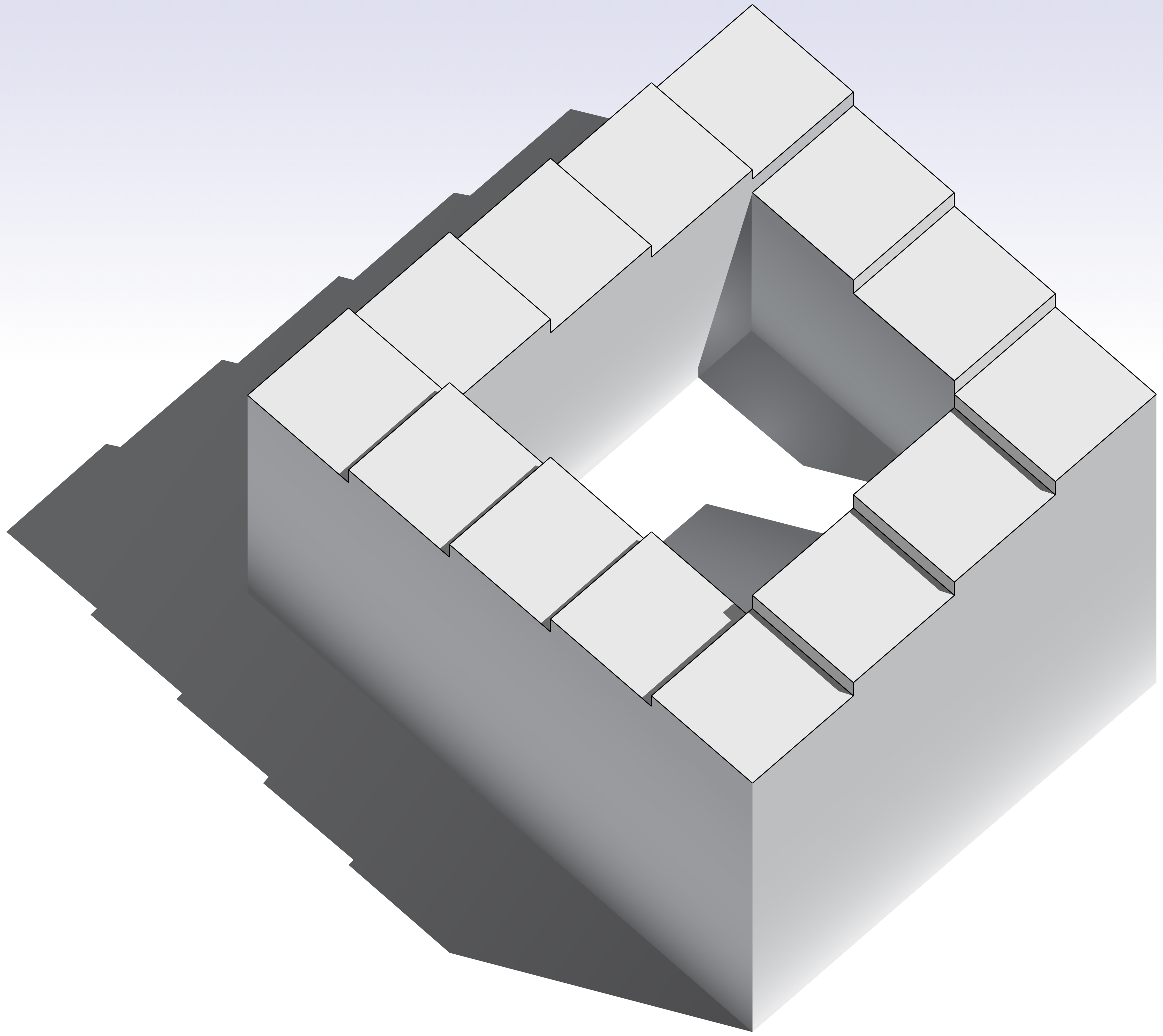
