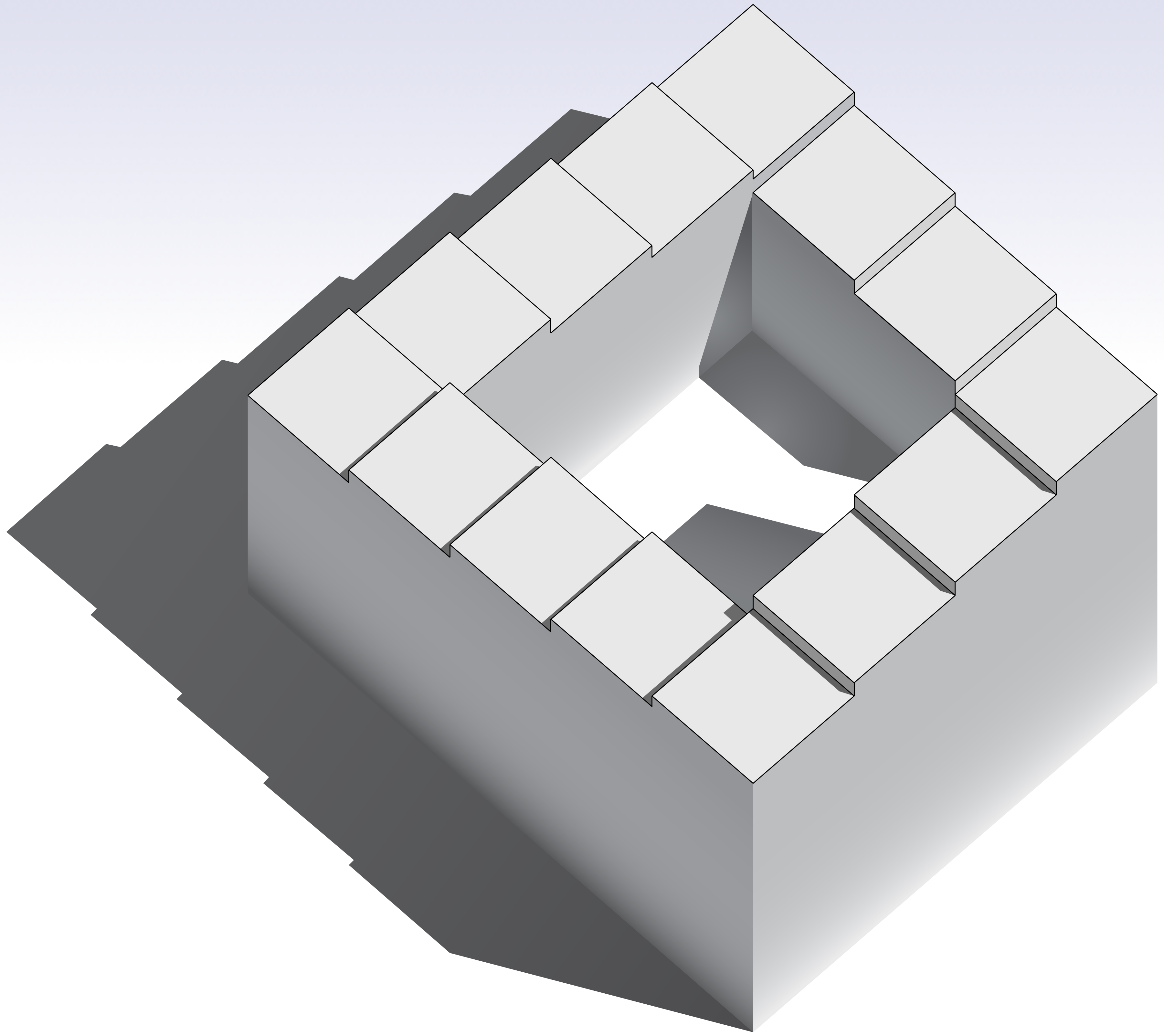
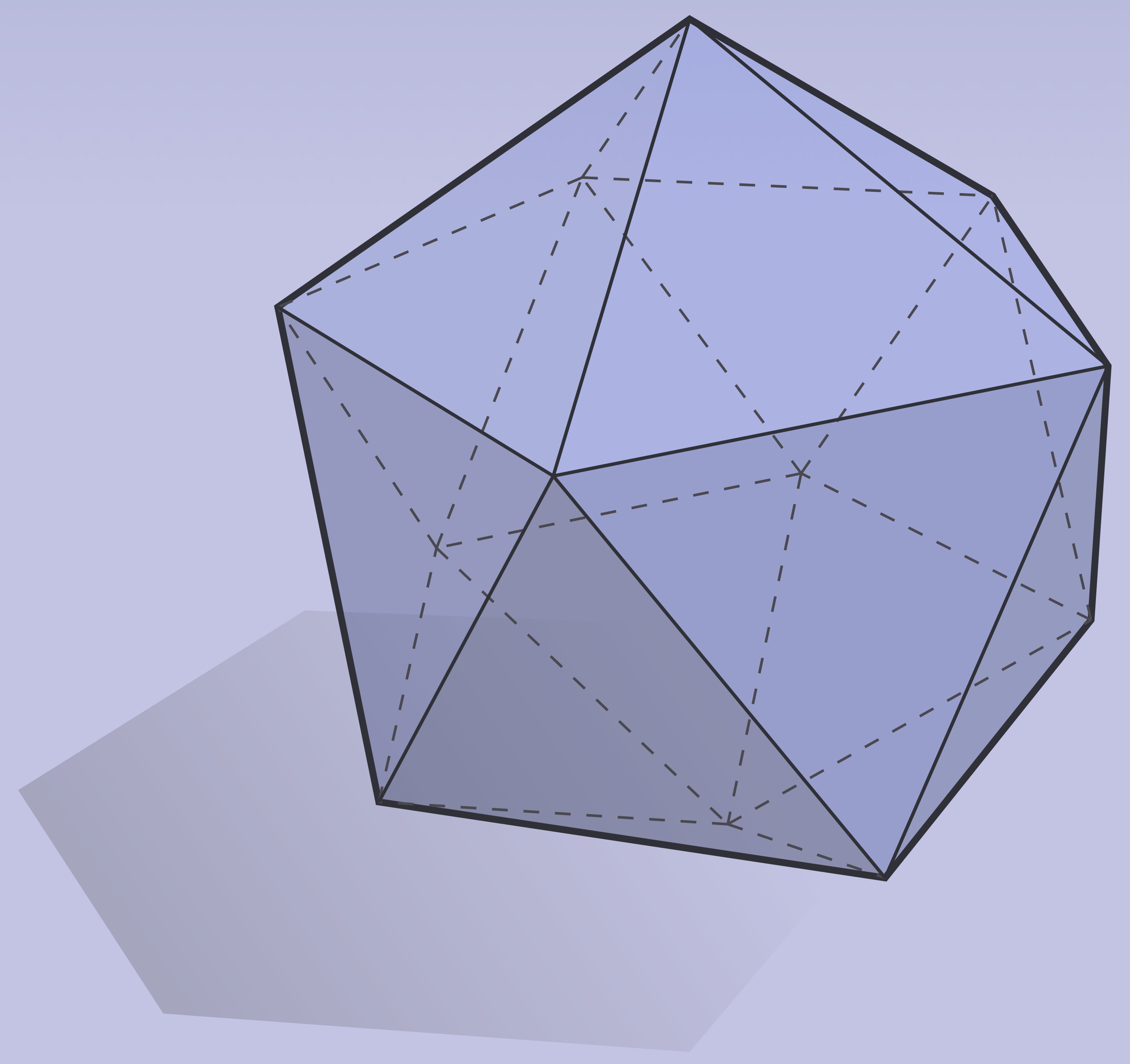


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

LECTURE 16:
DISCRETE CURVATURE II (VARIATIONAL)

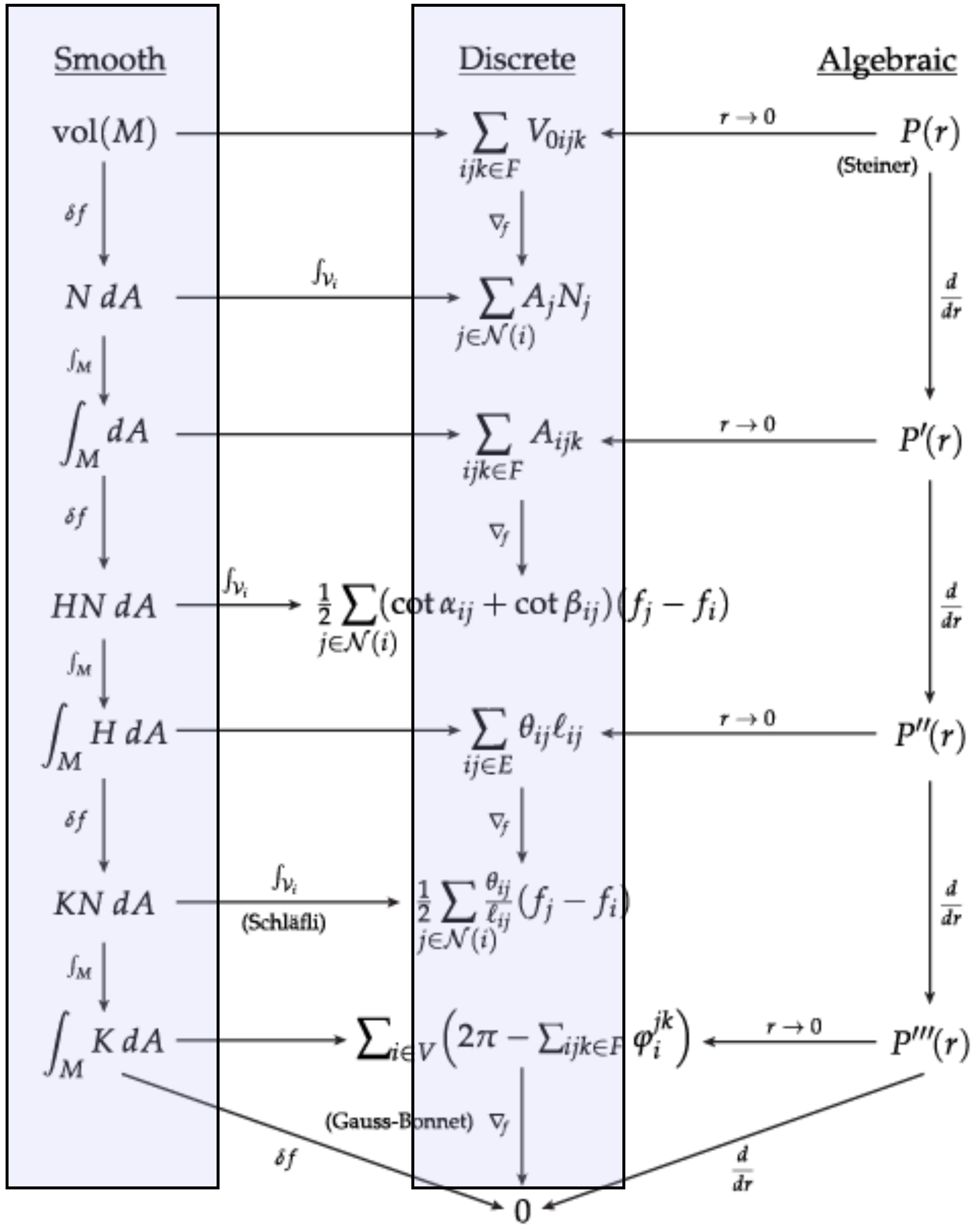


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017

A Unified Picture of Discrete Curvature

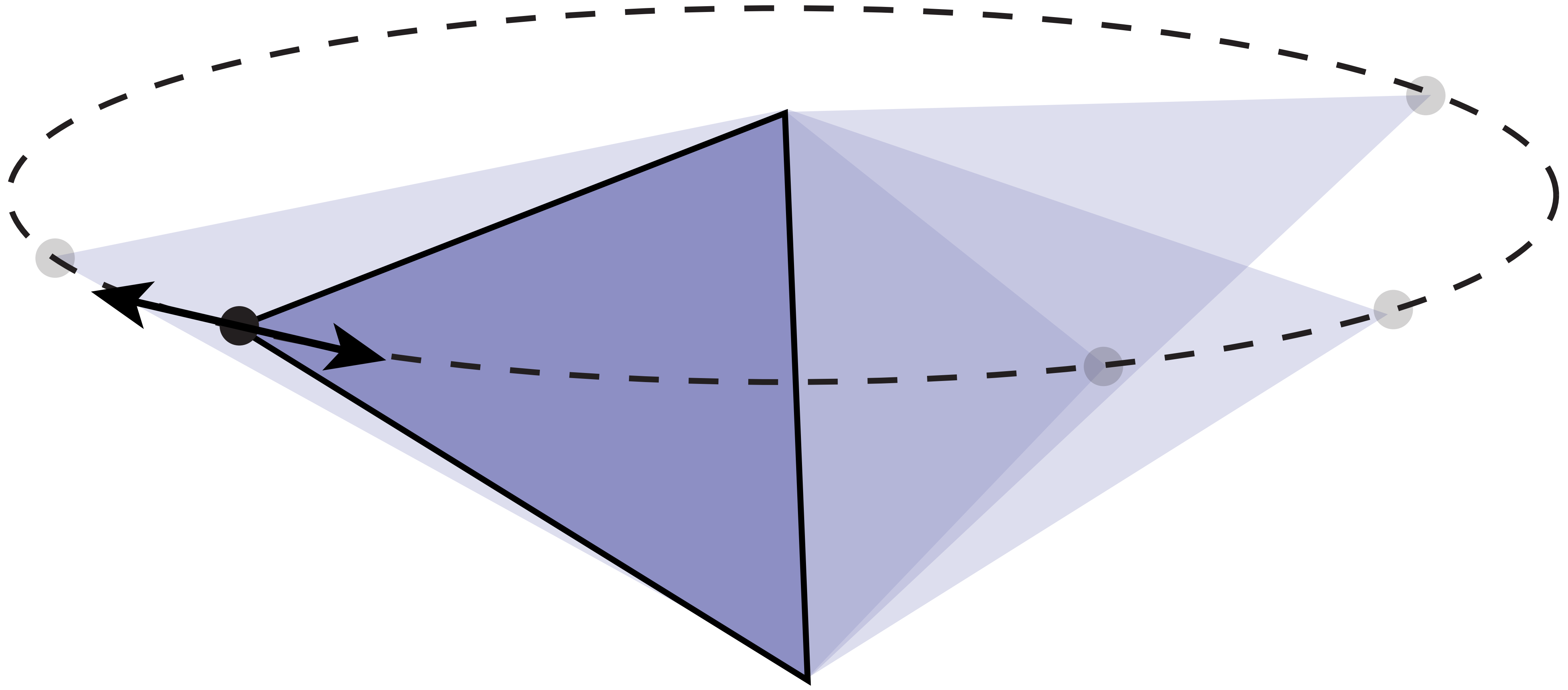
- By making some connections between smooth and discrete surfaces, we get a unified picture of many different discrete curvatures scattered throughout the literature
- To tell the full story we'll need a few pieces:
  - **geometric derivatives**
  - **Steiner polynomials**
  - **sequence of curvature variations**
  - **assorted theorems** (Gauss-Bonnet, Schläfli, $\Delta f = 2HN$)
- Start with *integral* viewpoint (1st lecture), then cover <u>*variational* viewpoint (2nd lecture).</u>



