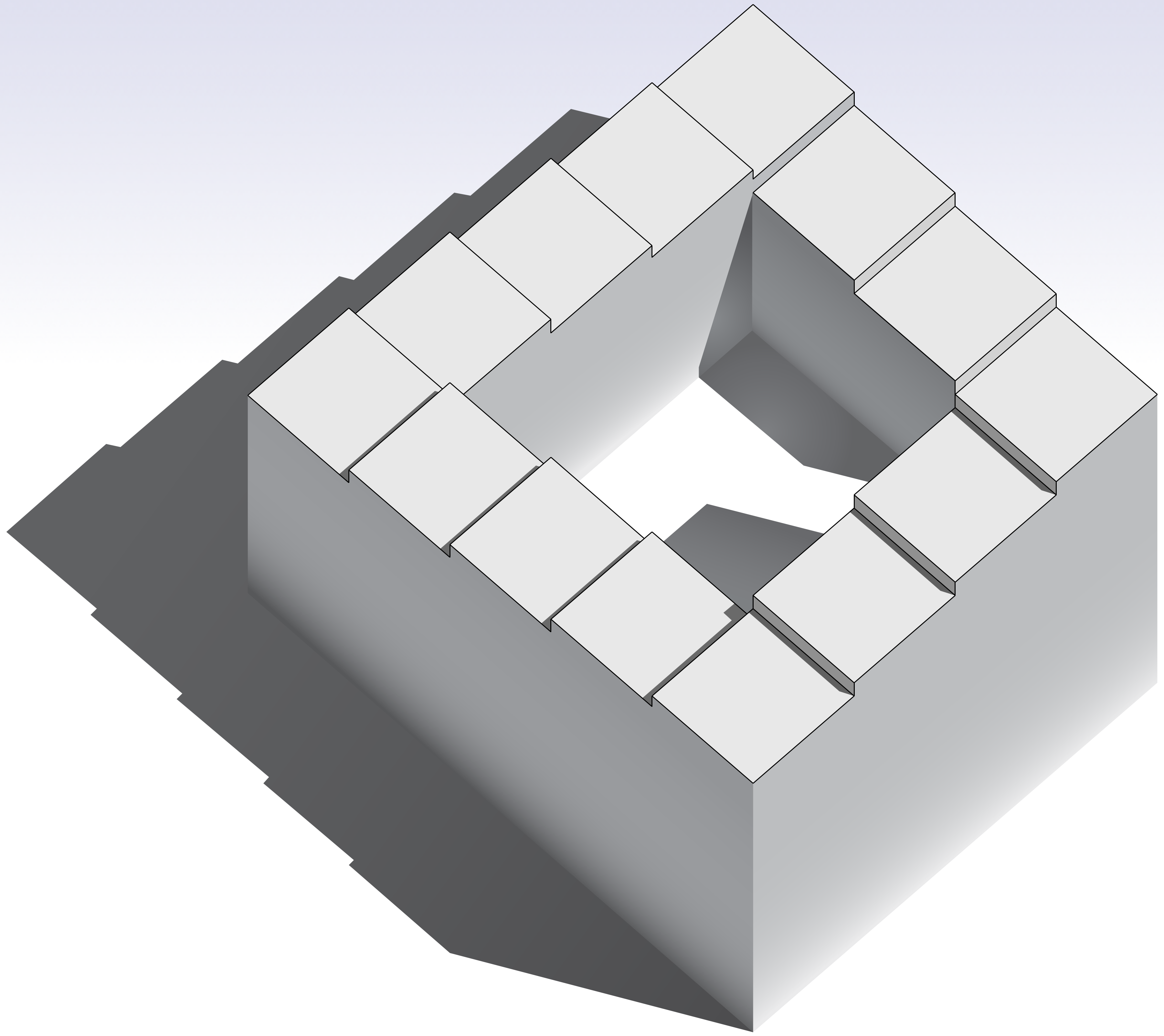
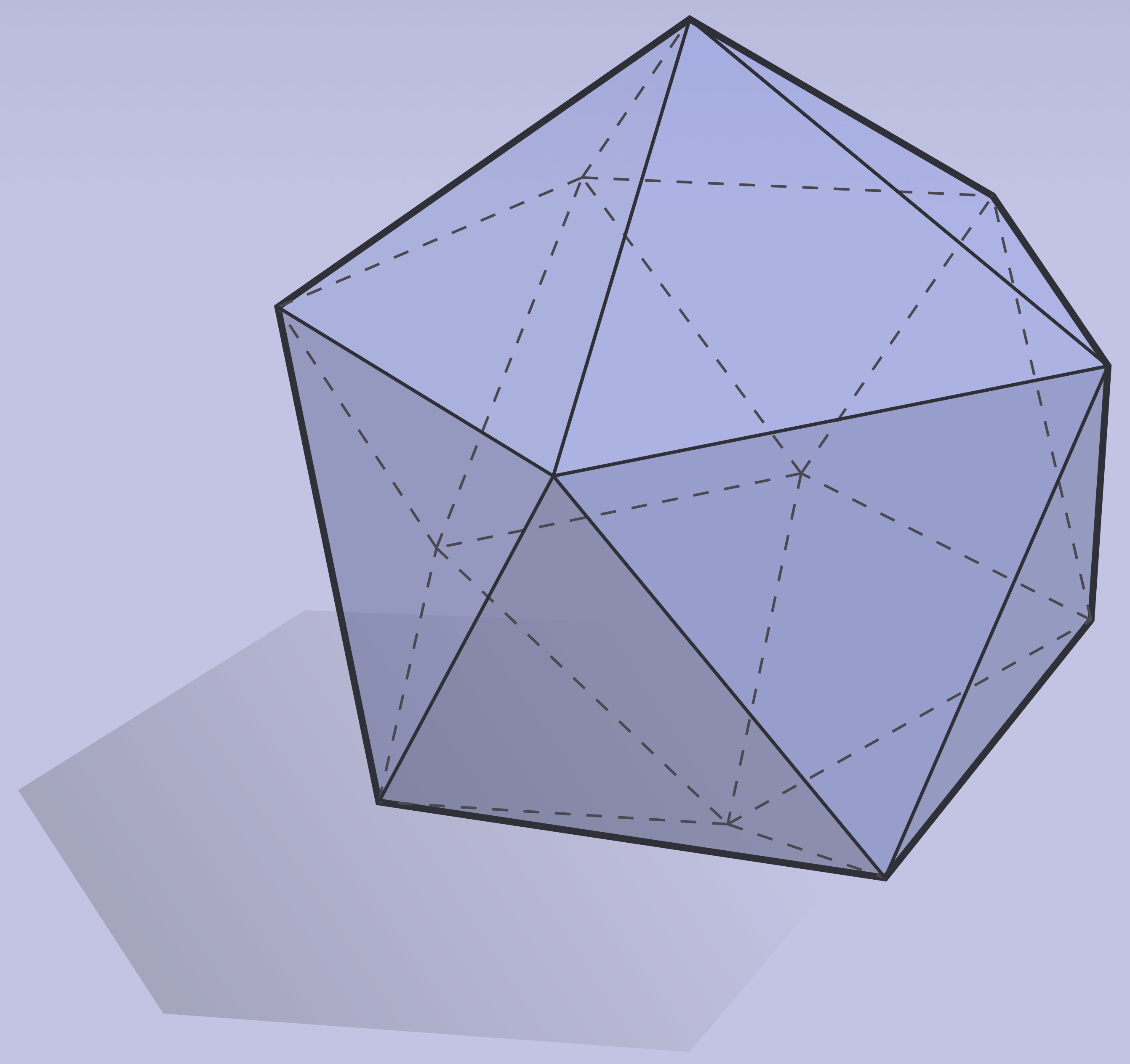


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 7:

EXTERIOR CALCULUS — INTEGRATION



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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Integration of Differential k-Forms

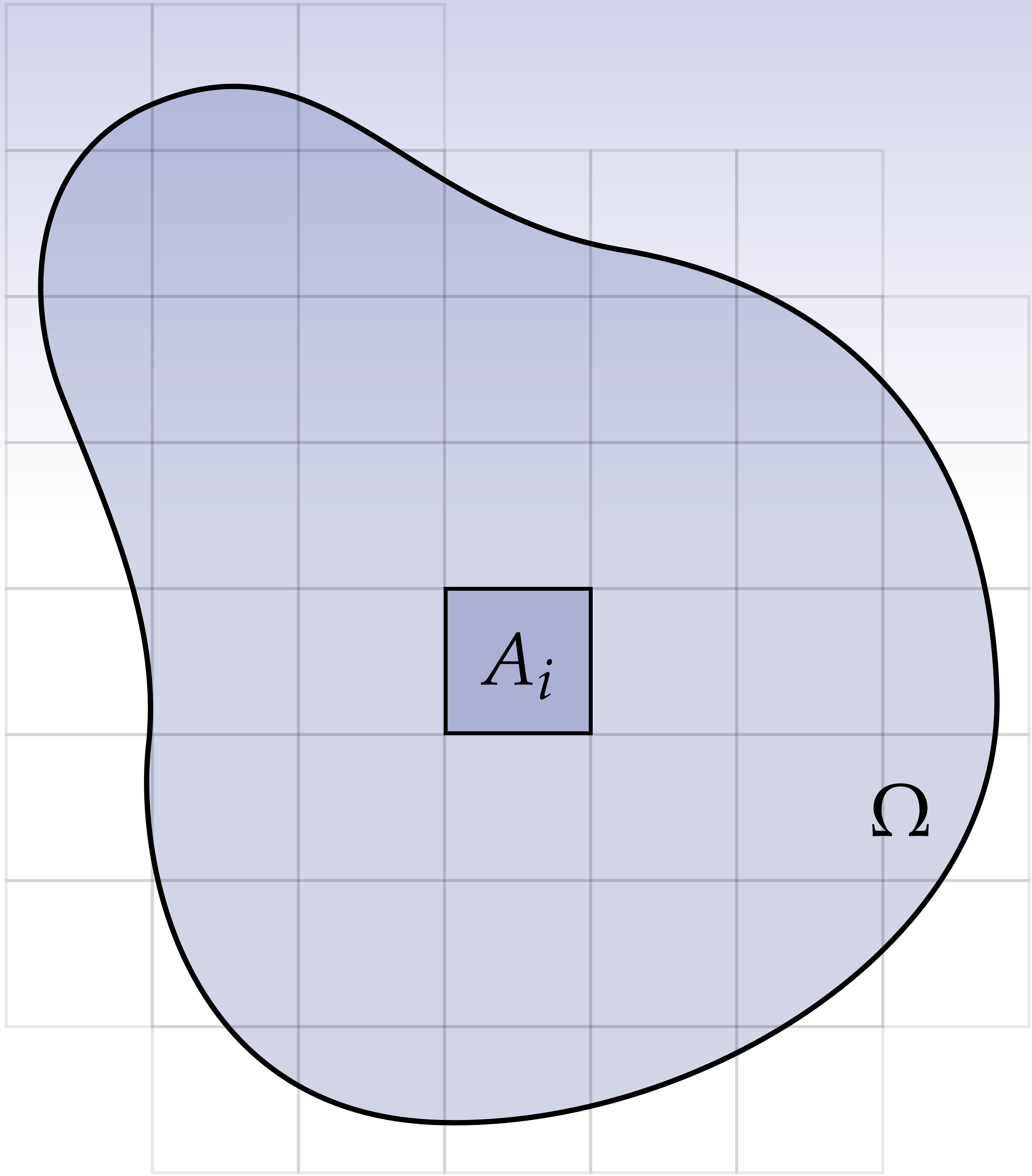
Integration and Differentiation

- Two big ideas in calculus:
  - **differentiation**
  - <u>**integration**</u>
  - linked by *fundamental theorem of calculus*
- Exterior calculus generalizes these ideas
  - **differentiation of *k*-forms** (exterior derivative)
  - <u>**integration of *k*-forms** (measure volume)</u>
  - linked by *Stokes' theorem*
- **Goal:** integrate differential forms over meshes to get *discrete exterior calculus (DEC)*

$$\boxed{\int_a^b f' dx = f(b) - f(a)}$$

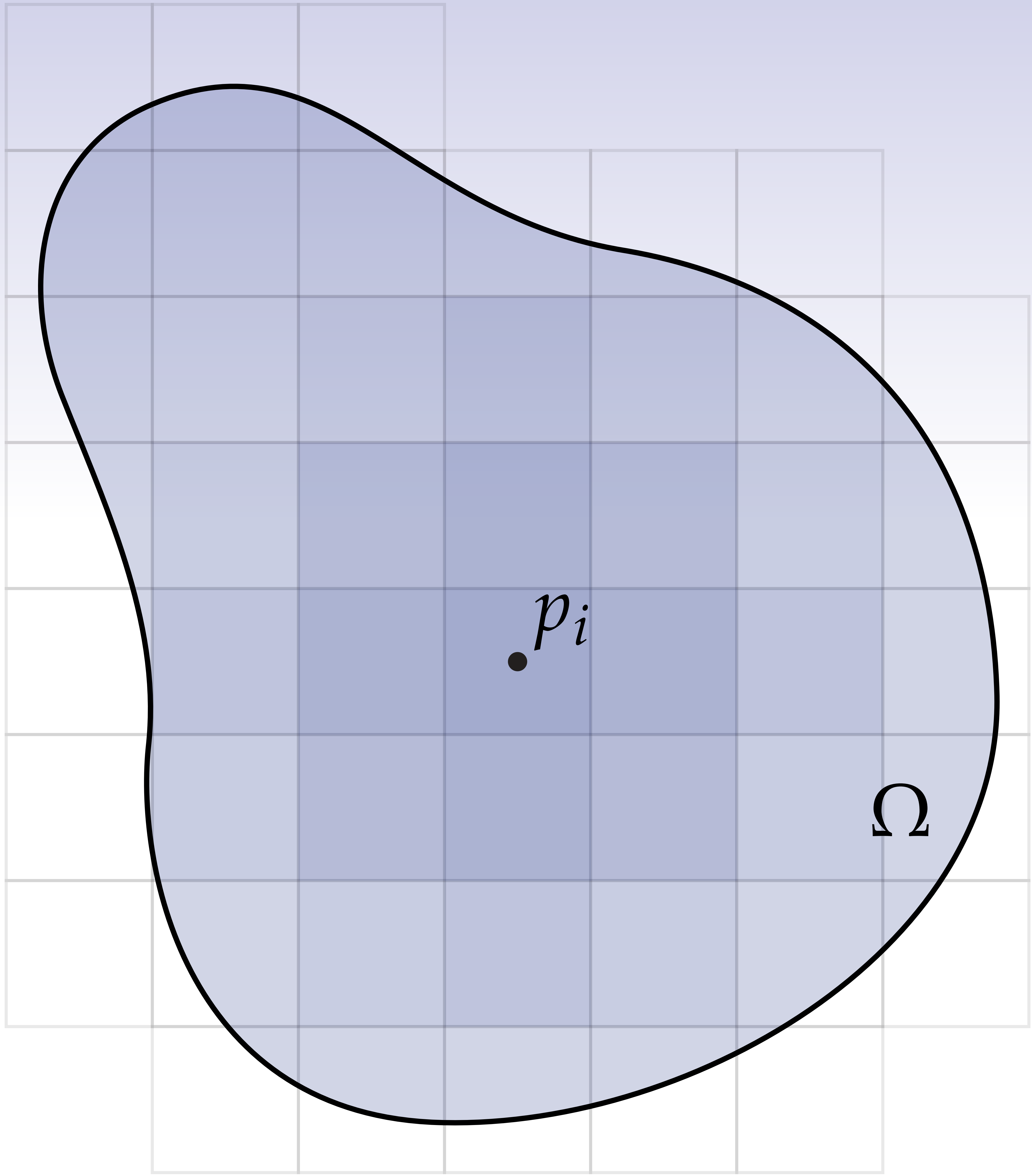
$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

