Written Assignment 1:
A First Look at Exterior Algebra and Exterior Calculus

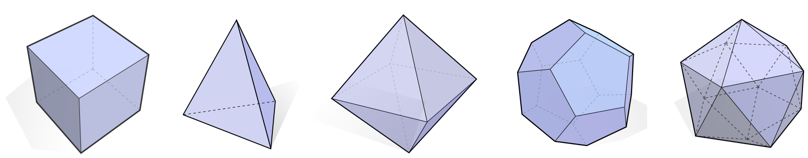
CMU 15-458/858 (Fall 2017)

Submission Instructions. Please submit your solutions to the exercises (whether handwritten, LaTeX, etc.) as a **single PDF file** by email to `Geometry.Collective@gmail.com`. Scanned images/photographs can be converted to a PDF using applications like *Preview* (on Mac) or a variety of free websites (*e.g.*, `http://imagetopdf.com`). Your submission email must include the string **DDG17A1** in the subject line. Your graded submission will (hopefully!) be returned to you at least one day before the due date of the next written assignment.

Grading. Please clearly show your work. Partial credit **will** be awarded for ideas toward the solution, so please submit your thoughts on an exercise even if you cannot find a full solution. **Note that you are required to complete only THREE exercises from each section!** You are of course welcome to do more. :-)

If you don't know where to get started with some of these exercises, just ask! A great way to do this is to leave comments on the course webpage under this assignment; this way everyone can benefit from your questions. We are glad to provide further hints, suggestions, and guidance either here on the website, via email, or in person. Office hours are still TBD, but let us know if youd like to arrange an individual meeting.

Late Days. Note that you have 5 no-penalty late days for the entire course, where a "day" runs from 6:00:00 PM Eastern to 5:59:59 PM Eastern the next day. No late submissions are allowed once all late days are exhausted. If you wish to claim one or more of your five late days on an assignment, please indicate which late day(s) you are using in your email submission. You must also draw **Platonic solids** corresponding to the late day(s) you are using (cube=1, tetrahedron=2, octahedron=3, dodecahedron=4, icosahedron=5). Use them wisely, as you cannot use the same polyhedron twice! If you are typesetting your homework on the computer, we have provided images that can be included for this purpose (in LaTeX these can be included with the `\includegraphics` command in the `graphicx` package).



1 2 3 4 5

Collaboration and External Resources. You are **strongly encouraged** to discuss all course material with your peers, including the written and coding assignments. You are especially encouraged to seek out new friends from other disciplines (CS, Math, Engineering, etc.) whose experience might complement your own. However, *your final work must be your own, i.e.*, direct collaboration on assignments is prohibited.

You are allowed to refer to any external resources *except* for homework solutions from previous editions of this course (at CMU and other institutions). If you use an external resource, cite such help on your submission. **If you are caught cheating, you will get a zero for the entire course.**

Warning! With probability 1, there are typos in this assignment. If *anything* in this handout does not make sense (or is blatantly wrong), let us know! We will be handing out extra credit for good catches. :-)

Format. This written assignment is intended to be a "crash course" in exterior algebra and exterior calculus in $\mathbb{R}^n$. To keep things simple, we'll mainly stick to the cases $n = 2$ or $n = 3$ (which are key for doing geometry in the plane or in three-dimensional space), but many of these ideas naturally generalize to any dimension n. Each subsection is divided into three parts:

(1) an intuition section where concepts are introduced informally with visualizations;

(2) an exercise section with calculation, proofs, and other exercises to cement the concepts; and

(3) a formal definition section where everything is laid out rigorously.

Note that it is **NOT** essential that you understand all the details in part (3) in order to do the homework exercises. These details are provided only as a reference, especially for those seeking more formal definitions.

> **Warning!** This assignment is closely connected to Chapter 3 of the course notes. However, Chapter 3 goes beyond "flat" spaces like the plane $\mathbb{R}^2$ and also discusses exterior calculus on spaces with curvature. As such, certain parts of Chapter 3 can be skipped for now (*e.g.*, the sharp and flat operators); we will eventually cover all this material in class.

1 Exterior Algebra in $\mathbb{R}^n$

In addition to the short readings outlined in the "Intuition" sections below, you may find it useful to look through the course slides on Exterior Calculus. Those interested in further details might consult the Wikipedia page on exterior algebra provides additional detail, though most of this material is **not** needed to do the exercises in this assignment.