Discrete Geometric Derivatives

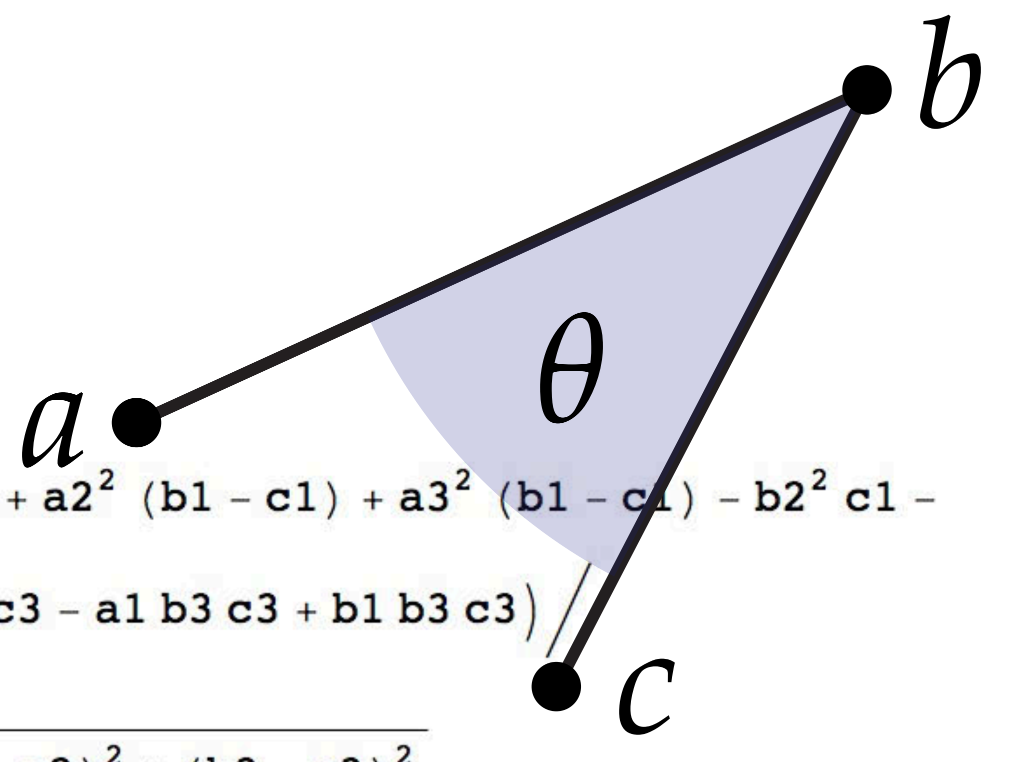
Discrete Geometric Derivatives

- Practical technique for calculating derivatives of discrete geometric quantities
- Basic question: *how does one geometric quantity change with respect to another?*
- *E.g.*, what's the gradient of triangle area with respect to the position of one of its vertices?
- **Don't** just grind out partial derivatives!
- **Do** follow a simple geometric recipe:
  - First, in which **direction** does the quantity change quickest?
  - Second, what's the **magnitude** of this change?
  - Together, direction & magnitude give us the gradient vector



Dangers of Partial Derivatives

- Why not just take derivatives *"the usual way?"*
  - usually takes way more work!
  - can lead to expressions that are
    - inefficient
    - numerically unstable
    - hard to interpret
- **Example:** gradient of angle between two segments (*b*,*a*), (*c*,*a*) w.r.t. coordinates of point *a*




```
In[58]:= a = {a1, a2, a3};
         b = {b1, b2, b3};
         c = {c1, c2, c3};
         θ = ArcCos[(a - b).(c - b) / (Sqrt[(a - b).(a - b)] Sqrt[(c - b).(c - b)])];
         FullSimplify[{∂a1 θ, ∂a2 θ, ∂a3 θ}]

Out[62]= {(a1 b2^2 + a1 b3^2 - a2 b2 (a1 + b1 - 2 c1) - a3 b3 (a1 + b1 - 2 c1) + a2^2 (b1 - c1) + a3^2 (b1 - c1) - b2^2 c1 -
    b3^2 c1 + a2 (a1 - b1) c2 - a1 b2 c2 + b1 b2 c2 + a3 (a1 - b1) c3 - a1 b3 c3 + b1 b3 c3) /
   (((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2)^(3/2) Sqrt[(b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2]
    Sqrt[1 - ((a1 - b1) (-b1 + c1) + (a2 - b2) (-b2 + c2) + (a3 - b3) (-b3 + c3))^2 /
      (((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2) ((b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2))]),
  (a3^2 b2 - a3 b2 b3 + b1 b2 c1 + a1^2 (b2 - c2) - a3^2 c2 - b1^2 c2 + 2 a3 b3 c2 - b3^2 c2 -
    a1 (a2 (b1 - c1) + b2 (b1 + c1) - 2 b1 c2) + a2 (b1 (b1 - c1) - (a3 - b3) (b3 - c3)) - a3 b2 c3 +
    b2 b3 c3) / (((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2)^(3/2) Sqrt[(b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2]
    Sqrt[1 - ((a1 - b1) (-b1 + c1) + (a2 - b2) (-b2 + c2) + (a3 - b3) (-b3 + c3))^2 /
      (((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2) ((b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2))]),
  (b3 (b1 c1 + (a2 - b2) (a2 - c2)) + a3 (b1 (b1 - c1) - (a2 - b2) (b2 - c2)) + a1^2 (b3 - c3) -
    (b1^2 + (a2 - b2)^2) c3 - a1 (a3 (b1 - c1) + b3 (b1 + c1) - 2 b1 c3)) /
   (((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2)^(3/2) Sqrt[(b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2]
    Sqrt[1 - ((a1 - b1) (-b1 + c1) + (a2 - b2) (-b2 + c2) + (a3 - b3) (-b3 + c3))^2 /
      (((a1 - b1)^2 + (a2 - b2)^2 + (a3 - b3)^2) ((b1 - c1)^2 + (b2 - c2)^2 + (b3 - c3)^2))])}
```

Geometric Derivation of Angle Derivative

- Instead of taking partial derivatives, let's break this calculation into two pieces:

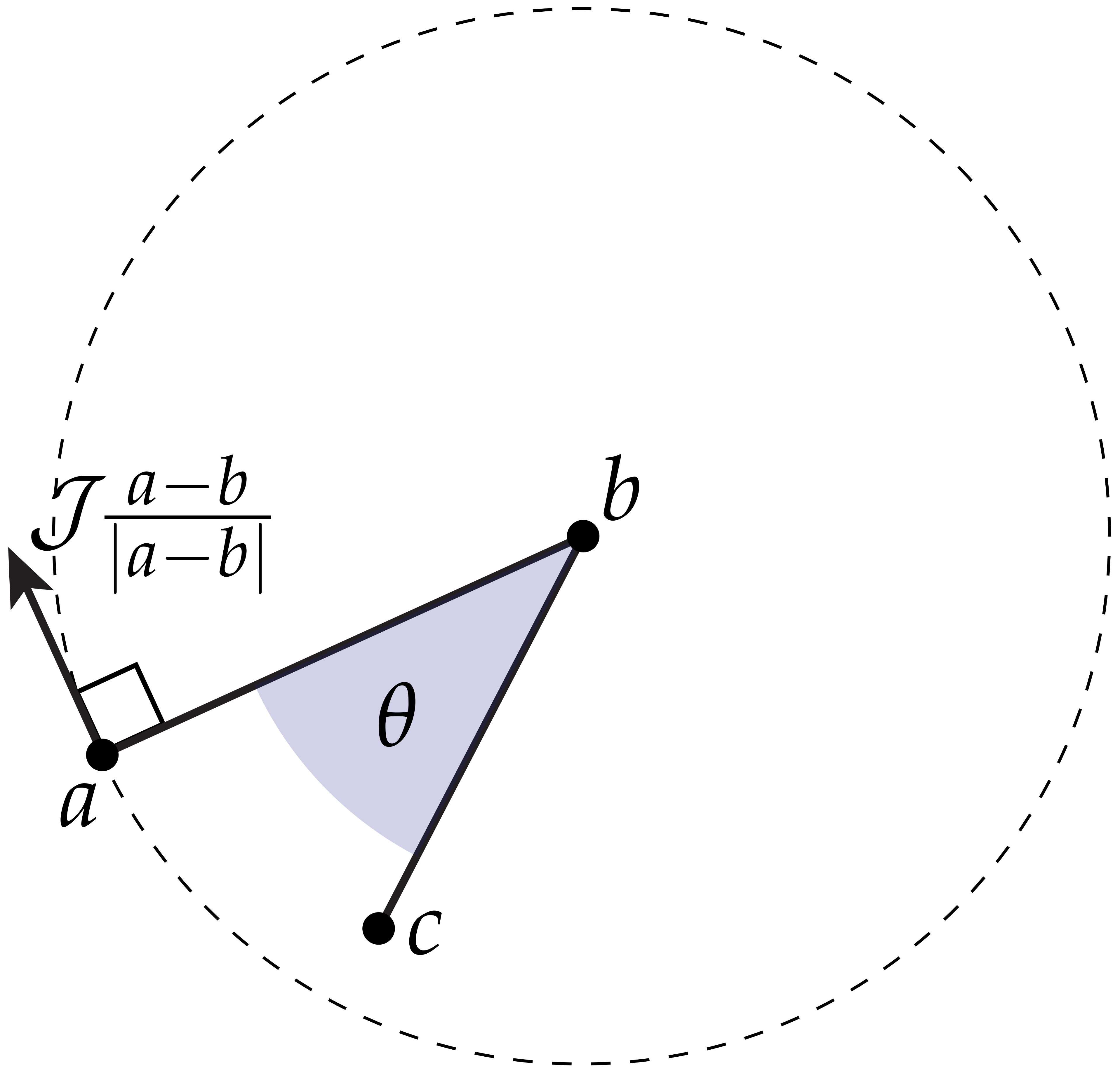
  1. (**Direction**) What direction can we move the point a to most quickly increase the angle θ?

     A: *Orthogonal to the segment ab.*

  2. (**Magnitude**) How much does the angle change if we move in this direction?

     A: *Moving around a whole circle changes the angle by 2π over a distance 2πr, where r = |b-a|. Hence, the instantaneous change is* 1/|*b-a*|.

- Multiplying the unit direction by the magnitude yields a final gradient expression.

