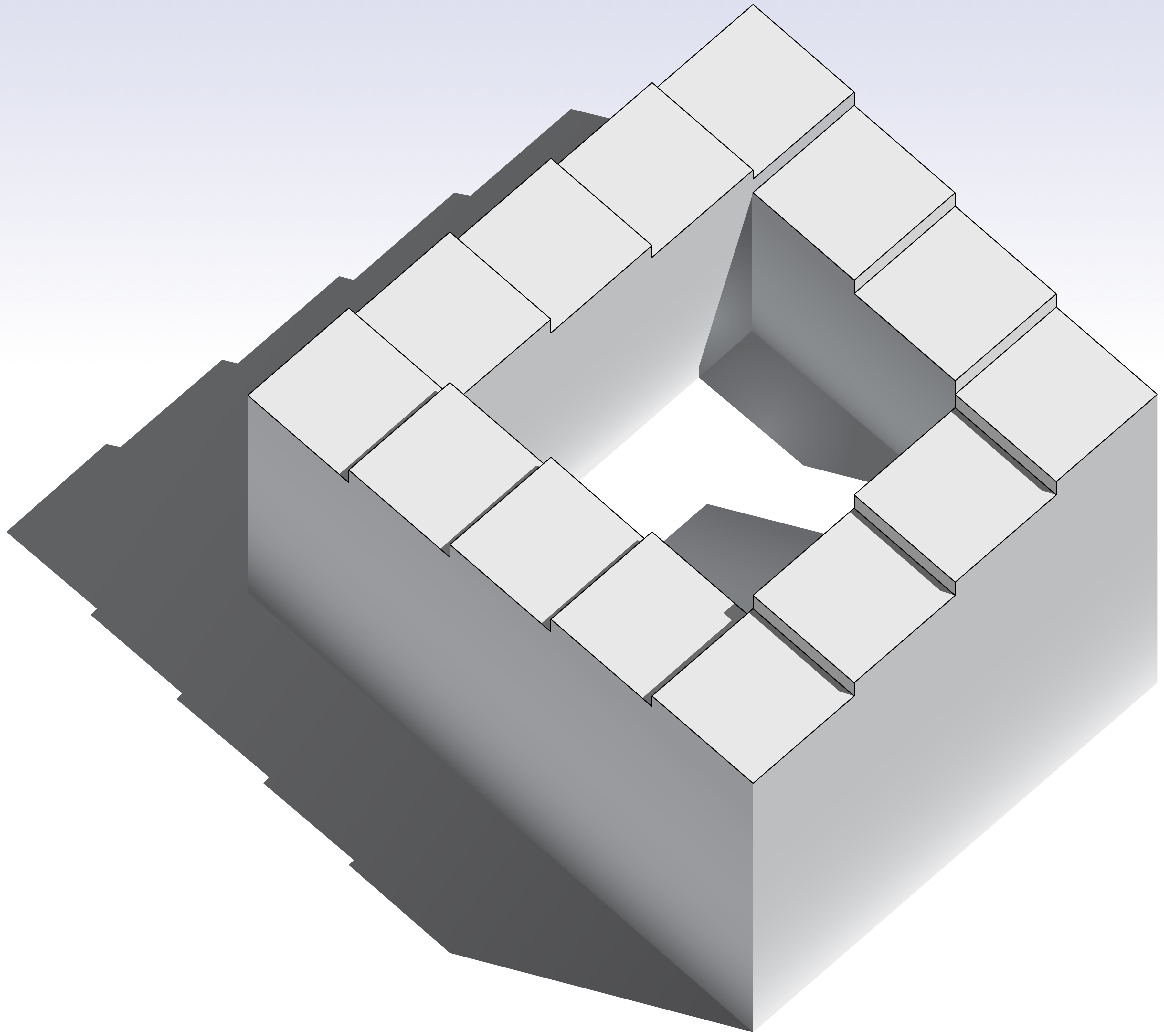
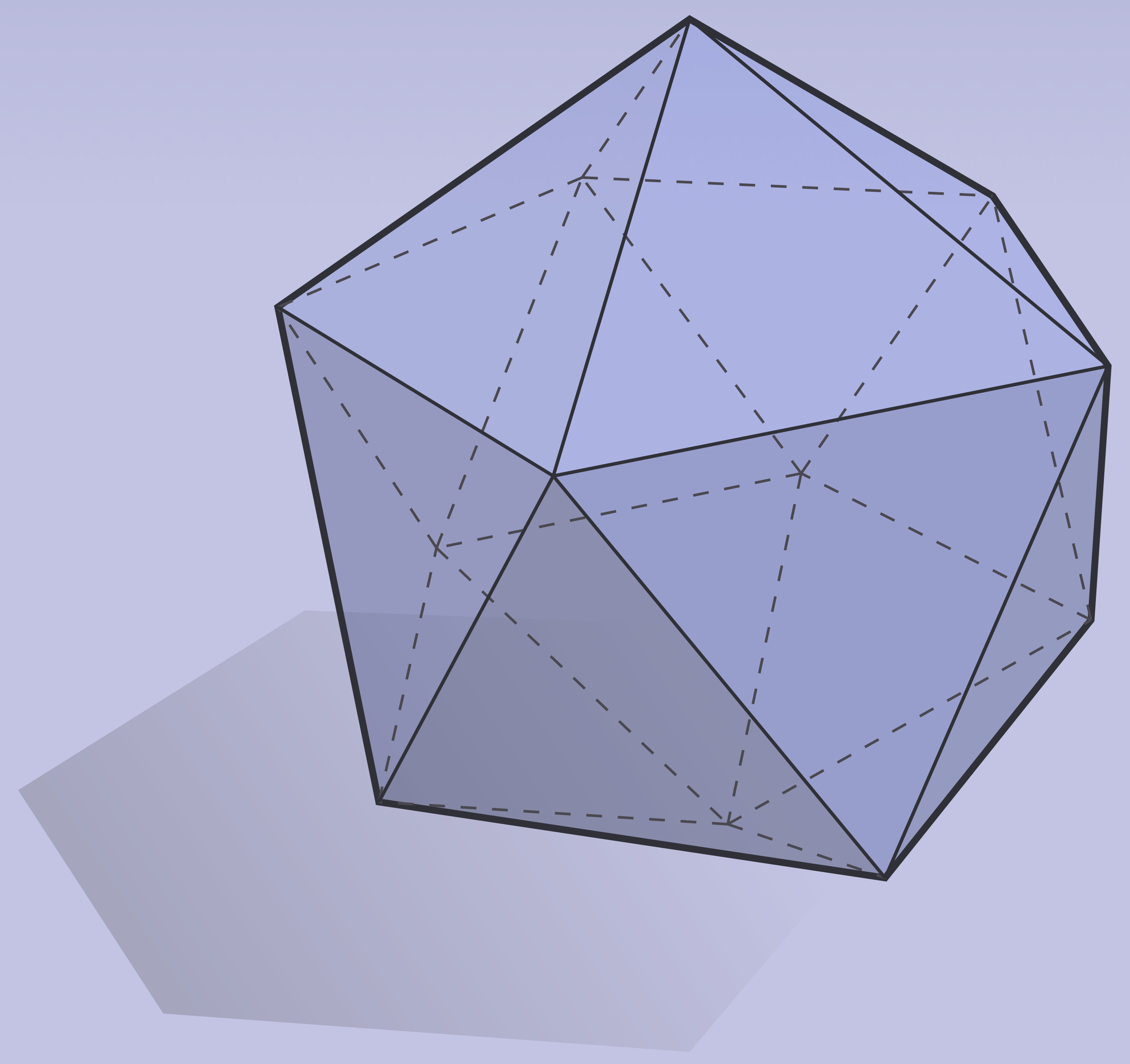


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 12:
SMOOTH SURFACES



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

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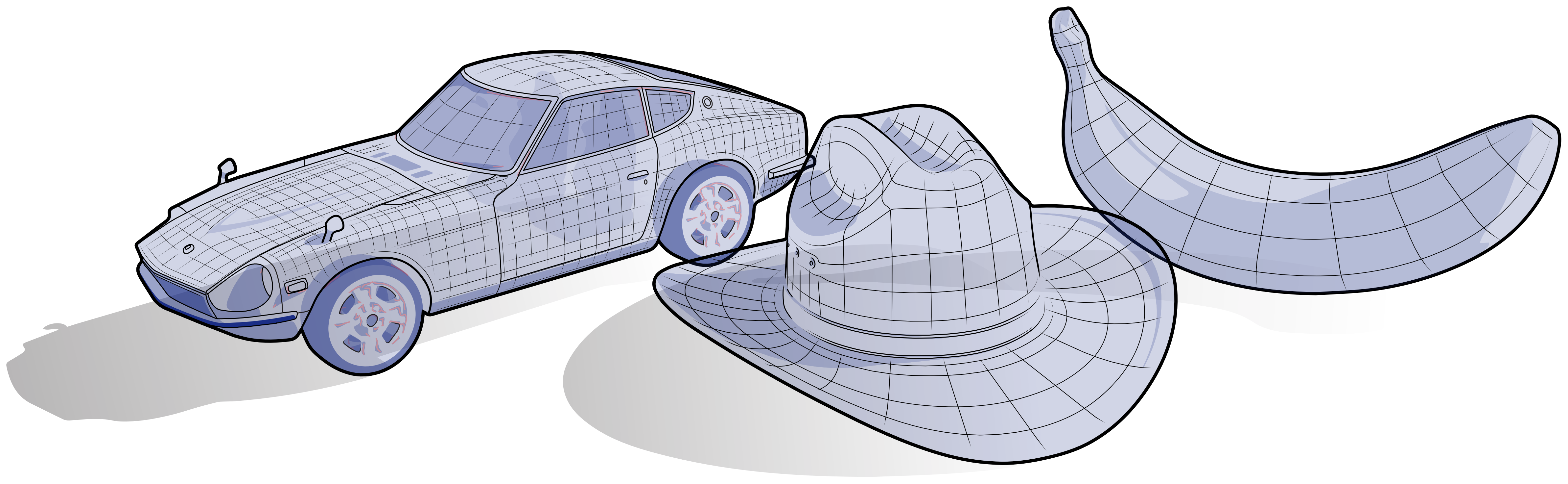
From Curves to Surfaces

- **Previously:** saw how to talk about 1D curves (both smooth and discrete)
- **Today:** will study 2D curved surfaces (both smooth and discrete)
  - Some concepts remain the same (*e.g.*, differential); others need to be generalized (*e.g.*, curvature)
  - Still use exterior calculus as our *lingua franca*

(Curves)

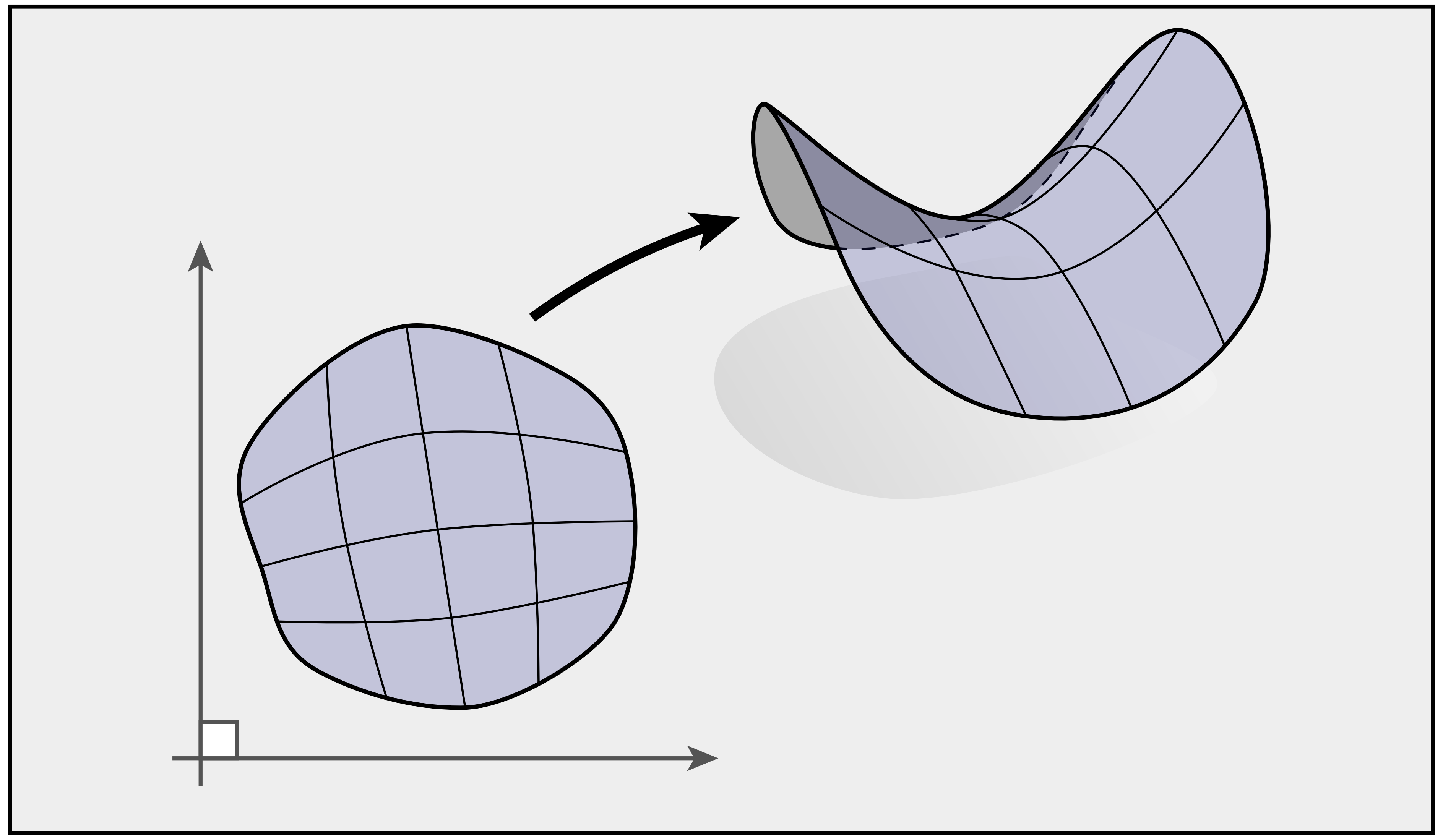


(Surfaces)

Surfaces—Local vs. Global View

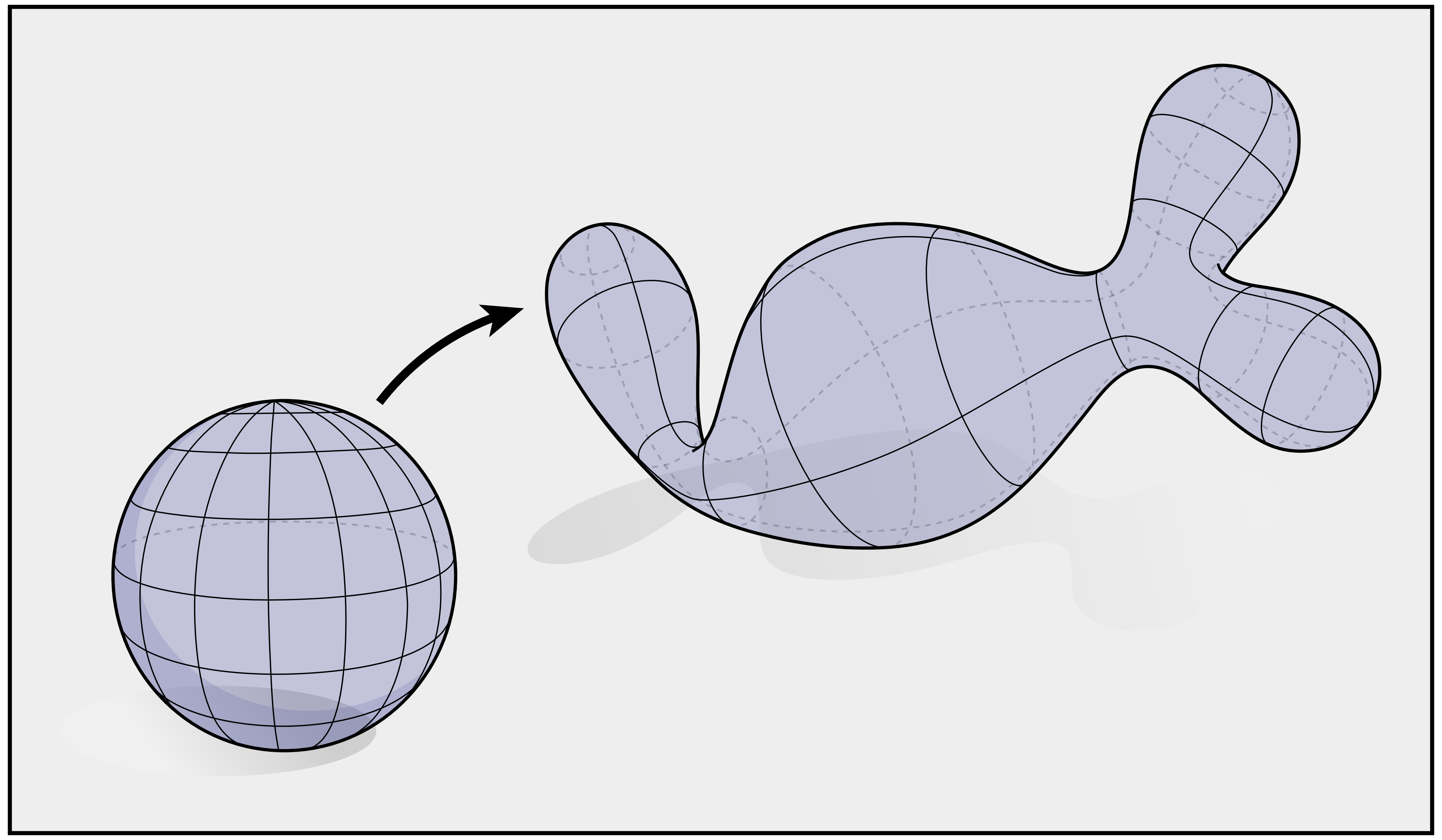
- So far, we've only studied exterior calculus in R^n

- Will therefore be easiest to think of surfaces expressed in terms of patches of the plane (**local picture**)

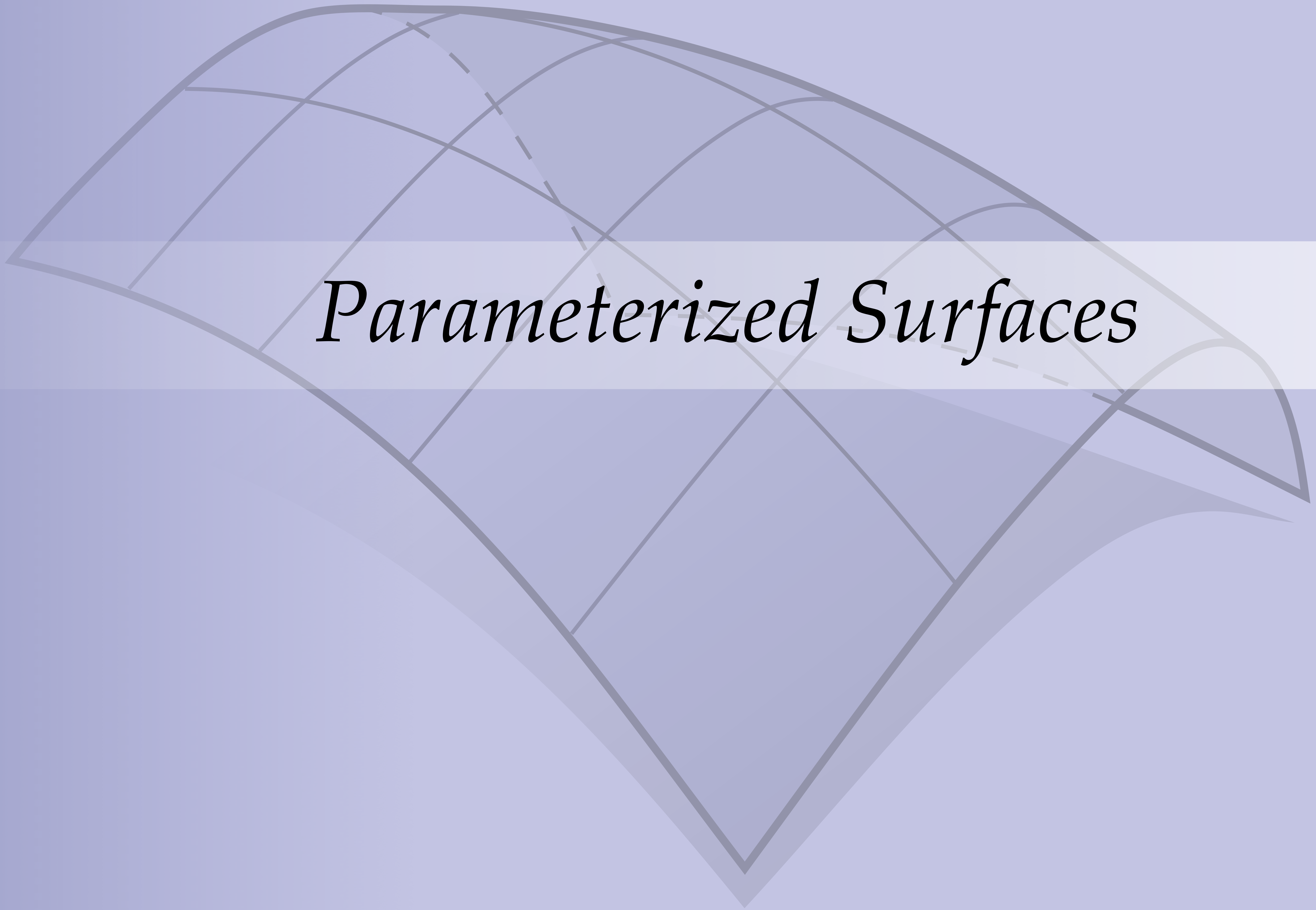


- Later, when we study* topology & smooth manifolds, we'll be able to more easily think about "whole surfaces" all at once (**global picture**).   (…*maybe)

- Global picture is *much* better model for **discrete** surfaces (meshes)…



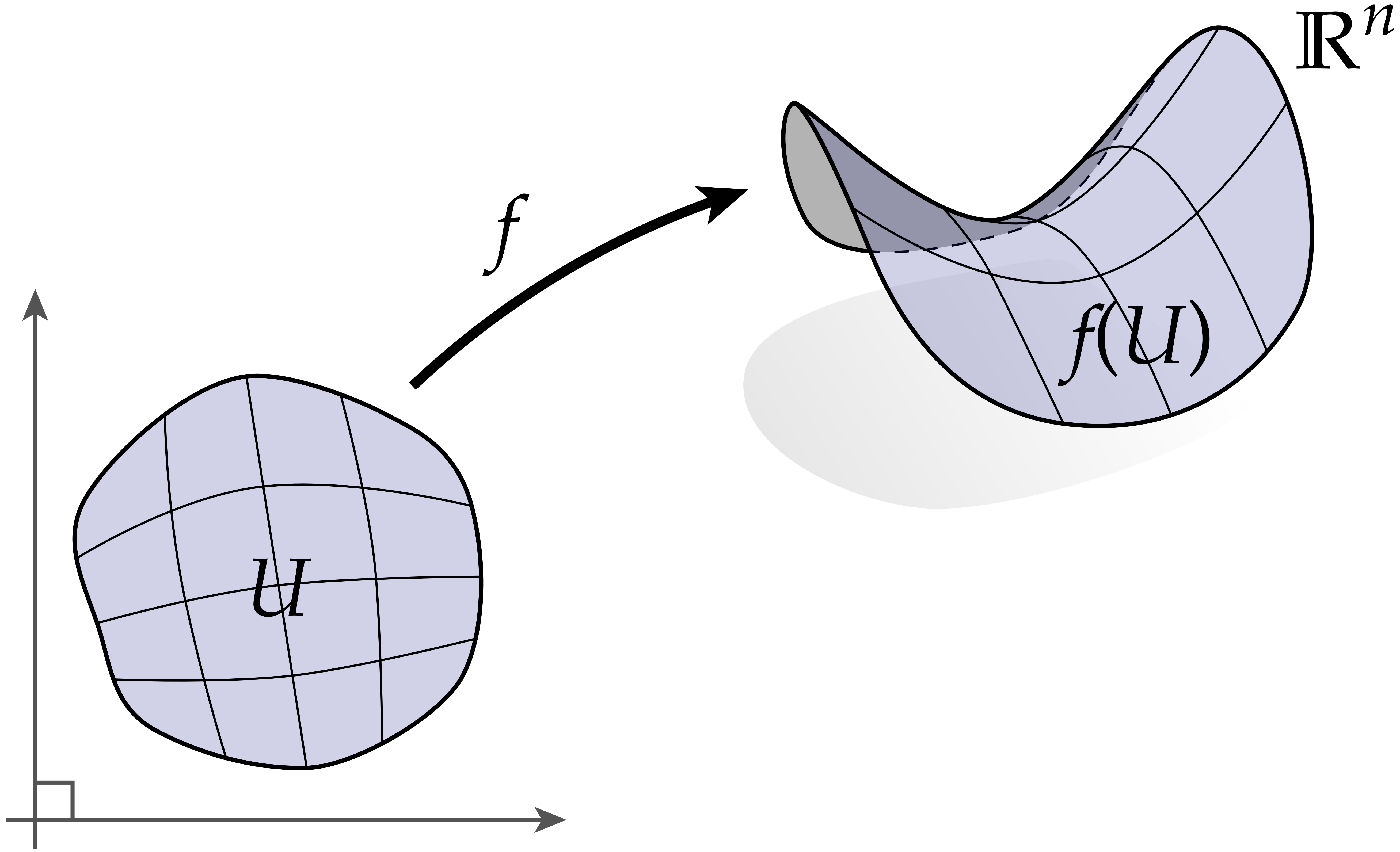
Parameterized Surfaces



Parameterized Surface

A **parameterized surface** is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$:

$$f : U \to \mathbb{R}^n$$



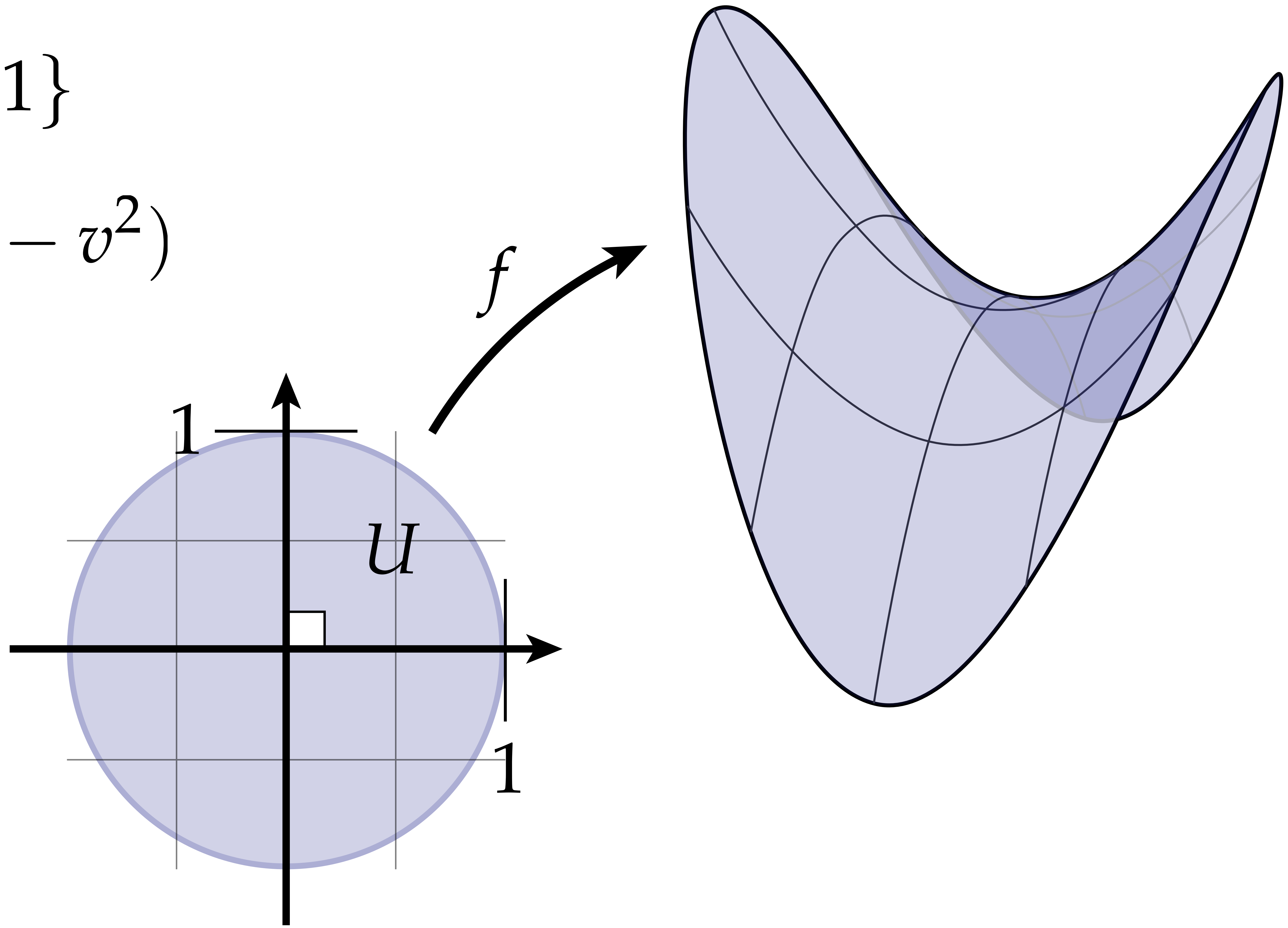
The set of points $f(U)$ is called the **image** of the parameterization.

