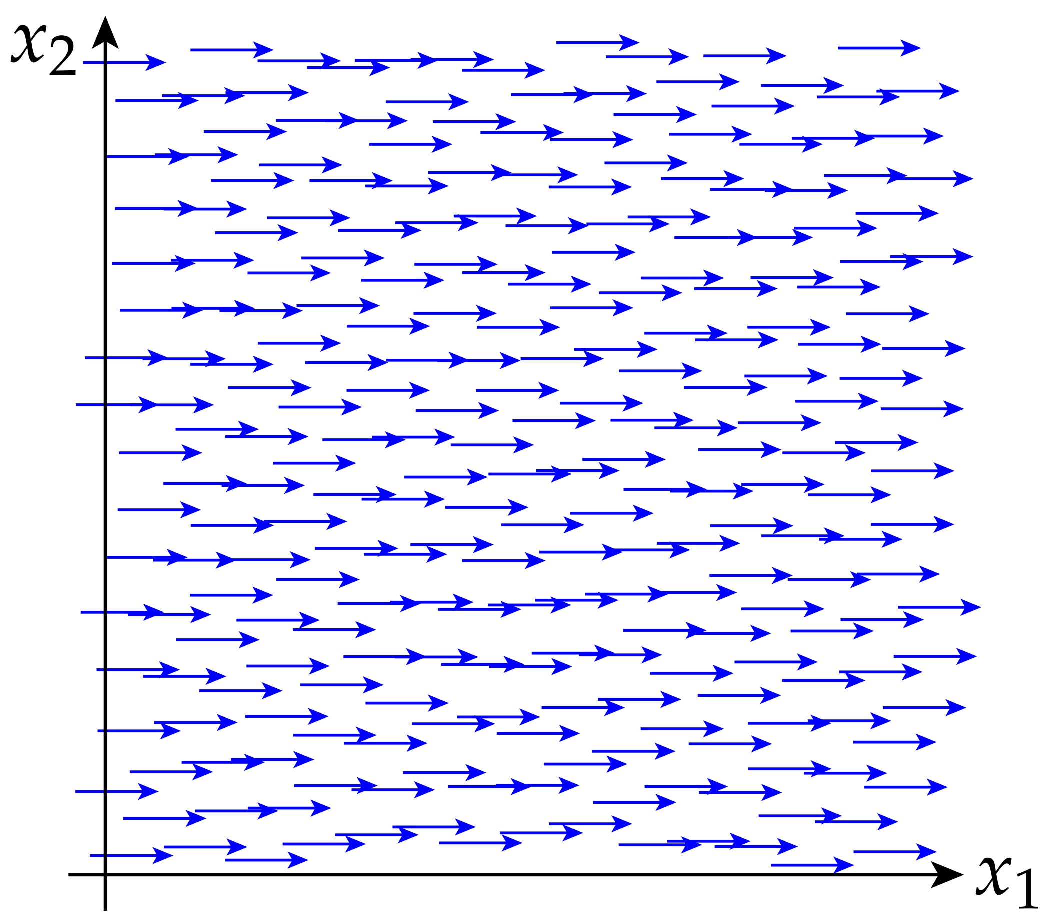
x_1 dx_1 x_2 dx_2

Key idea: derivative of each coordinate function yields a constant basis field.

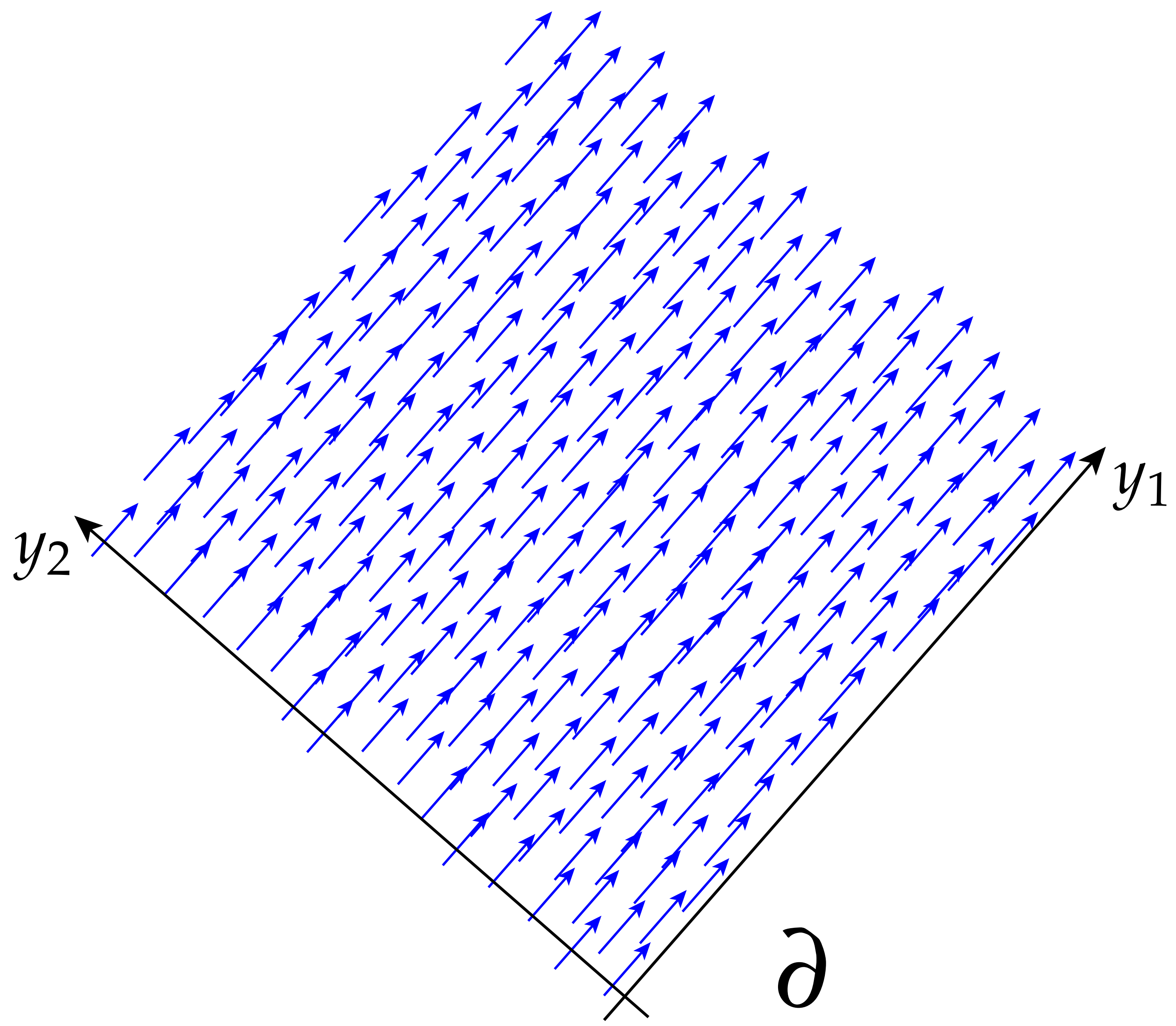
*We'll give a more precise meaning to "d" in a little bit.