1.1 Wedge product and k-vectors

1.1.1 Intuition

Read Chapter 3 of the course notes up to "3.1.3 The Hodge Star" as well as Example 1 in Section 3.2.

1.1.2 Exercises

Do **any three** of the following exercises. If you answer more than three problems, your highest three scores will be counted.

Exercise 1. Let $v = e_1 + 2e_2$ and $w = e_2 + 2e_3$ be 1-vectors in $\mathbb{R}^3$. Compute

(a) $v \wedge w$

(b) $w \wedge v$

(c) $v \wedge v$

Exercise 2. Consider the following 1-vectors in $\mathbb{R}^2$:

$$\begin{aligned} \alpha_0 &= e_1 + e_2 \\ \alpha_1 &= e_1 + 2e_2 \\ \alpha_2 &= e_1 + 4e_2 \end{aligned}$$

Compute

$$\alpha_0 \wedge \alpha_1 \wedge \alpha_2$$

and give an interpretation of the result.

Exercise 3. Let $u = e_1 + e_2 + e_3$ and $v = e_1 - e_2 + e_3$ be 1-vectors in $\mathbb{R}^3$. Compute both $u \wedge v$ and $u \times v$. What's the difference between these two quantities?

Exercise 4. Let $u = e_1 + e_2 - e_3$, $v = e_1 - e_2 + 2e_3$, and $w = 3e_1 + e_2$. Compute

(a) $u \wedge v + v \wedge w$

(b) $(u \wedge v) \wedge w$

1.1.3 Formal definitions

Real vector space. A *real vector space* is a set V together with binary operations.

$$+ : V \times V \to V \qquad \text{"addition"}$$
$$\cdot : \mathbb{R} \times V \to V \qquad \text{"scalar multiplication"}$$

which satisfy the following eight axioms for all $x, y, z \in V$ and $a, b \in \mathbb{R}$

$$\begin{aligned} &x + y = y + x && (ab) \cdot x = a \cdot (b \cdot x) \\ &(x + y) + z = x + (y + z) && 1 \cdot x = x \\ &\exists 0 \in V \text{ s.t. } x + 0 = 0 + x = x && a \cdot (x + y) = a \cdot x + a \cdot y \\ &\forall x, \exists \tilde{x} \in V \text{ s.t. } x + \tilde{x} = 0 && (a + b) \cdot x = a \cdot x + b \cdot x \end{aligned}$$

For brevity, the $\cdot$ is usually omitted (*e.g.*, $ax = a \cdot x$).

Basis and dimension. Let V be a vector space. A collection of vectors is *linearly independent* if no vector in the collection can be expressed as a linear combination of the others. A linearly independent collection of vectors $\{e_1, \ldots, e_n\}$ is a *basis* for V is every vector $v \in V$ can be expressed as

$$v = v_1 e_1 + \cdots v_n e_n$$

for some collection of coefficients $v_1, \ldots, v_n \in \mathbb{R}$. In this case, we say that V is *finite dimensional* with dimension n.

Canonical bases. In $\mathbb{R}^n$, the *canonical basis*, denoted by $e_1, \ldots, e_n$ is defined so that

$$\begin{aligned} e_1 &:= (1, 0, 0, \ldots, 0, 0) \\ e_2 &:= (0, 1, 0, \ldots, 0, 0) \\ &\vdots \\ e_n &:= (0, 0, 0, \ldots, 0, 1). \end{aligned}$$

Wedge product of k-vectors over $\mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the canonical ("usual") basis for $\mathbb{R}^n$. For each integer $0 \leq k \leq n$, let $\bigwedge^k$ denote an $\binom{n}{k}$-dimensional vector space with basis elements denoted by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for all possible sequences of indices $1 \leq i_1 < \cdots < i_k \leq n$, corresponding to all possible "axis-aligned" k-dimensional volumes. Elements of $\bigwedge^k$ are called *k-vectors*.