Parameterized Surface—Example

- As an example, we can express a *saddle* as a parameterized surface:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u,v,u^2 - v^2)$$

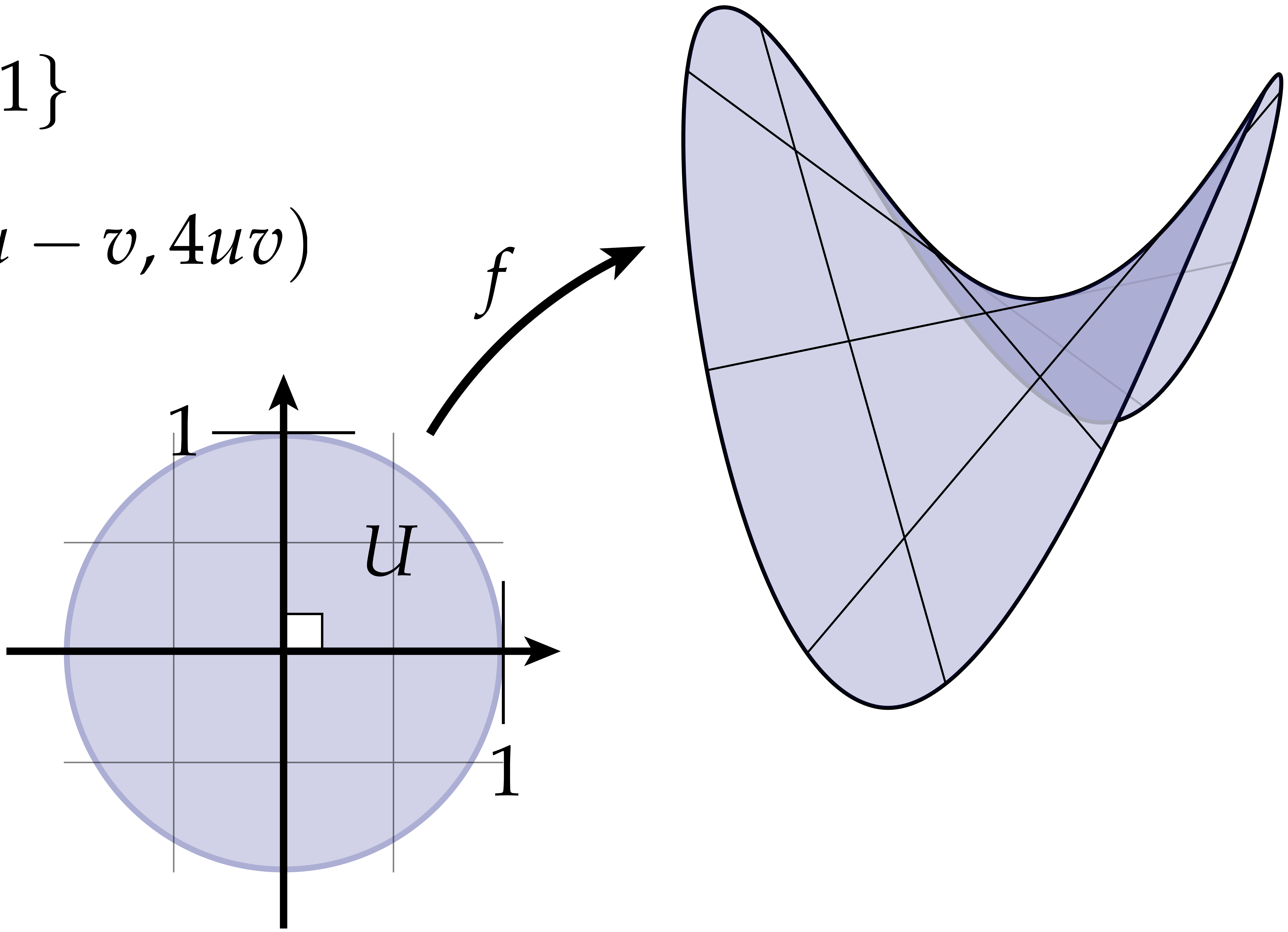


Reparameterization

- Many different parameterized surfaces can have the same image:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u+v, u-v, 4uv)$$

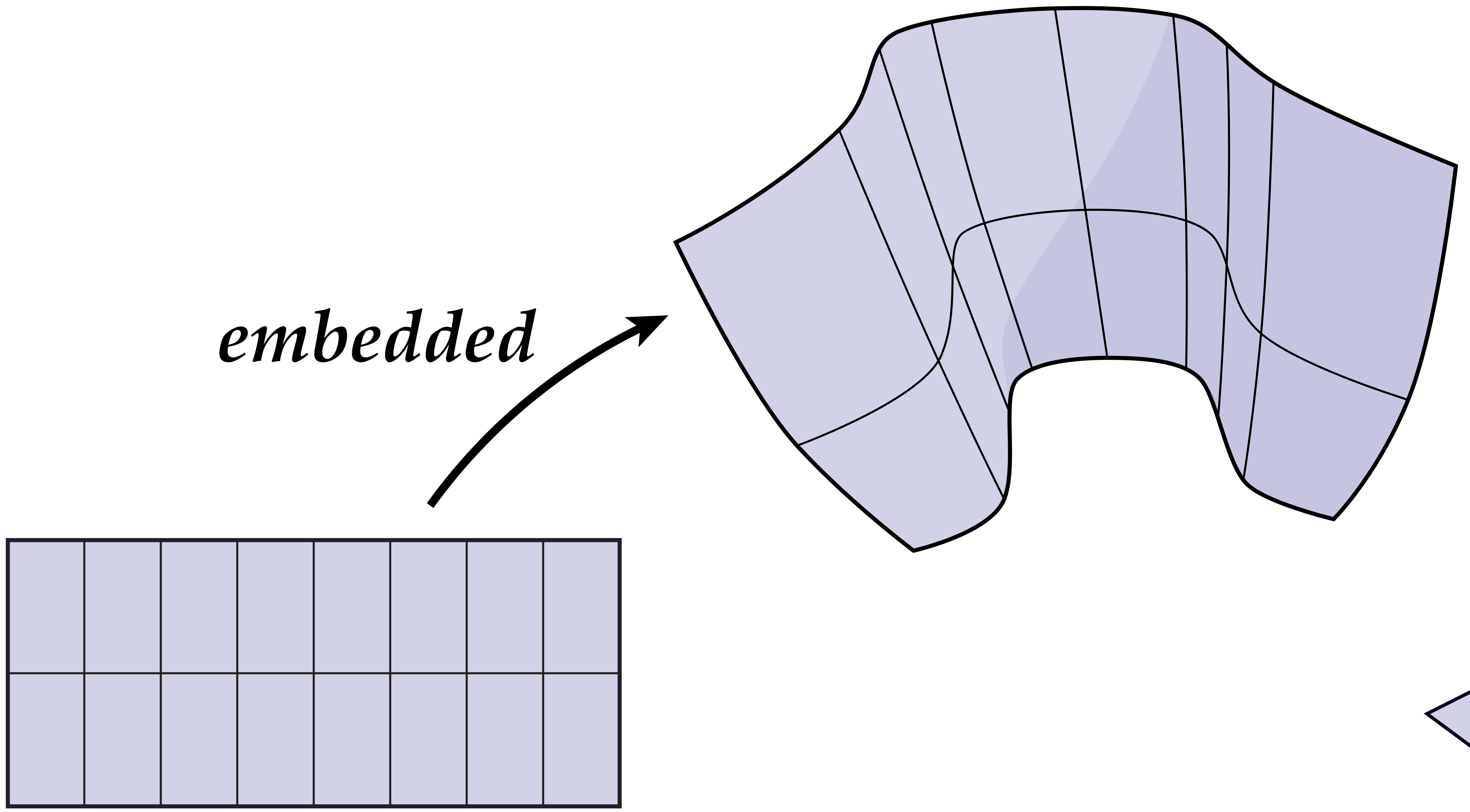


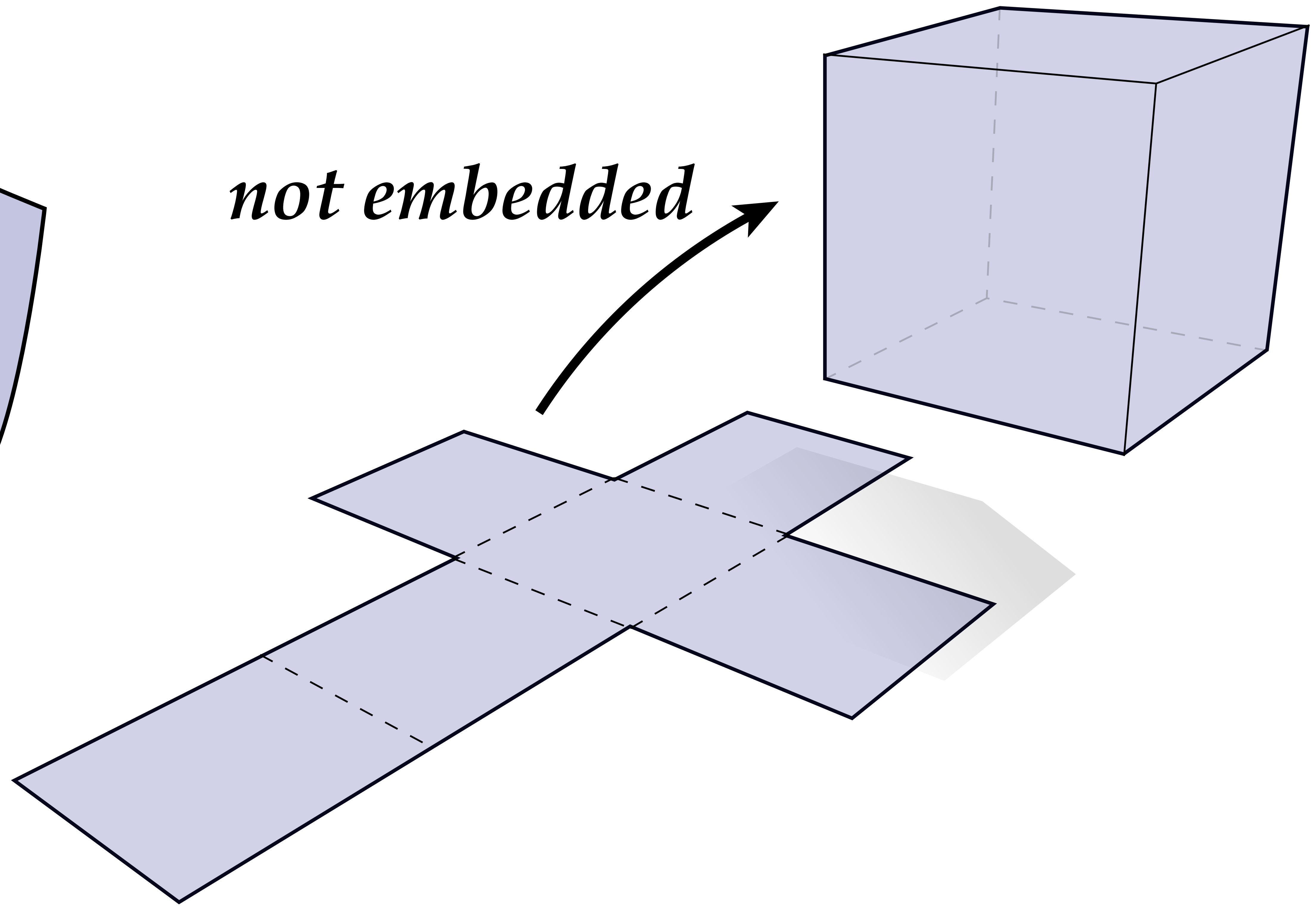
This *"reparameterization symmetry"* can be a major challenge in applications—*e.g.*, trying to decide if two parameterized surfaces (or meshes) describe the same shape.

Analogy: graph isomorphism

Embedded Surface

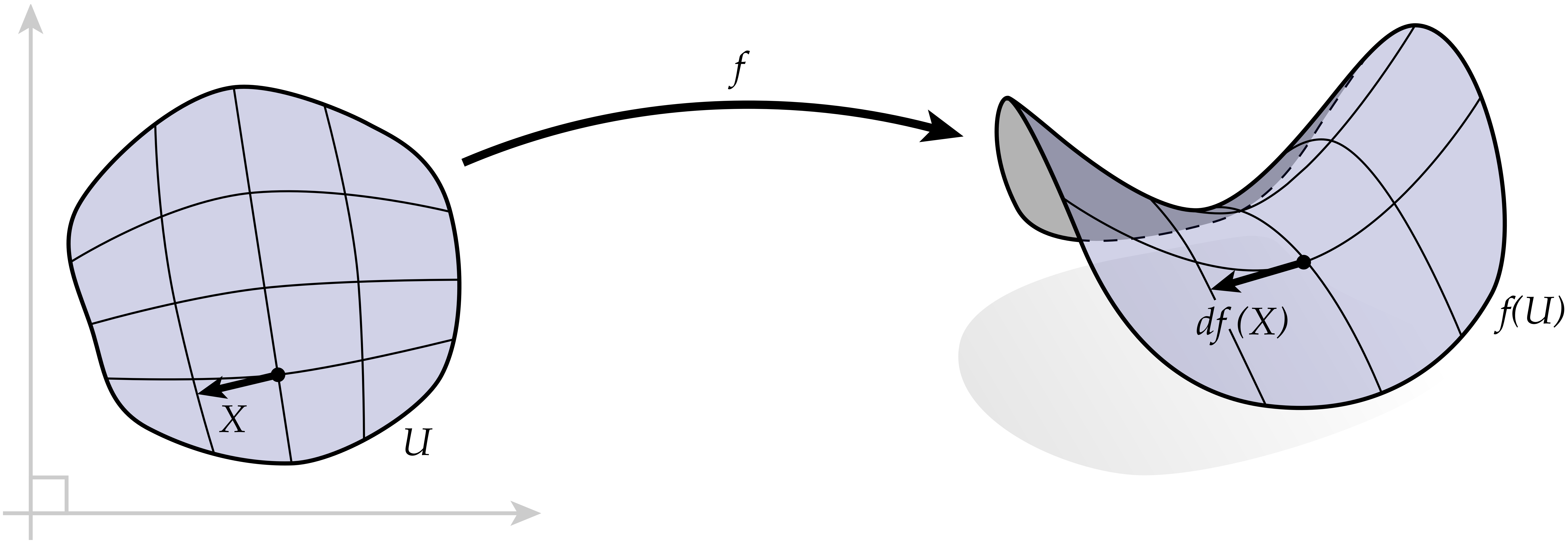
- Roughly speaking, an **embedded** surface does not self-intersect
- More precisely, a parameterized surface is an embedding if it is a continuous injective map, and has a continuous inverse on its image





Differential of a Surface

Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get mapped to vectors in space:



We say that *df "pushes forward"* vectors X into R^n, yielding vectors $df(X)$

Differential in Coordinates

In coordinates, the differential is simply the exterior derivative:

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u,v,u^2-v^2)$$

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv =$$

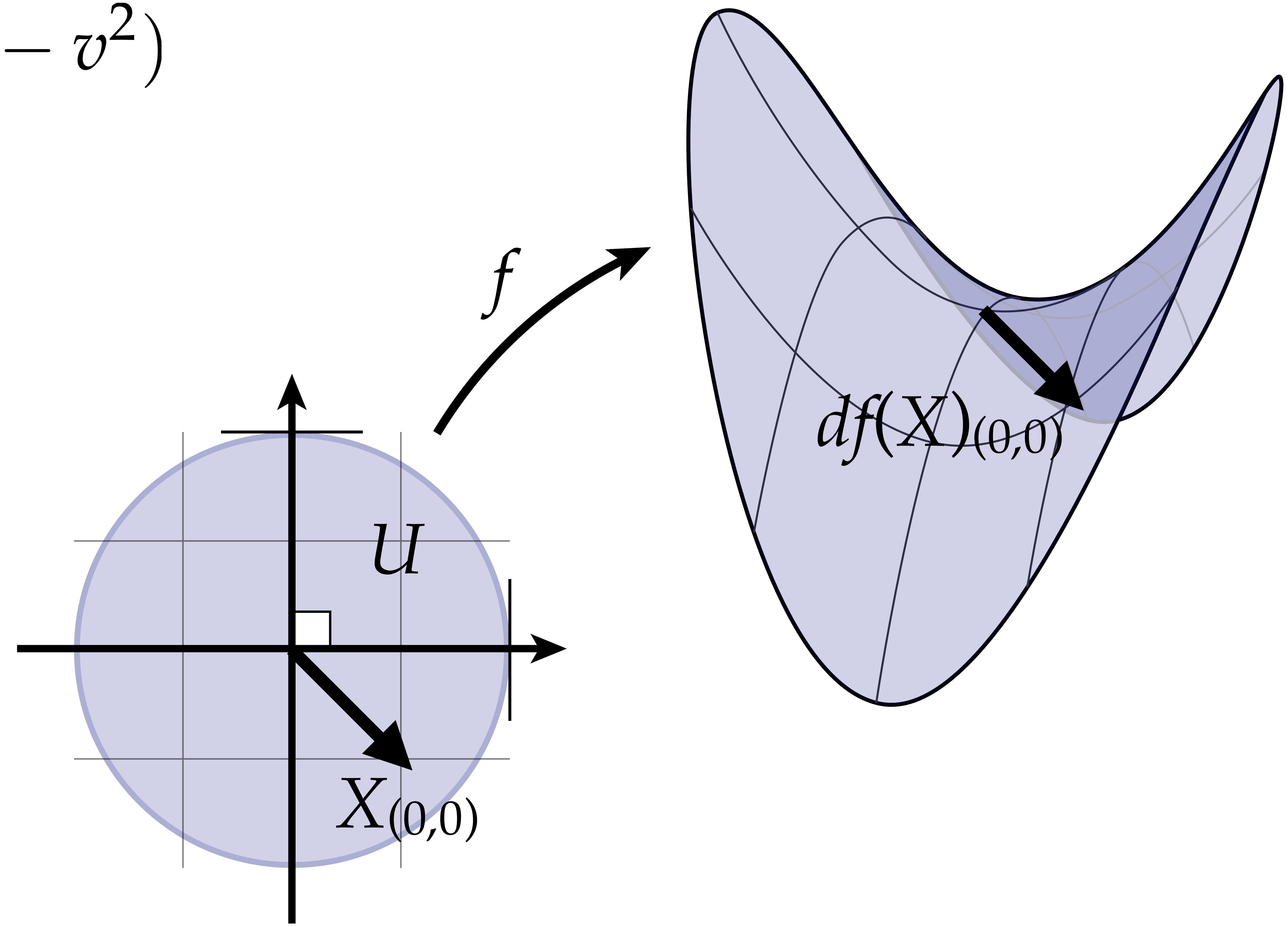
$$(1,0,2u)du + (0,1,-2v)dv$$

<u>Pushforward of a vector field:</u>

$$X := \tfrac{3}{4}\left(\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right)$$

$$df(X) = \tfrac{3}{4}(1,-1,2(u+v))$$

E.g., at $u{=}v{=}0$: $(\tfrac{3}{4}, -\tfrac{3}{4}, 0)$



Differential—Matrix Representation (Jacobian)

Definition. Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$, and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. Then the *Jacobian* of f is the matrix

$$J_f := \begin{bmatrix} \partial f^1/\partial x^1 & \cdots & \partial f^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial f^m/\partial x^1 & \cdots & \partial f^m/\partial x^n \end{bmatrix},$$

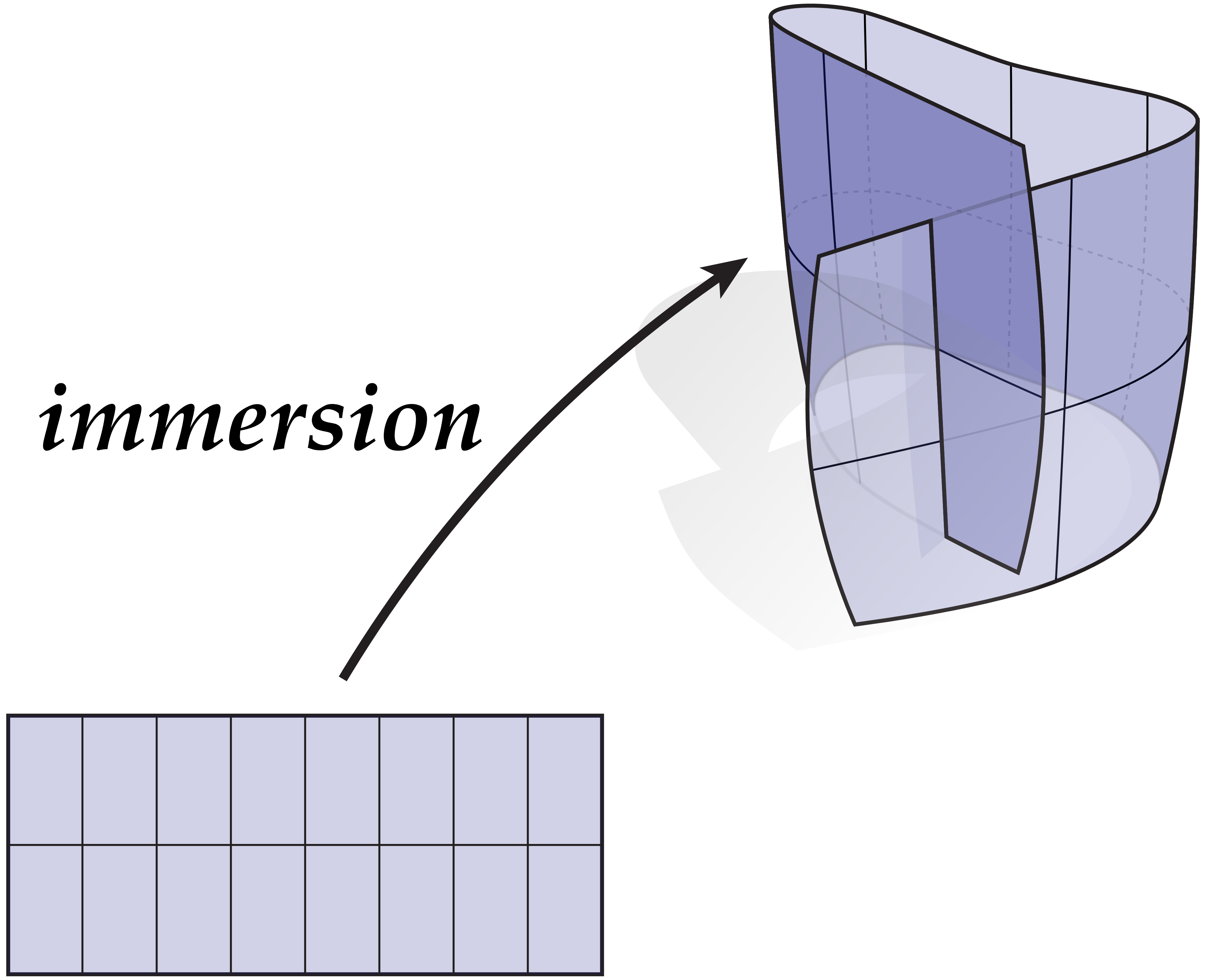
where $f^1, \ldots, f^m$ are the components of f w.r.t. some coordinate system on $\mathbb{R}^m$. This matrix represents the differential in the sense that $df(X) = J_f X$.