Coordinate Notation—Further Apologies

- There is at least one good reason for using this notation for basis fields
- Imagine a situation where we're working with two different coordinate systems:



$$\frac{\partial}{\partial x^1}$$



$$\frac{\partial}{\partial y^1}$$

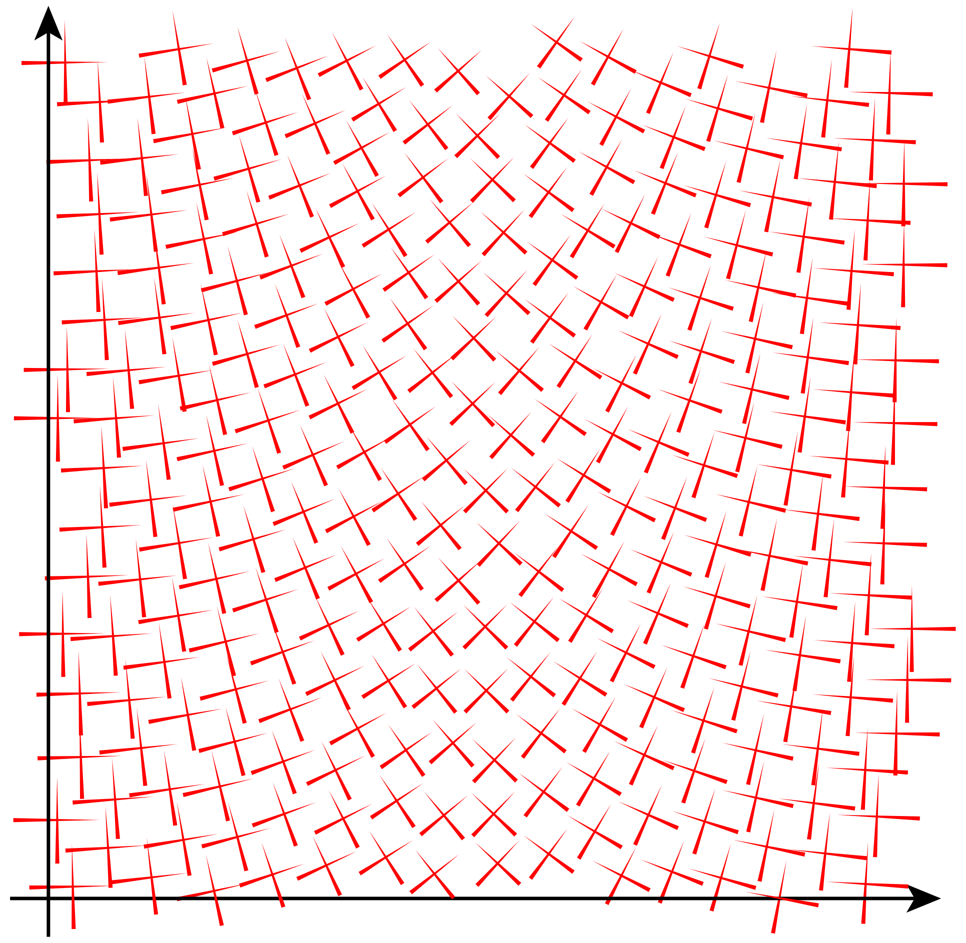
- Including the name of the coordinates in our name for the basis vector field (or basis differential 1-form) makes it clear which one we mean. Not true with e_i, X_i, etc.

Example: Hodge Star of Differential 1-form

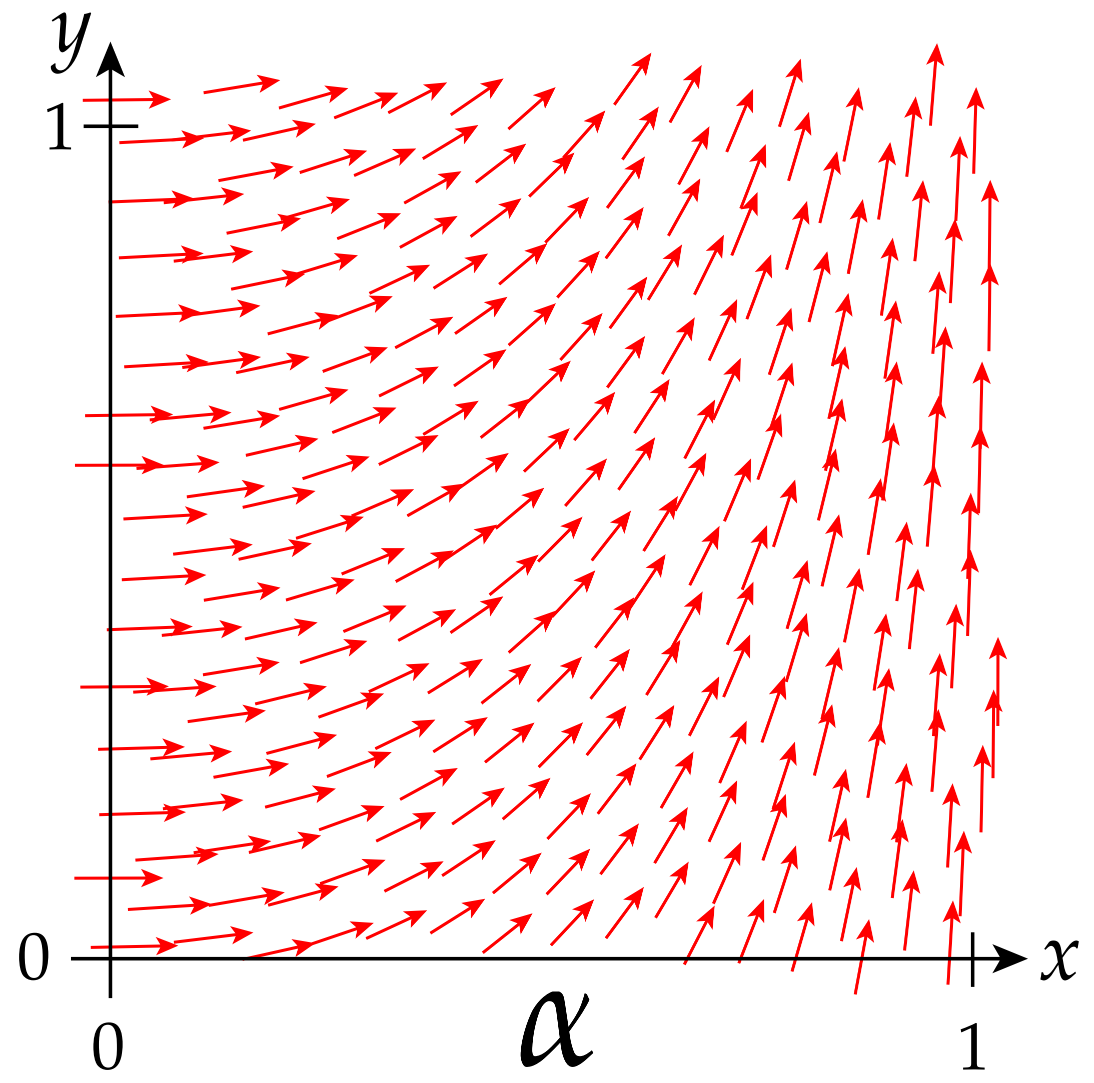
- Consider the differential 1-form $\alpha := (1-x)dx + xdy$
  - Use coordinates (x,y) instead of (x_1,x_2)
  - Notice this expression varies over space

Q: What's its Hodge star?

