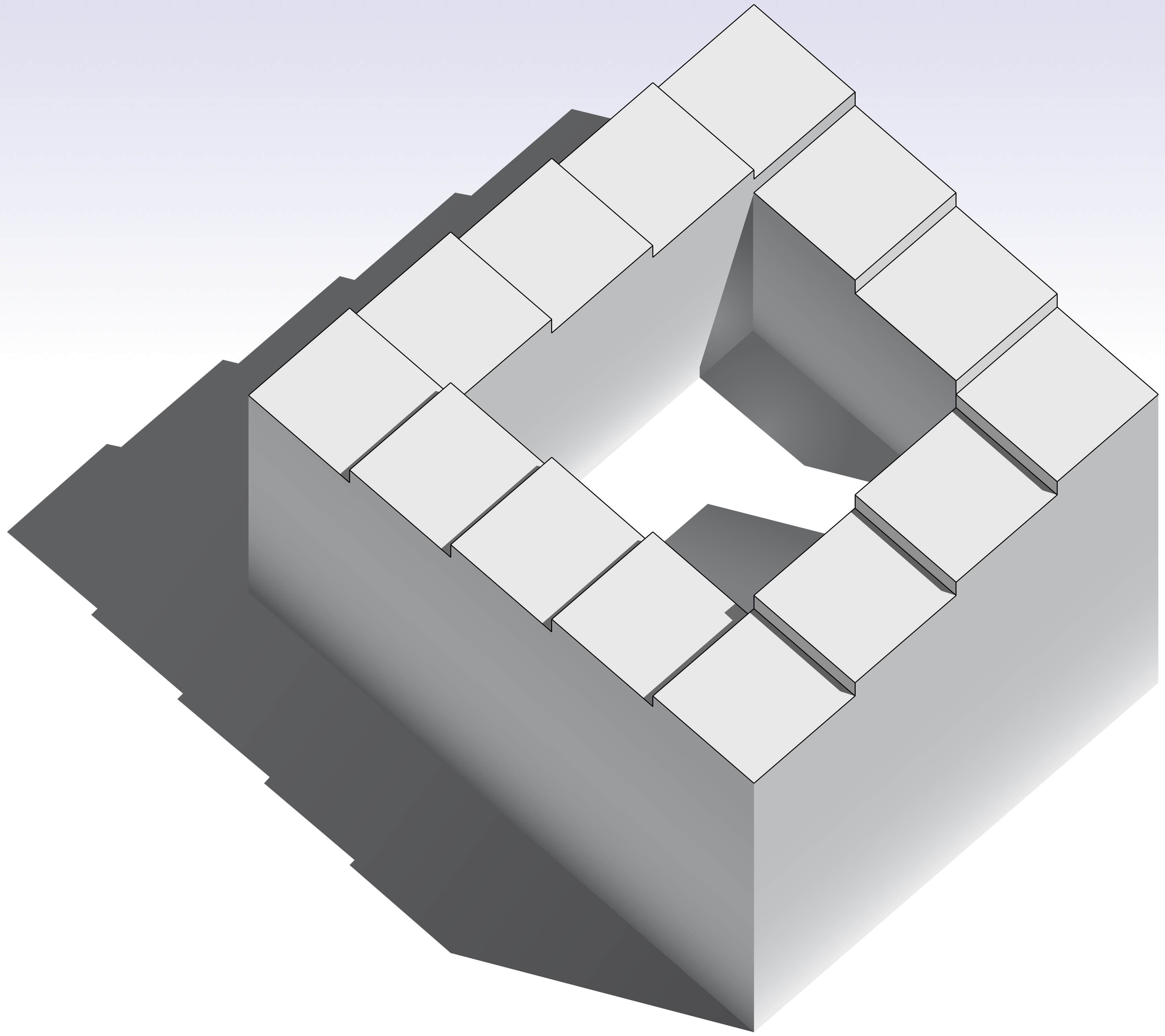
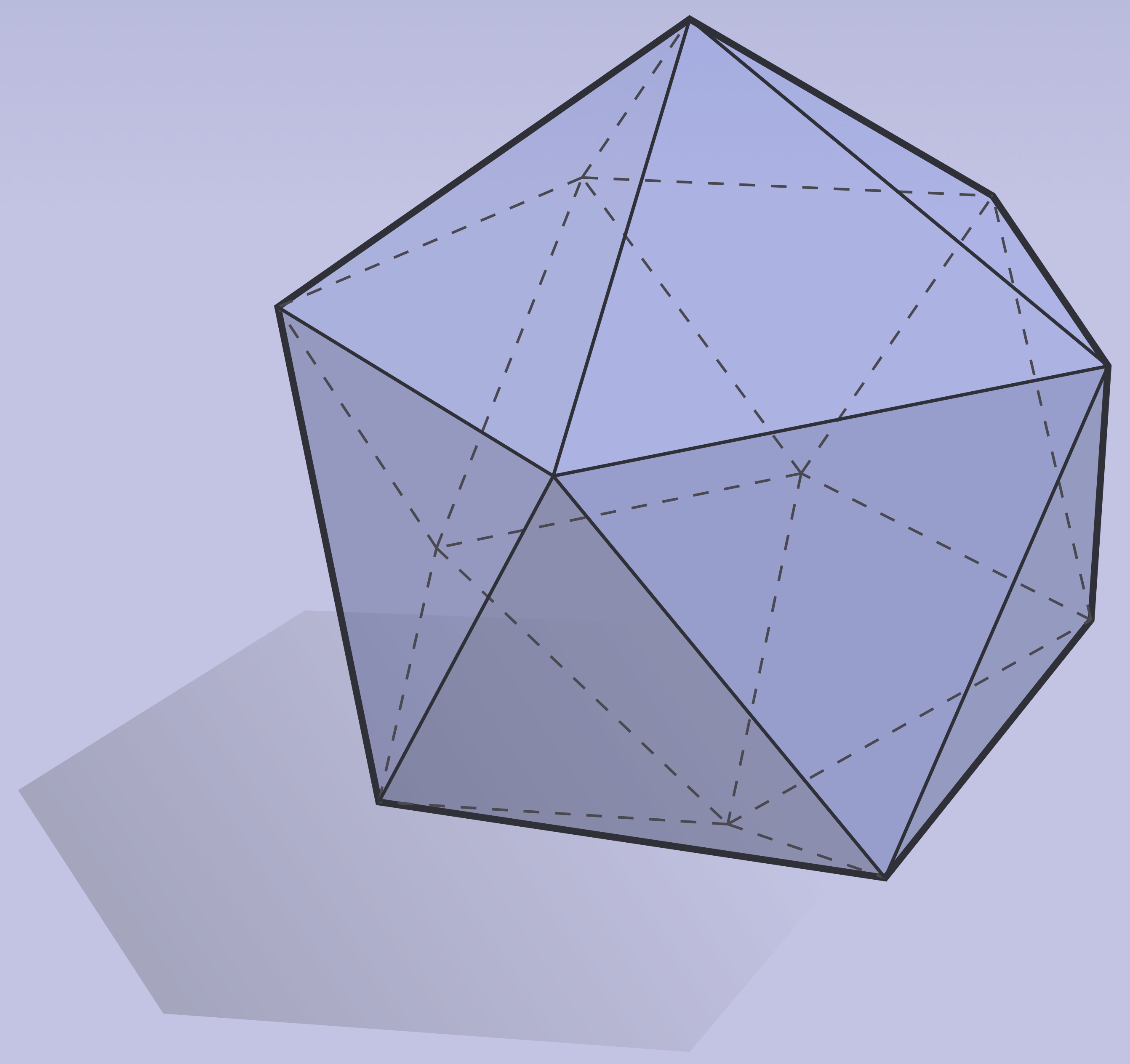


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 13:
Discrete Surfaces



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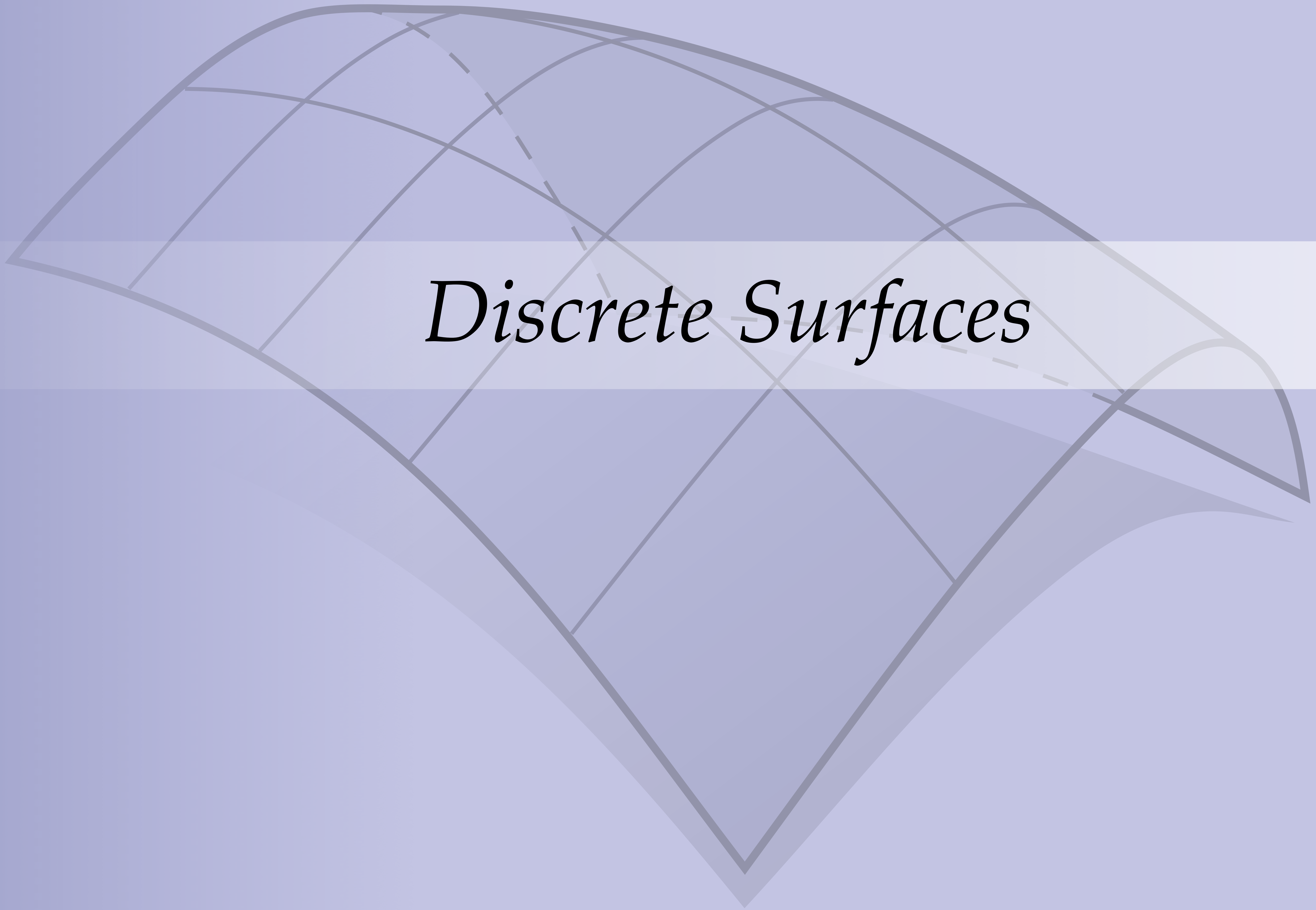
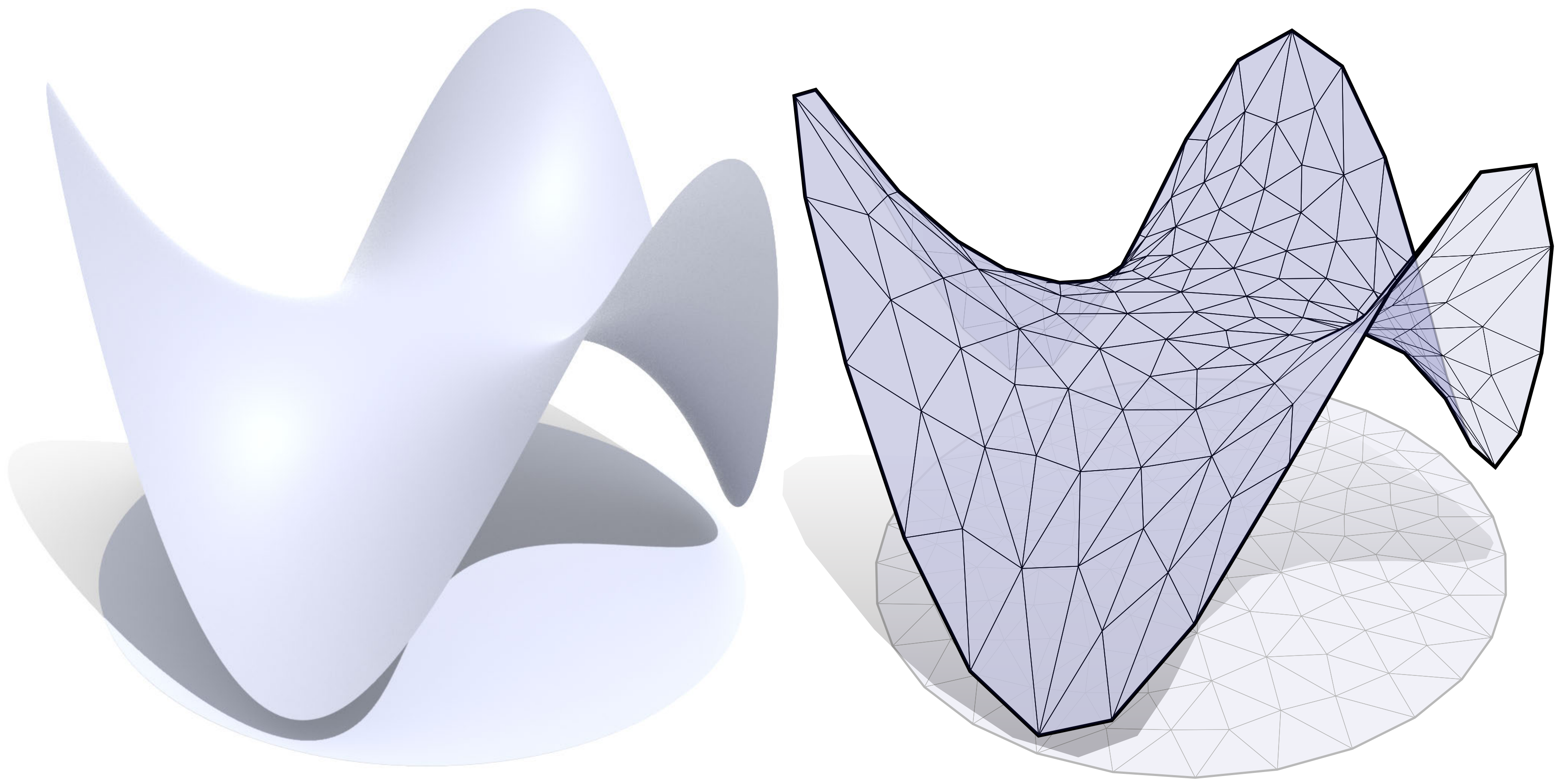
Discrete Surfaces