(In solid mechanics, also known as the *deformation gradient*.)

Note: does not generalize to infinite dimensions! (E.g., maps between functions.)

Immersed Surface

- A parameterized surface f is an *immersion* if its differential is nondegenerate, *i.e.*, if $df(X) = 0$ if and only if $X = 0$.



Intuition: no region of the surface gets "pinched"

Immersion — Example

Consider the standard parameterization of the sphere:

$$f(u,v) := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

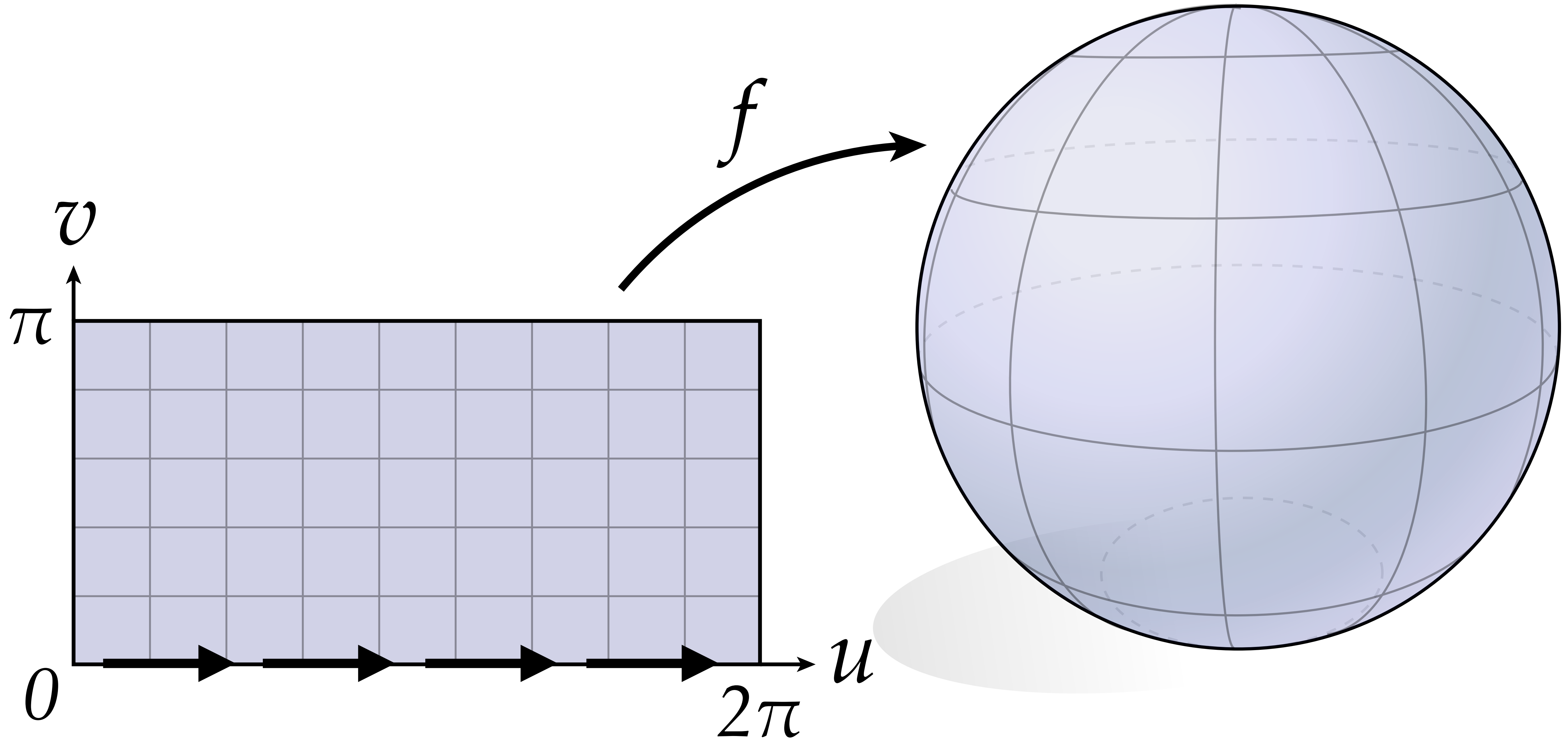
$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = \begin{matrix} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &) & du\ + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &) & dv \end{matrix}$$

Q: Is f an immersion?

A: No: when $v = 0$ we get

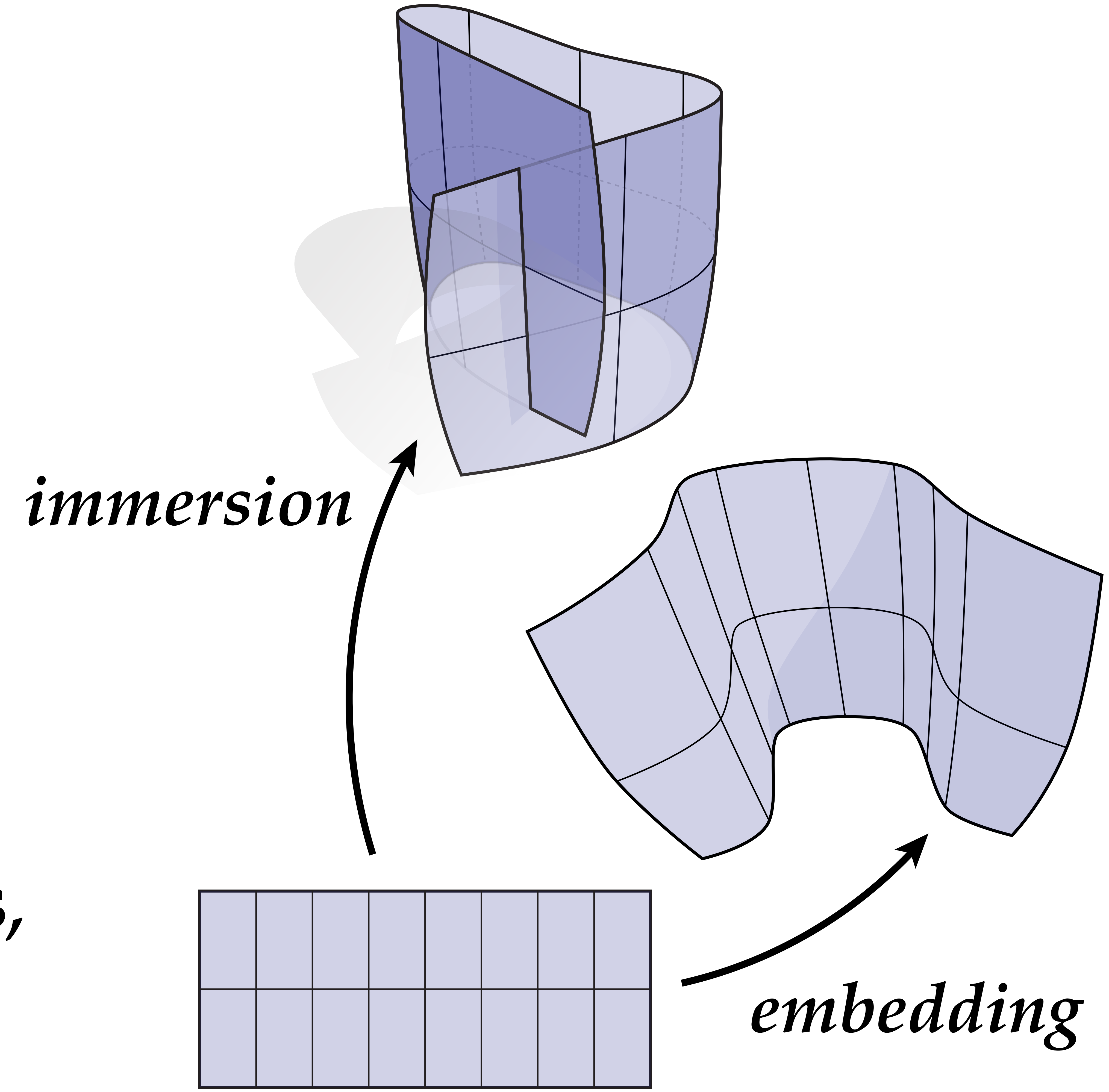
$$\begin{matrix} (& 0, & 0, & 0 &) & du\ + \\ (& \cos(u), & \sin(u), & -\sin(v) &) & dv \end{matrix}$$

Nonzero tangents mapped to zero!



Immersion vs. Embedding

- In practice, ensuring that a surface is globally embedded can be challenging
- Immersions are typically "nice enough" to define local quantities like tangents, normals, metric, etc.
- Immersions are also a natural model for the way we typically think about meshes: most quantities (angles, lengths, etc.) are perfectly well-defined, even if there happen to be self-intersections

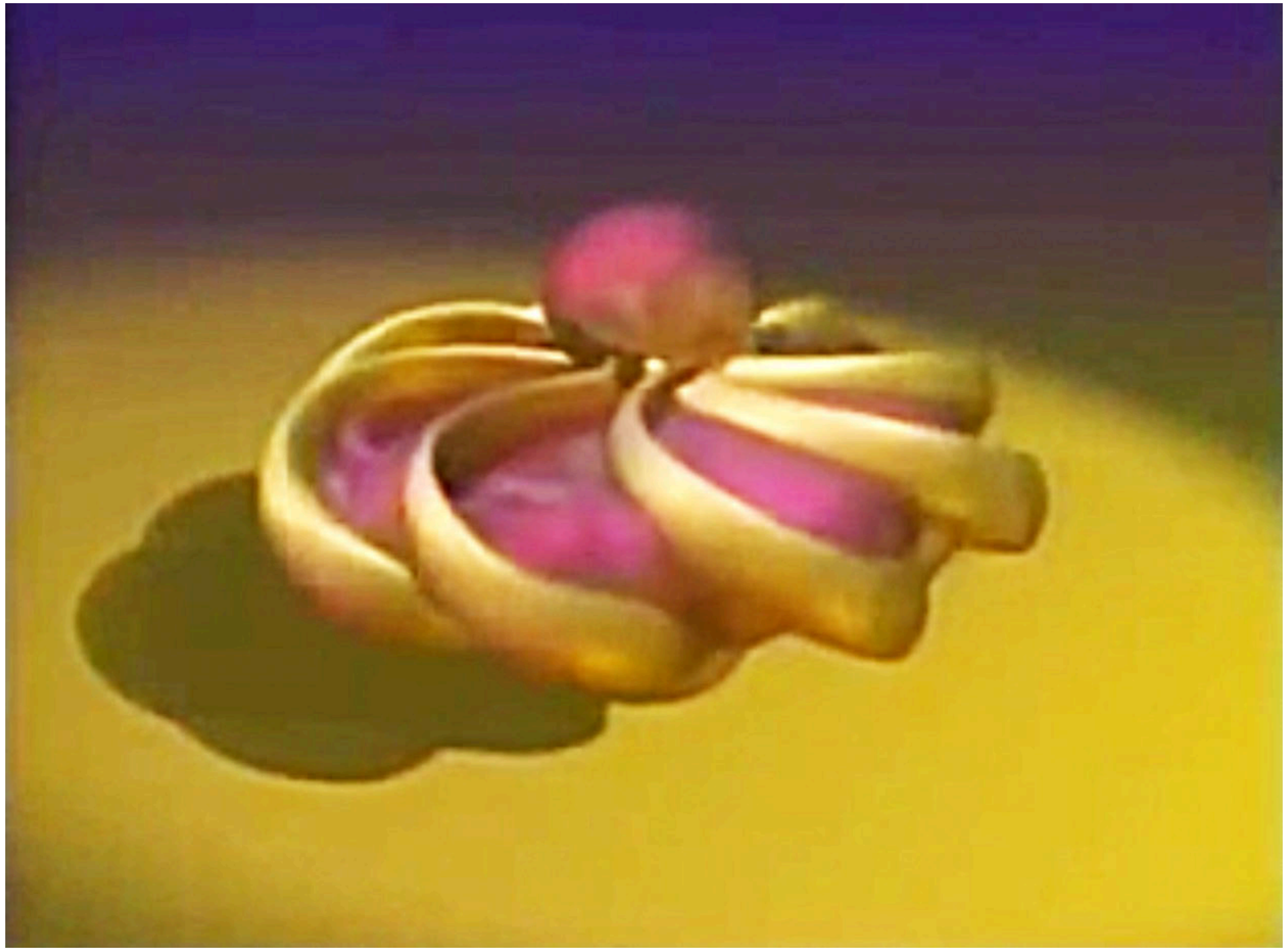


Circle Eversion

- Can you turn the circle inside-out, while remaining immersed?
- (Hint: we've already seen a theorem that says something about this question!)

Sphere Eversion

Turning a Sphere Inside-Out (1994)



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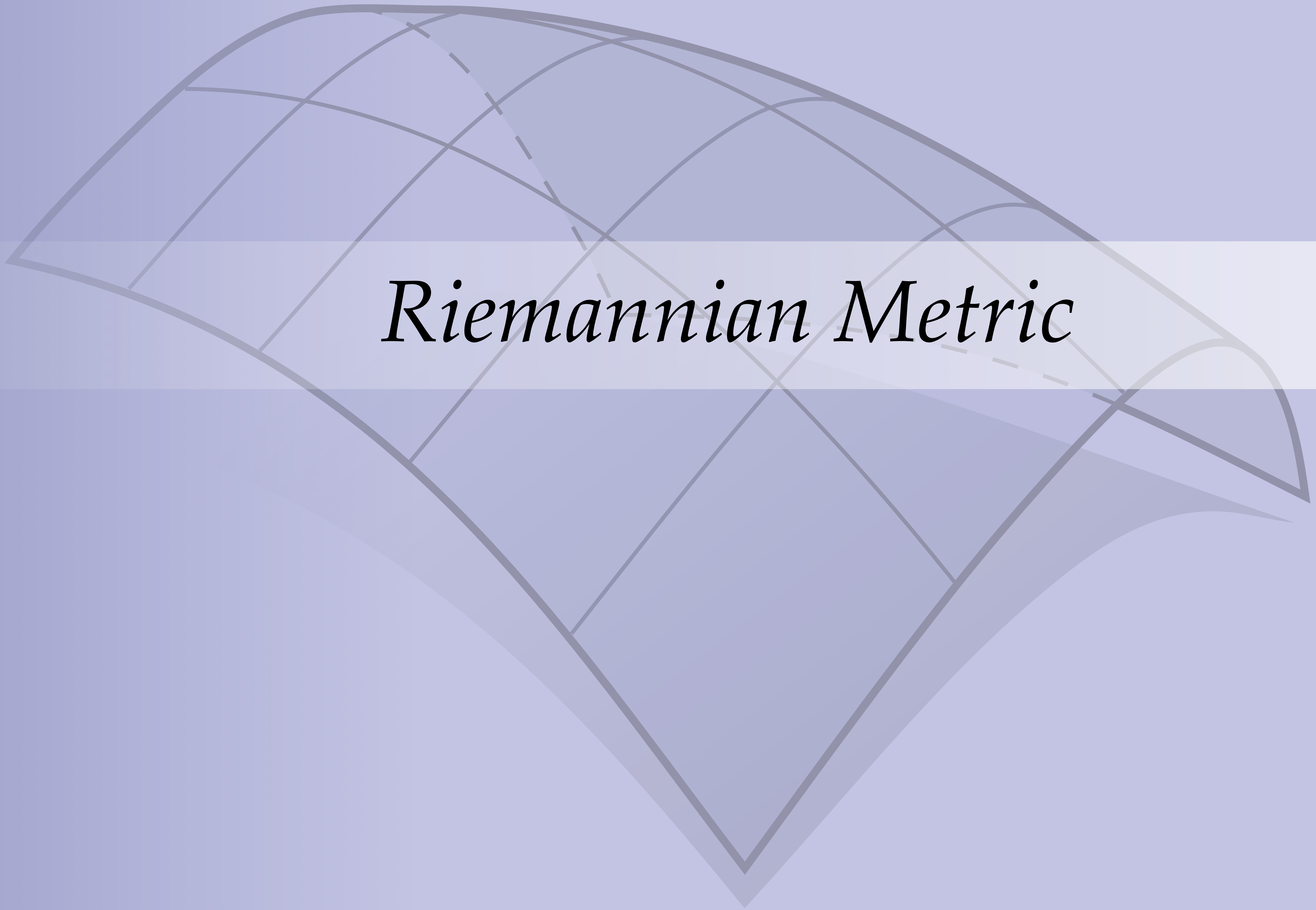
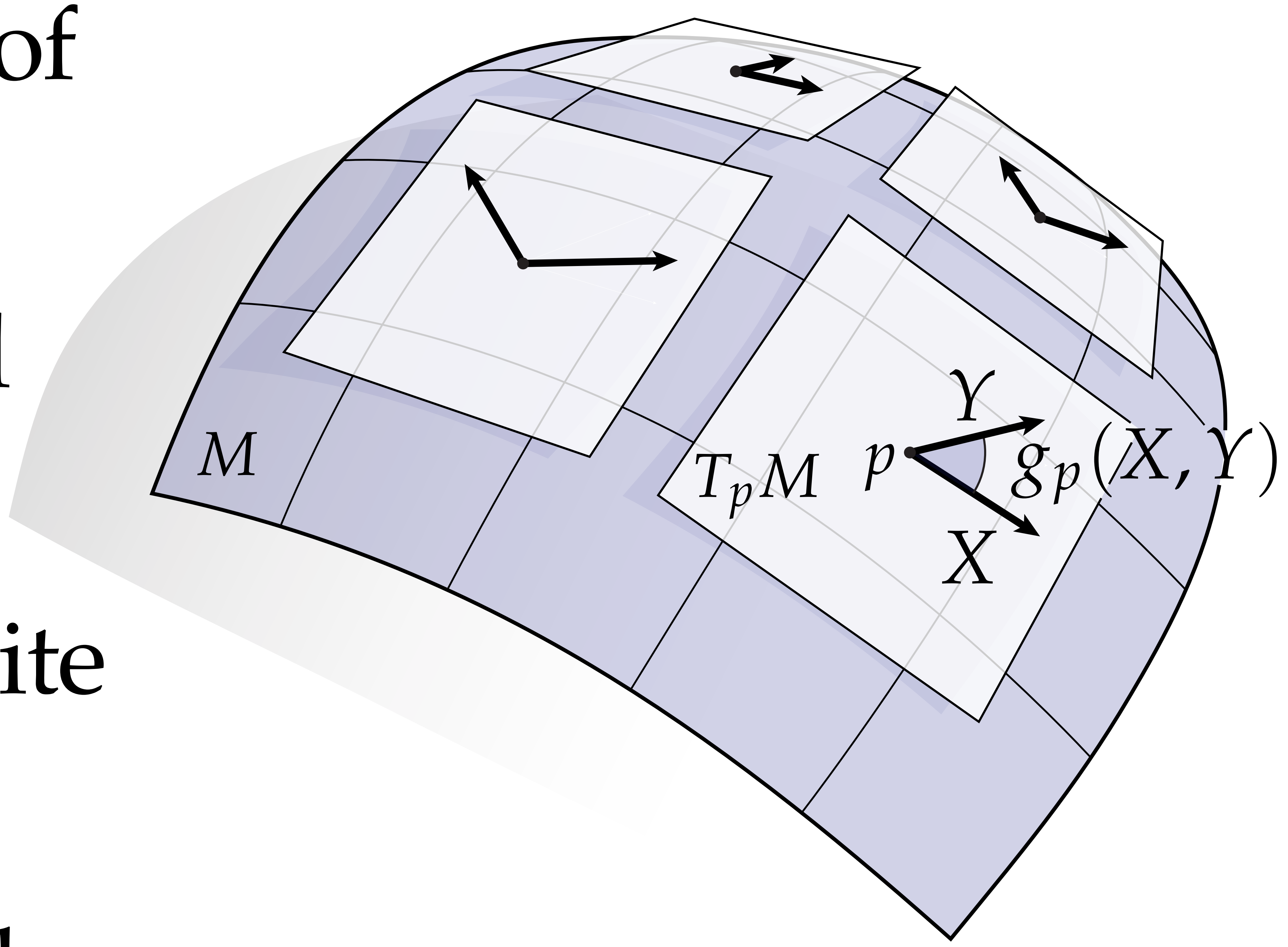
Riemannian Metric



Riemann Metric

- Many quantities on manifolds (curves, surfaces, *etc.*) ultimately boil down to measurements of *lengths* and *angles* of tangent vectors
- This information is encoded by the so-called *Riemannian metric**
- Abstractly: smoothly-varying positive-definite bilinear form
- For immersed surface, can (and will!) describe more concretely/geometrically

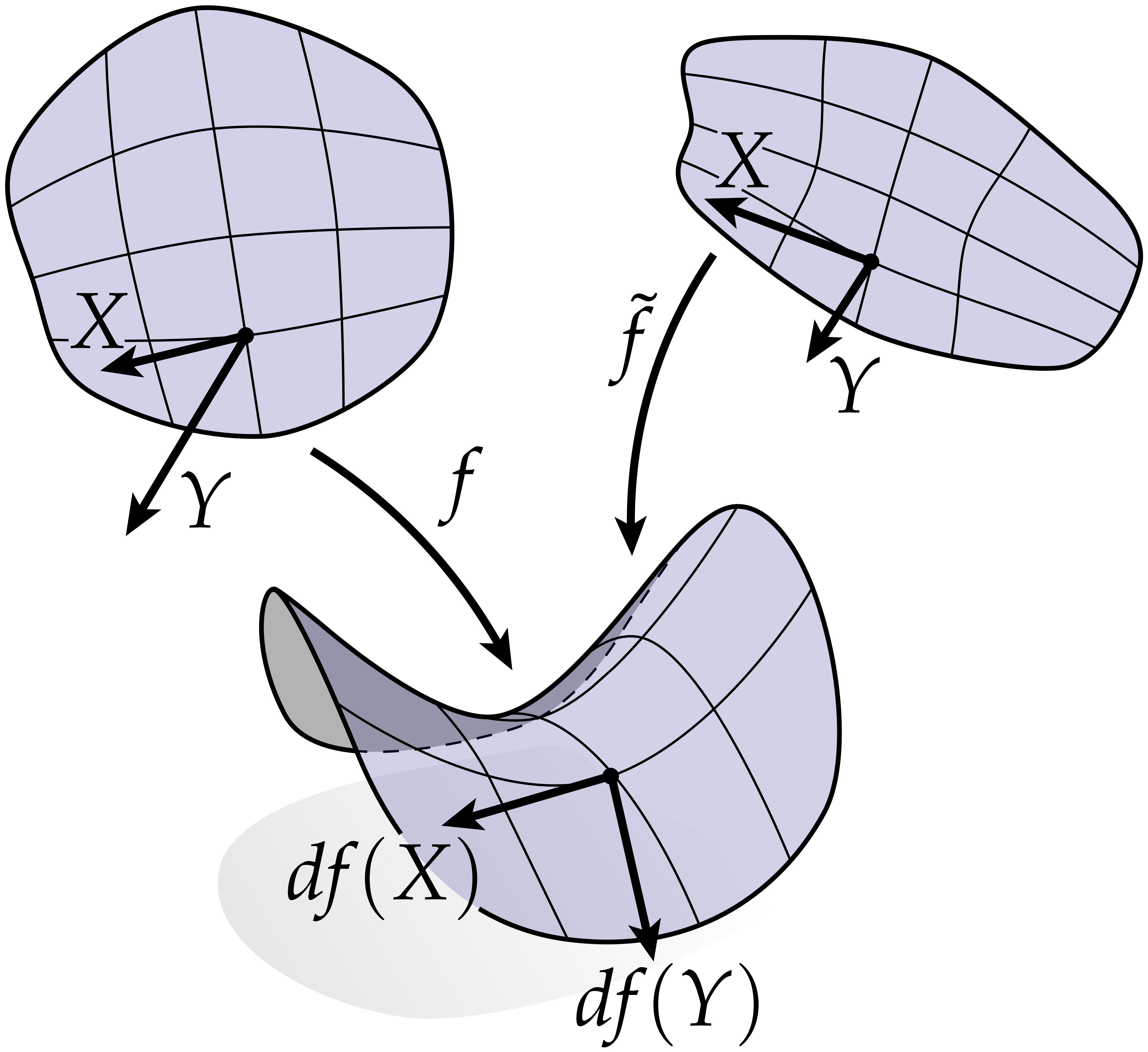


***Note:** *not* the same as a point-to-point distance metric $d(x,y)$

Metric Induced by an Immersion

- Given an immersed surface f, how should we measure inner product of vectors X, Y on its domain U?
- We should **not** use the usual inner product on the plane! (Why not?)
- Planar inner product tells us *nothing* about actual length & angle on the surface (and changes depending on choice of parameterization!)
- Instead, use **induced metric**