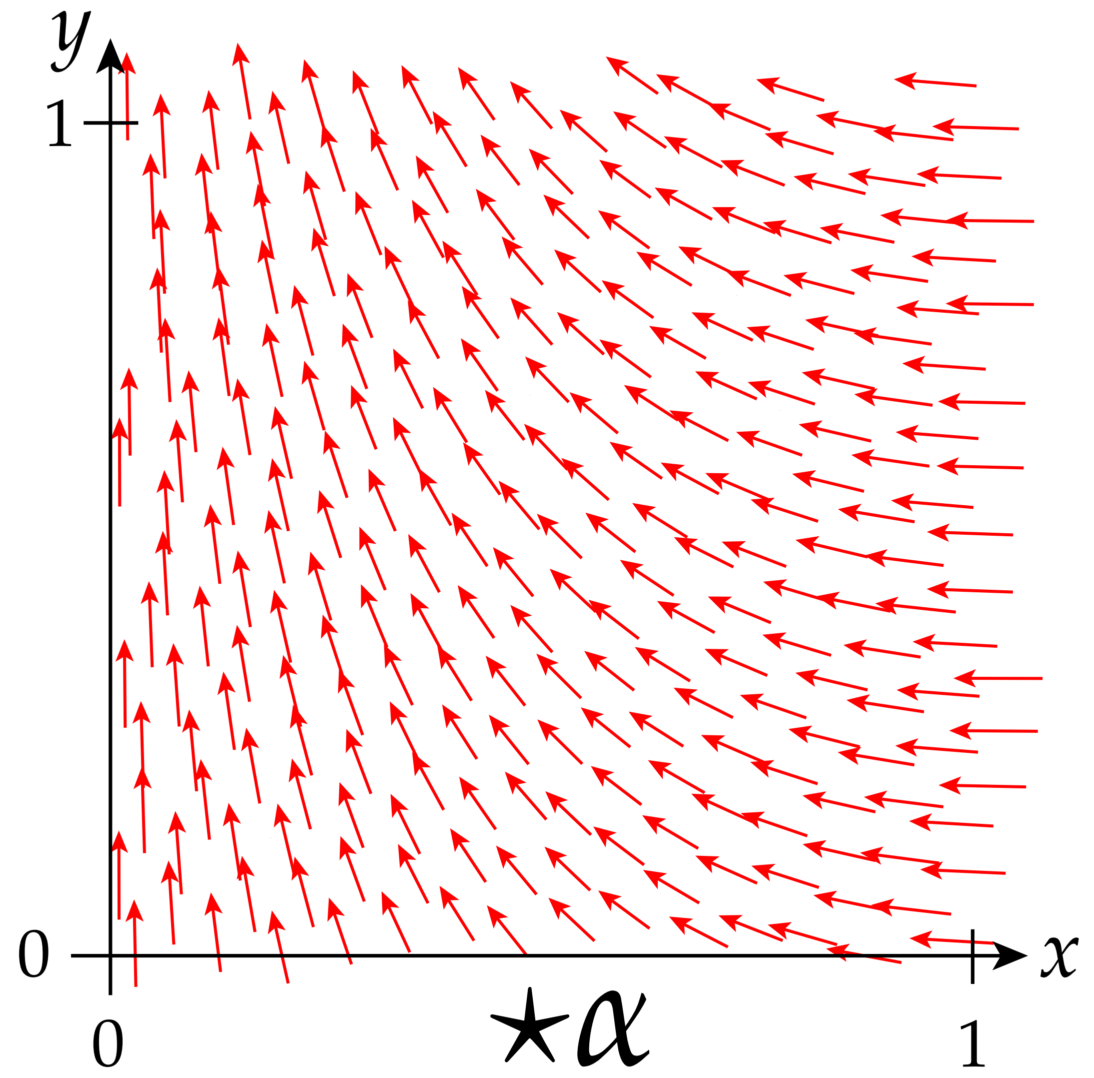
$$\begin{aligned}\star\alpha &= \star((1-x)dx) + \star(xdy) \\ &= (1-x)(\star dx) + x(\star dy) \\ &= (1-x)dy + -xdx\end{aligned}$$



$\alpha, \star\alpha$



α



$\star\alpha$

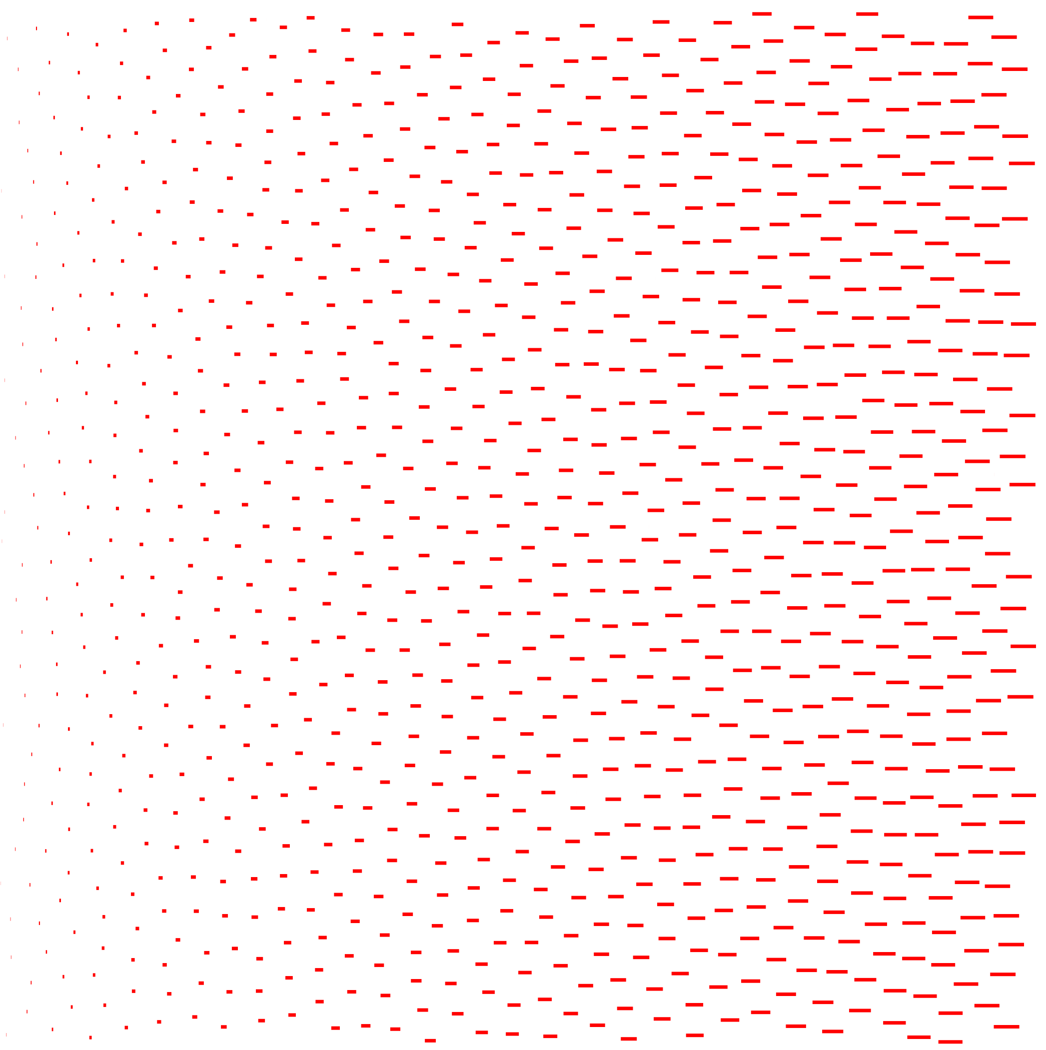
Recall that in 2D, 1-form Hodge star is quarter-turn.