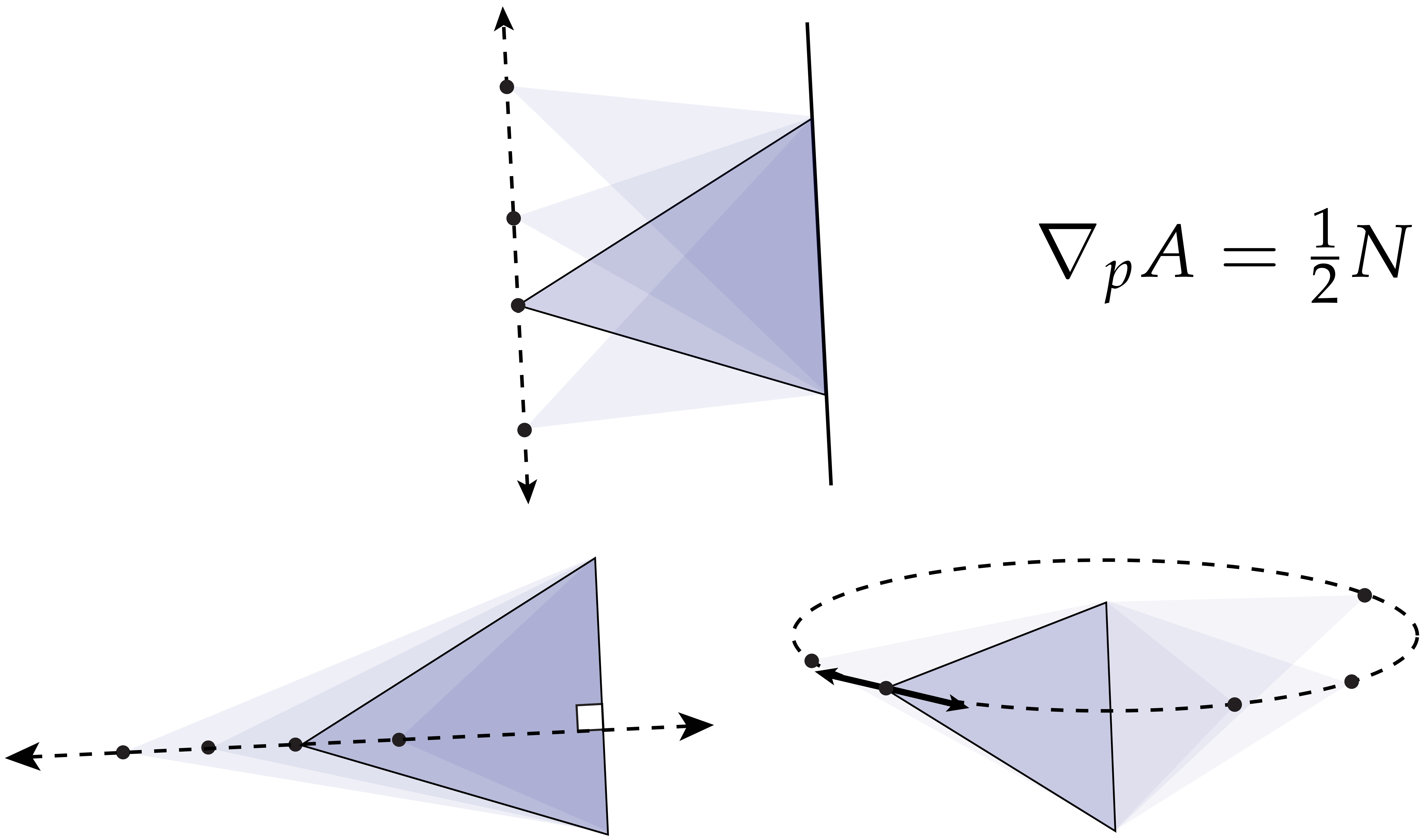


$$\nabla_a \theta = \mathcal{J}\frac{a-b}{|a-b|}^{2}$$

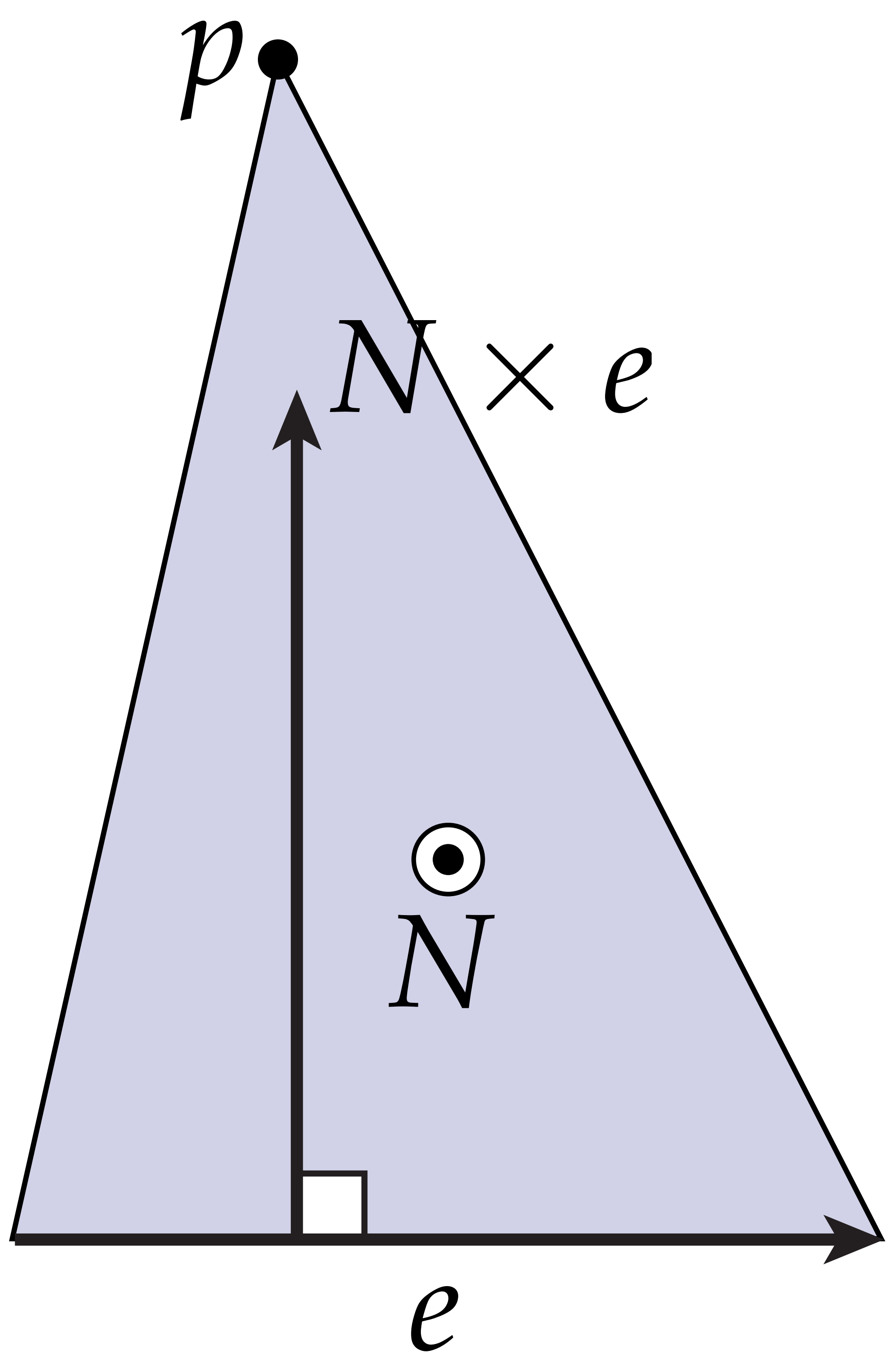
Gradient of Triangle Area

Q: What's the gradient of triangle area with respect to one of its vertices p?

A: Can express via its unit normal N and vector e along edge opposite p:

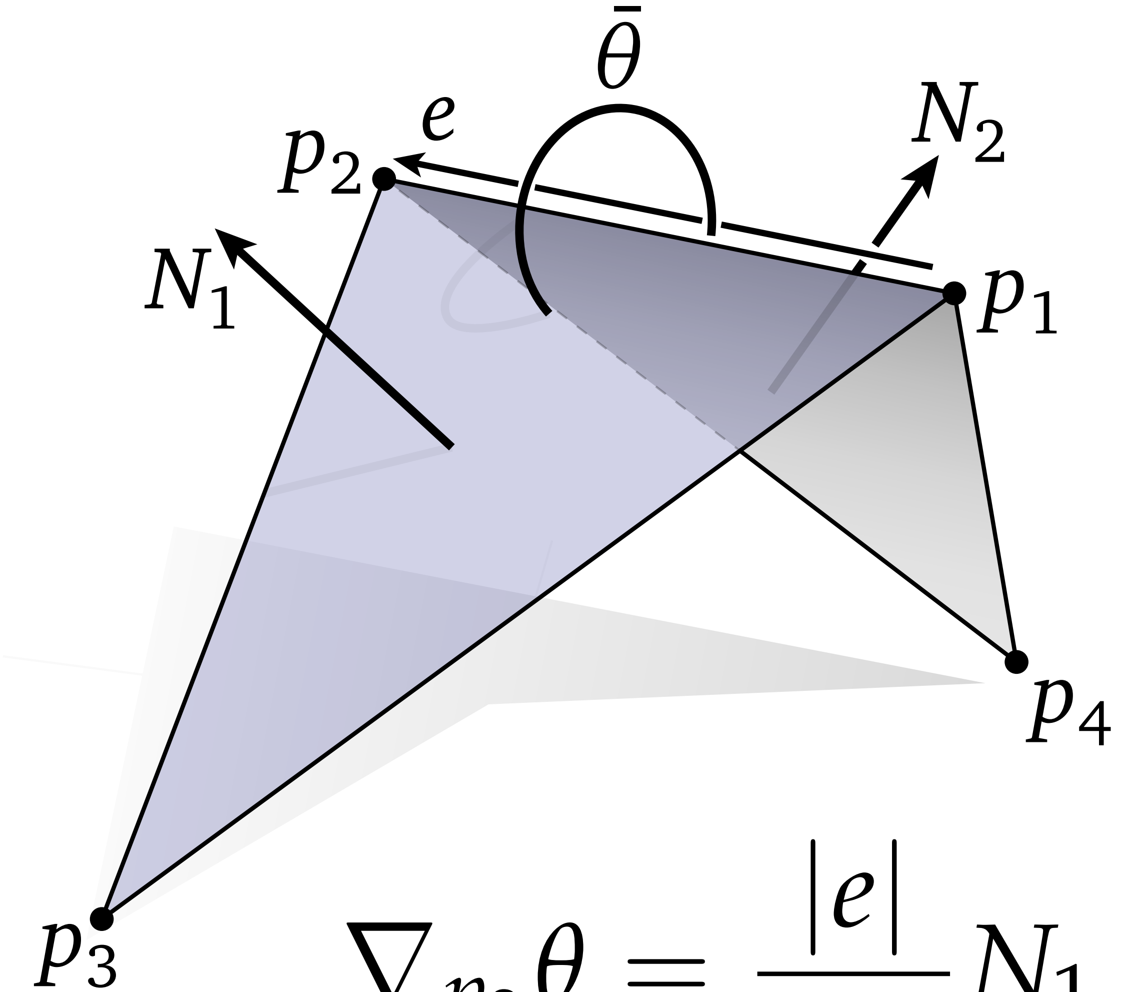


$$\nabla_p A = \frac{1}{2} N \times e$$

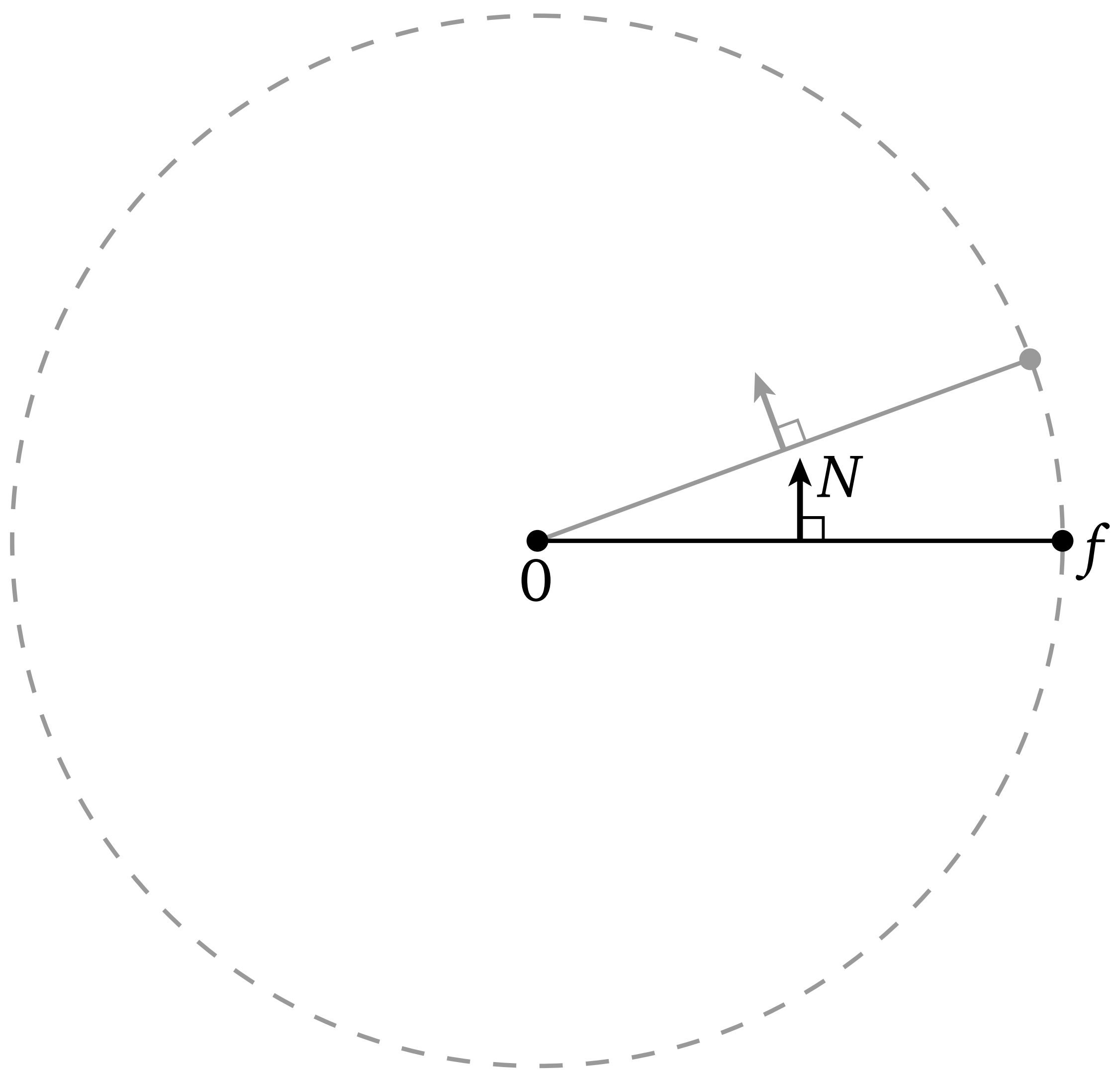


Geometric Derivation

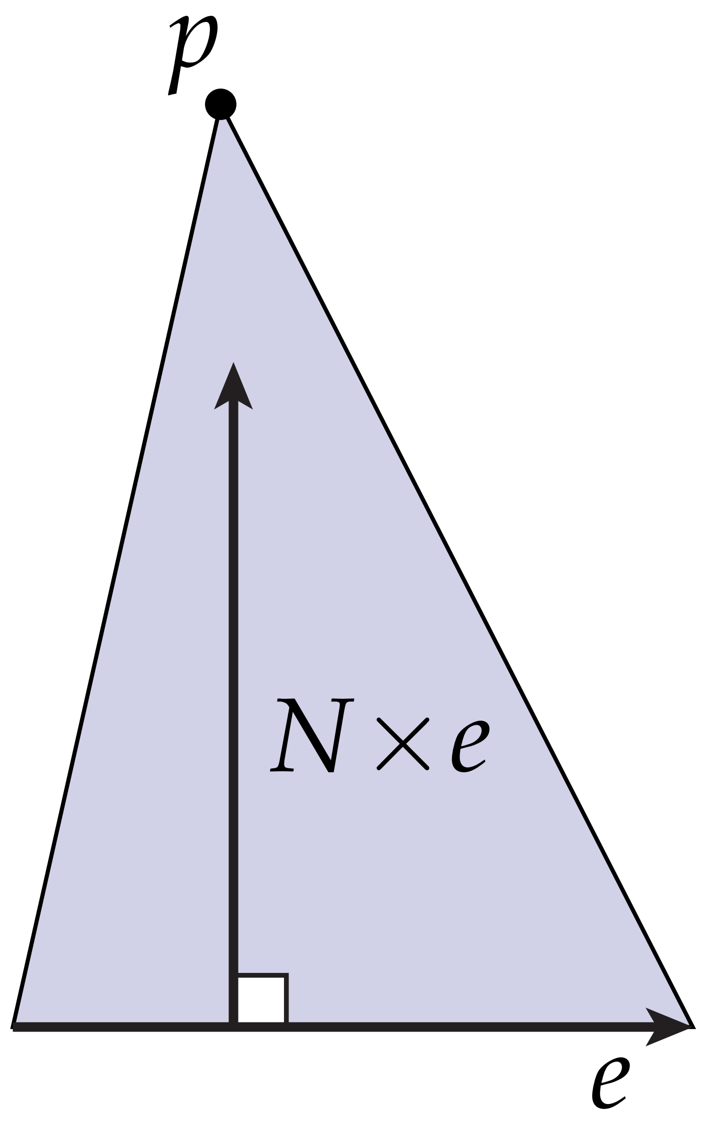
- In general, can lead to some pretty slick expressions (give it a try!)