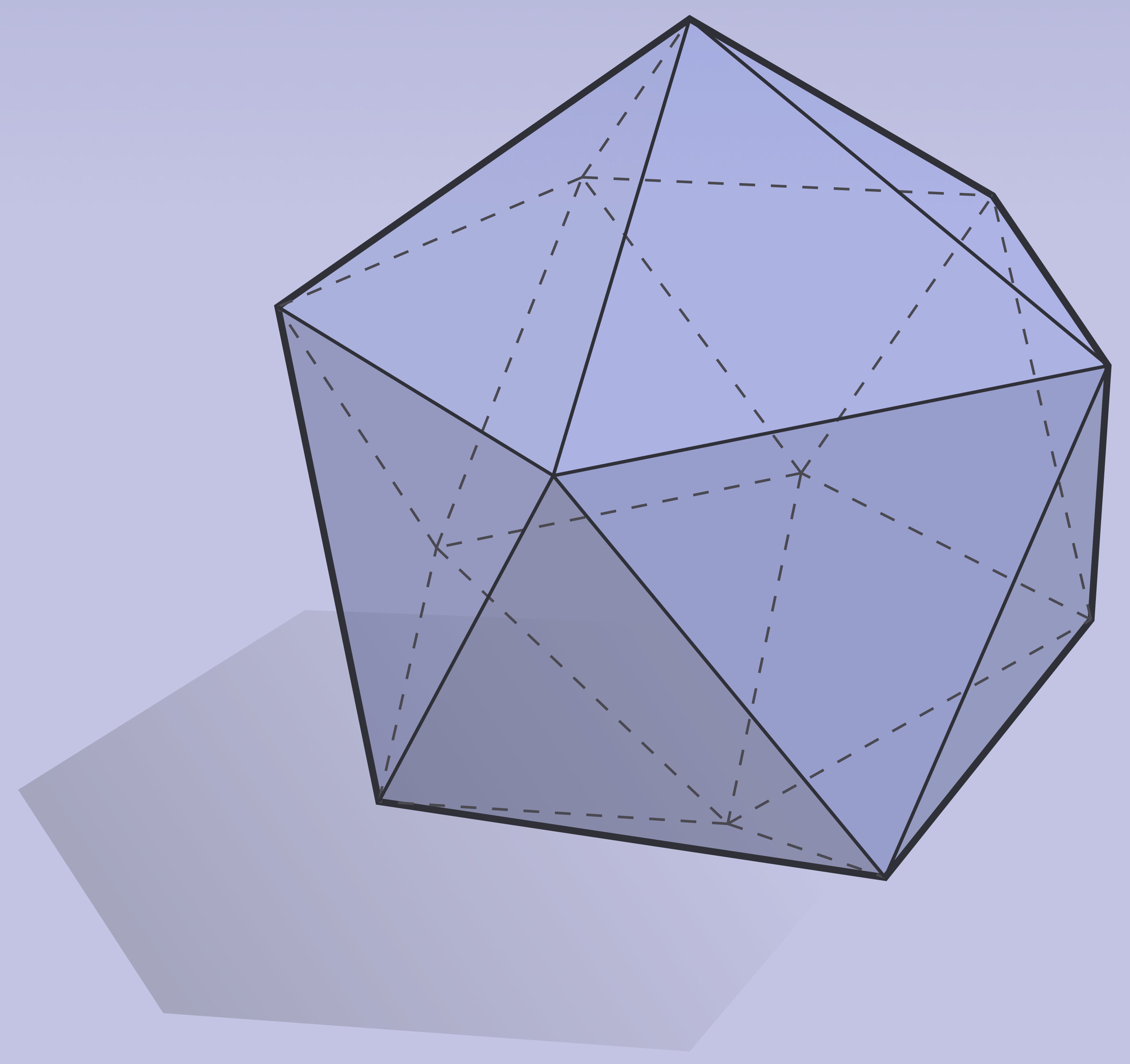