The *wedge product* is a bilinear map

$$\wedge_{k,l} : \bigwedge^k \times \bigwedge^l \to \bigwedge^{k+l}$$

uniquely determined by its action on basis elements; in particular, for any collection of *distinct* indices $i_1, \ldots, i_{k+l}$,

$$(e_{i_1} \wedge \cdots \wedge e_{i_k}) \wedge_{k,l} (e_{i_{k+1}} \wedge \cdots \wedge e_{i_{k+l}}) := \operatorname{sgn}(\sigma) e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_{k+l})},$$

where σ is a permutation that puts the indices of the two arguments in canonical (lexicographic) order, and $\text{sgn}(\sigma)$ is $+1$ if σ is an even permutation and -1 if σ is an odd permutation. Arguments with repeated indices are mapped to $0 \in \bigwedge^{k+l}$. For brevity, one typically drops the subscript on $\wedge_{k,l}$.

1.2 Hodge Star

1.2.1 Intuition

Read Section 3.1.3 and Section 3.2 of the course notes. See also the course slides, up until "Differential Forms."

1.2.2 Exercises

Do **any three** of the following exercises. If you answer more than three problems, your highest three scores will be counted.

Exercise 5. (Hodge star in different dimensions.)

(a) Assume we are working in $\mathbb{R}^2$, compute $\star e_1$.

(b) Assume we are working in $\mathbb{R}^3$, compute $\star e_1$.

(c) Why are the results of (a) and (b) different?

Exercise 6. Let $\alpha = e_1 + e_2 + e_3$, $\beta = e_1 - e_2 + 2e_3$, be 1-vectors in $\mathbb{R}^3$.

(a) Compute $\star\alpha$ and $\star\beta$

(b) Compute $\star(\alpha \wedge \beta)$.

(c) Compute $(\star\alpha) \wedge (\star\beta)$.

(d) Why do (b) and (c) have different answers?

Exercise 7. (Applying the Hodge star twice.) Let w be any 1-vector in $\mathbb{R}^n$.

(a) Show that if $n = 2$, then $\star(\star w) = -w$. Can you explain why, geometrically?

(b) Show that if $n = 3$, then $\star(\star w) = w$.

(c) (Extra credit) Show for all $n \geq 2$, that $\star(\star w) = (-1)^{n+1} w$.

(d) (Extra credit) If w were a k-vector in $\mathbb{R}^n$, what can you say about $\star(\star w)$?

Exercise 8. (Putting it all together.) In $\mathbb{R}^3$, let $\alpha = 2e_3$ and $\beta = e_1 - e_2$ be 1-forms, and let $\gamma = e_2 \wedge e_3$ be a 2-form.

(a) Compute $\alpha \wedge (\beta + \star\gamma)$.

(b) Compute $\star(\gamma \wedge \star(\alpha \wedge \beta))$.

1.2.3 Formal definitions

Hodge star The *Hodge star on k-vectors in $\mathbb{R}^n$* is a linear isomorphism

$$\star : \bigwedge^{k} \to \bigwedge^{n-k}$$

such that for any basis k-vector $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $\star\alpha$ is the unique basis $(n-k)$-vector times either $+1$ or -1 such that

$$\det(\alpha \wedge \star\alpha) = 1$$

where det denotes the determinant of the constituent 1-vectors (treated as column vectors) with respect to the inner product on $\mathbb{R}^n$.

Exterior algebra The collection of vector spaces $\bigwedge^k$ together with the maps $\wedge$ and $\star$ define an *exterior algebra* on V, sometimes known as a *graded algebra*.

2 Exterior Calculus in $\mathbb{R}^n$

2.1 Intuition

While discussing *exterior algebra*, we only needed to talk about individual k-vectors and their operators (*e.g.*, $\wedge$ and $\star$) on them. Now, we wish to discuss how k-vectors *change* across some domain (just as we might consider, say, the divergence or curl of a vector field).

Read Sections 3.3-3.7 of the course notes. Also keep reading the course slides up until "Differential Operators on Curved Domains."

2.2 Exercises

Do **any three** of the following exercises. If you answer more than three problems, your highest three scores will be counted.

Exercise 9. Recall that in exterior calculus we use $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ as a basis for vector fields on $\mathbb{R}^3$, and dx, dy, dz as a basis for differential 1-forms on $\mathbb{R}^3$.[1] Consider the differential 1-form α, which is defined by how it measures each of the three basis vectors:

$$\begin{aligned} \alpha(\tfrac{\partial}{\partial x}) &= 2z \\ \alpha(\tfrac{\partial}{\partial y}) &= 3x^2 \\ \alpha(\tfrac{\partial}{\partial z}) &= 5\cos(y) \end{aligned}$$

(a) Write α in terms of the basis $\{dx, dy, dz\}$.