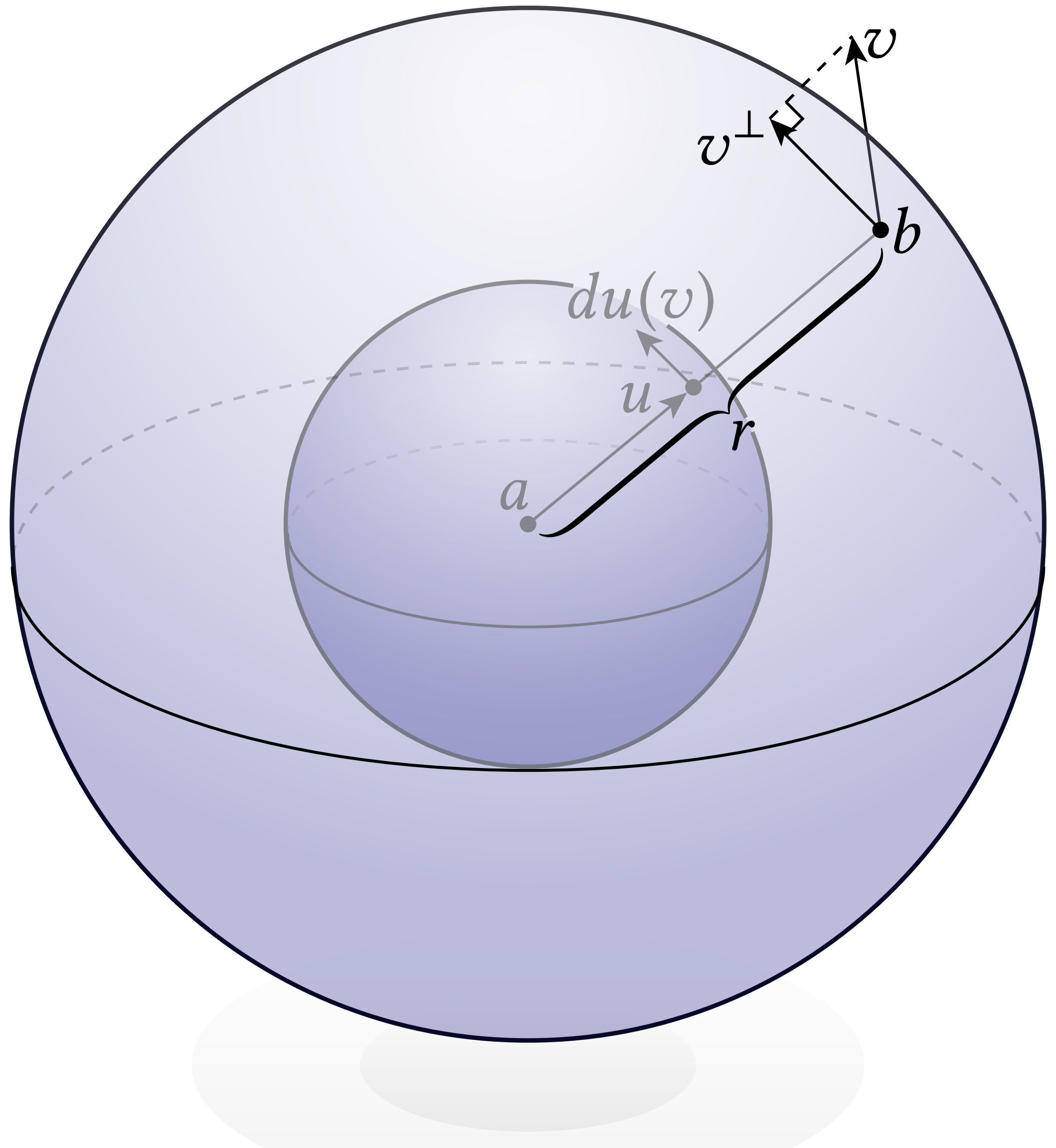
$$\nabla_{p_3}\theta = \frac{|e|}{2A_1}N_1$$



$$d_{f_i}N(X) = \frac{\langle N, X\rangle}{2A} e_i \times N$$



$$\nabla_p A = \tfrac{1}{2} N \times e$$



$$du(v) = \frac{v - \langle v, b-a\rangle (b-a)}{|b-a|^3}$$

Aside: Automatic Differentiation

- Geometric approach to differentiation greatly simplifies "small pieces" (gradient of a particular, angle, length, area, volume, …)
- For larger expressions that combine many small pieces, approach of *automatic differentiation* is extremely useful*
- Basically does nothing more than automate repeated application of chain rule
- Simplest implementation: use pair that store both a **value** and its **derivative**; operations on these tuples apply operation & chain rule

Example.

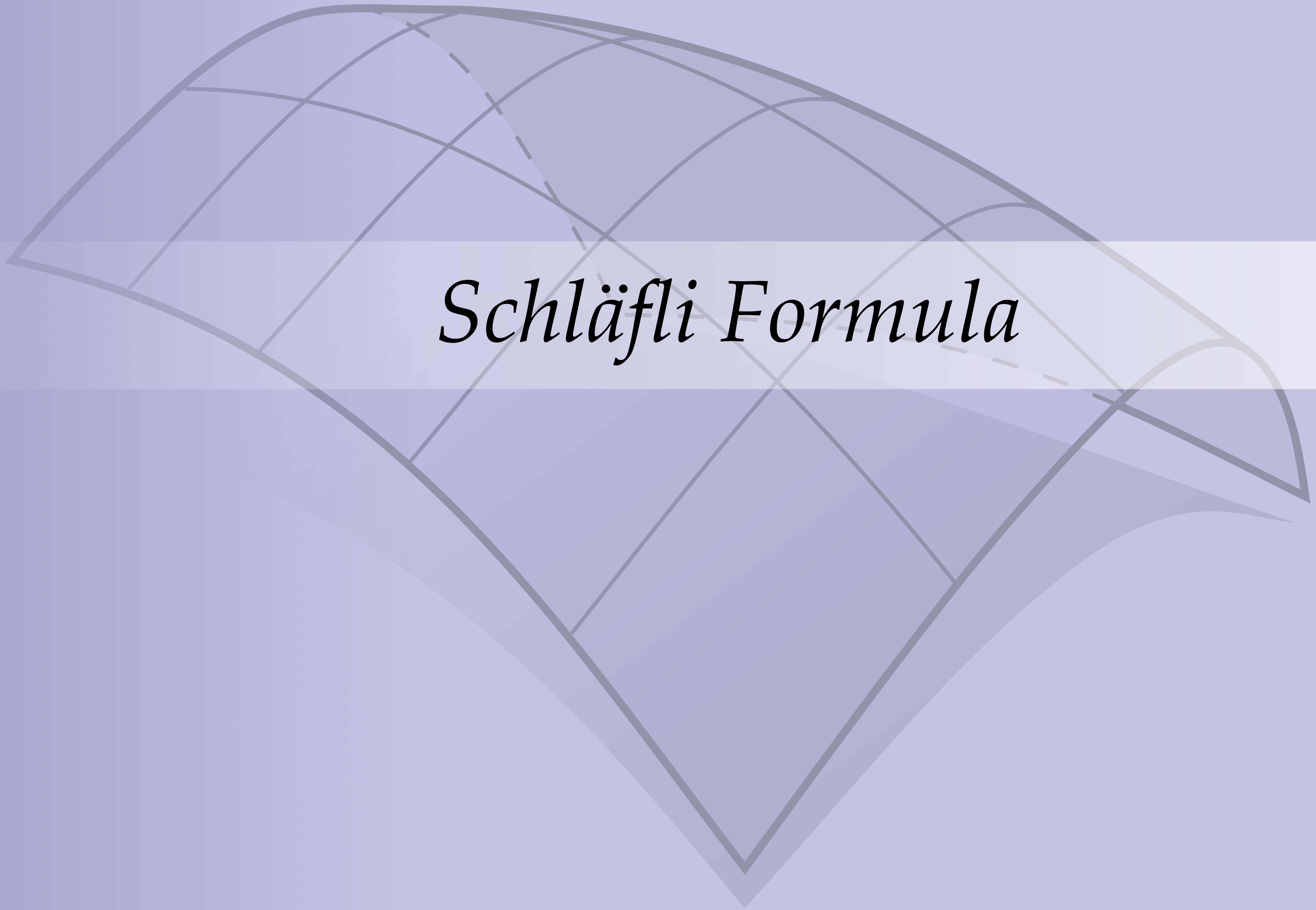
```
// define how multiplication and sine
// operate on (value,derivative) pairs
// (usually done by an existing library)
(a,a')*(b,b') := (a*b,a*b'+b*a')
sin((a,a')) := (sin(a),a'*cos(a))

// to evaluate a function and its
// derivative at a point, we first
// construct a pair corresponding to the
// identity function f(x) = x at the
// desired evaluation point
x = (5,1) // derivative of x w.r.t x is 1

// now all we have to do is type a
// function as usual, and it will yield
// the correct value/derivative pair
fx = sin(x*x) // (-0.132352, 9.91203)
```