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

Lecture 5:

Exterior Calculus in R^n



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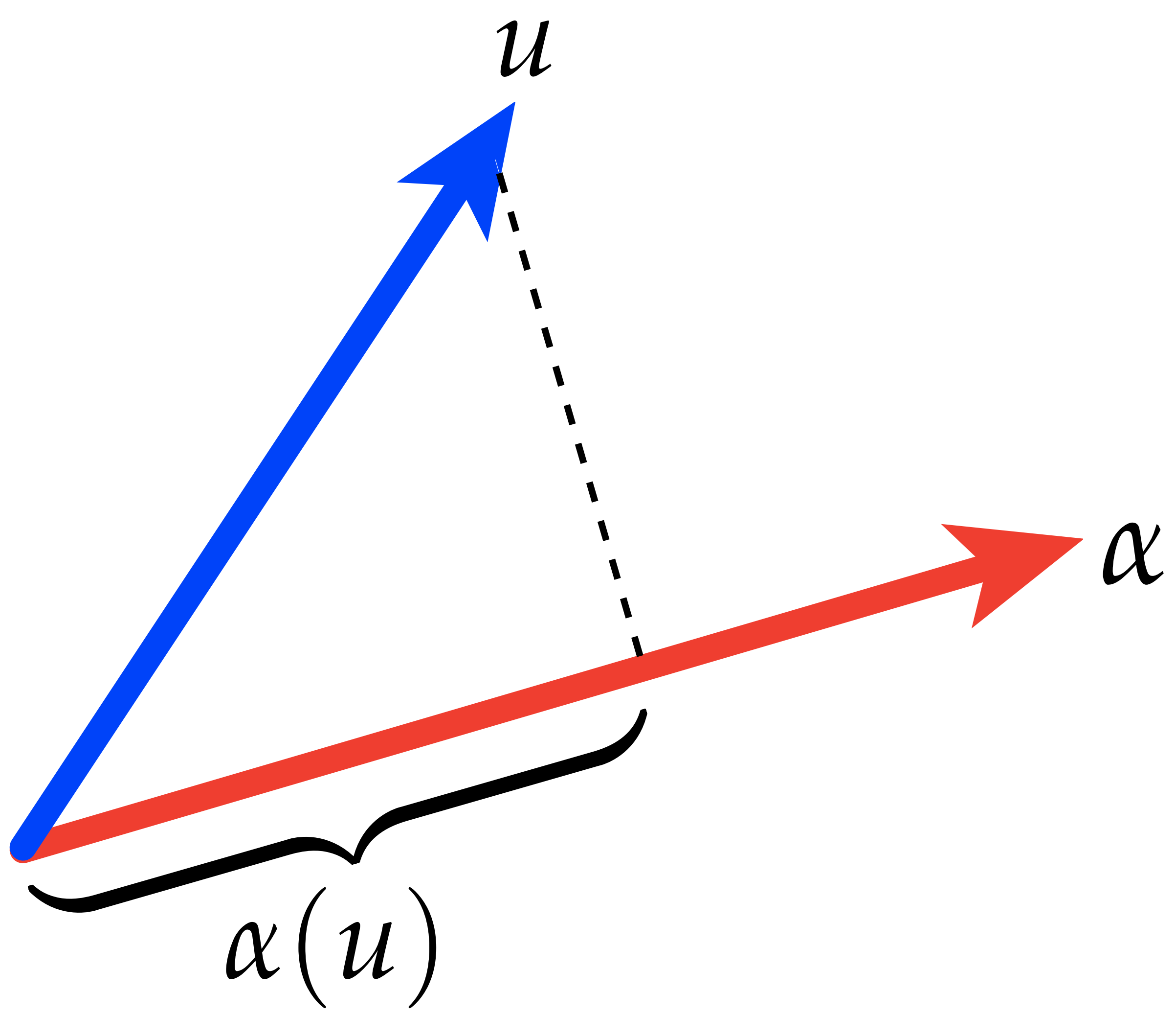
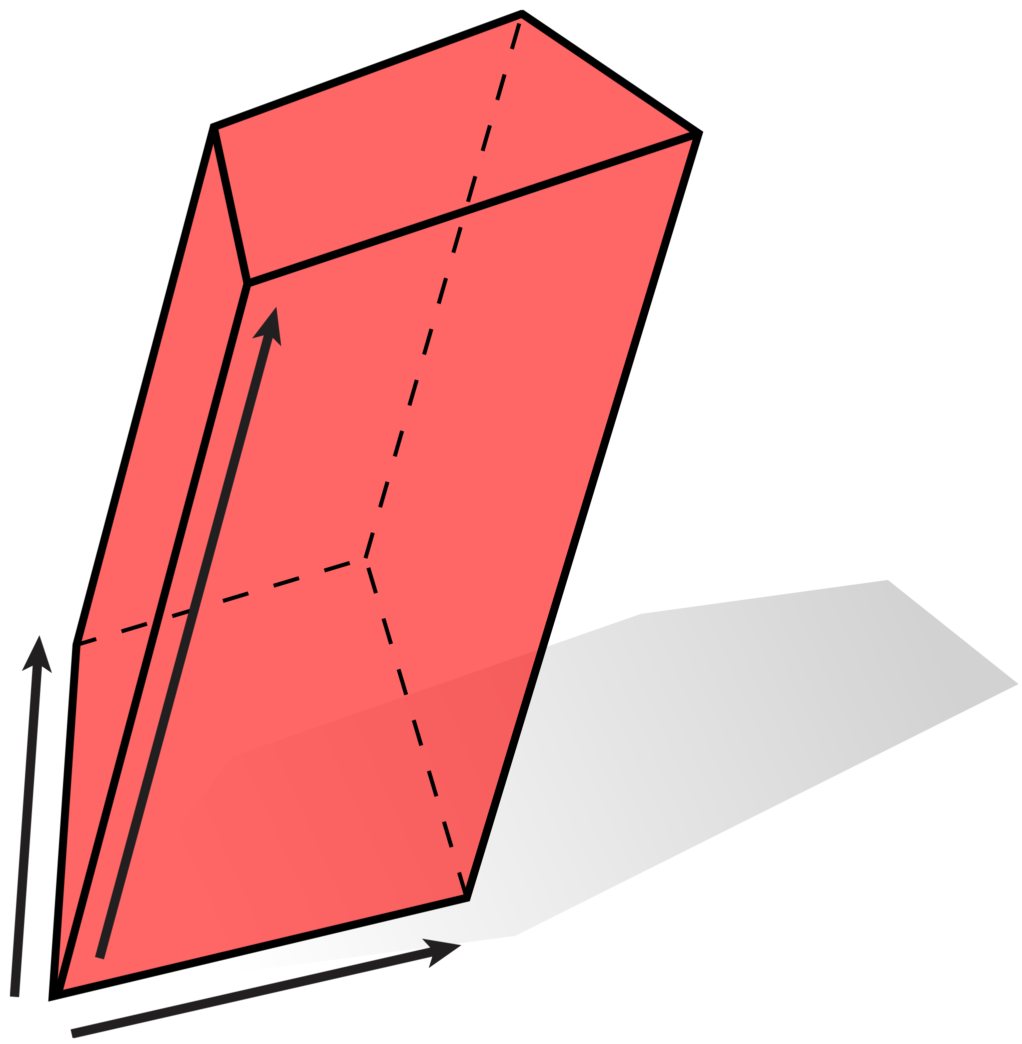
Exterior Calculus—Overview