So, when we overlay the two we get little crosses…

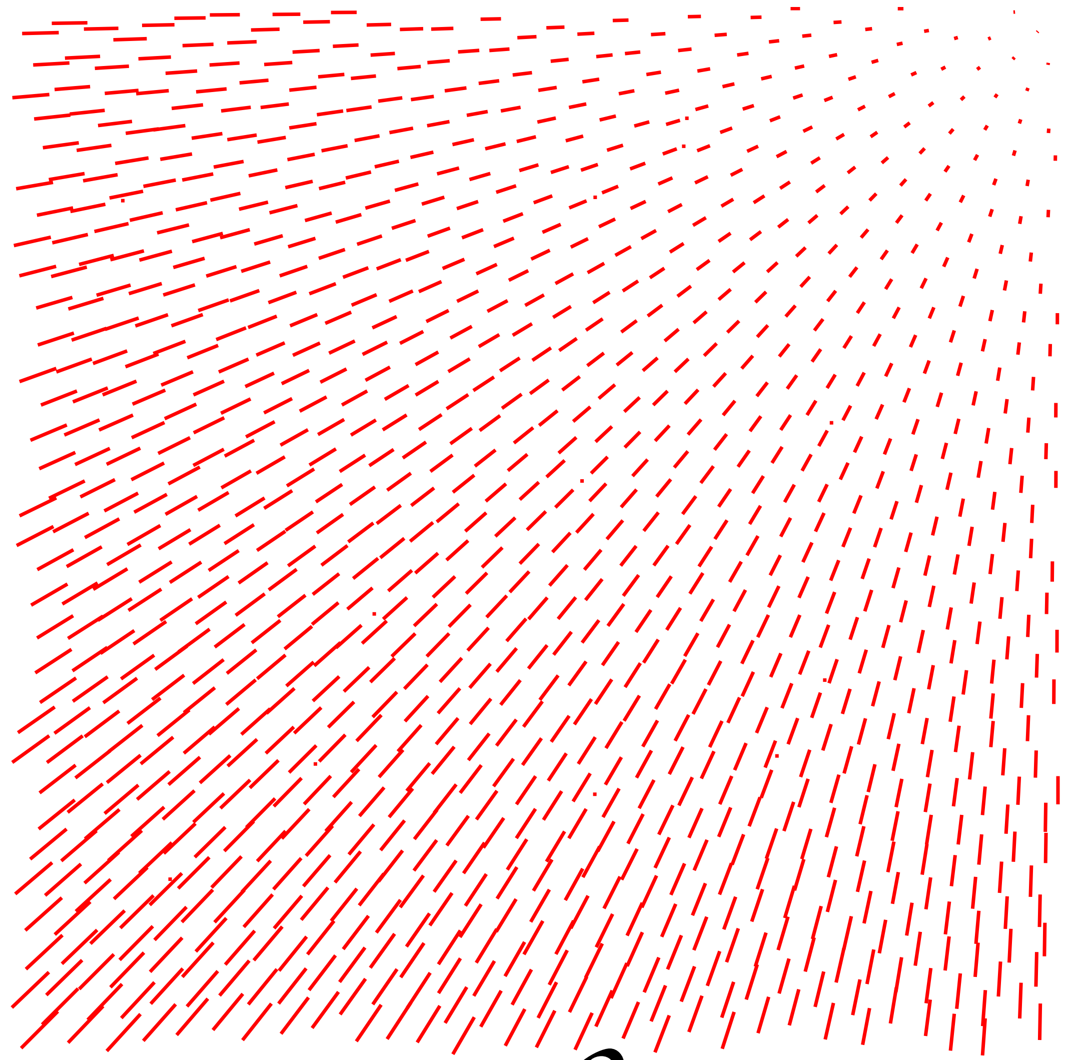
Example: Wedge of Differential 1-Forms

Consider the differential 1-forms*

$$\alpha := xdx, \qquad \beta := (1-x)dx + (1-y)dy$$



α

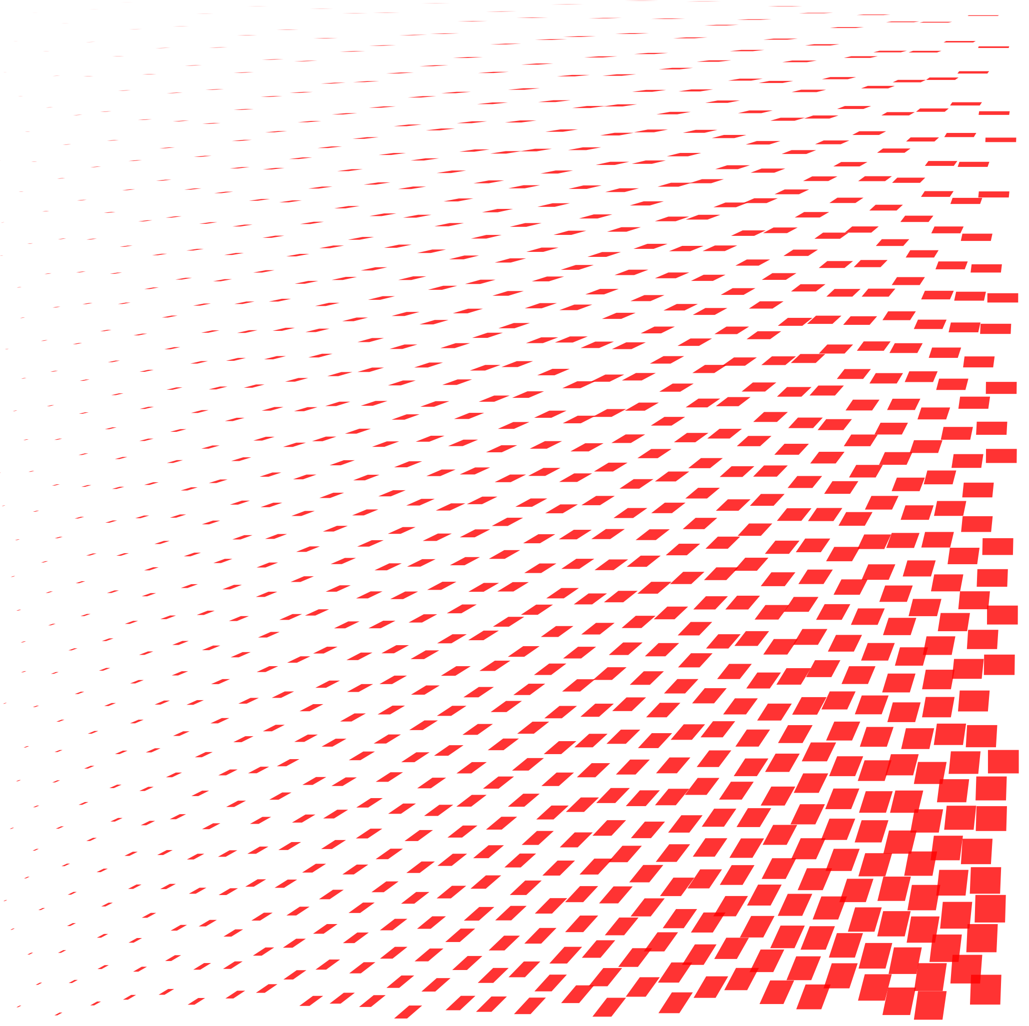


β

Q: What's their wedge product?

$$\begin{aligned}\alpha \wedge \beta &= (xdx) \wedge ((1-x)dx + (1-y)dy) \\ &= (xdx) \wedge ((1-x)dx) + (xdx) \wedge ((1-y)dy) \\ &= x(1-x)\cancelto{0}{dx \wedge dx} + x(1-y)dx \wedge dy \\ &= (x - xy)dx \wedge dy\end{aligned}$$

(What does the result **look** like?)



