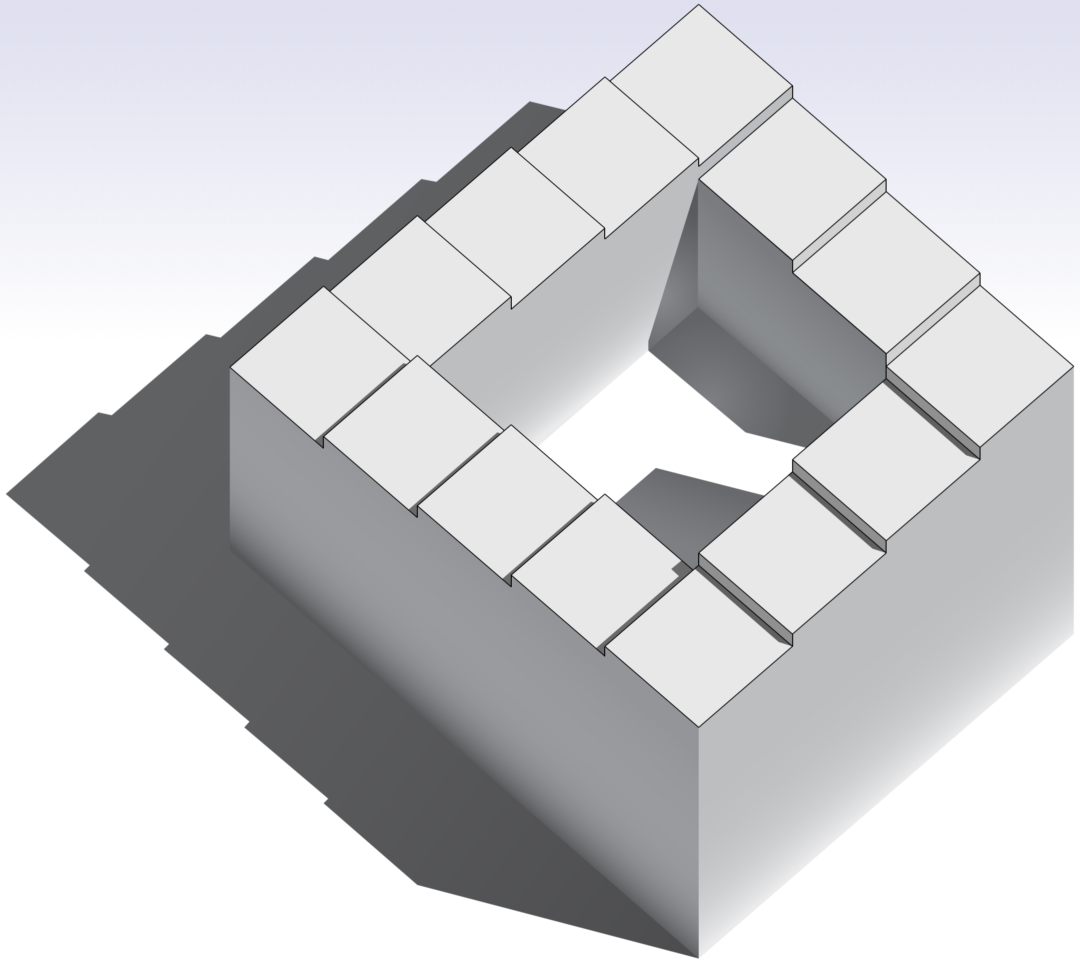
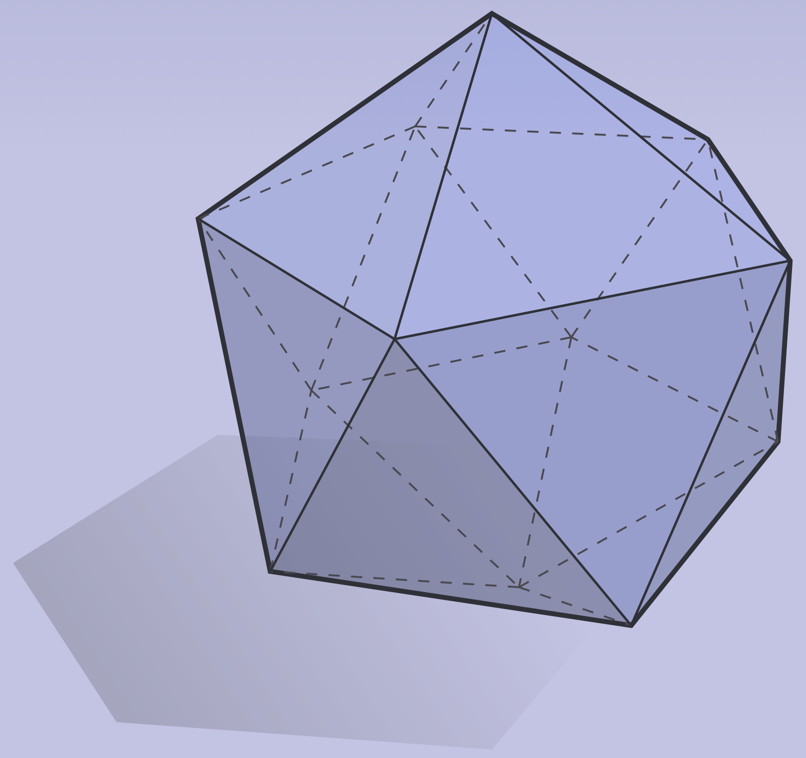


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 2:
COMBINATORIAL SURFACES



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

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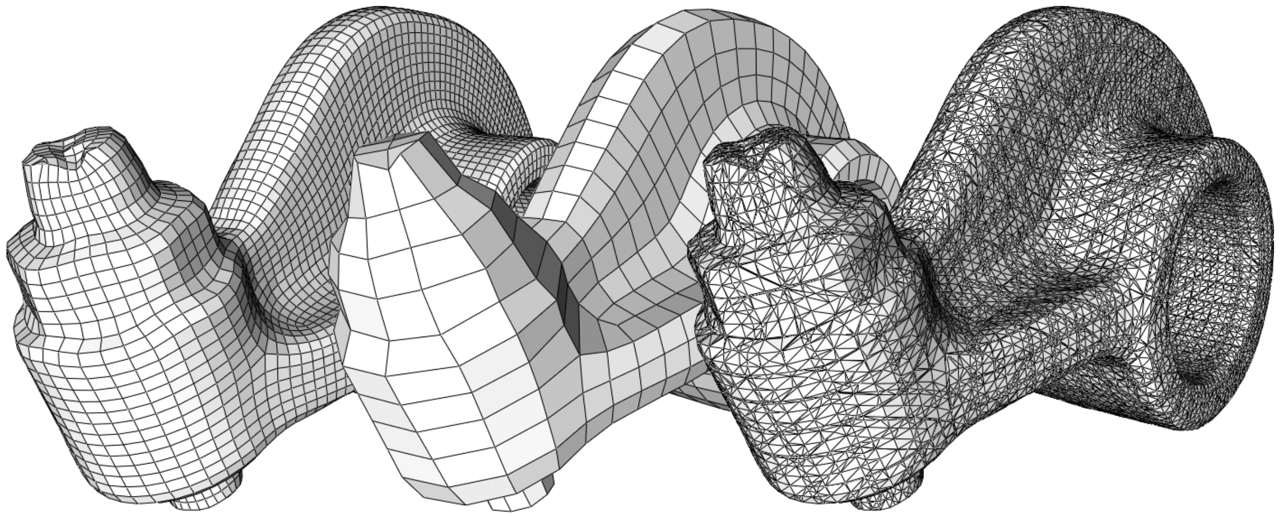
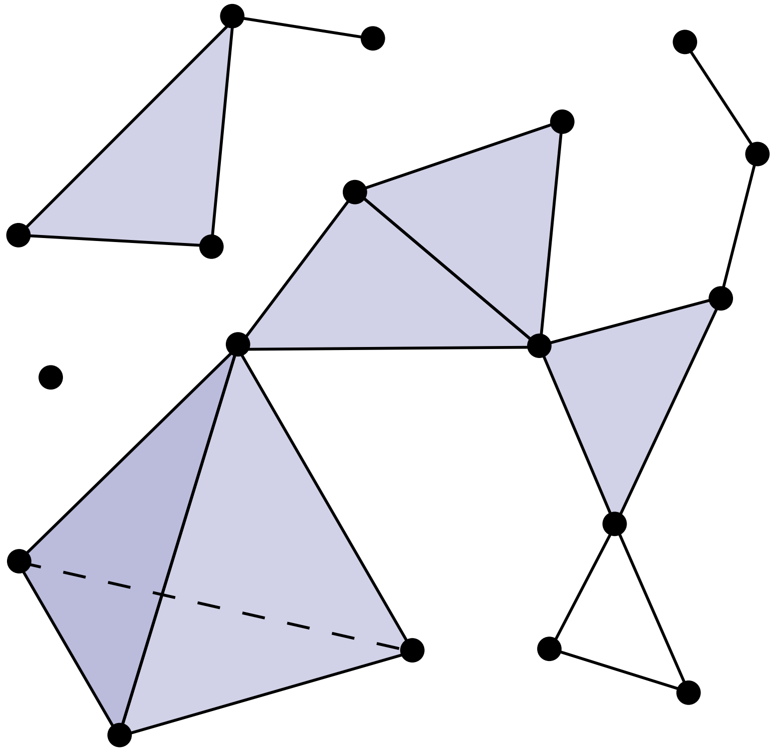
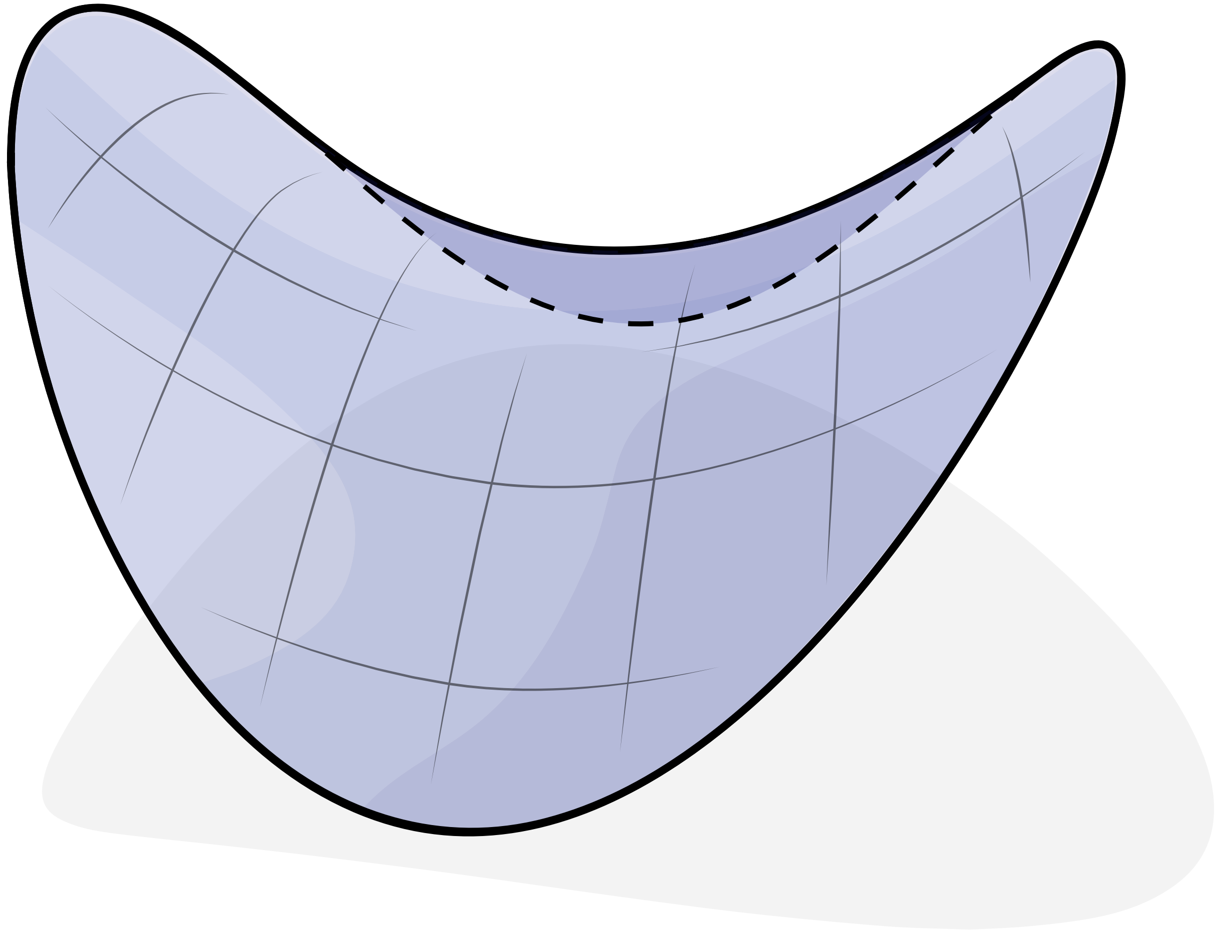
Today: What is a “Mesh?”

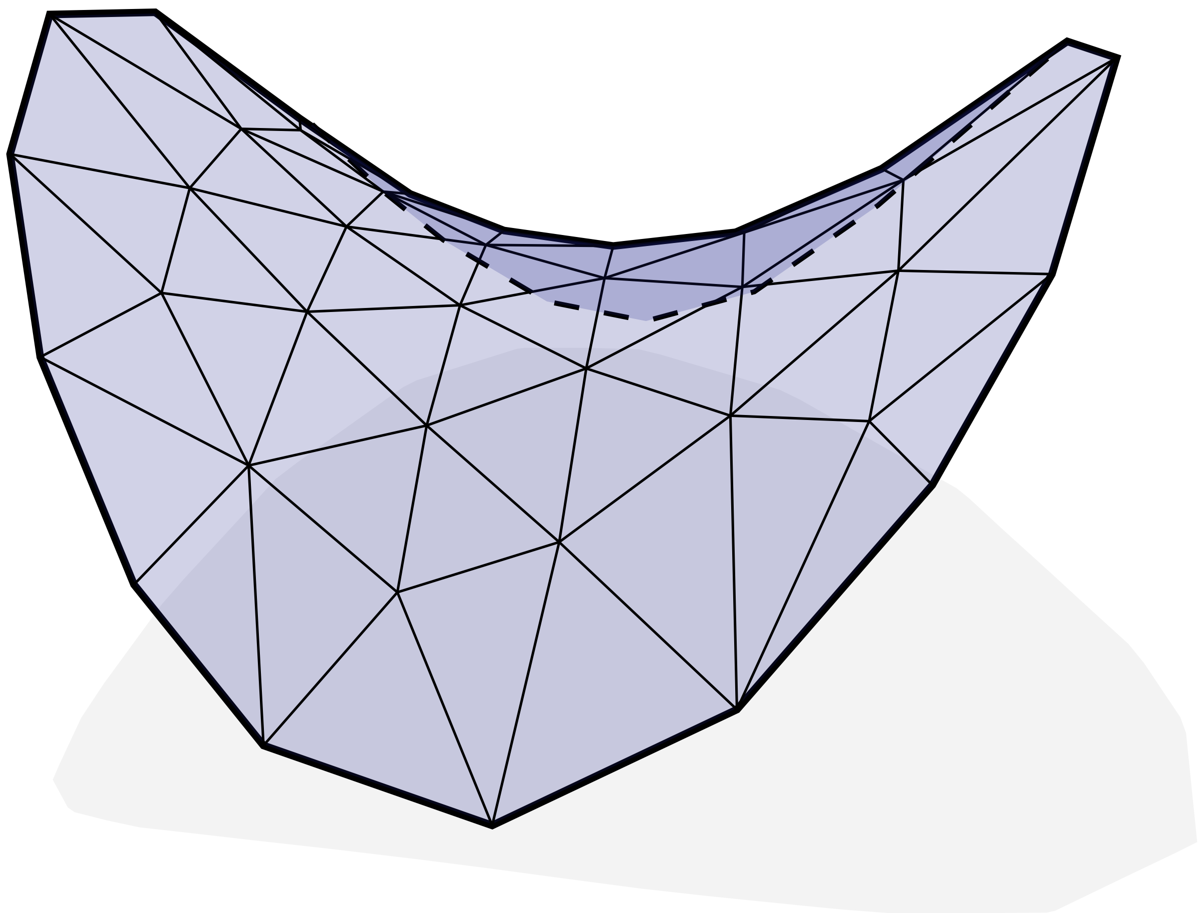
- Many possibilities…
- **Simplicial complex**
  - Abstract vs. geometric simplicial complex
  - Oriented, manifold simplicial complex
  - Application: *topological data analysis*
- **Cell complex**
  - Poincaré dual, discrete exterior calculus
- `Data structures:`
  - *adjacency list, incidence matrix, halfedge*





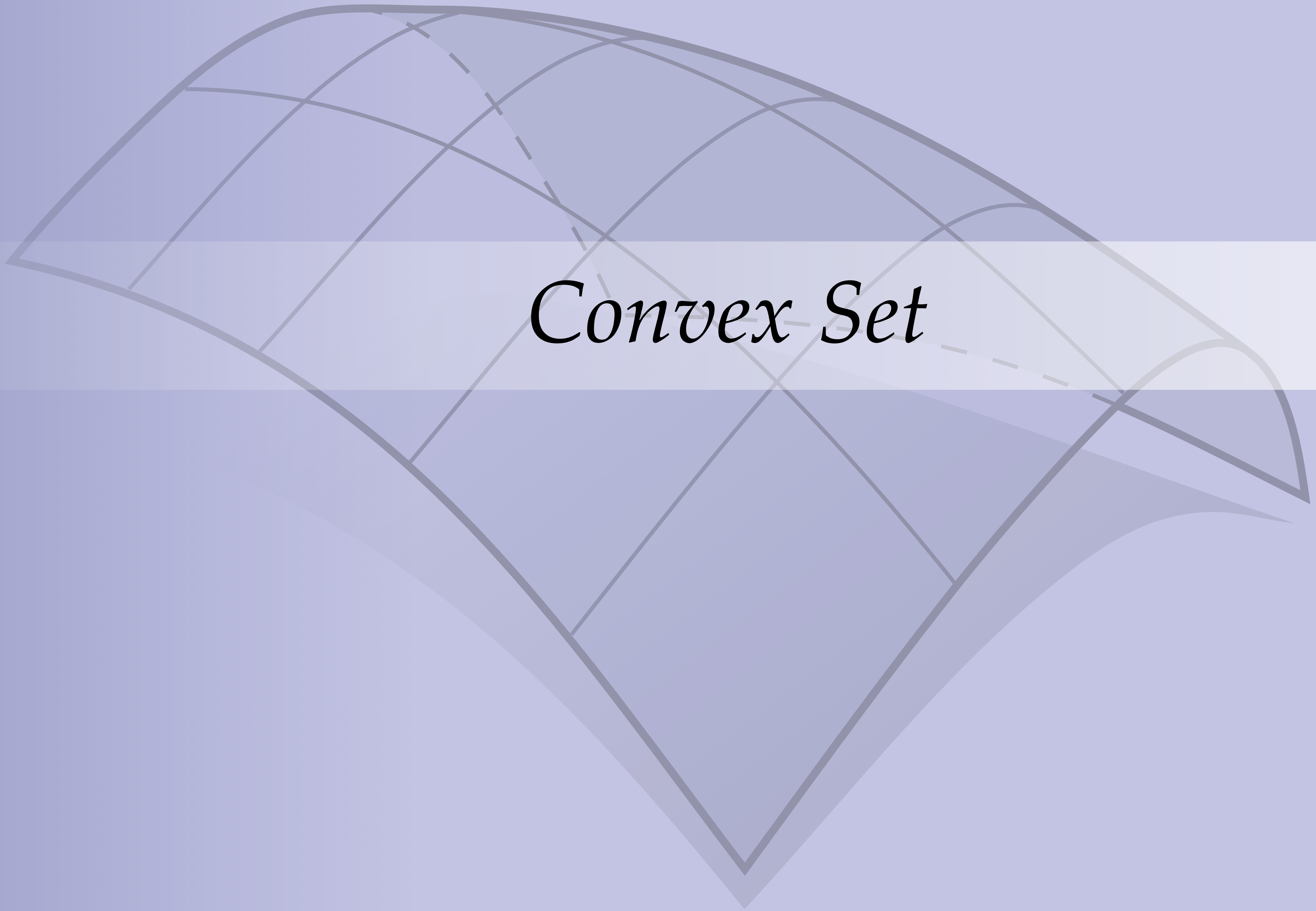
Connection to Differential Geometry?