Review—Integration of Area



$$\sum_i A_i \implies \int_\Omega dA$$

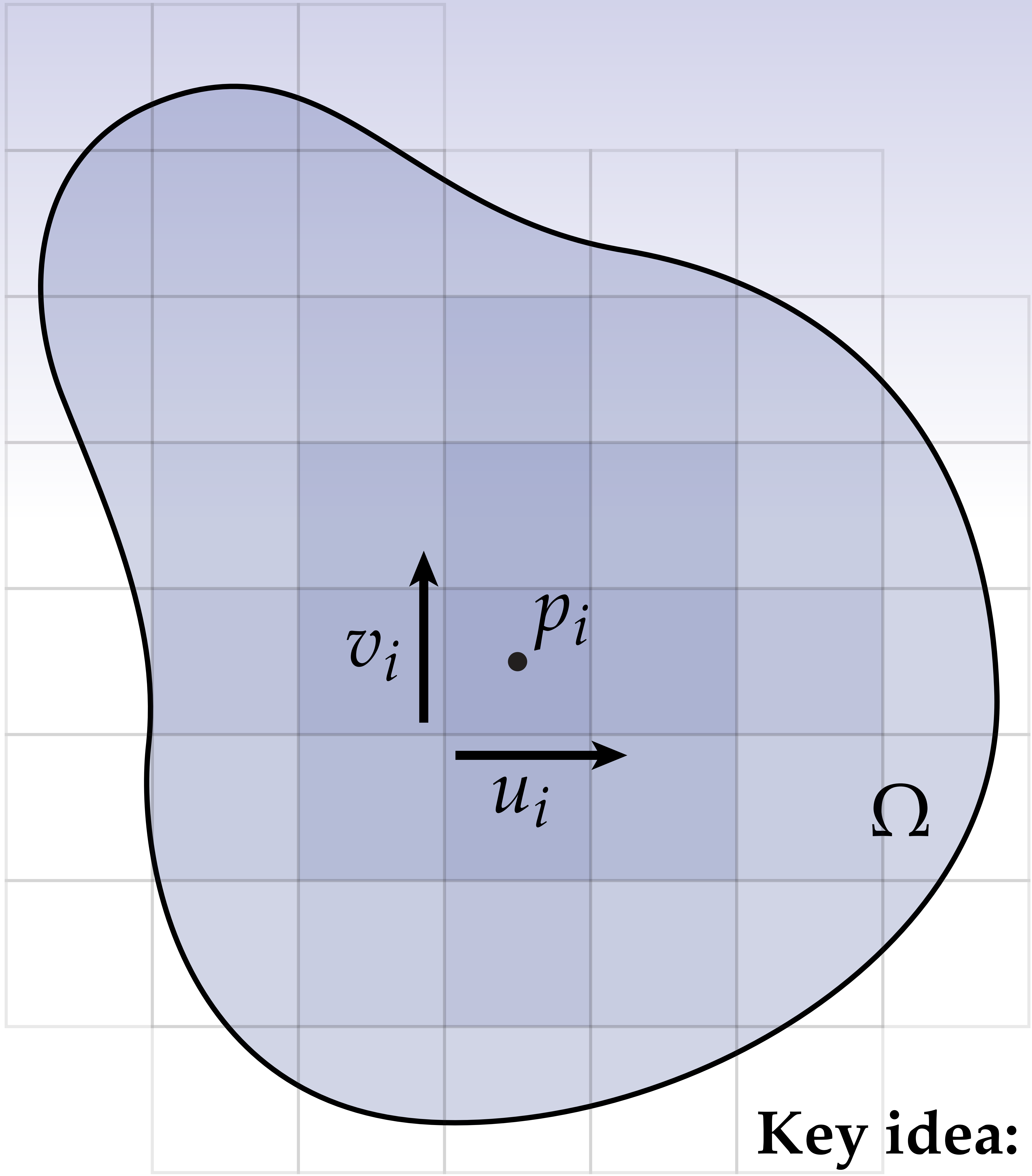
Review—Integration of Scalar Functions



$$\phi : \Omega \to \mathbb{R}$$

$$\sum_i A_i \phi(p_i) \implies \int_\Omega \phi \, dA$$

Integration of a 2-Form



ω — differential 2-form on Ω

$$\sum_i \omega_{p_i}(u_i, v_i) \implies \int_\Omega \omega$$

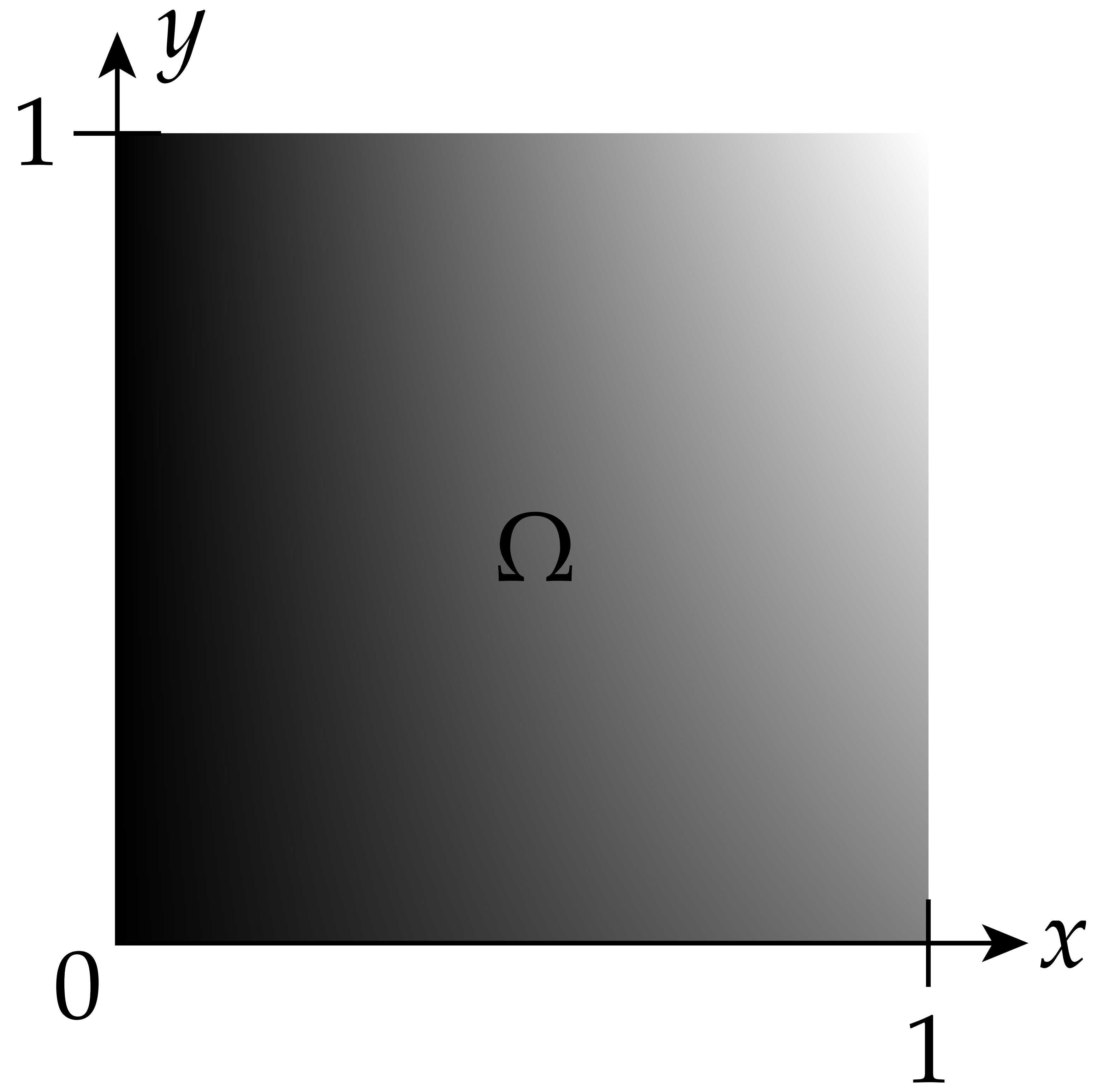
Key idea: integration *always* involves differential forms!

Integration of Differential 2-forms — Example

- Consider a differential 2-form on the unit square in the plane:

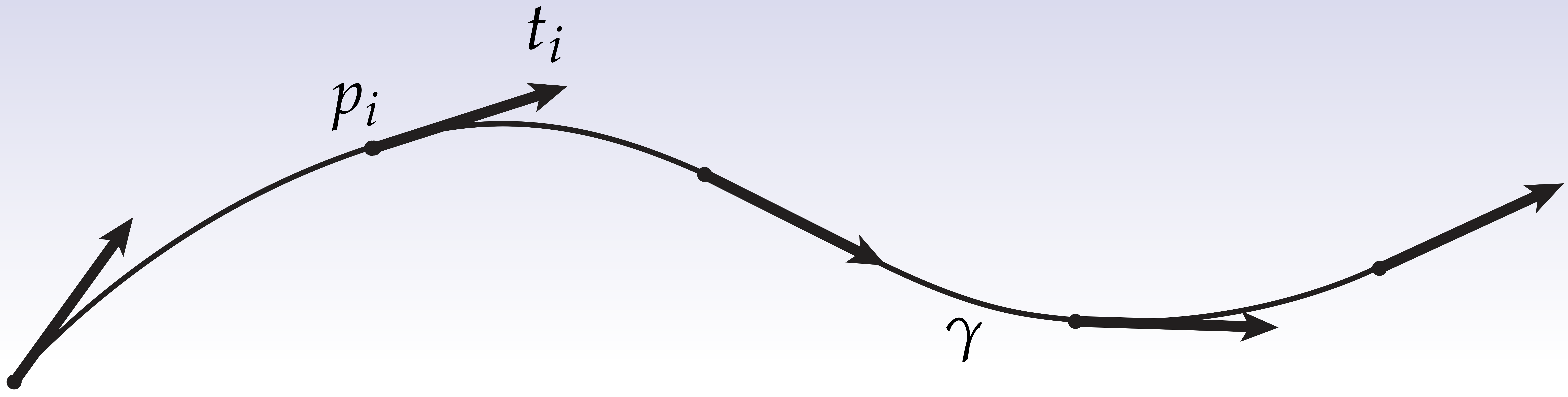
$$\omega := (x + xy)dx \wedge dy$$

$$\begin{aligned}\int_\Omega \omega &= \int_\Omega (x + xy)dx \wedge dy \\ &= \int_0^1 \int_0^1 (x + xy)dx \wedge dy \\ &= \cdots = \frac{3}{4}\end{aligned}$$



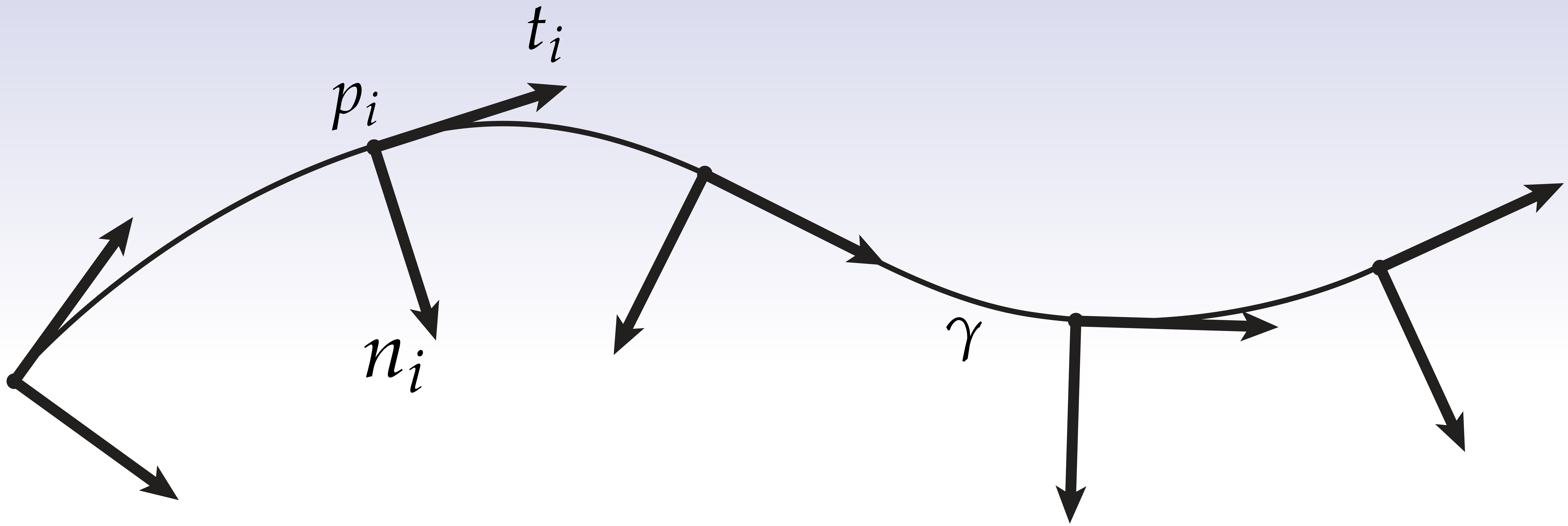
  - In this case, no different from usual “double integration” of a scalar function.

Integration on Curves



$$\int_\gamma \alpha \approx \sum_i \alpha_{p_i}(t_i)$$

Integration on Curves



$$\int_\gamma \star\alpha \approx \sum_i \star\alpha_{p_i}(t_i) = \sum_i \alpha_{p_i}(n_i)$$

Integration on Curves—Example

- Now consider a 1-form in the plane, which we will integrate over the unit circle:

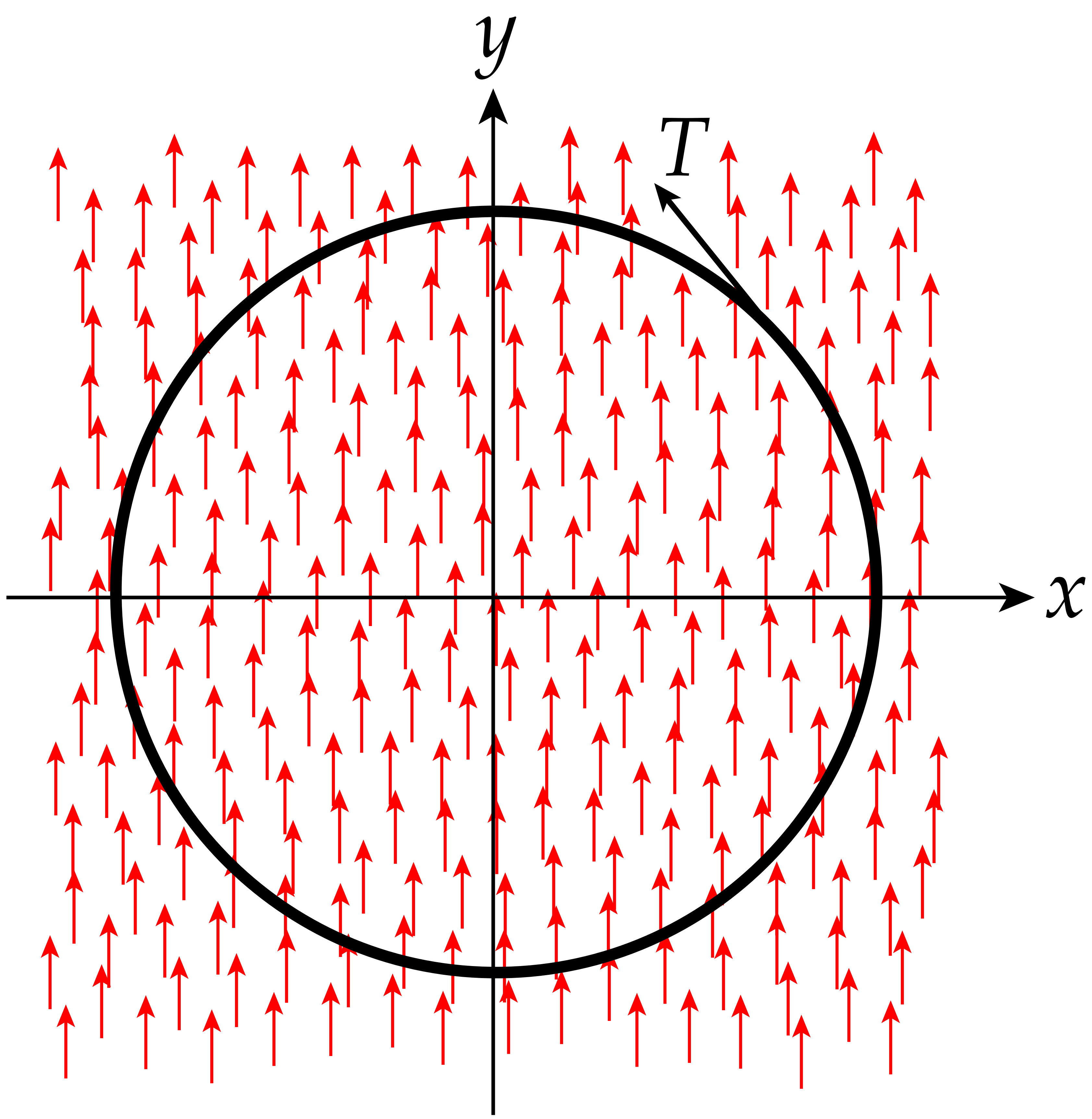
$$\int_{S^1} \alpha = \int_0^{2\pi} \alpha_{\gamma(s)}(T(s))\, ds =$$

$$\int_0^{2\pi} \alpha_\gamma(s)(-\sin(s)\tfrac{\partial}{\partial x} + \cos(s)\tfrac{\partial}{\partial y})ds =$$

$$\int_0^{2\pi} dy(-\sin(s)\tfrac{\partial}{\partial x} + \cos(s)\tfrac{\partial}{\partial y})ds =$$

$$\int_0^{2\pi} \cos(s)\, ds = 0$$

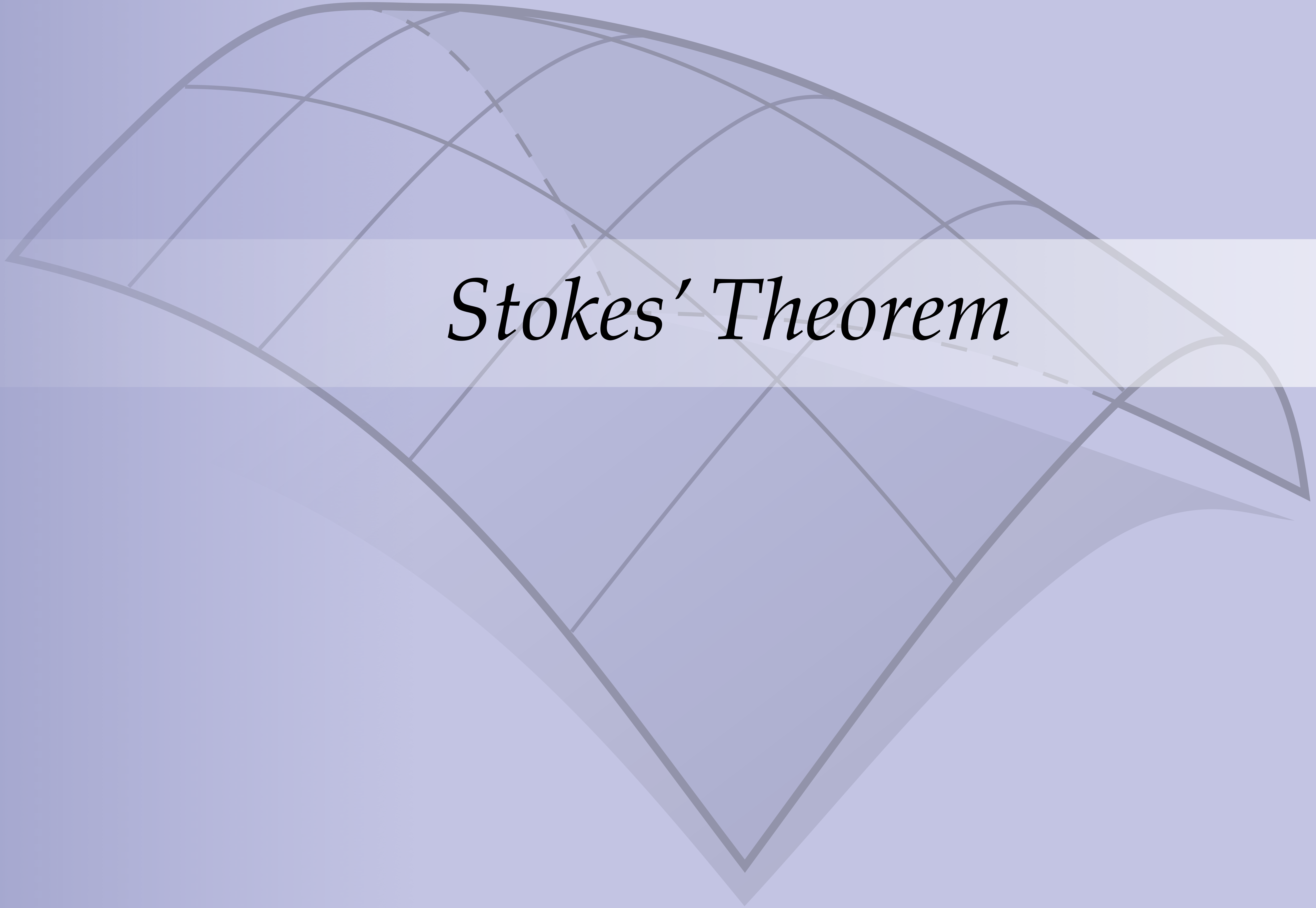
$\alpha = dy$



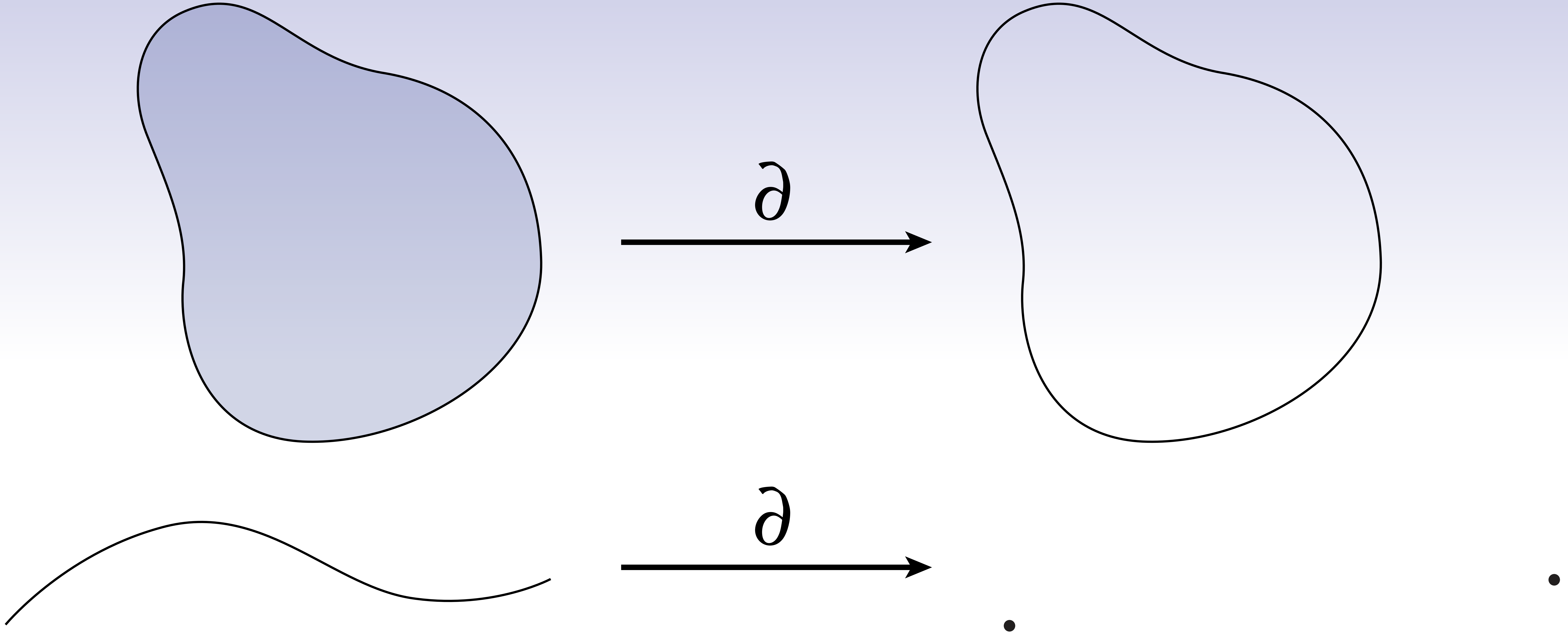
$\gamma : [0, 2\pi) \to \mathbb{R}^2; s \mapsto (\cos(s), \sin(s))$

(Why does this result make sense geometrically?)

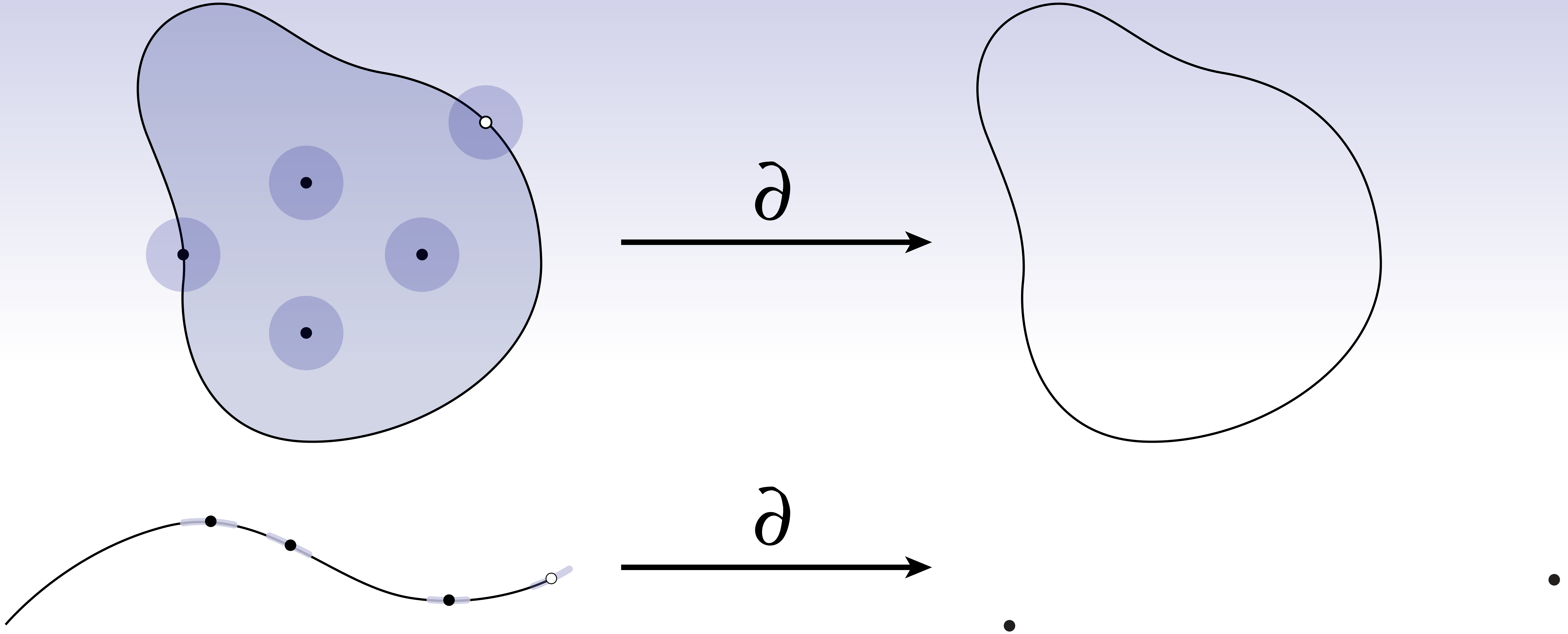
Stokes’ Theorem



Boundary



Boundary

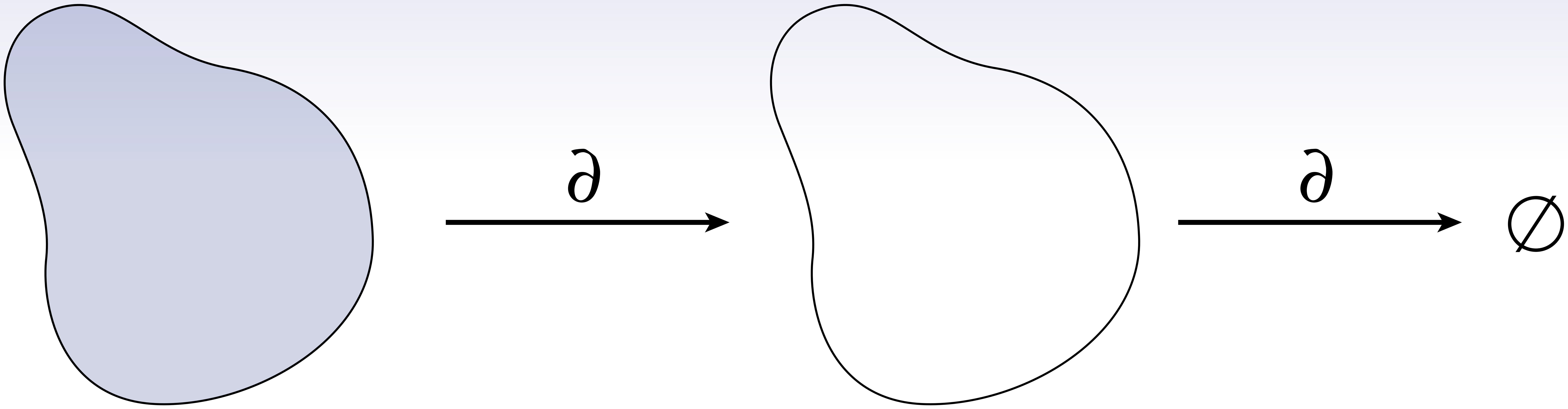


Basic idea: at an interior point p of a k-dimensional set the intersection of an open ball around p with the set looks like* an open k-ball; at a boundary point it doesn't.

*…is homeomorphic to, in the subspace topology.

