

## BOUNDARY INTEGRALS

### Area gradient

- vector mean curvature

$$D \subset S \quad \gamma = \partial D$$

$$\int_D \mathbf{H} dA = \oint_{\gamma} N \times df(X) d\ell$$

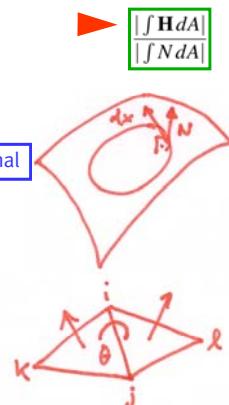
another normal

- discrete version

only makes sense  
as an integral,  
NEVER pointwise

$$\mathbf{H}_e = e \times N_1 - e \times N_2$$

$$|\mathbf{H}_e| = |e| 2 \sin \theta / 2$$



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## BOUNDARY INTEGRALS

### Area gradient

$$\mathbf{H}_p = \lim_{A \rightarrow 0} \frac{\nabla A}{A}$$

- vector mean curvature

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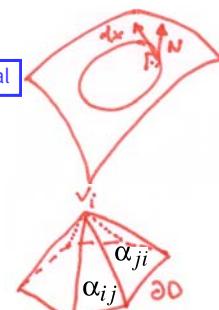
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only makes sense  
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$$2\mathbf{H}_i = \sum_j \mathbf{H}_{e_{ij}} = 2\nabla_i A$$

$$= \sum_j (\cot \alpha_{ij} + \cot \alpha_{ji})(p_i - p_j)$$



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## DISCRETE VERSIONS

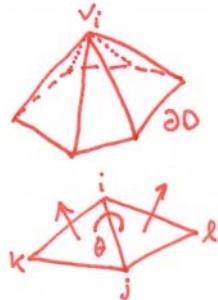
Just plug in

- vector area ( $\nabla \text{Vol}$ )
  - gradient w.r.t. vertex
  - cone neighborhood
- mean curvature

$$2H_e N = \int_{t_1, t_2} 2HN dA = e \wedge N_1 - e \wedge N_2$$

$$|H_e| = |e| \sin \theta / 2$$

$$4H_p N A = \nabla_p A$$



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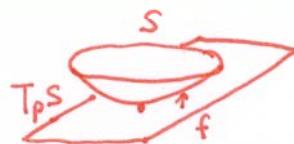
## LAPLACE - ( BELTRAMI )

Surface over tangent plane

- in eigen basis

$$H_p = \Delta_f f = \left( \frac{d^2}{du^2} + \frac{d^2}{dv^2} \right) f$$

principal curvature directions



- Laplace-Beltrami

$$\mathbf{H} = \Delta_f f$$

Laplace on the surface

...of the surface

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## STEINER POLYNOMIAL

And now for a totally different view

- consider convex polyhedron

- Steiner:  $\text{Vol}(N_t(P)) = \text{Vol}(P)$



$$\begin{aligned} &+t \text{Area } P \\ &+t^2/2 \int_P 2H dA \\ &+t^3/3 \int_P K dA \end{aligned}$$

- vertices?

$$2H_e = |e|\theta_e$$

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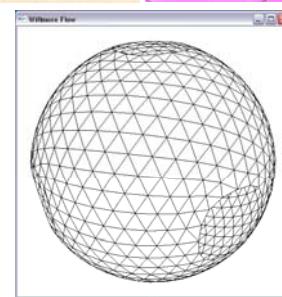
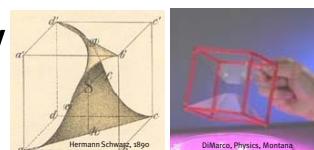
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## MINIMAL SURFACE