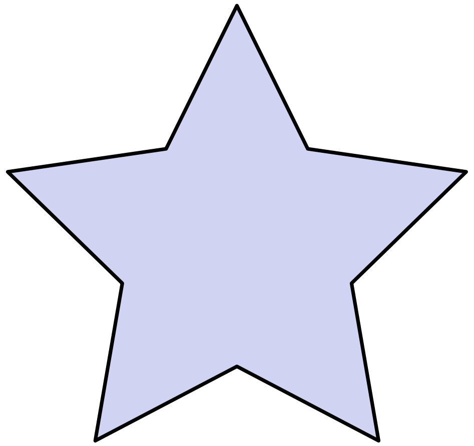
topological space $\longleftrightarrow$ abstract simplicial complex

Convex Set

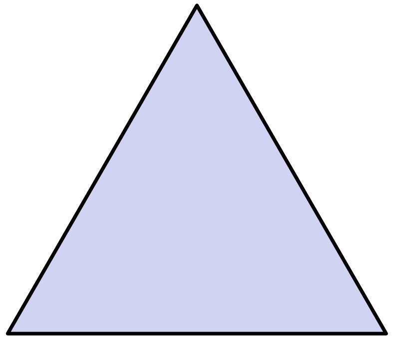


Convex Set—Examples

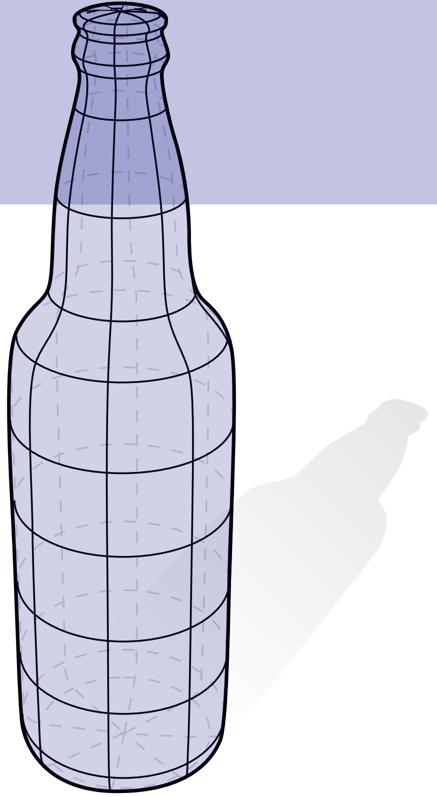
Which of the following sets are *convex*?



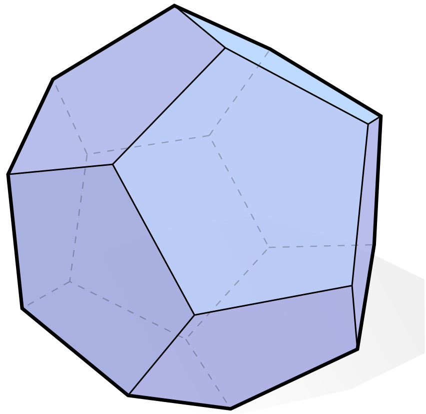
(A)



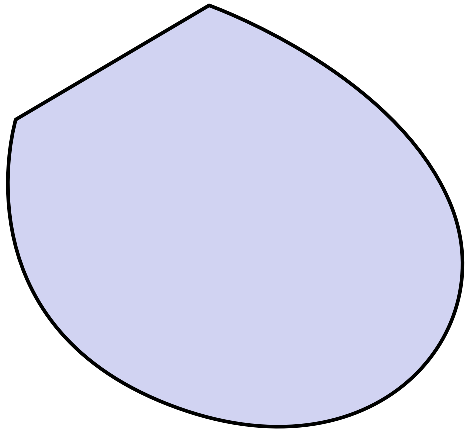
(B)



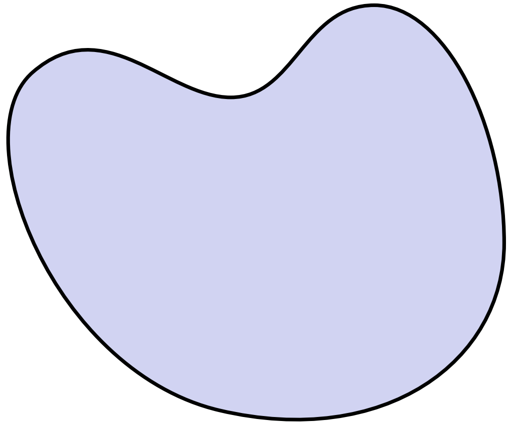
(C)



(D)



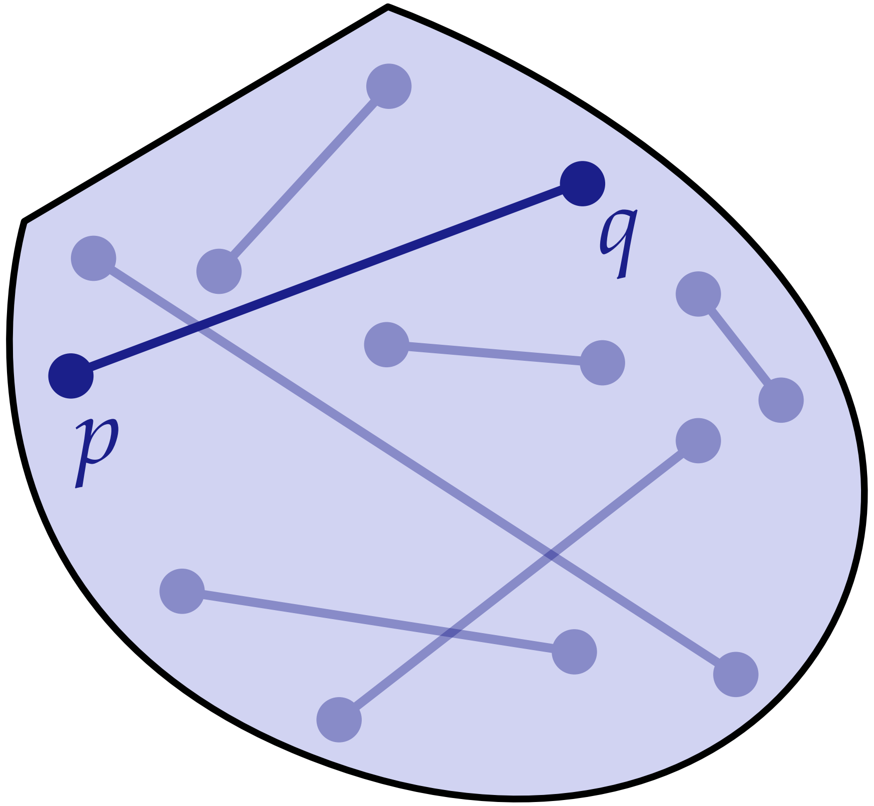
(E)



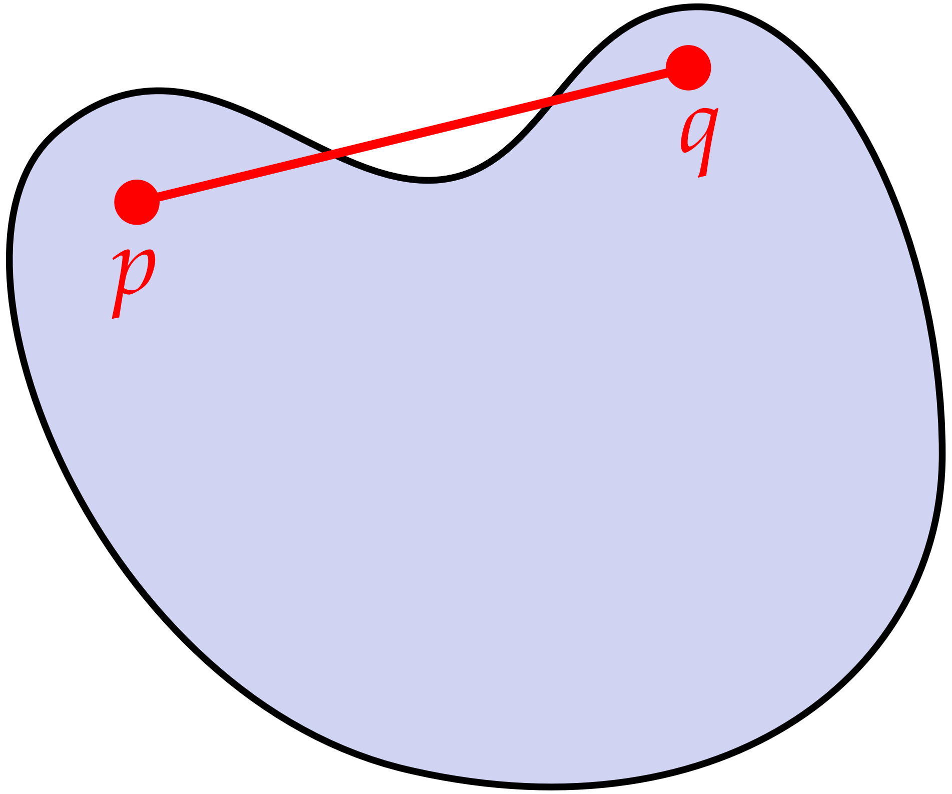
(F)

Convex Set

Definition. A subset $S \subset \mathbb{R}^n$ is *convex* if for every pair of points $p, q \in S$ the line segment between p and q is contained in S.

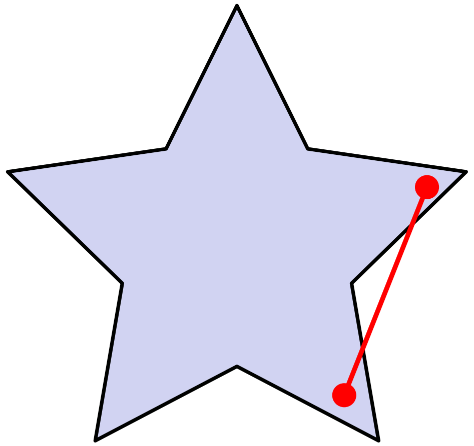


convex

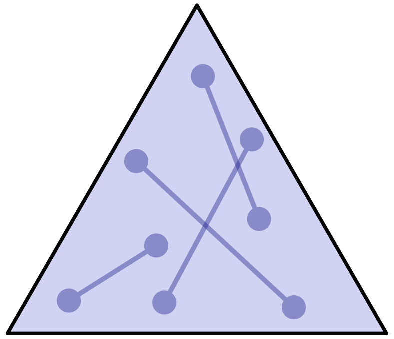


not convex

Convex Set—Examples



(A)



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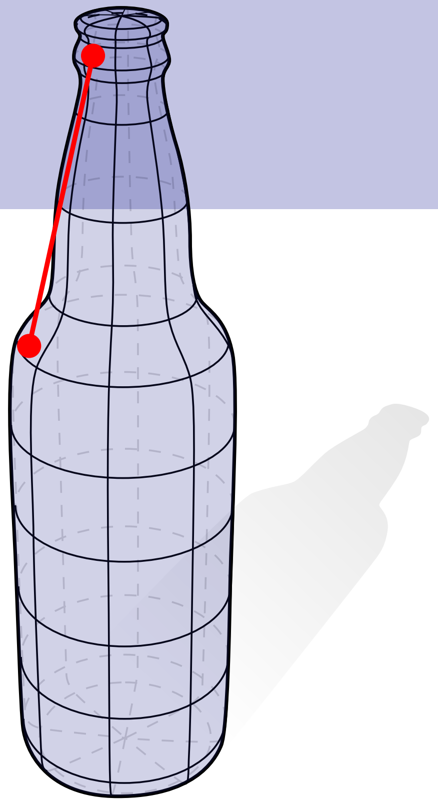
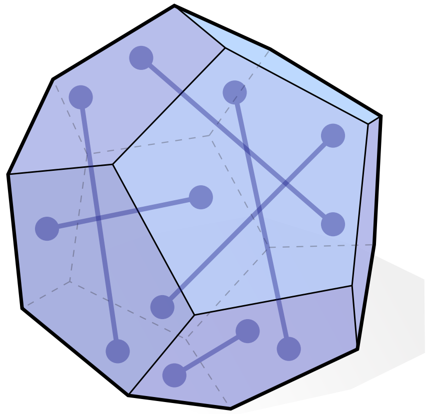
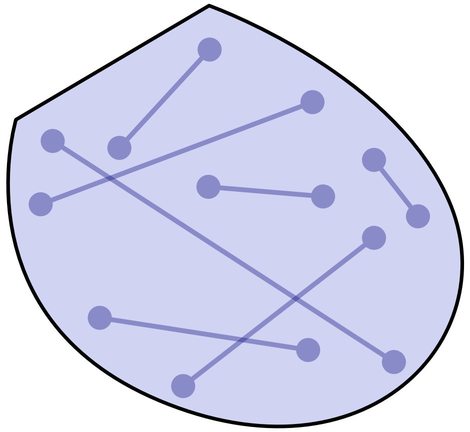


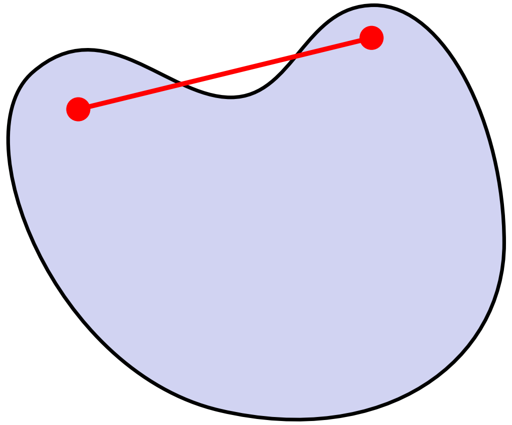
(C)



(D)



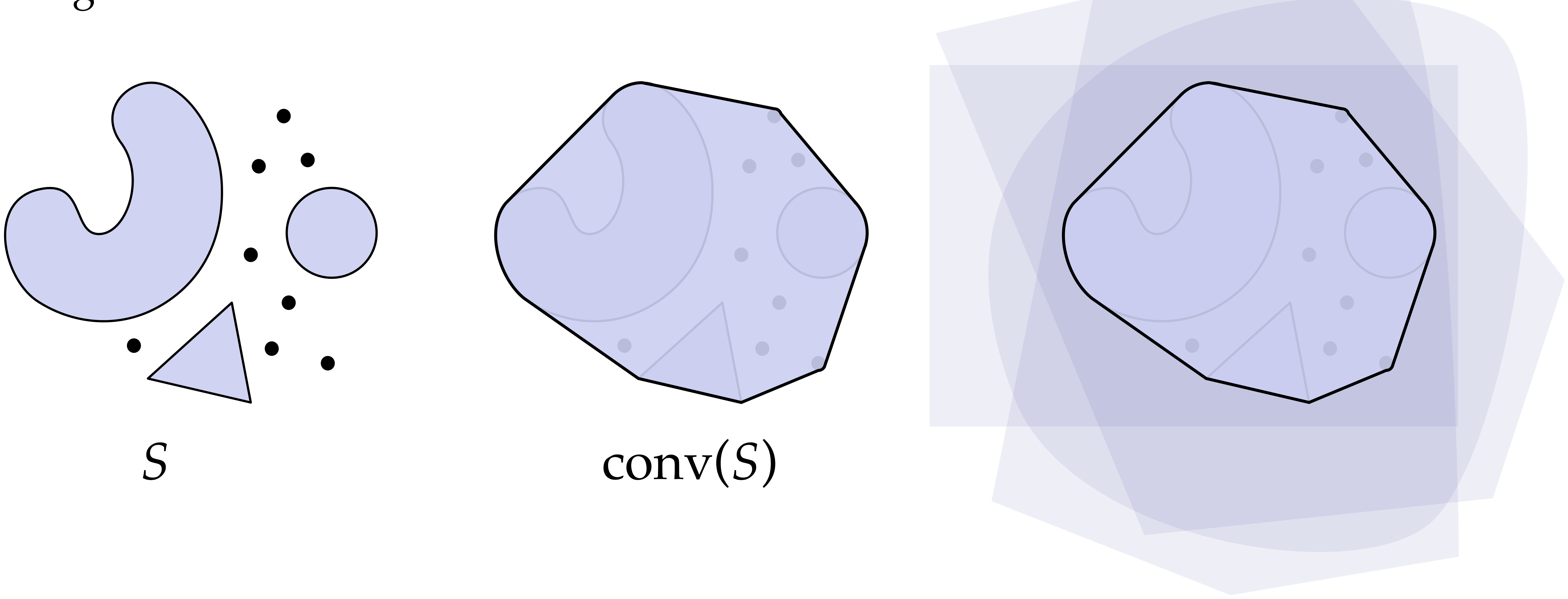
(E)



(F)