Boundary of a Boundary

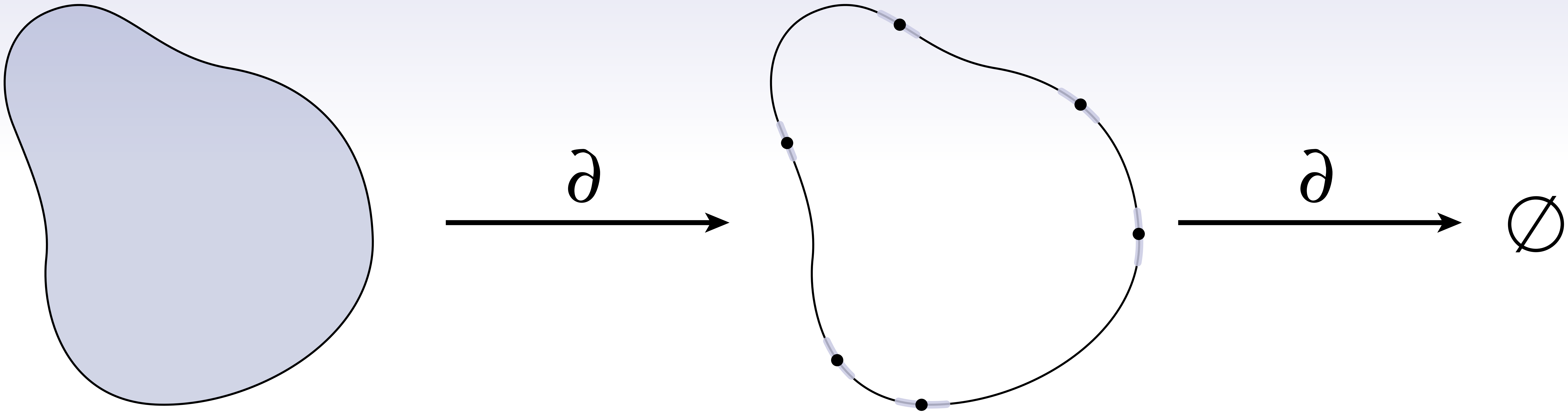
Q: Which points are in the boundary of the boundary?



A: No points! Boundary of a boundary is always *empty*.

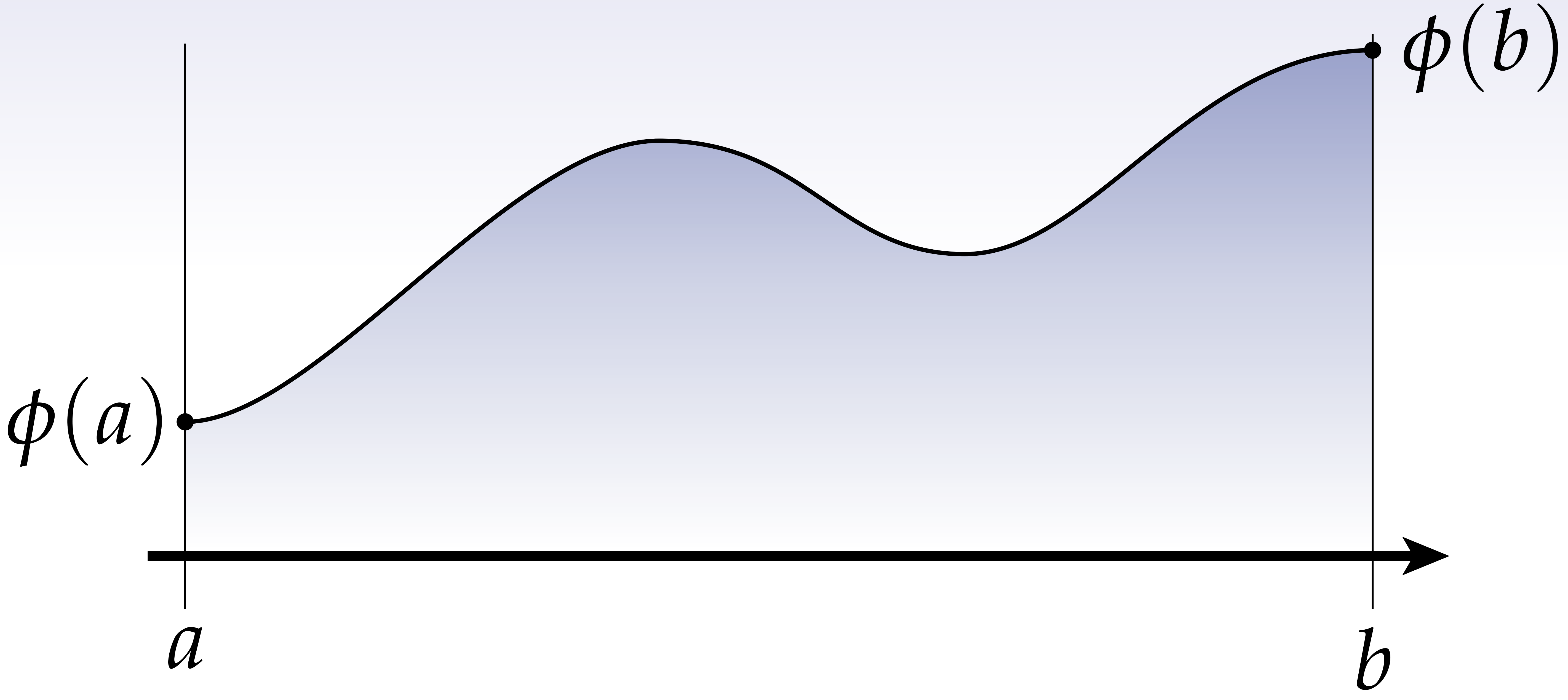
Boundary of a Boundary

Q: Which points are in the boundary of the boundary?



A: No points! Boundary of a boundary is always *empty*.

Review: Fundamental Theorem of Calculus



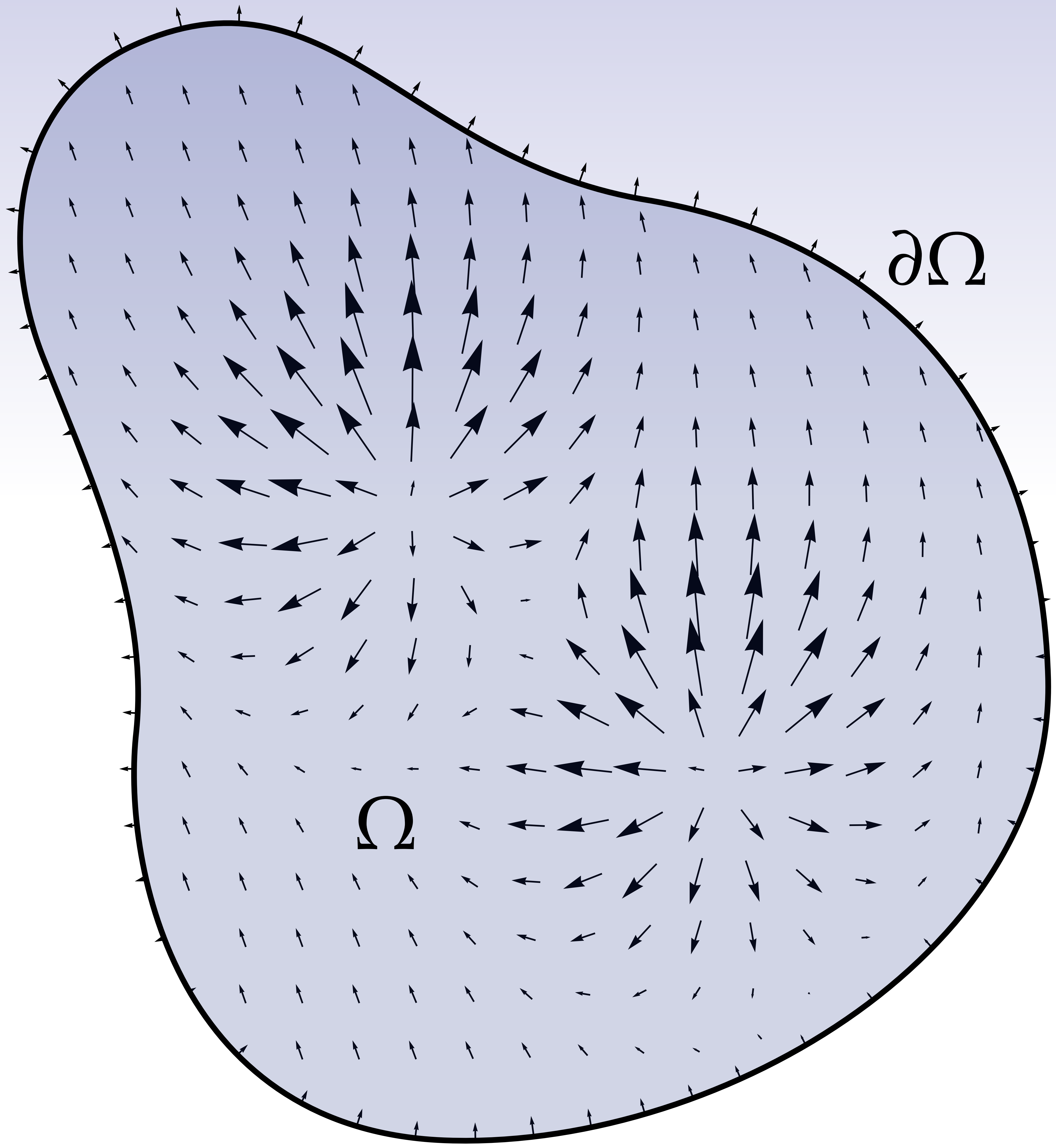
$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a)$$

Stokes' Theorem

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

Analogy: fundamental theorem of calculus

Example: Divergence Theorem



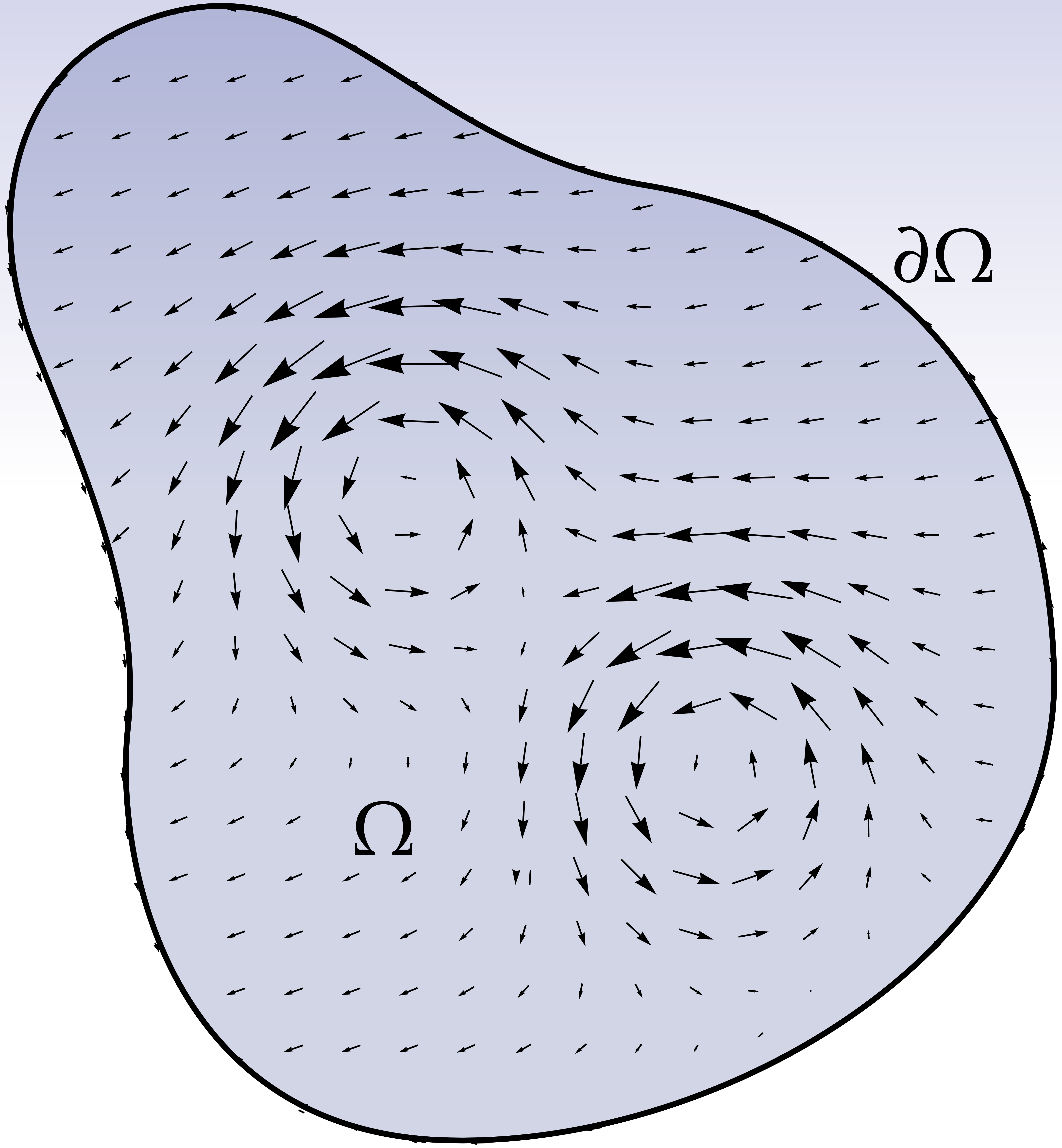
$$\int_\Omega \nabla \cdot X \, dA = \int_{\partial\Omega} n \cdot X \, d\ell$$

$$\Longleftrightarrow$$

$$\int_\Omega d \star \alpha = \int_{\partial\Omega} \star\alpha$$

What goes in, must come out!

Example: Green's Theorem



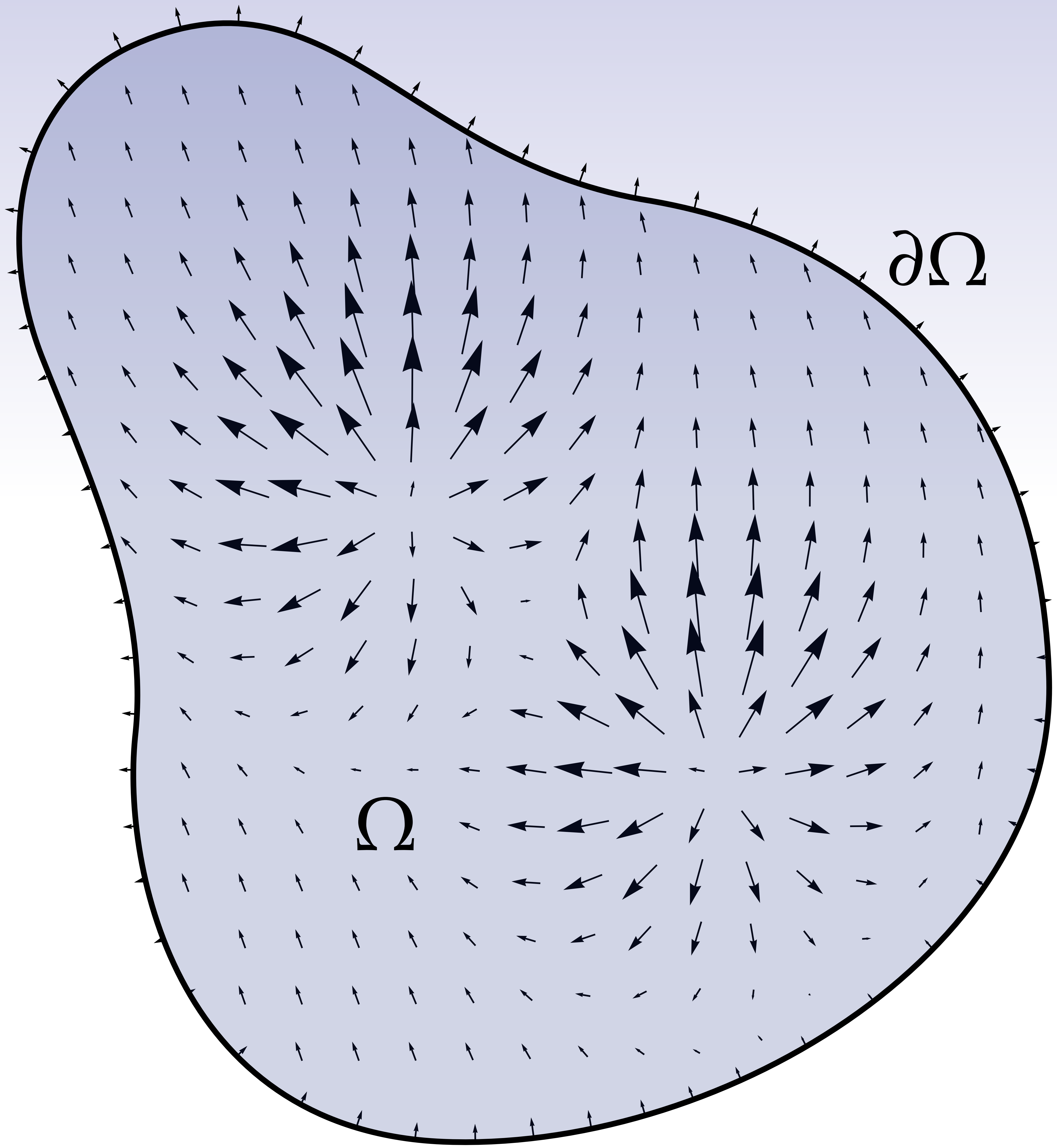
$$\int_{\Omega} \nabla \times X \, dA = \int_{\partial\Omega} t \cdot X \, d\ell$$

$$\Longleftrightarrow$$

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

What goes around comes around!