*More recently known as *backpropagation*

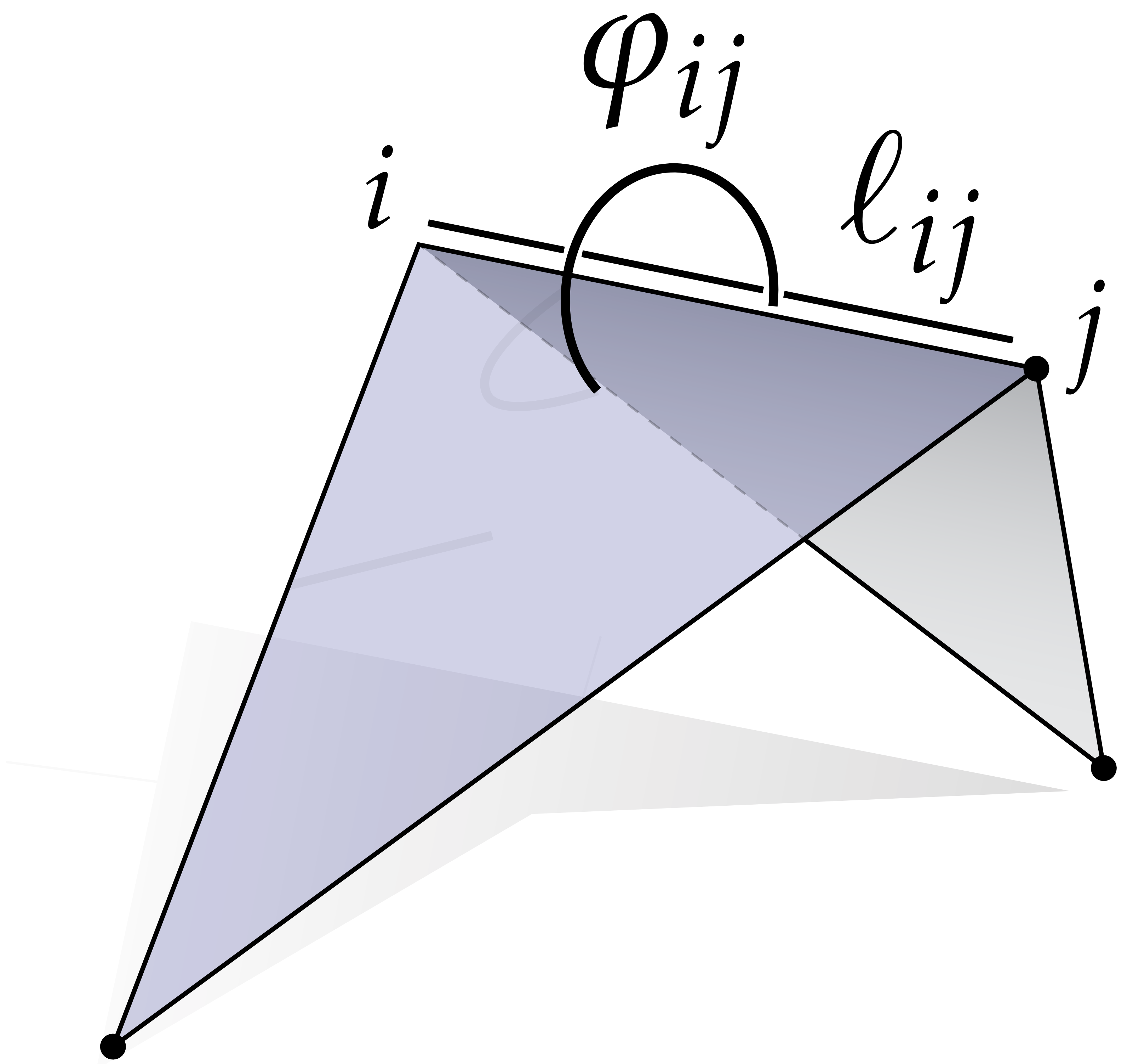
Schläfli Formula



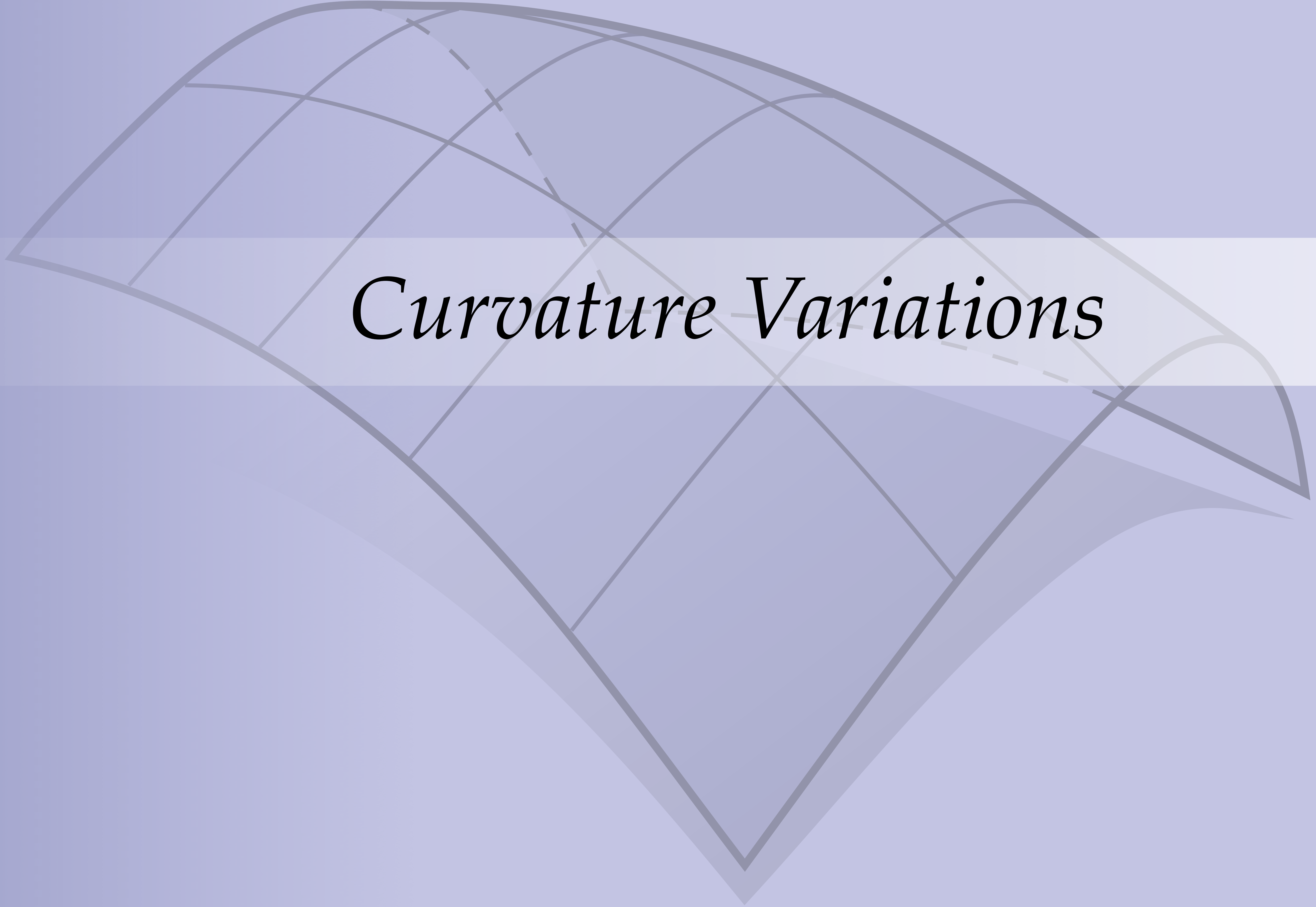
Schläfli Formula

- Consider a closed polyhedron in R^3 with edge lengths l_{ij} and dihedral angles φ_{ij}. Then for any motion of the vertices,

$$\boxed{\sum_{ij \in E} \ell_{ij} \tfrac{d}{dt} \varphi_{ij} = 0}$$



Curvature Variations



Sequence of Variations (Smooth)

For a smooth surface $f : M \longrightarrow R^3$ (without boundary), let

$$\text{volume}(f) := \frac{1}{3}\int_M N \cdot f \, dA \qquad \text{mean}(f) := \int_M H \, dA$$

$$\text{area}(f) := \int_M dA \qquad \text{Gauss}(f) := \int_M K \, dA = 2\pi\chi$$

How can we move the surface so that each of these quantities changes as quickly as possible? Remarkably enough…

$$\delta\, \text{volume}(f) = 2N$$

$$\delta\, \text{area}(f) = 2HN$$

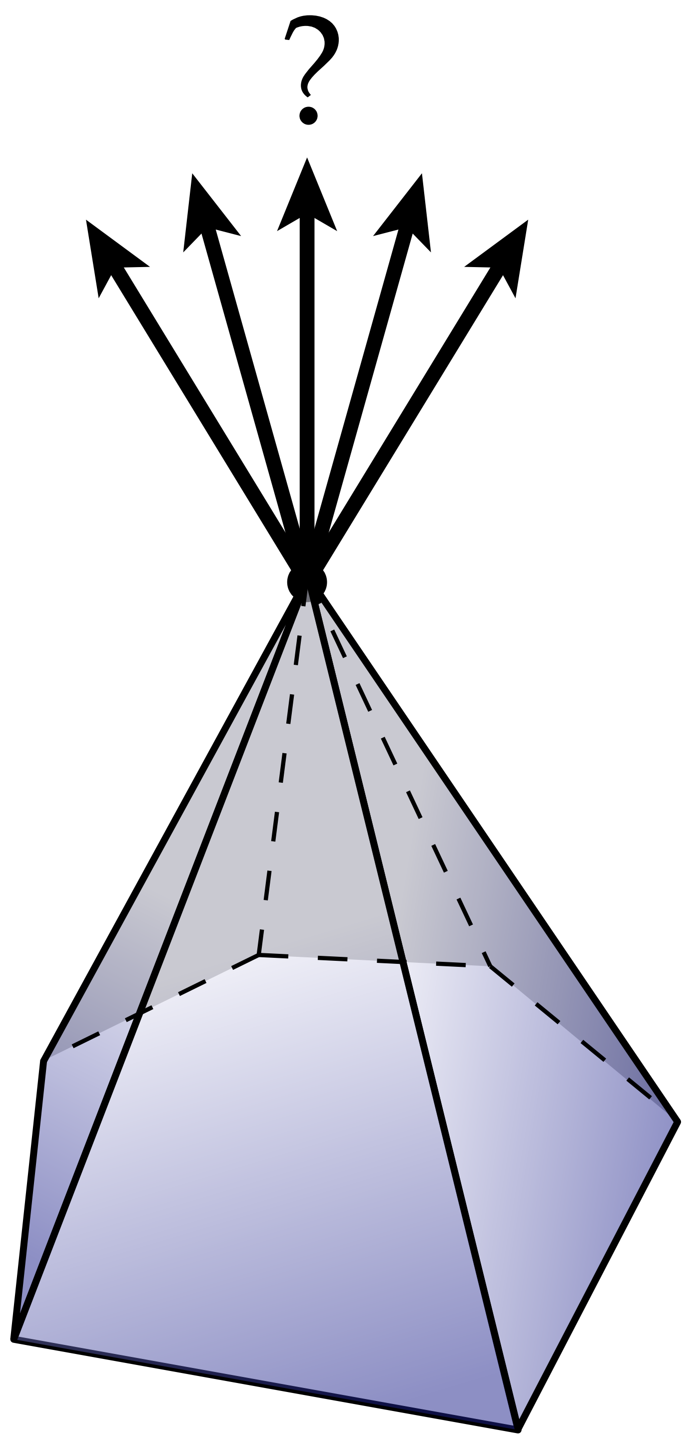
$$\delta\, \text{mean}(f) = 2KN$$

$$\delta\, \text{Gauss}(f) = 0$$

$$\boxed{\text{volume} \xrightarrow{\delta f} \text{area} \xrightarrow{\delta f} \text{mean} \xrightarrow{\delta f} \text{Gauss} \xrightarrow{\delta f} 0}$$

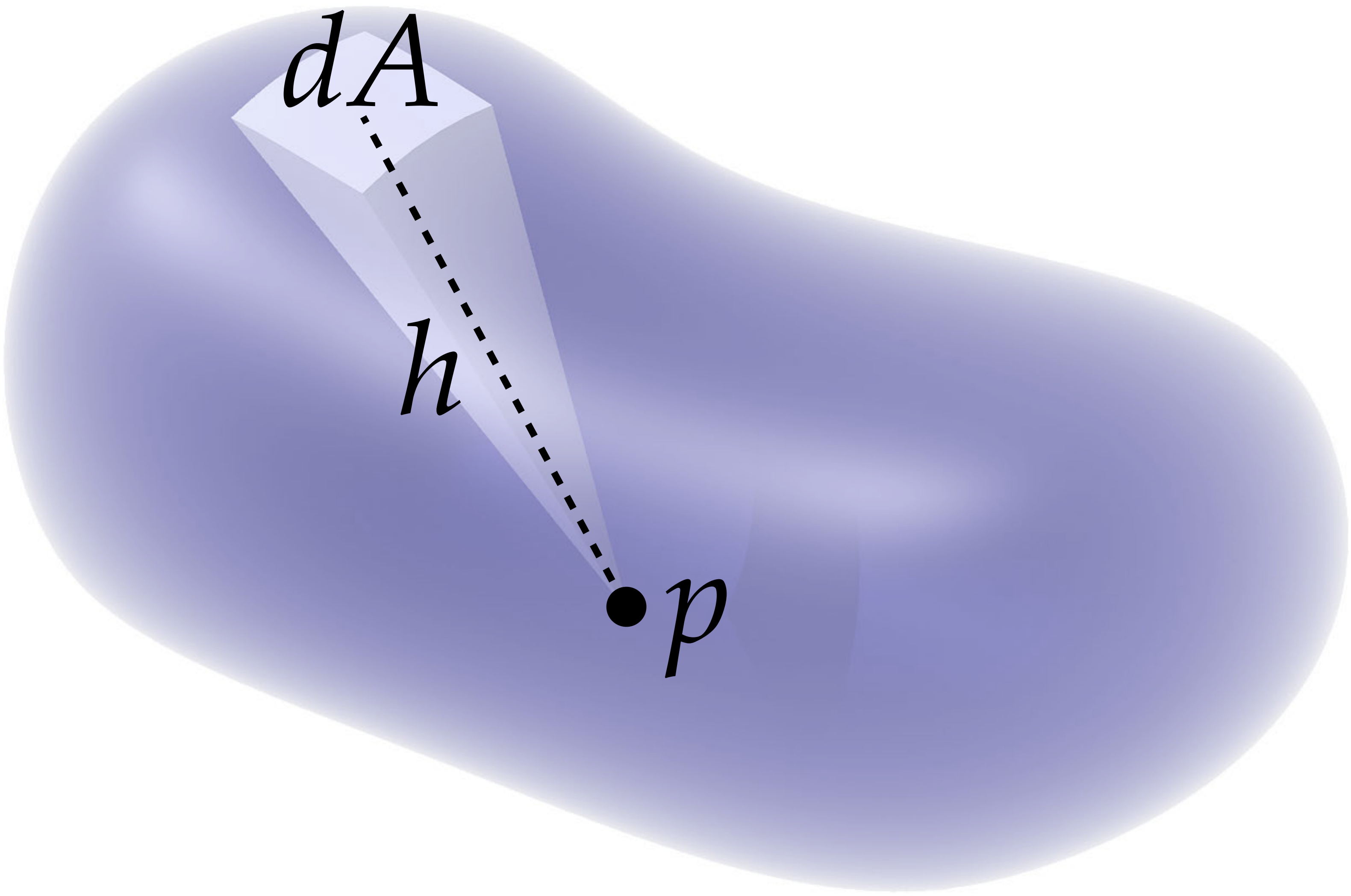
Discrete Normal via Volume Variation

- Recall that we still don't have a clear definition for discrete normals at *vertices*, where the surface is not differentiable
- However, in the smooth setting we know that the normal is equal to (half) the volume gradient
- **Idea:** Since volume is perfectly well-defined for a discrete surface, why not use volume gradient to *define* vertex normals?
- Now just need to calculate the gradient of volume with respect to motion of one of the vertices, which we can do using our "geometric approach"…