Convex Hull

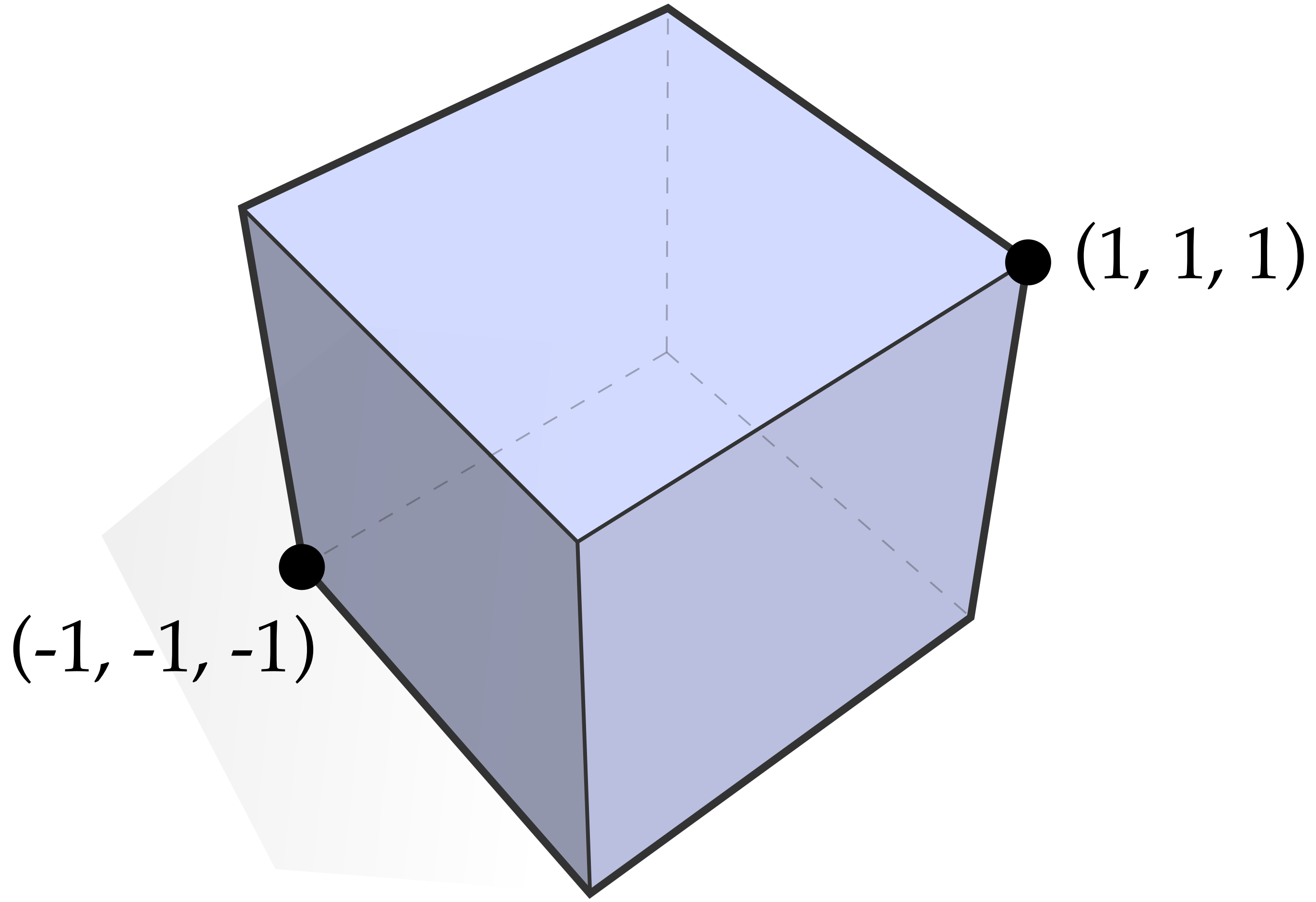
Definition. For any subset $S \subset \mathbb{R}^n$, its convex hull $\text{conv}(S)$ is the smallest convex set containing S, or equivalently, the intersection of all convex sets containing S.

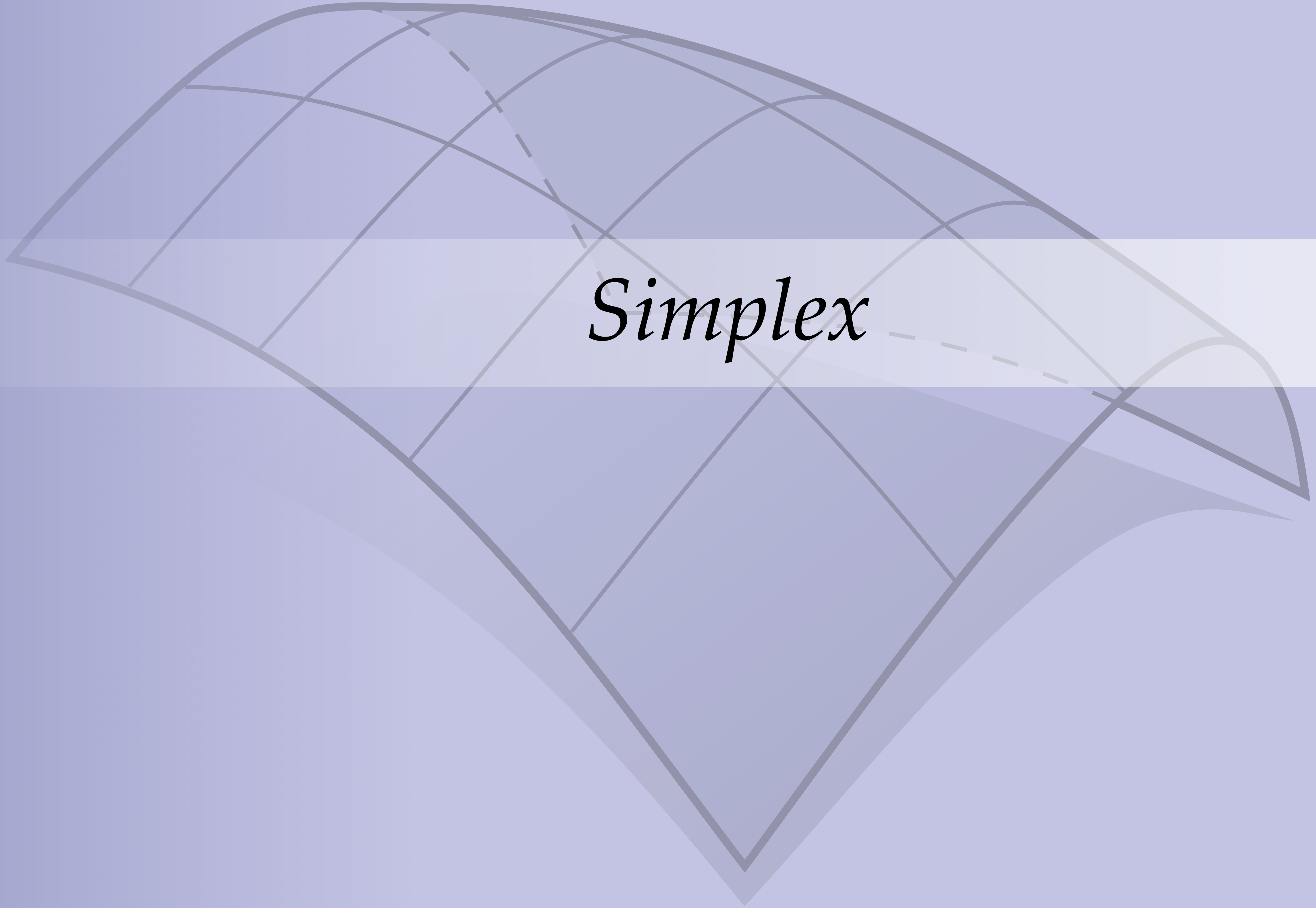


Convex Hull—Example

Q: What is the convex hull of the set $S := \{(\pm 1, \pm 1, \pm 1)\} \subset \mathbb{R}^3$?

A: A cube.

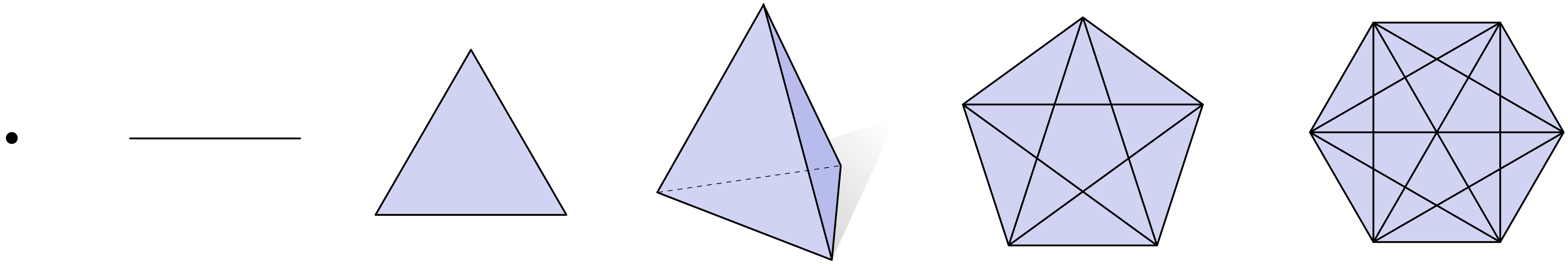




Simplex

Simplex — Basic Idea

Roughly speaking, a *k-simplex* is a point, a line segment, a triangle, a tetrahedron…

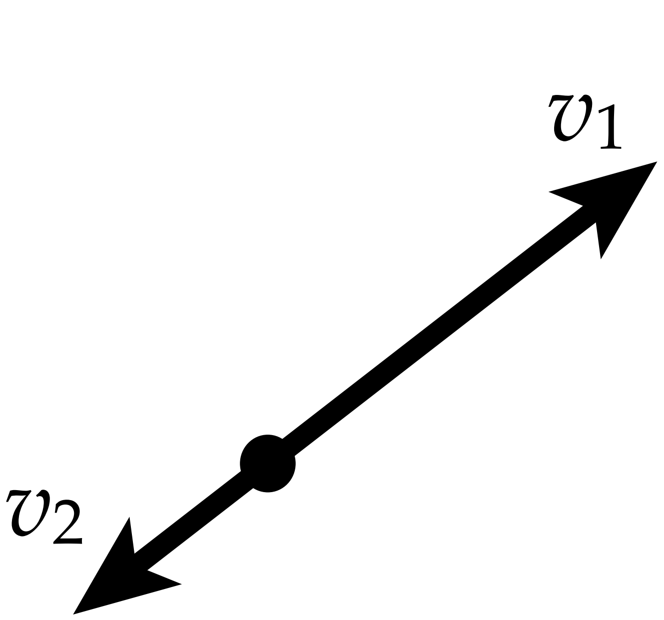


$k = 0$ $k = 1$ $k = 2$ $k = 3$ $k = 4$ $k = 5$

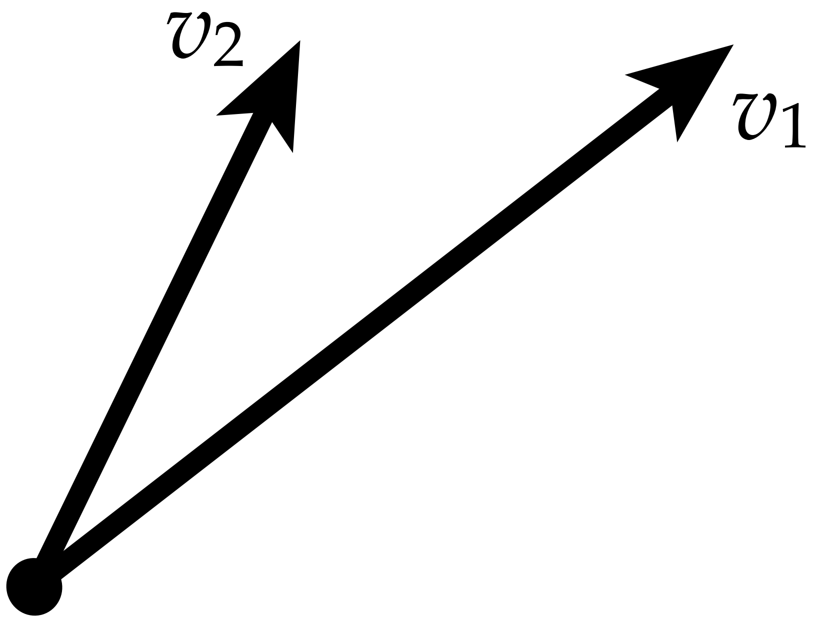
(…a bit hard to draw for $k \geq 4$!)

Linear Independence

Definition. A collection of vectors $v_1, \ldots, v_n$ is *linearly independent* if no vector can be expressed as a linear combination of the others, *i.e.*, if there is no collection of coefficients $a_1, \ldots, a_n \in \mathbb{R}$ such that $v_j = \sum_{i \neq j} a_i v_i$ (for any v_j).



linearly dependent

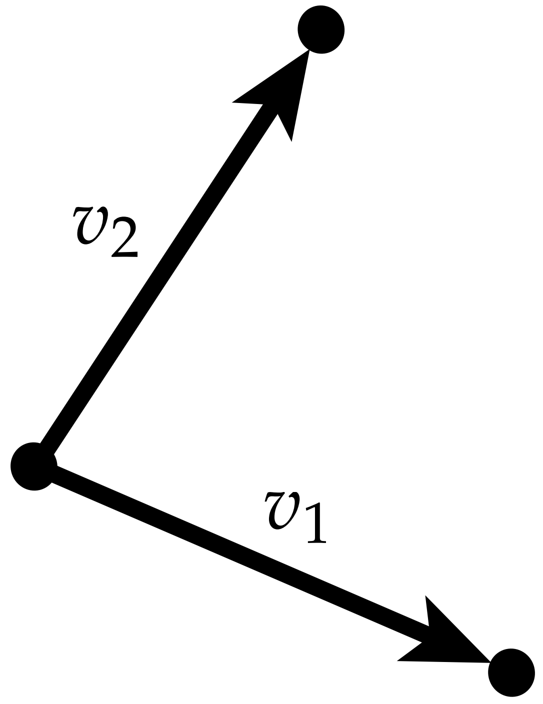


linearly independent

Affine Independence

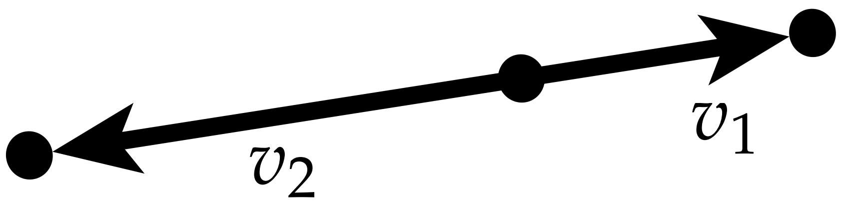
Definition. A collection of points $p_0, \ldots, p_k$ are *affinely independent* if the vectors $v_i := p_i - p_0$ are linearly independent.

(A)



affinely independent

(B)

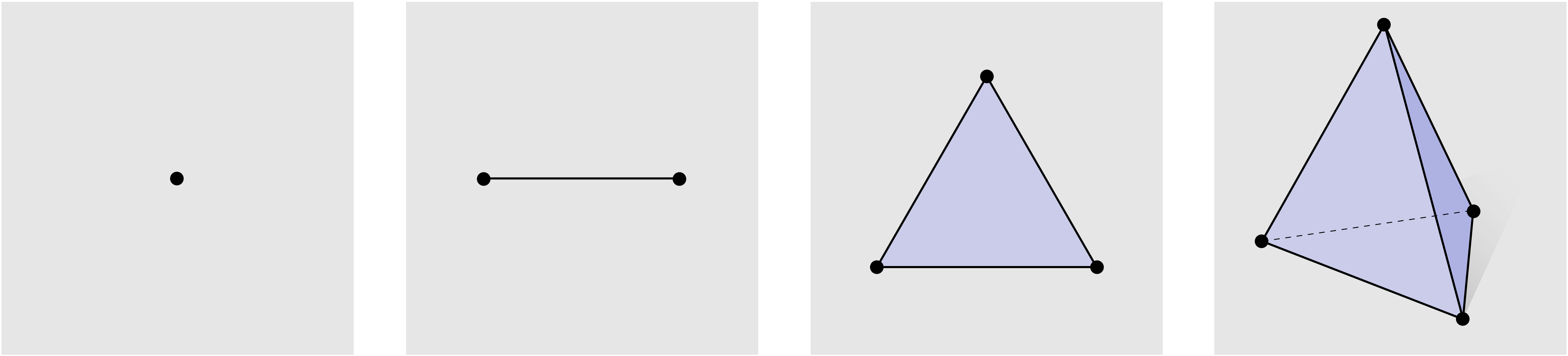


affinely dependent

(Colloquially: might say points are in *"general position"*.)

Simplex — Geometric Definition

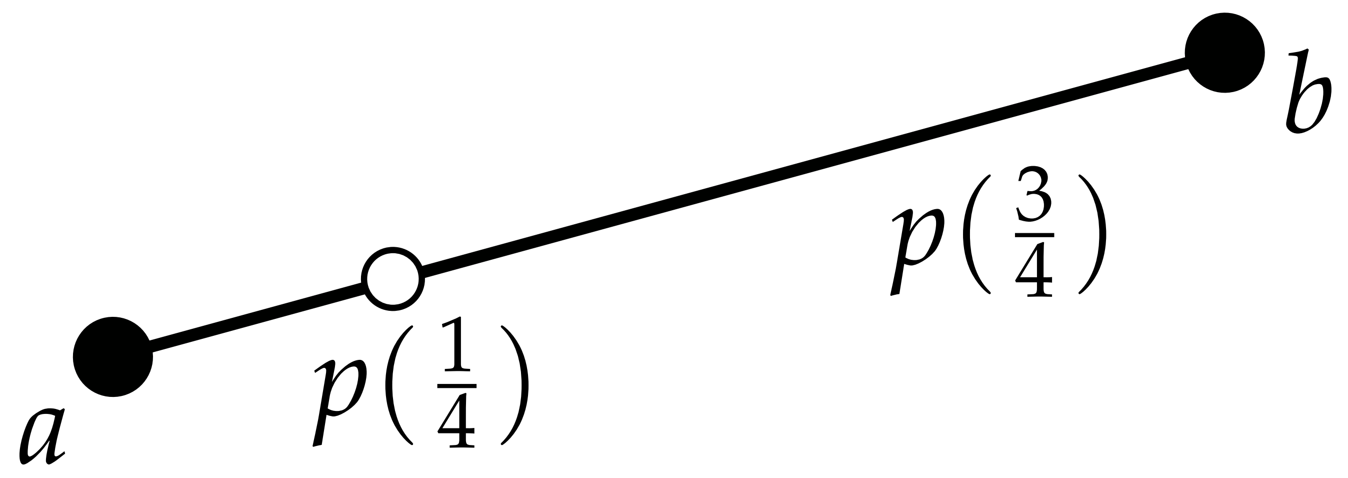
Definition. A *k-simplex* is the convex hull of $k+1$ affinely-independent points, which we call its *vertices*.