Stokes' Theorem

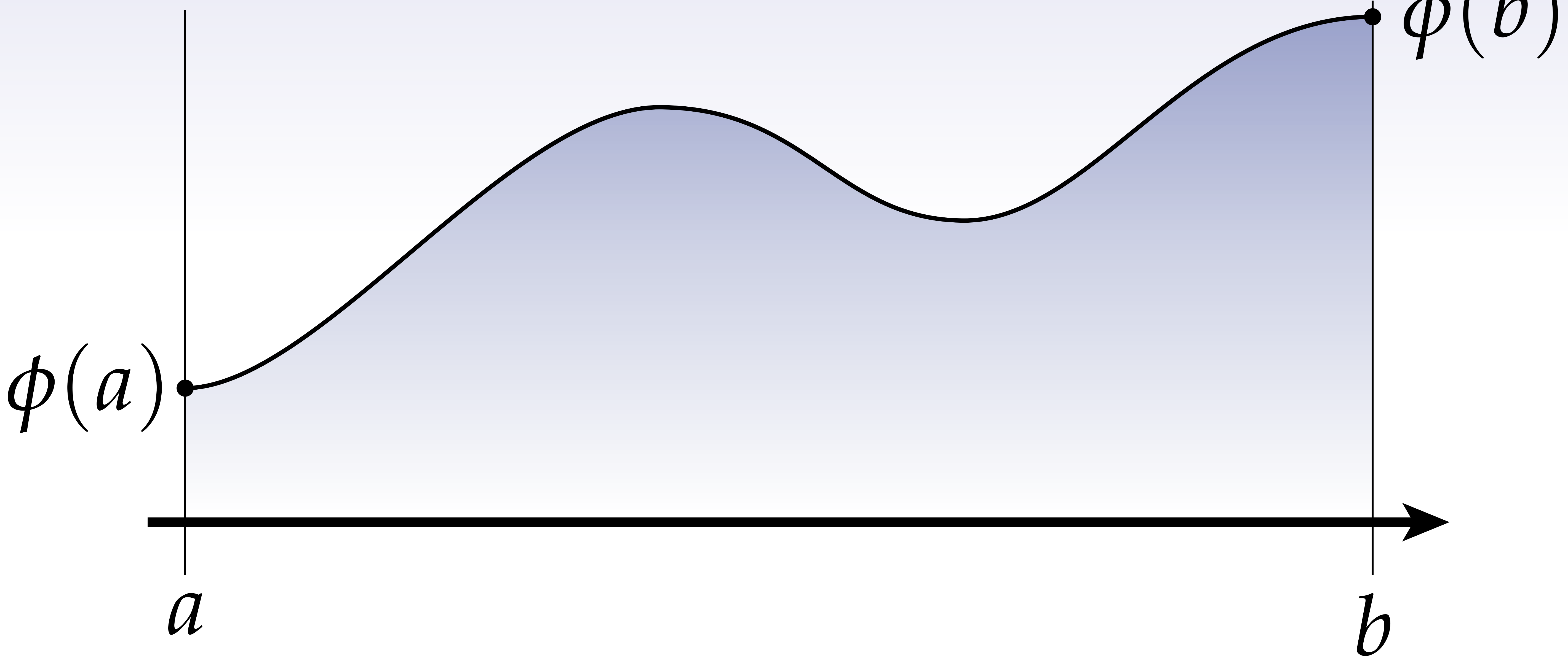


$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

"The change we see on the outside is purely a function of the change within."

—Zen koan

Fundamental Theorem of Calculus & Stokes'

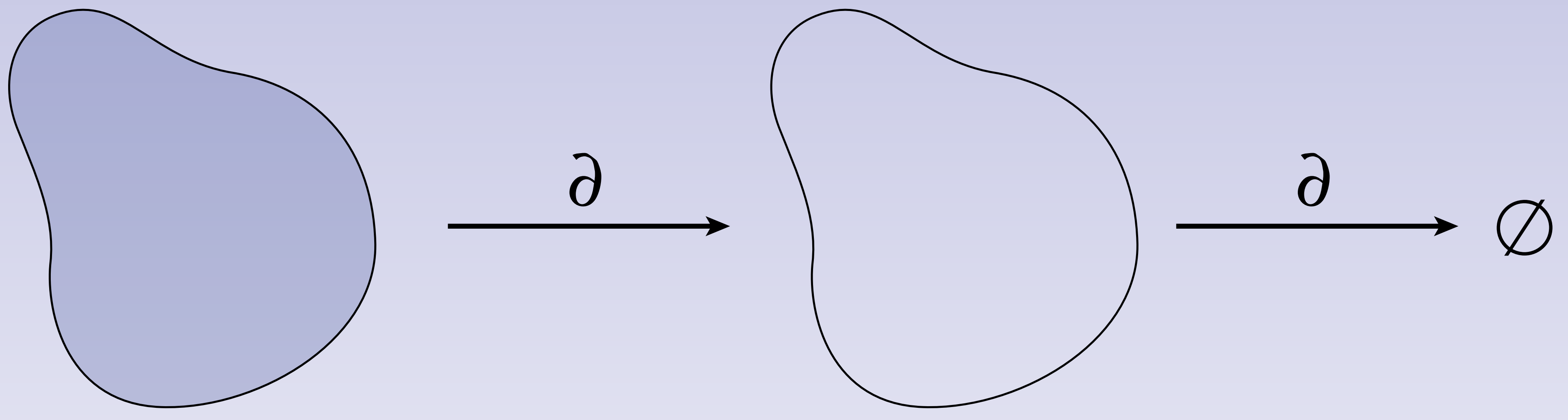


$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a)$$

$$\Longleftrightarrow$$

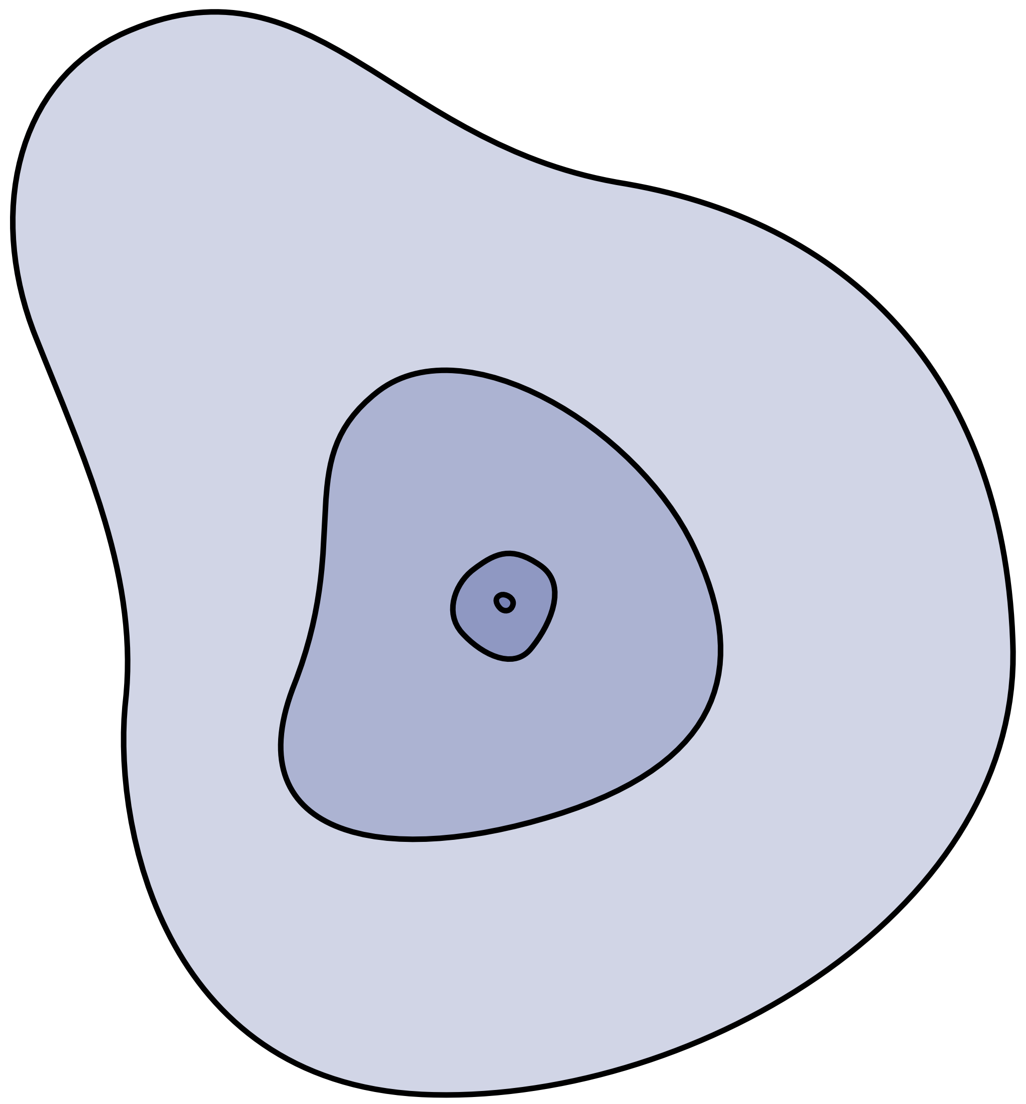
$$\int_{[a,b]} d\phi = \int_{\partial [a,b]} \phi$$

Why is $d \circ d = 0$?



$$\int_\Omega dd\phi = \int_{\partial\Omega} d\phi = \int_{\underbrace{\partial\partial\Omega}_{\emptyset}} \phi = 0$$

...for *any* Ω (no matter how small!)



Why is $d \circ d = 0$?

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

"behaves like gradient for functions"

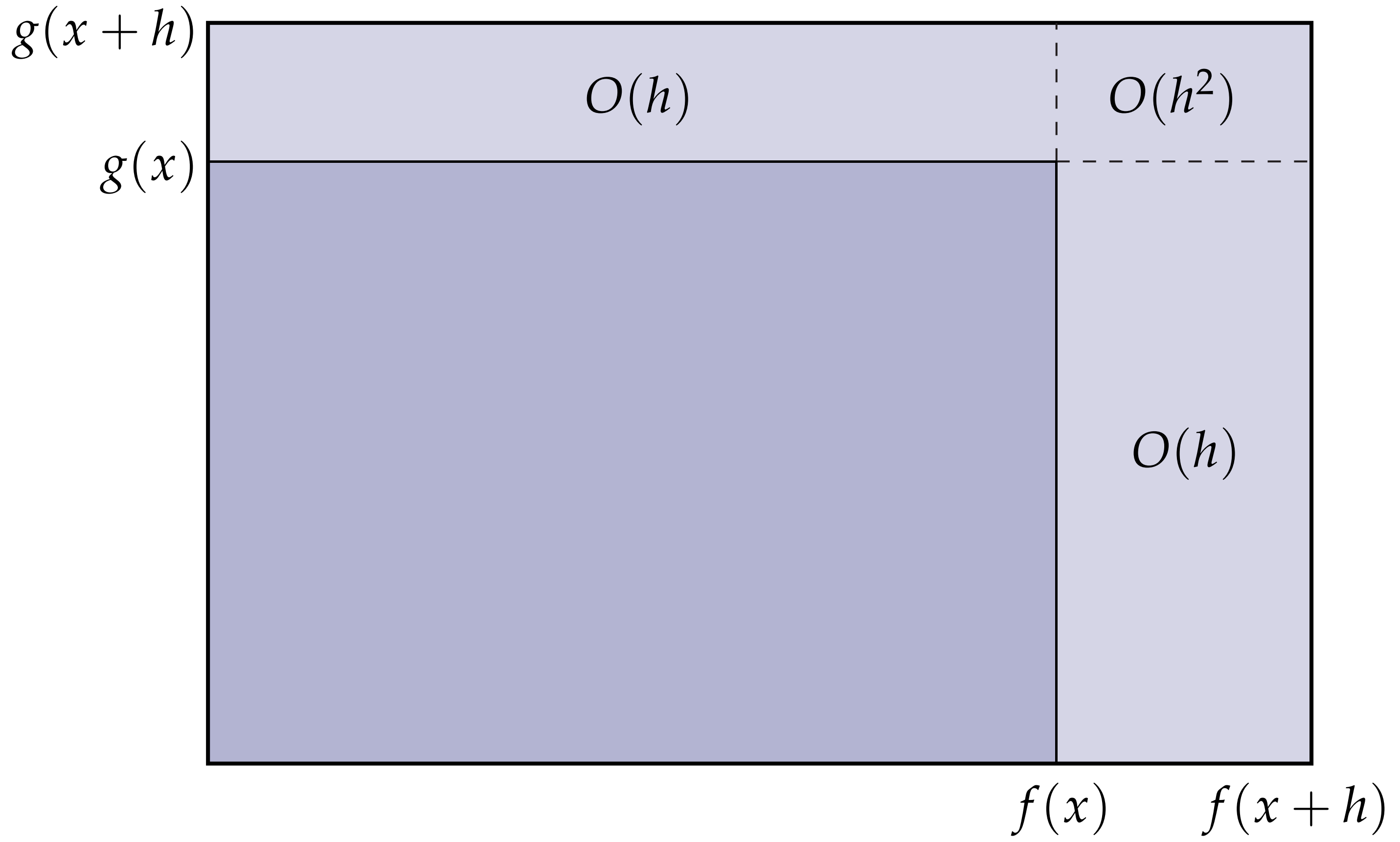
differential $$d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \cdots + \frac{\partial \phi}{\partial x^n} dx^n$$

product rule $$d(\alpha \wedge \beta) = d\alpha \wedge \beta (-1)^k \alpha \wedge d\beta$$