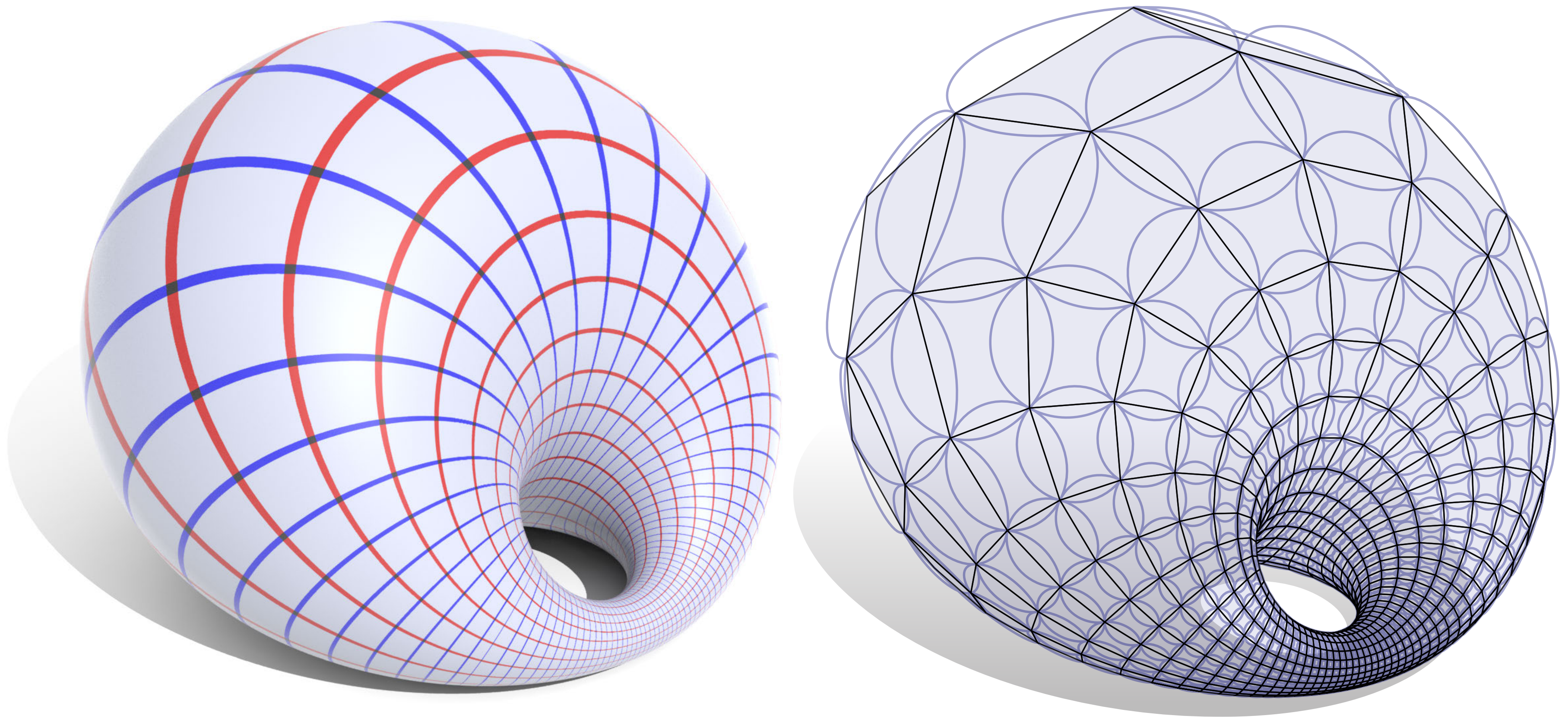


Discrete Models of Surfaces

- Two primary models of surfaces in discrete differential geometry:
- **Simplicial**
  - surfaces are simplicial 2-manifolds
  - natural fit with discrete exterior calculus



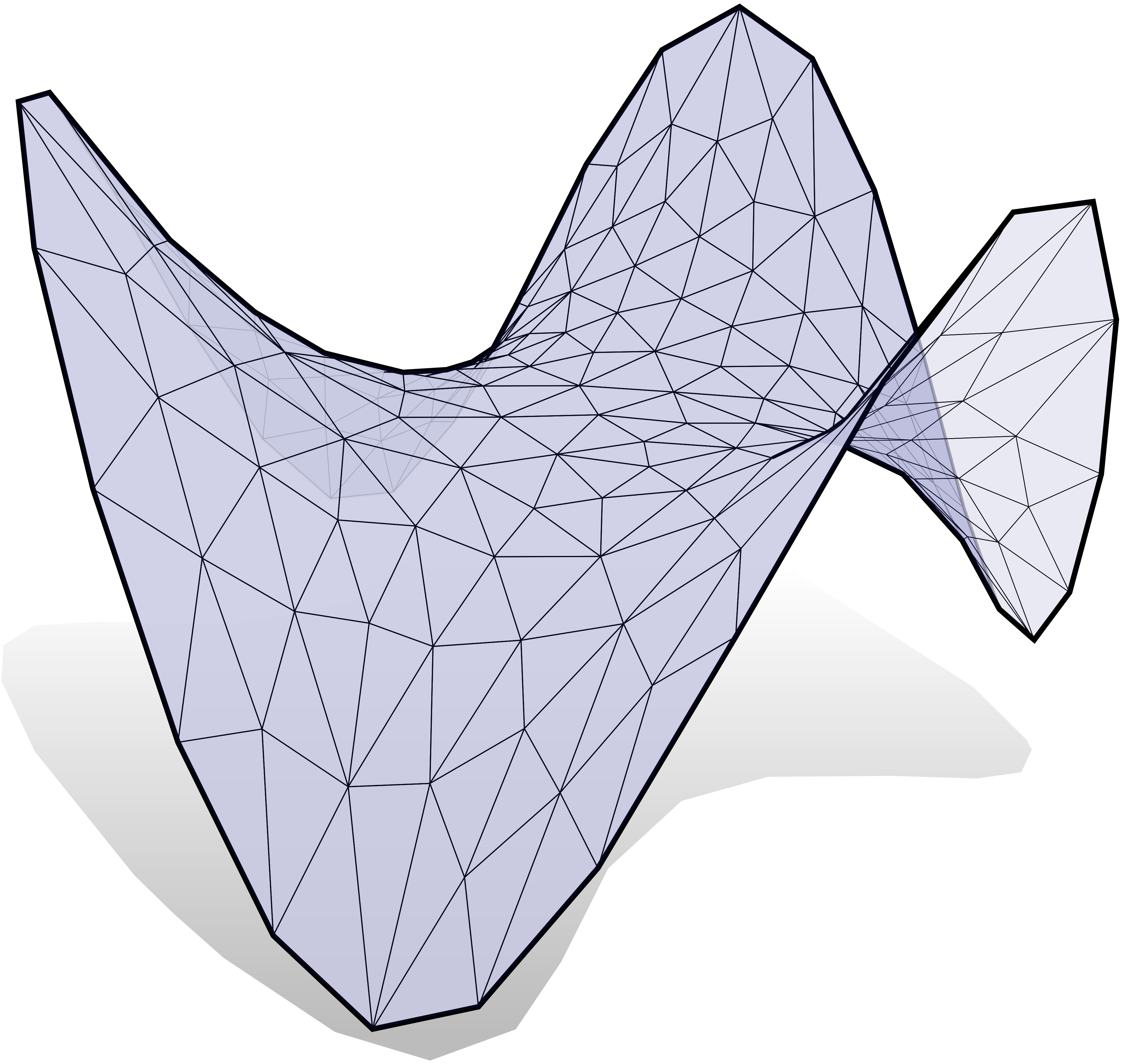
- **Nets**
  - surfaces are piecewise integer lattices
  - natural fit with *discrete integrable systems*



- Simplicial surfaces more common in applications; focus of our course

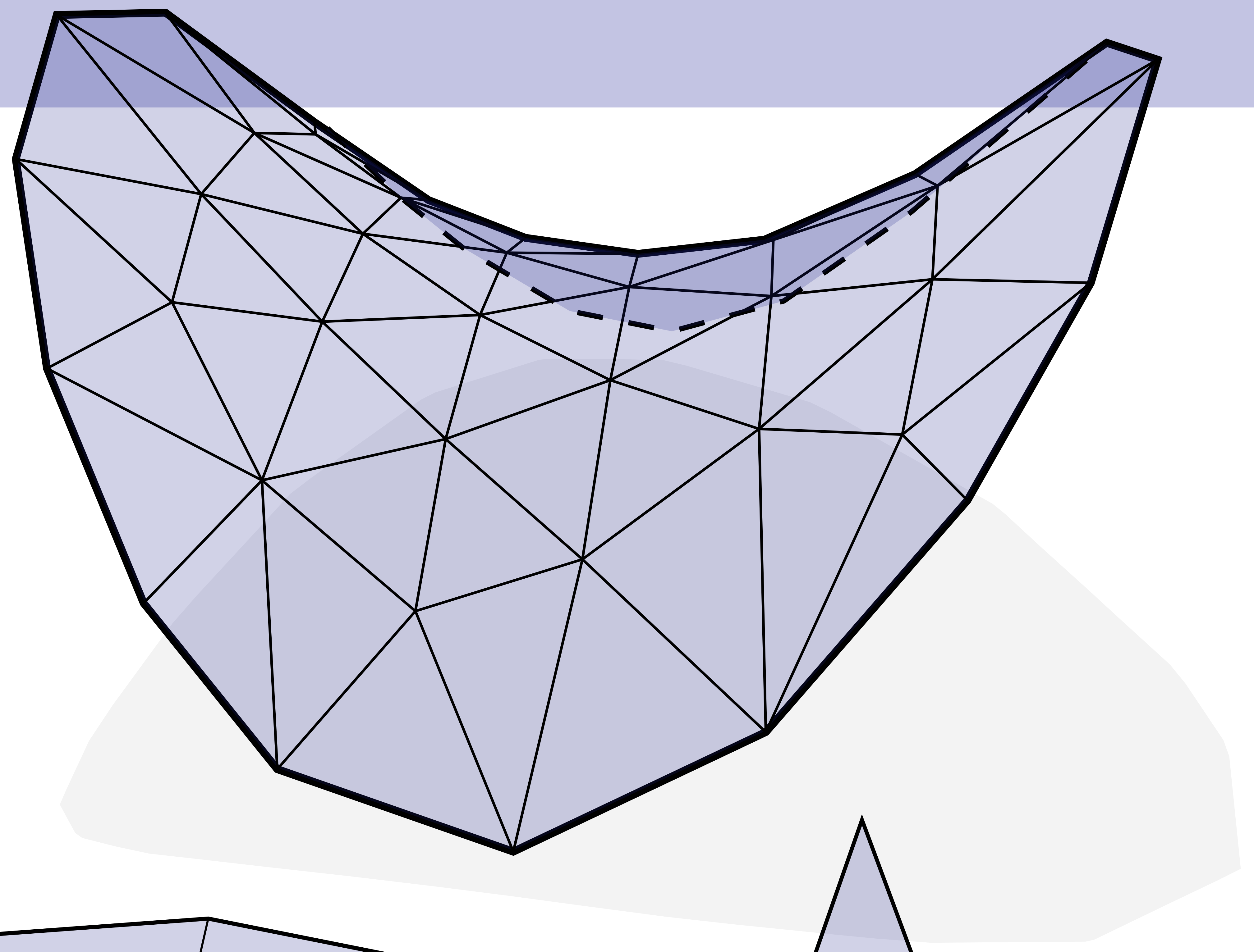
Simplicial Surface—Short Story

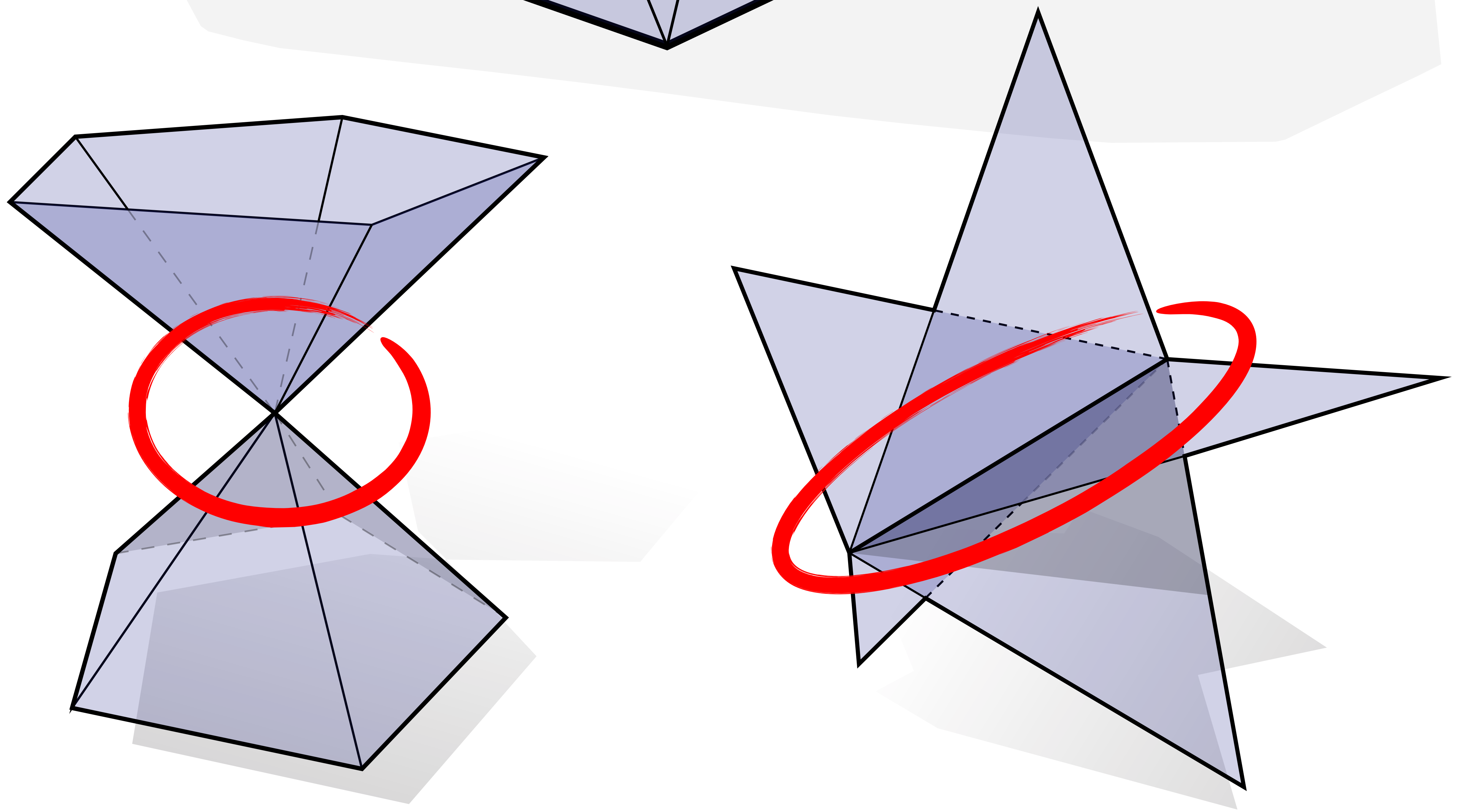
- Loosely speaking, a **simplicial surface** is *"just a triangle mesh"*
- But, being more careful about definitions will allow us to connect "triangle meshes" to concepts from differential geometry
- As with smooth surfaces, will also add some conditions that make life easier. E.g.,
  - mesh connectivity is *manifold*
  - vertex coordinates describe a *simplicial immersion*



Abstract Simplicial Surface

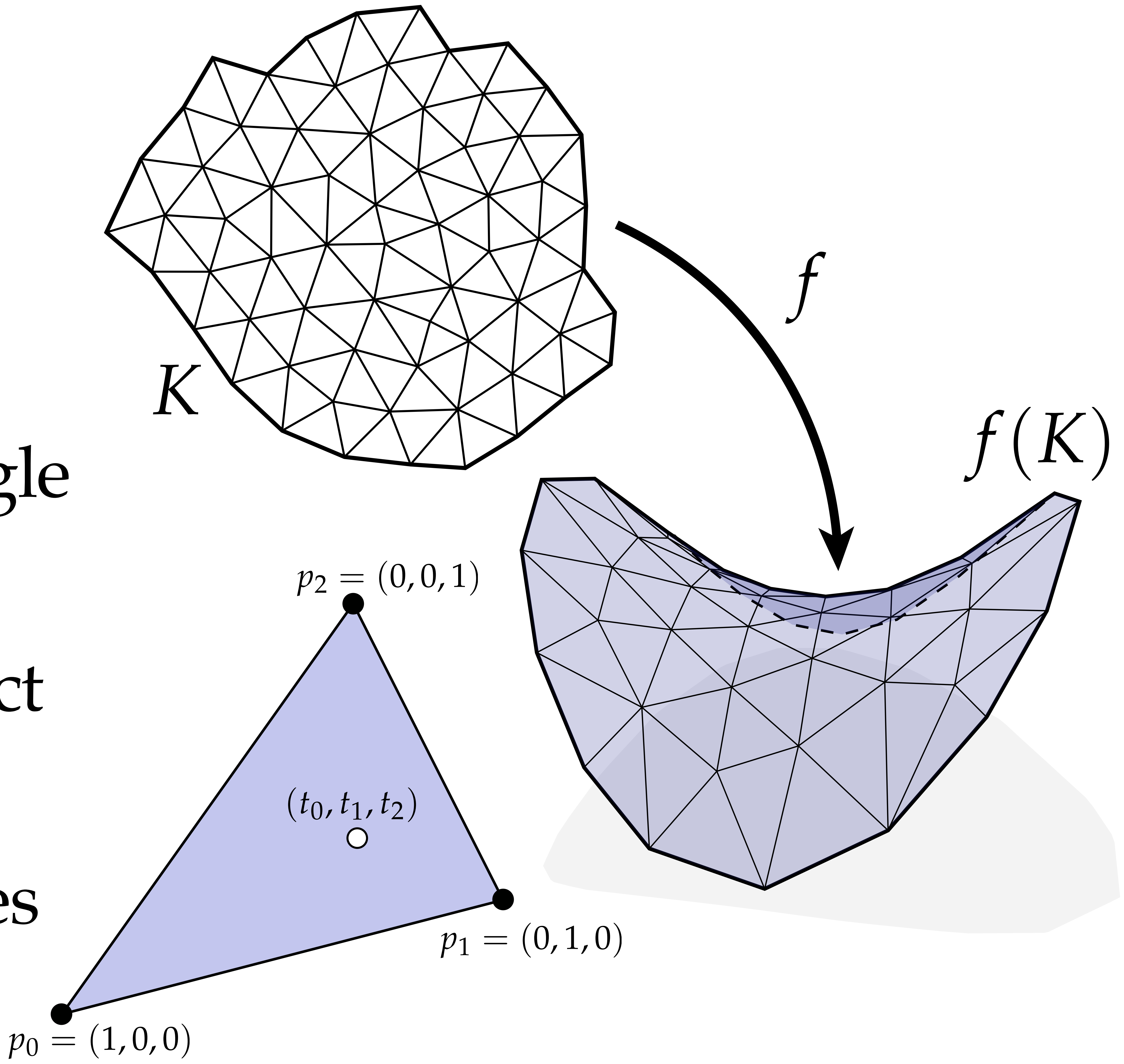
- An **(abstract) simplicial surface** is a manifold simplicial 2-complex
  - highest-degree simplices are triangles
  - every edge contained in two triangles (or one, along boundary)
  - every vertex contained in a single edge-connected cycle of triangles (or path, along boundary)
- Will typically denote by $K=(V,E,F)$
- *No "shape"—just connectivity*