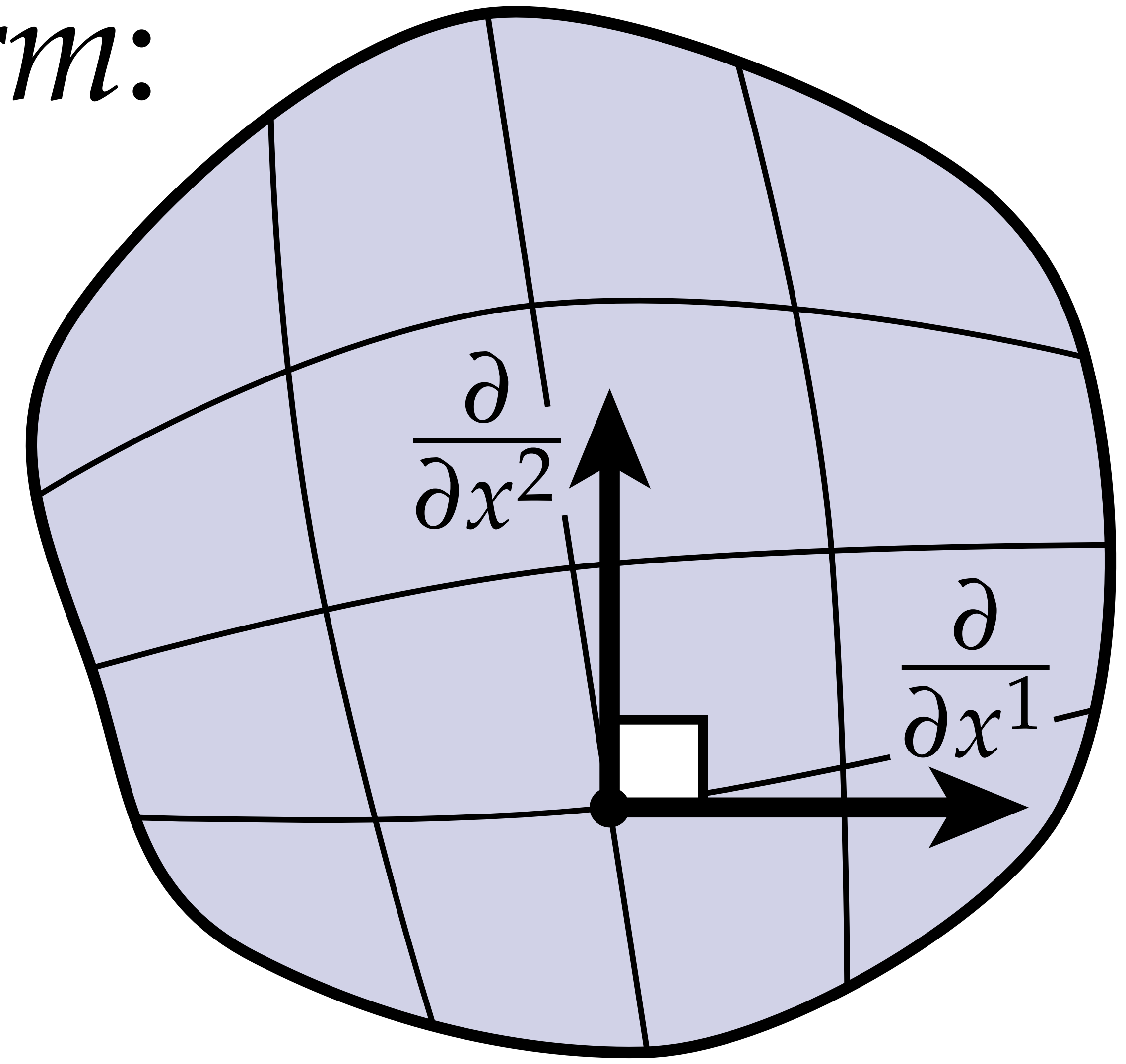
$$g(X,Y) := \langle df(X), df(Y) \rangle$$



Key idea: must account for "stretching"

Induced Metric—Matrix Representation

- Metric is a bilinear map from a pair of vectors to a scalar, which we can represent as a 2x2 matrix $\mathbf{I}$ called the *first fundamental form*:



$$g(X, Y) = X^T \mathbf{I} Y$$

$$\Rightarrow \mathbf{I}_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle df\left(\frac{\partial}{\partial x^i}\right), df\left(\frac{\partial}{\partial x^j}\right)\right\rangle$$

- Alternatively, can express first fundamental form via Jacobian:

$$g(X, Y) = \langle df(X), df(Y)\rangle = (J_f X)^\mathsf{T}(J_f Y) = X^\mathsf{T}(J_f^\mathsf{T} J_f)Y$$

$$\Rightarrow \mathbf{I} = J_f^\mathsf{T} J_f$$

Induced Metric—Example

Can use the differential to obtain the induced metric:

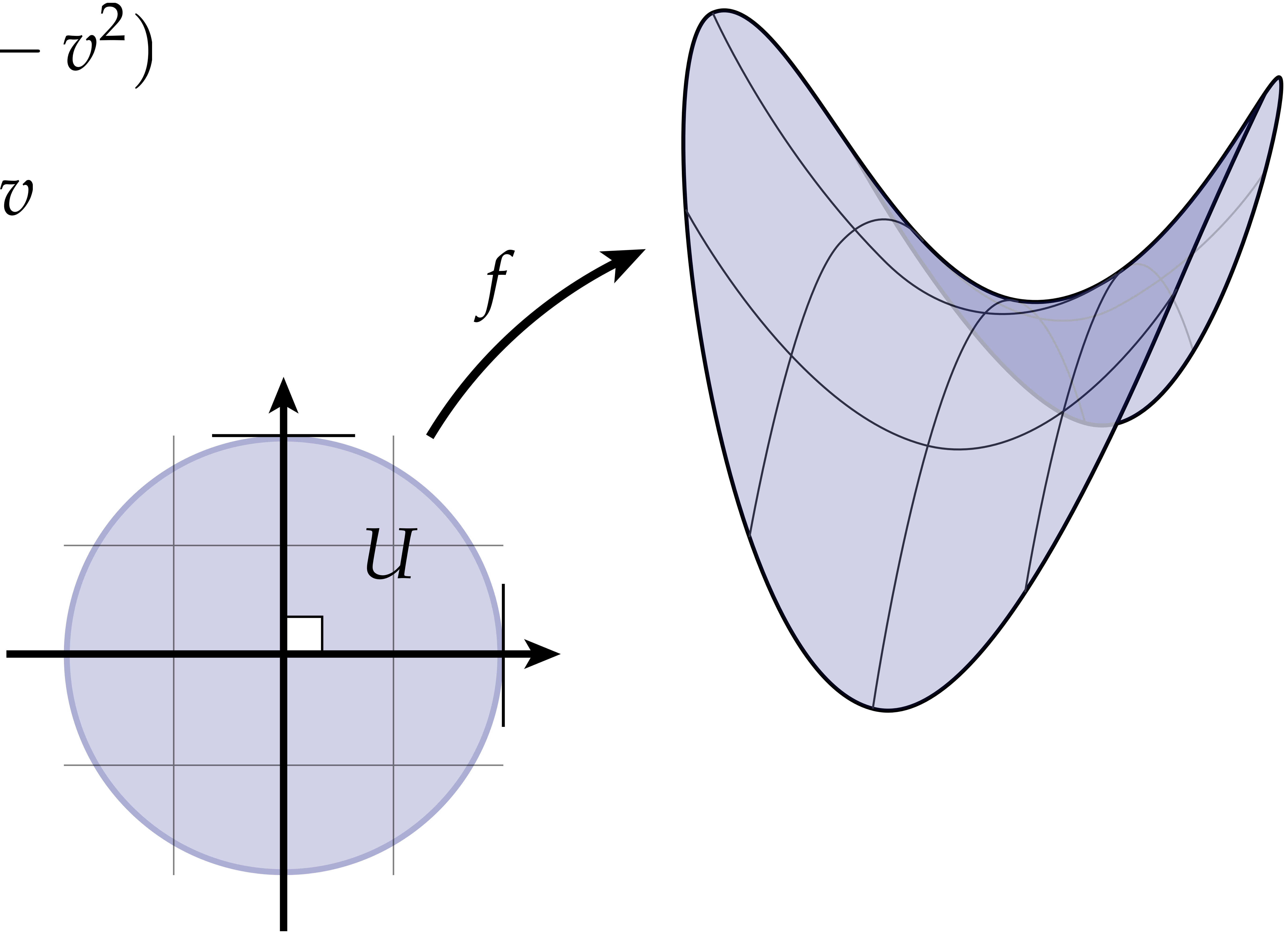
$$f : U \to \mathbb{R}^3;\ (u, v) \mapsto (u, v, u^2 - v^2)$$

$$df = (1, 0, 2u)du + (0, 1, -2v)dv$$

$$J_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

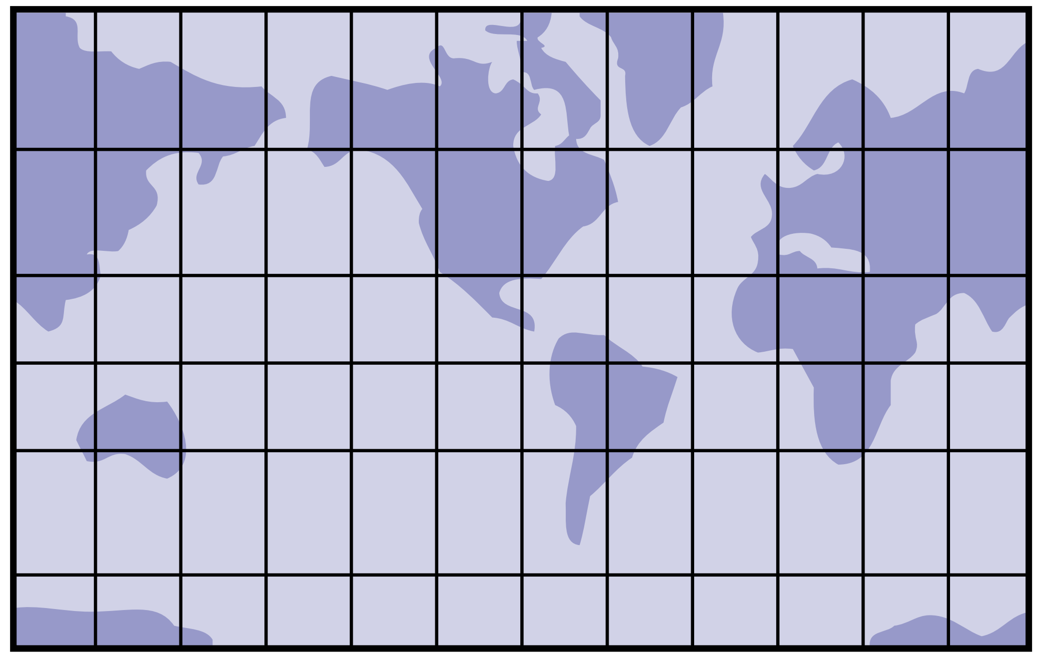
$$\mathbf{I} = J_f^\top J_f$$

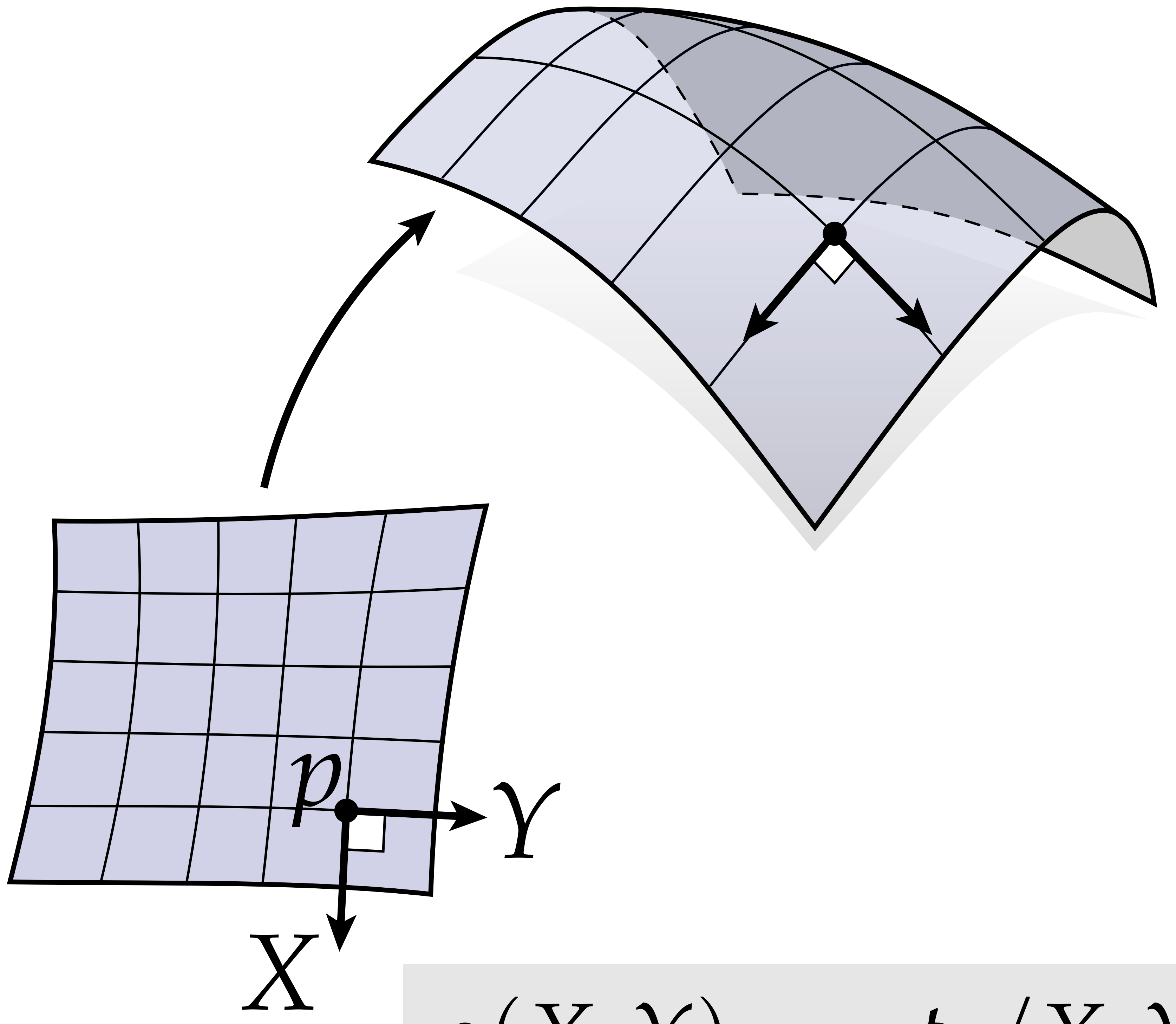
$$= \begin{bmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{bmatrix}$$



Conformal Coordinates

- As we've just seen, there can be a complicated relationship between length & angle on the domain (2D) and the image (3D)
- For curves, we simplified life by using an *arc-length* or *isometric* parameterization: lengths on domain are identical to lengths along curve
- For surfaces, usually not possible to preserve all *lengths* (e.g., globe). Remarkably, however, can always preserve *angles* (**conformal**)
- Equivalently, a parameterized surface is *conformal* if at each point the induced metric is simply a positive rescaling of the 2D Euclidean metric





$$g(X,Y)_p = \phi_p \langle X, Y \rangle$$

Example (Enneper Surface)

Consider the surface

$$f(u,v) := \begin{bmatrix} uv^2 + u - \frac{1}{3}u^3 \\ \frac{1}{3}v\left(v^2 - 3u^2 - 3\right) \\ (u-v)(u+v) \end{bmatrix}$$

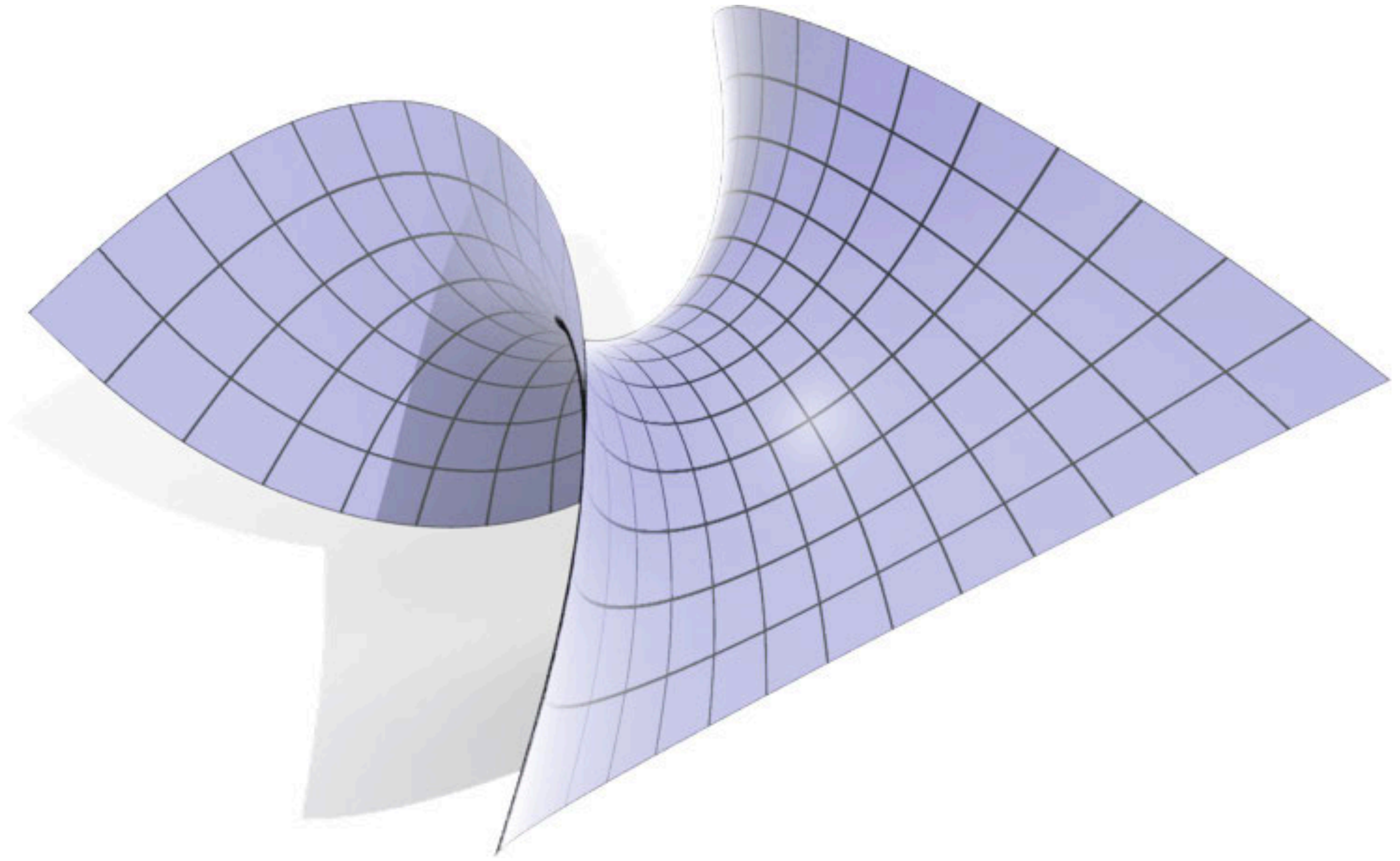
Its Jacobian matrix is

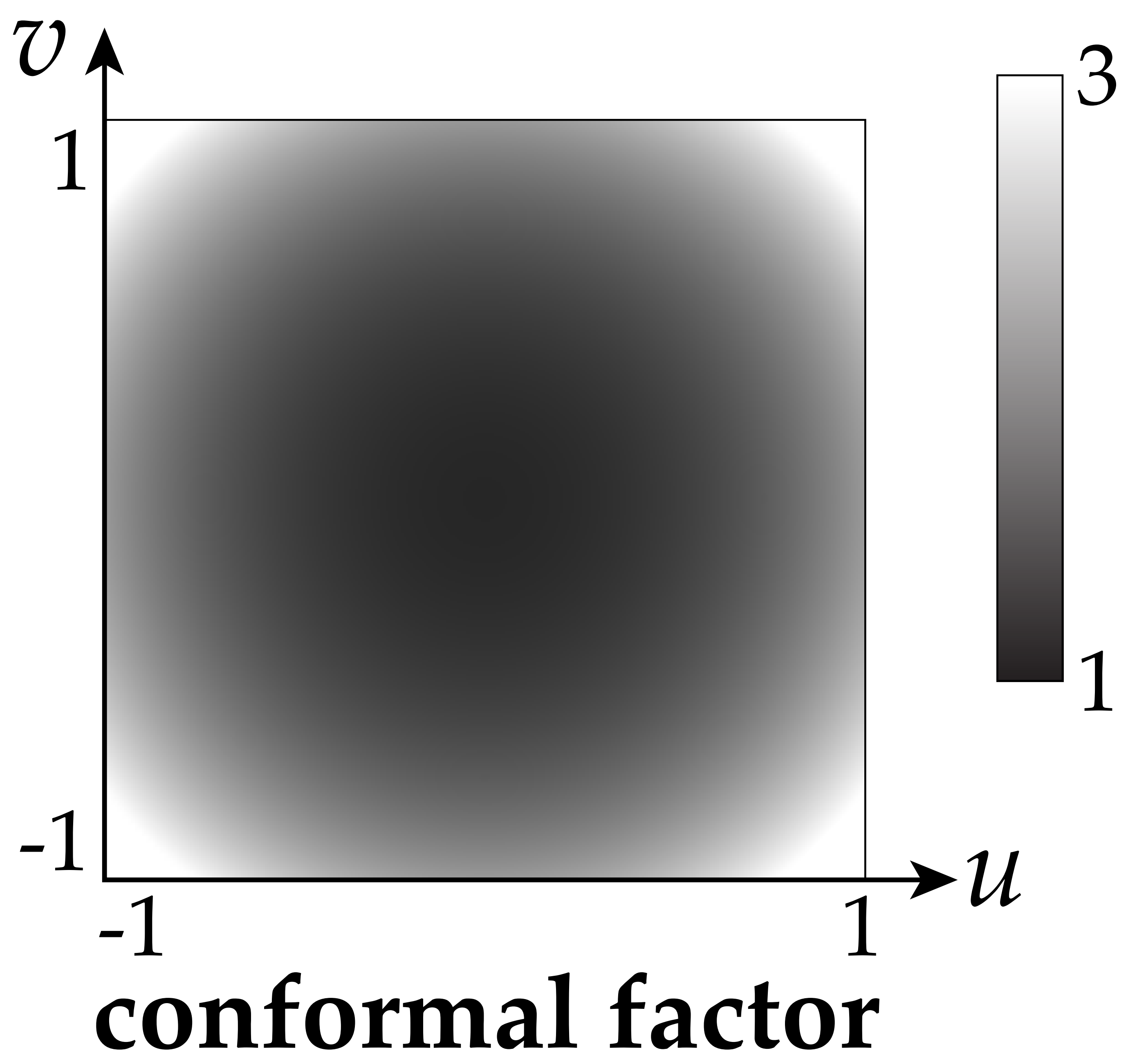
$$J_f = \begin{bmatrix} -u^2+v^2+1 & 2uv \\ -2uv & -u^2+v^2-1 \\ 2u & -2v \end{bmatrix}$$

Its metric then works out to be just a scalar function times the usual metric of the Euclidean plane:

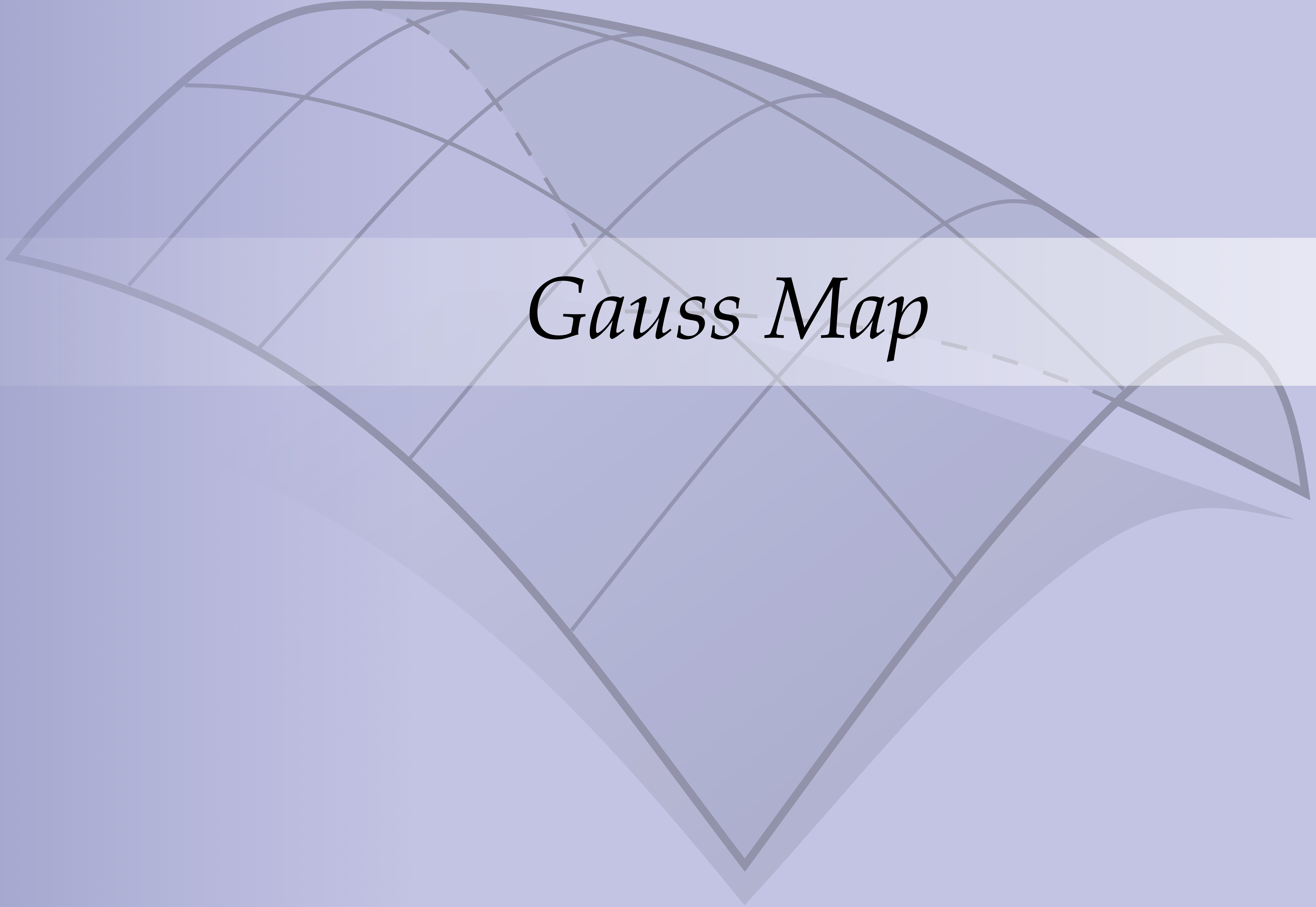
$$\mathbf{I} = J_f^T J_f = \left(u^2+v^2+1\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This function is called the *conformal scale factor*.