Volume Enclosed by a Smooth Surface

- What's the volume enclosed by a *smooth* surface f?
- One way: pick any point p, integrate volume of "infinitesimal pyramids" over the surface
- For a pyramid with base area b and height h, the volume is $V = bh/3$ (no matter what shape the base is)
- For our infinitesimal pyramid, the height is the distance from the surface f to the point p along the normal direction: $h = (f - p) \cdot N$
- The area of the base is just the infinitesimal surface area dA. Now we just integrate…



$$\frac{1}{3}\int_M (f - p) \cdot N \, dA =$$

$$\frac{1}{3}\int_M f \cdot N \, dA - p \cdot \cancelto{0}{\int_M N \, dA} =$$

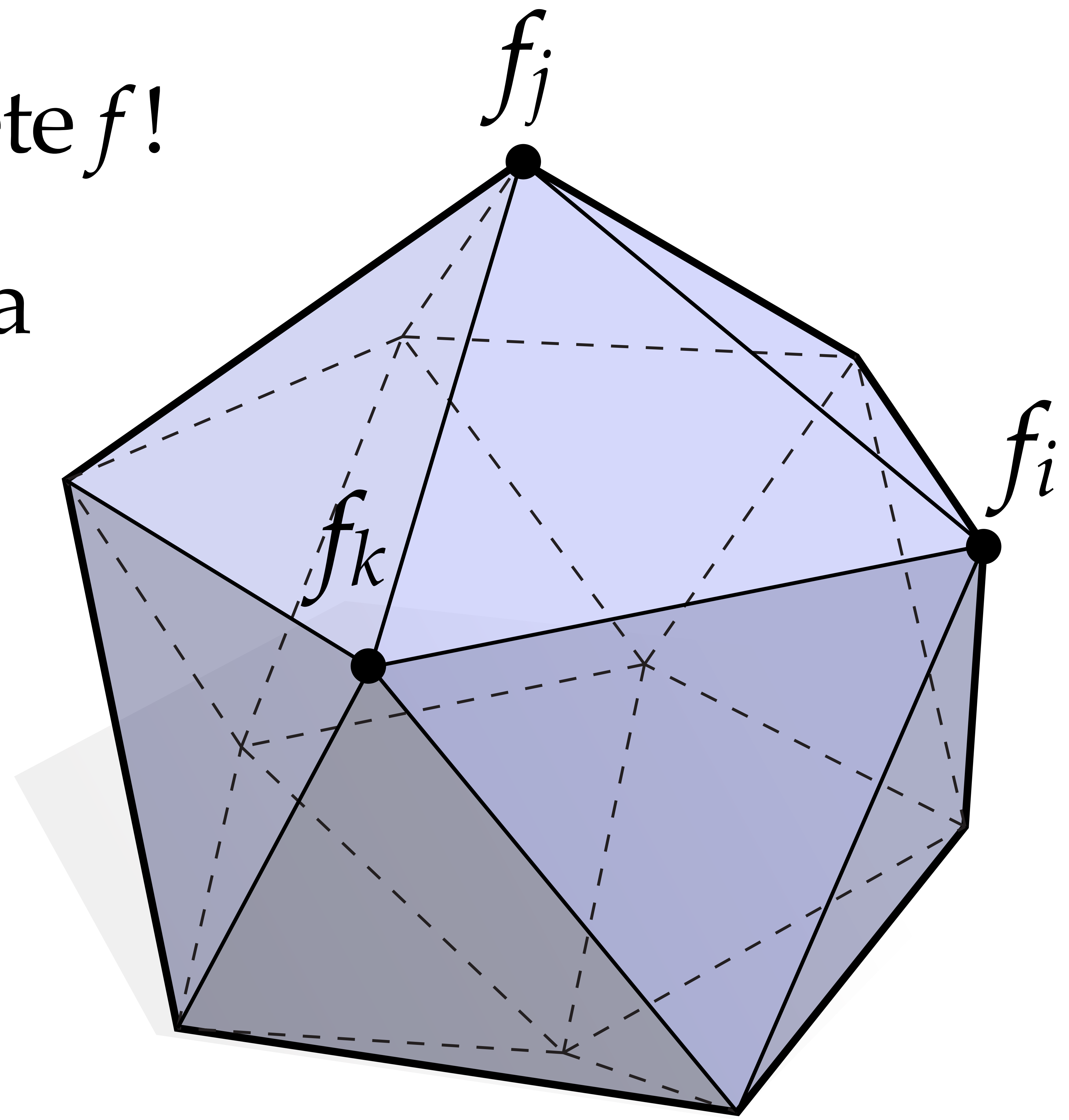
$$\boxed{\frac{1}{3}\int_M f \cdot N \, dA}$$

Notice: doesn't depend on choice of point p!

Volume Enclosed by a Discrete Surface

- What's the volume enclosed by a *discrete* surface?
- Simply apply our smooth formula to a discrete f!
- **<u>Exercise.</u>** Show that the volume enclosed by a simplicial surface can be expressed as

$$\text{volume}(f) = \frac{1}{6} \sum_{ijk \in F} f_i \cdot (f_j \times f_k)$$



Discrete Volume Gradient

- Taking the gradient of enclosed volume with respect to the position f_i of some vertex i should now give us a notion of vertex normal:

$$\nabla_{f_i}\text{volume}(f) = \tfrac{1}{6}\nabla_{f_i}\sum_{ijk\in F} f_i \cdot (f_j \times f_k) = \frac{1}{6}\sum_{ijk\in F} f_j \times f_k$$

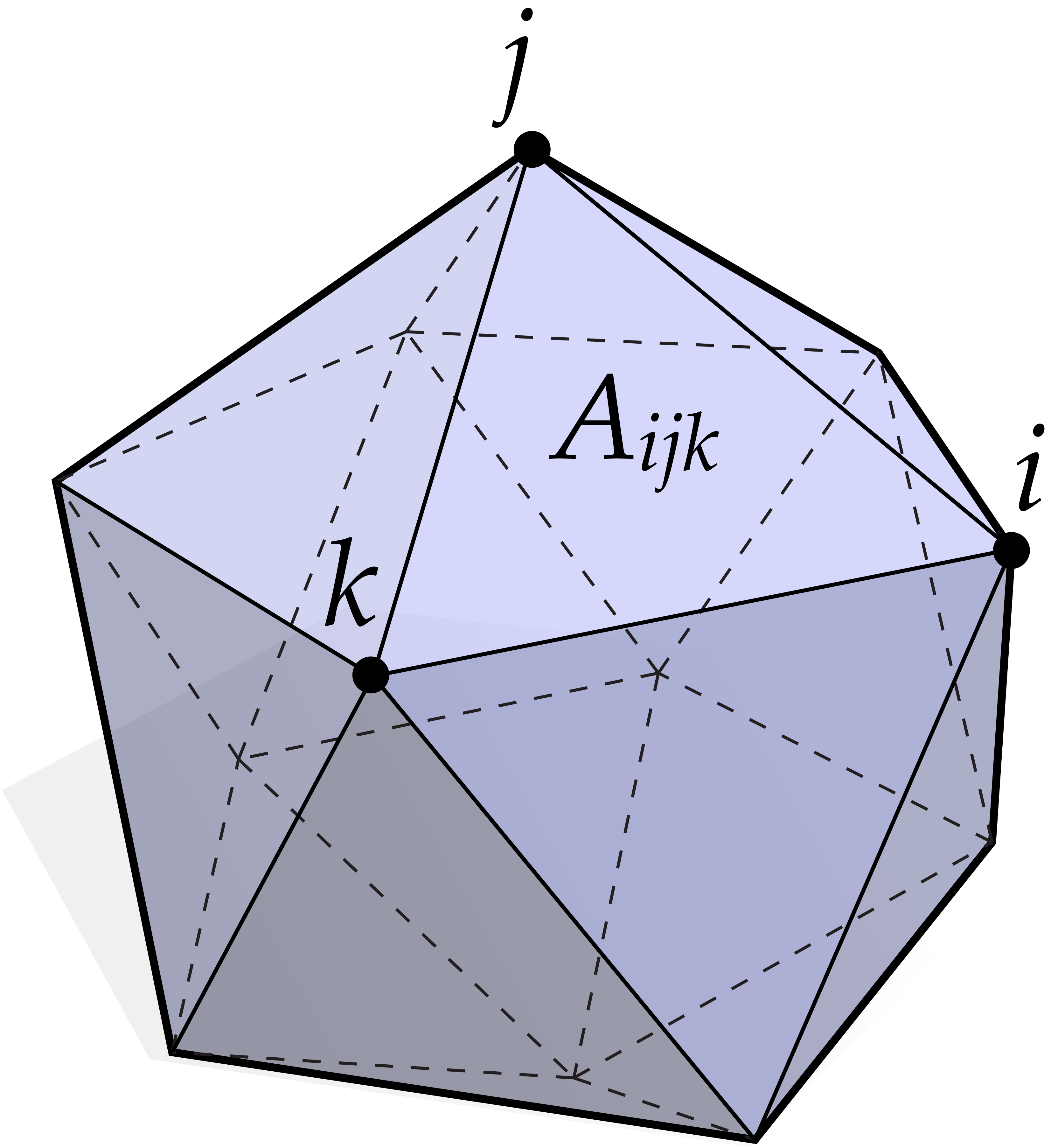
- But wait—this expression is the same as the discrete area vector!
- In other words: taking the gradient of discrete volume gave us exactly the same thing as integrating the normal over the dual cell.
- Agrees with the first expression in our sequence of variations:

$$\delta\text{volume}(f) = N$$

Total Area of a Discrete Surface

- Total area of a discrete surface is simply the sum of the triangle areas:

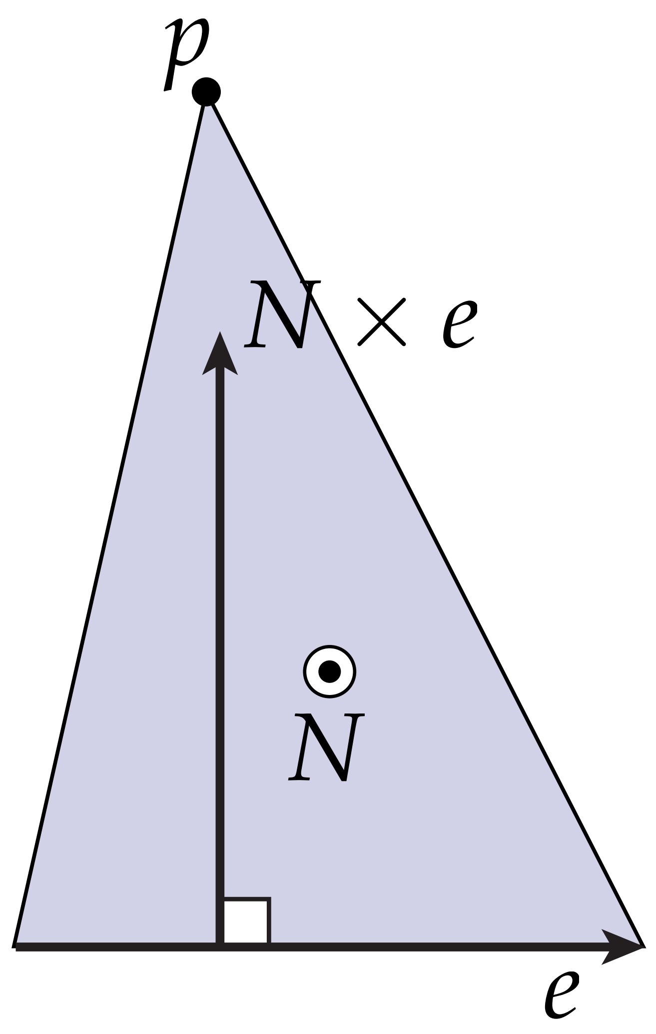
$$\text{area}(f) := \sum_{ijk \in F} A_{ijk}$$



Q: Suppose f is not a discrete immersion. Is area well-defined? Differentiable?

Discrete Area Gradient

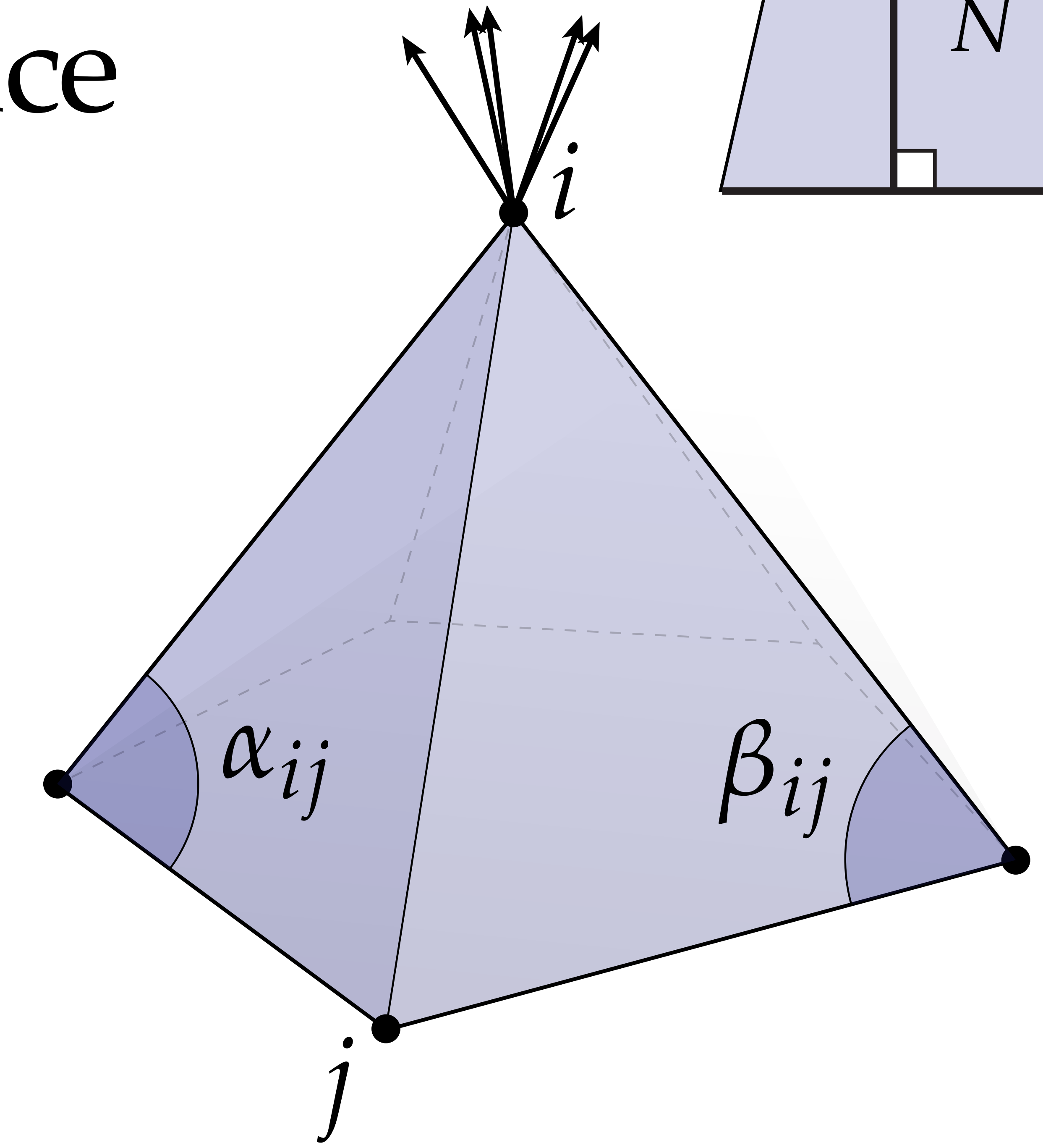
- Recall that the gradient of triangle area with respect to position p of a vertex is just half the normal cross the opposite edge:



$$\nabla_p A = \frac{1}{2} N \times e$$

- By summing contribution of all triangles touching a given vertex, can show that gradient of total surface area with respect to vertex coordinate f_i is

$$\nabla_{f_i} \text{area}(f) = \sum_{ij} \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$