Simplicial Map

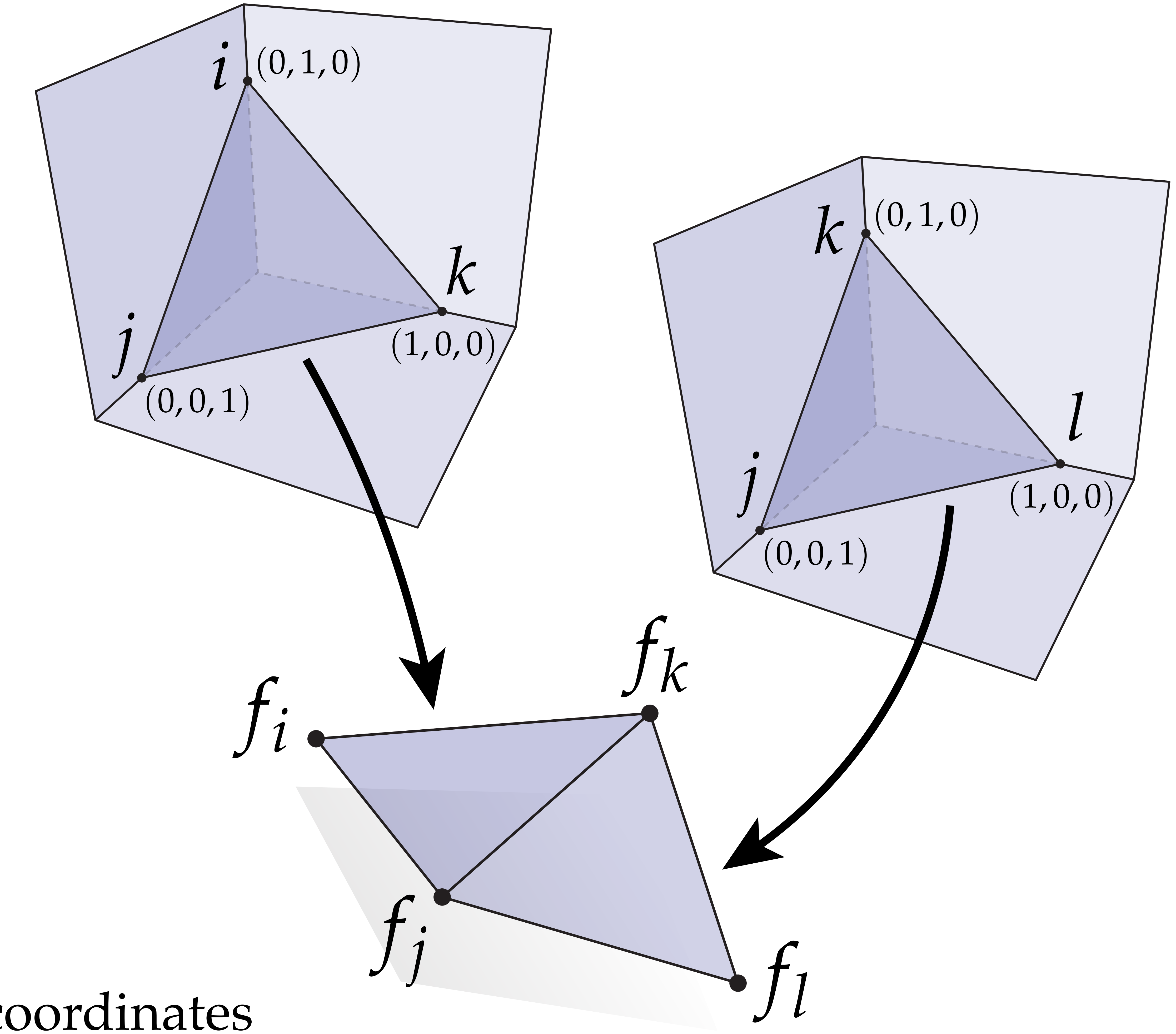
- How do we give a "shape" to an abstract simplicial surface?
- Assign coordinates f_i to each vertex (discrete R^n-valued 0-form)
- Linearly interpolate over each triangle via *barycentric coordinates*
- Image of each simplex in our abstract surface is now a simplex in R^n
- Any map from simplices to simplices is called a **simplicial map**



Simplicial Map, continued

- What's really going on here? E.g., what's the domain of our map f?
- Abstract simplicial complex is just a set of subsets… How do we talk about points "inside" a simplex?
- Barycentric coordinates effectively associate each abstract simplex with a a copy of the *standard simplex*
- Domain of f is then the (disjoint) union of all these simplices, "glued" together along shared edges*

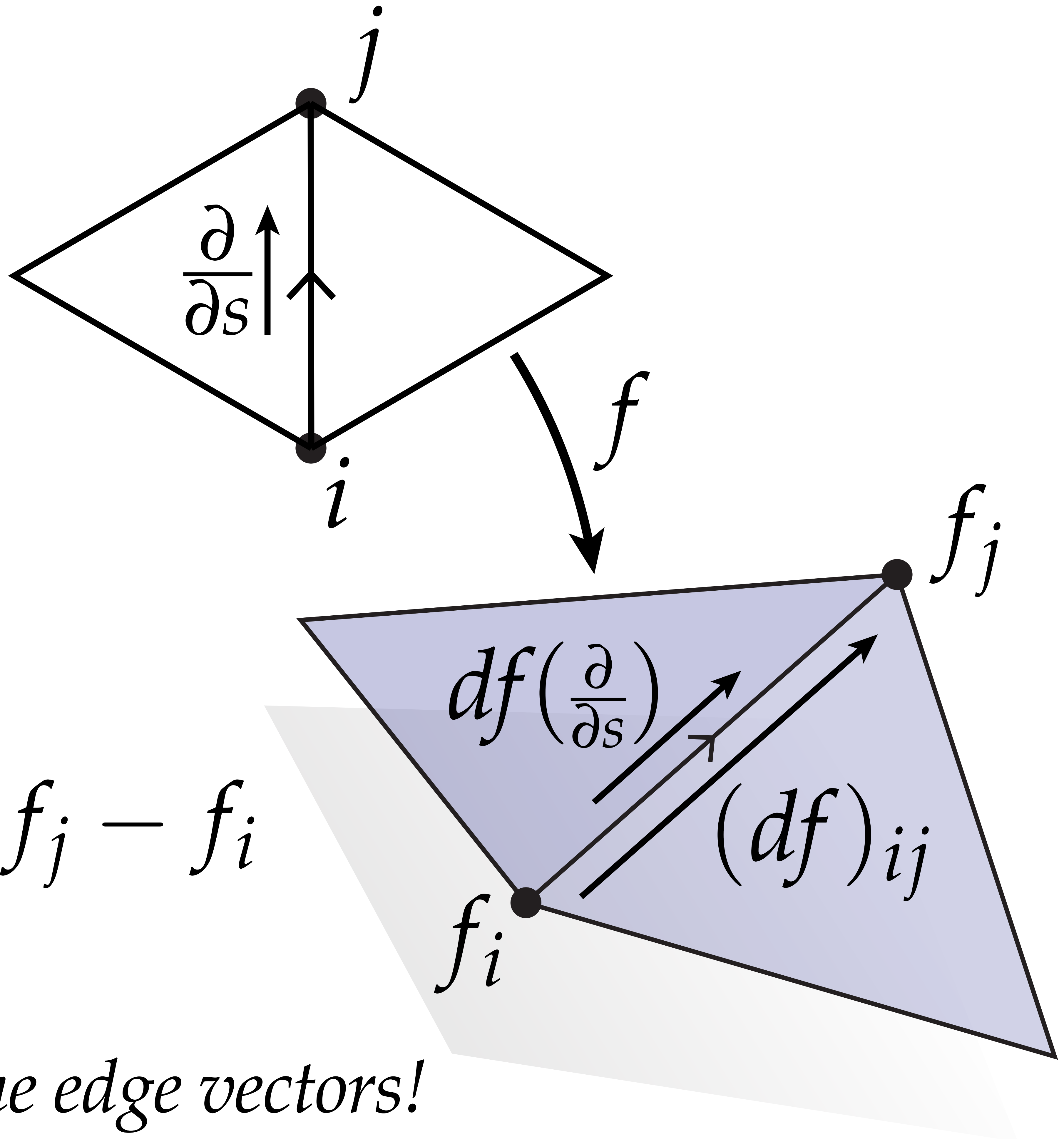
$K = \{ \{i,j,k\}, \{j,k,l\}, \{i,j\}, \{j,k\}, \{k,i\}, \ldots\}$



*Formally: quotient space w.r.t. equivalence on barycentric coordinates

Discrete Differential

- Map f is given by a discrete, R^n-valued 0-form
- **Discrete differential** df is just discrete exterior derivative
- What does it mean, geometrically?
- Recall that a discrete 1-form represents the integral of a smooth 1-form over a 1-simplex*:



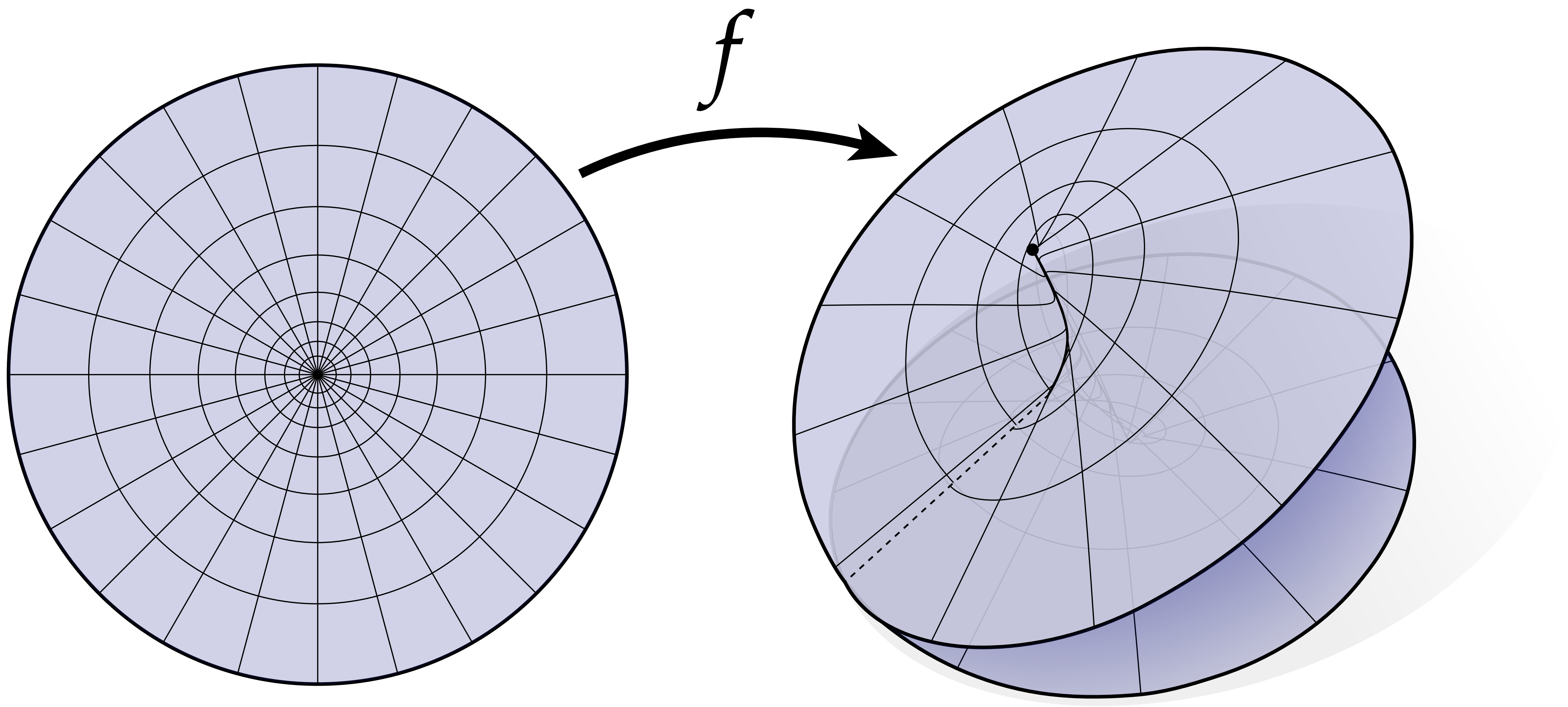
$$(df)_{ij} := \int_{\sigma_{ij}} df\left(\tfrac{d}{ds}\right)\, ds = \int_{\sigma_{ij}} df = \int_{\partial\sigma_{ij}} f = f_j - f_i$$

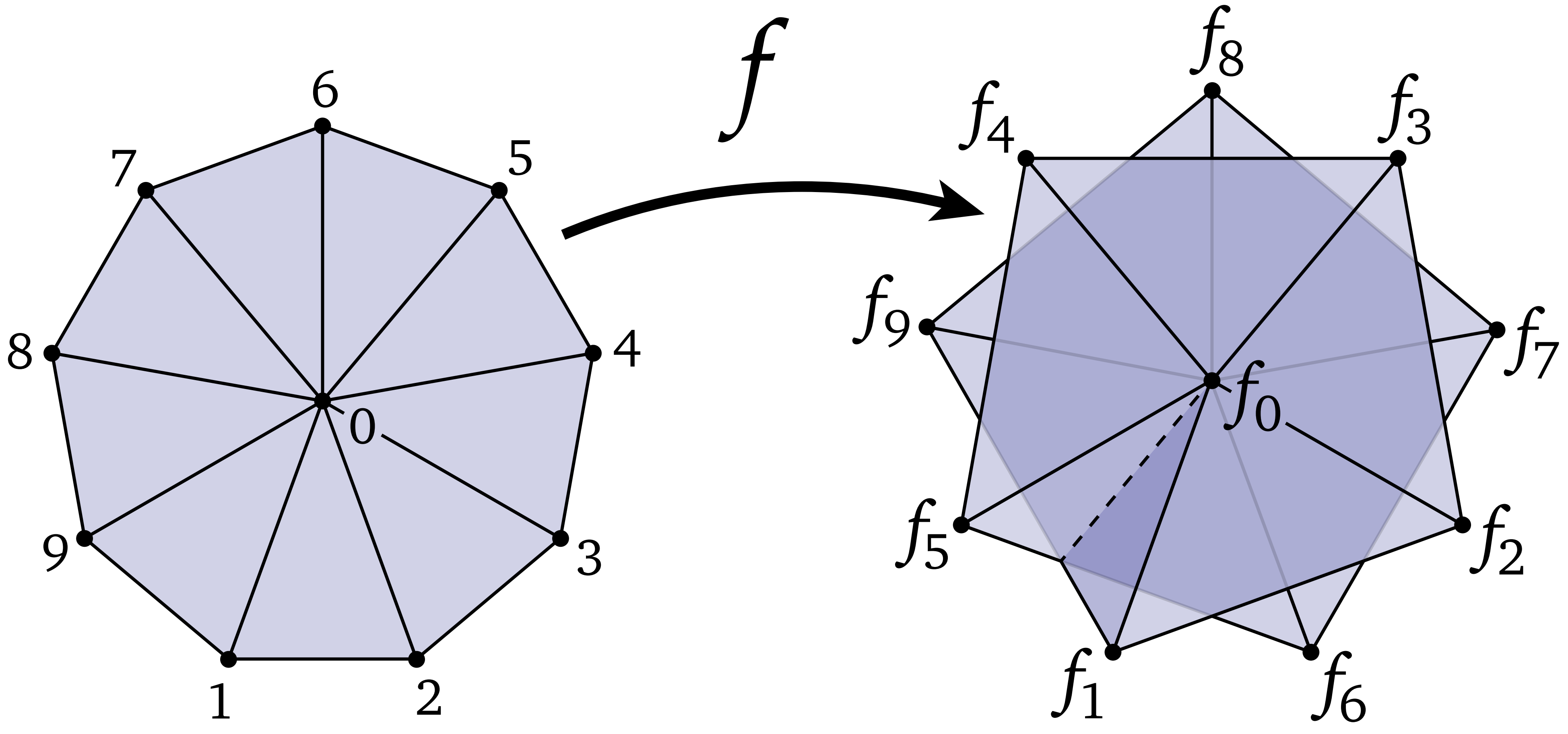
- In other words, *discrete differential is nothing more than the edge vectors!*
- Like any other 1-form, antisymmetric w.r.t. orientation: $df_{ji} = -df_{ij}$

