Parameterizations

What is a parameterization?

- function from some region $\Omega \subset \mathbb{R}^2$ to the embedded surface $M \subset \mathbb{R}^3$

$$S(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

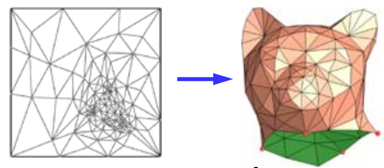


image from Sander et al. 2002

- we go the *other* way around
- how to measure distortion?

Measuring Distortion

Dirichlet energy of a map [Pinkall/Polthier '93]

- harmonic param: $\Delta_S \mathbf{u} = 0 \quad \mathbf{u}|_{\partial\Omega} = \mathbf{u}_0$

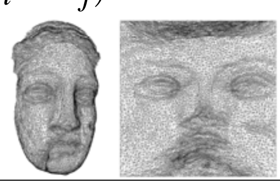
$$E_D(\mathbf{u}) = \int_S (\nabla_S \mathbf{u})^2 \, dA$$

- minimizer is discrete harmonic

$$0 = \sum_j (\cot \alpha_{ij} + \cot \alpha_{ji})(\mathbf{u}_i - \mathbf{u}_j)$$

angles in mesh

texture coords.



- need to fix boundary

Harmonic Map

Properties of minimizer

- link with area of triangle??

$$\boxed{\left\langle \frac{\partial s}{\partial u_1}, \frac{\partial s}{\partial u_2} \right\rangle = 0}$$

$$\boxed{\left|\frac{\partial s}{\partial u_1}\right| = \left|\frac{\partial s}{\partial u_2}\right|}$$

conditions for u
to be conformal

$$\begin{aligned} A &= \frac{1}{2}\int_T \left|\frac{\partial s}{\partial u_1} \times \frac{\partial s}{\partial u_2}\right| du \\ &\leq \frac{1}{2}\int_T \left|\frac{\partial s}{\partial u_1}\right| \left|\frac{\partial s}{\partial u_2}\right| du \\ &\leq \frac{1}{4}\int_T \left(\frac{\partial s}{\partial u_1}\right)^2 + \left(\frac{\partial s}{\partial u_2}\right)^2 du \end{aligned}$$

Conformal Energy

Minimize non-orthogonality

$$\begin{aligned} E_{LSCM}(\mathbf{u}) &= \frac{1}{2}\int_S |\nabla u_1^{\perp} - \nabla u_2|^2 \, dA \\ &= \frac{1}{2}\int_S |\nabla u_1^{\perp}|^2 + |\nabla u_2|^2 - 2\nabla u_1^{\perp} \cdot \nabla u_2 \, dA \\ &= \frac{1}{2}\int_S |\nabla \mathbf{u}|^2 - 2\nabla u_1 \times \nabla u_2 \, dA \\ &= E_D(\mathbf{u}) - A(\mathbf{u}) \end{aligned}$$

In the Complex Plane

Affine maps between triangles

- two triangles uniquely determine

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

- decompose linear part uniquely

$$A = \tfrac{1}{2}(A + JAJ) + \tfrac{1}{2}(A - JAJ)$$

- using c ... r ... $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Symmetric trace-free

Anti-symmetric

$$A(z) = \mu\bar{z} + \lambda z$$

Maps between Triangles

Holomorphic parameterization

- differential is stretch rotation

$$A(z) = \mu\bar{z} + \lambda z$$

stretch reflection

stretch rotation

$$\lambda = \partial f = \tfrac{1}{2}(df(X) - J\,df(JX))$$

$$\mu = \bar{\partial} f = \tfrac{1}{2}(df(X) + J\,df(JX))$$

$$|df|^2 = 2(|\mu|^2 + |\lambda|^2)$$

Using It…

Energies

- Dirichlet

$$E_D(f) = \frac{1}{2}\int |df|^2\, dA = \sum_{t_{ijk}} (|\mu|^2 + |\lambda|^2) A_{ijk}$$

- Holomorphic

$$E_H(f) = \frac{1}{2}\int |\bar{\partial} f|^2\, dA = \sum_{t_{ijk}} |\mu|^2 A_{ijk}$$

- fun fact

$$A(f) = \sum_{t_{ijk}} (|\lambda|^2 - |\mu|^2) A_{ijk}$$

$$\det df = |\lambda|^2 - |\mu|^2$$

$$2E_H(f) = E_D(f) - A(f)$$

Computing It

(Anti-)Holomorphic part

$$\mu = \frac{i}{4A_{ijk}}(f_i p_{jk} + f_j p_{ki} + f_k p_{ij})$$

$$\lambda = \frac{-i}{4A_{ijk}}(f_i \bar{p}_{jk} + f_j \bar{p}_{ki} + f_k \bar{p}_{ij})$$

- compare with

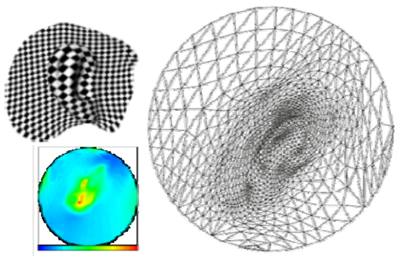
$$\operatorname{grad} f = \frac{1}{2A} N \times (f_i p_{jk} + f_j p_{ki} + f_k p_{ij})$$

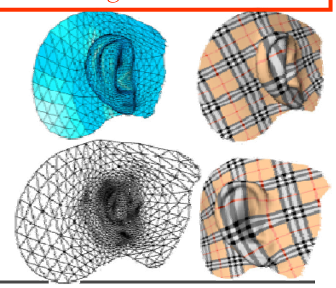
Discrete Conformal

Minimizer of conformal energy

- $E_C(\mathbf{u}) = E_D(\mathbf{u}) - \mathrm{Area}(\mathrm{Range}(\mathbf{u}))$

Fixed boundary: Dirichlet condition | $\nabla E_C(\mathbf{u}) = 0$ | Free boundary: match gradient of area





Little Aside

What's the param of a flat mesh?

- itself...

Notion of Barycentric Coordinates

- vertex can be "reconstructed" by a linear combo of its neighbors
- try it on cot formula...
  - can you think of other weights?

Recap

Invariants as overarching theme

- shape does not depend on Euclidean motions
  - metric and curvatures
- smooth continuous notions to discrete notions
  - variational formulations
  - careful: generally only as averages

Tools

Operators we have now

- volume gradient: notion of normal
- area gradient: notion of normal
  - also: mean curvature
- smoothing, parameterization, editing (bi-Laplace-Beltrami)

Harmonic

Flow

$\Delta_S^2 \mathbf{S} = 0$ G^1 boundaries to make smooth bump

Conformal

Down the Line

Approach so far

- essentially linear: PL mesh...
- same equations can be derived with
  - DEC: discrete exterior calculus
    - coming right up!