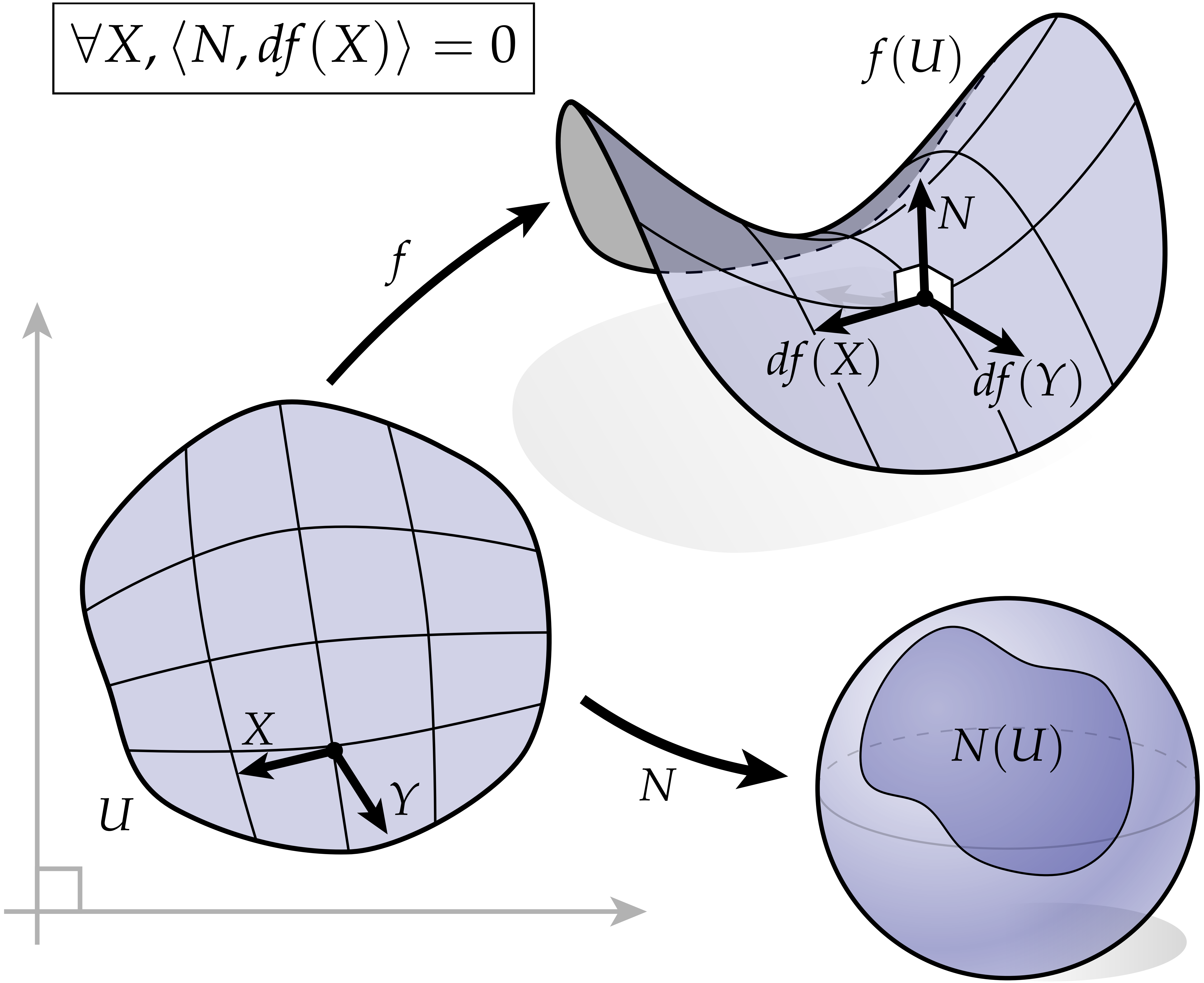
Gauss Map



Gauss Map

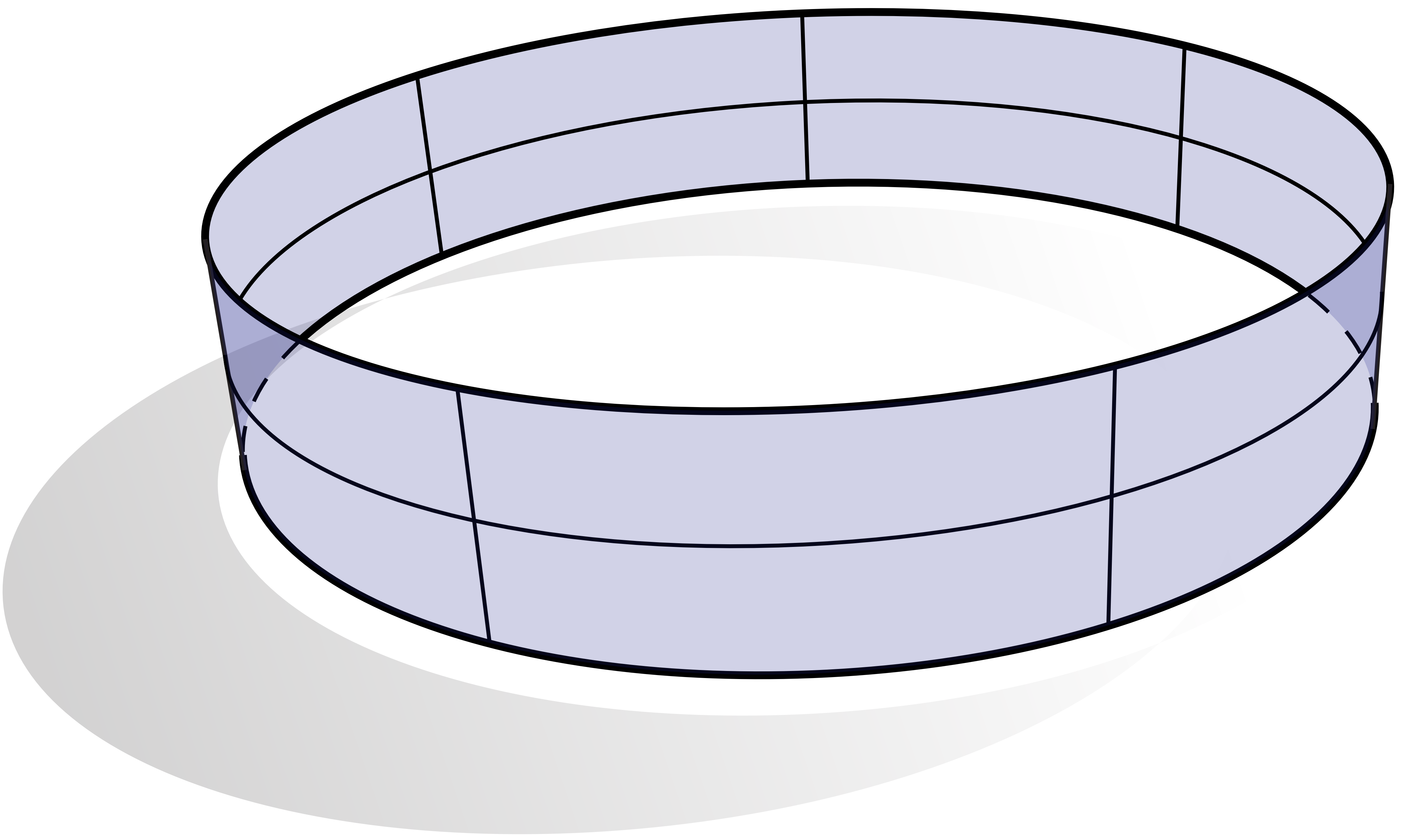
- A vector is **normal** to a surface if it is orthogonal to all tangent vectors
- **Q:** Is there a *unique* normal at a given point?
- **A:** No! Can have different magnitudes/directions.
- The **Gauss map** is a *continuous* map taking each point on the surface to a *unit* normal vector
- Can visualize Gauss map as a map from the surface to the unit sphere

$$\forall X, \langle N, df(X) \rangle = 0$$

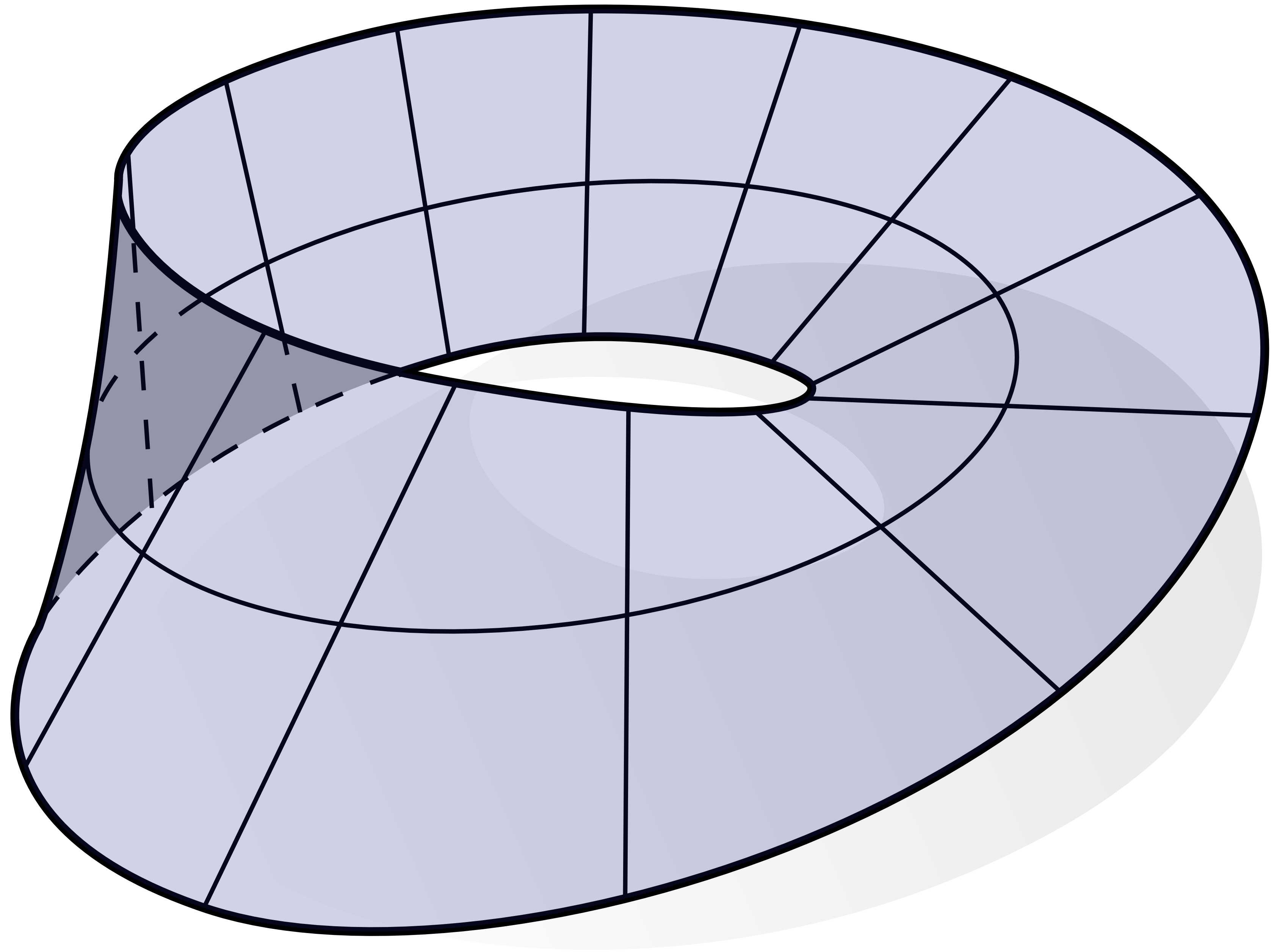


Orientability

Not every surface admits a Gauss map (globally):



orientable



nonorientable

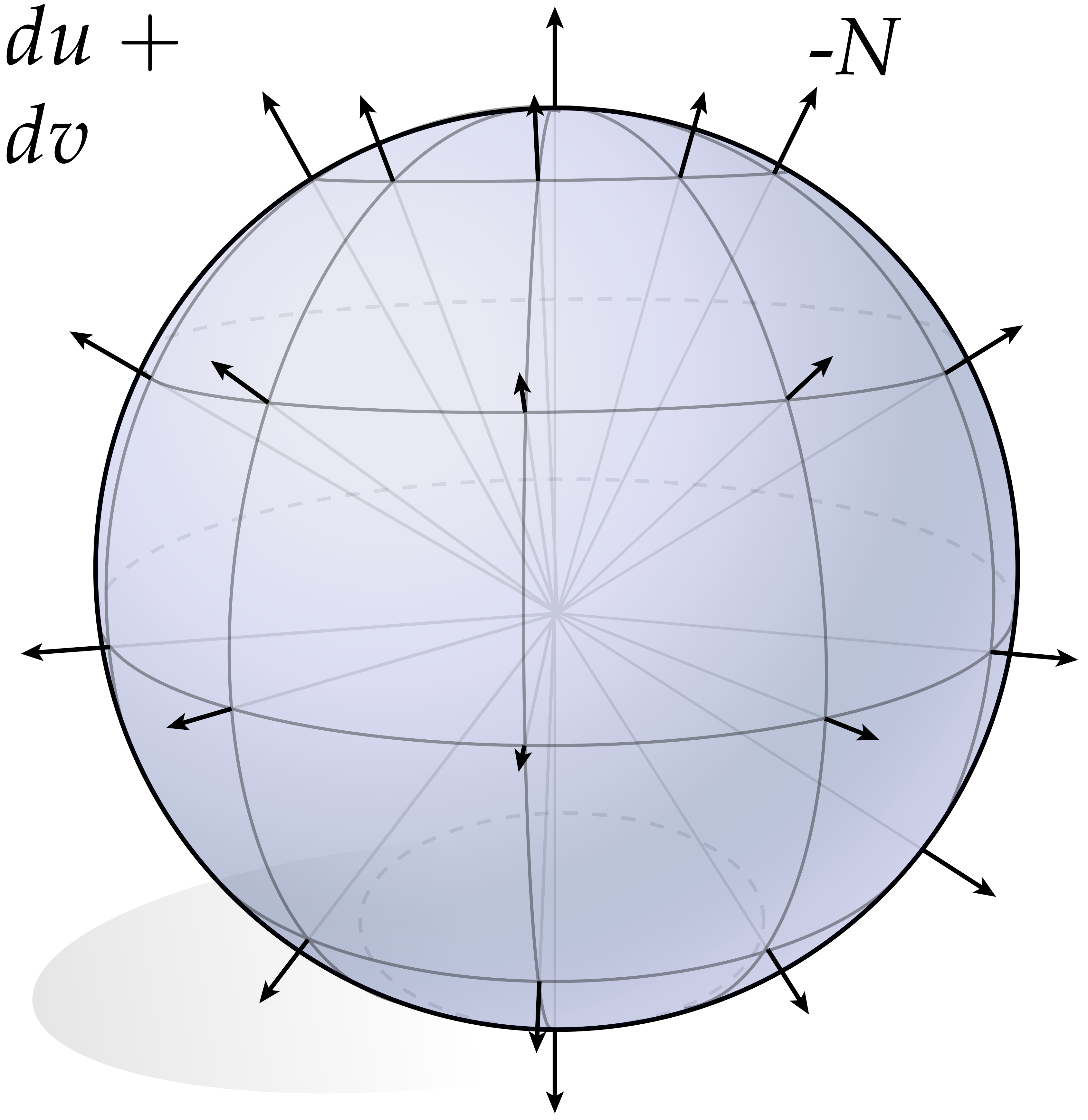
Gauss Map—Example

Can obtain unit normal by taking the cross product of two tangents*:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{matrix} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &) & du + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &) & dv \end{matrix}$$

$$df(\tfrac{\partial}{\partial u}) \times df(\tfrac{\partial}{\partial v}) = \begin{bmatrix} -\cos(u)\sin^2(v) \\ -\sin(u)\sin^2(v) \\ -\cos(v)\sin(v) \end{bmatrix}$$



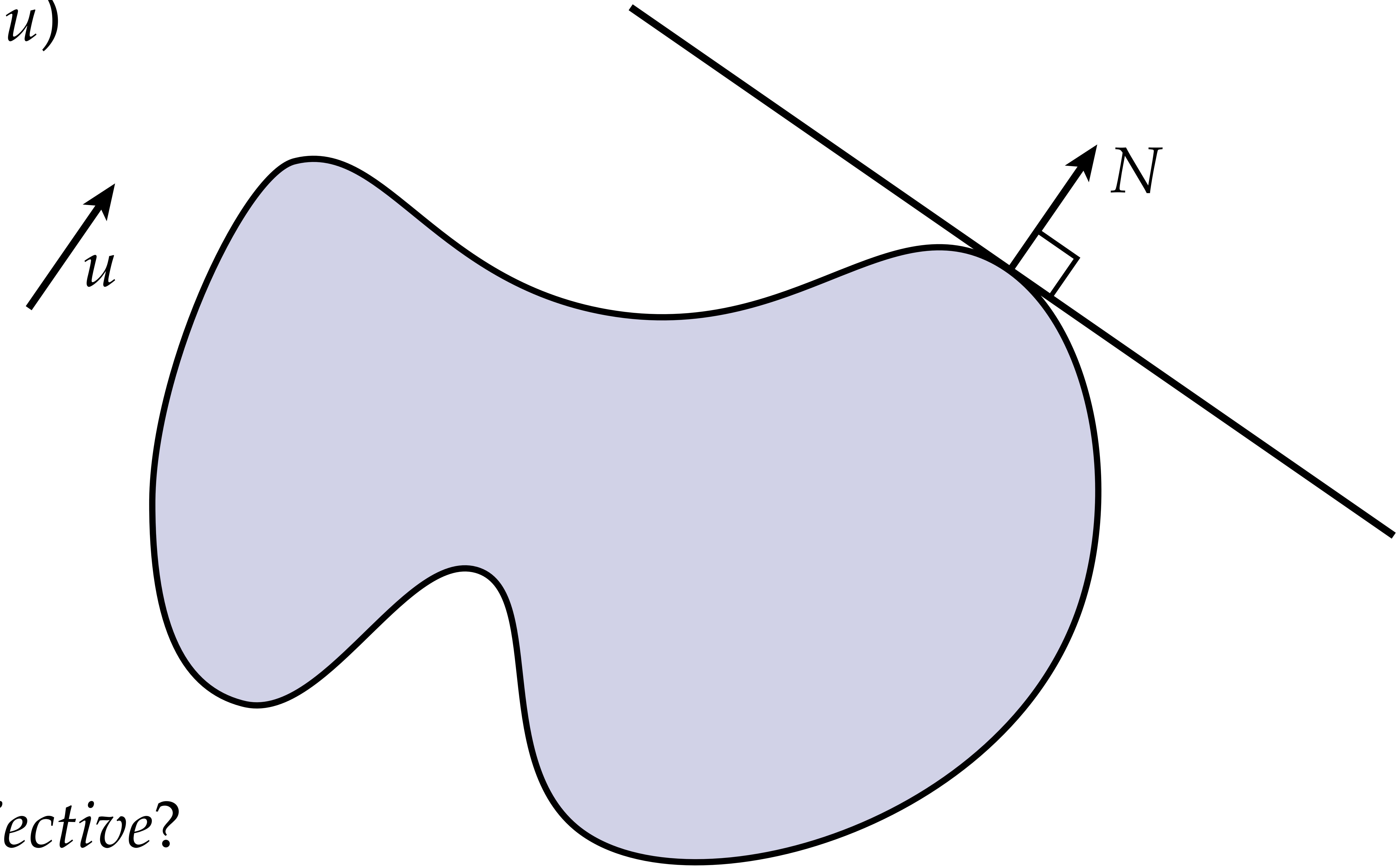
To get *unit* normal, divide by length. In this case, can just notice we have a constant multiple of the sphere itself:

$$\Rightarrow N = -f$$

*Must not be parallel!

Surjectivity of Gauss Map

- Given a unit vector u, can we always find some point on a surface that has this normal? ($N = u$)
- Yes! **Proof** (Hilbert):



Q: Is the Gauss map *injective*?

Vector Area

- Given a little patch of surface Ω, what's the "average normal"?
- Can simply integrate normal over the patch, divide by area:

$$\frac{1}{\text{area}(\Omega)} \int_\Omega N\, dA$$

- Integrand $N\,dA$ is called the **vector area**. (Vector-valued 2-form)
- Can be easily expressed via exterior calculus*:

$$\begin{aligned} df \wedge df(X,Y) = df(X) \times df(Y) - df(Y) \times df(X) = \\ 2df(X) \times df(Y) = \\ 2NdA(X,Y) \end{aligned}$$

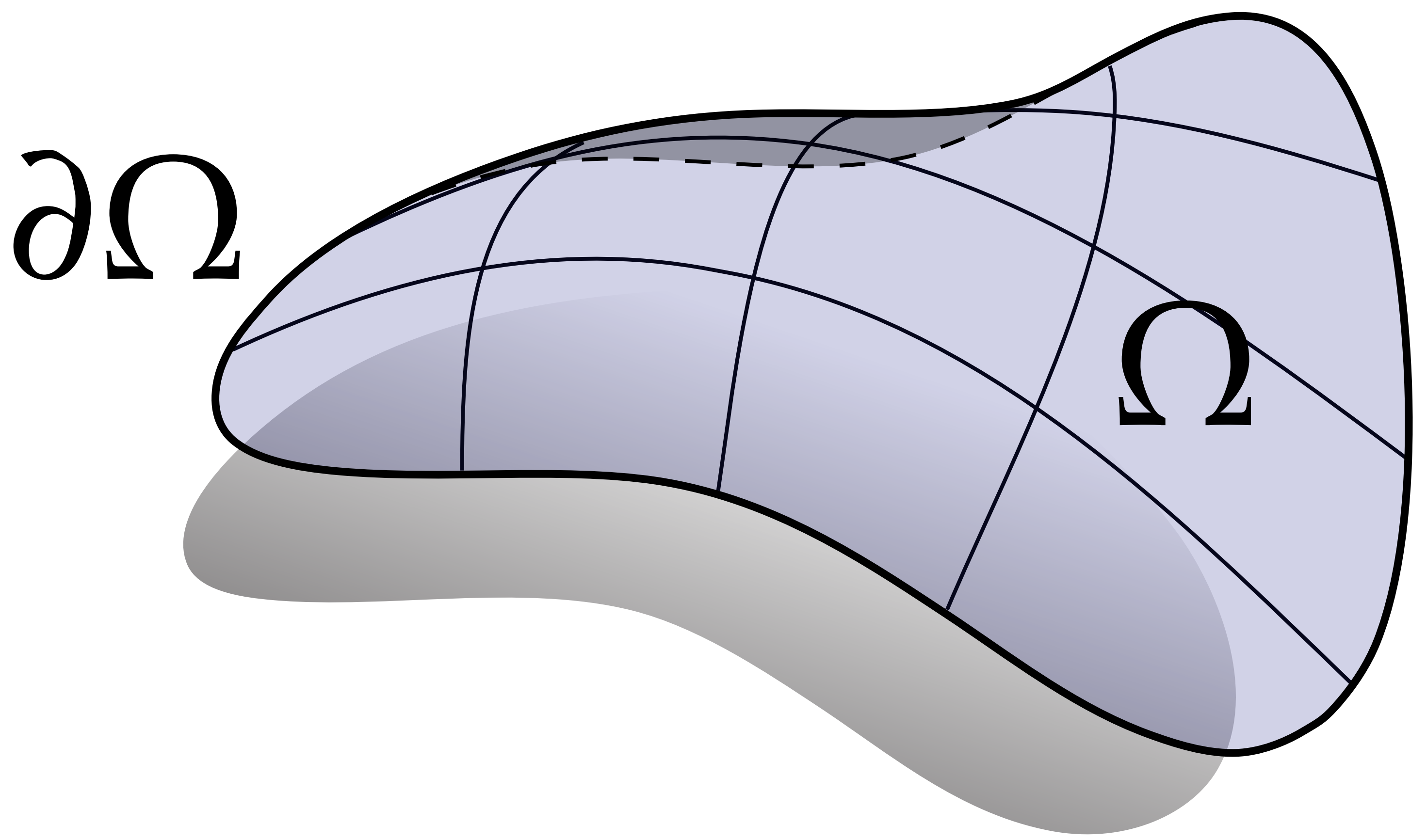
$$\implies \boxed{\mathcal{A} = \tfrac{1}{2} df \wedge df}$$