*Here we can imagine σ_{ij} is the standard 1-simplex

Discrete Immersion

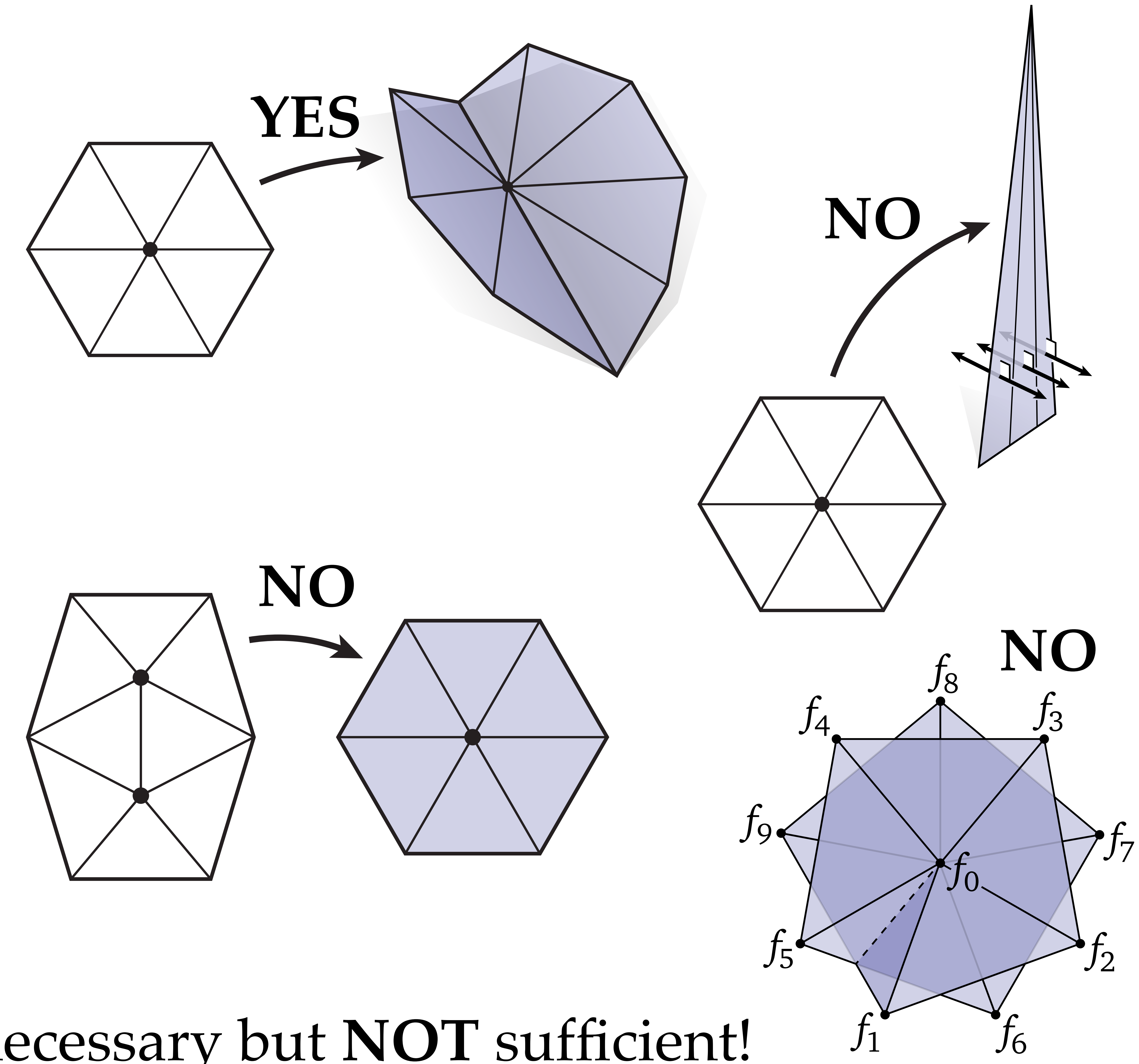
- In smooth setting, a map *f* is an immersion if differential is nondegenerate, i.e., if it maps nonzero vectors to nonzero vectors
- In discrete setting, a nondegenerate (discrete) differential just means no zero edge lengths
- Doesn't faithfully capture important features of smooth immersions! *E.g.*, no *branch points*





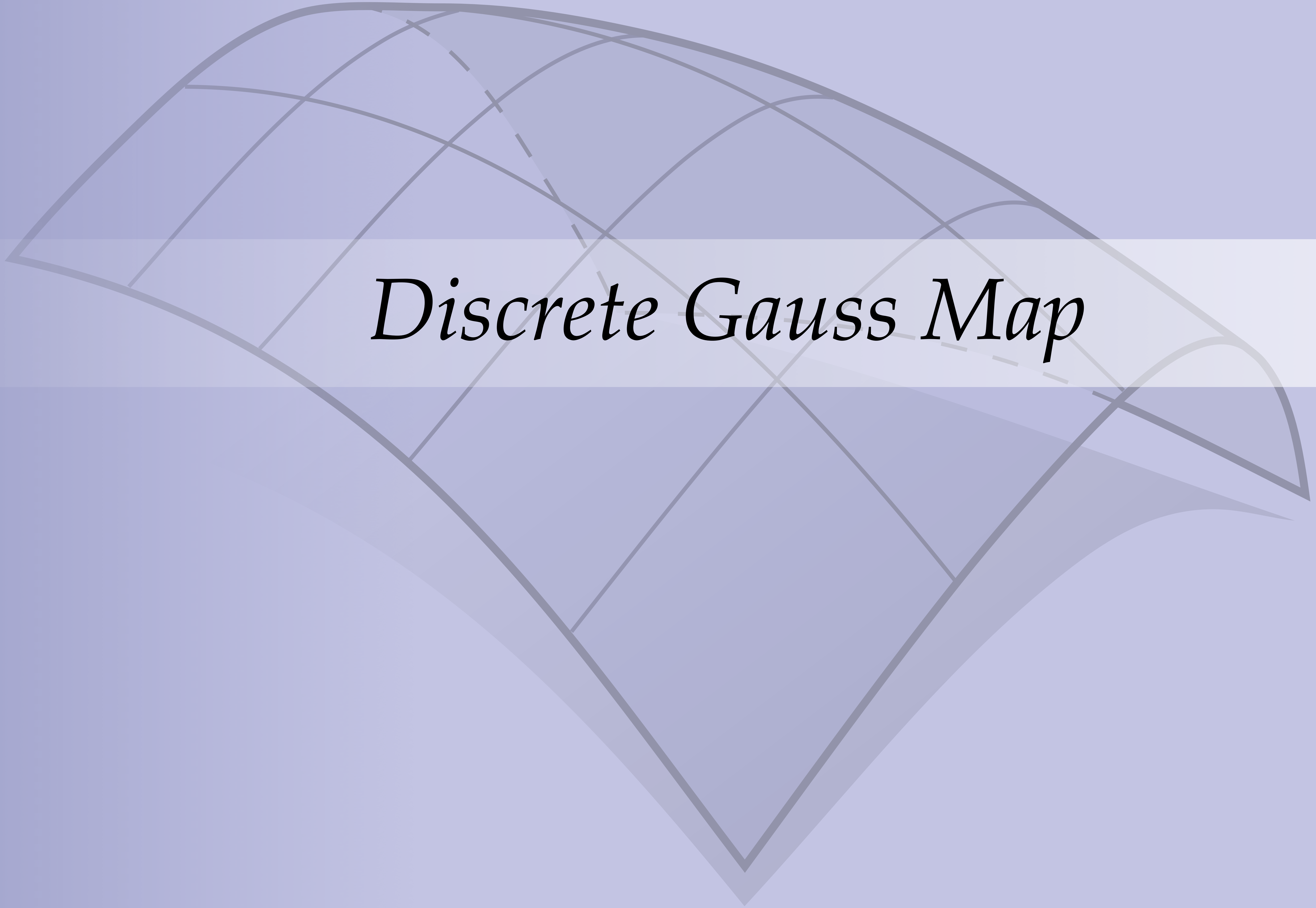
Simplicial Immersion

- In smooth setting, a map f is an immersion if its *differential* df is injective
- In the discrete setting, a simplicial map f is a **discrete immersion** if the *map itself* is locally injective
- **Fact.** A simplicial map is locally injective if and only if every vertex star is embedded



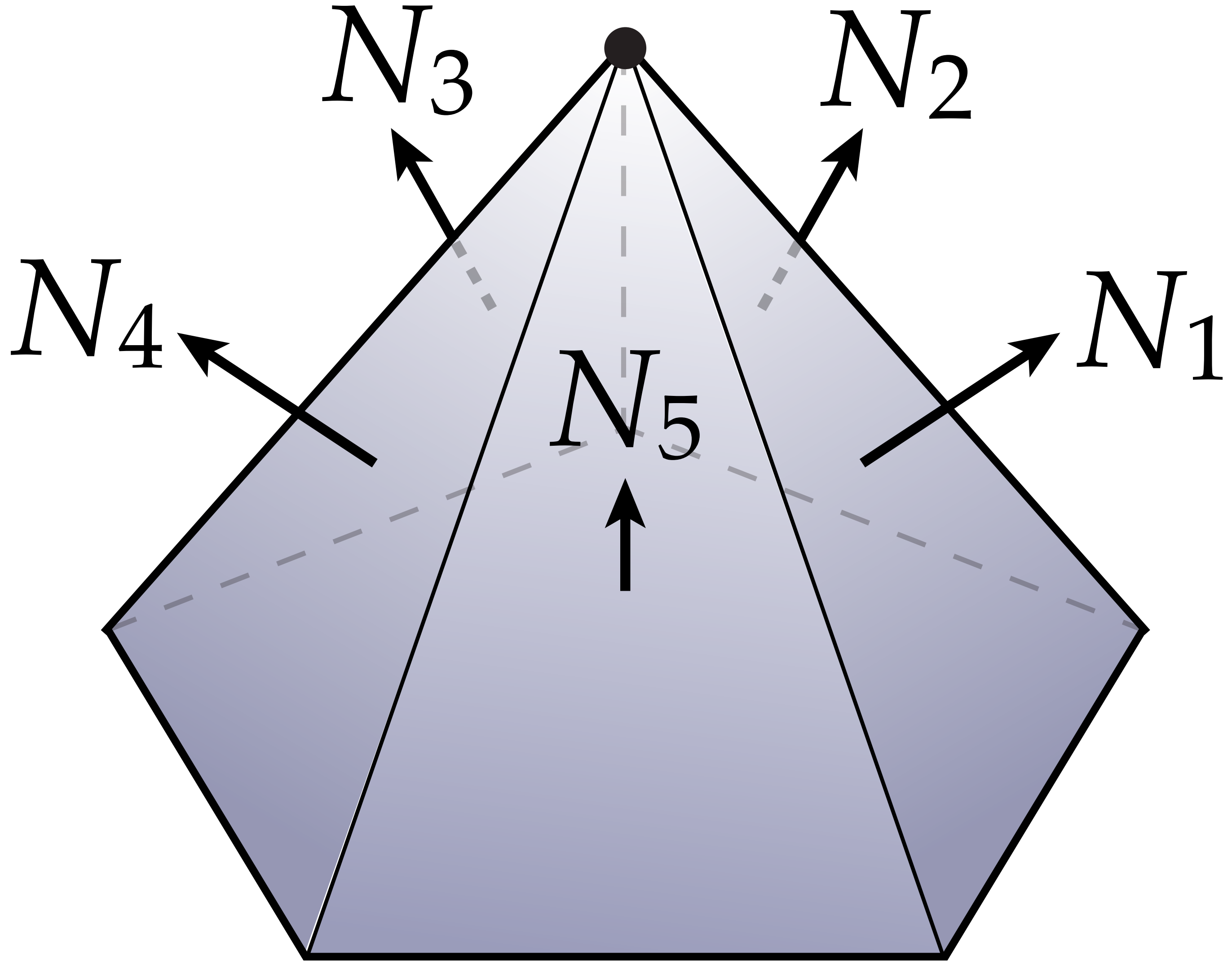
Note: "no degenerate elements/angles" is necessary but **NOT** sufficient!