Barycentric Coordinates—1-Simplex

- We can describe a simplex more explicitly using *barycentric coordinates*.
- For example, a 1-simplex is comprised of all weighted combinations of the two points where the weights sum to 1:

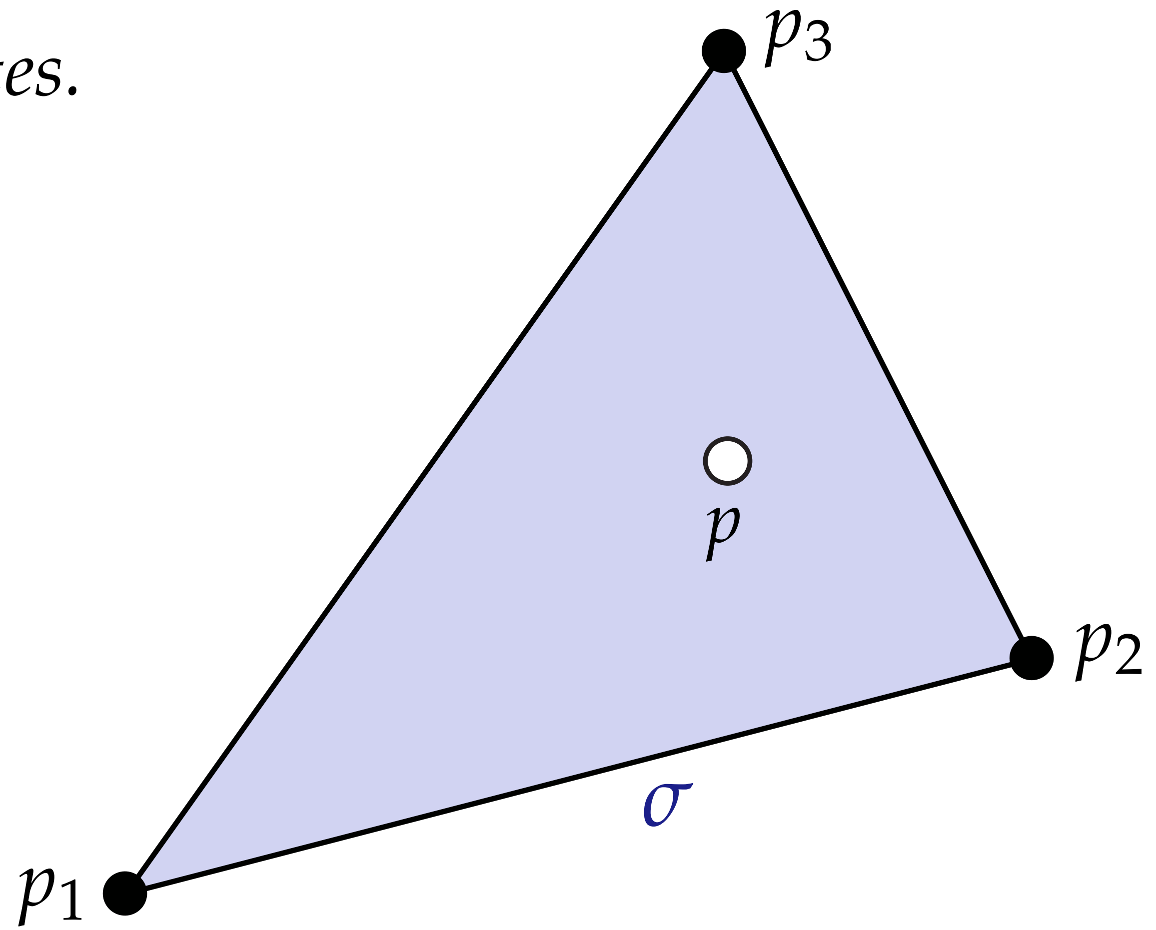
$$p(t) := (1-t)a + tb, \quad t \in [0,1]$$



Barycentric Coordinates—k-Simplex

- More generally, any point p in a k-simplex σ can be expressed as a (nonnegative) weighted combination of the vertices, where the weights sum to 1.
- The weights t_i are called the *barycentric coordinates.*

$$\sigma = \left\{ \sum_{i=0}^{k} t_i p_i \;\middle|\; \sum_{i=0}^{k} t_i = 1,\ t_i \geq 0\ \forall i \right\}$$



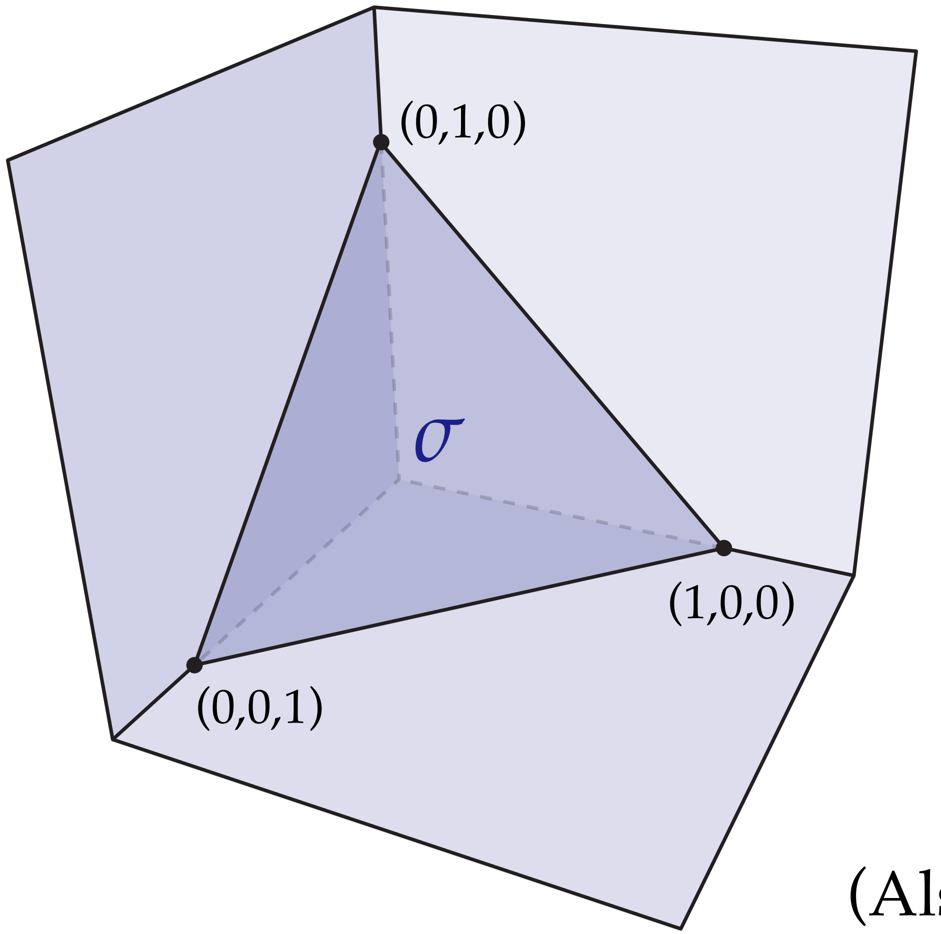
$$p = \frac{2}{10} p_1 + \frac{4}{10} p_2 + \frac{4}{10} p_3$$

(Also called a "convex combination.")

Simplex — Example

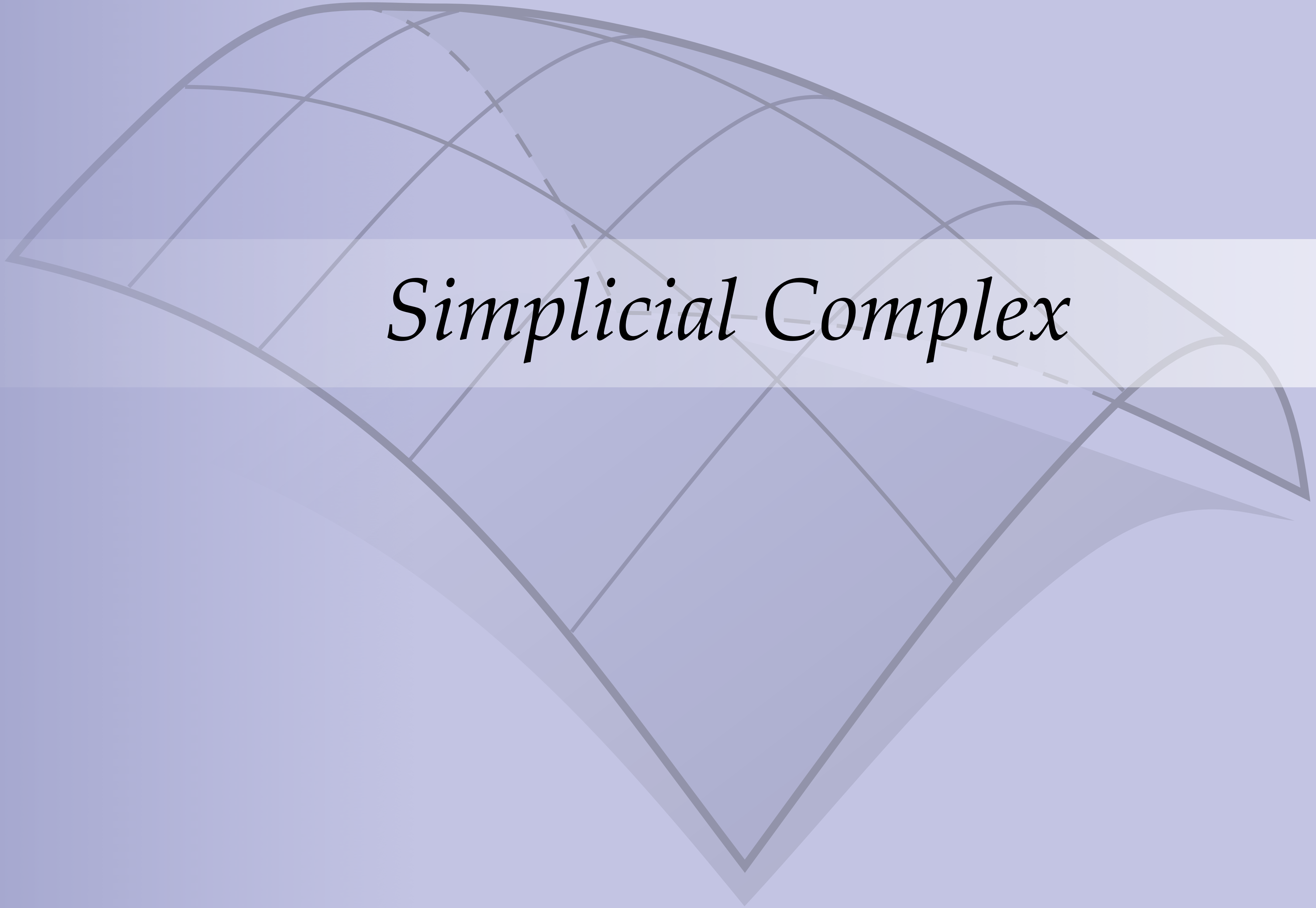
Definition. The *standard n-simplex* is the collection of points

$$\sigma := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0\ \forall i \right\}.$$



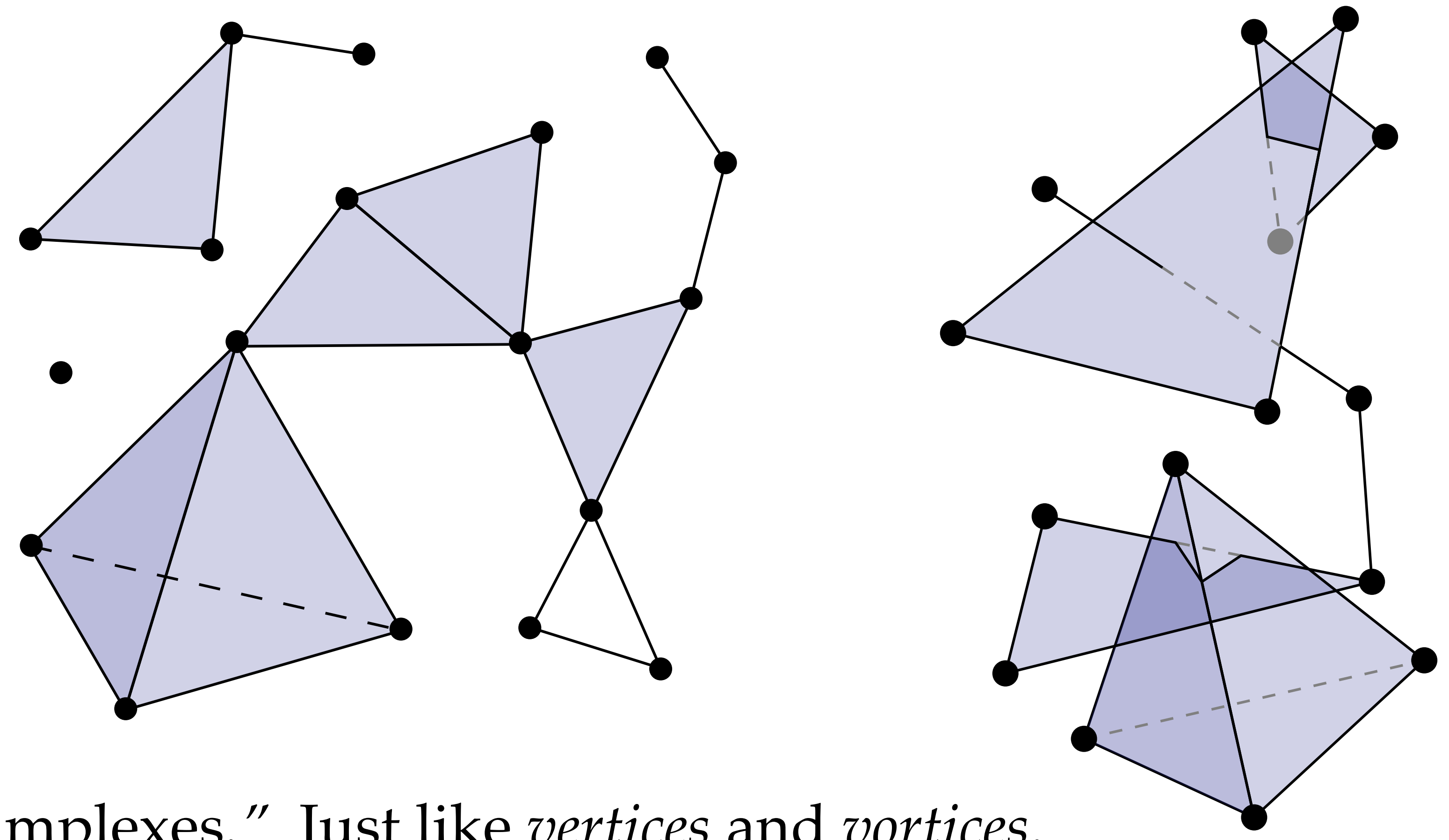
(Also known as the “probability simplex.”)

Simplicial Complex



Simplicial Complex—Rough Idea

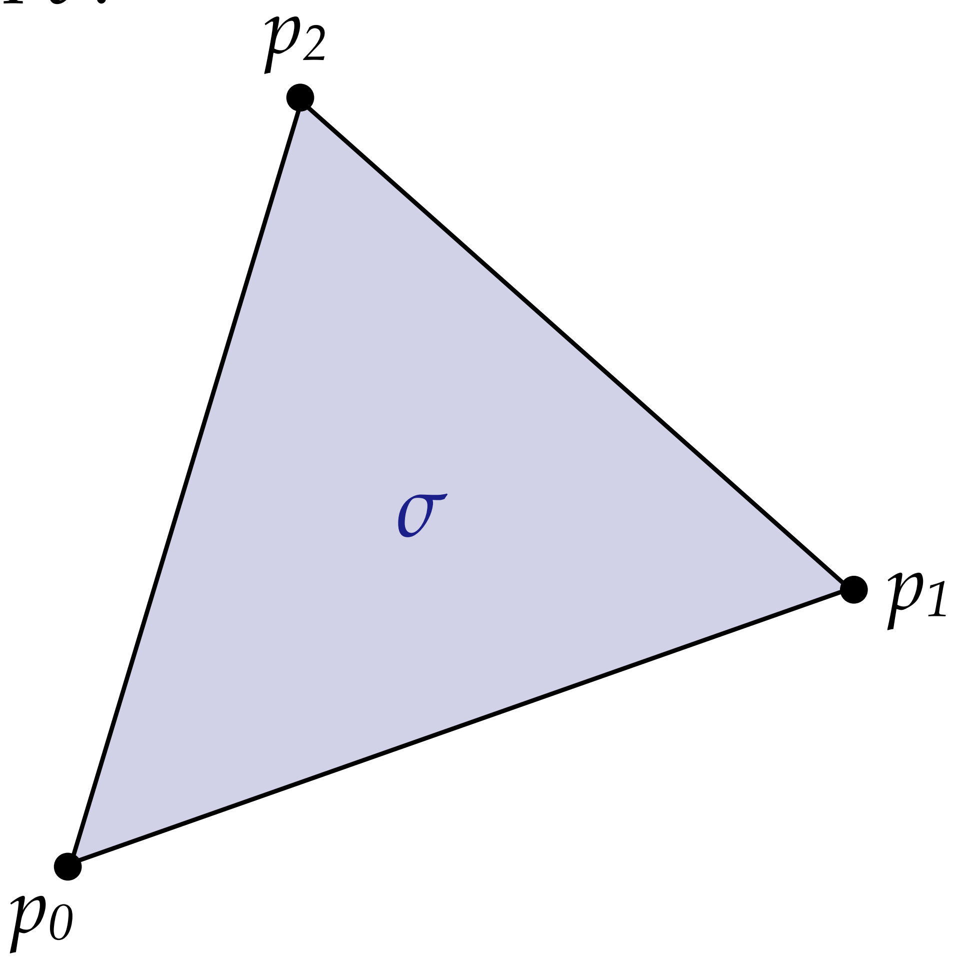
- Roughly speaking, a *simplicial complex* is "a bunch of simplices*"
  - …but with some specific properties that make them easy to work with.
- Also have to resolve some basic questions—*e.g.*, how can simplices intersect?