(b) Evaluate α at the point $p = (1, 2, 3) \in \mathbb{R}^3$ on the constant vector field $U = 3\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

(c) What is $-\alpha$?

Exercise 10. Let $U = \frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and $V = 3\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ be vector fields on $\mathbb{R}^3$. Let $\alpha = xdy$ and $\beta = dx + dz$ be differential 1-forms. Compute

(a) What kind of object is $\alpha(U)$?

(b) $\alpha(U), \alpha(V), \beta(U), \beta(V)$.

(c) $(\alpha \wedge \beta)(U, V)$.

(d) $(\alpha \wedge \beta)(V, U)$.

Exercise 11. Compute in $\mathbb{R}^3$

(a) $(\star[d(e^y dx + \sin(z)dz)]) \wedge dz$.

(b) $d[\star(d(dx \wedge z^2 dy)) + \star(xyzdx \wedge dz \wedge dy)]$.

[1] **Important:** Even though these bases may *look* like derivatives, you should *try to forget that they have anything to do with derivatives for now.* Just treat them like any other basis you have ever worked with (*e.g.*, e_1, e_2, e_3).

Exercise 12. Coderivative. The *coderivative* of a differential k-form α in $\mathbb{R}^n$, denoted by $\delta\alpha$ is defined as

$$\delta\alpha = \star(d(\star\alpha)),$$

(As discussed in Section 3.4 of the course notes, the coderivative is related to the *divergence* operator from vector calculus when $k = 1$ and $n = 3$.)

(a) If α is a differential 0-form on $\mathbb{R}^n$, explain why $\delta\alpha = 0$.

(b) More generally, if α is a differential k-form on $\mathbb{R}^n$, explain why $\delta\alpha$ is a differential $(k-1)$-form.

(c) Consider the differential 1-form $\alpha = e^y\,dx + (x+y)^2\,dy$ on $\mathbb{R}^2$. Compute $\delta\alpha$.

Exercise 13. *k*-form Laplacian The general k-form Laplacian is defined as

$$\Delta = \delta d + d\delta = \star d \star d + d \star d \star .$$

(a) Why is the second term of Δ (that is, $d \star d\star$) unnecessary when applied to differential 0-forms?

(b) In vector calculus, the Laplacian of a scalar function ϕ is just the sum of second partial derivatives $\frac{\partial^2\phi}{\partial x_1^2} + \cdots + \frac{\partial^2\phi}{\partial x_n^2}$. Using this expression, compute the Laplacian of the scalar function $\phi(x,y) = xy + 2y^2$ over $\mathbb{R}^2$.

(c) Compute the Laplacian $\Delta\phi$ of the same function from part (b), this time using the expression from exterior calculus. (Show your work!)

(d) Compute $\Delta\alpha$ for the 1-form $\alpha = xdx + zdy - ydz$ in $\mathbb{R}^3$.

Exercise 14. Exactness. A common misconception when people first learn exterior calculus is that the differential d behaves like a derivative in single-variable calculus. For example, in $\mathbb{R}^1$, the function $f(x) = x^3$ has the property that $\frac{d^2}{dx^2}f = 6x$, but the 0-form $\alpha(x) = x^3$ has the property that

$$d(d(\alpha)) = 0.$$

Why are these different? *Answer however you see fit.*

2.3 Formal definitions

Linear functions and dual vector space. Let V be a real vector space V. A function $\alpha : V \to \mathbb{R}$ is *linear* if for all $u, v \in V$ and $c \in \mathbb{R}$,

$$\alpha(u+v) = \alpha(u) + \alpha(v)$$
$$\alpha(cu) = c\alpha(u),$$

The *dual vector space* V^* is the space of all linear functions $\alpha : V \to \mathbb{R}$ with the operations

$$(\alpha+\beta)(u) = \alpha(u) + \beta(u)$$
$$(c\alpha)(u) = c(\alpha(u)).$$

We call the elements of V^* to be *forms*. For $(\mathbb{R}^n)*$, the dual of $\mathbb{R}^n$, the basis is denoted as $dx^1, \ldots, dx^n$ so that for all $i, j \in \{1, \ldots, n\}$,

$$dx^i(e_j) := \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Wedge product of k-forms over $\mathbb{R}^n$. Let $dx^1, \ldots, dx^n$ be the canonical basis for $(\mathbb{R}^n)^*$. For each integer $0 \le k \le n$, let Ω^k denote an $\binom{n}{k}$-dimensional vector space with basis elements denoted by

$dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ for all possible sequences of indices $1 \leq i_1 < \cdots < i_k \leq n$, corresponding to all possible "axis-aligned" k-dimensional volumes. Elements of Ω^k are called *k-forms*.

The *wedge product* is a bilinear map

$$\wedge_{k,l} : \Omega^k \times \Omega^l \to \Omega^{k+l}$$

uniquely determined by its action on basis elements; in particular, for any collection of *distinct* indices $i_1, \ldots, i_{k+l}$,

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge_{k,l} (dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{k+l}}) := \operatorname{sgn}(\sigma) dx^{\sigma(i_1)} \wedge \cdots \wedge dx^{\sigma(i_{k+l})},$$

where σ is a permutation that puts the indices of the two arguments in canonical (lexicographic) order. Arguments with repeated indices are mapped to $0 \in \Omega^{k+l}$. As before, for brevity, one typically drops the subscript on $\wedge_{k,l}$.

Furthermore, these k-forms are also functions. For every $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Omega^1$ and $v_1, \ldots, v_k \in \mathbb{R}^n$ we have that

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)}).$$

where S_k is the set of permutations on k elements.

Hodge star. The Hodge star on k-forms is completely analogous with the corresponding basis of $dx^1, \ldots, dx^n$.

Differential. Recall that Ω^k is the space of k-forms on $\mathbb{R}^n$. The *k-form differential* $d_k : \Omega^k \to \Omega^{k+1}$ is the unique linear map satisfying the following three properties.

- **Differential.** If $k = 0$, for all $\phi \in \Omega^0$ and $X \in \mathbb{R}^n$, $d_0\phi(X) = D_X\phi$, the directional derivative in the direction X (*e.g.*the one from multivariable calculus).
- **Product rule.** If α is a k-form and β is an ℓ-form then

$$d_{k+\ell}(\alpha \wedge \beta) = d_k\alpha \wedge \beta + (-1)^k \alpha \wedge d_\ell \beta.$$

- **Exactness.** $d_{k+1} \circ d_k = 0$.

3 Discrete Exterior Calculus

3.1 Intuition

Read Section 3.8 of the course notes, and follow the slides from "Discrete Exterior Calculus" up until "The Poisson Equation."

To give a taste of what discrete exterior calculus is like, we introduce the concept in $\mathbb{R}^2$. The discrete versions of the operators d and $\star$ are given as in the reading (also see the "Formal definitions" in this section). Another important operator to discretize is the wedge product ($\wedge$), which we discuss only in the "Formal definitions" section (a rare exception: you *will* need to read this section to do one of the exercises).

3.2 Exercises

Do **any three** of the following exercises. If you answer more than three problems, your highest three scores will be counted.

All of these exercises take place in $\mathbb{R}^2$.

Exercise 15. Integration practice. Consider the (continuous) 1-form $\alpha = 2dx + xdy$, and the two points $A = (0,0)$ and $B = (1,1)$ in the plane.[2]

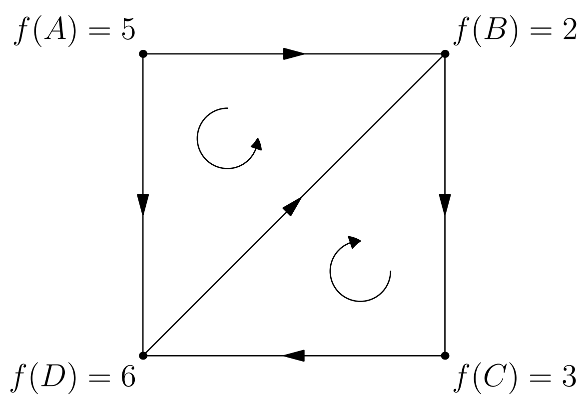
(a) Integrate α over the oriented edge (A, B) to get the discrete 1-form $\hat{\alpha}(A, B)$.