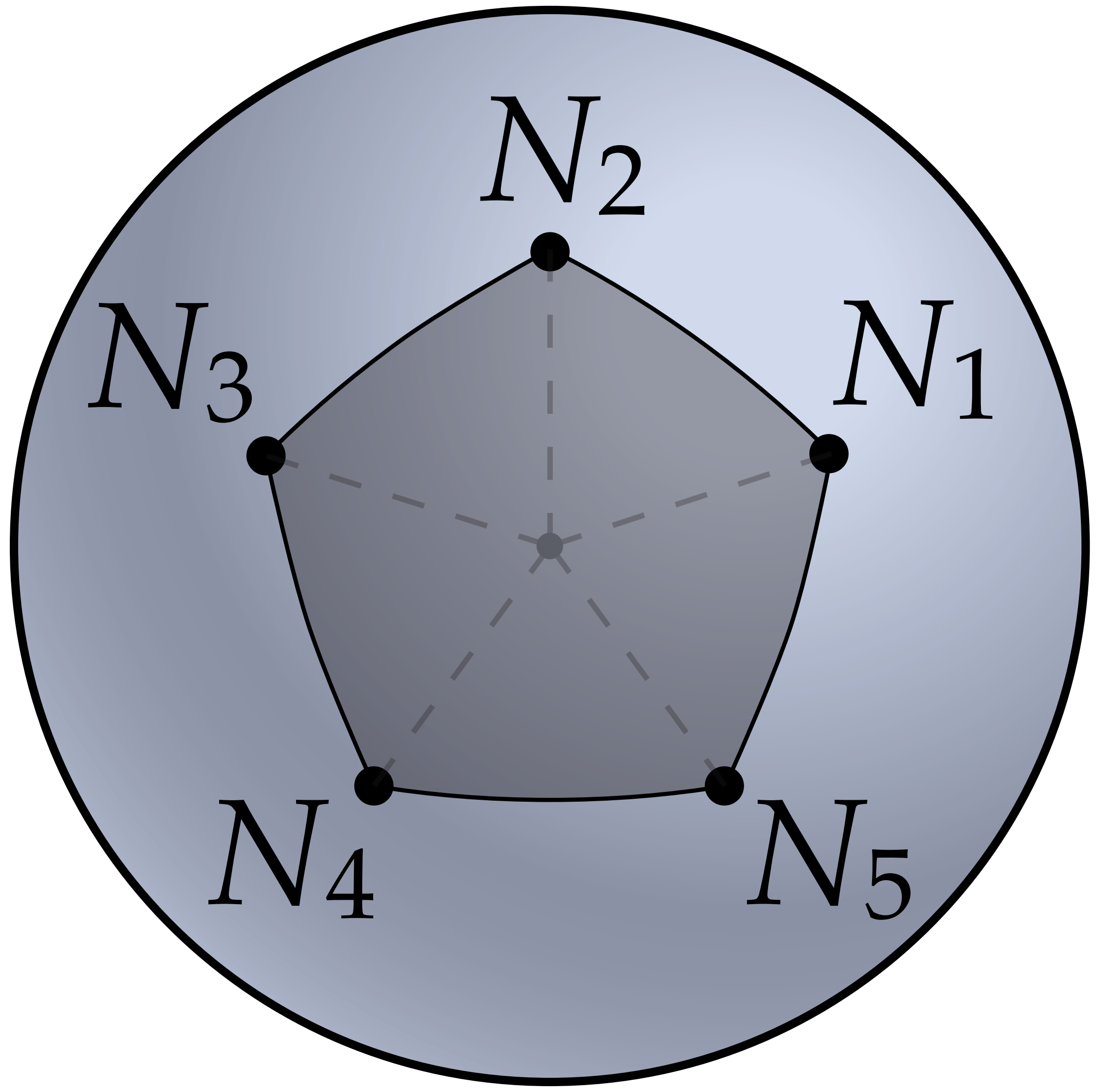
Discrete Gauss Map



Discrete Gauss Map

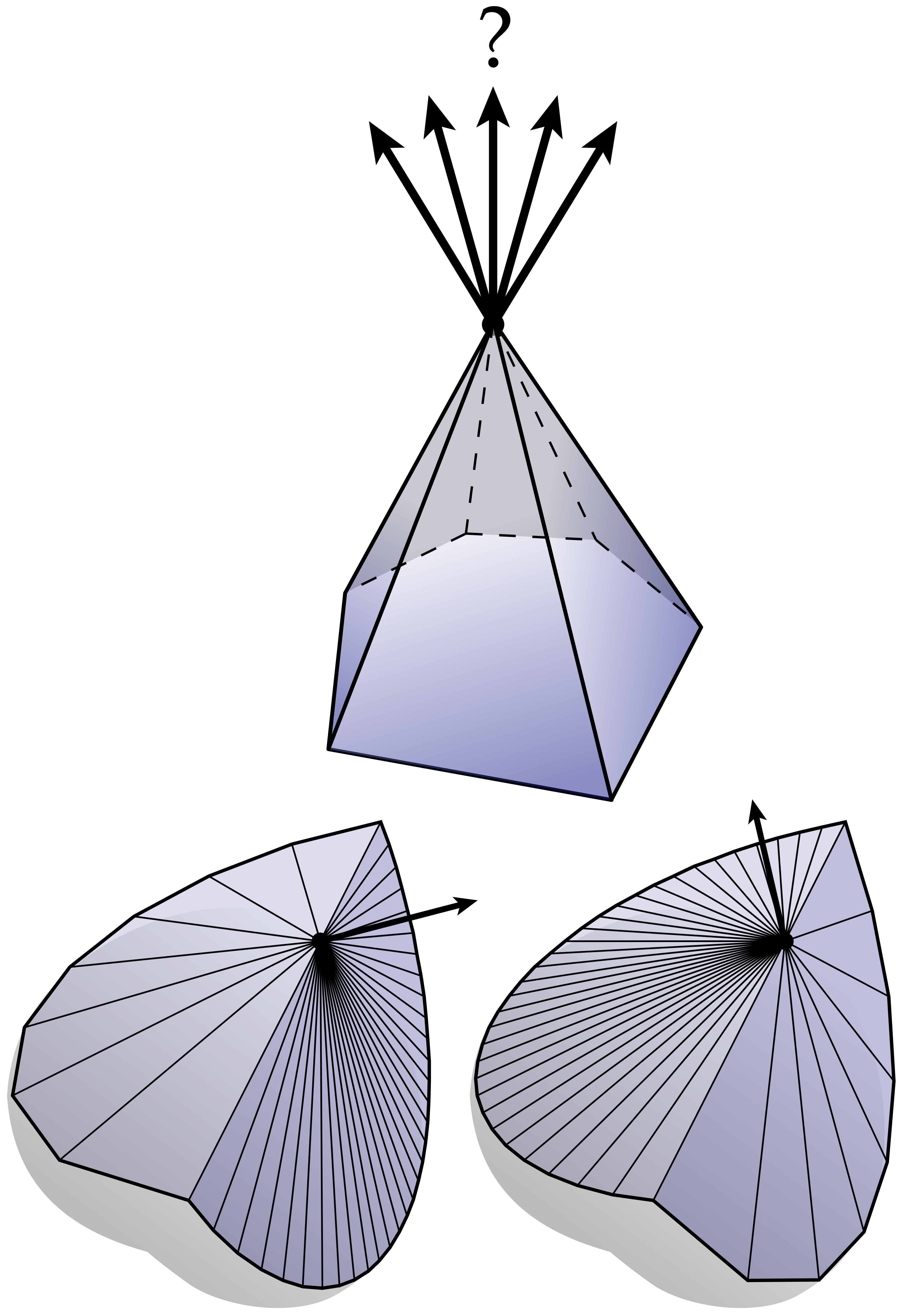
- For a discrete immersion, the Gauss map is simply the triangle normals
- Most naturally viewed as a dual discrete R^3-valued 0-form (vector per triangle)
- Visualize as points on the unit sphere
- Connecting adjacent normals by arcs corresponds to family of normals orthogonal to edge





Discrete Vertex Normal?

- Discrete Gauss map still doesn't define normals at vertices (or edges)
- Can take ad-hoc approach, but may behave poorly
- E.g., uniformly averaging face normals yields results that depend on tessellation rather than geometry
- Better approach: start in the smooth setting & apply principled discretization



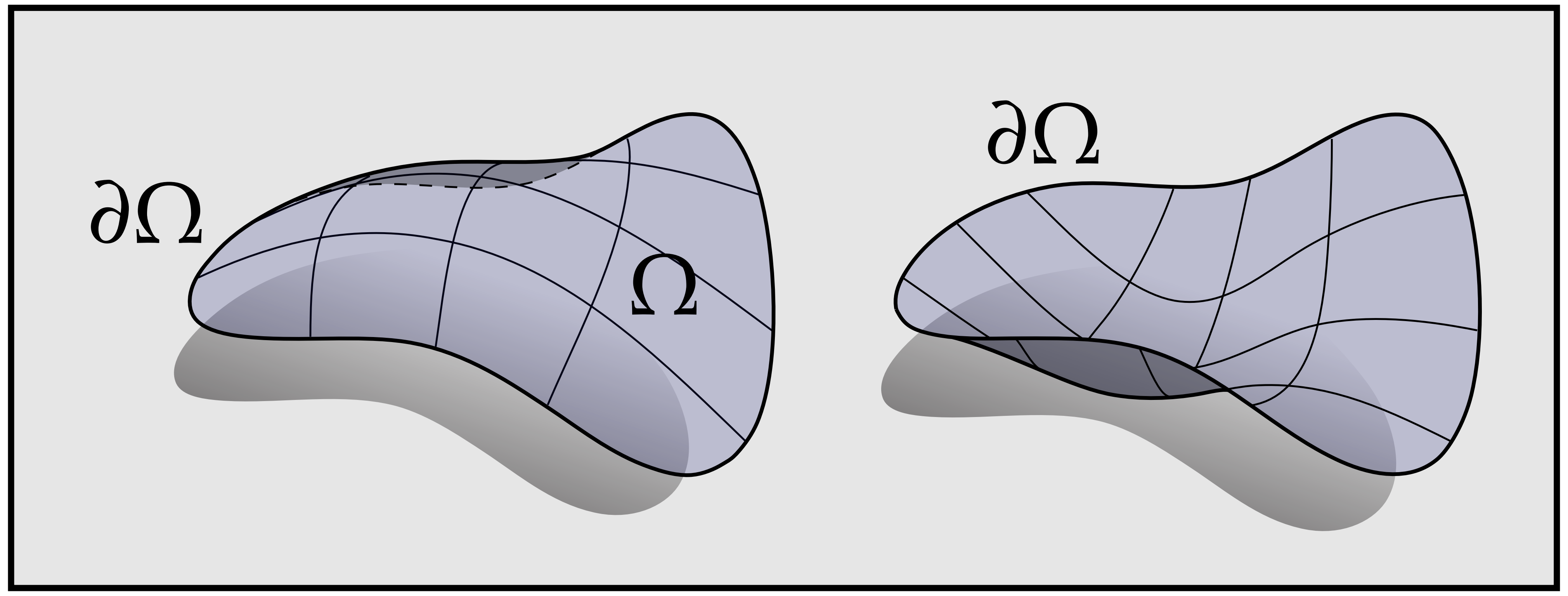
Discrete Vector Area

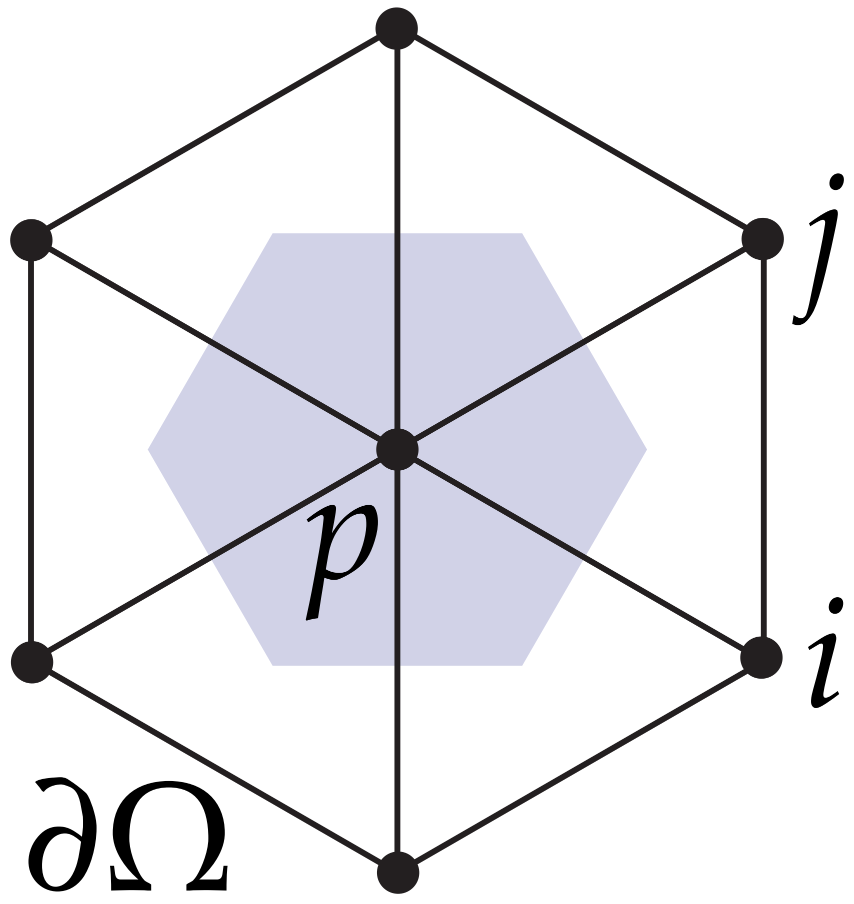
- Recall smooth vector area: $\int_\Omega N\,dA = \frac{1}{2}\int_\Omega df \wedge df = \frac{1}{2}\int_{\partial\Omega} f \times df$
- **Idea:** Integrate NdA over dual cell to get normal at vertex p

$$\frac{1}{3}\int_\Omega N\,dA = \frac{1}{6}\int_{\partial\Omega} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\int_{e_{ij}} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\frac{f_i+f_j}{2}\times(f_j - f_i) = \boxed{\frac{1}{6}\sum_{ij\in\partial\Omega} f_i \times f_j}$$



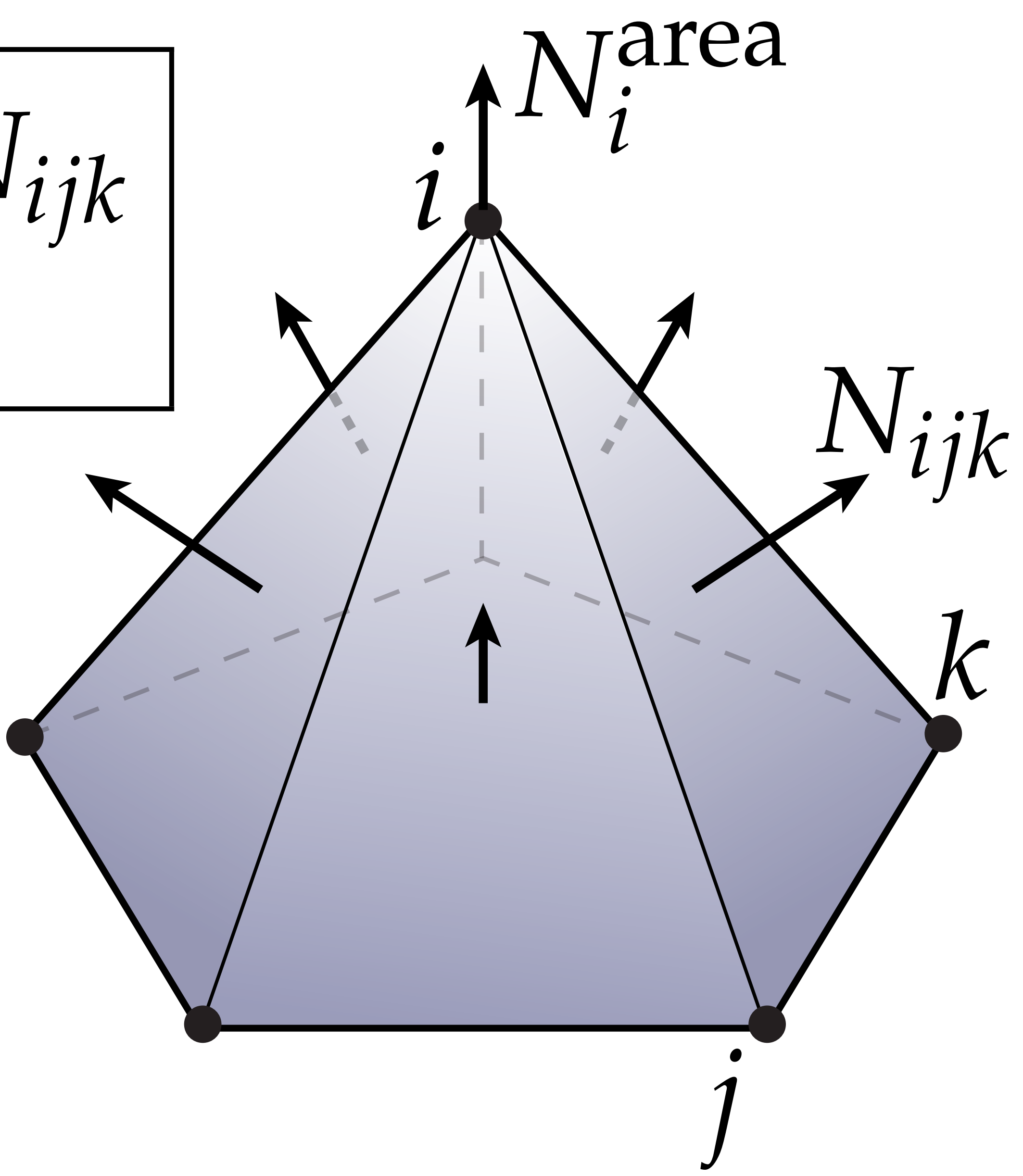


Q: What kind of quantity is the final expression? Does that matter?

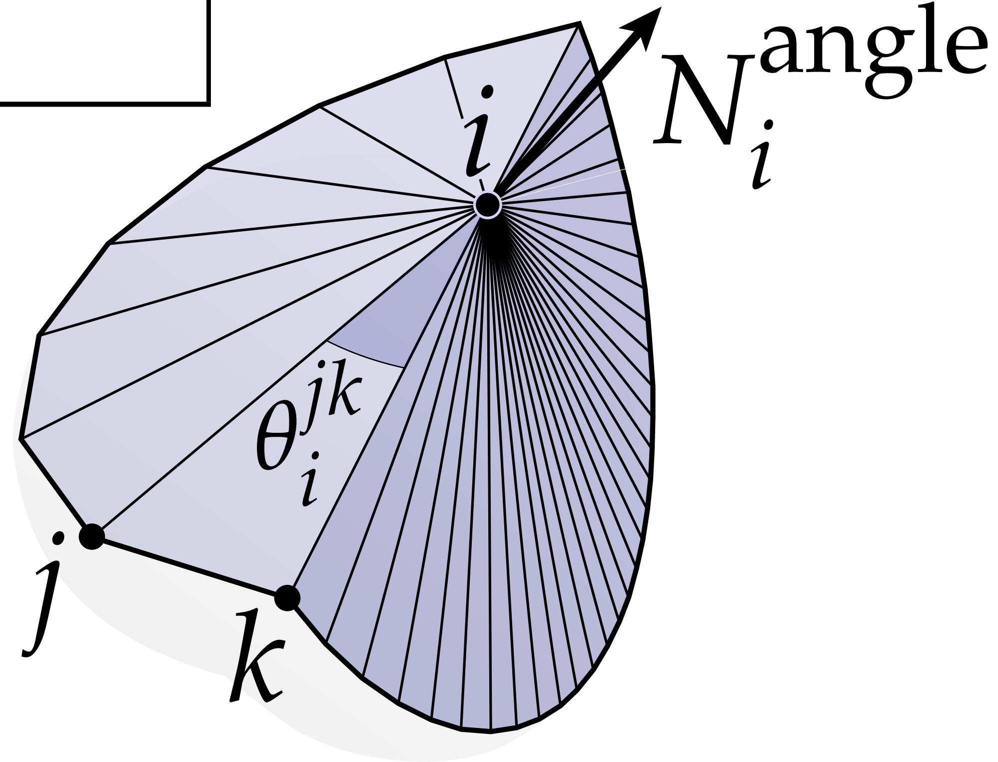
Other Natural Definitions

- area-weighted vertex normal
  - sum of triangle normals times triangle areas
  - smooth setting: volume variation gives $N\,dA$
- angle weighted vertex normal
  - sum of triangle normals times interior angles
  - gives same result, independent of triangulation
- …Please, just anything but uniformly weighted!