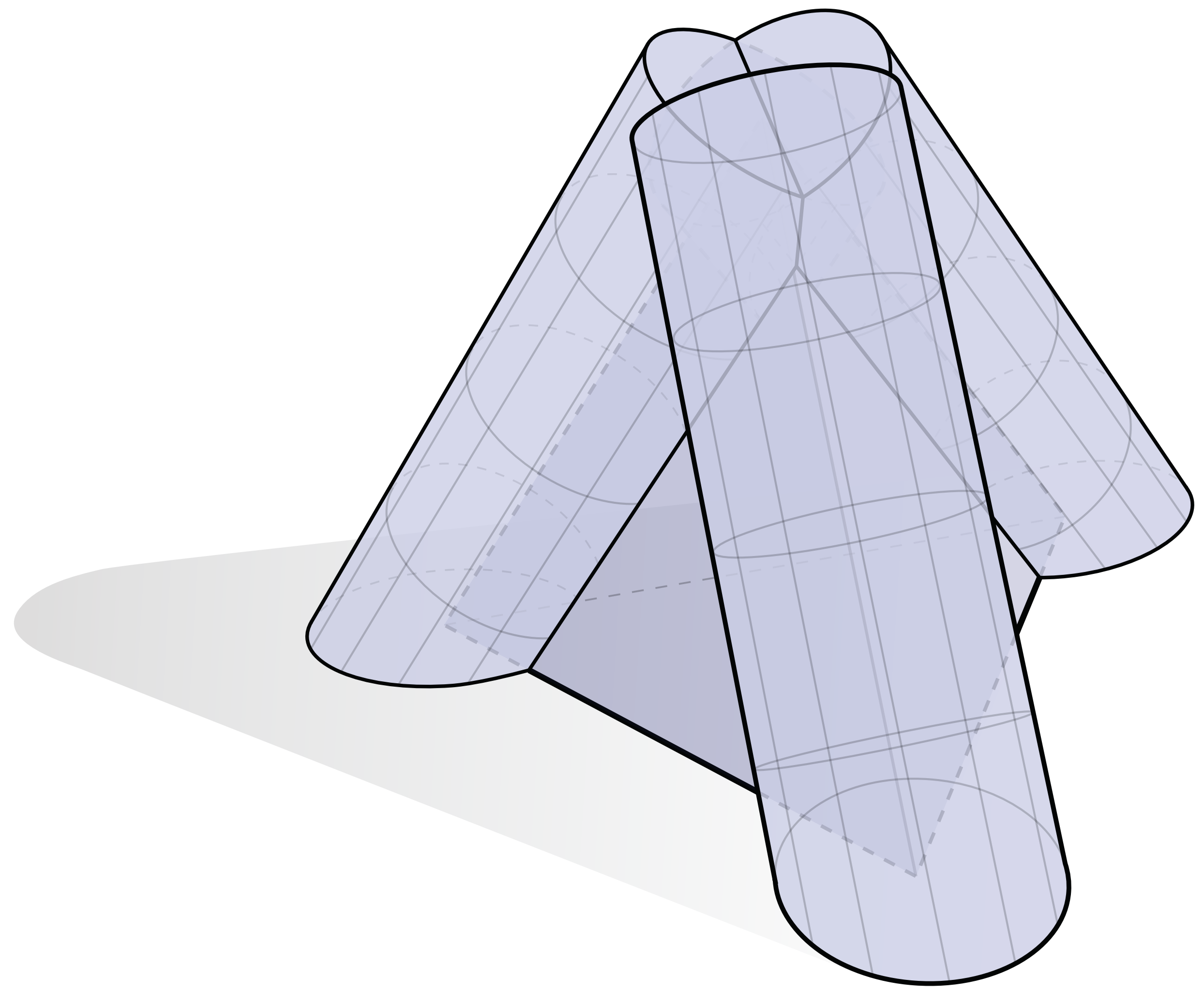


- Agrees with second expression in our sequence:

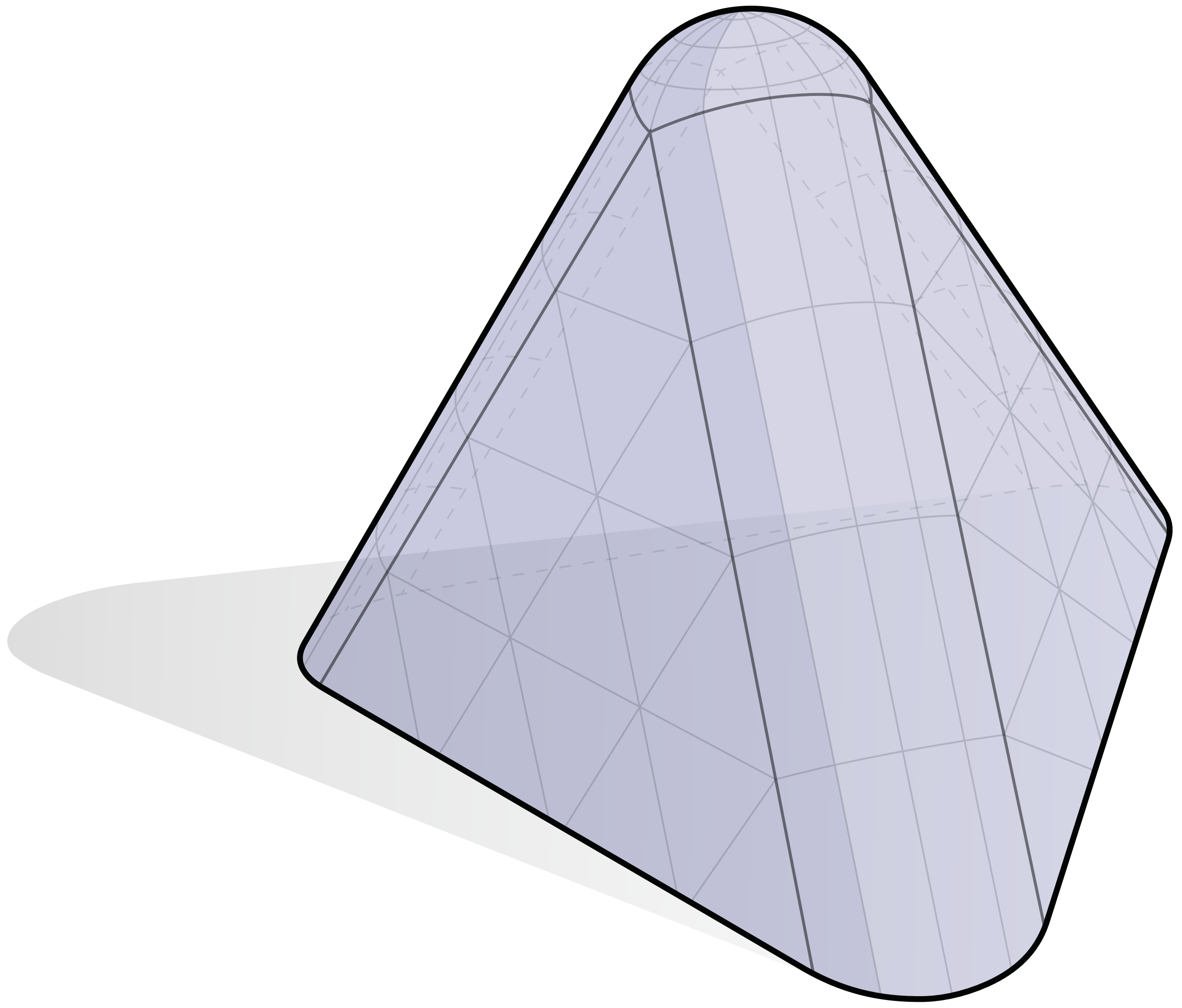
$$\delta \text{area}(f) = HN = \frac{1}{2} \Delta f$$

Total Mean Curvature of a Discrete Surface

- From our Steiner polynomial, we know the total mean curvature of a discrete surface is



$$\text{mean}(f) = \frac{1}{2} \sum_{ij \in E} \ell_{ij} \varphi_{ij}$$



(In fact, total volume and area used for the previous two calculations also agree with Steiner polynomial…)

Discrete Mean Curvature Gradient

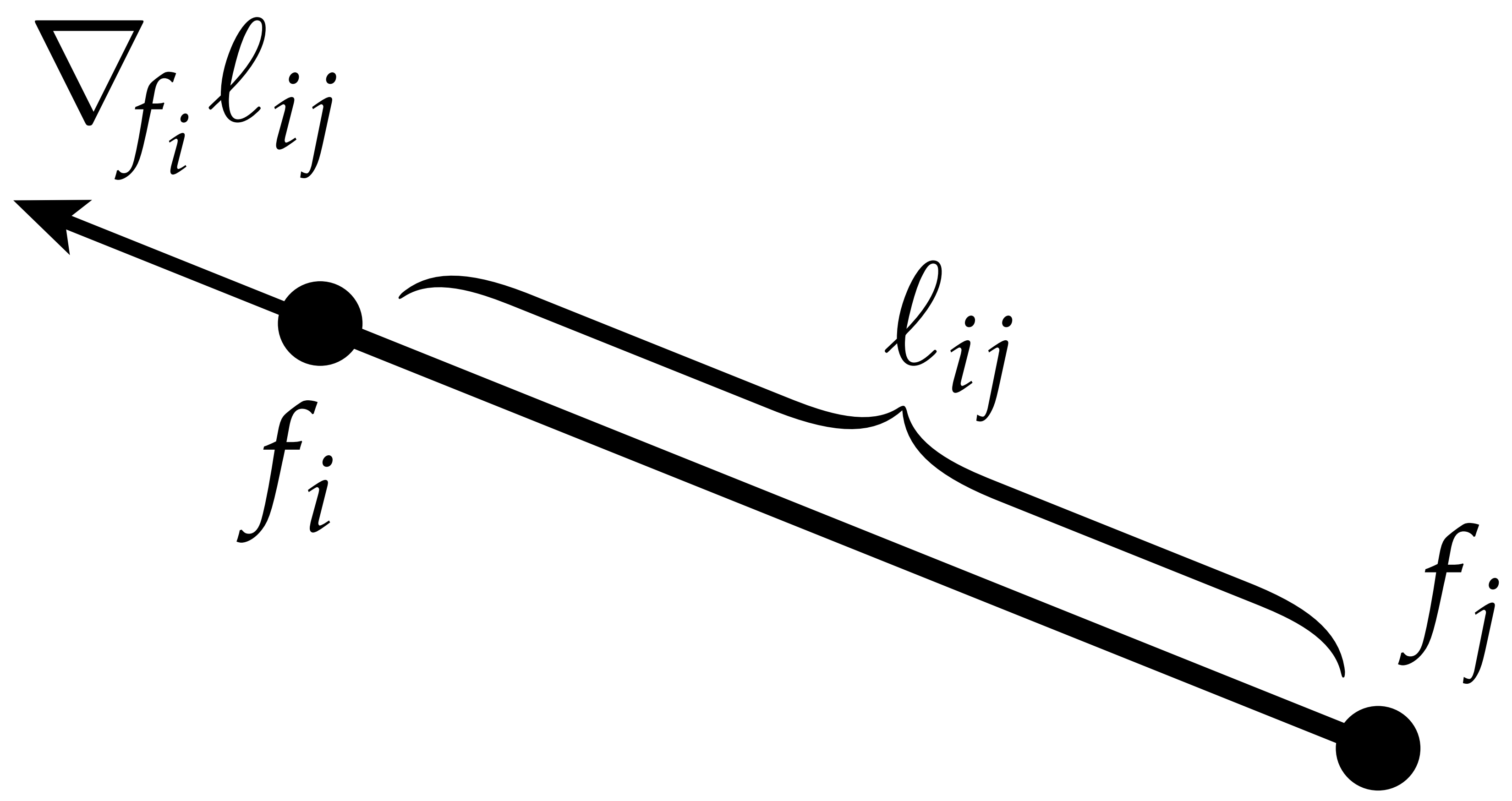
- What's the gradient of total mean curvature with respect to a particular vertex position f_i?

$$\nabla_{f_i}\text{mean}(f) = \frac{1}{2}\sum_{ij\in E}\nabla_{f_i}(\ell_{ij}\varphi_{ij}) =$$

$$\frac{1}{2}\sum_{ij\in E}(\nabla_{f_i}\ell_{ij})\varphi_{ij} + \cancel{\ell_{ij}(\nabla_{f_i}\varphi_{ij})} =$$