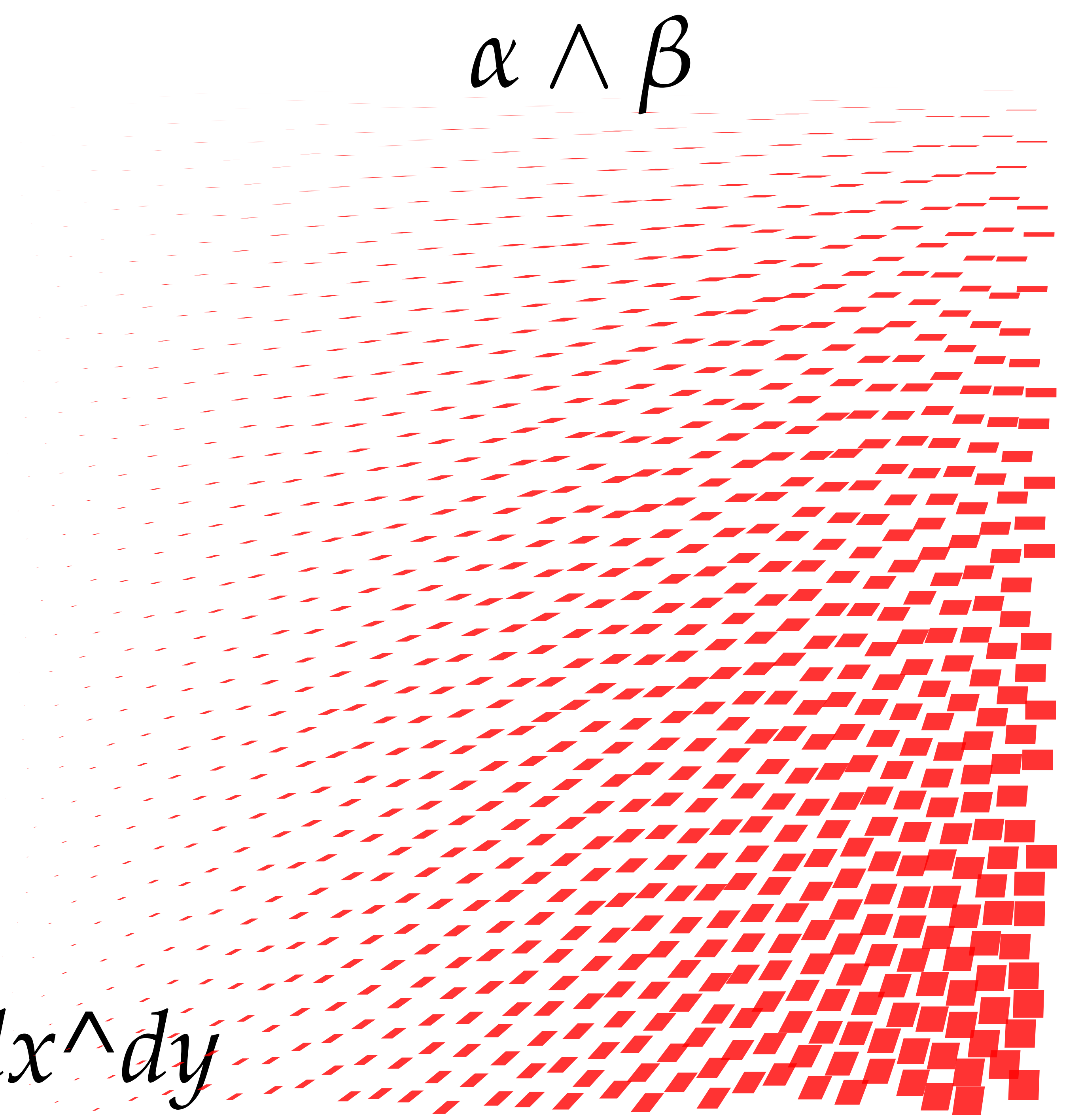
$\alpha \wedge \beta$

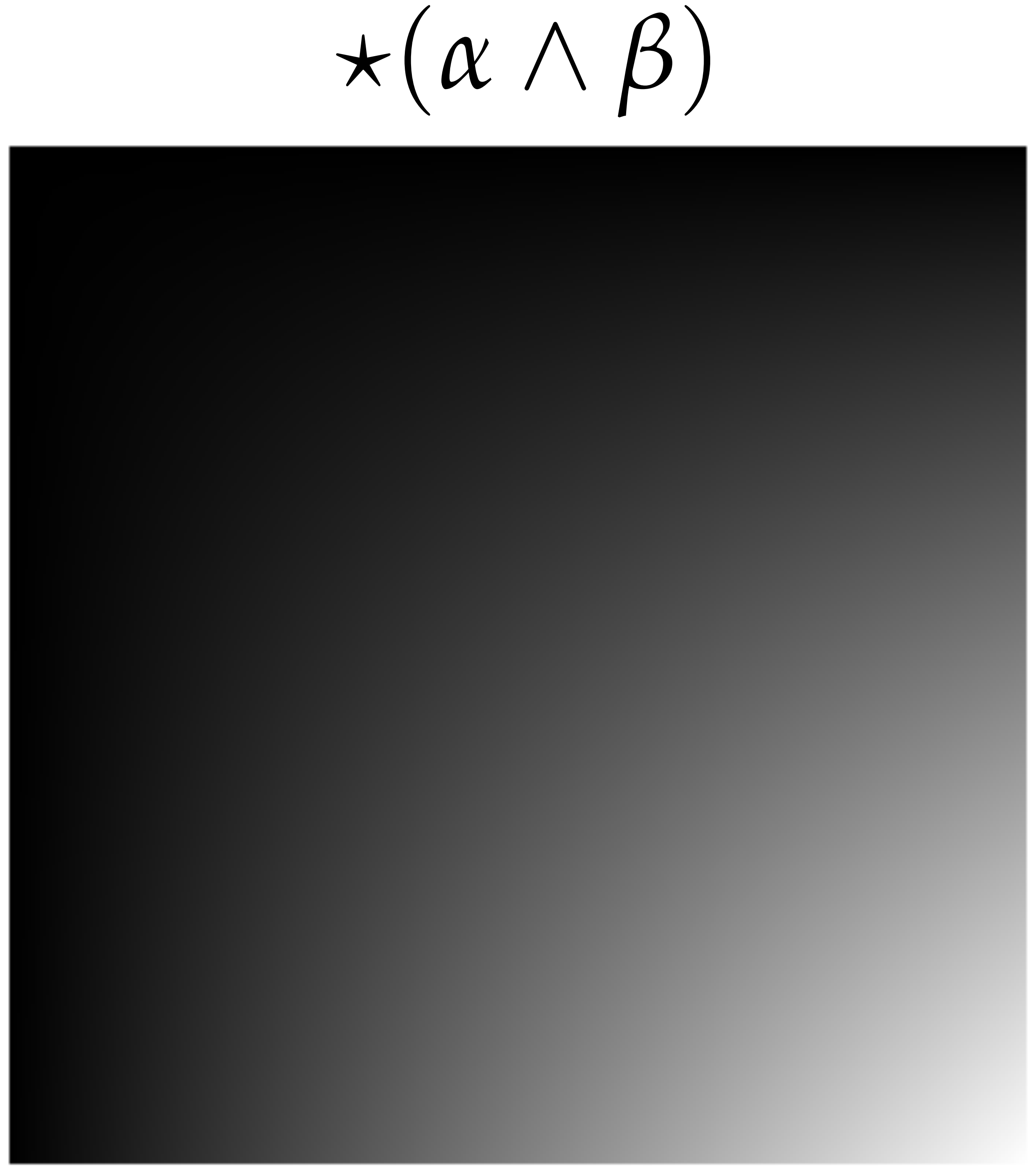
*All plots in this slide (and the next few slides) are over the unit square [0,1] x [0,1].

Volume Form / Differential n-form

- Our picture has little parallelograms
- But what information does our differential 2-form actually encode?

$$\alpha \wedge \beta = (x - xy)dx \wedge dy$$





- Has magnitude (x-xy), and "direction" dx^dy
- But in the plane, *every* differential 2-form will be a multiple of dx^dy!
  - More precisely, some positive scalar function times dx^dy, which measures *unit area*
- In n-dimensions, any *positive* multiple of $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is called a *volume form*.
  - Provides some meaningful (i.e., nonzero, nonnegative) notion of volume.