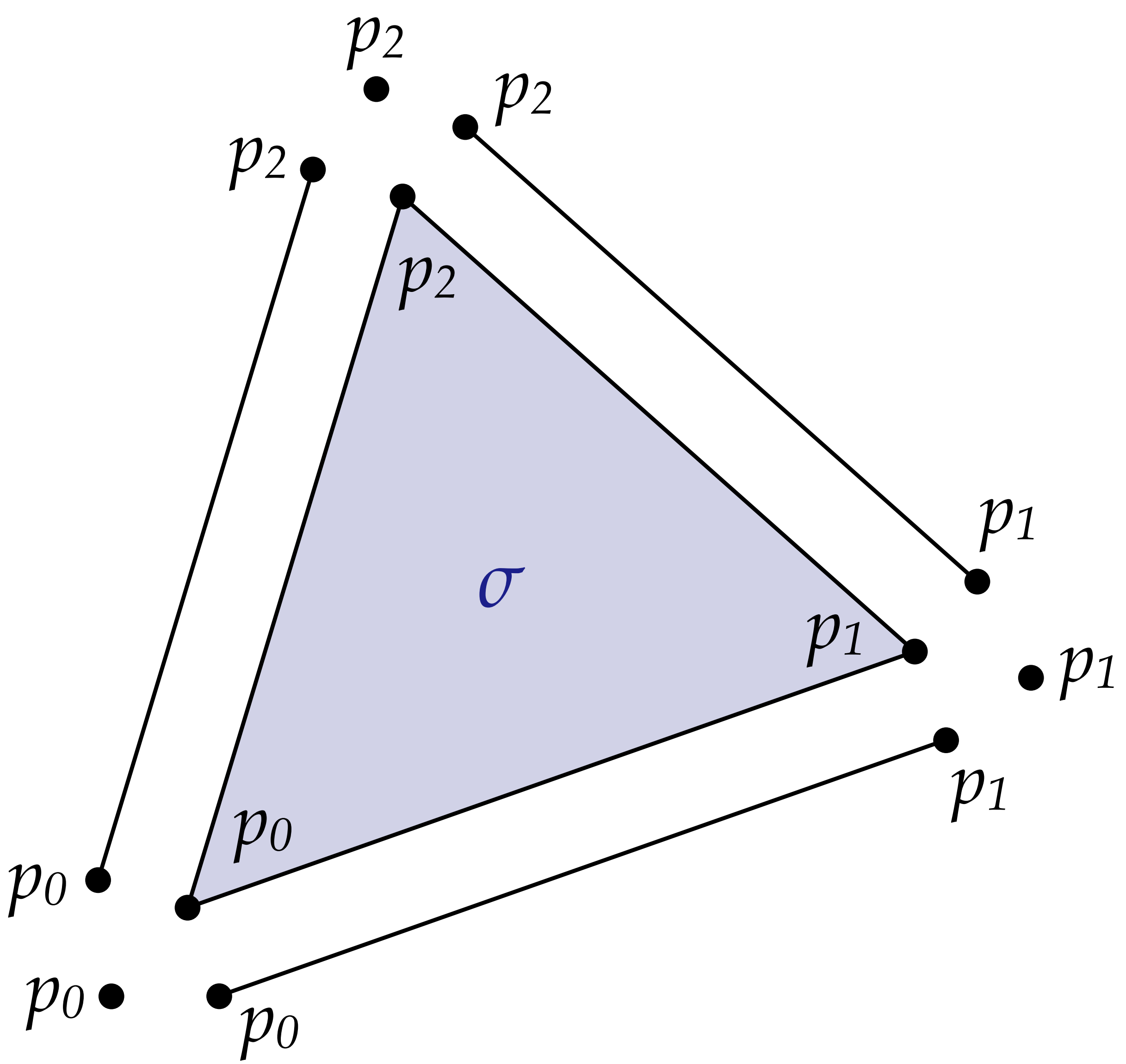


*Plural of simplex; not "simplexes." Just like *vertices* and *vortices*.

Face of a Simplex

Definition. A *face* of a simplex σ is any simplex whose vertices are a subset* of the vertices of σ.





Q: Anything missing from this picture?

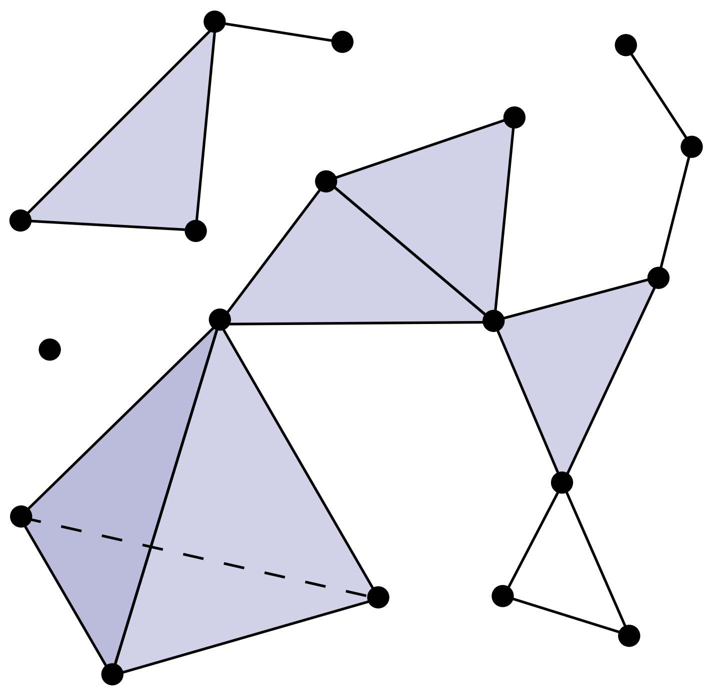
A: Yes—formally, the *empty set* $\emptyset$.

*Doesn't have to be a *proper* subset, *i.e.*, a simplex is its own face.

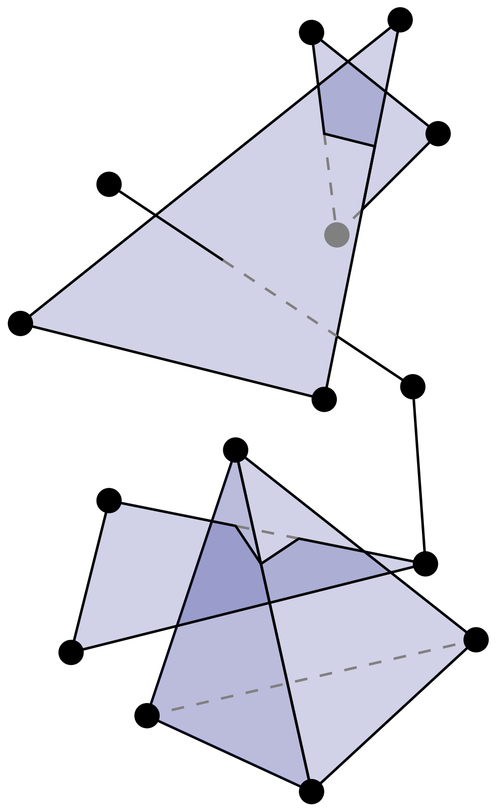
Simplicial Complex—Geometric Definition

Definition. A *(geometric) simplicial complex* is a collection of simplices where

- the intersection of any two simplices is a simplex in the complex, and
- every face of every simplex in the complex is also in the complex

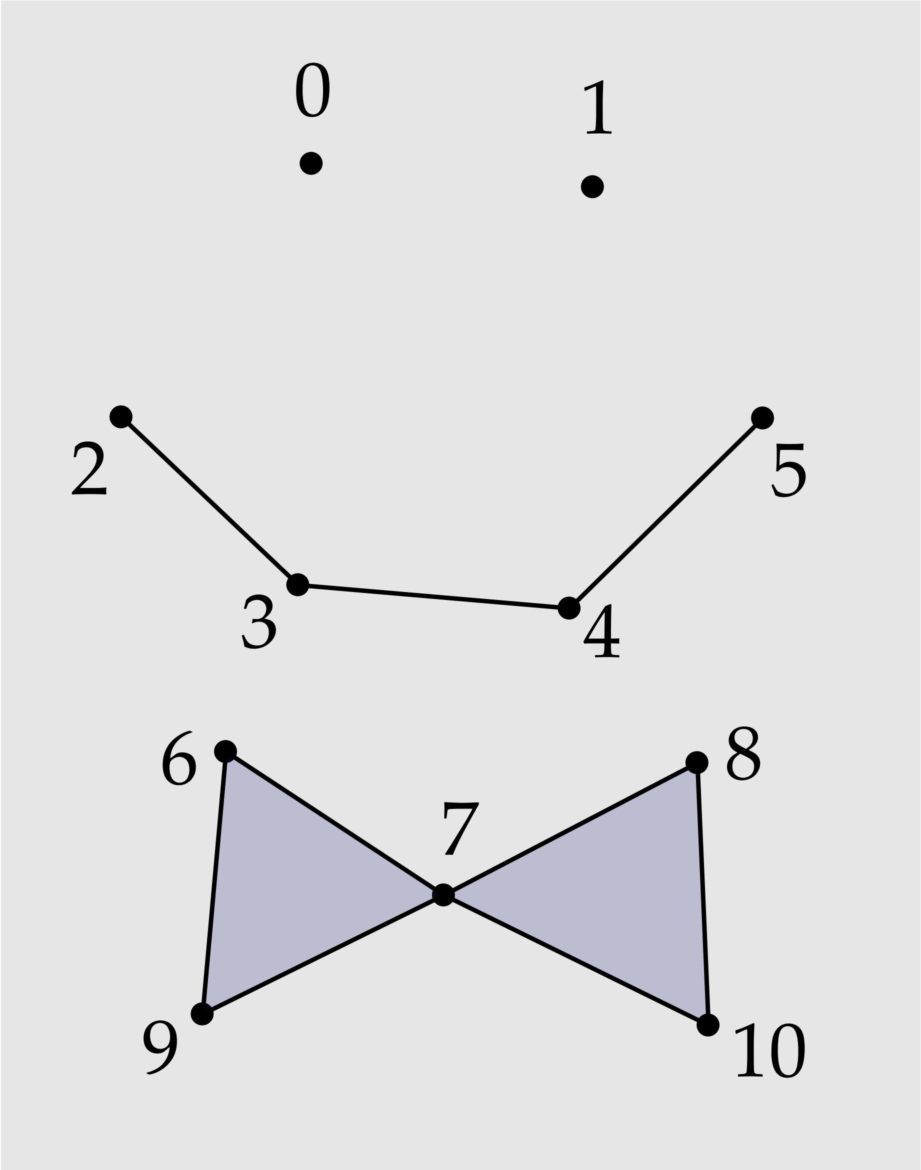


simplicial complex



not a <u>geometric</u> simplicial complex…

Simplicial Complex — Example



Q: What are all the simplices?

A: {6,7,9} {7,10,8}

{2,3} {3,4} {4,5} {6,7} {7,9} {9,6} {7,8} {8,10} {10,7}

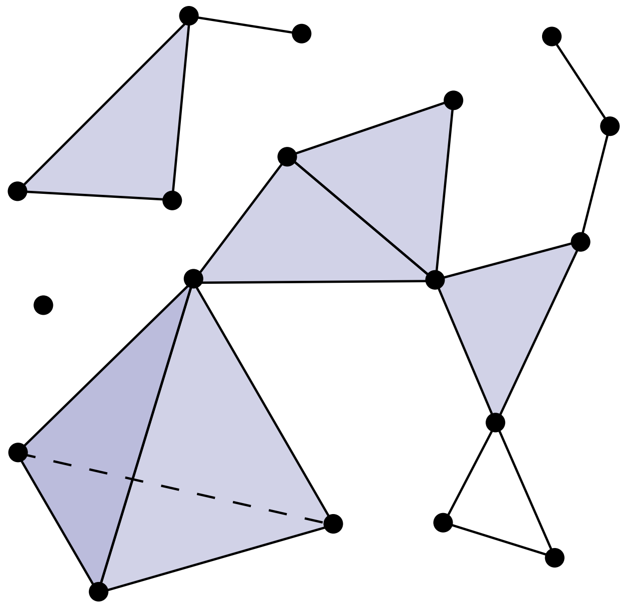
{0} {1} {2} {3} {4} {5} {6} {7} {8} {9} {10}

ø

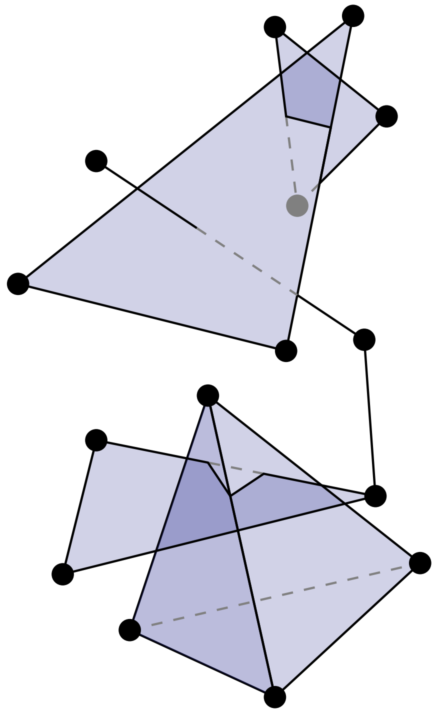
Notice: didn't really say anything about geometry here…

Abstract Simplicial Complex

Definition. Let S be a collection of sets. If for each set $\sigma \in S$ all subsets of σ are contained in S, then S is an *abstract simplicial complex*. A set $\sigma \in S$ of size $k+1$ is an *(abstract) simplex*.



geometric simplicial complex



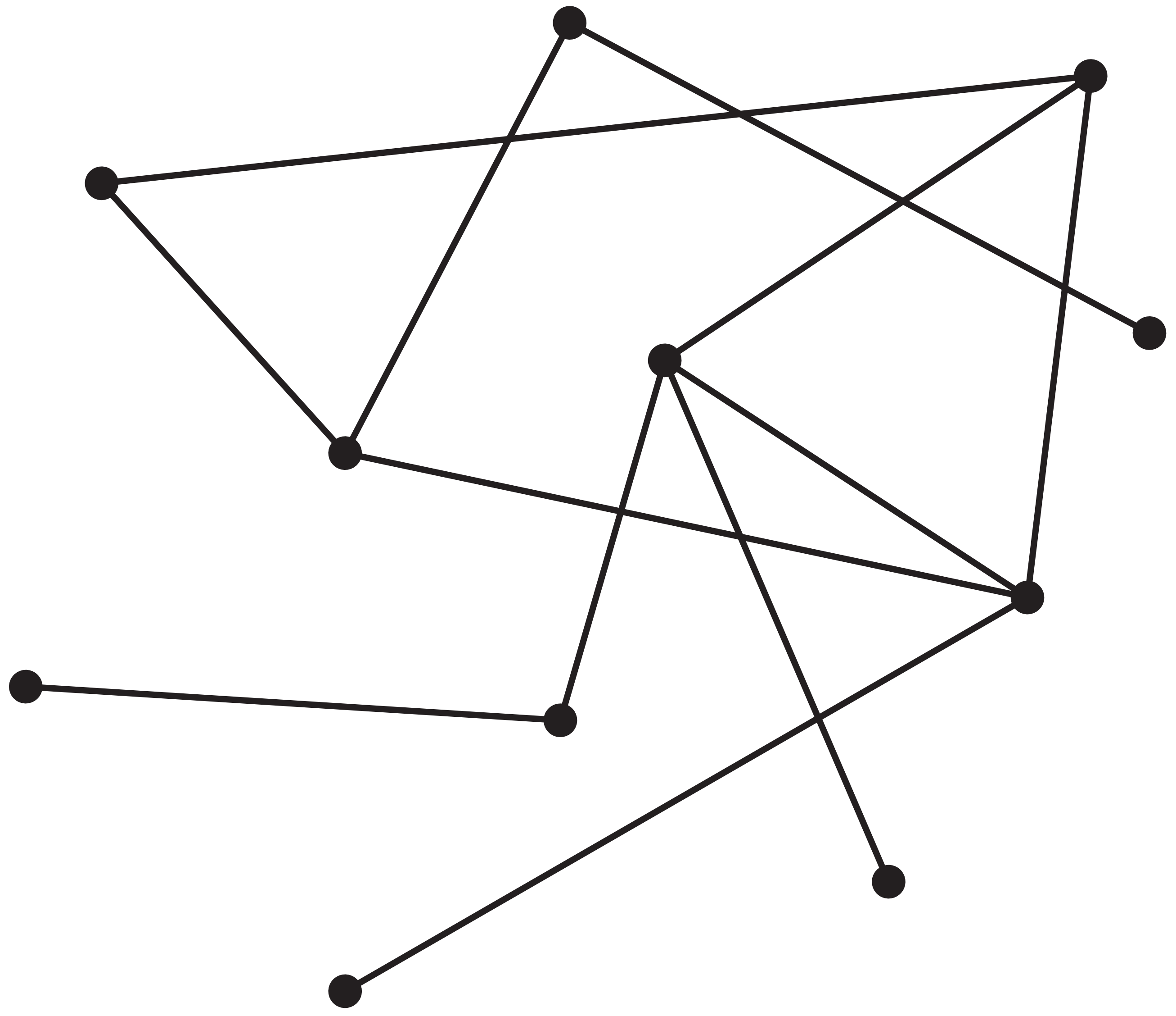
abstract simplicial complex*

- Only care about how things are *connected*, not how they are arranged geometrically.
- Provides our discrete analogue for a *topological space*

*…visualized by mapping it into R^3.

Abstract Simplicial Complex—Graphs

- Any *(undirected) graph* G = *(V,E)* is an abstract simplicial (1-)complex
  - 0-simplices are vertices
  - 1-simplices are edges

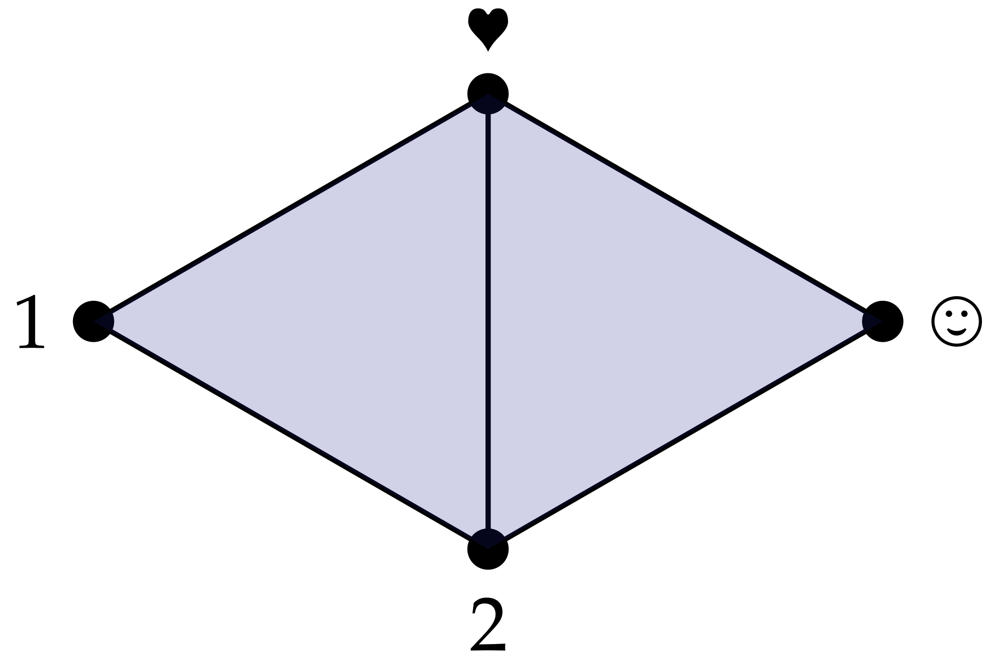


Abstract Simplicial Complex — Example

Example: Consider the set
$S := \{\{1, 2, ♥\}\}, \{2, ♥, ☺\}, \{1, 2\}, \{2, ♥\}, \{♥, 1\}, \{2, ☺\}, \{♥, ☺\}, \{1\}, \{2\}, \{♥\}, \{☺\}, \emptyset\}$

Q: Is this set an abstract simplicial complex? If so, what does it look like?