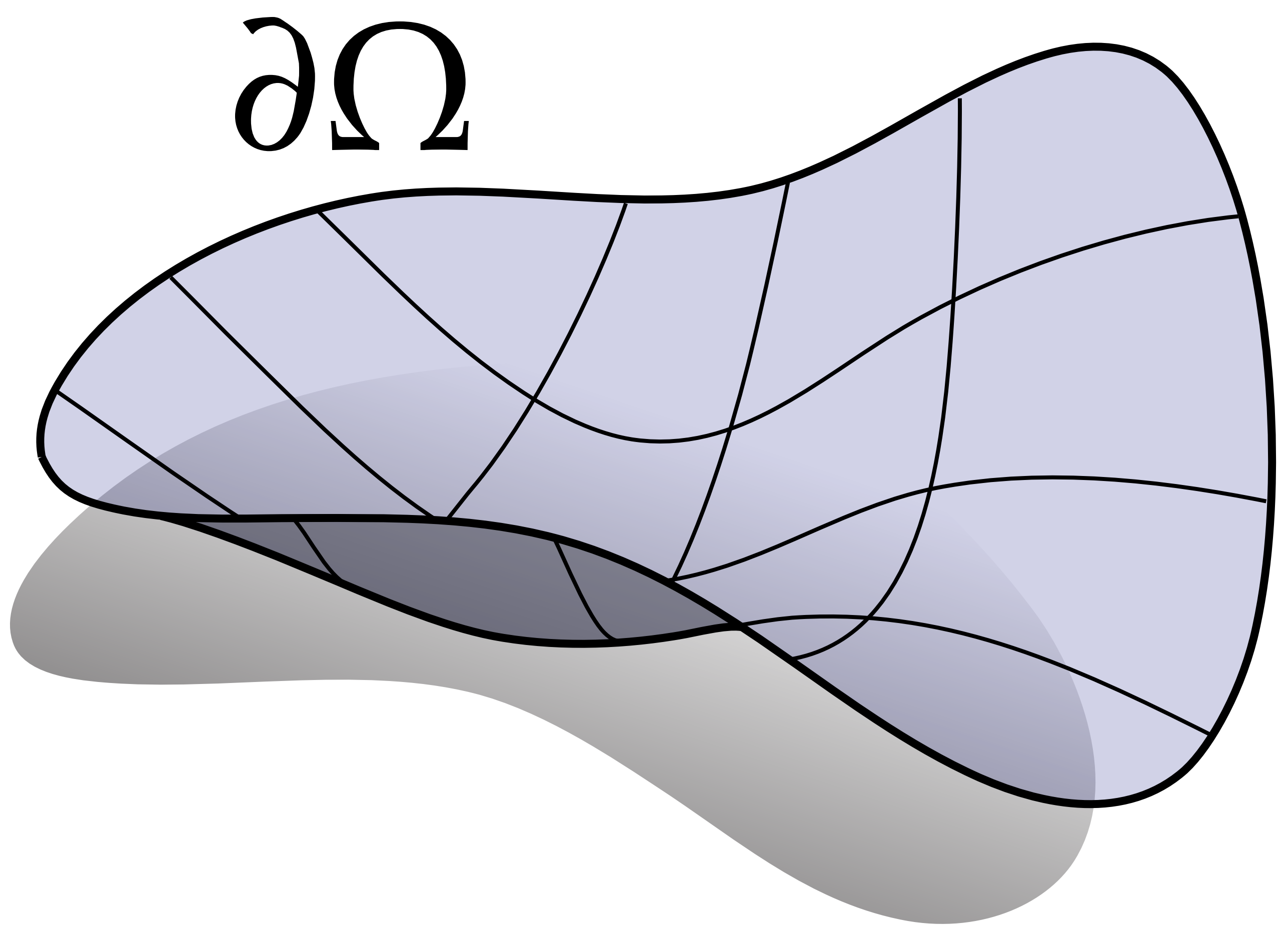
Vector Area, continued

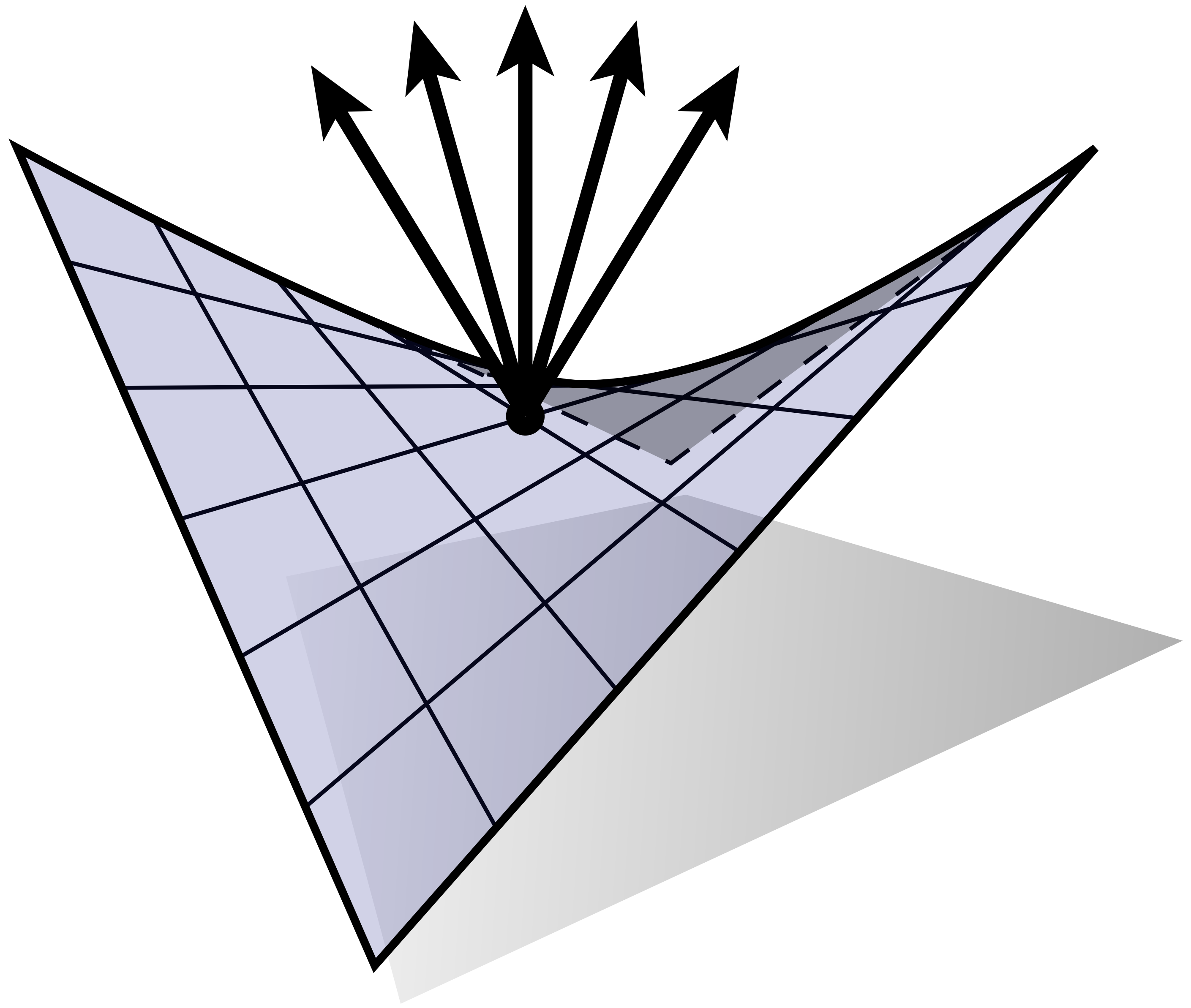
- By expressing vector area this way, we make an interesting observation:

$$2\int_\Omega N\,dA = \int_\Omega df\wedge df = \int_\Omega d(f\,df) = \int_{\partial\Omega} f\,df = \int_{\partial\Omega} f(s)\times df(T(s))\,ds$$

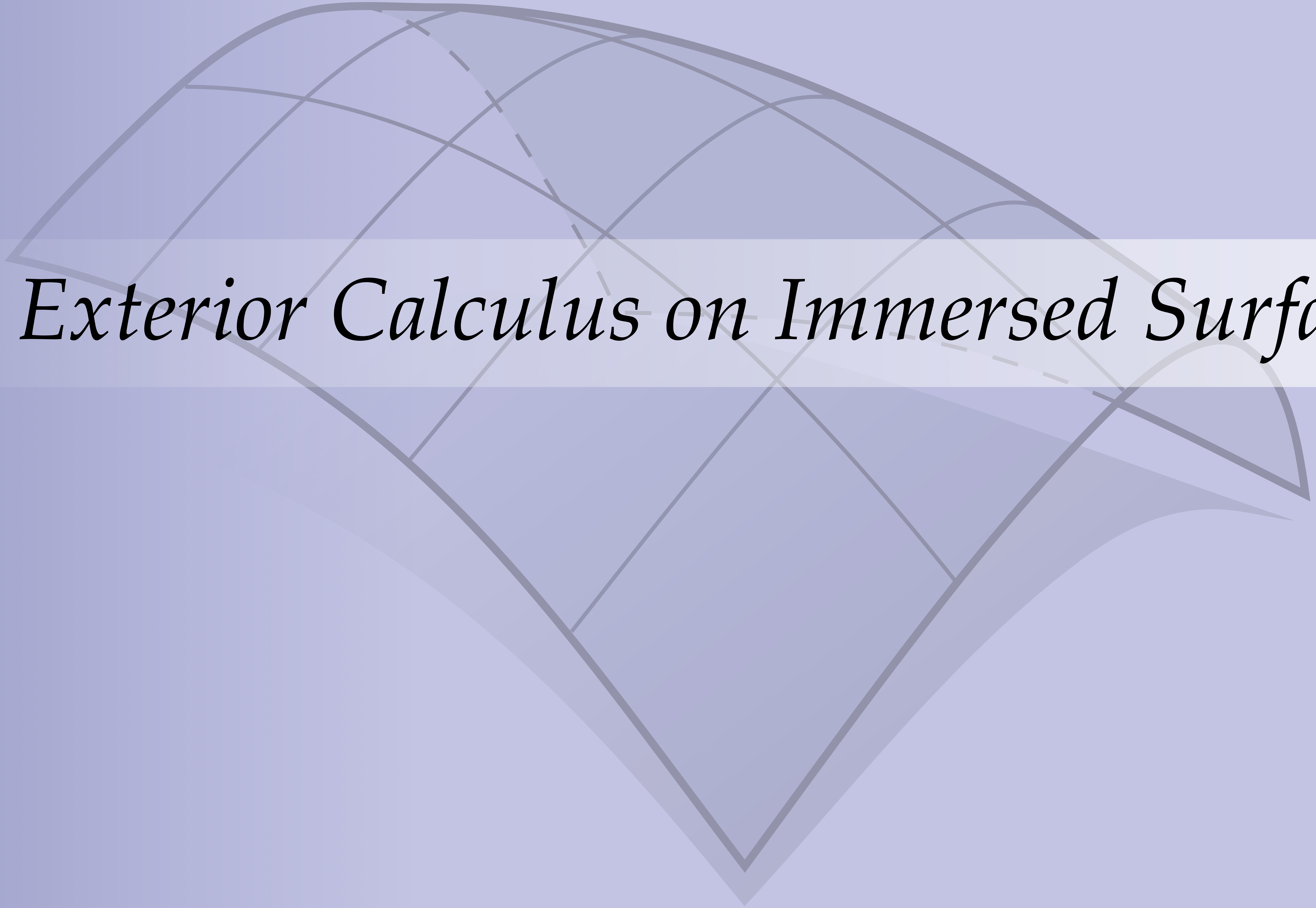
- Hence, vector area is the same for any two patches w/ same boundary
- Can define "normal" given **only** boundary (*e.g.*, nonplanar polygon)
- **Corollary:** *integral of normal vanishes for any closed surface*





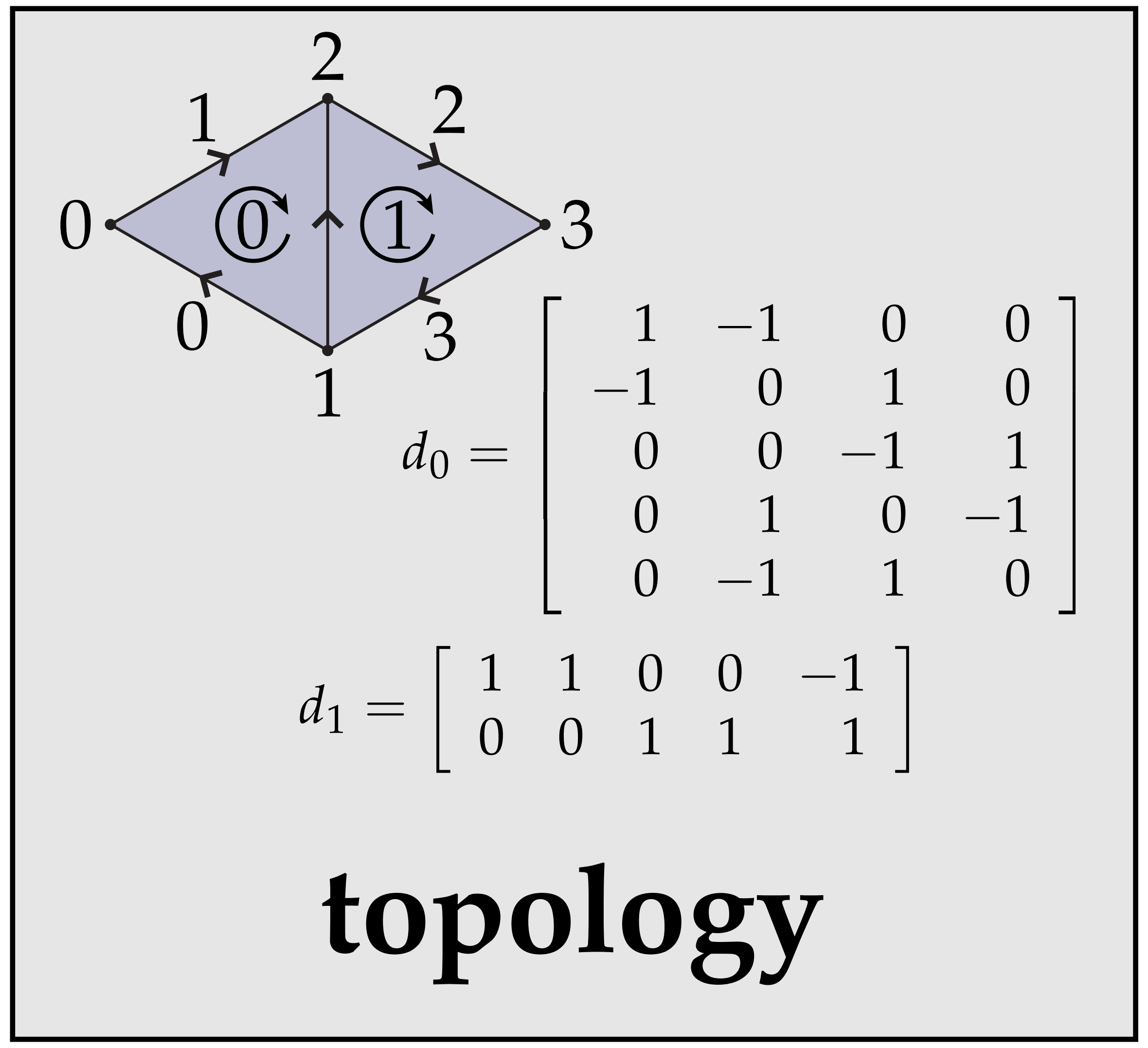


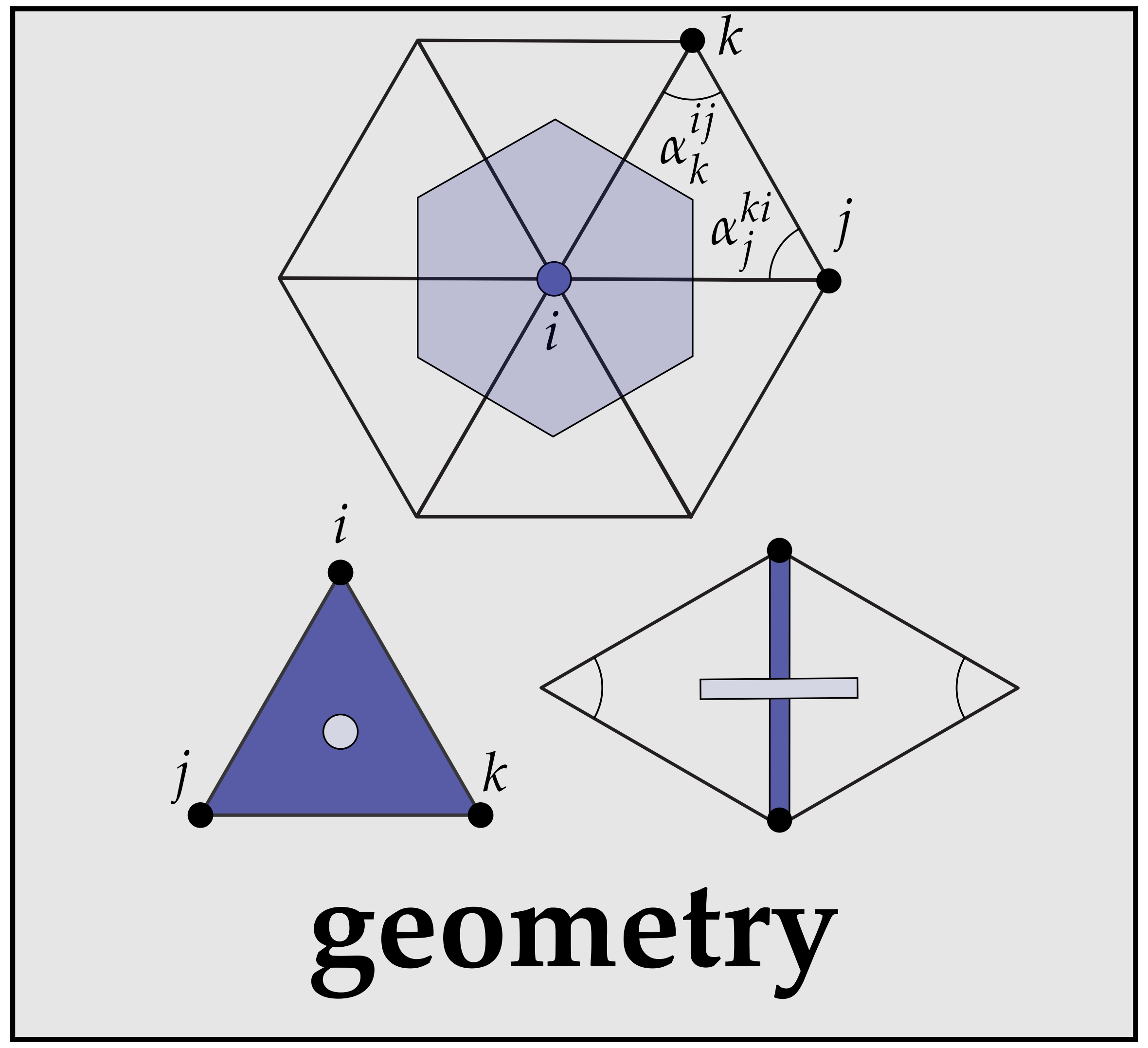
Exterior Calculus on Immersed Surfaces



Exterior Calculus on Curved Domains

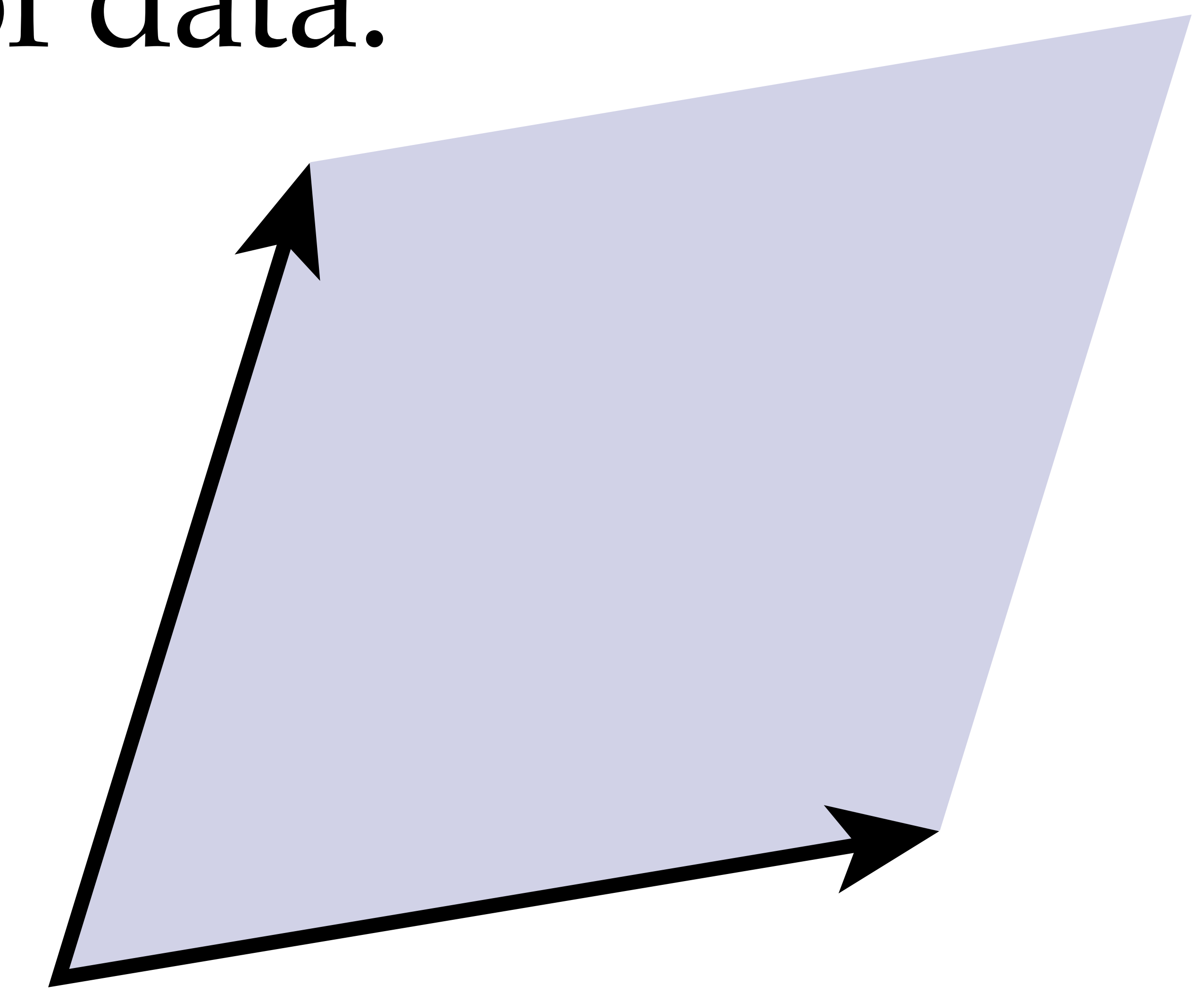
- Initial study of differential forms was in **flat** Euclidean R^n
- How do we do exterior calculus on **curved** spaces?
- Recall that operators nicely “split up” topology & geometry:
  - **(topology)** wedge product ($\wedge$), exterior derivative (d)
  - **(geometry)** Hodge star ($\star$)
- For instance, discrete d uses only mesh connectivity **(topology)**; discrete $\star$ involves only ratios of volumes **(geometry)**
- Therefore, to get exterior calculus to work with curved spaces, we just need to figure out what the Hodge star looks like!
- Traditionally taught from abstract **intrinsic** point of view; we’ll start with the concrete **extrinsic** picture (which fewer people know… but is more directly relevant for real applications!)