- *Previously:*
  - **1-form**—linear measurement of a vector
  - ***k*-form**—multilinear measurement of volume
  - **differential *k*-form**—*k*-form at each point

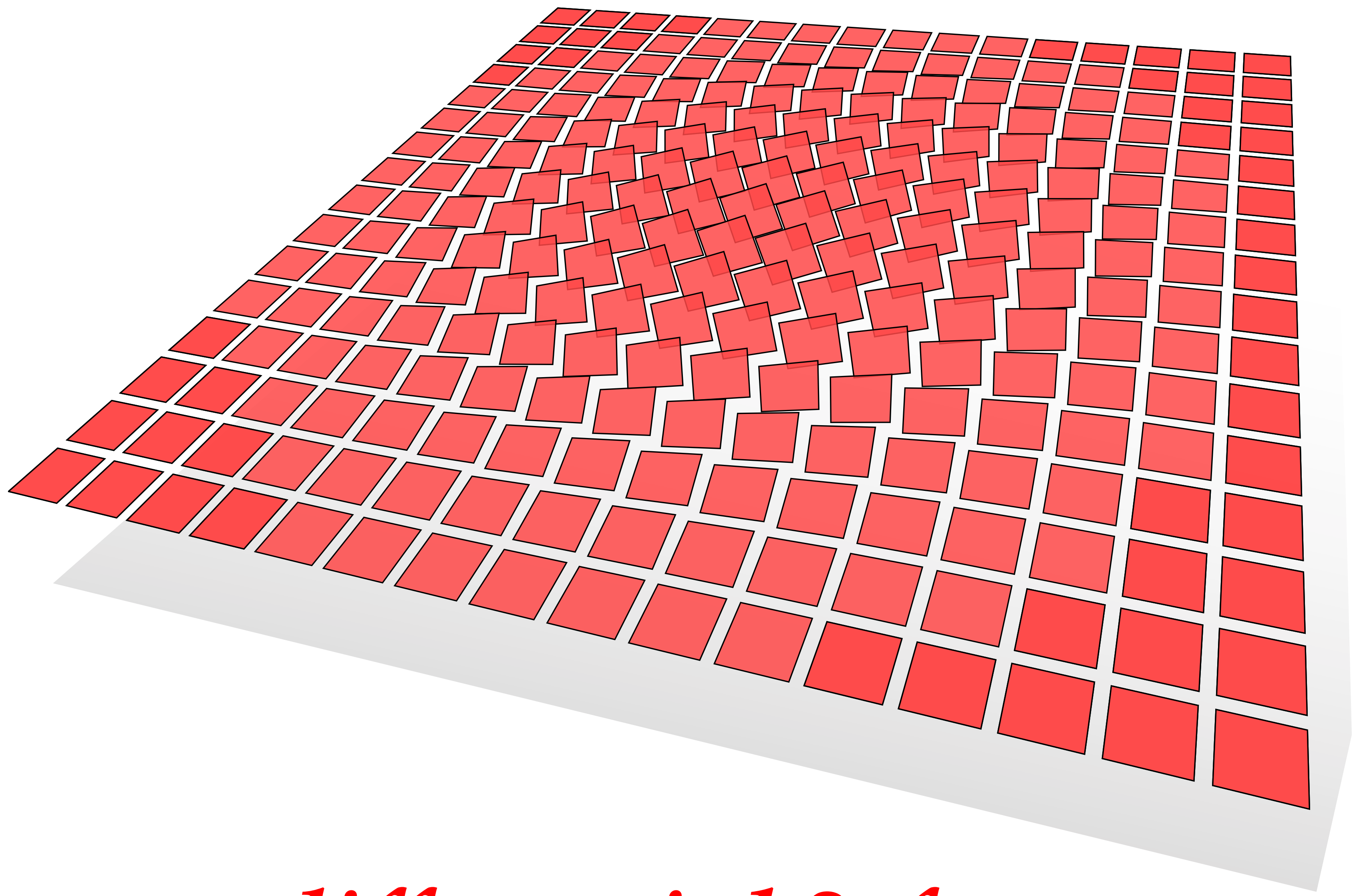
- *Today:* ***exterior calculus***
  - how do *k*-forms *change*?
  - how do we *integrate* *k*-forms?