A: Yes—it's a pair of 2-simplices (triangles) sharing a single edge:



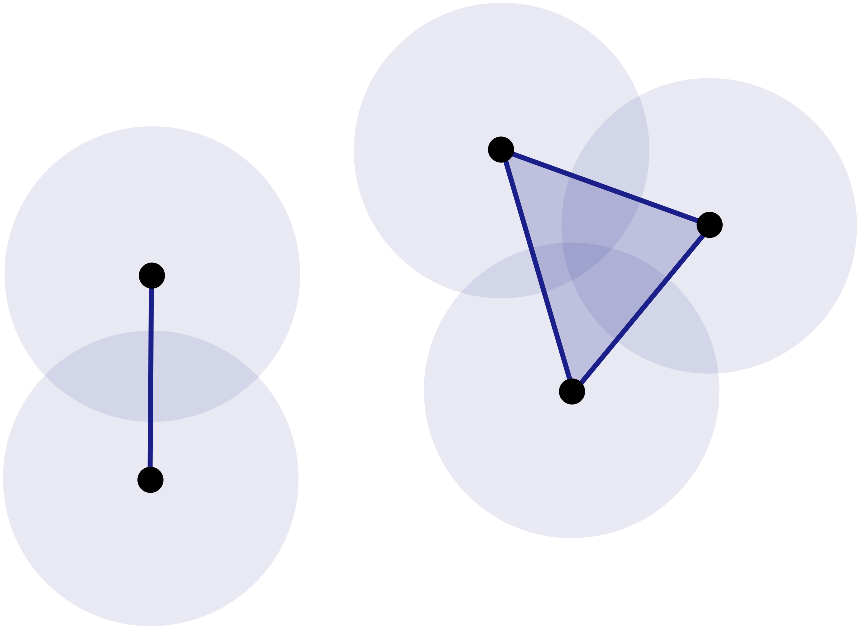
Vertices no longer have to be points in space; can represent anything at all.

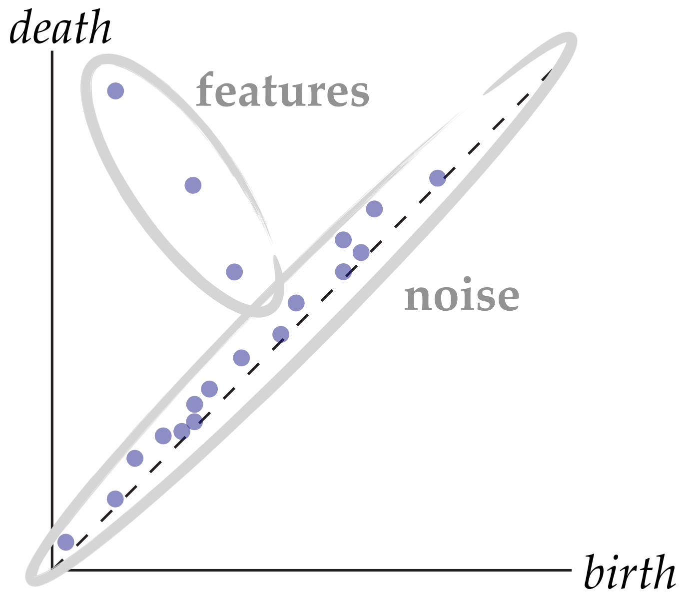
Application: Topological Data Analysis

Forget (mostly) about geometry—try to understand data in terms of *connectivity*.

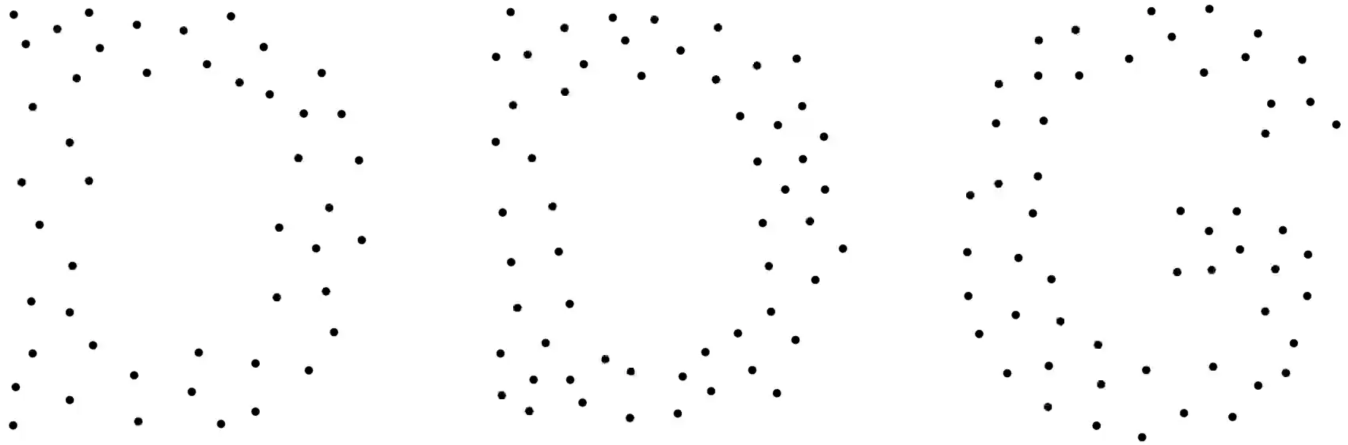
E.g., *persistent homology*:

- "grow" balls around each point
- connect (k+1) overlapping balls into k-simplex
- track "birth" and "death" of features like connected components, holes, etc.
- features that persist for a long time are likely "real"
- features that quickly appear, then disappear are likely noise



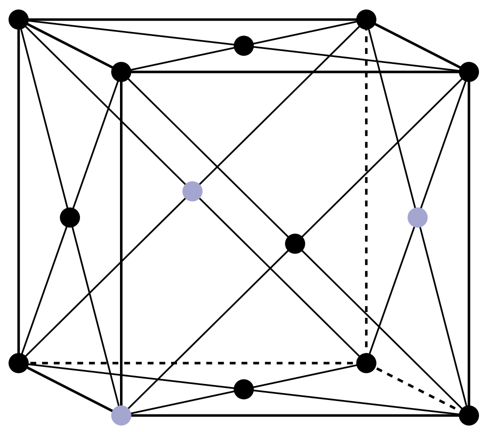


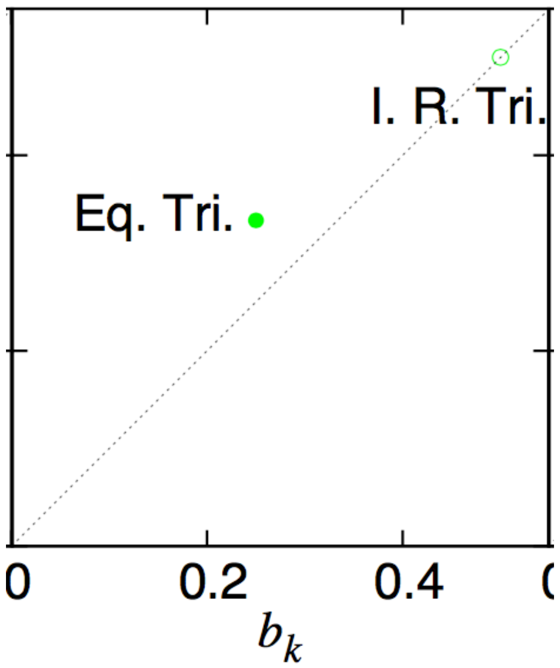
persistence diagram



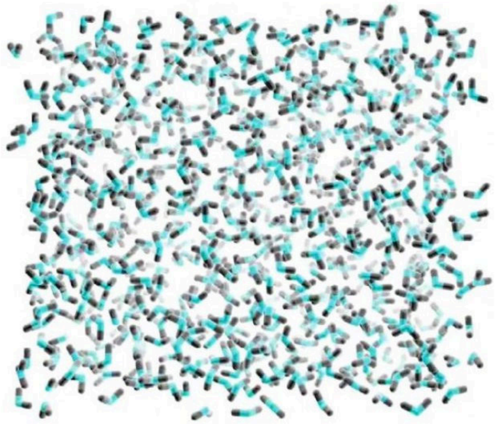
Example: Material Characterization via Persistence

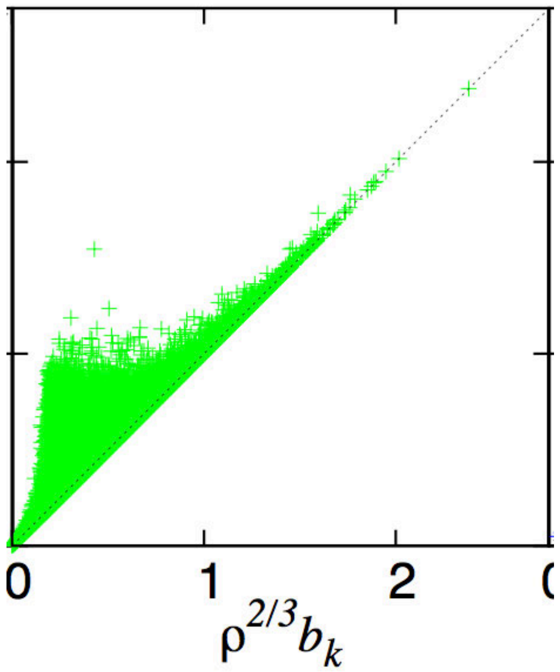
Regular



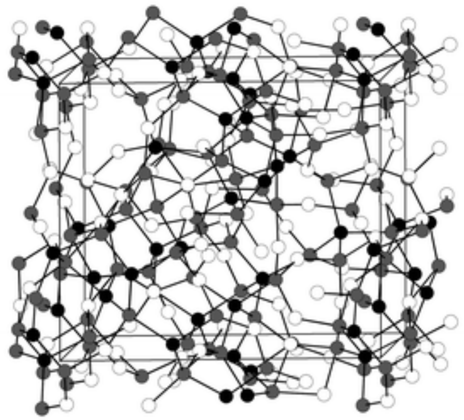


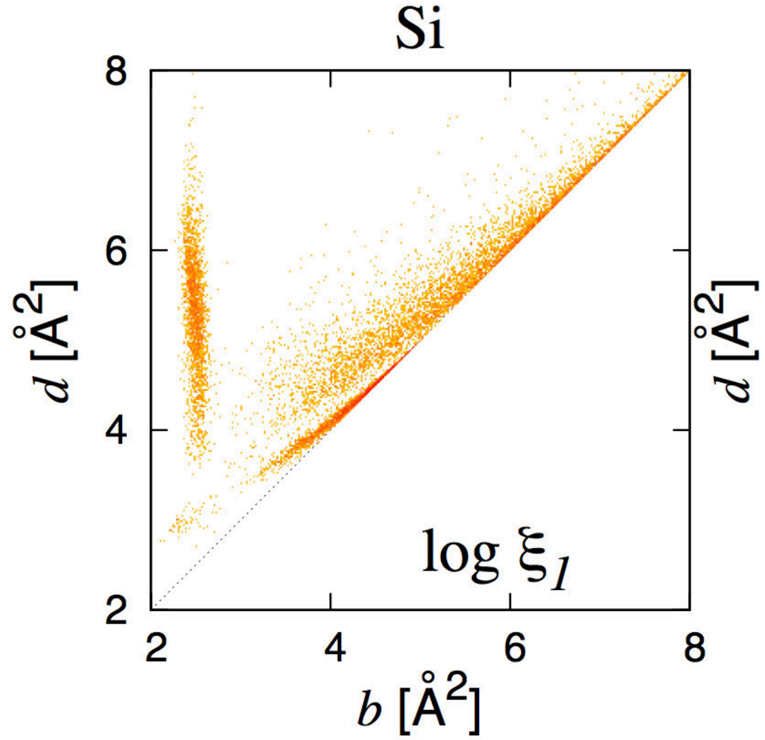
Random





Glass

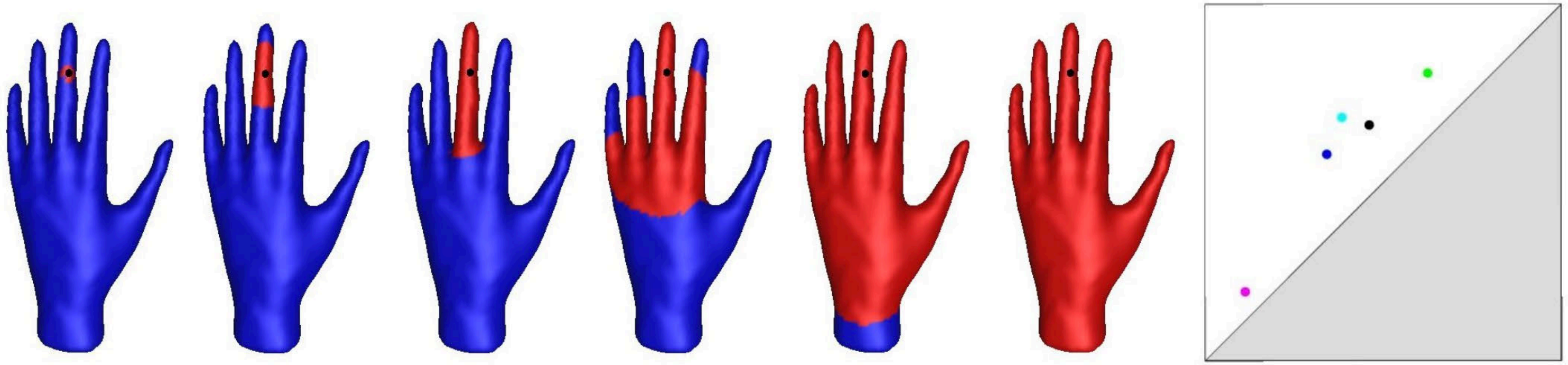




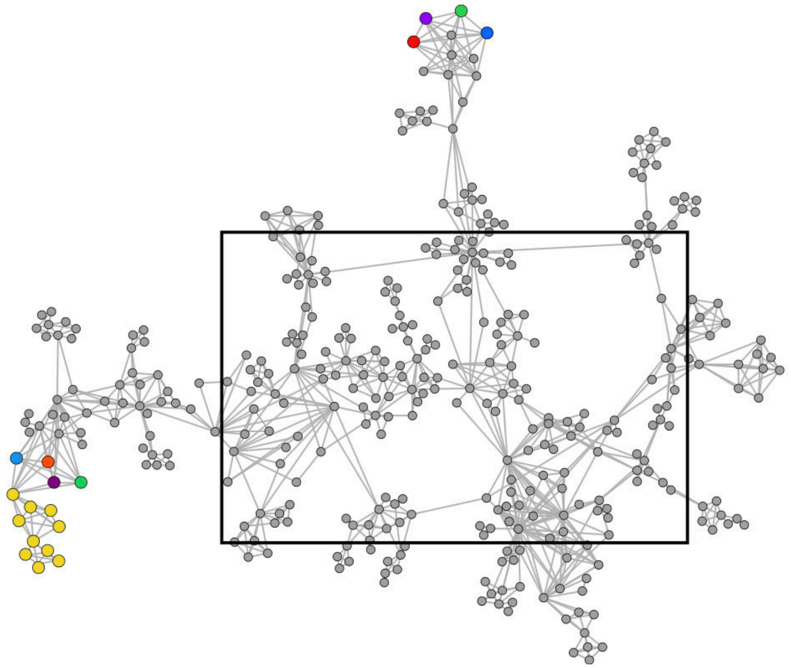
Nakamura et al, *"Persistent Homology and Many-Body Atomic Structure for Medium-Range Order in the Glass"*

Persistent Homology—More Applications

M. Carrière, S. Oudot, M. Ovsjanikov, *"Stable Topological Signatures for Points on 3D Shapes"*



C. Carstens, K. Horadam,
"Persistent Homology of Collaboration Networks "



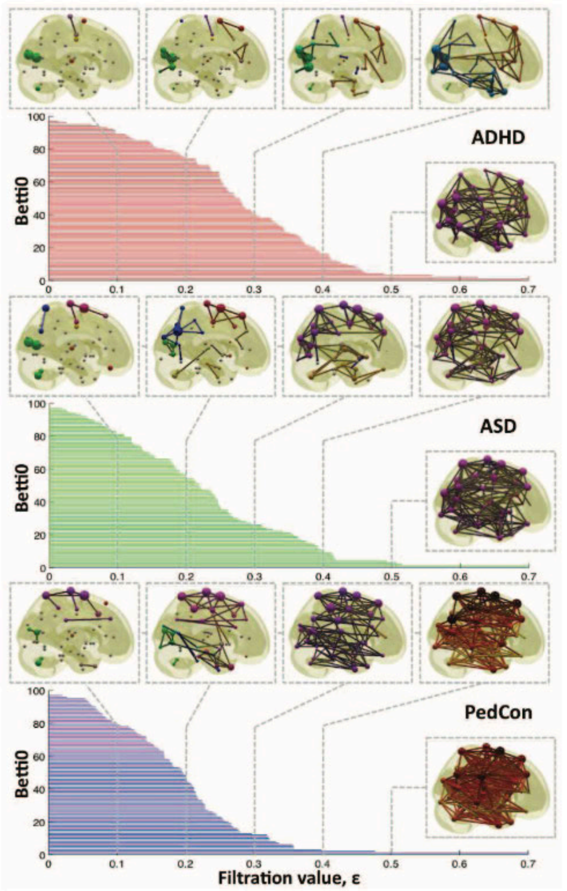


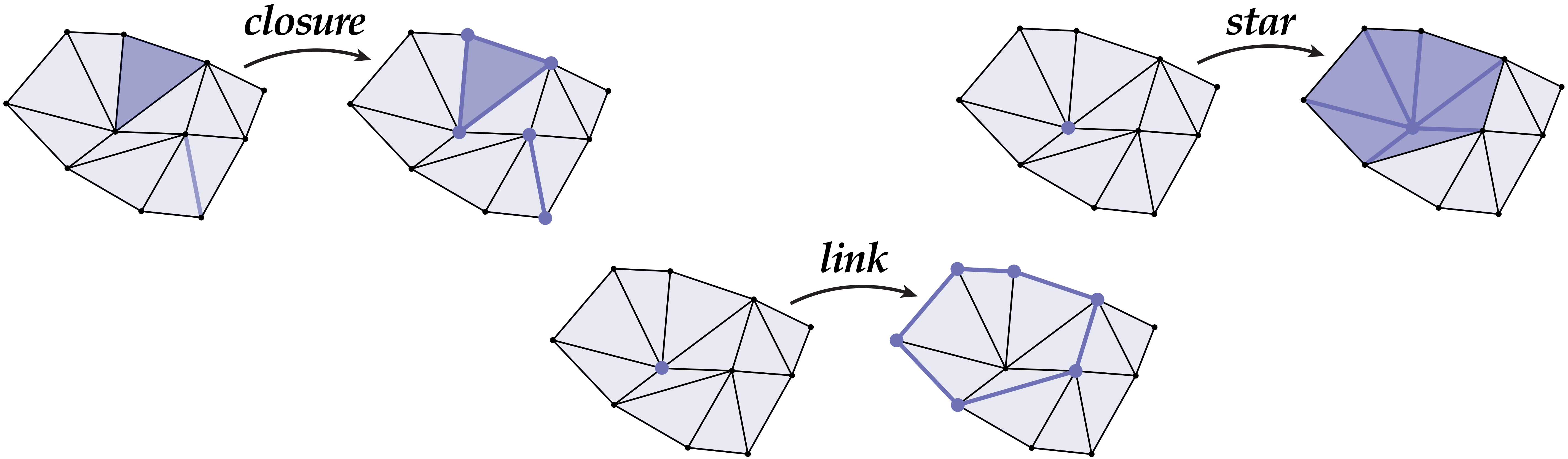
Fig. 4. Barcode of the 0-th Betti number.

