Curvature of Curves

Not just normal curvature

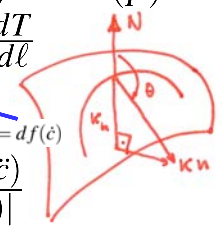
normal component of curvature of curve

- curve in surface $\kappa_n(p) = -\kappa(p)\cos\theta$

$$\gamma = f(c) \quad T = \frac{df(\dot{c})}{|df(\dot{c})|} \quad \kappa n = \frac{dT}{d\ell}$$

$$\kappa_n = -N \cdot \kappa n = -N \cdot \frac{dT}{dt}\frac{dt}{d\ell} \quad \frac{d\ell}{dt} = df(\dot{c})$$

$$df(\dot{c})\kappa_n = -N \cdot \frac{dT}{dt} = -\frac{N \cdot df(\ddot{c})}{|df(\dot{c})|}$$

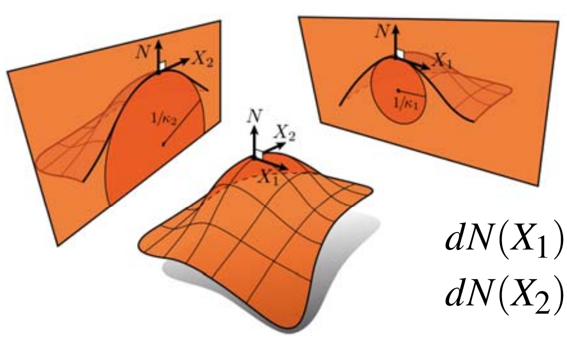


- all curves with same tangent vector have same normal curvature!

Principal Curvatures

Self-adjoint operator

- complete set of eigenvalues/vectors



$$dN(X_1) = \kappa_1 df(X_1)$$
$$dN(X_2) = \kappa_2 df(X_2)$$

Invariants

Gaussian and mean curvature

- determinant and trace only

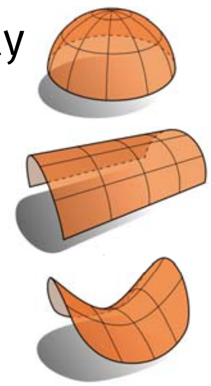
geometric mean $\det S = \kappa_1 \kappa_2 = K$

arithmetic mean $\operatorname{tr} S = \kappa_1 + \kappa_2 = 2H$

- eigenvalues/vectors

$$\kappa_n(V(\theta)) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

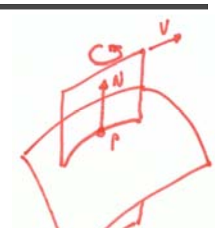
$V(\theta) = X\cos\theta + X^{\perp}\sin\theta$



Curvatures

Integral representations

- smooth setting



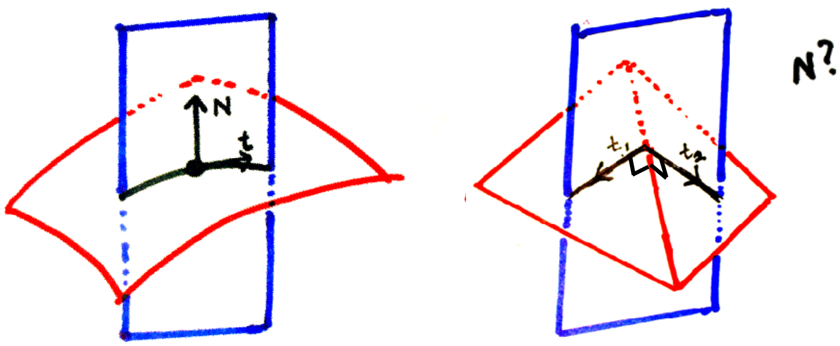
$|X| = |Y| = 1 \quad X \perp Y$

$$H_p = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\,d\theta$$

$$K_p = \lim_{A\to 0} \frac{A_G}{A} \qquad \boxed{K|df(X)\times df(Y)|}$$

$$\frac{\int_{\mathbb{S}^2_A} d\mathbb{S}^2}{\int_{S_A} dA} = \frac{\int |dN(X)\times dN(Y)|}{\int |df(X)\times df(Y)|}$$

Discrete Setup?



Gaussian Curvature

On a mesh

- can't take the limit… $K_p = \lim_{A \to 0} \frac{A_G}{A}$
  - average does make sense

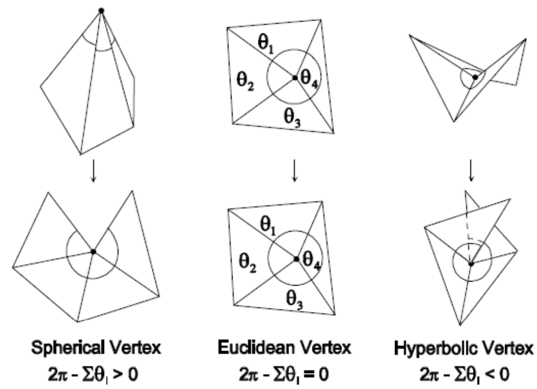
$$\int_A K_p \approx A K_p \approx A_G$$

only makes sense as an integral, NEVER pointwise

$$2\pi - \sum_{t_{pjk}} \alpha_{jk}$$

Discrete Gauss curvature at a vertex

A Good Definition?



developable...

Scalar Mean Curvature

Integral representation

- variation along a vector field

$$\partial_V \mathrm{Vol} = \int \langle V, N \rangle \, dA$$

$$\boxed{H = \frac{\partial_V \mathrm{Area}}{\partial_V \mathrm{Vol}}}$$

for constant V move out of integral

$$\boxed{\frac{|\int \mathbf{H}\, dA|}{|\int N\, dA|} \to H_p}$$

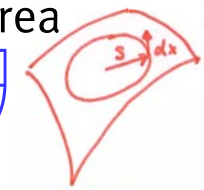
$$\partial_V \mathrm{Area} = \int \langle V, \underline{\mathbf{H}} \rangle \, dA$$

Boundary Integrals

$\frac{|\int \mathbf{H}\, dA|}{|\int N\, dA|}$

Vector area

- volume gradient: vector area



$$D \subset S \qquad \gamma = \partial D$$

another normal

$$\int_D N\, dA = 1/2 \oint_\gamma f \times df(X) d\ell = \mathbf{A}_\gamma$$

- discrete version

only makes sense as an integral, NEVER pointwise

$$3\mathbf{A}_i = 1/2 \sum p_j \times p_{j+1}$$

area weighted triangle normals