Boundary Integrals

Area gradient

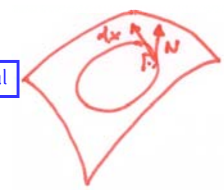
$\frac{|\int \mathbf{H}\,dA|}{|\int N\,dA|}$

- vector mean curvature

$$D \subset S \quad \gamma = \partial D$$

another normal

$$\int_D \mathbf{H}\,dA = \oint_\gamma N \times df(X)\,d\ell$$

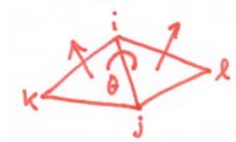


- discrete version

only makes sense as an integral, NEVER pointwise

$$\mathbf{H}_e = e \times N_1 - e \times N_2$$

$$|\mathbf{H}_e| = |e| 2 \sin\theta/2$$



Boundary Integrals

Area gradient

$\mathbf{H}_p = \lim_{A\to 0} \frac{\nabla A}{A}$

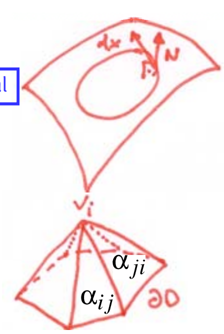
- vector mean curvature

$$D \subset S \quad \gamma = \partial D$$

another normal

$$\int_D \mathbf{H}\,dA = \oint_\gamma N \times df(X)\,d\ell$$

- discrete version



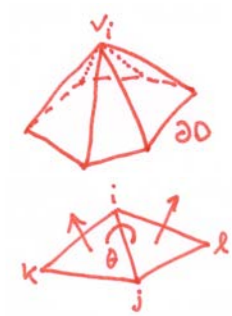
only makes sense as an integral, NEVER pointwise

$$2\mathbf{H}_i = \sum_j \mathbf{H}_{e_{ij}} = 2\nabla_i A$$
$$= \sum_j (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j)$$

Discrete Versions

Just plug in

- vector area (∇ Vol)
  - gradient w.r.t. vertex
  - cone neighborhood
- mean curvature



$$2H_eN = \int_{t_1,t_2} 2HN\,dA = e \wedge N_1 - e \wedge N_2$$

$|H_e| = |e| \sin\theta/2$ $\qquad$ $4H_pNA = \nabla_p A$

Laplace-(Beltrami)

Surface over tangent plane

- in eigen basis

$$H_p = \Delta_f f = \left(\frac{d^2}{du^2} + \frac{d^2}{dv^2}\right) f$$

principal curvature directions

- Laplace-Beltrami $\mathbf{H} = \Delta_f f$

Laplace on the surface

...of the surface

Steiner Polynomial

And now for a totally different view

- consider convex polyhedron
- Steiner: $\mathrm{Vol}\,(N_t(P)) = \mathrm{Vol}\,(P) + t\,\mathrm{Area}\,P + t^2/2 \int_P 2H\,dA + t^3/3 \int_P K\,dA$

- vertices?

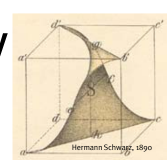
$2H_e = |e|\theta_e$

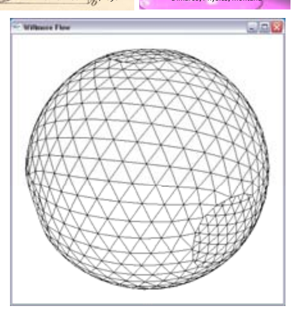
Minimal Surface

Minimum area energy

- minimal surface

$$E_A = \int_S 1\,dA$$





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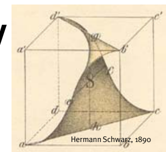

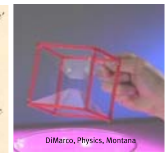

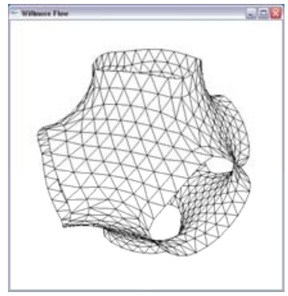

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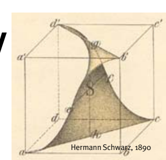