Applying a Differential 1-Form to a Vector Field

- The whole point of a differential 1-form is to measure vector fields. So let's do it!

$$\alpha := x dx$$

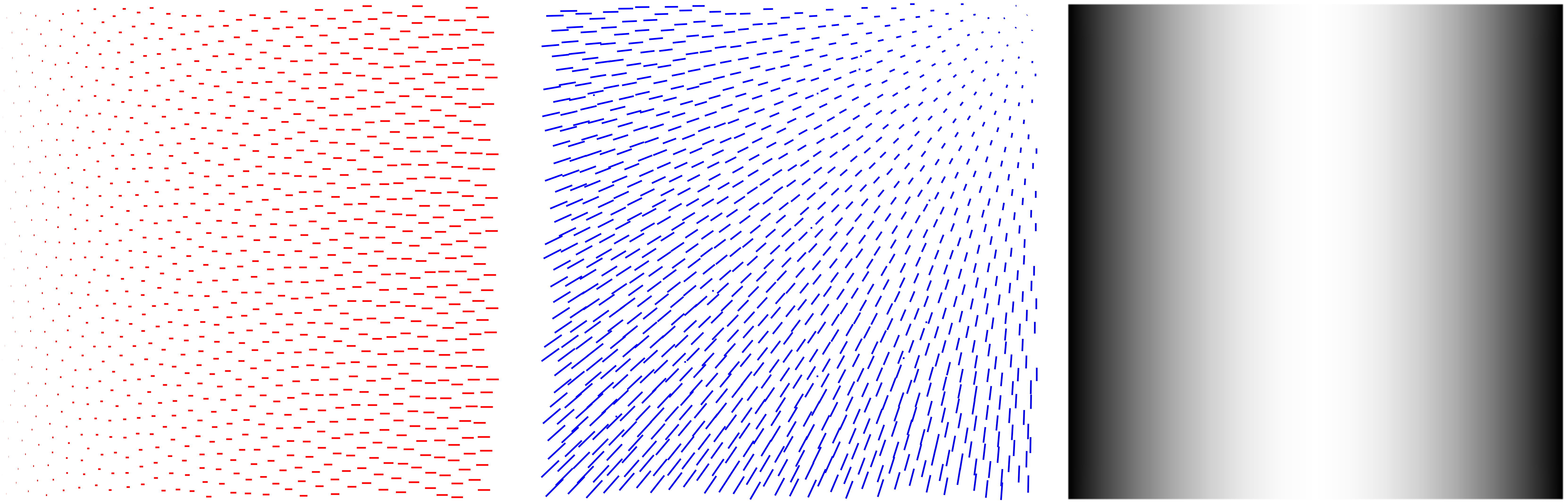
$$X := (1-x)\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y}$$

$$\alpha(X) = (x dx)\left((1-x)\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y}\right)$$

$$= (x dx)\left((1-x)\frac{\partial}{\partial x}\right) + (x dx)\left((1-y)\frac{\partial}{\partial y}\right)$$

$$= (x - x^2)\underbrace{dx(\tfrac{\partial}{\partial x})}_{1} + (x - xy)\underbrace{dx(\tfrac{\partial}{\partial y})}_{0}$$