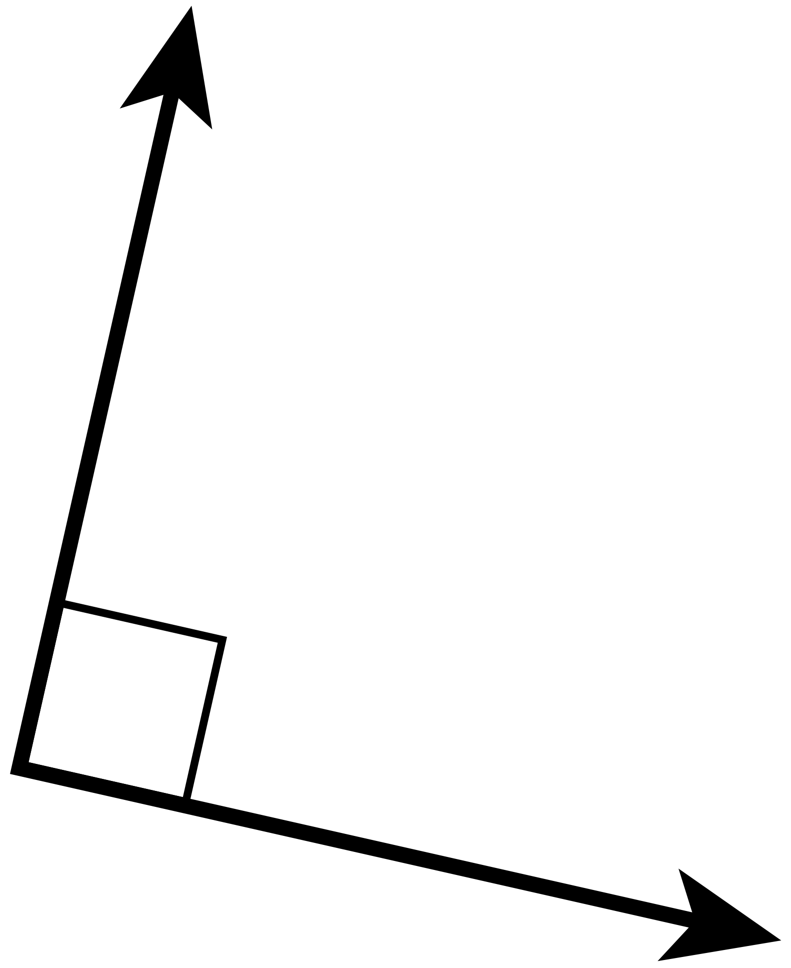




Exterior Calculus on Immersed Surfaces

- For surface immersed in 3D, just need two pieces of data:
  - **Area form**—*"how big is a given region?"*
    - lets us define Hodge star on 0/2-forms
    - can express via cross product in R^3
  - **Complex structure**—*"how do we rotate by 90°?"*
    - lets us define Hodge star on 1-forms
    - can express via cross product w/ surface normal
- All of this data also determined by induced metric

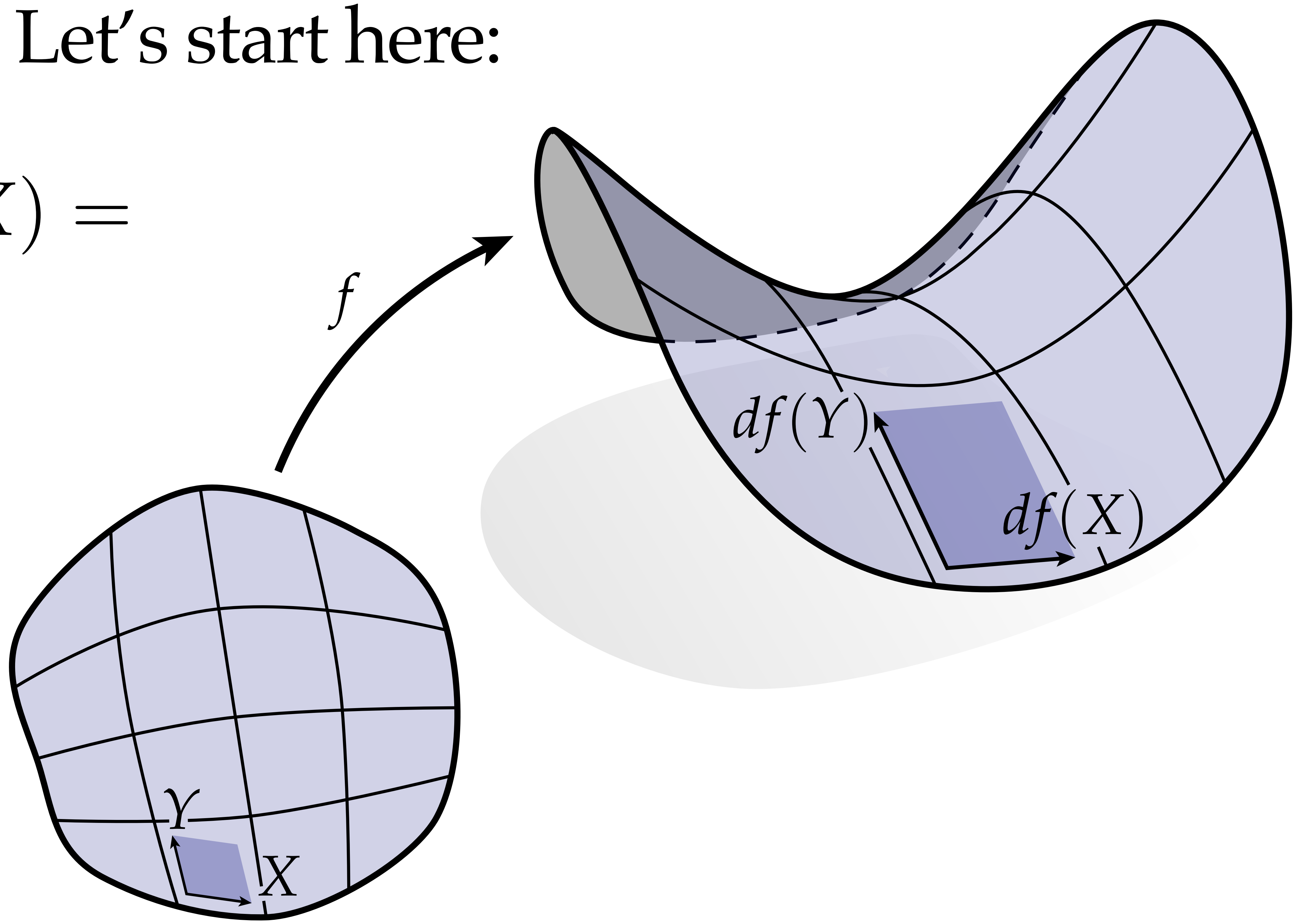




Induced Area 2-Form

- What signed area should we associate with a pair of vectors X, Y on the domain?
- Not just their cross product! Need to account for "stretching" caused by immersion f
- What's the signed area of the stretched vector? Let's start here:

$$df \wedge df(X,Y) = df(X) \times df(Y) - df(Y) \times df(X) = 2df(X) \times df(Y)$$



Since $df(X)$ and $df(Y)$ are *tangent*, we get

$$df \wedge df(X,Y) = 2NdA(X,Y)$$

where dA is the area 2-form on $f(M)$. Hence,

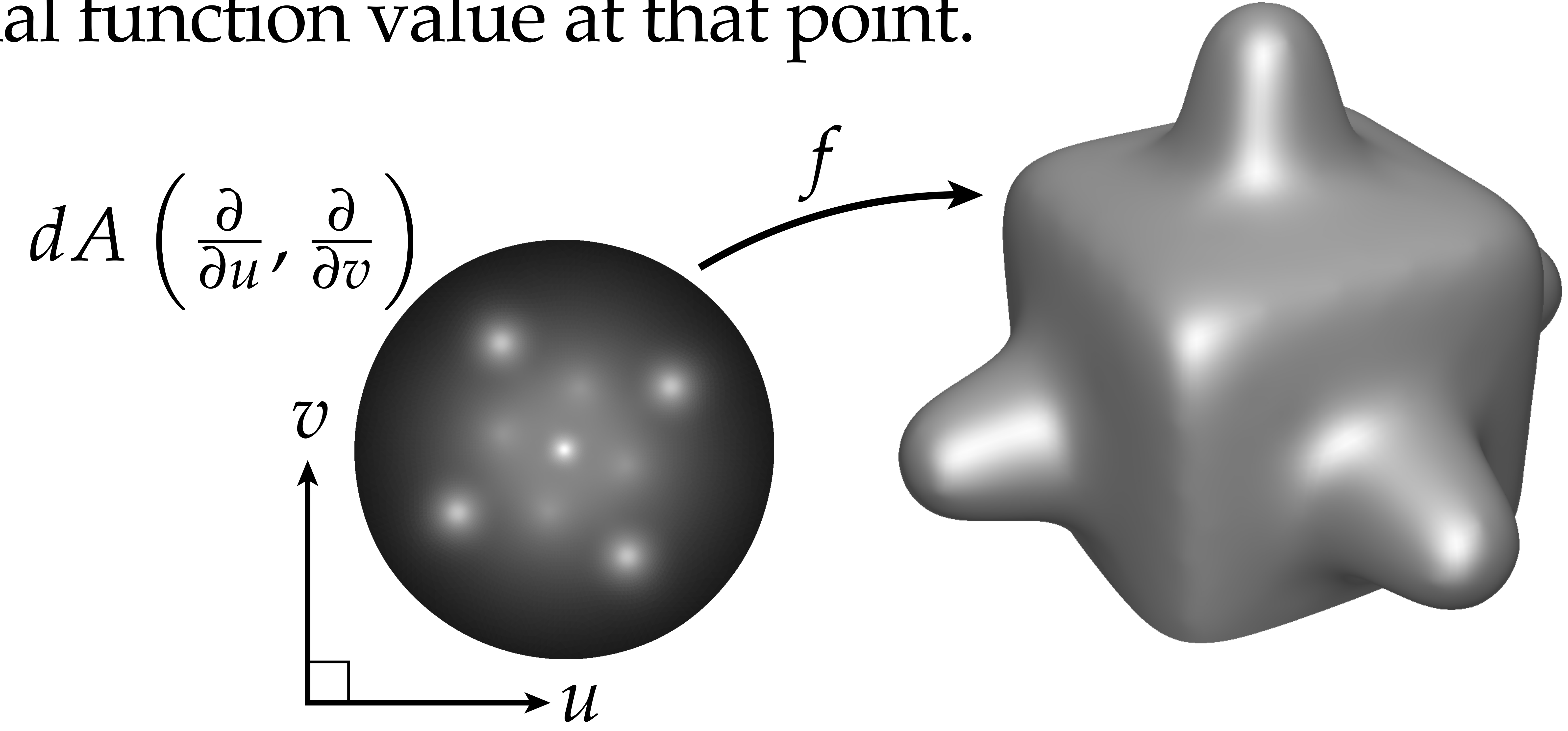
$$\boxed{dA = \frac{1}{2}\langle N, df \wedge df \rangle}$$

Induced Hodge Star on 0-Forms

- Given the area 2-form dA, can easily define Hodge star on 0-forms:

$$\phi \stackrel{\star}{\longmapsto} \phi\, dA$$

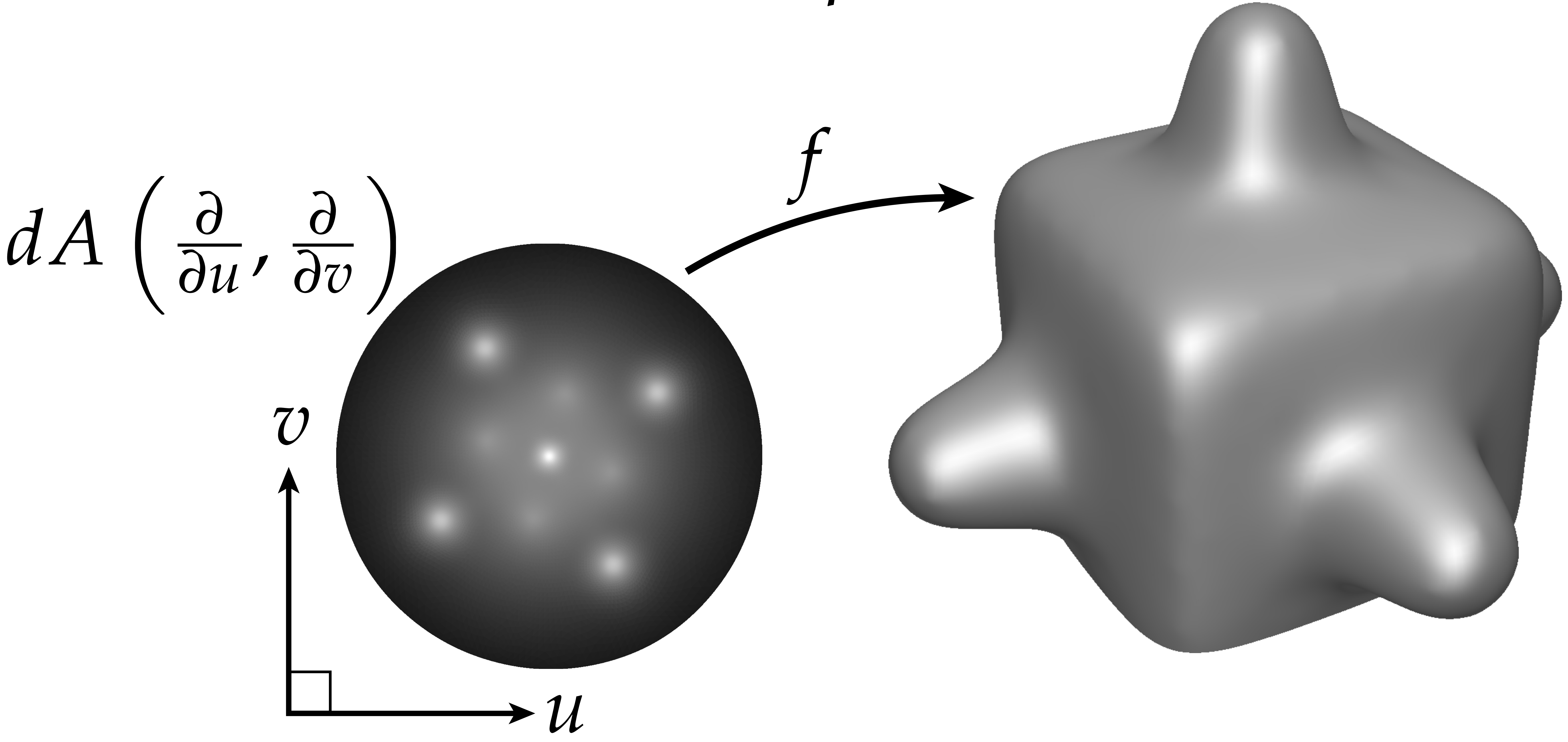
- **Meaning?** Applying this new 2-form to a unit area *on the surface* yields the original function value at that point.



Induced Hodge Star on 2-Forms

- To get the 2-form Hodge star, we just go the other way
- Suppose ω is a 2-form on $f(M)$. Then its Hodge dual is the unique 0-form ϕ such that

$$\omega = \phi\, dA$$



Complex Structure

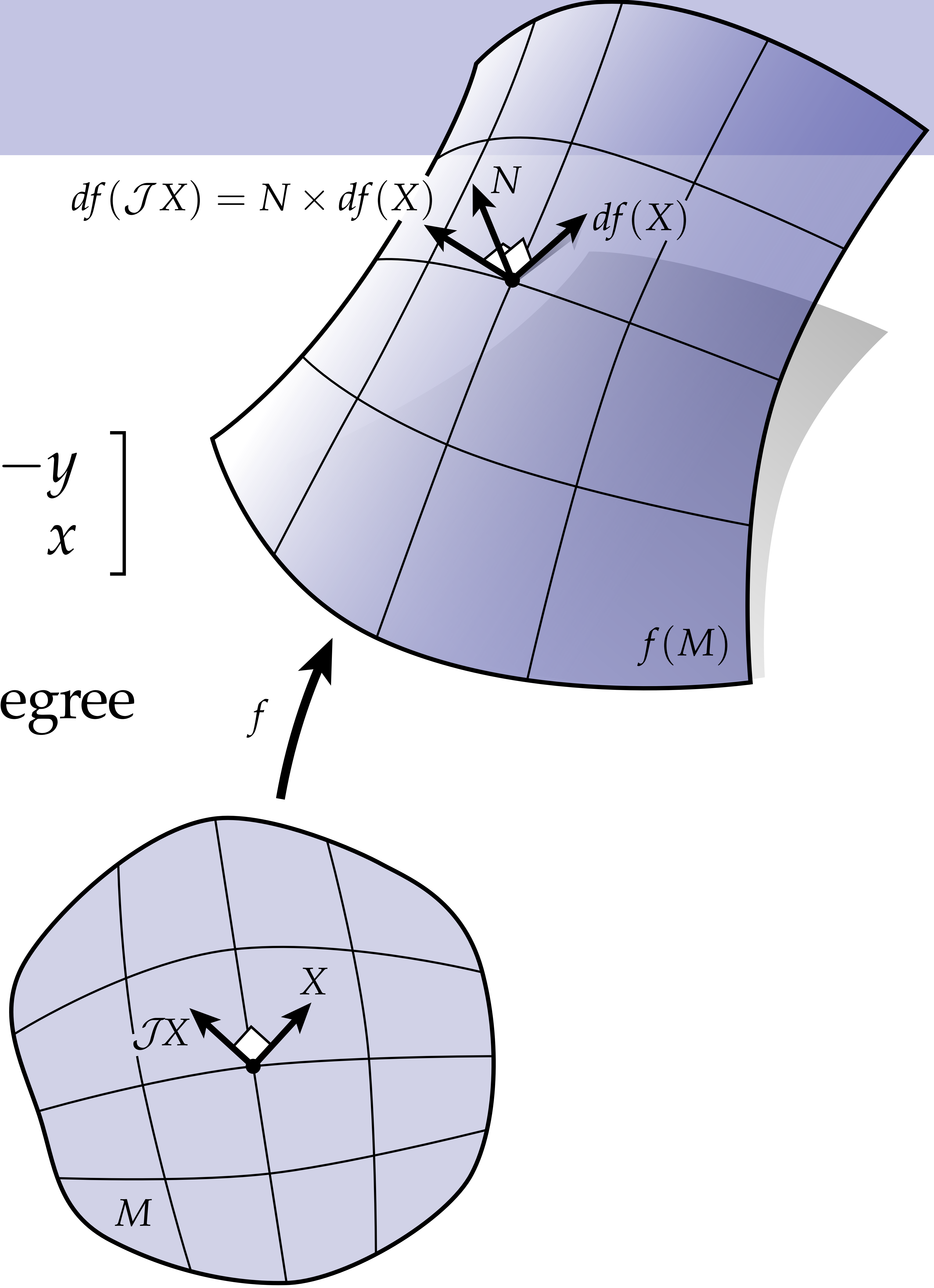
- The *complex structure** tells us how to rotate by 90°
- In R^2, we just replace (x,y) with $(-y,x)$:

$$\mathcal{J}_{\mathbb{R}^2} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \mathcal{J}_{\mathbb{R}^2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

- For a surface immersed in R^3, we can express a 90-degree rotation via a cross product with the unit normal N:

$$df(\mathcal{J}_f X) := N \times df(X)$$

- This relationship uniquely determines $\mathcal{J}_f$
- An immersion is conformal if and only if $\mathcal{J}_f = \mathcal{J}_{\mathbb{R}^2}$



*Sometimes called *linear complex structure*; same thing for surfaces.

Complex Structure in Coordinates

- Suppose we want to explicitly compute the linear complex structure*
- Similar strategy to shape operator: solve a matrix equation for $\mathcal{J}$

$$\widehat{N} := \begin{bmatrix} 0 & -N_z & N_y \\ N_z & 0 & -N_x \\ -N_y & N_x & 0 \end{bmatrix}$$

cross product w/ normal

$(N \times u = \widehat{N}u)$

$$A := \begin{bmatrix} \partial f_x/\partial u & \partial f_x/\partial v \\ \partial f_y/\partial u & \partial f_y/\partial v \\ \partial f_z/\partial u & \partial f_z/\partial v \end{bmatrix}$$

Jacobian

$$J := \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

complex structure

$$df(\mathcal{J}X) = N \times df(X) \implies \boxed{J = (A^TA)^{-1}(A^T\hat{N}A)}$$

***Note:** not something you do much in practice, but may help make definition feel more concrete…

Induced Hodge Star on 1-Forms

- Recall that for a 1-form α in the plane, applying $\star\alpha$ to a vector X is the same as applying α to a 90-degree rotation of X:

$$\star_{\mathbb{R}^2}\alpha(X) = \alpha(\mathcal{J}_{\mathbb{R}^2} X)$$

- For 1-forms on an immersed surface f, we instead want to apply a 90-degree rotation with respect to the surface itself:

$$\star_f\alpha(X) = \alpha(\mathcal{J}_f X)$$

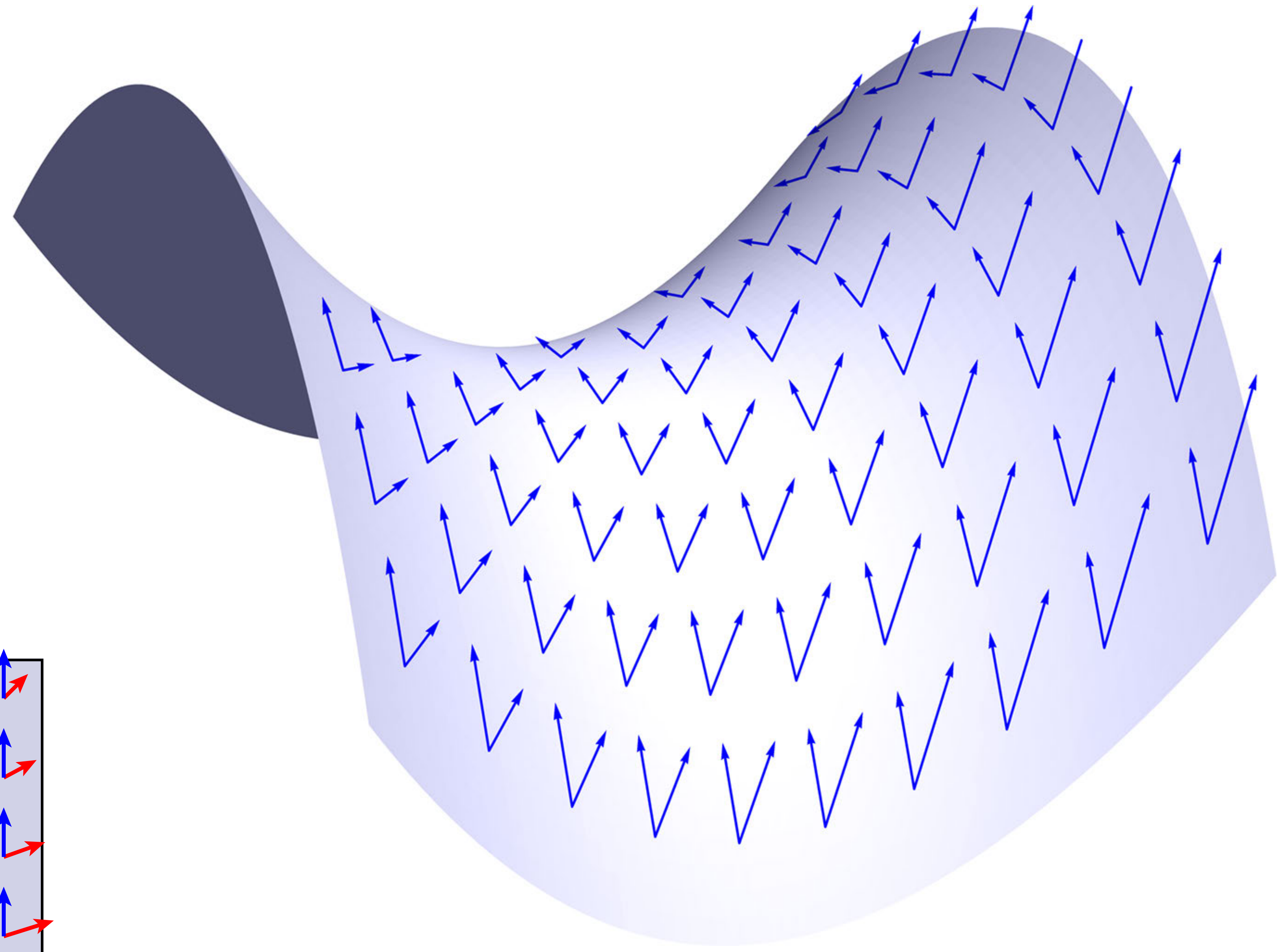
- At this point we have everything we need to do calculus on curved surfaces: 0-, 1-, and 2-form Hodge star. (Will see more general/abstract/intrinsic definitions for n-manifolds later on.)

Sharp and Flat on a Surface

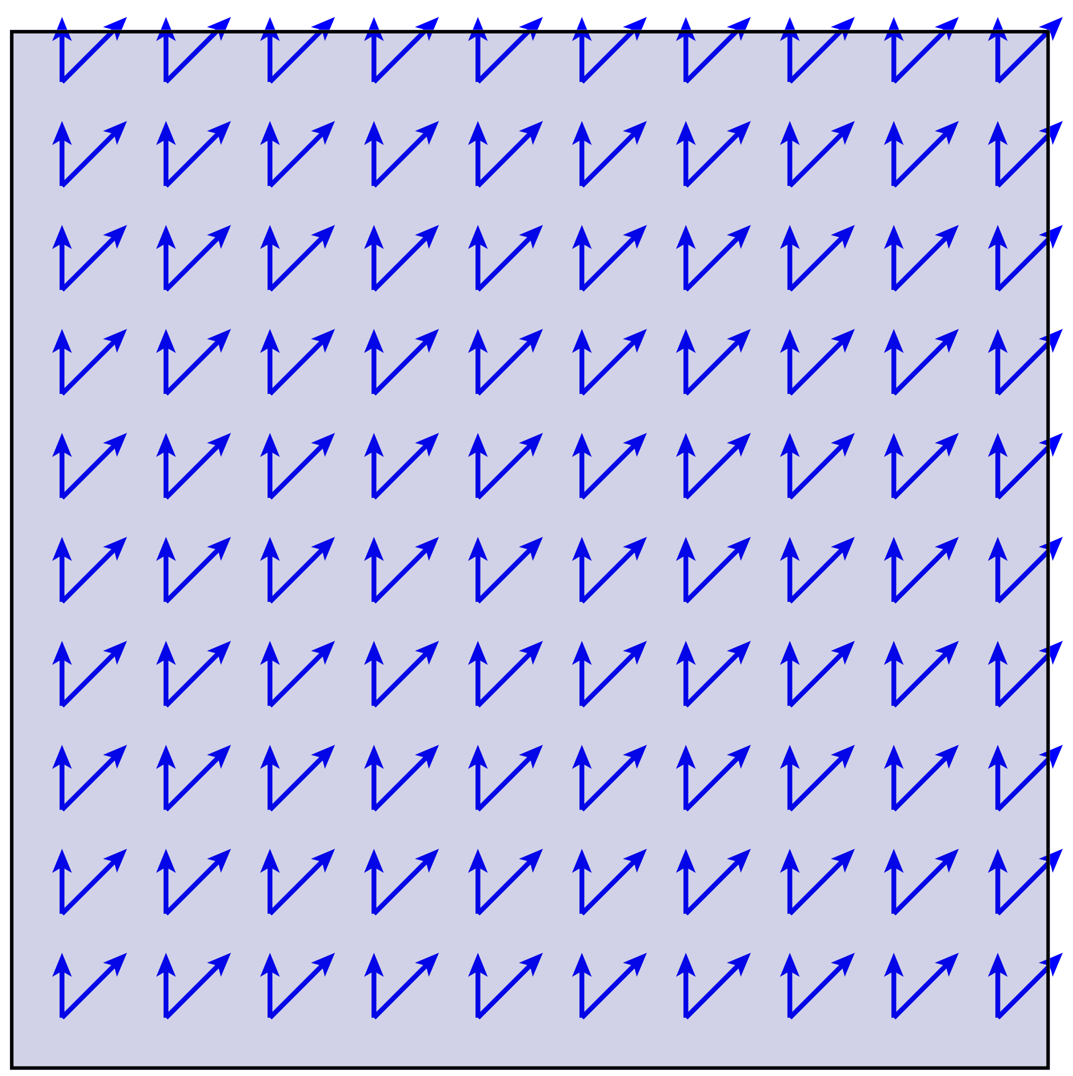
- Can use induced metric to translate between vector fields and 1-forms:

$$X^\flat(Y) := g(X,Y) \qquad g(\alpha^\sharp, Y) := \alpha(Y)$$

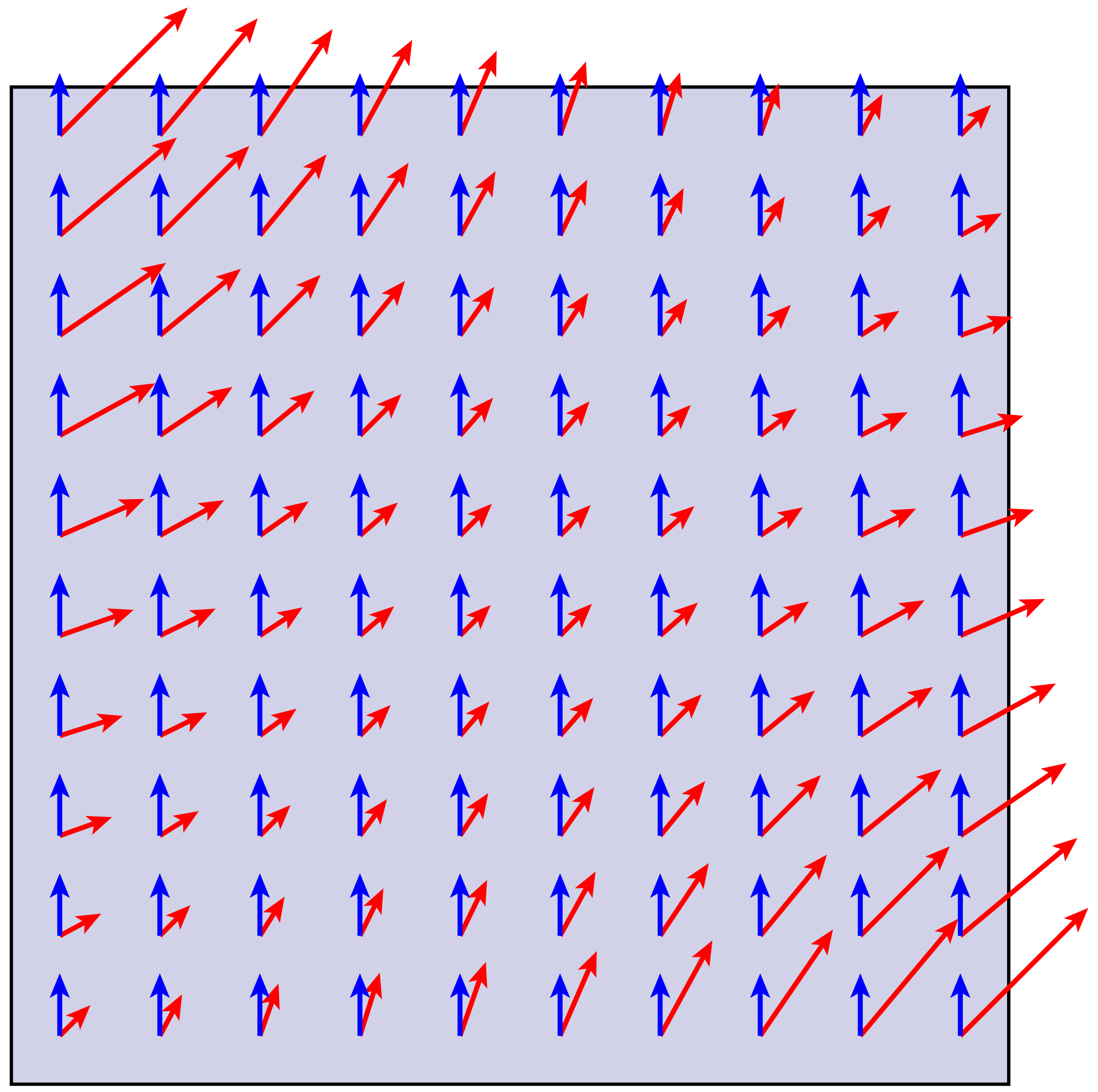
- No longer just a trivial "transpose" (as in Euclidean R^n)
- E.g., flat correctly encodes inner product on surface