~~exactness~~ $$\cancel{d \circ d = 0}$$

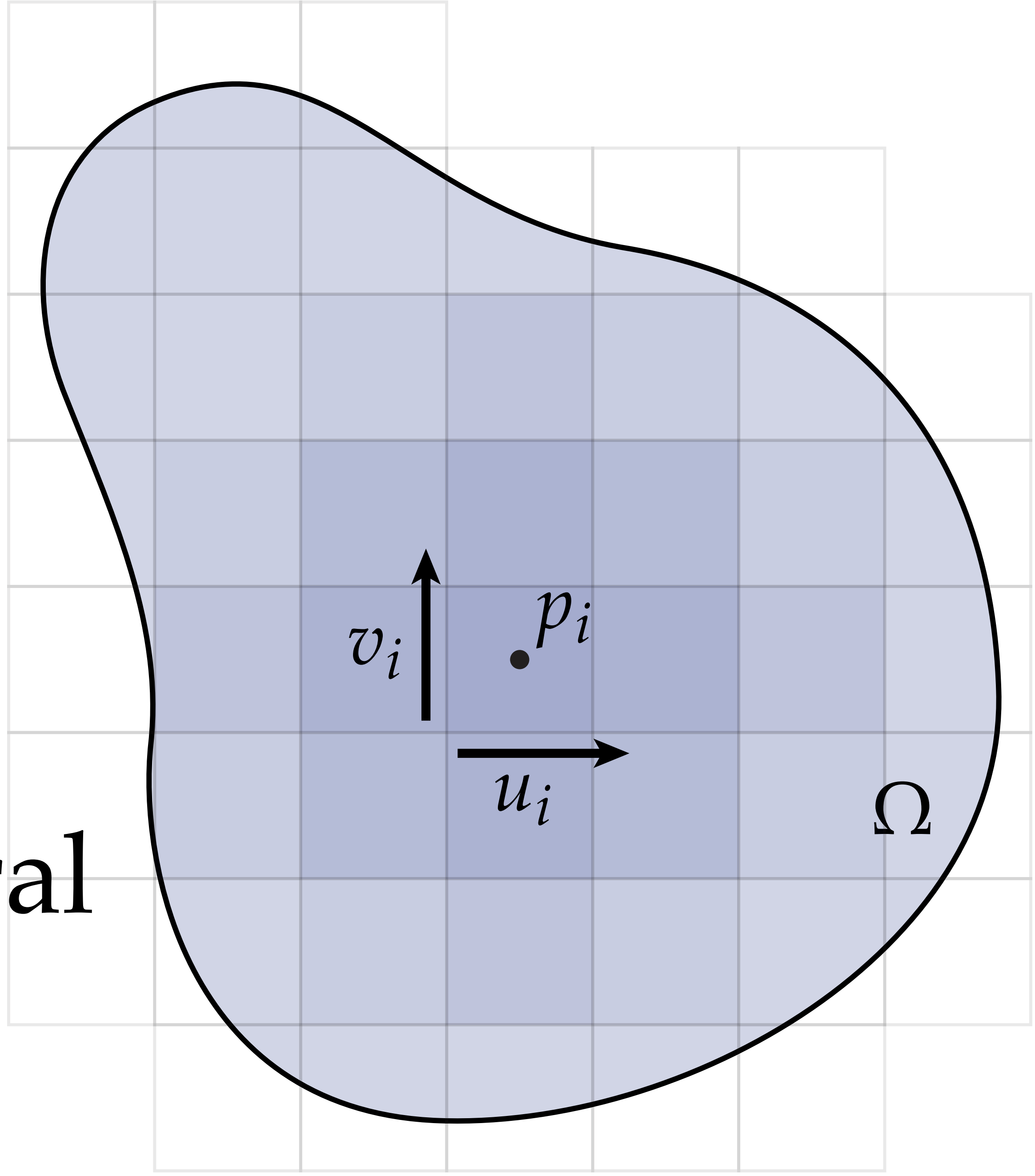
Stokes' theorem $$\int_\Omega d\alpha = \int_{\partial\Omega} \alpha$$

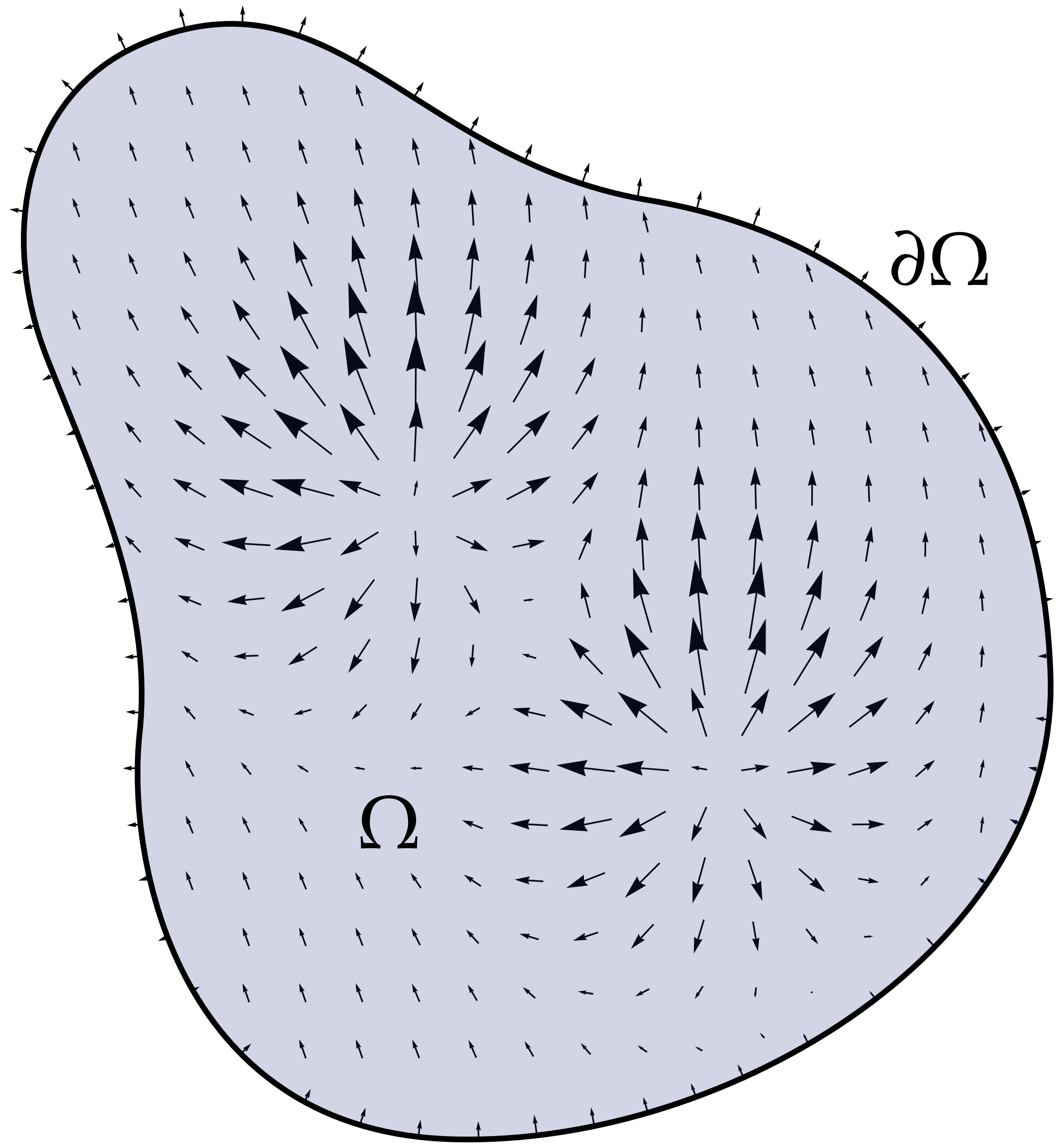
what goes in, must come out!



Integration & Stokes' Theorem - Summary

- Integration
  - break domain into small pieces
  - measure each piece with *k*-form
- Stokes' theorem
  - convert region integral to boundary integral
  - super useful—lets us "skip" a derivative
  - special cases: divergence theorem, Green's theorem, fundamental theorem of calculus, Cauchy's integral theorem… and *many more!*
  - Gets used *over and over* again in geometric computing
    - finite element methods, boundary element methods, …
    - <u>**discrete exterior calculus**</u>





$$\int_M d\alpha = \int_{\partial M} \alpha$$

Inner Product on Differential k-Forms

Inner Product—Review

- Recall that a *vector space* V is any collection of "arrows" that can be added, scaled, …
- **Q:** What's an *inner product* on a vector space?
- **A:** Loosely speaking, a way to talk about lengths, angles, etc., in a vector space
- More formally, a symmetric positive-definite bilinear map:

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$$

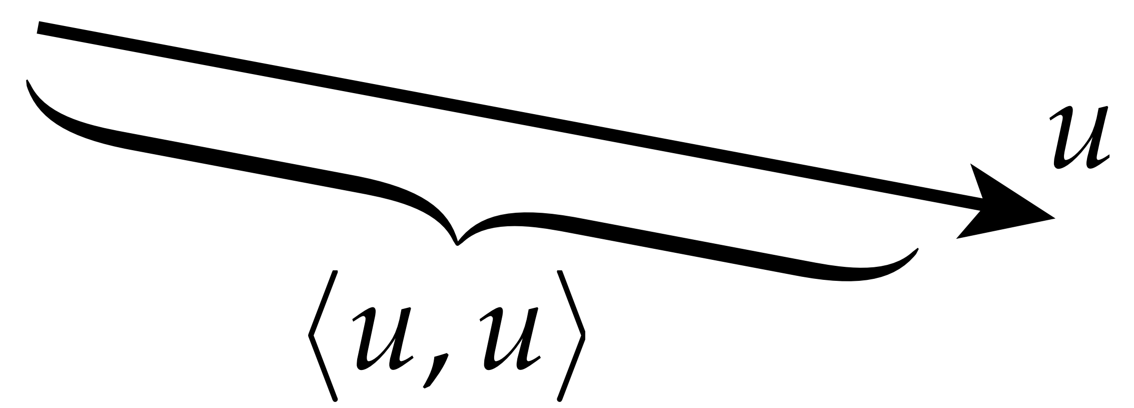
$$\langle u, v \rangle = \langle v, u \rangle$$

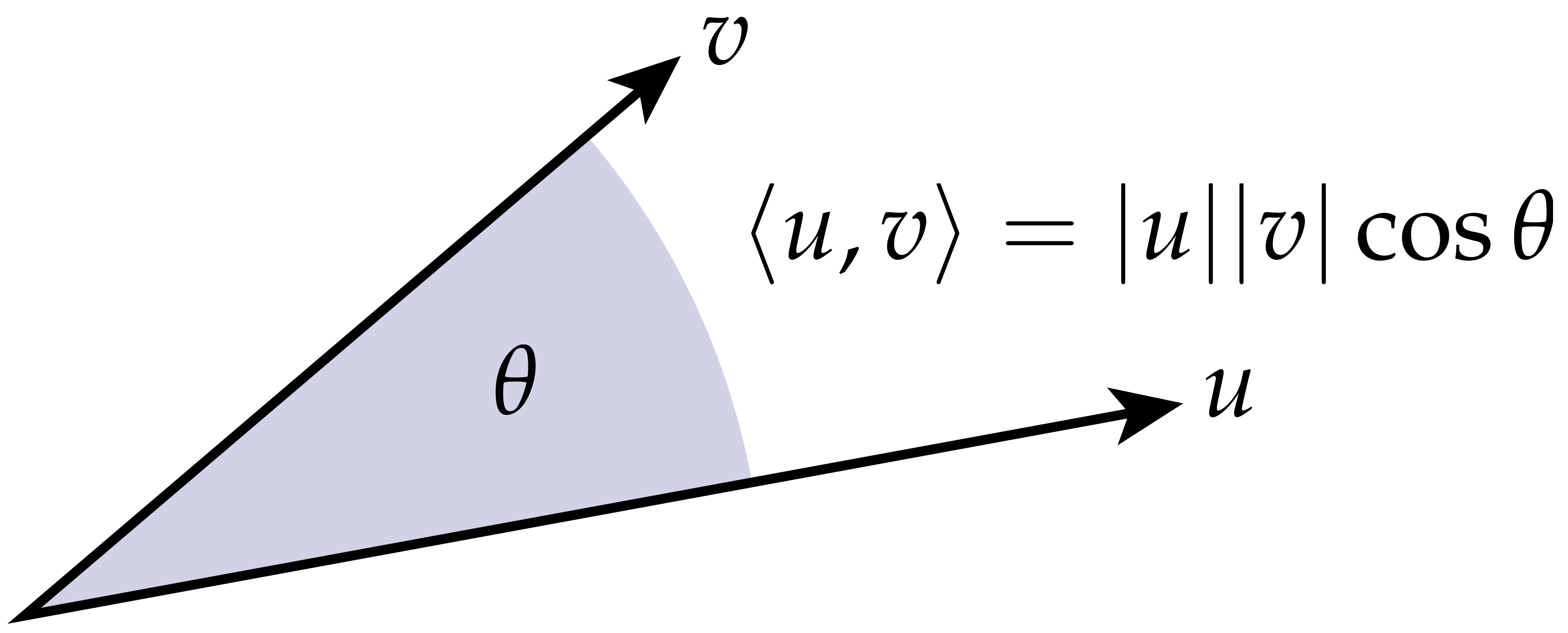
$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle au, v \rangle = a \langle u, v \rangle$$

$$\langle u, u \rangle \geq 0; \ \langle u, u \rangle = 0 \iff u = 0$$

for all vectors u,v,w in V and scalars a.





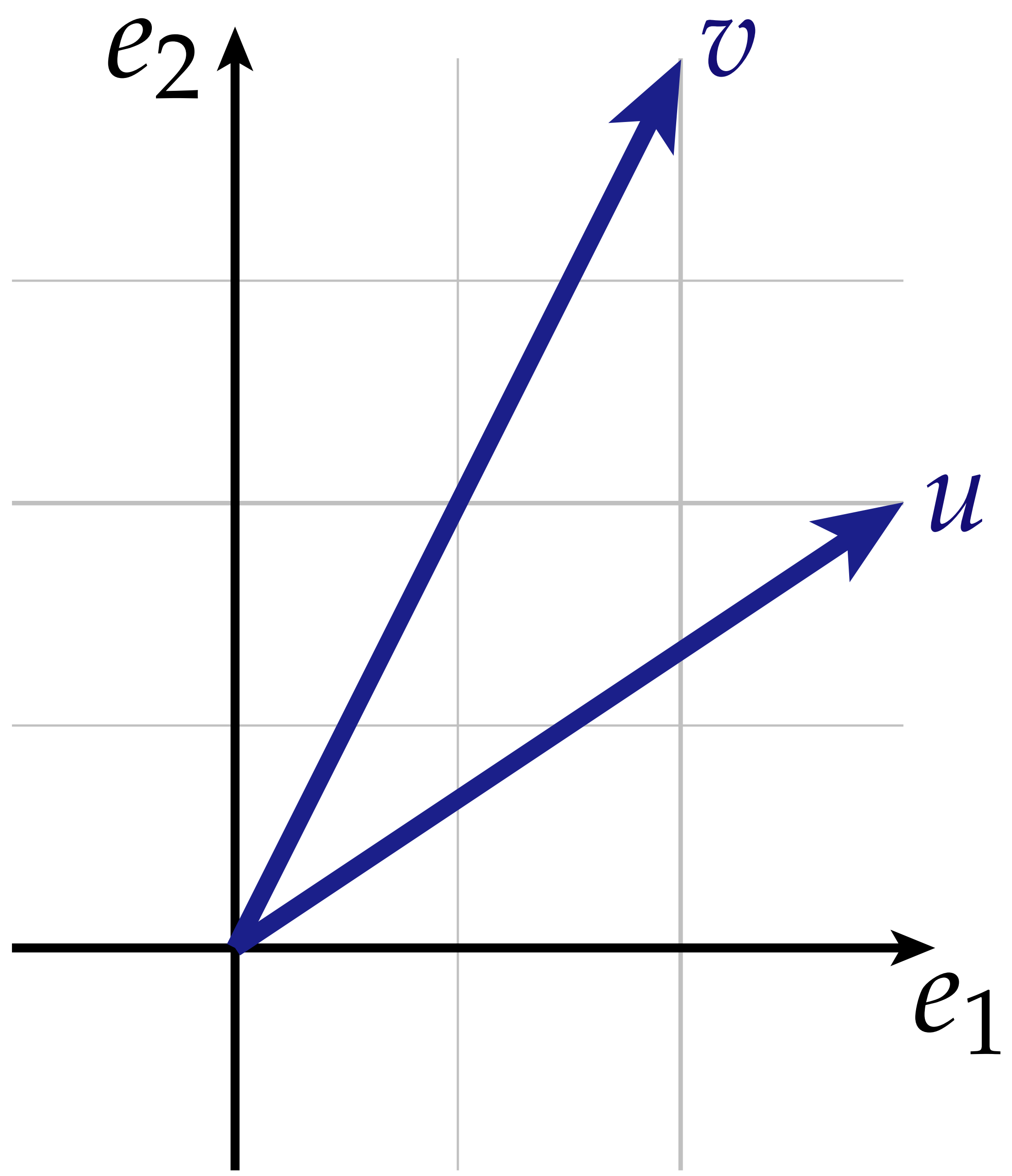
(Geometric interpretation of these rules?)