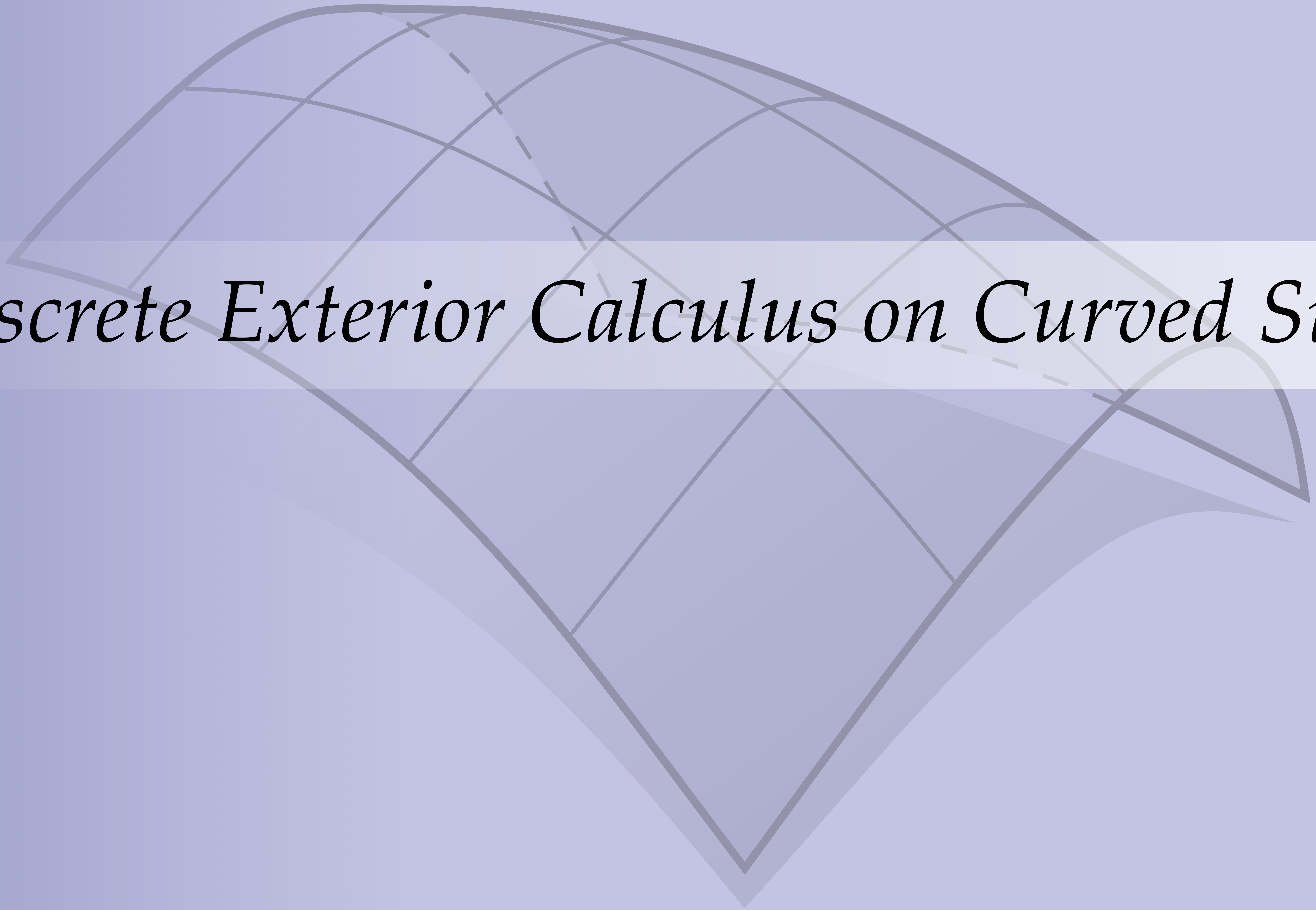
$$\boxed{\sum_{ijk} A_{ijk} N_{ijk}}$$



$$\boxed{\sum_{ijk} \theta_i^{jk} N_{ijk}}$$

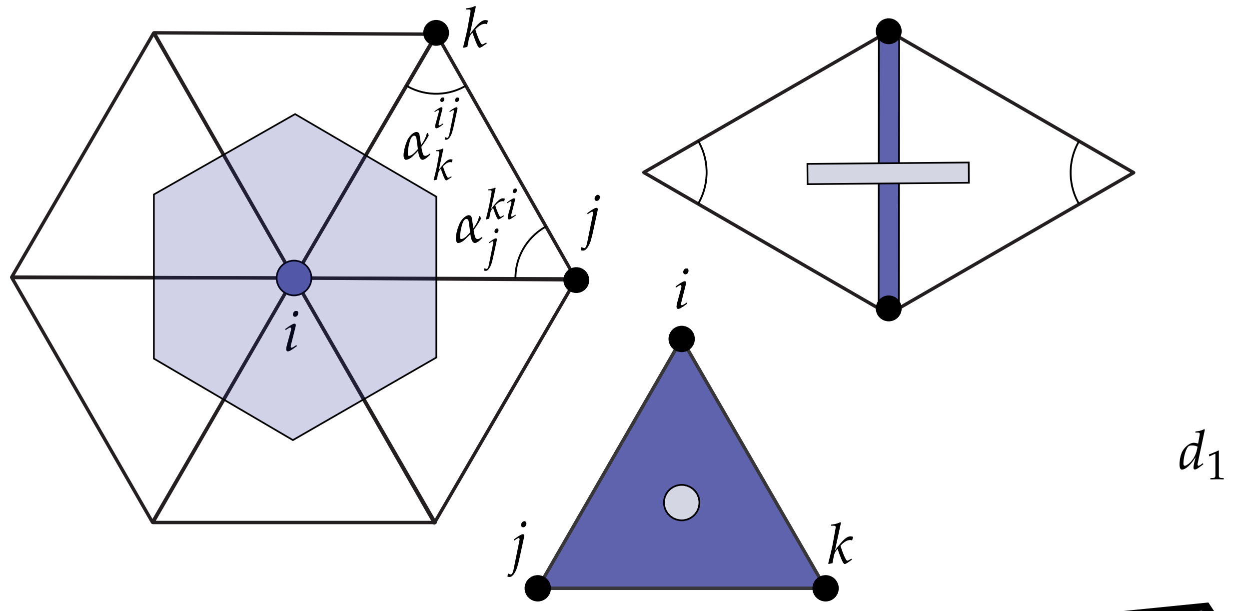


Discrete Exterior Calculus on Curved Surfaces



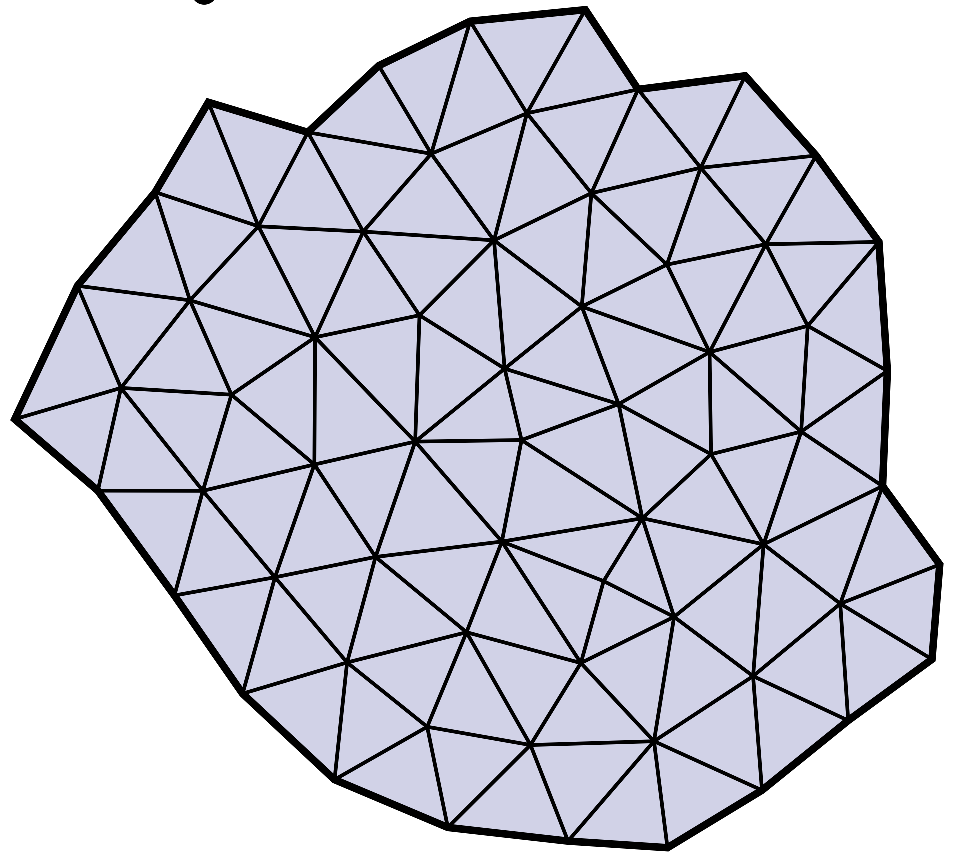
Discrete Exterior Calculus on Curved Surfaces

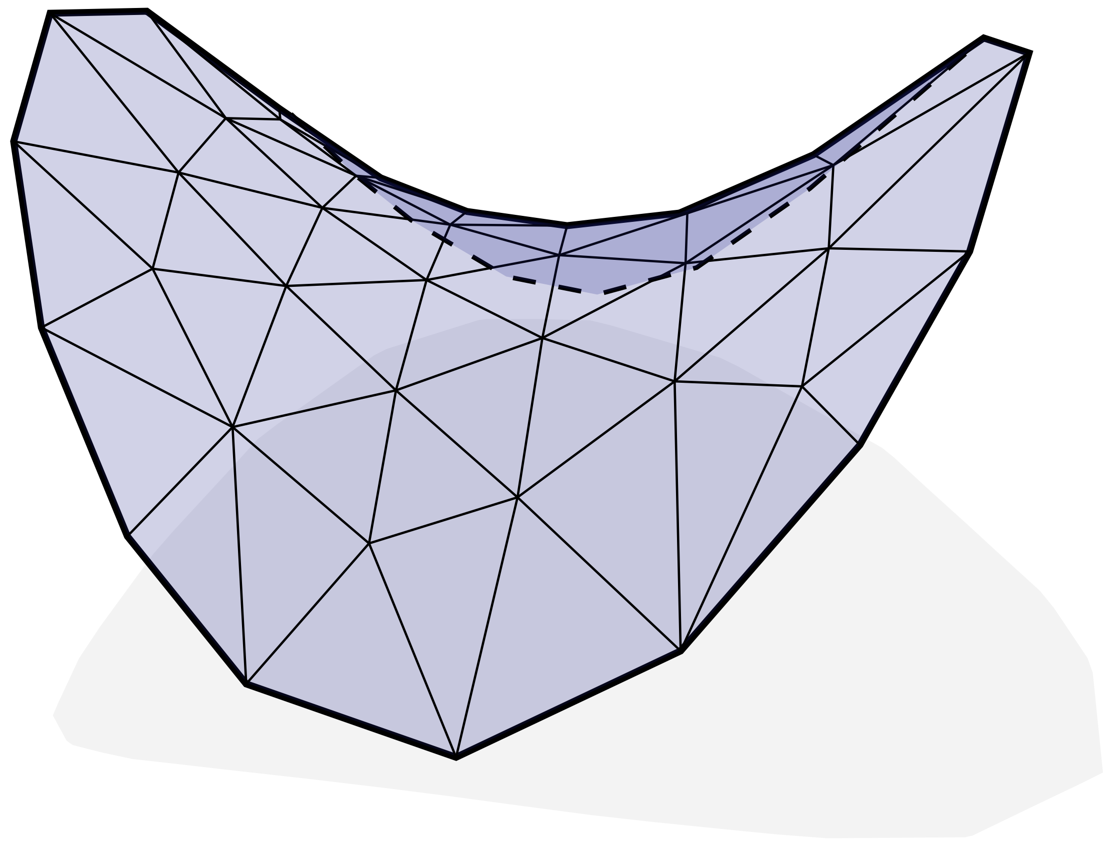
- In the smooth setting, we first defined exterior calculus in R^n, then saw how to augment it to work on curved surfaces
- Key observation: only need to change the Hodge star, which encodes all the metric information (length, angle, area, …)
- For simplicial surfaces in R^3, life is in a sense even easier since each simplex is already flat!
- Still need to think just a *little* about how to define the discrete Hodge star…



$$d_0 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

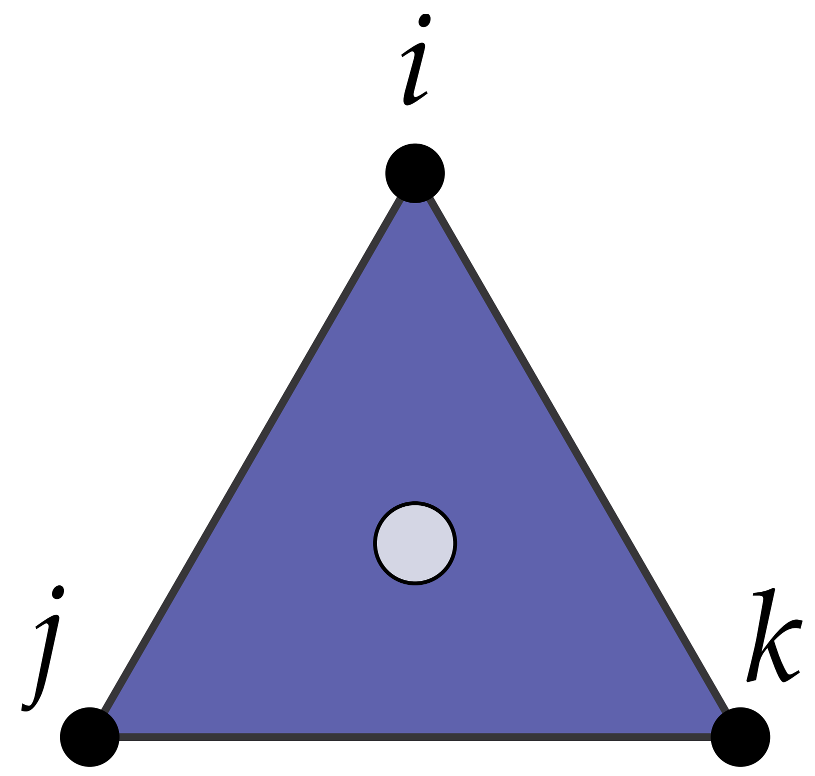
$$d_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$