H. Lee, M. Chung, H. Kang, B. Kim, D. Lee
"Discriminative Persistent Homology of Brain Networks"

…*and much more* (identifying patients with breast cancer, classifying players in basketball, new ways to compress images, *etc.*)

Anatomy of a Simplicial Complex

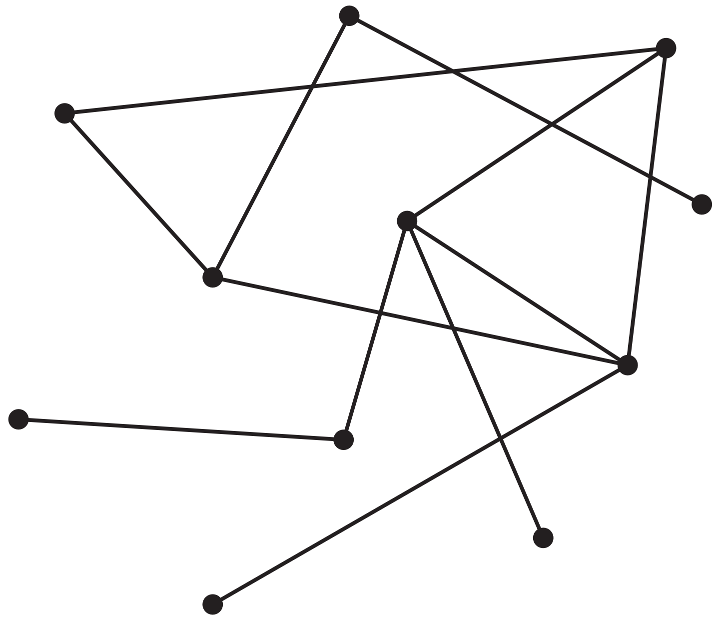
- **Closure:** smallest simplicial complex containing a given set of simplices
- **Star:** union of simplices containing a given subset of simplices
- **Link:** closure of the star minus the star of the closure



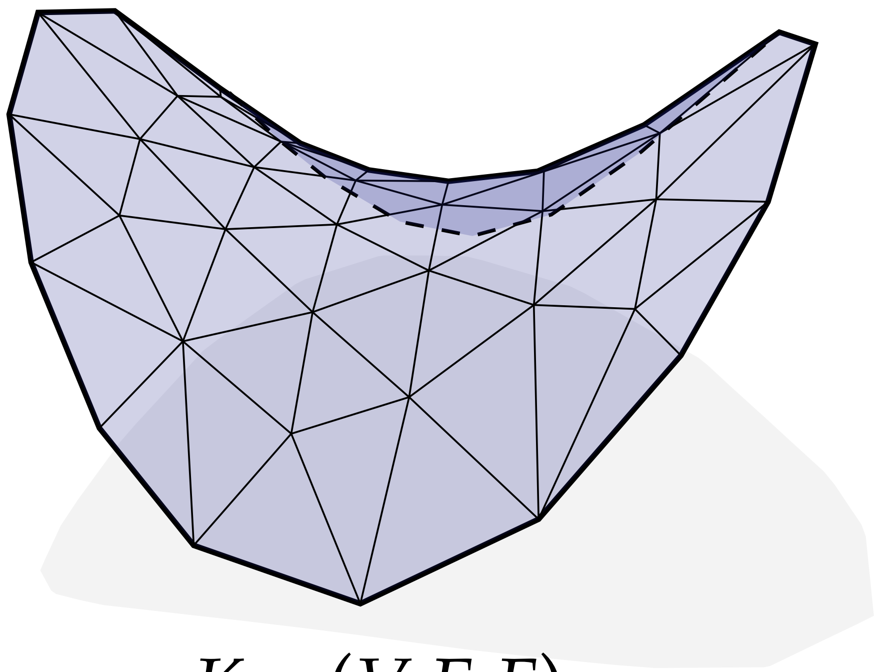
Vertices, Edges, and Faces

Some notation:

- For simplicial **1-complexes** (graphs) we often write $G = (V,E)$
- For simplicial **2-complexes** (triangle meshes) we often write $K = (V,E,F)$
  - V = vertices
  - E = edges
  - F = faces*
  - (K is for *“komplex!”*)



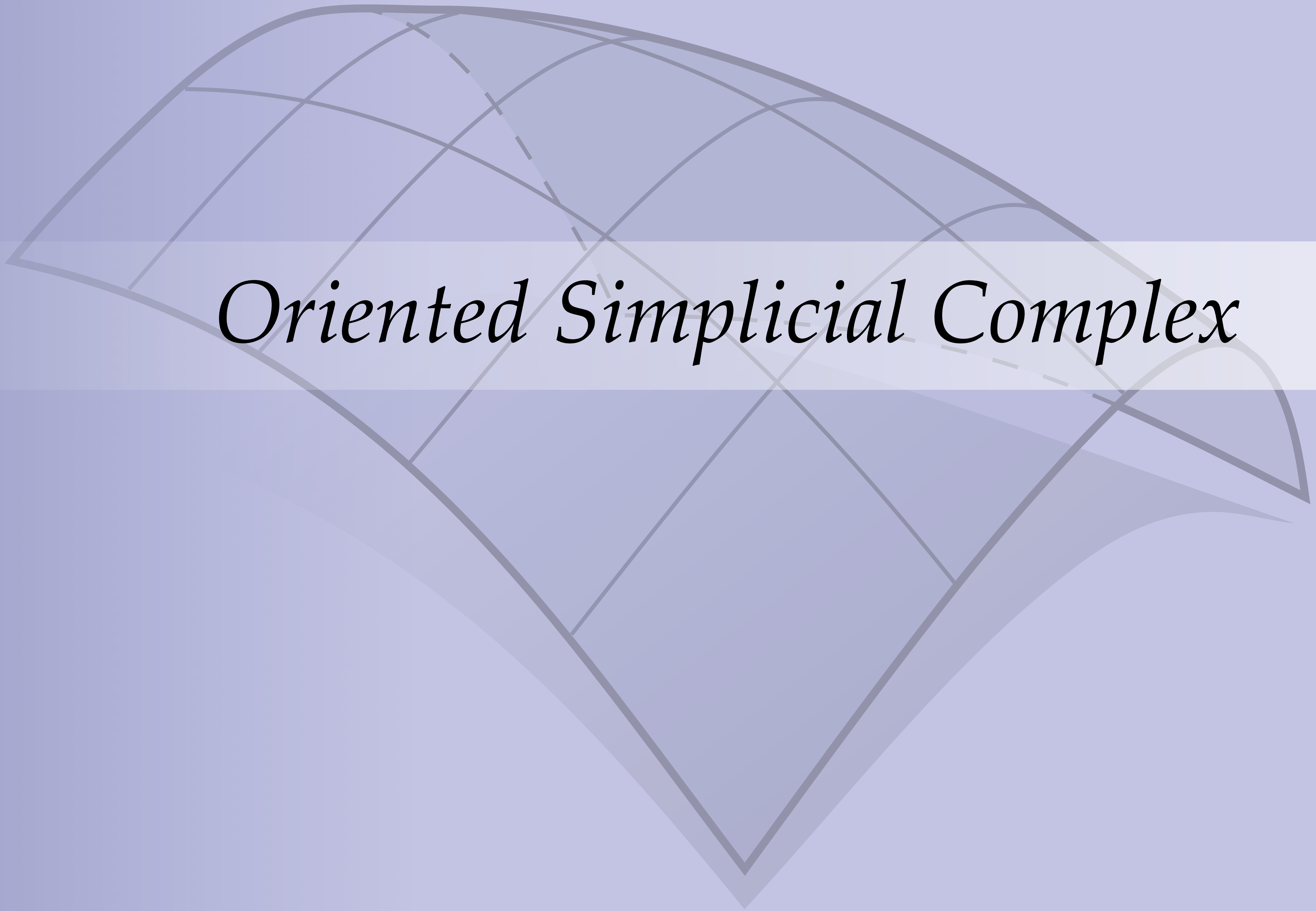
$G = (V,E)$



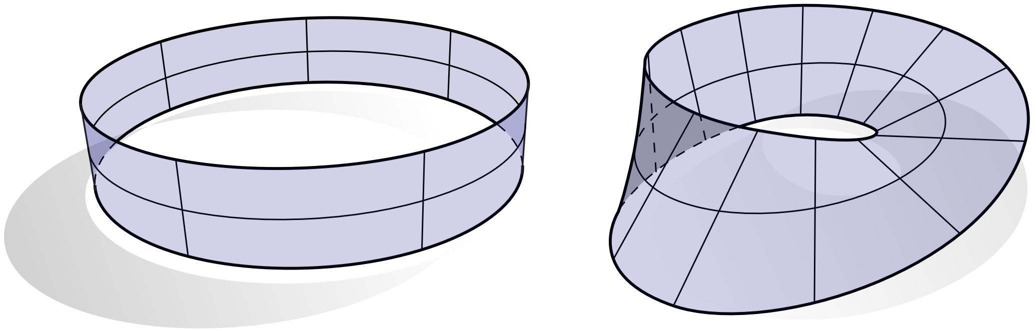
$K = (V,E,F)$

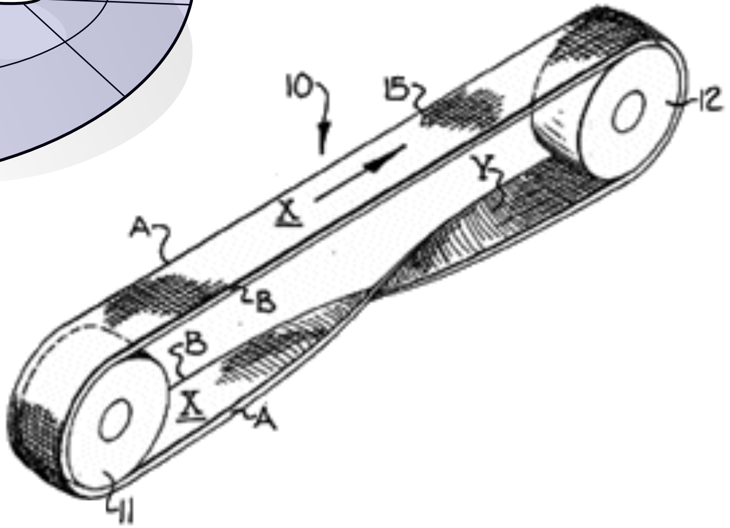
*Not to be confused with the generic *face* of a simplex…

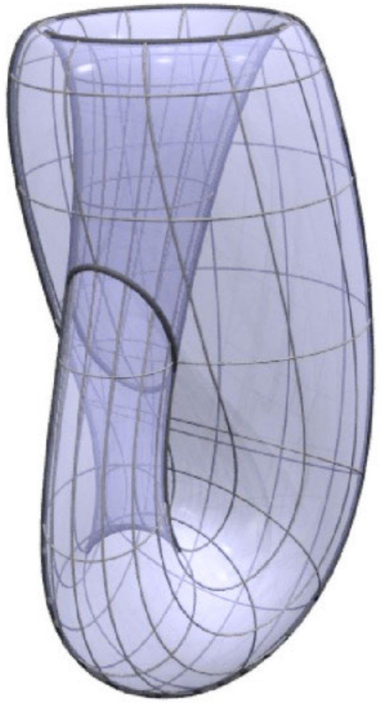
Oriented Simplicial Complex

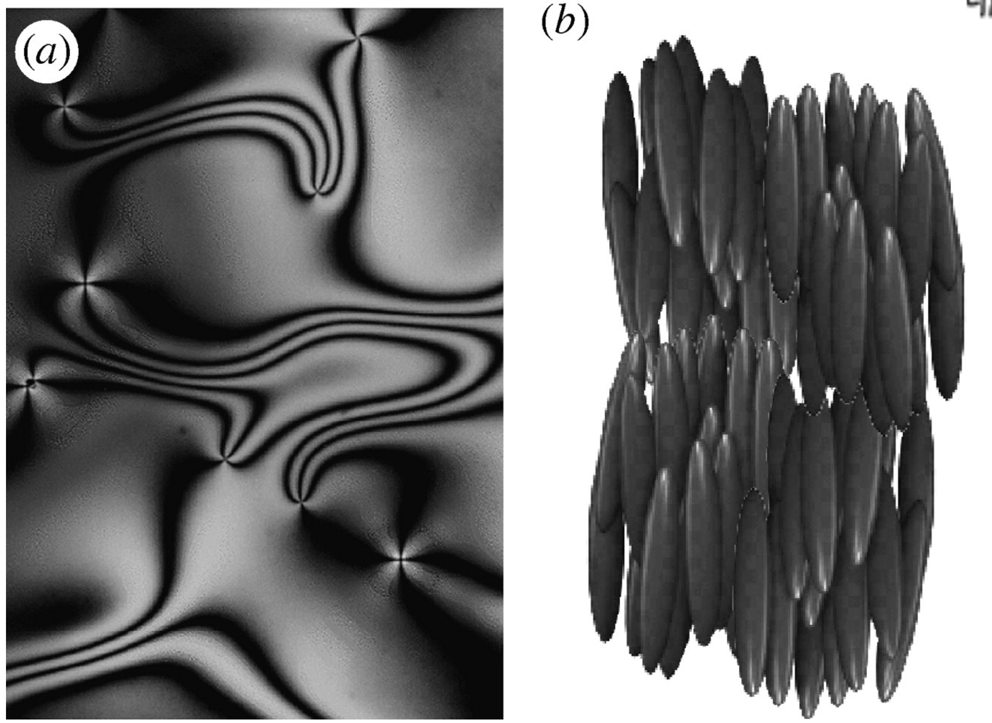


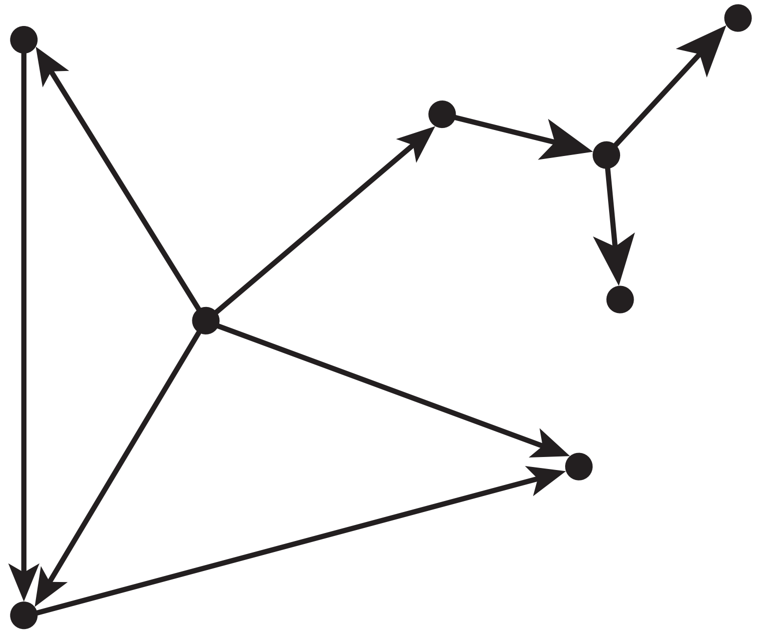
Orientation — Visualized





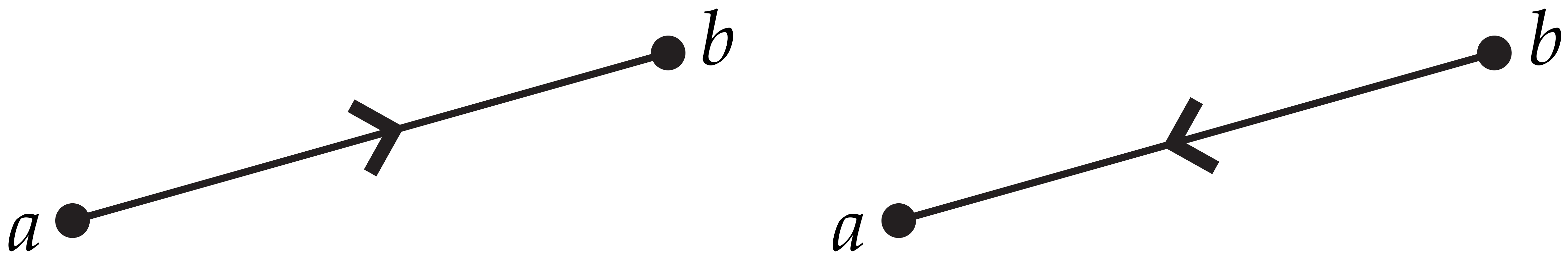






Orientation of a 1-Simplex

- Basic idea: does a 1-simplex $\{a,b\}$ go from a to b or from b to a?
- Instead of unordered set $\{a,b\}$, now have *ordered tuple* (a,b) or (b,a)