Minimum area energy

- minimal surface

$$E_A = \int_S 1 dA$$



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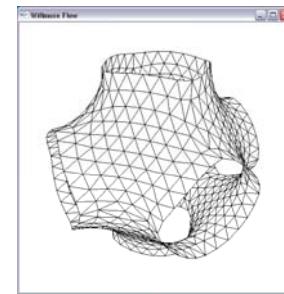
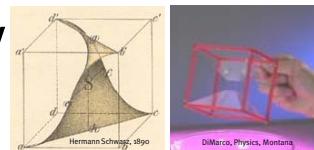
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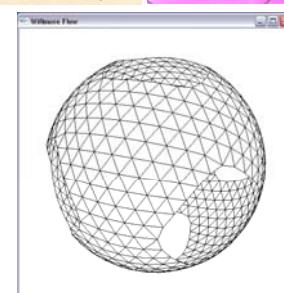
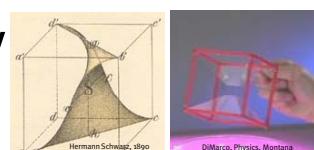
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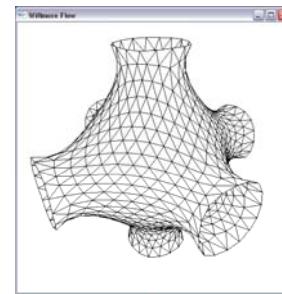
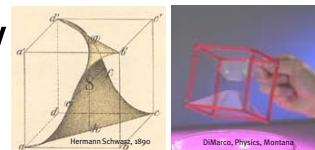
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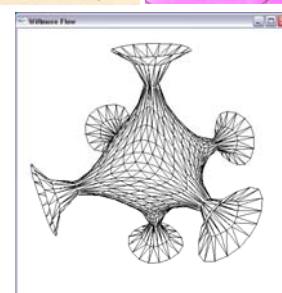
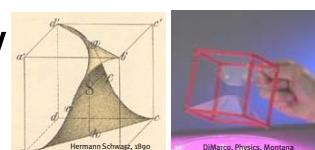
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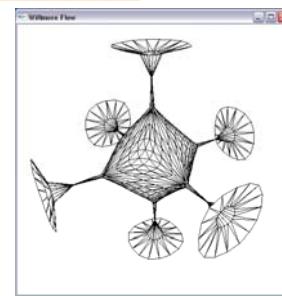
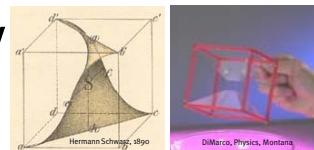
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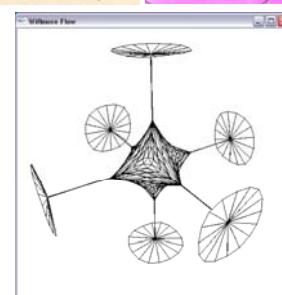
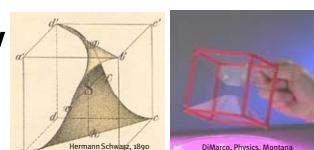
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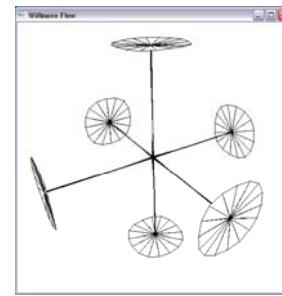
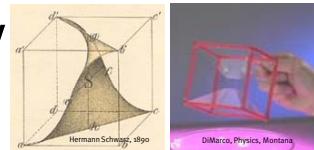
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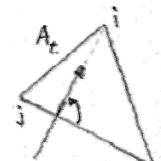
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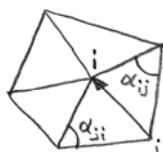
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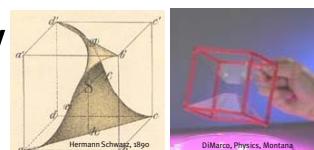


$$E_A = \int_S 1 dA$$

$$2\partial_i A_{tijk} = R^{\pi/2} (p_k - p_j)$$



$$\sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji}) (p_i - p_j) = 0$$



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## MEAN CURVATURE FLOW

### Laplace-Beltrami

- Dirichlet energy

$$\min \int (\nabla u)^2 \rightsquigarrow \Delta u = 0 \\ u|_{\partial\Omega} = u_0$$

- on surface

$$\begin{aligned}\partial_t p_i &= -\mathbf{H}_i / 2A_i \\ &= -1/4A_i \sum_{e_{ij}} (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j)\end{aligned}$$

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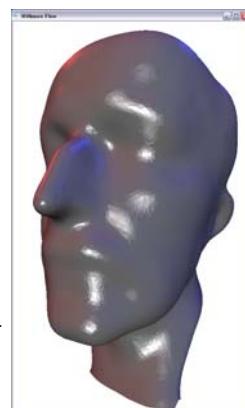
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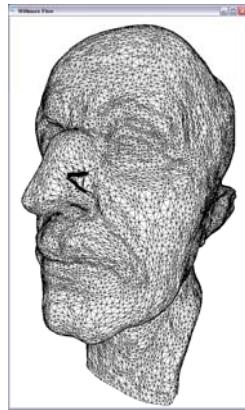
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## CONVERGENCE?

Can be tricky...

- see Cohen-Steiner paper
- think about chinese lanterns...
  - Schwarz's example a good one to keep in mind

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