(b) Integrate α over the oriented edge (B, A) to get the discrete 1-form $\hat{\alpha}(B, A)$.

(c) How do these two discrete quantities relate?

Exercise 16. Exactness. Consider any triangle mesh (V, E, F). Prove that $d_1 \circ d_0 = 0$. That is, every discrete differential 0-form when differentiated twice becomes 0.

Exercise 17. Discrete operator practice. Consider the following triangulated region below.



Let $V = \{A, B, C, D\}$ be the set of vertices. Assume they have coordinates

$$A = (0,1); B = (1,1); C = (1,0); D = (0,0).$$

There is a function $f : V \to \mathbb{R}$ on vertices (*i.e.*, a discrete 0-form) whose values are given in the figure above. The arrow on each edge and face indicates its orientation.

(a) Letting d denote the *discrete* exterior derivative, what kind of object is df? (*i.e.*, a 0-form, 1-form, etc.)

(b) What are the domain and range of df?

(c) Compute df.

(d) Compute $d(df)$.

Exercise 18. Discrete wedge practice. Refer to the same setup as the previous exercise. Define a second discrete 0-form $h : V \to \mathbb{R}$ with values

$$h(A) = -3, h(B) = 0, h(C) = 2, h(D) = 3$$

(a) Compute $f \wedge_{0,0} h$

(b) Compute $(df) \wedge_{1,0} h$

(c) Compute $[d((df) \wedge_{1,0} h)] \wedge_{2,0} h$.

(d) Compute $(df) \wedge_{1,1} (dh)$.

[2]The recipe for integrating a 1-form over an edge is:

1. Compute the unit vector along the edge (being careful about orientation).
2. Apply the 1-form to the unit vector.
3. Integrate the resulting expression over the edge.

The final integral (in Step 3) should not look much different from what you did in your intro calculus/vector calculus class.

Exercise 19. Commutativity of d. Refer to the same setup as the previous two exercises. Consider the (continuous) 0-form $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g = y^2(x + 2y).$$

(a) Discretize g; that is, evaluate g at each vertex in V. Call these values $\hat{g} : V \to \mathbb{R}$.

(b) Compute the (continuous) differential dg.

(c) Compute the (discrete) differential $d\hat{g}$.

(d) Integrate the 1-form found in (b) over each edge of the triangulation.

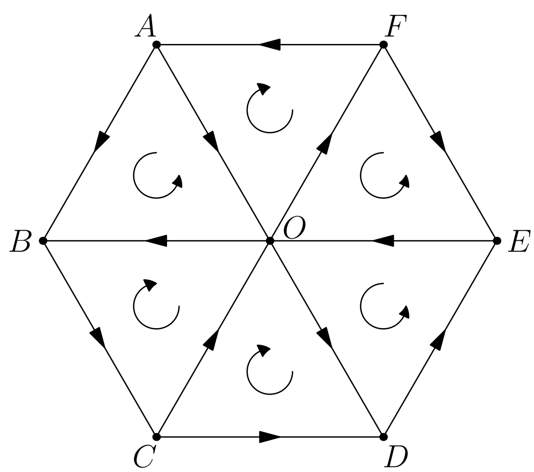
(e) Why are the answers to (c) and (d) the same?

Exercise 20. Matrix representations. Continue using the same setup from the previous exercises. Algorithmically, 0-forms, 1-forms, and 2-forms are often represented as vectors of length $|V|$, $|E|$, and $|F|$, respectively. The differential and other operators can then be represented as matrices, as discussed in the course notes.

(a) Write out the $|E| \times |V|$ matrix representing d_0.

(b) Write out the $|F| \times |E|$ matrix representing d_1.

Exercise 21. Discrete Hodge star practice. Consider the following mesh of 6 equilateral triangles with side-length 1.