$$= x - x^2$$

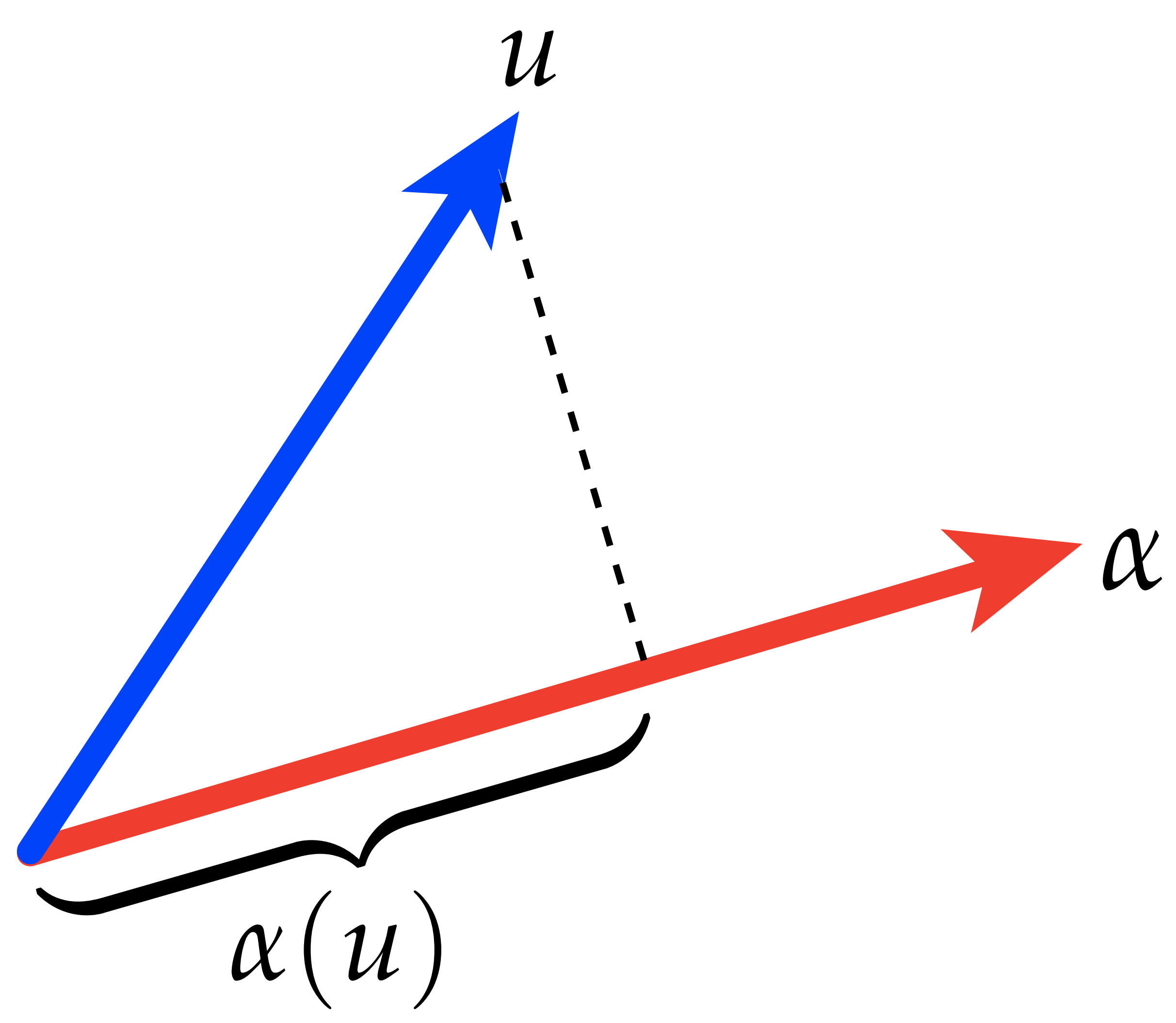


α $\quad$ X $\quad$ $\alpha(X)$

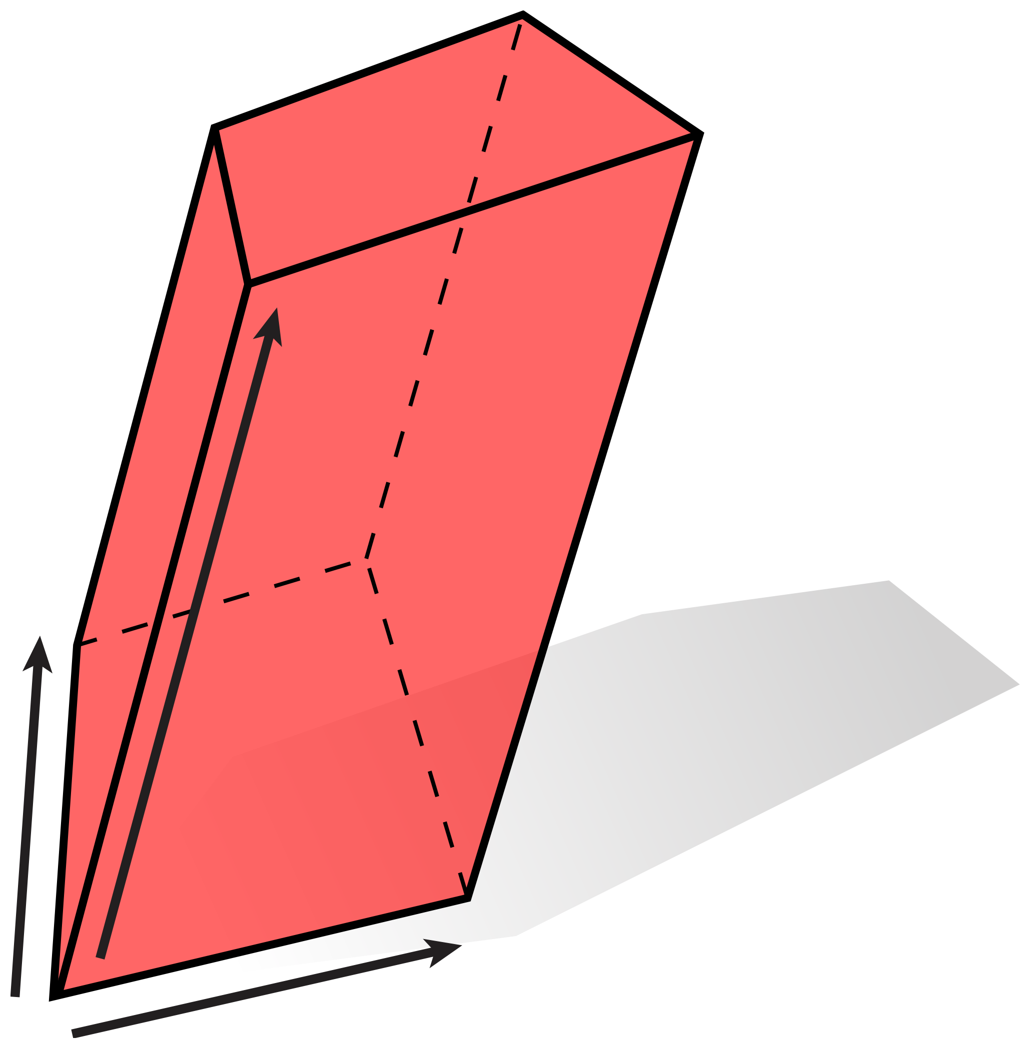
(Kind of like a dot product…)

Differential Forms in R^n - Summary

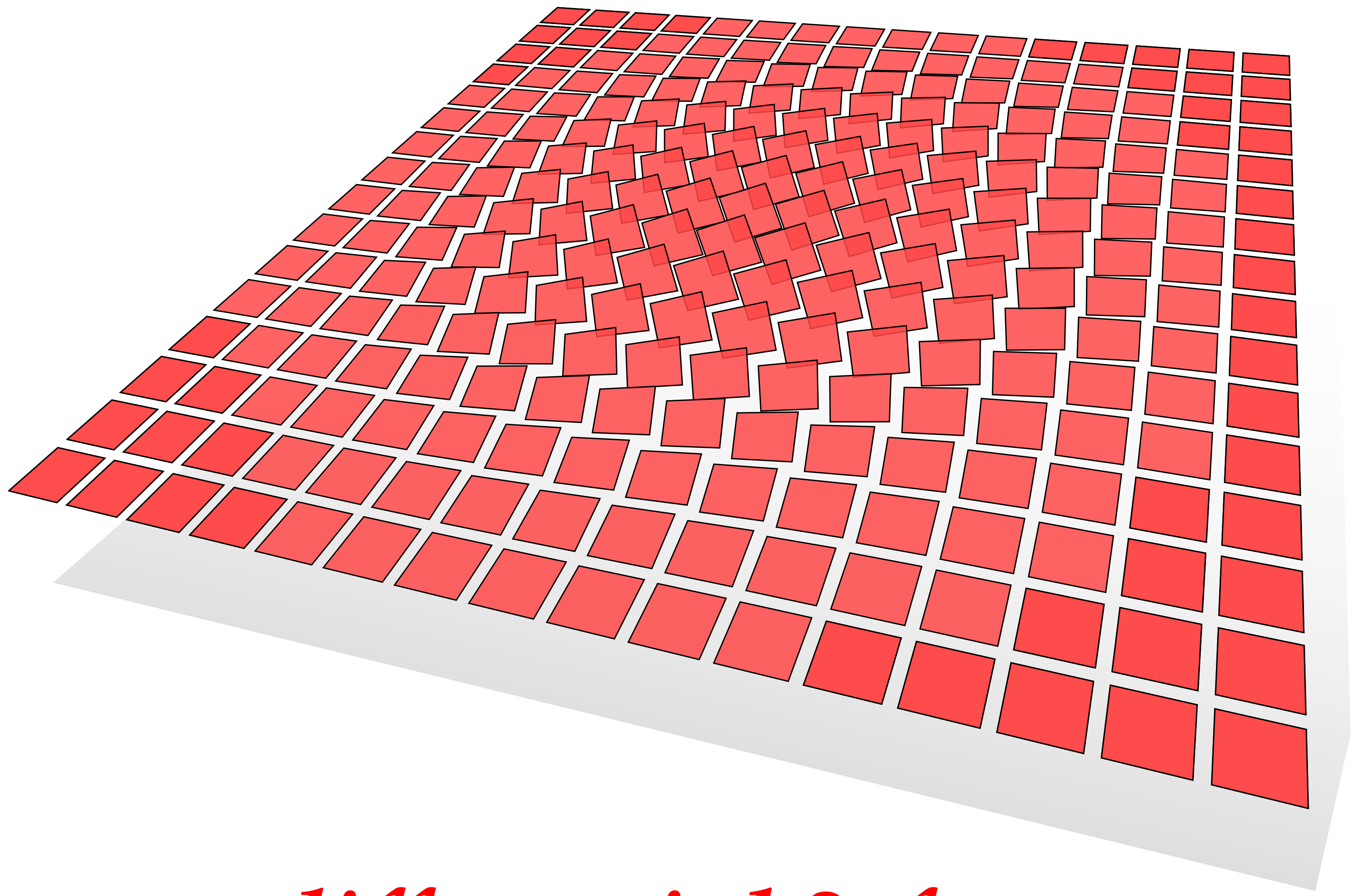
- Started with a vector space V (*e.g.*, R^n)
  - ***(1-forms)*** Dual space V^* of covectors, i.e., linear measurements of vectors
  - ***(k-forms)*** Wedge together k covectors to get a measurement of k-dim. volumes
  - **(*differential k-forms*)** Put a k-form at each point of space



1-form



3-form



differential 2-form

Exterior Algebra & Differential Forms—Summary

	primal	**dual**
vector space	vectors	covectors
exterior algebra	k-vectors	k-forms
spatially-varying	k-vector fields	differential k-forms

Where Are We Going Next?

GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

Linear algebra: "little arrows" (vectors)

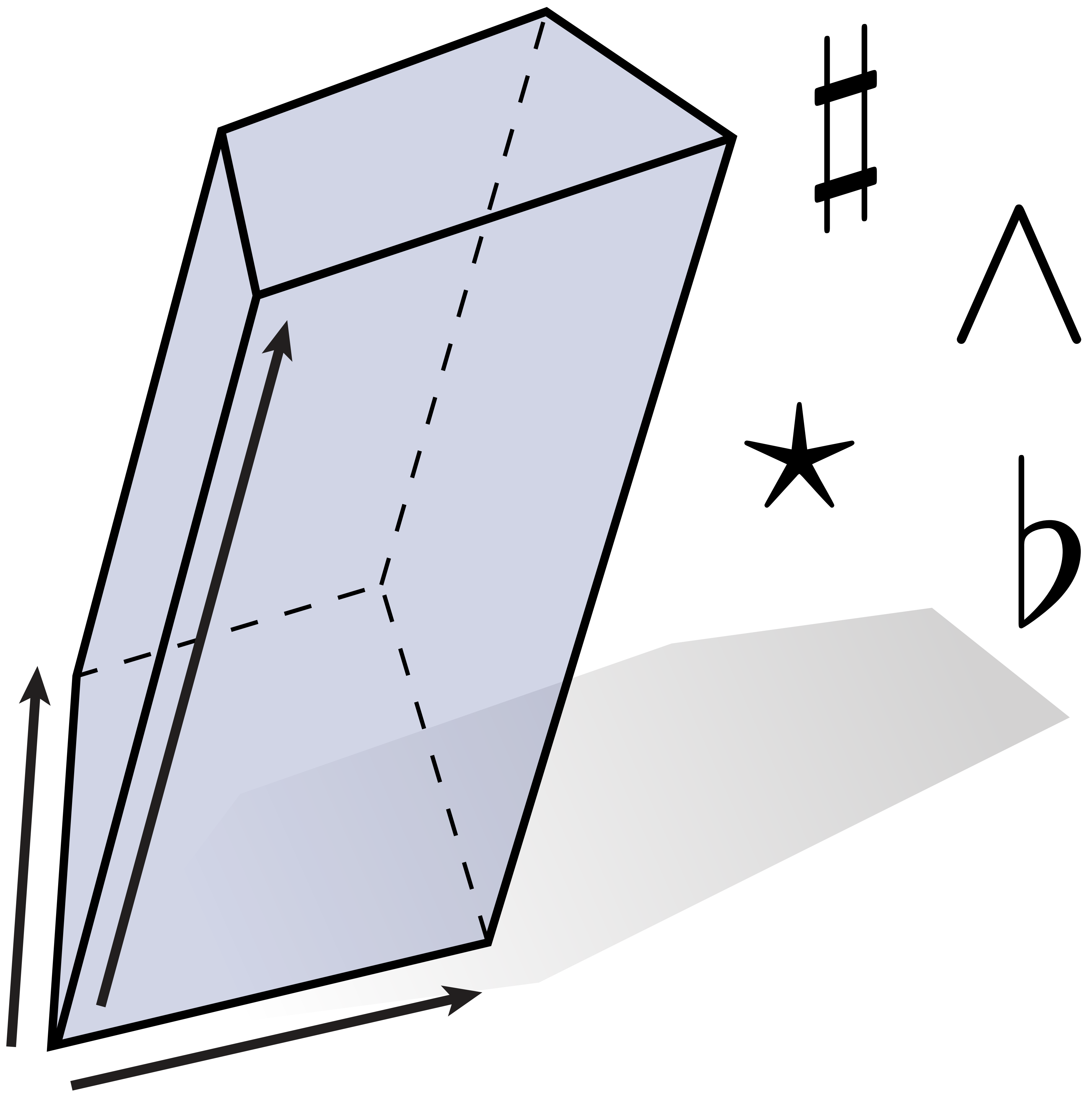
Vector Calculus: how do vectors *change?*

Next few lectures:

Exterior algebra: "little volumes" (k-vectors)

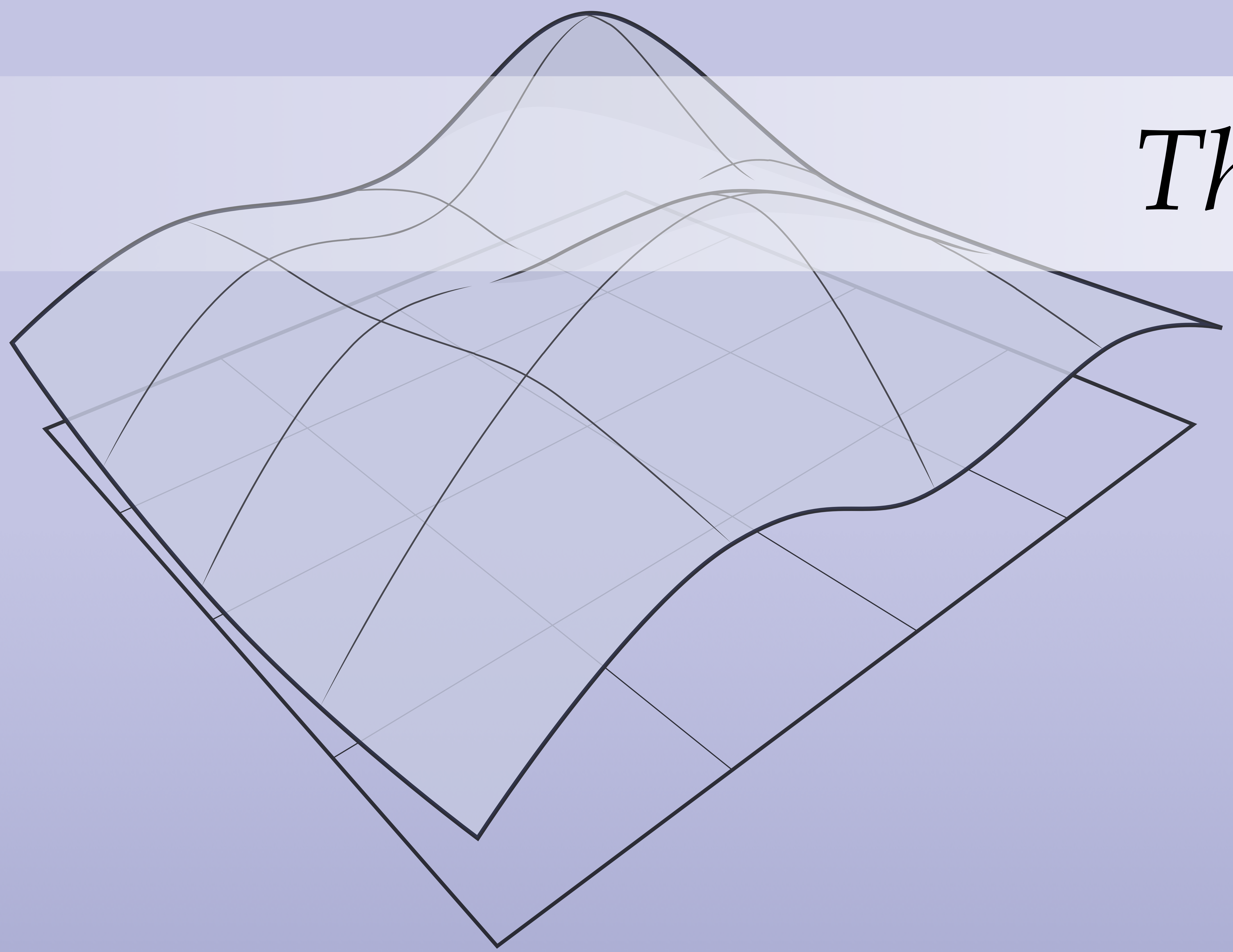
Exterior calculus: how do k-vectors change?

DEC: how do we do all of this on meshes?



Basic idea: replace vector calculus with computation on meshes.

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY
AN APPLIED INTRODUCTION