$df(X) \cdot df(Y)$



$X \cdot Y \neq df(X) \cdot df(Y)$



$X^\flat(Y) = df(X) \cdot df(Y)$

Metric, Area Form, and Complex Structure

- Riemannian metric on a surface can be decomposed into area form, complex structure:

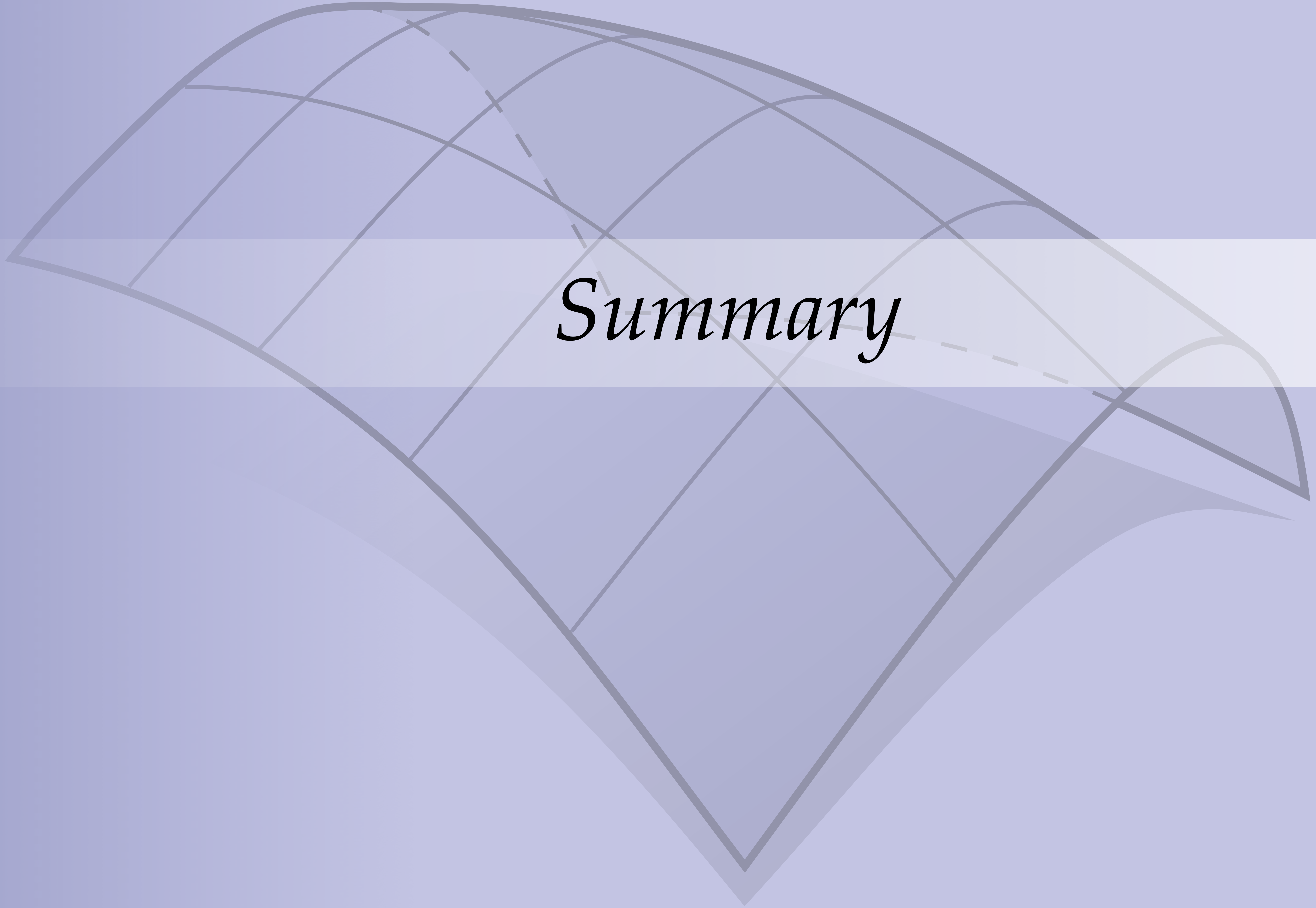
complex structure

$$g(X,Y) = dA(X,JY)$$

metric   area form

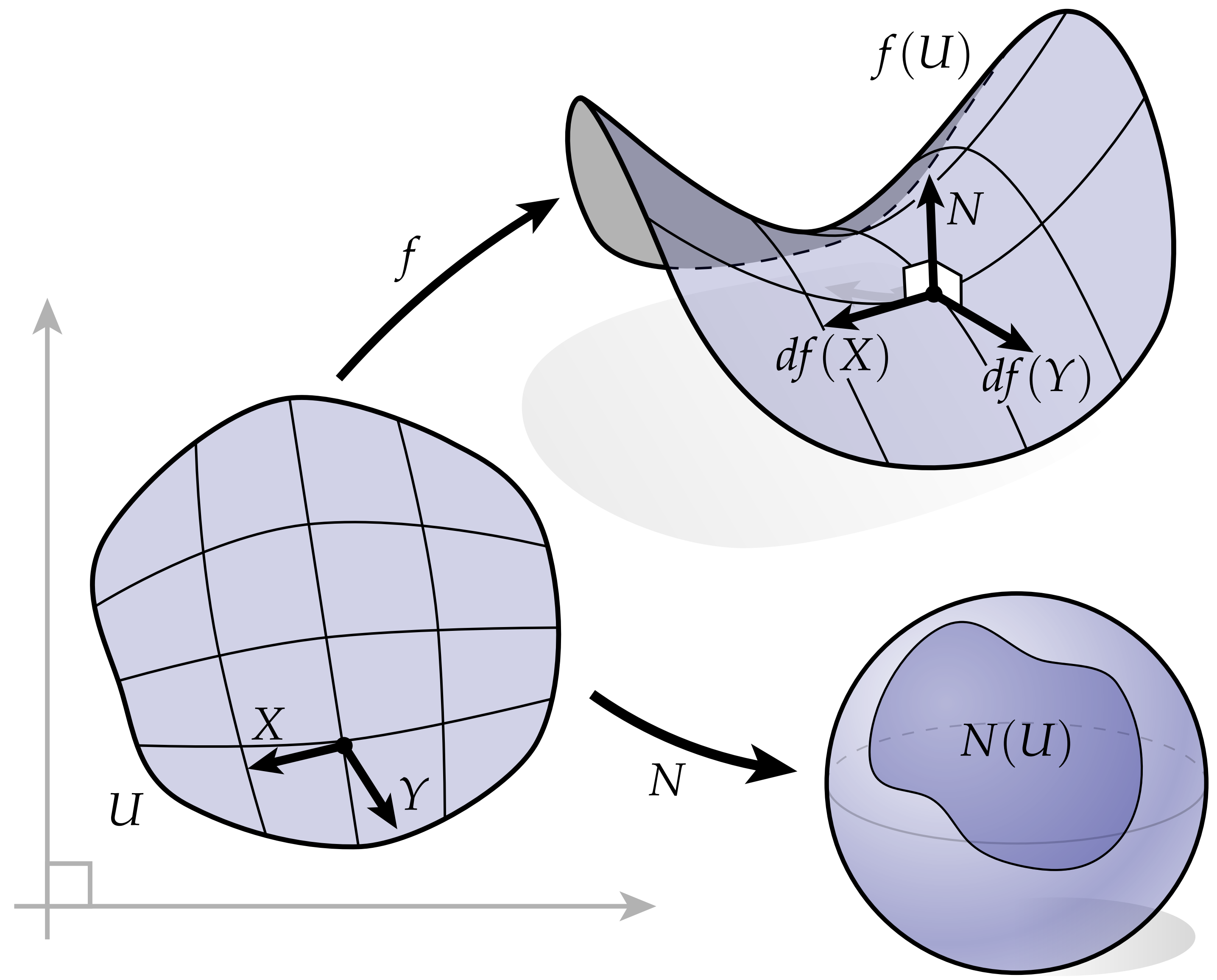
Q: In the plane, how is this relationship related to the cross product, dot product, and 90-degree rotation?

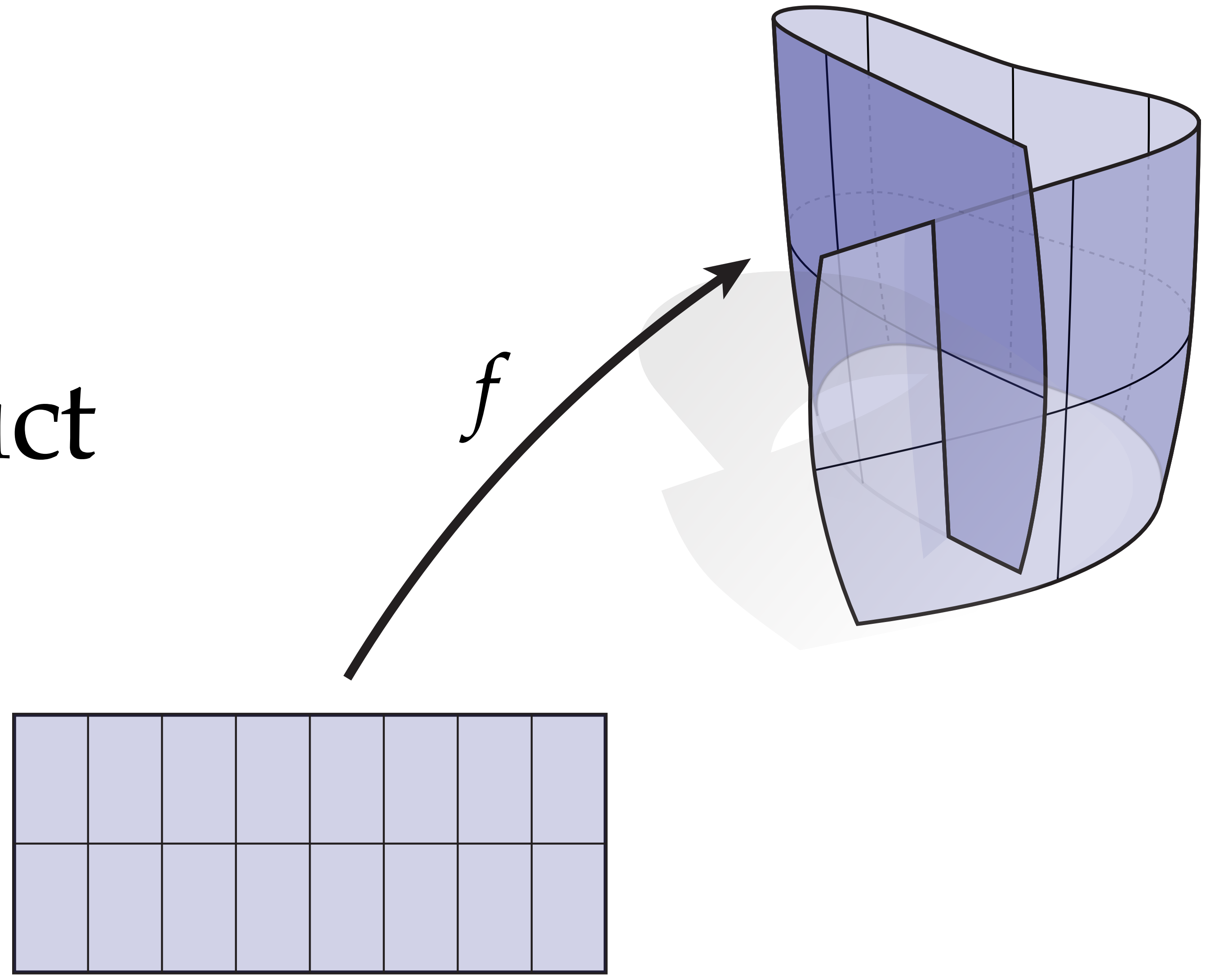
Summary



Smooth Surfaces — Summary

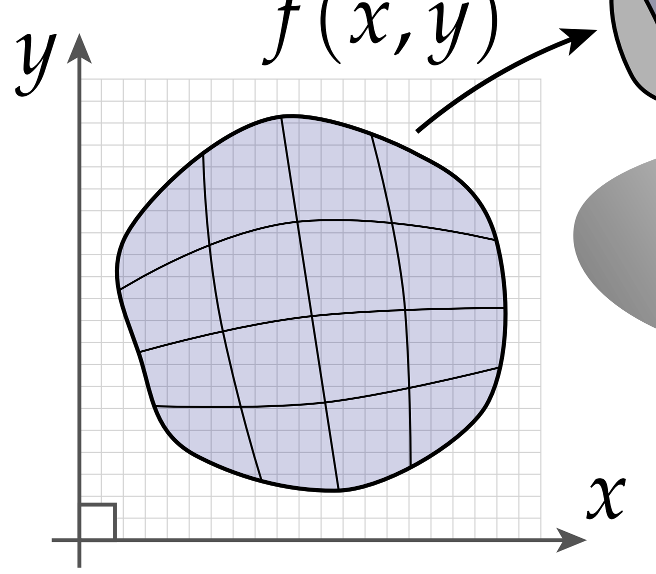
- Can describe shape a surface patch via a function $f : U \longrightarrow R^3$
  - embedded if no self-intersection, preserves global topology
  - **exterior calculus:** R^3-valued differential 0-form on U
- Differential $df : TU \longrightarrow TR^3$ "pushes forward" tangent vectors
  - $df(X)$ "stretches out" tangent vector X
  - surface is immersed if df is nondegenerate ($df(X) \neq 0$ for $X \neq 0$)
  - **exterior calculus:** R^3-valued differential 1-form
- Induced metric $g(X,Y) = \langle df(X), df(Y) \rangle$ gives "true" inner product
- Normal described by a function $N : U \longrightarrow R^3$ (Gauss map)
  - can also be viewed as a map to the sphere

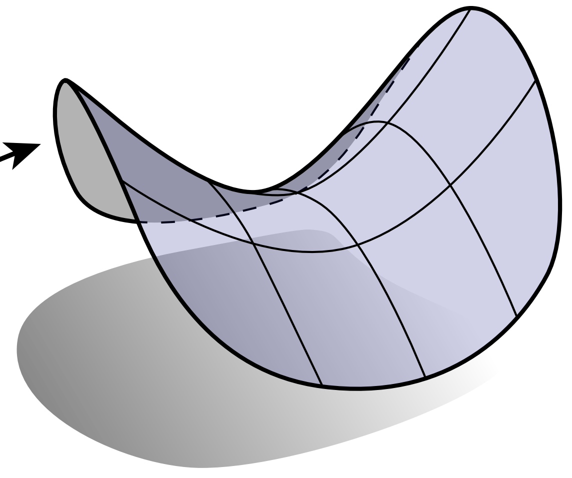


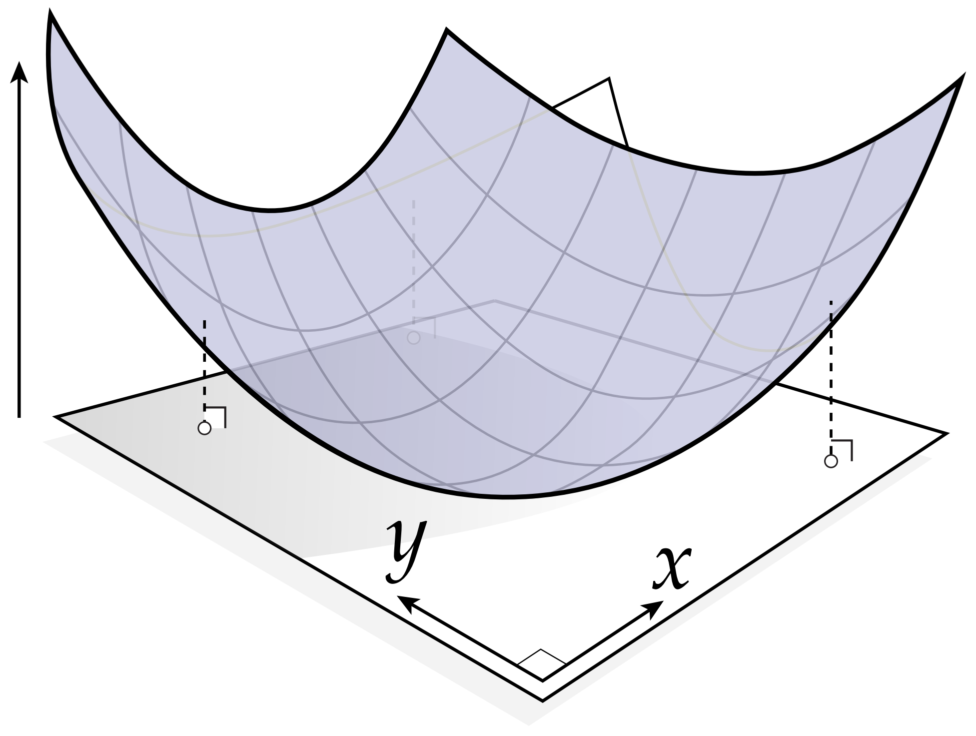


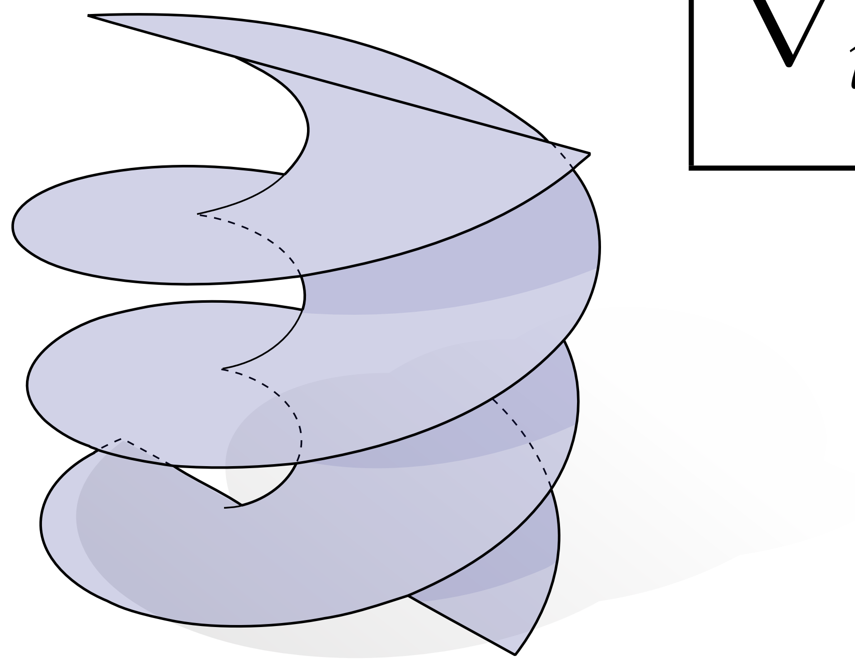
Only Scratched the Surface!

- Many ways to express the geometry of a surface:
  - height function over tangent plane
  - local parameterization
  - Christoffel symbols — coordinates/indices
  - **differential forms** — “coordinate free”
  - moving frames — change in *adapted frame*
  - Riemann surfaces (*local*); Quaternionic functions (*global*)
- Each dialect provides additional power—and can lead to totally different *algorithms!*
- Some references on web to further reading…





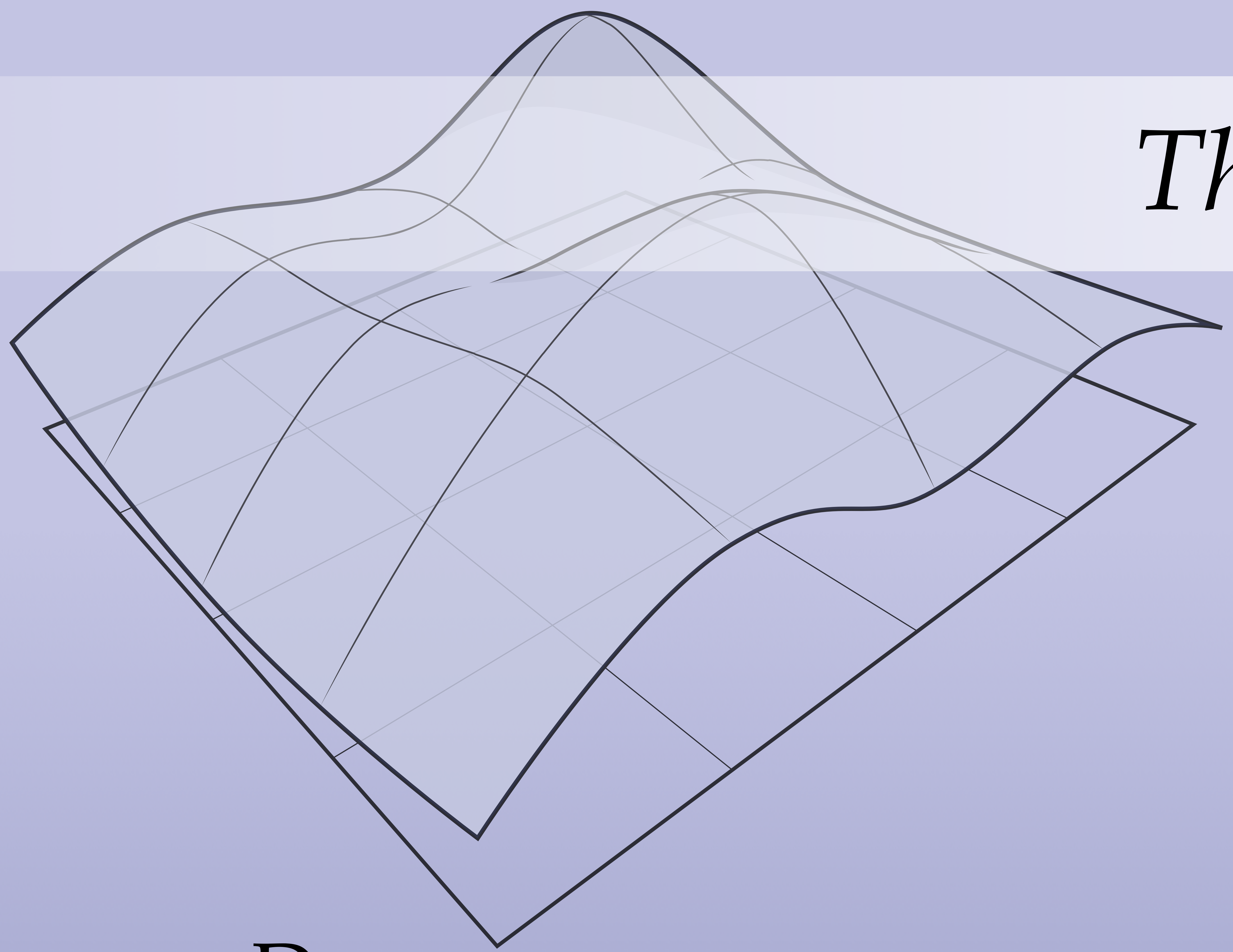




$$\nabla_i e_j = \Gamma_{ij}^k e_k$$

$$\begin{aligned} d\sigma &= \omega \wedge \sigma \\ d\omega &= \omega \wedge \omega \end{aligned}$$

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858