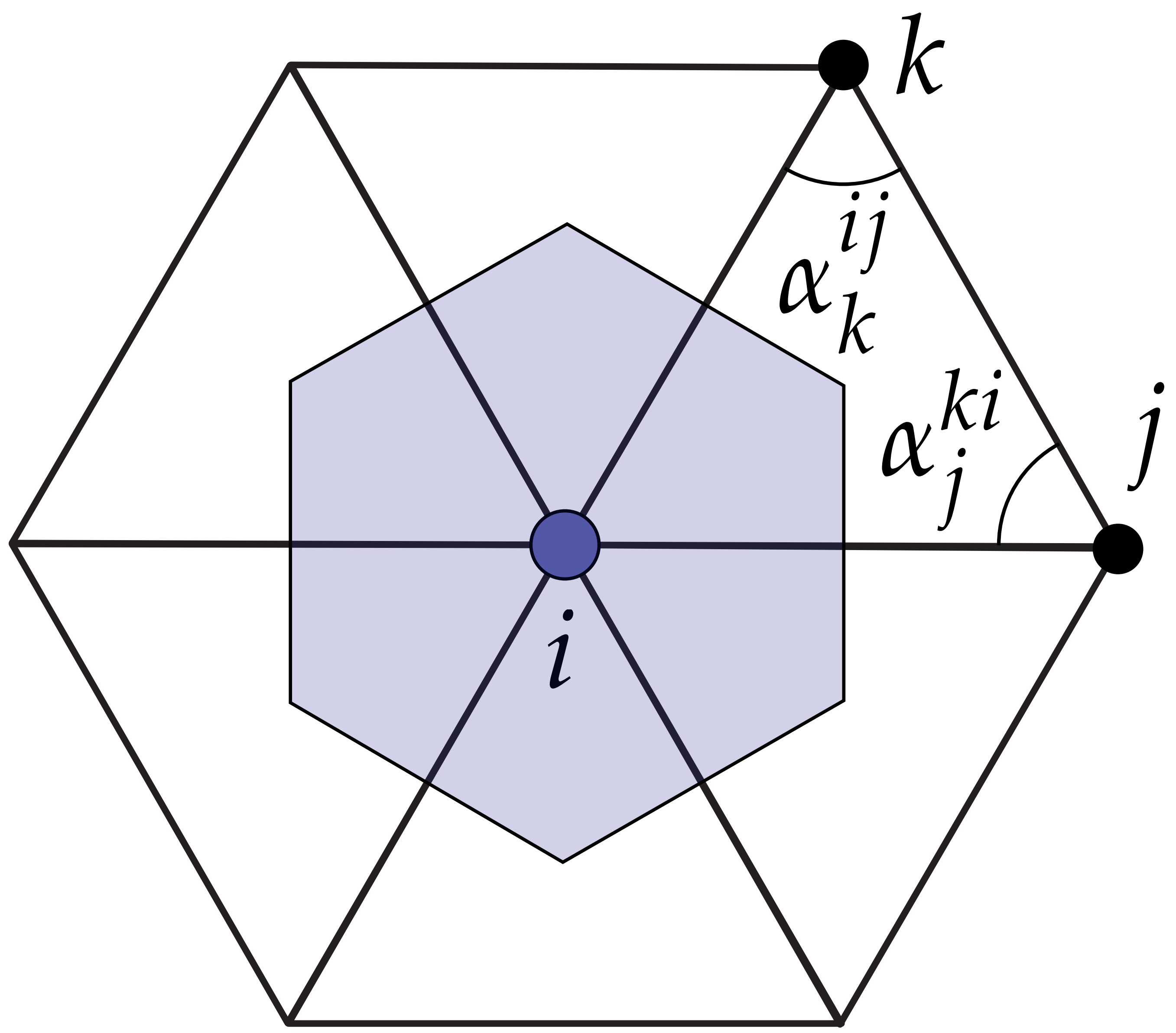
Diagonal Hodge Star on a Surface

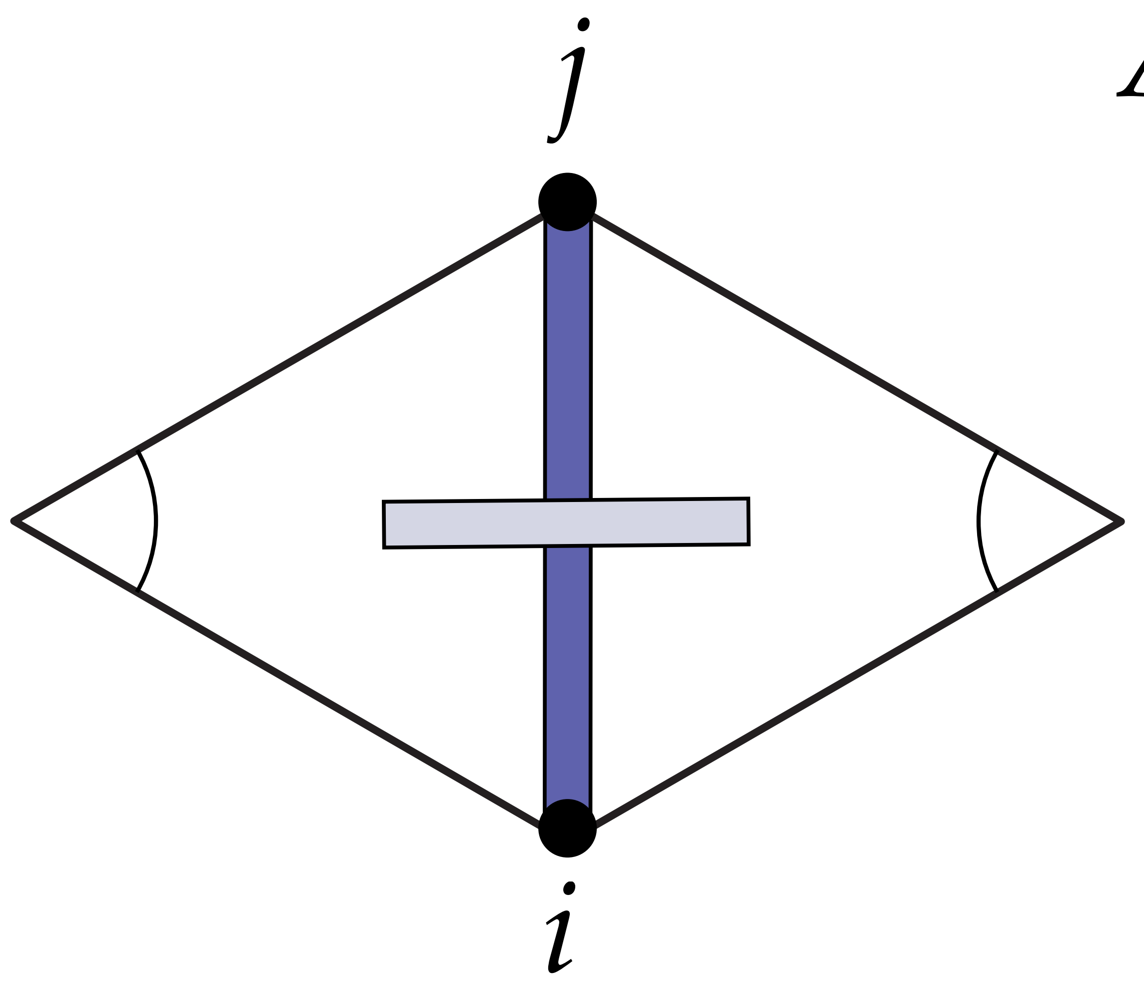
- Recall that on a simplicial surface, we can discretize the Hodge star via diagonal matrices storing volume ratios (given by formulas below)
- **Q:** What happens if our mesh is no longer flat?



$$\frac{1}{A_{ijk}} = \frac{1}{\sqrt{s(s-\ell_{ij})(s-\ell_{jk})(s-\ell_{ki})}}$$

$$s = \tfrac{1}{2}(\ell_{ij} + \ell_{jk} + \ell_{ki})$$





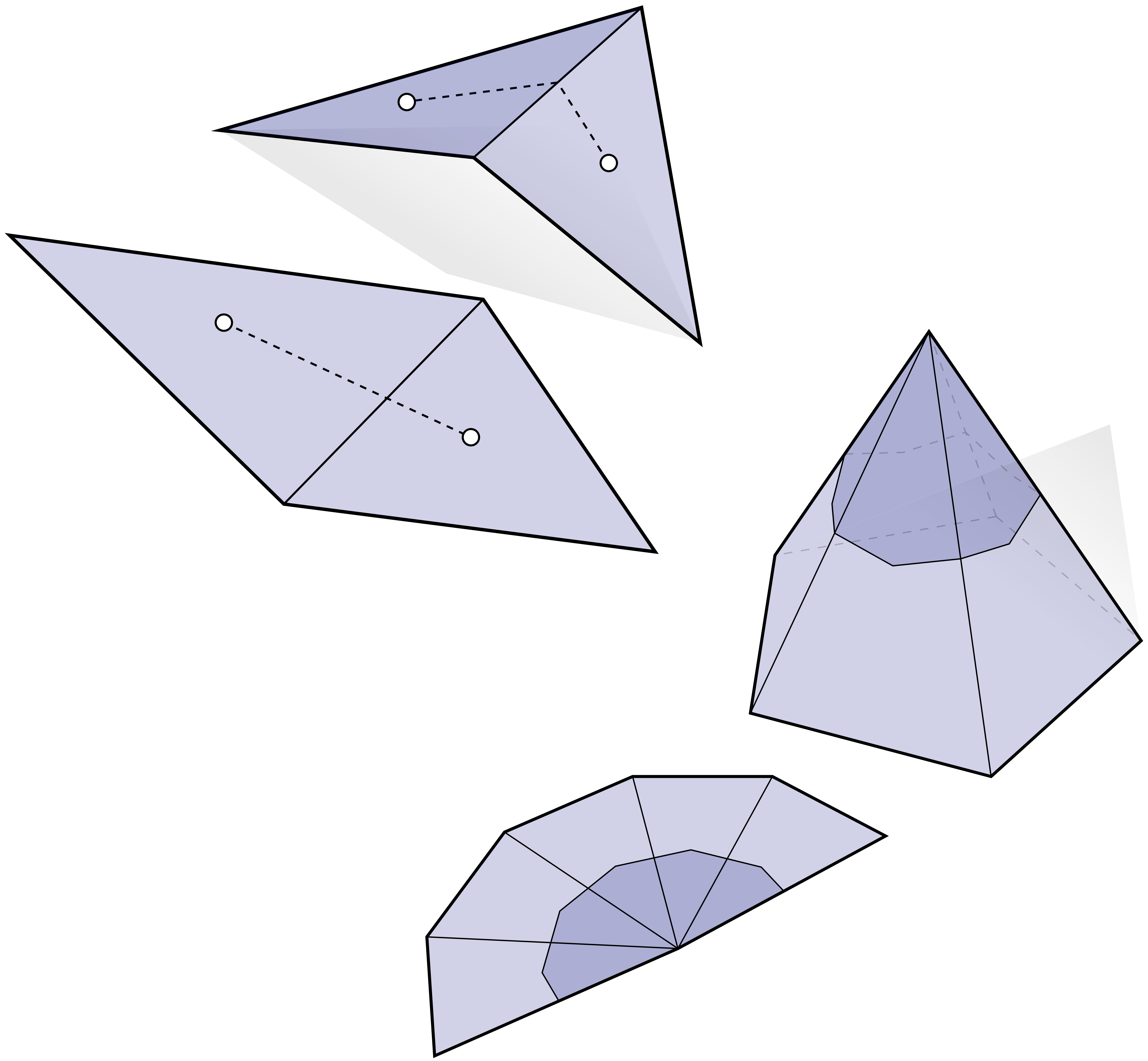
$$\frac{A_{\text{dual}}}{1} = \frac{1}{8} \sum_{ijk \in F} (\ell_{ij}^2 \cot \alpha_k^{jk} + \ell_{ik}^2 \cot \alpha_j^{ki})$$

Diagonal Hodge Star on a Curved Surface

- **A:** Nothing changes! As long we have a discrete immersion, we can still apply the same formulas—which depend only on *primal lengths* and *interior angles*

- In the case of the 1-form Hodge star, we are effectively taking a length ratio involving the dual distance "along" the surface

- Importantly, this means that our DEC operators are purely *intrinsic*: depends only on data that can be measured by an observer "crawling along the surface"



Discrete Laplace-Beltrami Operator

- As a result, we can immediately build discrete differential operators for *curved* surfaces by just composing our existing discrete exterior derivative and discrete Hodge star operators
- For instance, the Laplacian on 0-forms now becomes something known as the *Laplace-Beltrami operator* (which we'll talk **much** more about later!)
- Using our expressions for the discrete Hodge star, can write the discrete Laplace-Beltrami operator via the famous *cotan formula:*

$$\boxed{(\Delta u)_i = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)}$$

Recovery of Discrete Surfaces

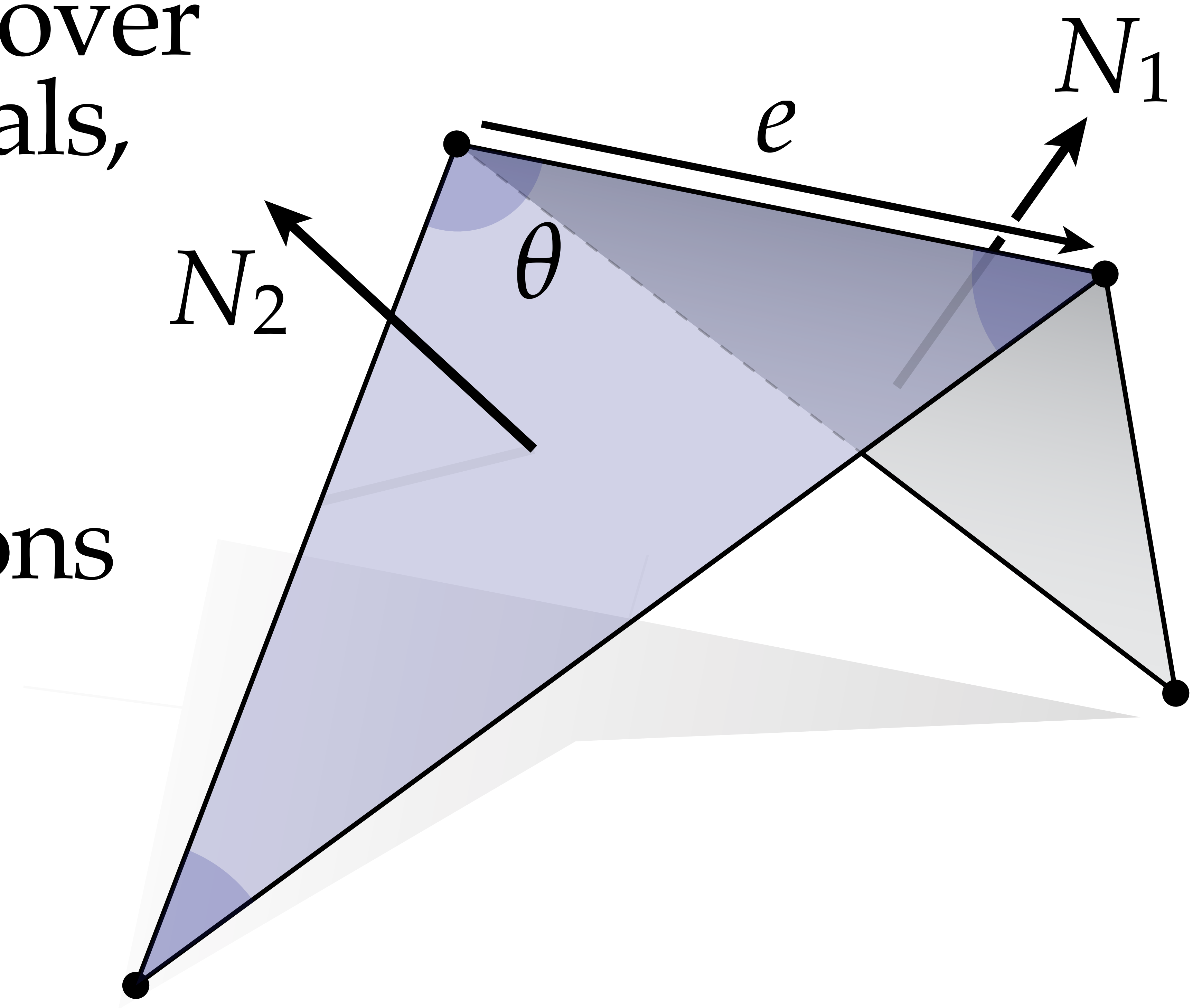
Recovery of Discrete Surfaces

- In a variety of situations, we've seen that shape can be recovered (up to rigid motions) via "indirect" measurements (curvatures, *etc.*)
  - **Plane curves** can be recovered from their curvature (exterior angle)
  - **Space curves** can be recovered from their curvature and torsion
  - **Smooth surfaces** can be recovered from 1st & 2nd fundamental form
  - **Convex surfaces** can be recovered from Riemannian metric…

Q: What data is sufficient to describe a *discrete* surface?

