Discrete

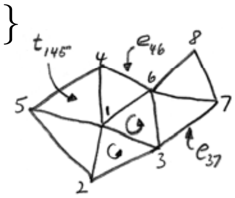
Setup

"pointers" "floats"

- topology & geometry
- simplicial complex: "triangle mesh"
  - 2-manifold $K = \{V, E, T\}$

    $V = \{v_i\}$ $E = \{e_{ij}\}$ $T = \{t_{ijk}\}$
  - Euler characteristic

    $F - E + V = 2(1-g) = \chi$



A First Example

Gaussian curvature

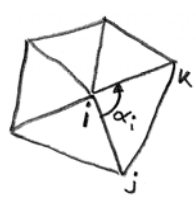
- Gauss-Bonnet

  $2\pi\chi = \int_S K\, dA$
- discrete

  $K_i = 2\pi - \sum_{t_{ijk}} \alpha_i$

  $\sum_i K_i = 2\pi\chi$

$K > 0$ $K = 0$ $K < 0$

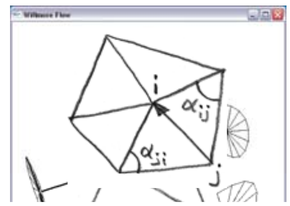


Geometric Flow (Area)

Minimize area energy

- minimal surface: area gradient flow

$$E_A = \int_S 1\, dA \qquad S_t = -\nabla E_A$$



$$2\partial_{v_i} A_{t_{ijk}} = R^{\pi/2}(v_k - v_j)$$

$$\partial_t v_i = -\nabla_{v_i} E_A = -\frac{1}{2}\sum_{e_{ij}}(\cot\alpha_{ij} + \cot\alpha_{ji})(v_i - v_j)$$

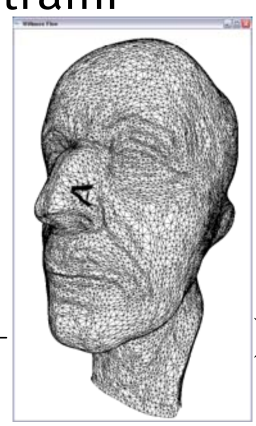
Mean Curvature Flow

Laplace and Laplace-Beltrami

- Dirichlet energy

$$\begin{matrix} \Delta u = 0 \\ u|_{\partial\Omega} = u_0 \end{matrix} \rightsquigarrow \min \int (\nabla u)^2$$

- on surface



$$\partial_t v_i = -(H\vec{n})_i = -\frac{1}{4A_i}\sum_{e_{ij}}(\cot\alpha_{ij} + \quad)$$

Parameterization

Harmonic function

- from surface to region in $\mathbb{R}^2$

$$u : S \rightarrow \mathbb{R}^2$$

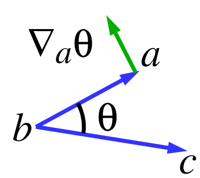
$$\Delta_S u = 0$$

- linear system
  - boundaries
    - Dirichlet/Neumann

Geometry-based Approach

Benefits

- everything is geometric
  - often more straightforward
  - tons of indices: forbidden!



The story is not finished...

- still many open questions
  - in particular: numerical analysis

What Matters?

Structure preservation!

Accuracy
Speed
Size

- symmetry groups
  - rigid bodies: Euclidean group
  - fluids: diffeomorphism group
  - conformal geometry: Möbius group
- many more: symplectic, invariants, Stokes' theorem, de Rham complex, etc. (pick your favorite...)

Themes for This Class

What characterizes structure(s)?

- dealing with shape
  - Euclidean invariance

- dealing with physics
  - conservation/balance laws

- importance of measures
  - mass, area, curvature, flux, circulation

Things to Cover

Differential Geometry

- curves and surfaces (volumes)
- computational topology
- (discrete) exterior calculus
- geometric flows
- parameterization
- discrete connections