Euclidean Inner Product—Review

- Most basic inner product: inner product of two vectors in Euclidean R^n
- Just sum up the product of components:

$$u = u^1 e_1 + \cdots + u^n e_n$$
$$v = v^1 e_1 + \cdots + v^n e_n$$

$$\langle u, v \rangle := \sum_{i=1}^{n} u^i v^i$$



<u>Example.</u>

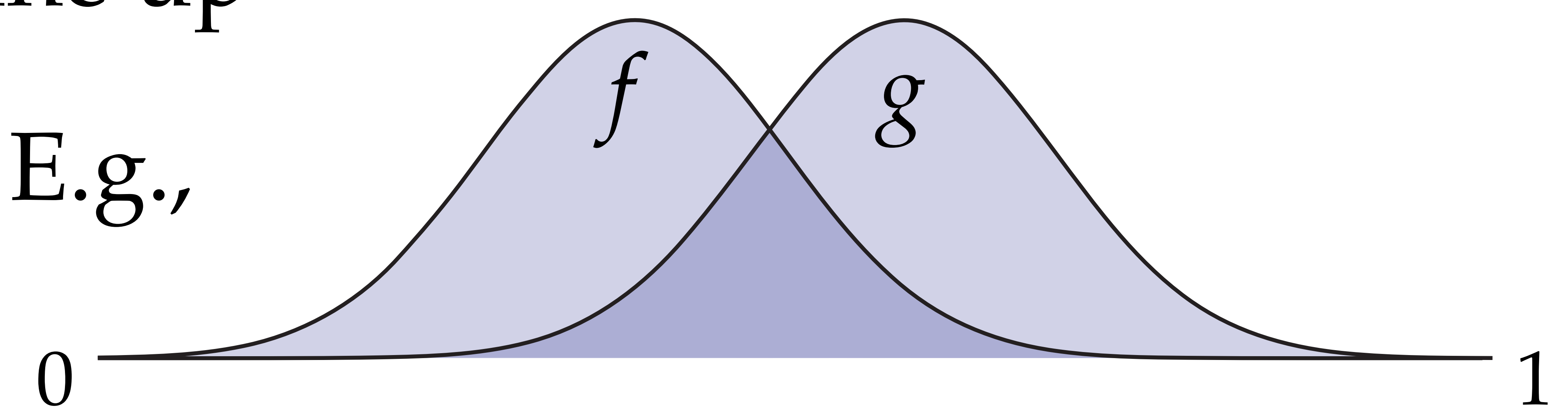
$$u = 3e_1 + 2e_2$$
$$v = 2e_1 + 4e_2$$
$$\langle u, v \rangle = 3 \cdot 2 + 2 \cdot 4 = 14$$

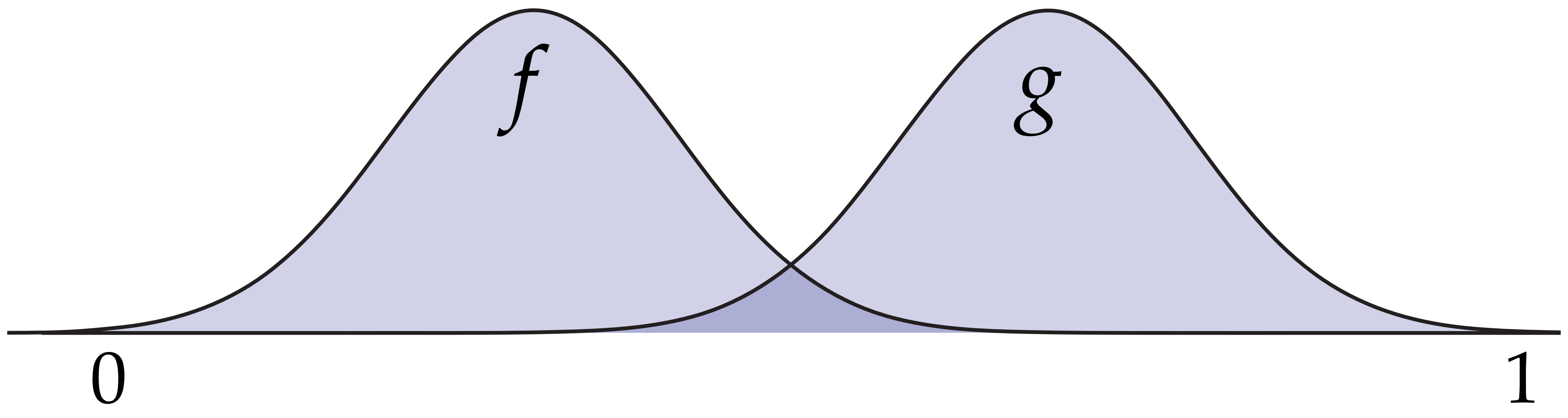
(Does this operation satisfy all the requirements of an inner product?)

L^2 Inner Product of Functions / 0-forms

- Remember that in many situations, *functions* are also vectors
- What does it mean to measure the inner product between functions?
- Want some notion of how well two functions "line up"
- One idea: mimic what we did for vectors in R^n. E.g.,

$$f : [0,1] \to \mathbb{R}$$
$$g : [0,1] \to \mathbb{R}$$
$$\langle\langle f, g \rangle\rangle := \int_0^1 f(x)g(x)dx$$





- Called the L² inner product. (**Note:** *f* and *g* must each be square-integrable!)
- Does this capture notion of "lining up"? Does it obey rules of inner product?

Inner Product on k-Forms

Definition. Let $\alpha, \beta \in \Omega^k$ be any two differential k-forms. Their (L^2) inner product is defined as*

$$\langle\langle \alpha, \beta \rangle\rangle := \int_\Omega \star\alpha \wedge \beta$$

Q: What happens when k=0?

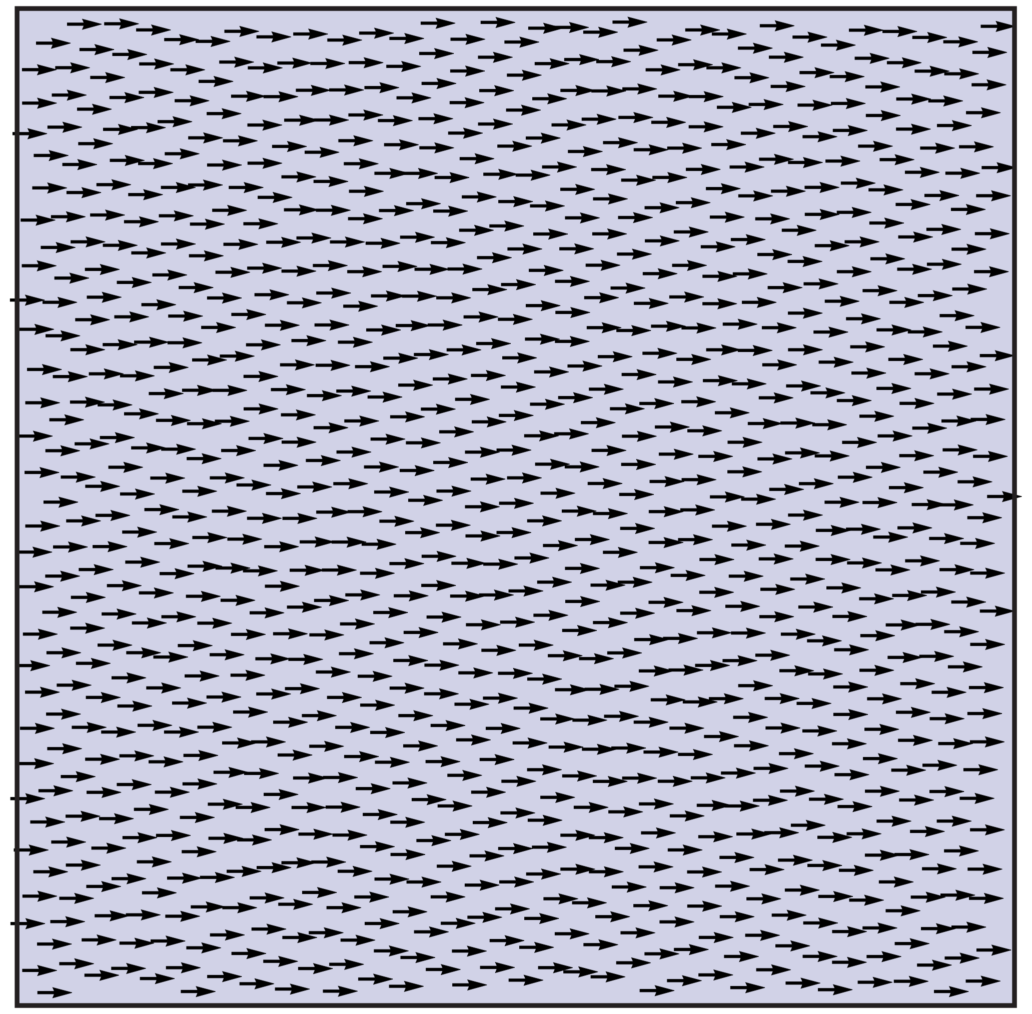
A: We just get the usual L^2 inner product on functions.

Q: What's the degree (k) of the integrand? Why is that important?

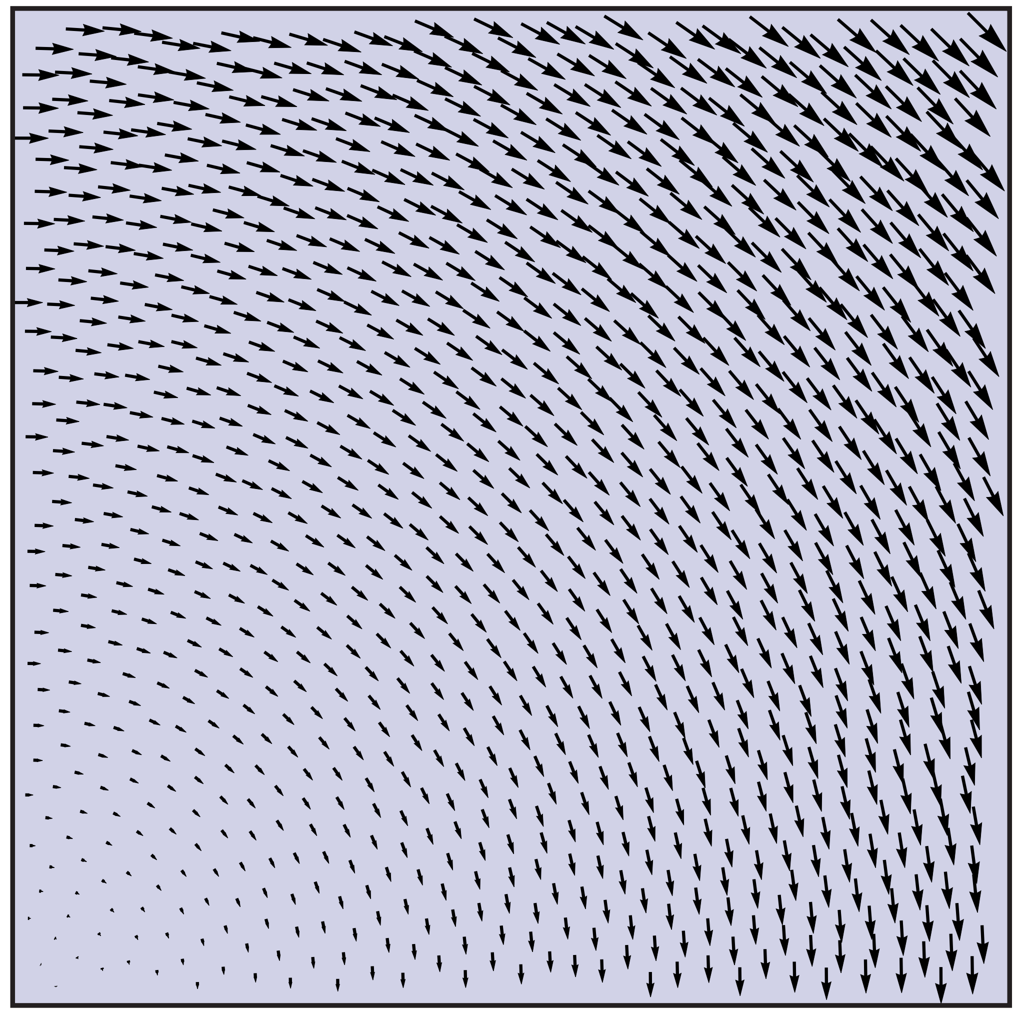
A: Integrand is always an n-form, which is the only thing we can integrate in n-D!

*Some authors define the integrand as $\alpha \wedge \star\beta$; our convention is consistent with the convention that in 2D the 1-form Hodge star is a *counter*-clockwise rotation.

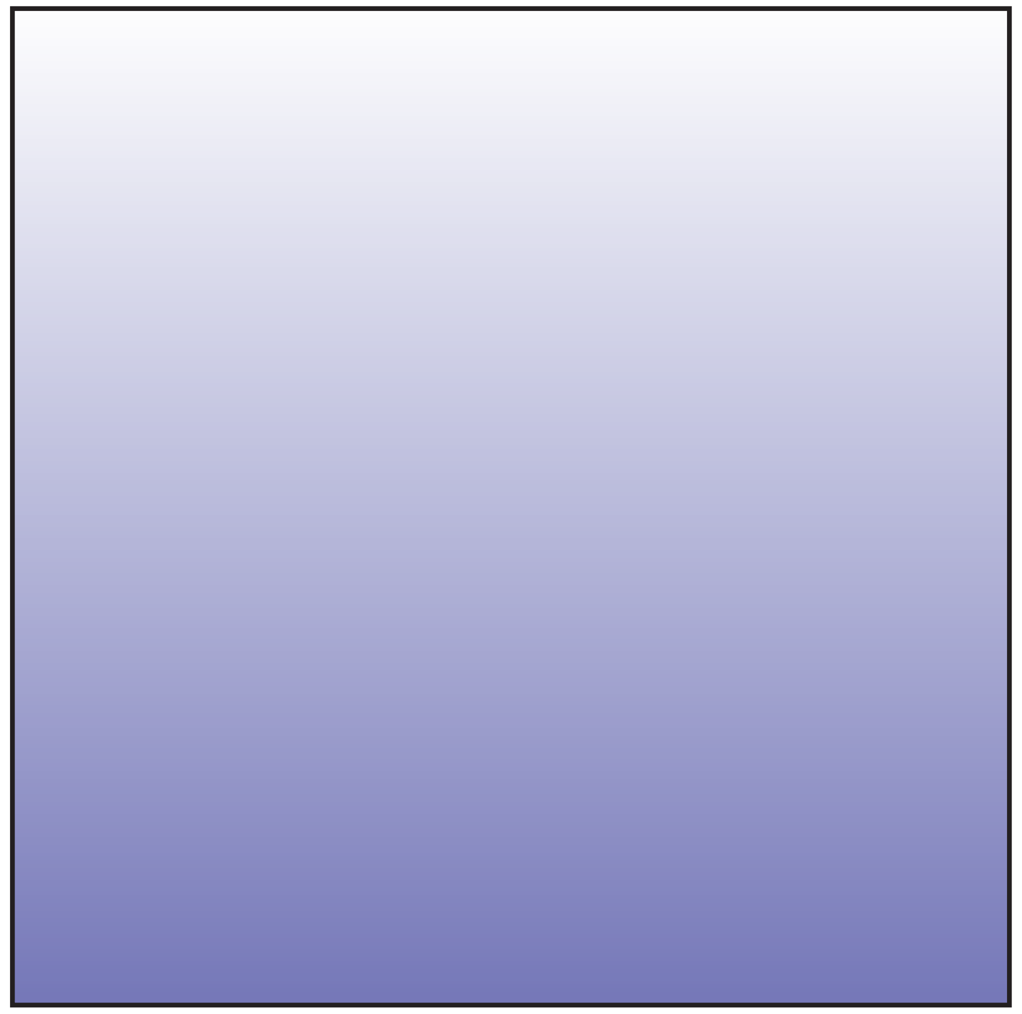
Inner Product of 1-Forms—Example



α



β



$\star\alpha \wedge \beta$

Example. Consider two 1-forms on the unit square $[0,1] \times [0,1]$ given by

$$\begin{aligned} \alpha &:= du, \\ \beta &:= v\,du - u\,dv. \end{aligned}$$

Their inner product is

$$\langle\langle \alpha, \beta \rangle\rangle = \int_0^1 \int_0^1 (\star\alpha) \wedge \beta =$$