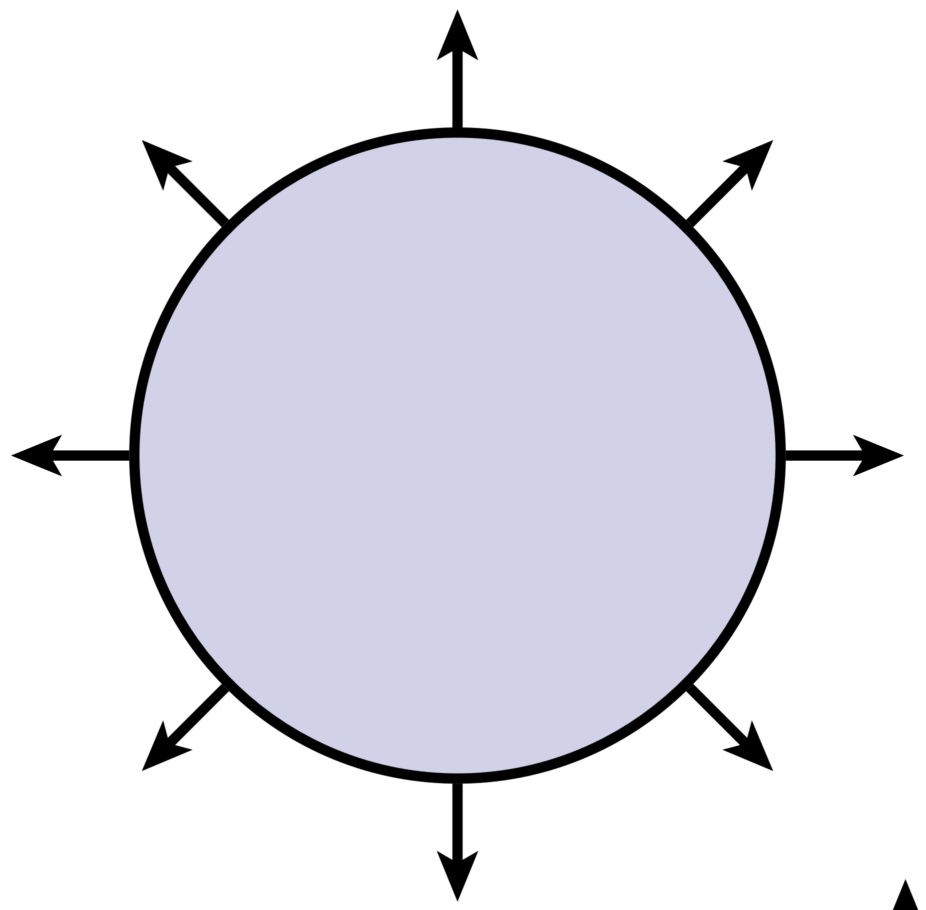
Surface Recovery from Discrete Gauss Map

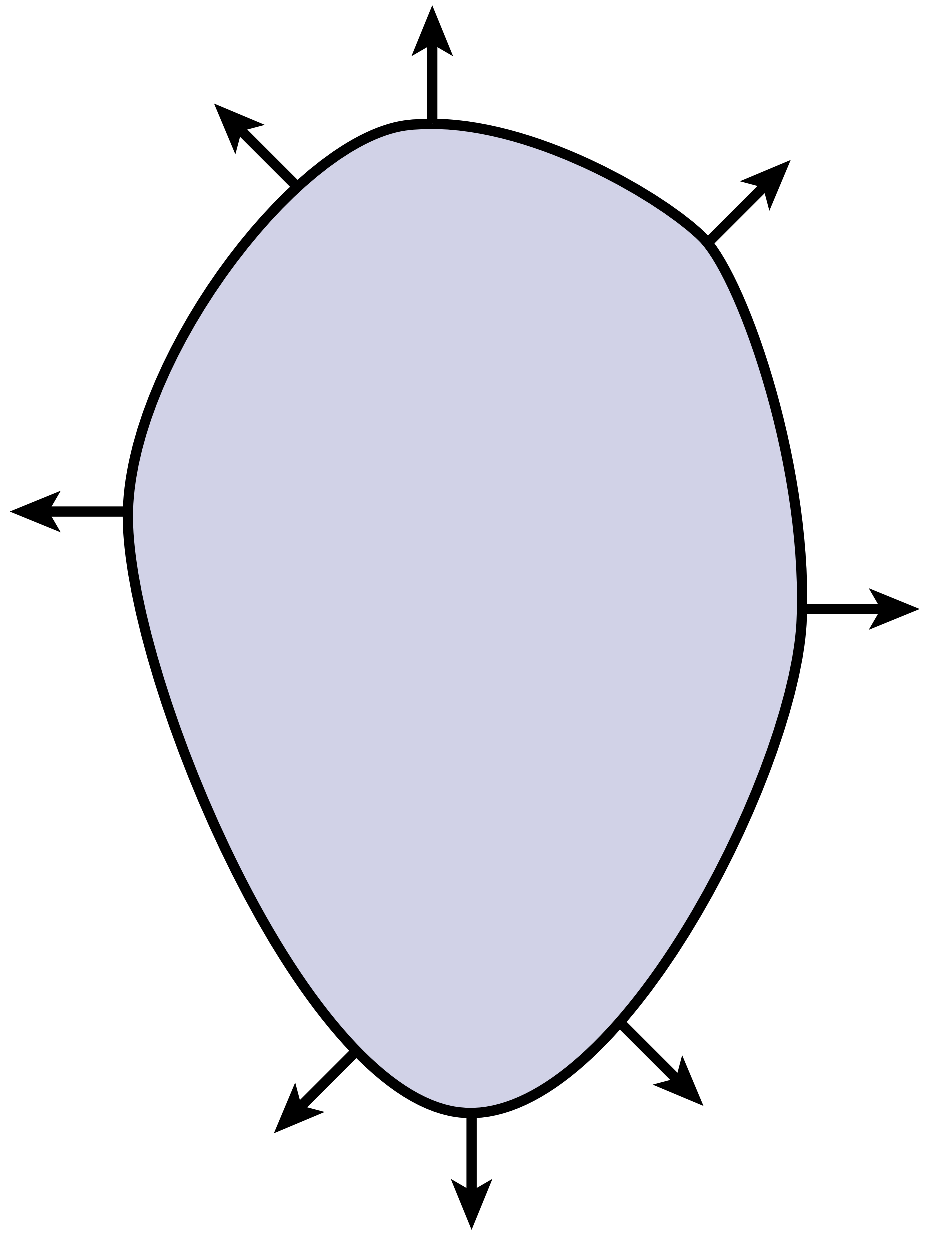
- **Q:** Given only discrete Gauss map, can we recover the immersion? (*I.e.*, given only triangle normals, can we get vertex positions?)
- **A:** Yes! Basic recipe:
  - Cross product of normals gives edge directions
  - Dot product of edges gives interior angles
  - Angles + normals give triangles up to scale; normals give orientation
  - Build triangles one-by-one and "glue" together
- **Q:** Does this recipe *always* work?



Shape Recovery from Smooth Gauss Map?

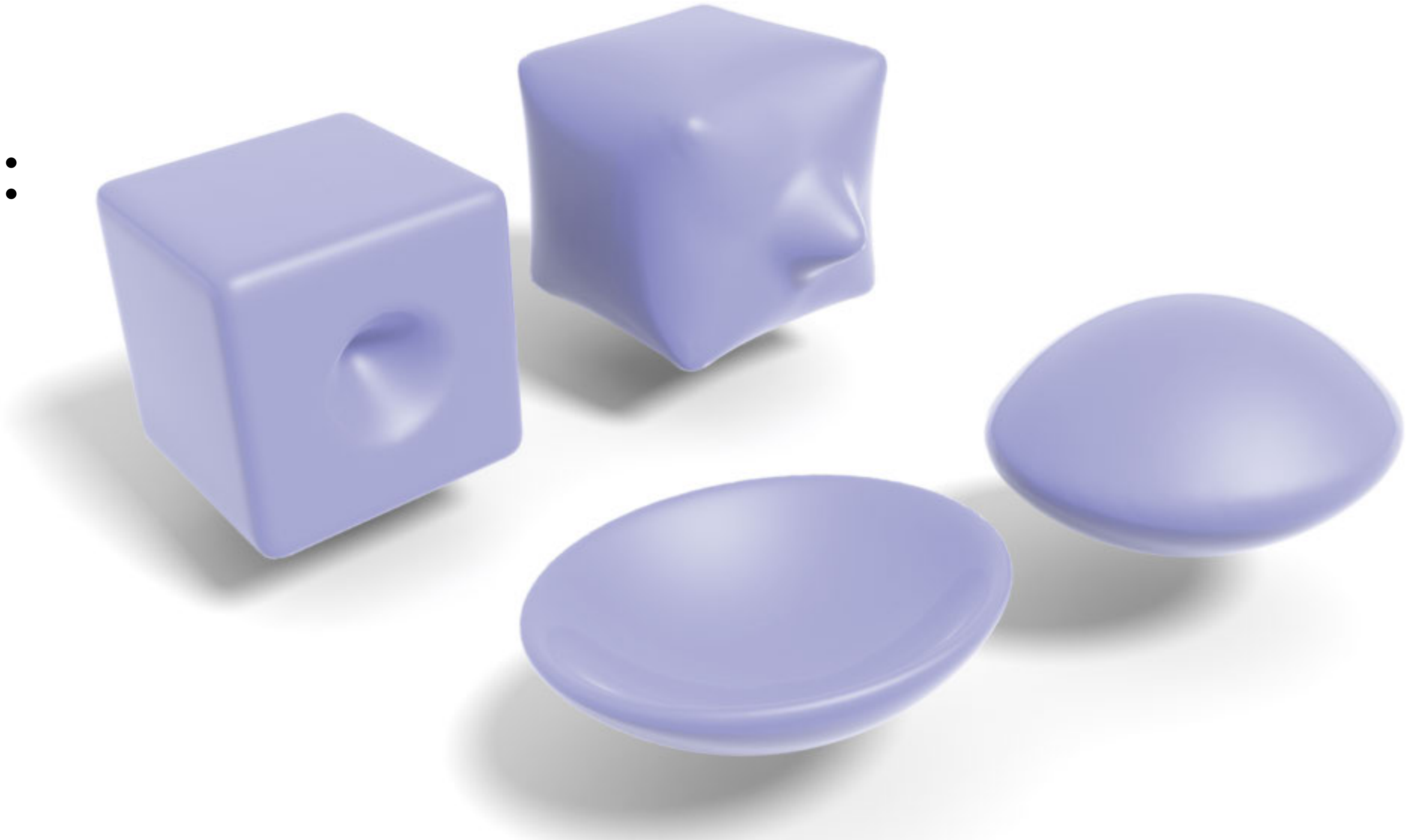
- **Q:** Is it *strange* that we can recover a discrete surface from Gauss map? Can we do something similar in the smooth setting?
- Consider a simpler case: Gauss map on a *curve*
- $N(s) := (\cos(s), \sin(s))$
- **Problem:** unless we know curve is arc-length parameterized, N is the Gauss map of *any* convex curve! Need additional data (parameterization)
- Similar story for convex discrete curves, or convex smooth surfaces
- So why don't we need additional data for a *discrete surface*?





Recovery from Metric

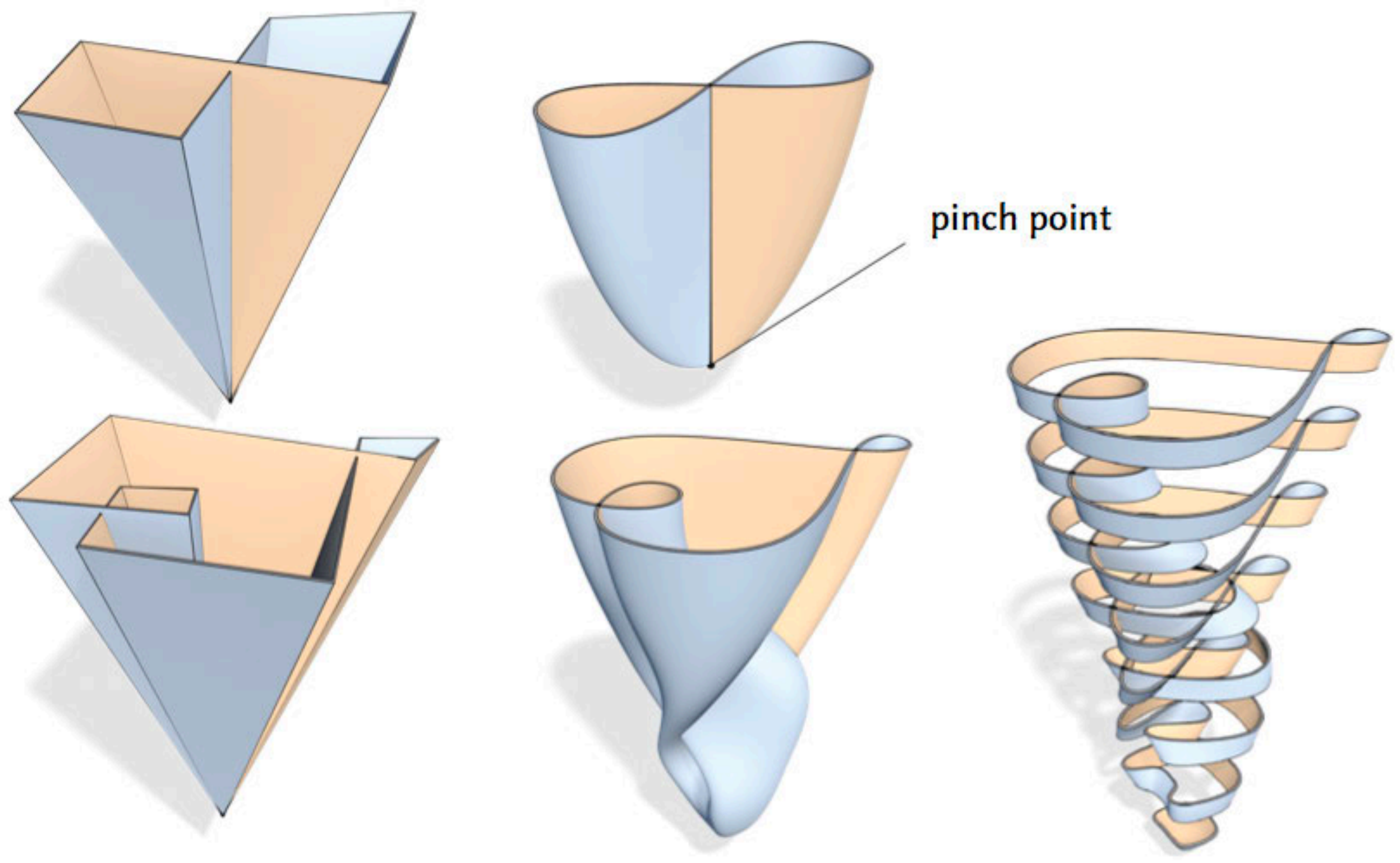
- **Theorem.** (Cohn-Vossen) Smooth convex surface is uniquely determined (up to rigid motions) by its *Riemannian metric*.
- **Theorem.** (Alexandrov-Connelly) A convex polyhedron is uniquely determined by its *edge lengths*.
- Not always true in nonconvex case:



Recovery From Discrete Metric

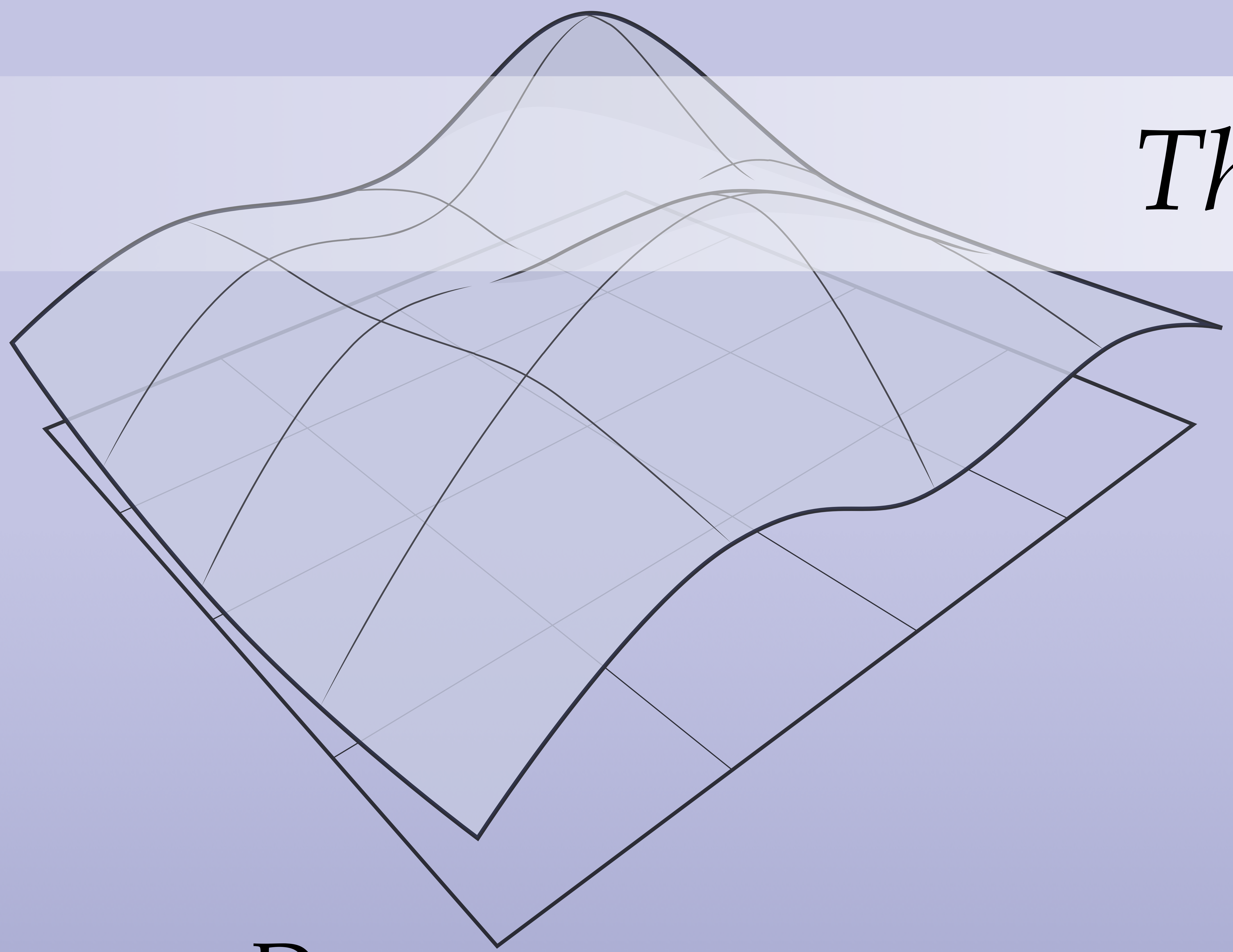
Algorithm: Shape from Metric

- Recent algorithm (*approximately!*) recovers surface from edge lengths
  - Chern et al, "Shape from Metric" (2018)
  - Nice read if you want to get deeper into discrete surfaces: discrete immersion, discrete spin structure…



http://page.math.tu-berlin.de/~chern/projects/ShapeFromMetric/

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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