- Why do we care? *Eventually* will be useful for recording integrals…

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

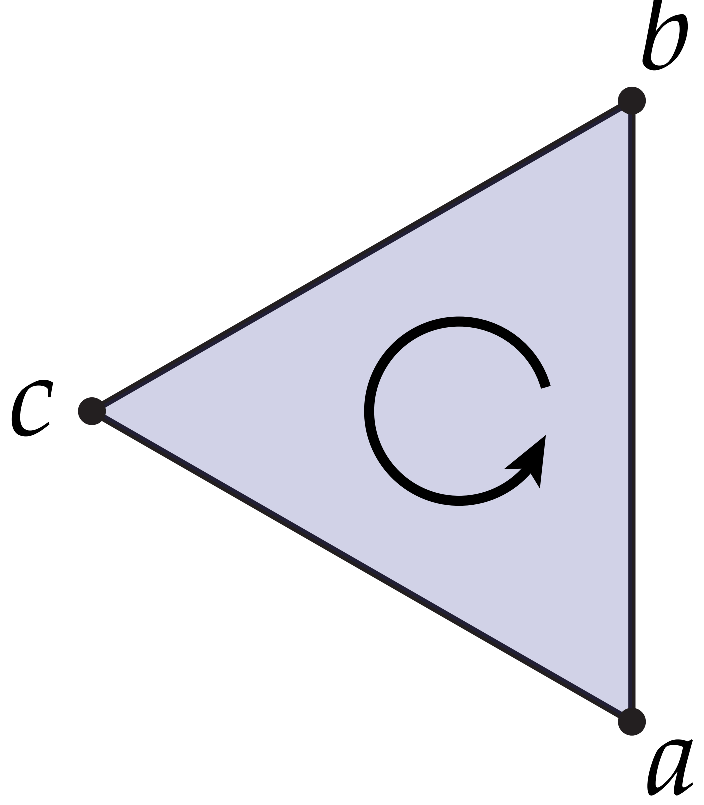
Orientation of a 2-Simplex

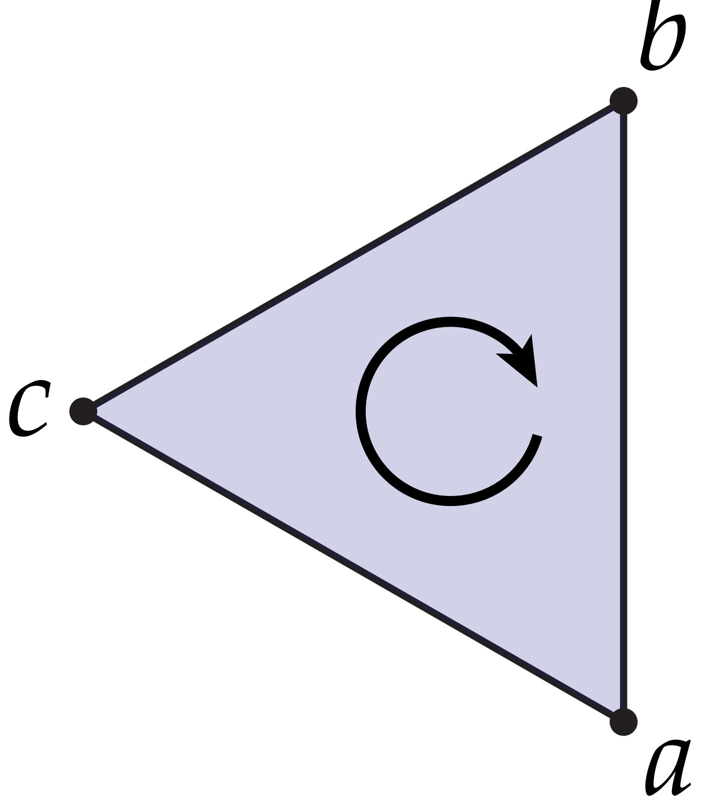
For a 2-simplex, orientation given by "winding order" of vertices:

(a,b,c)

(b,c,a)

(c,a,b)





(a,c,b)

(c,b,a)

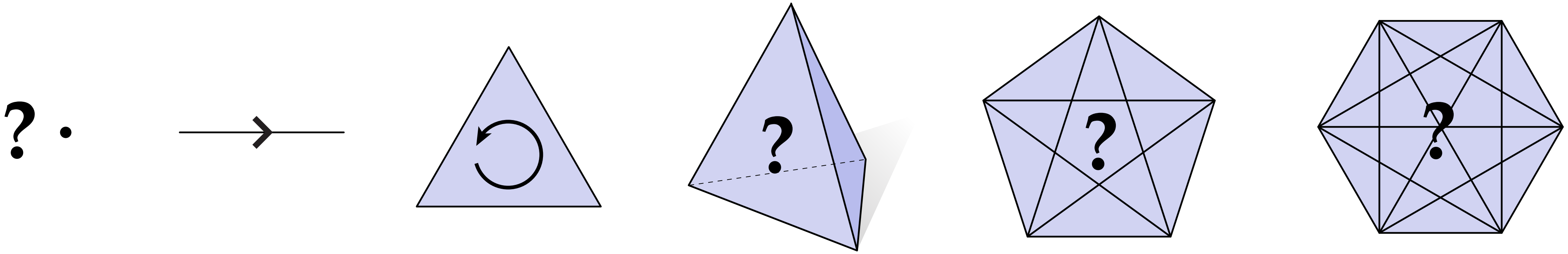
(b,a,c)

Hence, an *oriented 2-simplex* can be specified by a 3-tuple.

(Circular shifts describe the same oriented 2-simplex.)

Oriented k-Simplex

How do we define orientation in general? (Hard to draw arrows…)



Similar idea to orientation for 2-simplex:

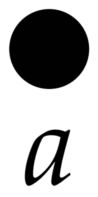
Definition. An oriented k-simplex is an ordered tuple, up to even permutation.

Hence, always* two orientations: *even* or *odd* permutations of vertices.

Convention: even permutations of (0,…,k) **“positive”**; otherwise **“negative.”**

Oriented 0-Simplex?

Q: What's the orientation of a single vertex?



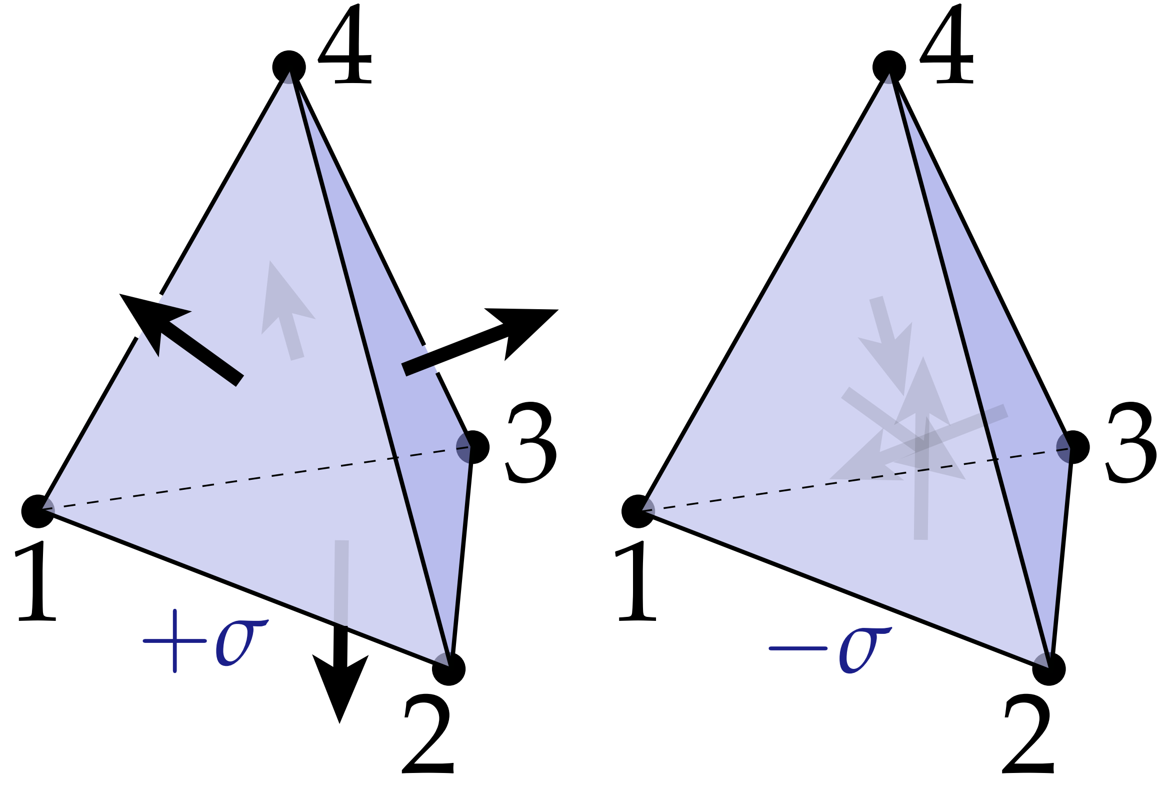
A: Only one permutation of vertices; only one orientation! (Positive):

$$(a)$$

Oriented 3-Simplex

Hard to draw pictures as k gets large!

But still easy to apply definition:



even / positive		**odd / negative**	
$(1,2,3,4)$	$(3,1,2,4)$	$(1,2,4,3)$	$(3,1,4,2)$
$(1,3,4,2)$	$(3,2,4,1)$	$(1,3,2,4)$	$(3,2,1,4)$
$(1,4,2,3)$	$(3,4,1,2)$	$(1,4,3,2)$	$(3,4,2,1)$
$(2,1,4,3)$	$(4,1,3,2)$	$(2,1,3,4)$	$(4,1,2,3)$
$(2,3,1,4)$	$(4,2,1,3)$	$(2,3,4,1)$	$(4,2,3,1)$
$(2,4,3,1)$	$(4,3,2,1)$	$(2,4,1,3)$	$(4,3,1,2)$

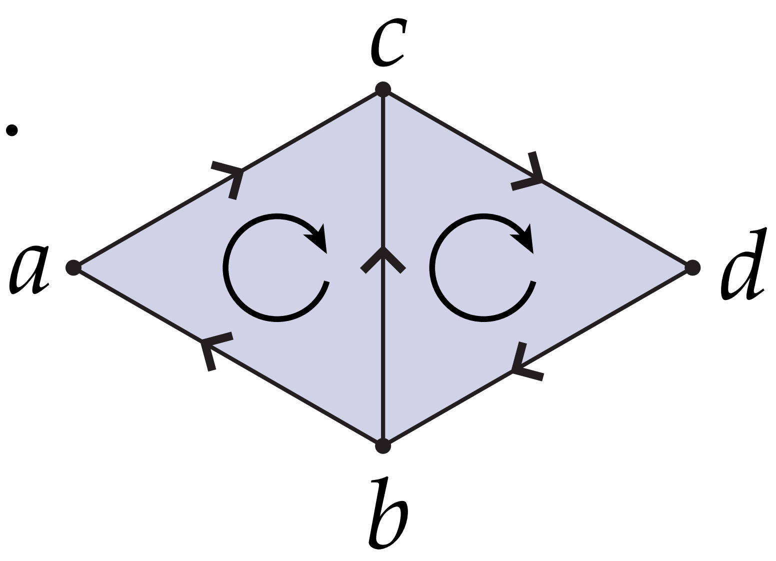
…much easier, of course, to just pick a single representative.

E.g., $+\sigma := (1,2,3,4)$, and $-\sigma := (1,2,4,3)$.

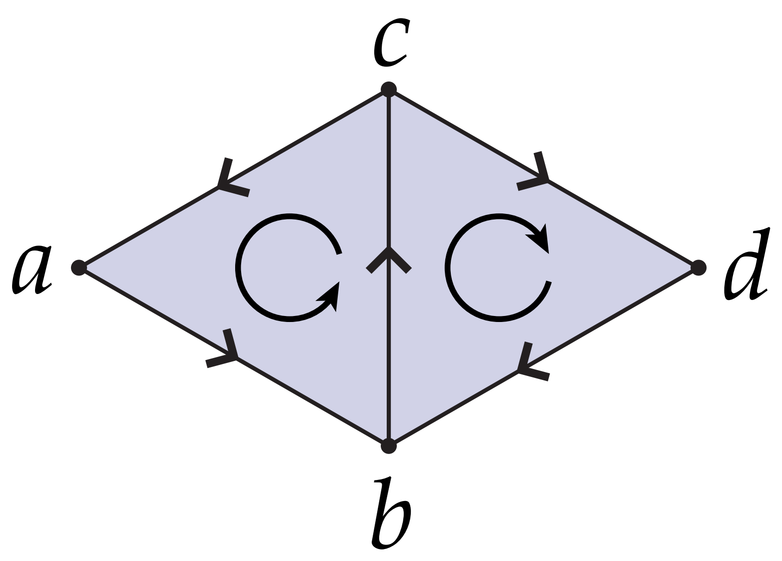
Oriented Simplicial Complex

Definition. An *orientation* of a simplex is an ordering of its vertices up to even permutation; one can specify an oriented simplex via one of its representative ordered tuples. An *oriented simplicial complex* is a simplicial complex where each simplex is given an ordering.

Example.



$$\{\emptyset, (a), (b), (c), (d), (a,c), (b,a), (b,c), (c,d), (d,b), (a,c,b), (b,c,d)\}$$

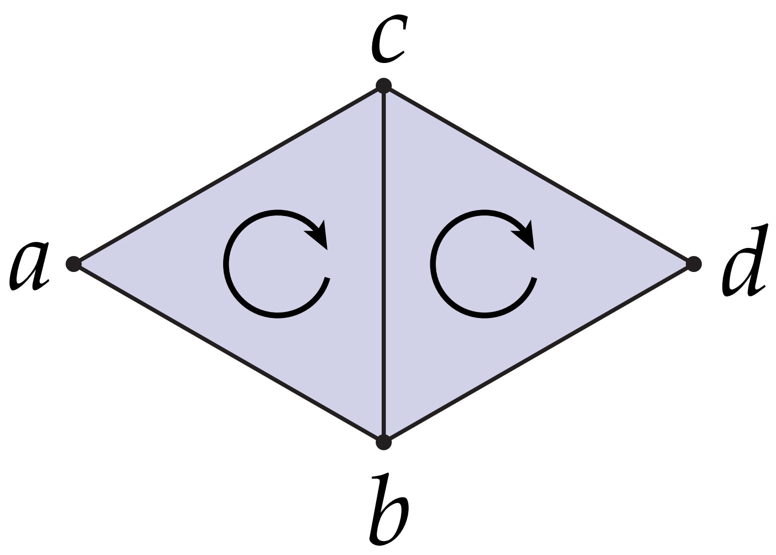


$$\{\emptyset, (a), (b), (c), (d), (c,a), (a,b), (b,c), (c,d), (d,b), (a,b,c), (b,c,d)\}$$

Relative Orientation

Definition. Two distinct oriented simplices have the same *relative orientation* if the two (maximal) faces in their intersection have **opposite** orientation.

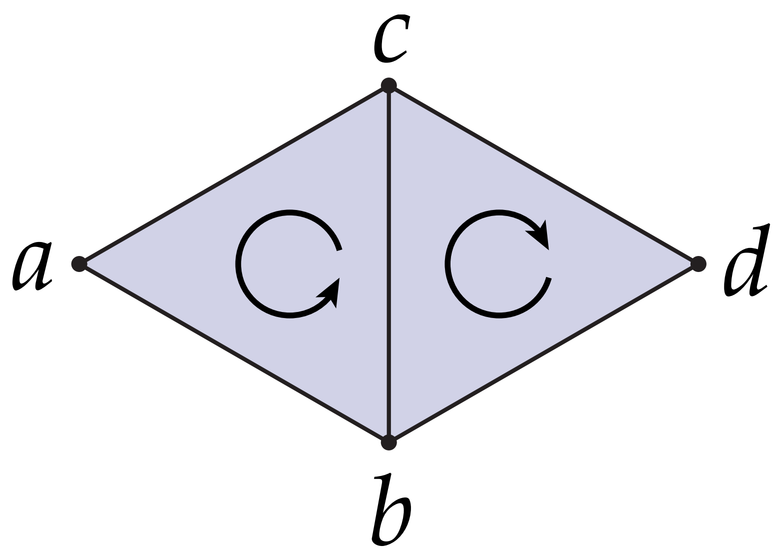
<u>**Example:**</u> Consider two triangles that intersect along an edge:



same relative orientation

$$(a,c,b) \quad \Rightarrow \quad (c,b)$$
$$(b,c,d) \quad \Rightarrow \quad (b,c)$$

$$(c,b) = -(b,c)$$

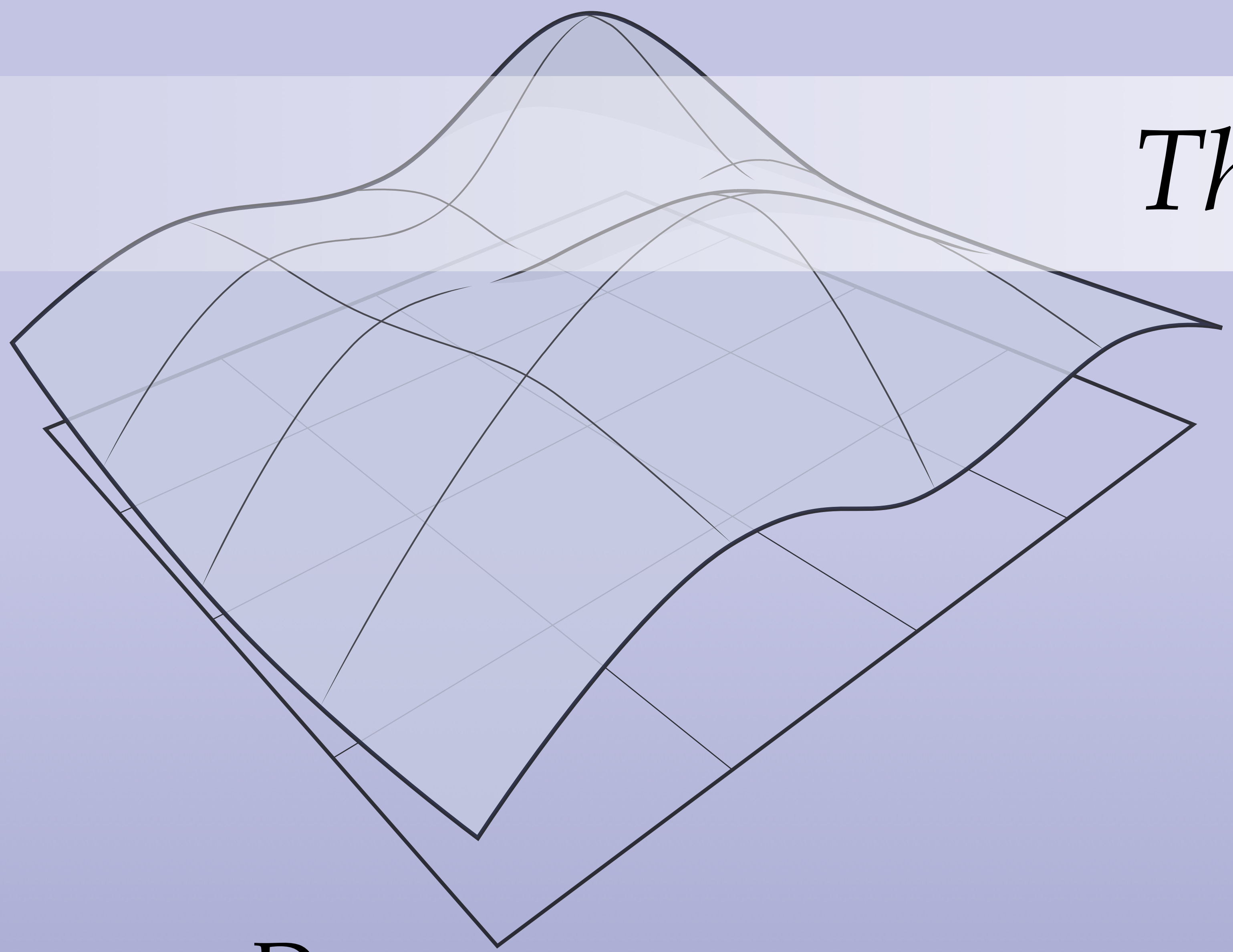


different relative orientation

$$(a,b,c) \quad \Rightarrow \quad (b,c)$$
$$(b,c,d) \quad \Rightarrow \quad (b,c)$$

$$(b,c) = +(b,c)$$

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

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