Consider the three forms α_0, α_1, α_2 such that

$$\alpha_0(A) = 1 \quad \alpha_0(B) = 2 \quad \alpha_0(C) = 3 \quad \alpha_0(D) = 4 \quad \alpha_0(E) = 5 \quad \alpha_0(F) = 6 \quad \alpha_0(O) = 7$$

$$\alpha_1(A,O) = 2 \quad \alpha_1(O,B) = -5 \quad \alpha_1(C,O) = 3 \quad \alpha_1(O,D) = 1 \quad \alpha_1(E,O) = -3 \quad \alpha_1(O,F) = -2 \quad \alpha_1(A,B) = 4$$
$$\alpha_1(B,C) = 5 \quad \alpha_1(C,D) = 3 \quad \alpha_1(D,E) = -2 \quad \alpha_1(F,E) = 1 \quad \alpha_1(F,A) = 0$$

$$\alpha_2(A,B,O) = 3 \quad \alpha_2(C,B,O) = -2 \quad \alpha_2(D,C,O) = 1 \quad \alpha_2(D,E,O) = 0 \quad \alpha_2(E,F,O) = -1 \quad \alpha_2(A,F,O) = -2$$

(a) Draw[3] the dual mesh. You do *not* need to draw boundary elements.

(b) What kind of discrete forms are $\star_0\alpha_0$, $\star_1\alpha_1$, and $\star_2\alpha_2$?

(c) Compute $\star_0\alpha_0$. (The area of each triangle is $\frac{\sqrt{3}}{4}$.)

(d) Compute $\star_1\alpha_1$.

(e) Compute $\star_2\alpha_2$.

(f) Write down a matrix representation of $\star_1$. (Ignore boundary elements.) The rows should represent edges of the primal mesh, and the columns should represent edges of the dual mesh.

(g) Write down a matrix representation of $\star_2$.

3.3 Formal definitions

3.3.1 Triangle meshes and their duals

A *triangle mesh in* $\mathbb{R}^2$ is a collection of triangles which are connected. (There are other assumptions, but this is rigorous enough for the current assignment.) Each edge and face will have an orientation.

The vertices are denoted as V, the oriented edges are denoted as E, and the oriented faces are denoted as F. For notation, we denote vertices with a single symbol, like v; oriented edges with an ordered pair, like (u, v) is oriented from u to v; and oriented faces with an ordered triple, like (u, v, w) is oriented from u to v to w.

The *dual mesh* also has vertices $V^\star$, edges $E^\star$, and faces $F^\star$. Each *dual vertex* $f^\star \in V^\star$ is the *circumcenter* of some face in $f \in F$. (The circumcenter is the unique point equidistant from each vertex of the face.) Each *dual edge* $e^\star \in E^\star$ connects the circumcenters of two faces sharing the edge $e \in E$, and each *dual face* $v^\star \in F^\star$ is the region bounded by the dual edges surrounding $v \in V$. Like with the *primal* (original) mesh, each dual edge and dual face has an orientation. Note that the dual faces are not necessarily triangles (see section 3.8.4 of the course notes).

3.3.2 Discrete forms

Let Ω^0 be the set of functions $f : V \to \mathbb{R}$, the *discrete* 0*-forms*. Likewise, let Ω^1 be the set of functions $\alpha : E \to \mathbb{R}$, the *discrete* 1*-forms* and let Ω^2 be the set of functions $\beta : F \to \mathbb{R}$, the *discrete* 2*-forms*. The discrete 1-forms and 2-forms also satisfy the following equations.

$$\begin{aligned}\forall (u, v) \in E, \alpha(u, v) &= -\alpha(v, u)\\ \forall (u, v, w) \in F, \beta(u, v, w) &= \beta(v, w, u) = \beta(w, u, v)\\ &= -\beta(v, u, w) = -\beta(w, v, u) = -\beta(u, w, v).\end{aligned}$$

Let $\Omega^{\star 0}$ be the set of functions $f : V^\star \to \mathbb{R}$, the *discrete dual* 0*-forms*. Likewise, let $\Omega^{\star 1}$ be the set of functions $\alpha : E^\star \to \mathbb{R}$, the *discrete dual* 1*-forms* and let $\Omega^{\star 2}$ be the set of functions $\beta : F^\star \to \mathbb{R}$, the *discrete dual* 2*-forms*. Like their primal counterparts, the discrete dual 1-forms and 2-forms change sign when the orientation is changed:

$$\forall (u, v) \in E^\star, \alpha(u, v) = -\alpha(v, u)$$
$$\forall (u_1, \ldots, u_k) \in F^\star \text{ and } \sigma = S_k, \beta(u_1, \ldots, u_k) = \operatorname{sgn}(\sigma)\beta(u_{\sigma(1)}, \ldots u_{\sigma(k)}).$$

[3] Either a hand-drawn or computer-drawn image is acceptable.