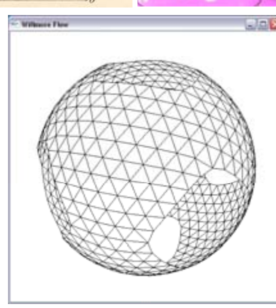

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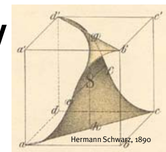

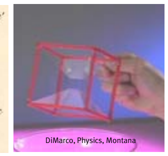

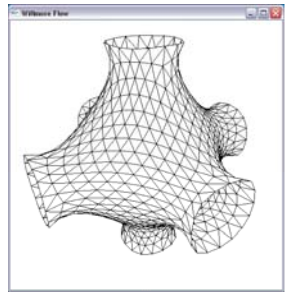

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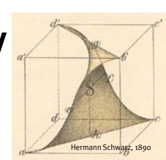

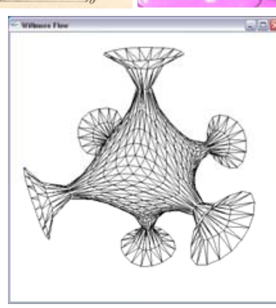

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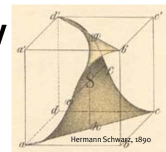

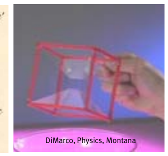

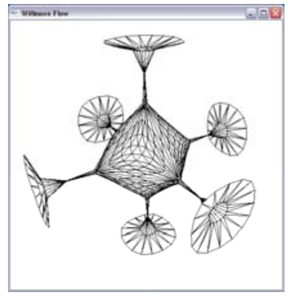

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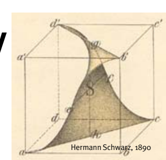

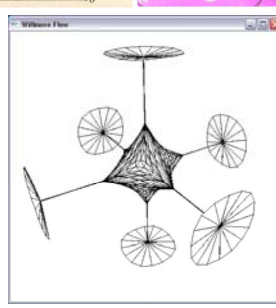

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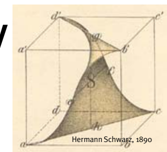

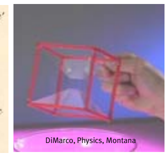

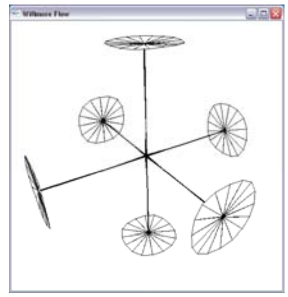

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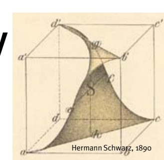

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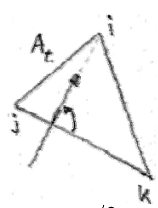
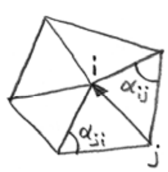


$$2\partial_i A_{t_{ijk}} = R^{\pi/2}(p_k - p_j)$$



$$\sum_{e_{ij}} (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j) = 0$$

Mean Curvature Flow

Laplace-Beltrami

- Dirichlet energy

$$\min \int (\nabla u)^2 \rightsquigarrow \begin{array}{l} \Delta u = 0 \\ u|_{\partial\Omega} = u_0 \end{array}$$

- on surface

$$\begin{aligned} \partial_t p_i &= -\mathbf{H}_i / 2A_i \\ &= -1/4A_i \sum_{e_{ij}} (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j) \end{aligned}$$

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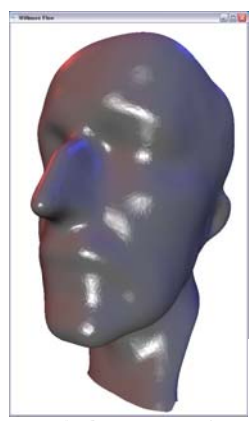

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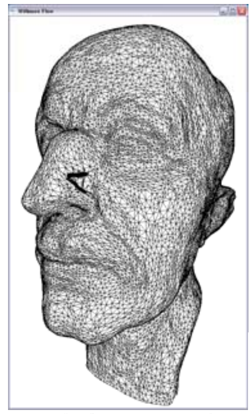

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Convergence?

Can be tricky...

- see Cohen-Steiner paper
- think about chinese lanterns...
  - Schwarz's example a good one to keep in mind