0 (Schläfli)

$$\frac{1}{2}\sum_{ij\in E}\frac{\varphi_{ij}}{\ell_{ij}}(f_i - f_j)$$



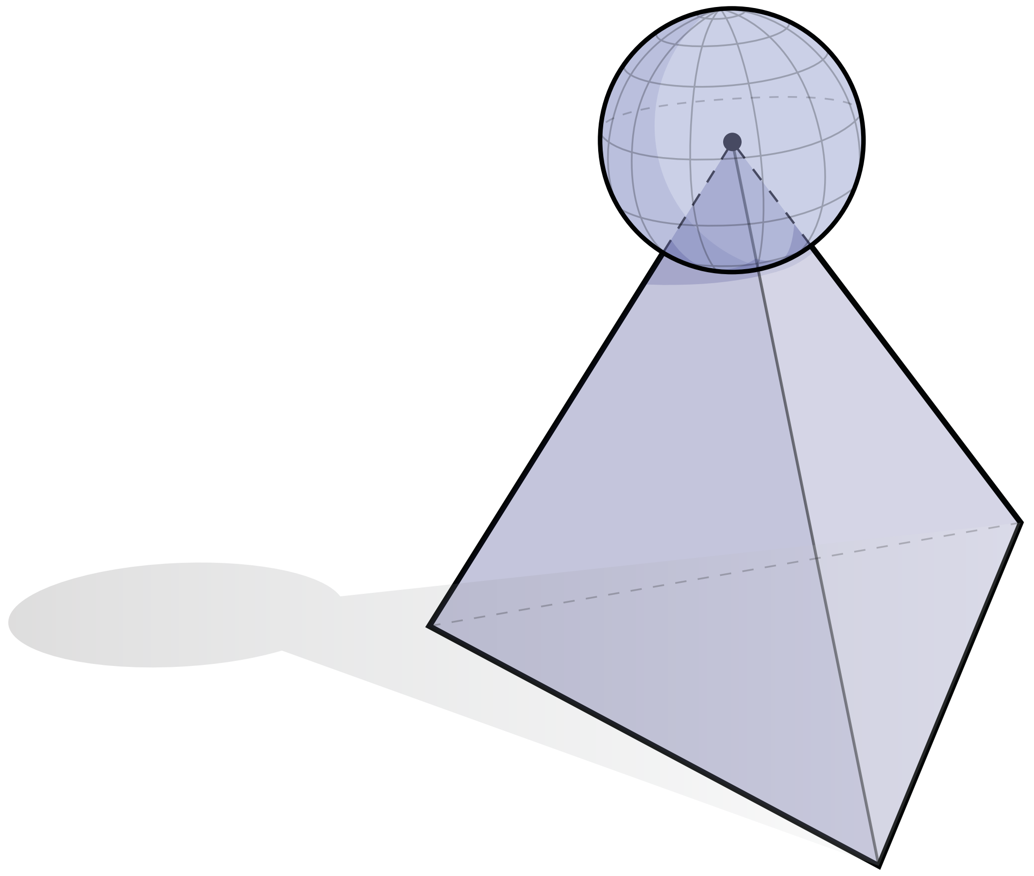
- Agrees with third expression in our sequence:

$$\delta\text{mean}(f) = KN$$

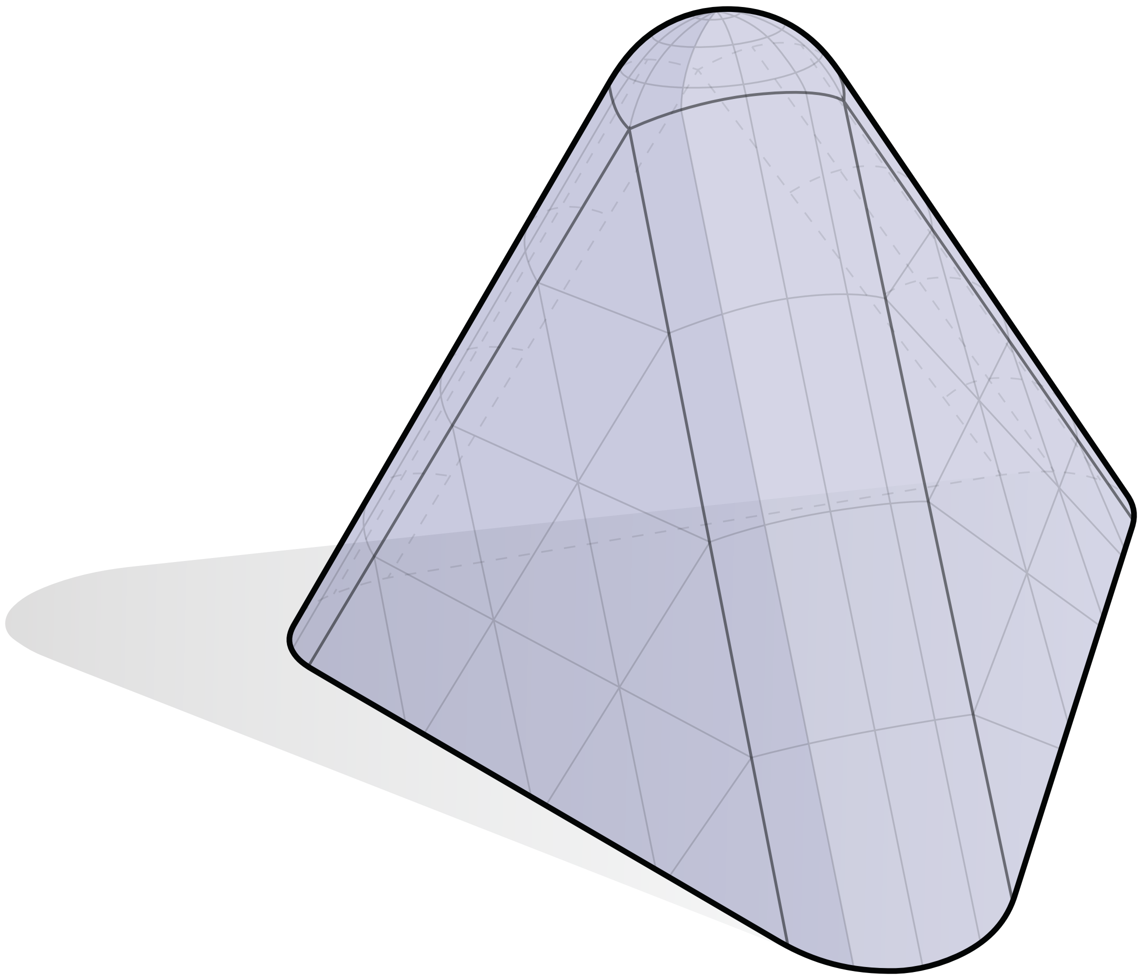
Total Gauss Curvature

- Total Gauss curvature of a discrete surface is sum of angle defects:

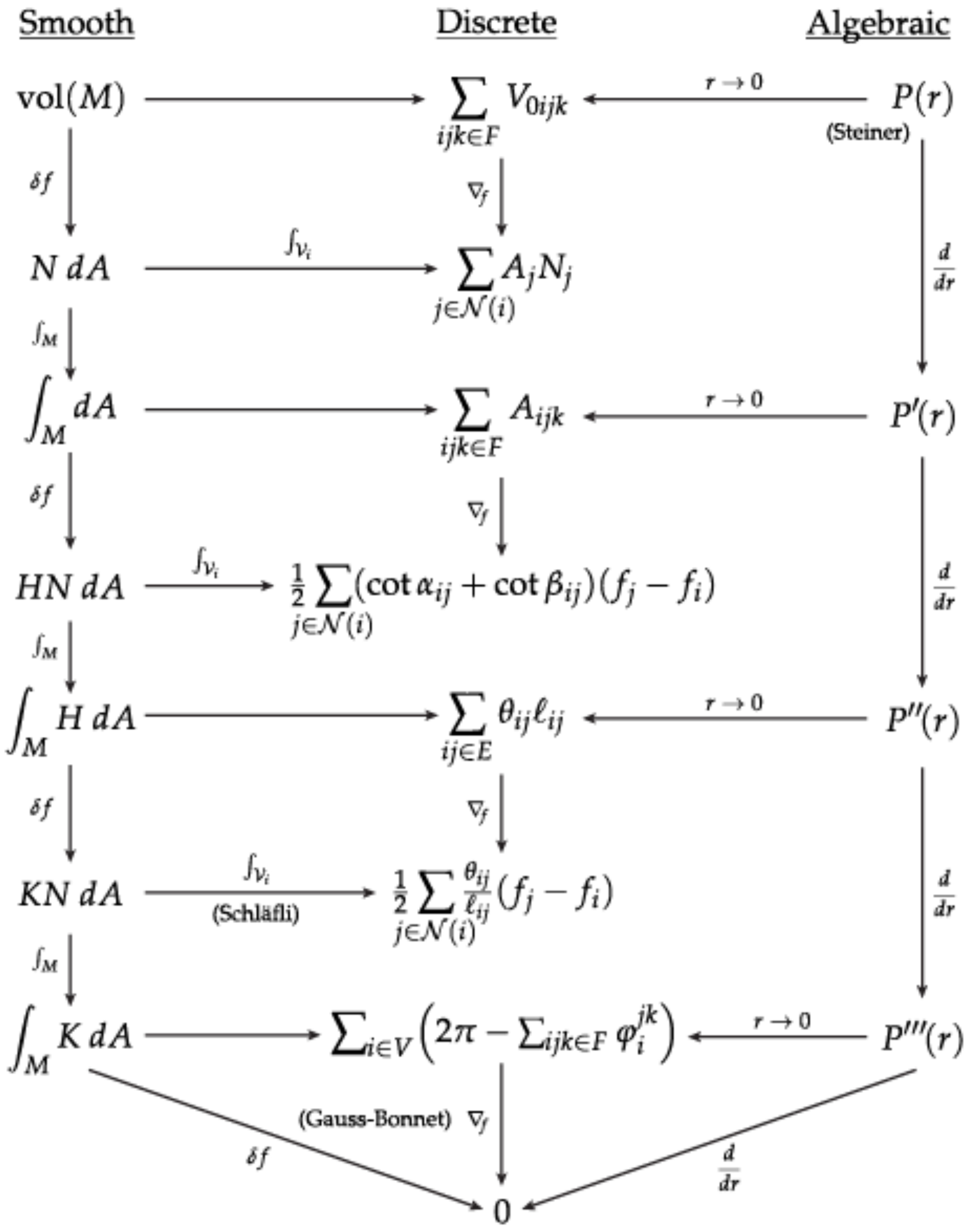
$$\mathrm{Gauss}(f) = \sum_{i \in V} \left(2\pi - \sum_{ijk} \theta_i^{jk} \right)$$



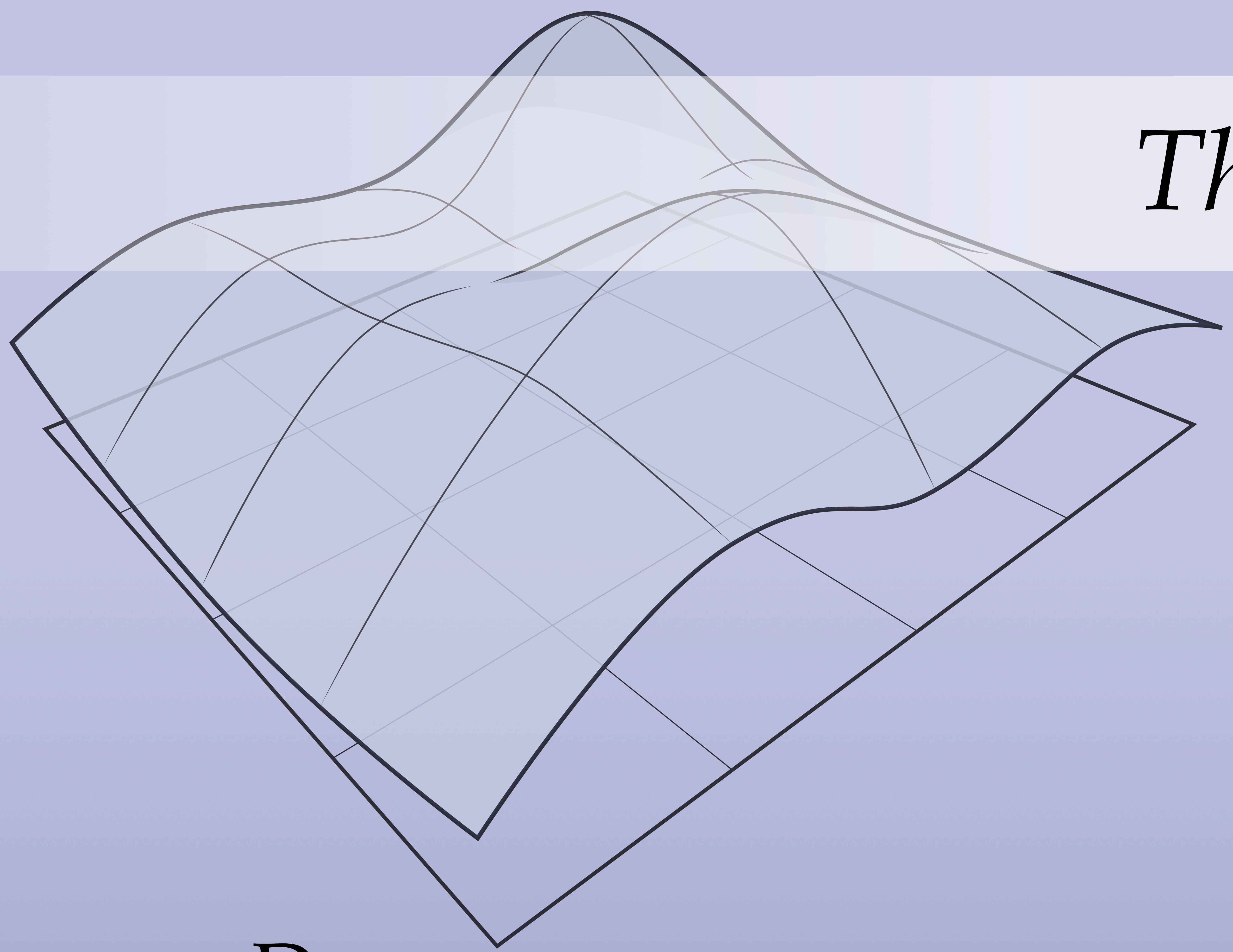
- From (discrete) Gauss-Bonnet theorem, we know this sum is always equal to just $2\pi\chi = 2\pi(V\text{-}E\text{+}F)$
- Gradient with respect to motion of any vertex is therefore *zero*—sequence ends here!



Discrete Curvature—Panoramic View



Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017