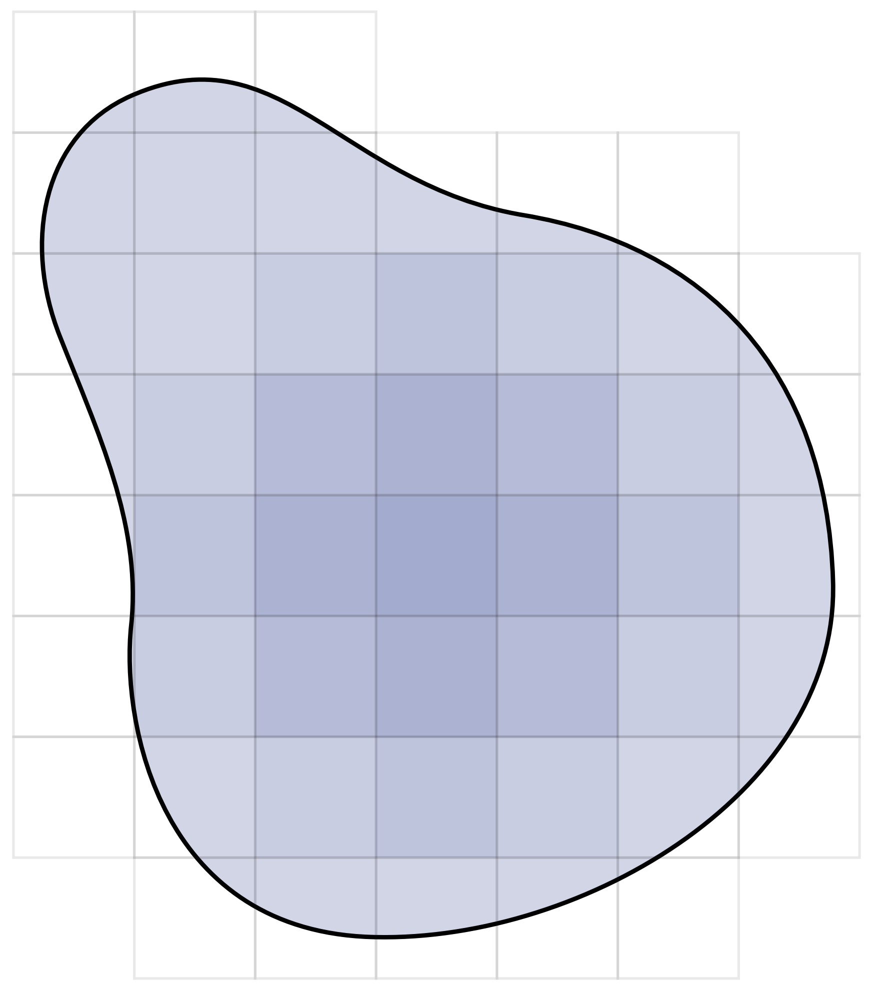
$$\int_0^1 \int_0^1 dv \wedge (v\,du - u\,dv) =$$

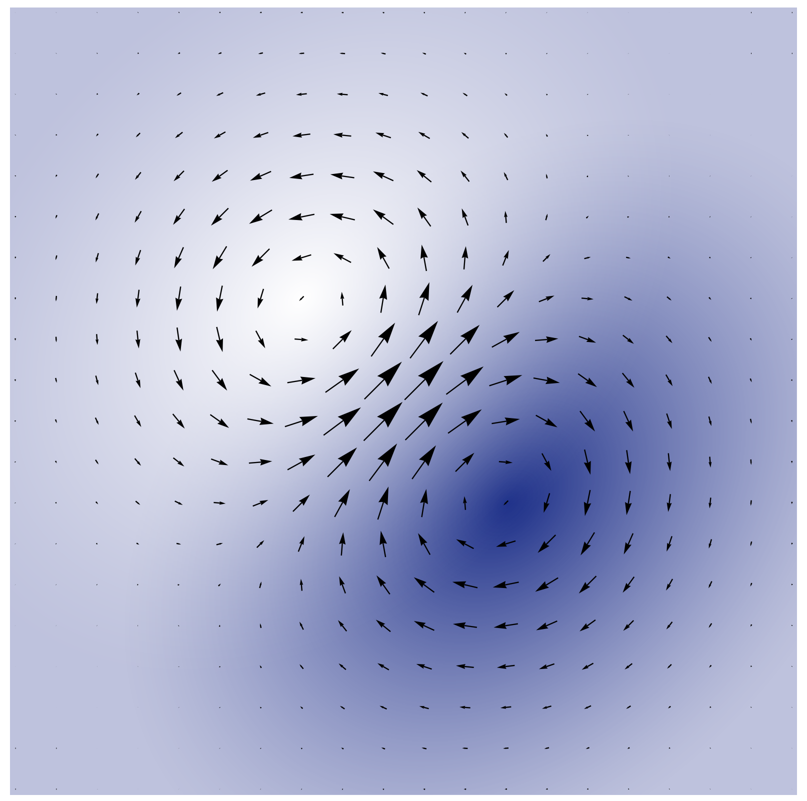
$$-\int_0^1 \int_0^1 v\,du \wedge dv = -\frac{1}{2}.$$

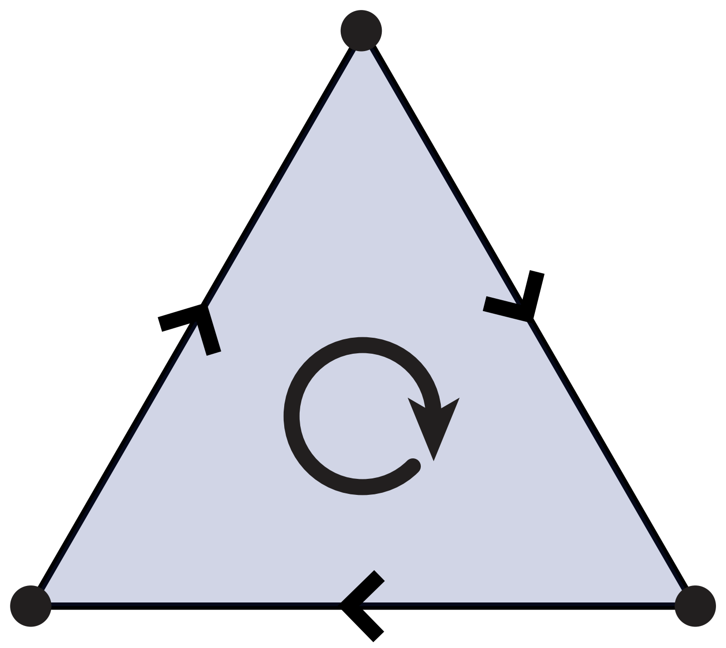
Summary

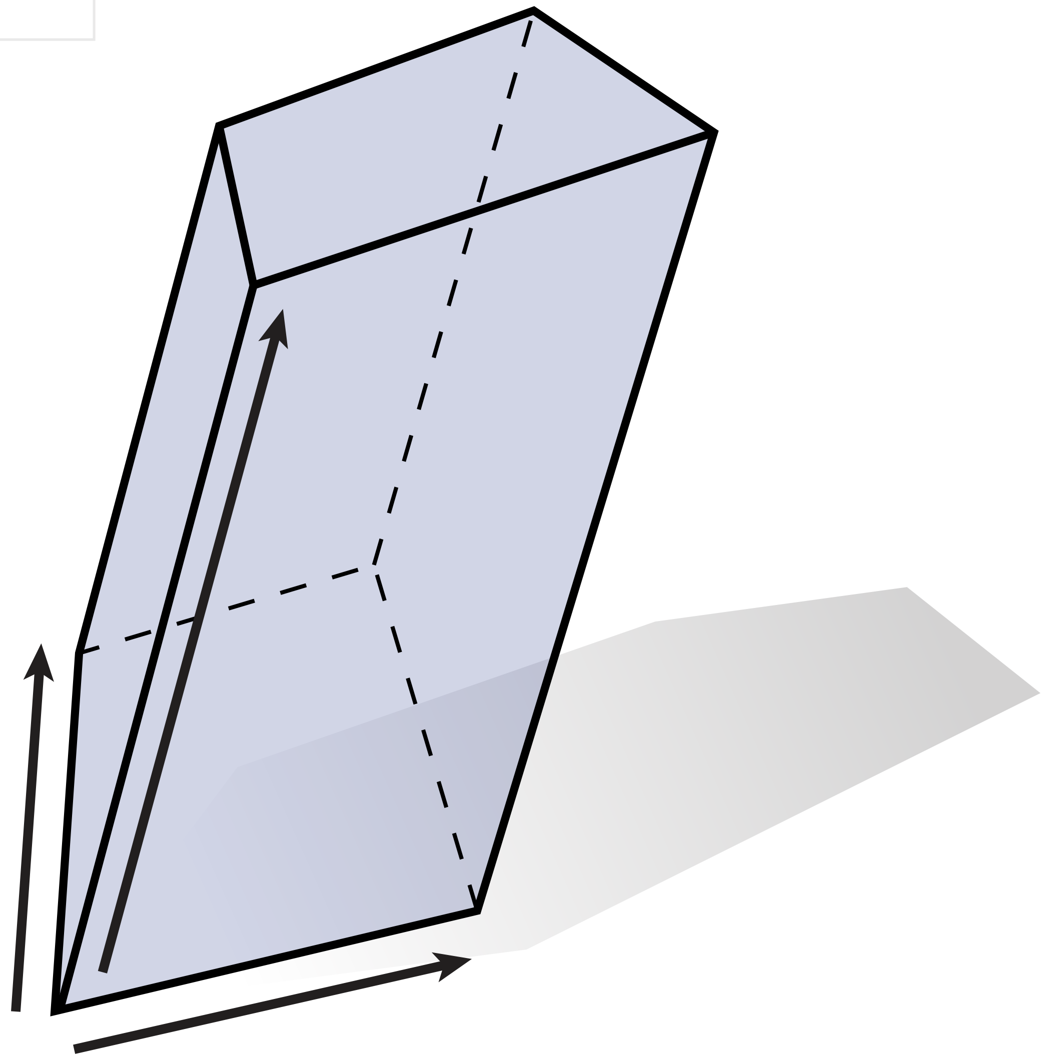
Exterior Calculus—Summary

- **What we’ve seen so far:**
- *Exterior algebra*: language of volumes (*k*-vectors)
- *k-form:* measures a k-dimensional volume
- *Differential forms: k*-form at each point of space
- *Exterior calculus*: differentiate/integrate forms
- *Simplicial complex*: mesh made of vertices, edges, triangles…
- **Next up:**
  - Put all this machinery together
  - *Integrate* to get discrete exterior calculus (DEC)

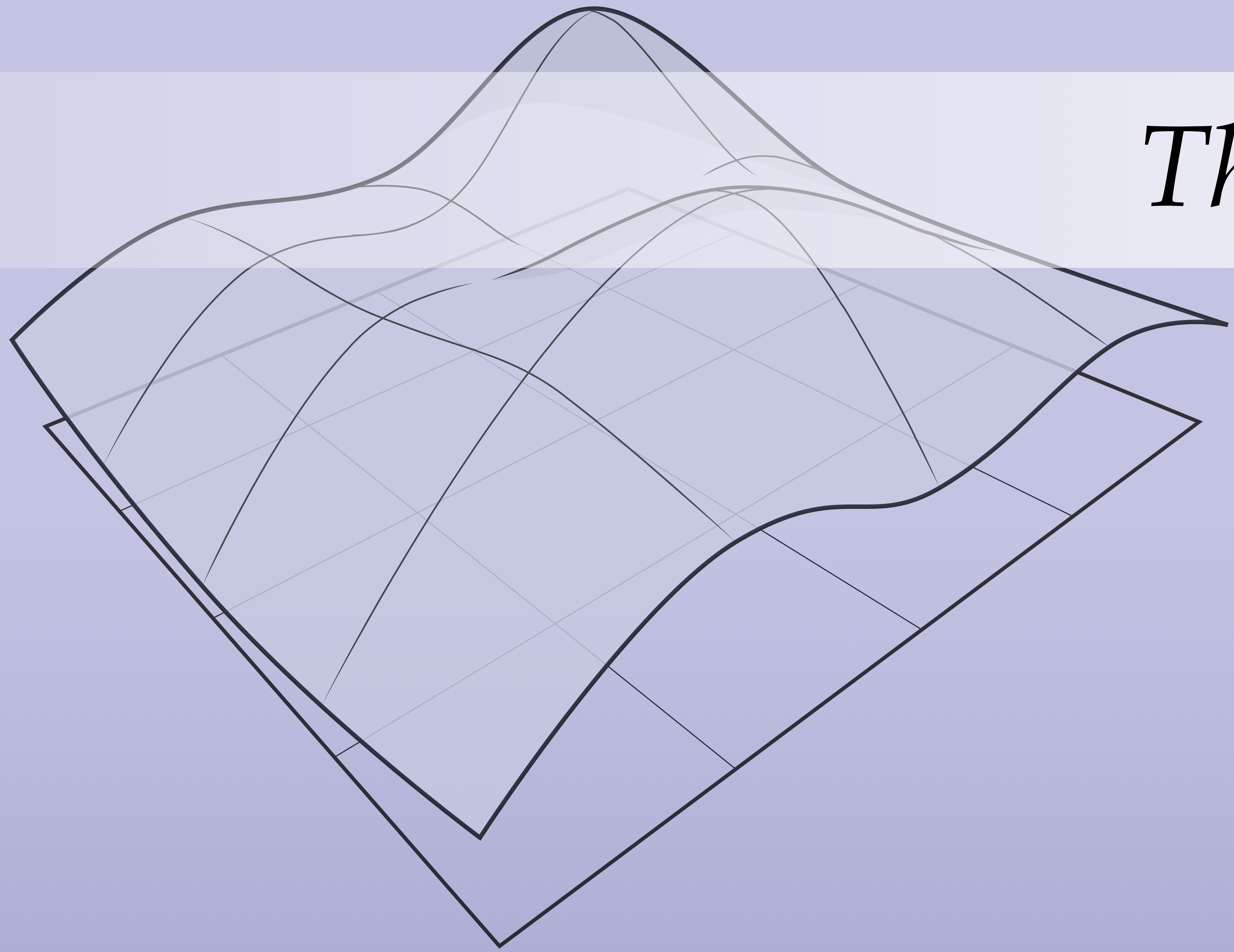








Thanks!



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