3.3.3 Discrete differential

The discrete differential for triangle meshes in $\mathbb{R}^2$ is a pair of maps $d_0 : \Omega_0 \to \Omega_1$ and $d_1 : \Omega_1 \to \Omega_2$. Such that for each $f \in \Omega_0$, $\alpha \in \Omega_1$, $(u,v) \in E$ and $(u,v,w) \in F$,

$$(d_0 f)(u,v) := f(v) - f(u)$$
$$(d_1\alpha)(u,v,w) := \alpha(u,v) + \alpha(v,w) + \alpha(w,u).$$

3.3.4 Discrete Hodge star

There are three Hodge stars $\star_0, \star_1, \star_2$.

The first one, $\star_0 : \Omega^0 \to \Omega^{\star 2}$ maps a primal 0-form $\alpha_0 : V \to \mathbb{R}$ to a dual 2-form $\star_0\alpha_0 : F^\star \to \mathbb{R}$. For all $v^\star \in F^\star$,

$$(\star_0\alpha_0)(v^\star) := |\text{Area}(v^\star)|\alpha_0(v),$$

The second one, $\star_1 : \Omega^1 \to \Omega^{\star 1}$ maps a primal 1-form $\alpha_1 : E \to \mathbb{R}$ to a dual 1-form $\star_1\alpha_1 : E^\star \to \mathbb{R}$. For all $e^\star inE^\star$,

$$(\star_1\alpha_1)(e^\star) := \frac{|\text{Length}(e^\star)|}{|\text{Length}(e)|}\alpha_1(e).$$

Finally, $\star_2 : \Omega^2 \to \Omega^{\star 0}$ maps a primal 2-form $\alpha_2 : F \to \mathbb{R}$ to a dual 0-form $\star_2\alpha_2 : V^\star \to \mathbb{R}$. For all $f^\star \in V^\star$,

$$(\star_2\alpha_2)(f^\star) := \frac{1}{|\text{Area}(f)|}\alpha_2(f).$$

3.3.5 Discrete wedge

There are four wedge products: $\wedge_{0,0}, \wedge_{1,0}, \wedge_{2,0}, \wedge_{1,1}$.

$\wedge_{0,0} : \Omega^0 \times \Omega^0 \to \Omega^0$ is simply the pointwise product. For all $f, g \in \Omega^0$ and $v \in V$

$$(f \wedge_{0,0} g)(v) := f(v)g(v).$$

$\wedge_{1,0} : \Omega^1 \times \Omega^0 \to \Omega^1$ is a product of the form at the edge with the average with the 0-forms at the ends. For all $\alpha \in \Omega^1$, $f \in \Omega^0$, and $(u,v) \in E$

$$(\alpha \wedge_{1,0} f)(u,v) := \alpha(u,v)\frac{f(u) + f(v)}{2}.$$

$\wedge_{2,0} : \Omega^2 \times \Omega^0 \to \Omega^2$. For all $\beta \in \Omega^2$, $f \in \Omega^0$, and $(u,v,w) \in F$

$$(\beta \wedge_{2,0} f)(u,v,w) := \beta(u,v,w)\frac{f(u) + f(v) + f(w)}{3}.$$

$\wedge_{1,1} : \Omega^1 \times \Omega^1 \to \Omega^2$. For all $\alpha_1, \alpha_2 \in \Omega^1$ and $(u,v,w) \in F$

$$\begin{aligned}(\alpha_1 \wedge_{1,1} \alpha_2)(u,v,w) :=& \frac{1}{6}[\alpha_1(u,v)\alpha_2(v,w) - \alpha_1(v,w)\alpha_2(u,v) \\ &+ \alpha_1(v,w)\alpha_2(w,u) - \alpha_1(w,u)\alpha_2(v,w) \\ &+ \alpha_1(w,u)\alpha_2(u,v) - \alpha_1(u,v)\alpha_2(w,u)].\end{aligned}$$

Any other discrete wedge product in $\mathbb{R}^2$ returns 0. (Can you see why?)