1-form



3-form



differential 2-form

Integration and Differentiation

- Two big ideas in calculus:
  - **differentiation**
  - **integration**
  - linked by *fundamental theorem of calculus*

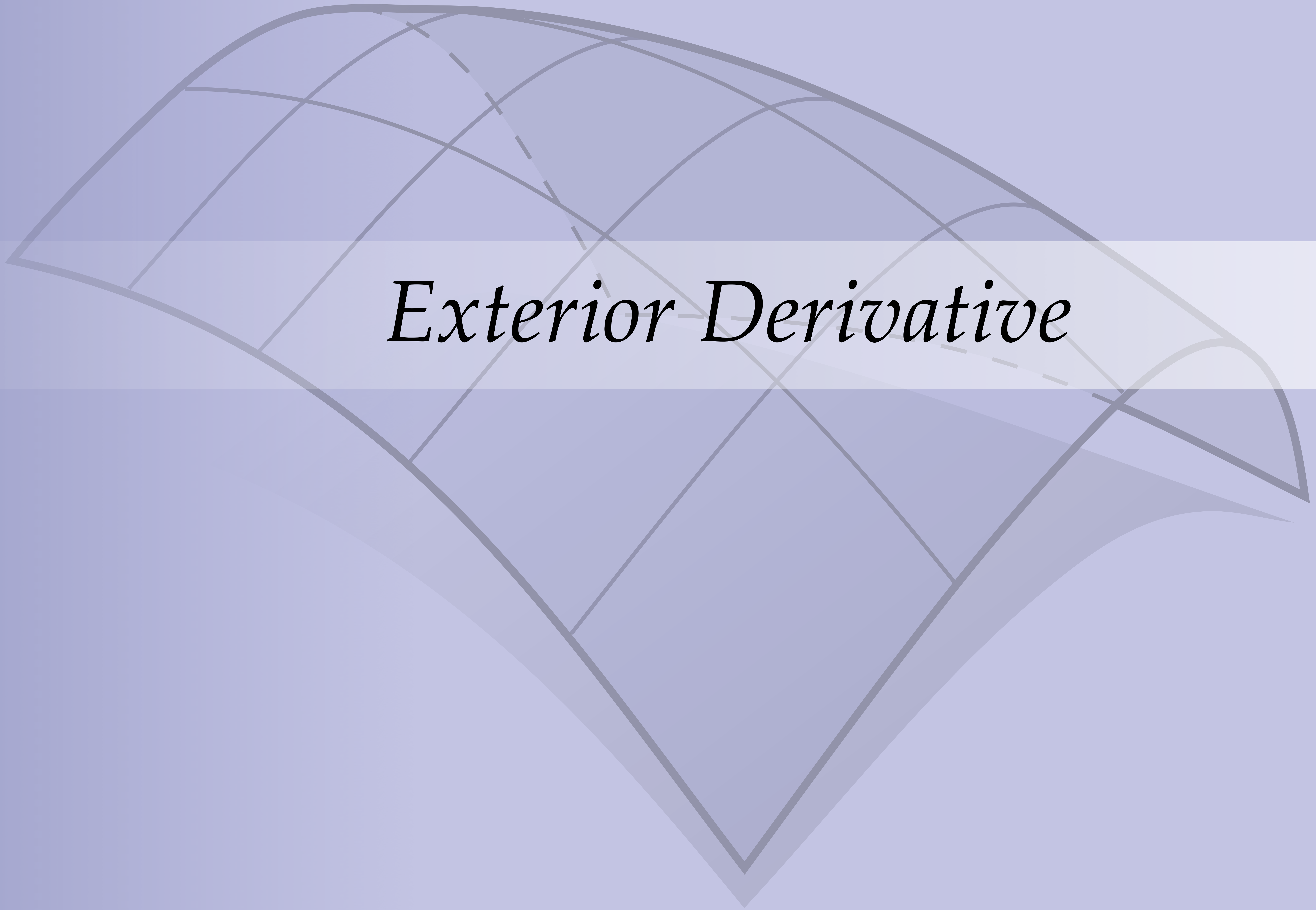
$$\int_a^b f' dx = f(b) - f(a)$$

- Exterior calculus generalizes these ideas
  - **differentiation of *k*-forms** (exterior derivative)
  - **integration of *k*-forms** (measure volume)
  - linked by *Stokes' theorem*

$$\int_M d\alpha = \int_{\partial M} \alpha$$

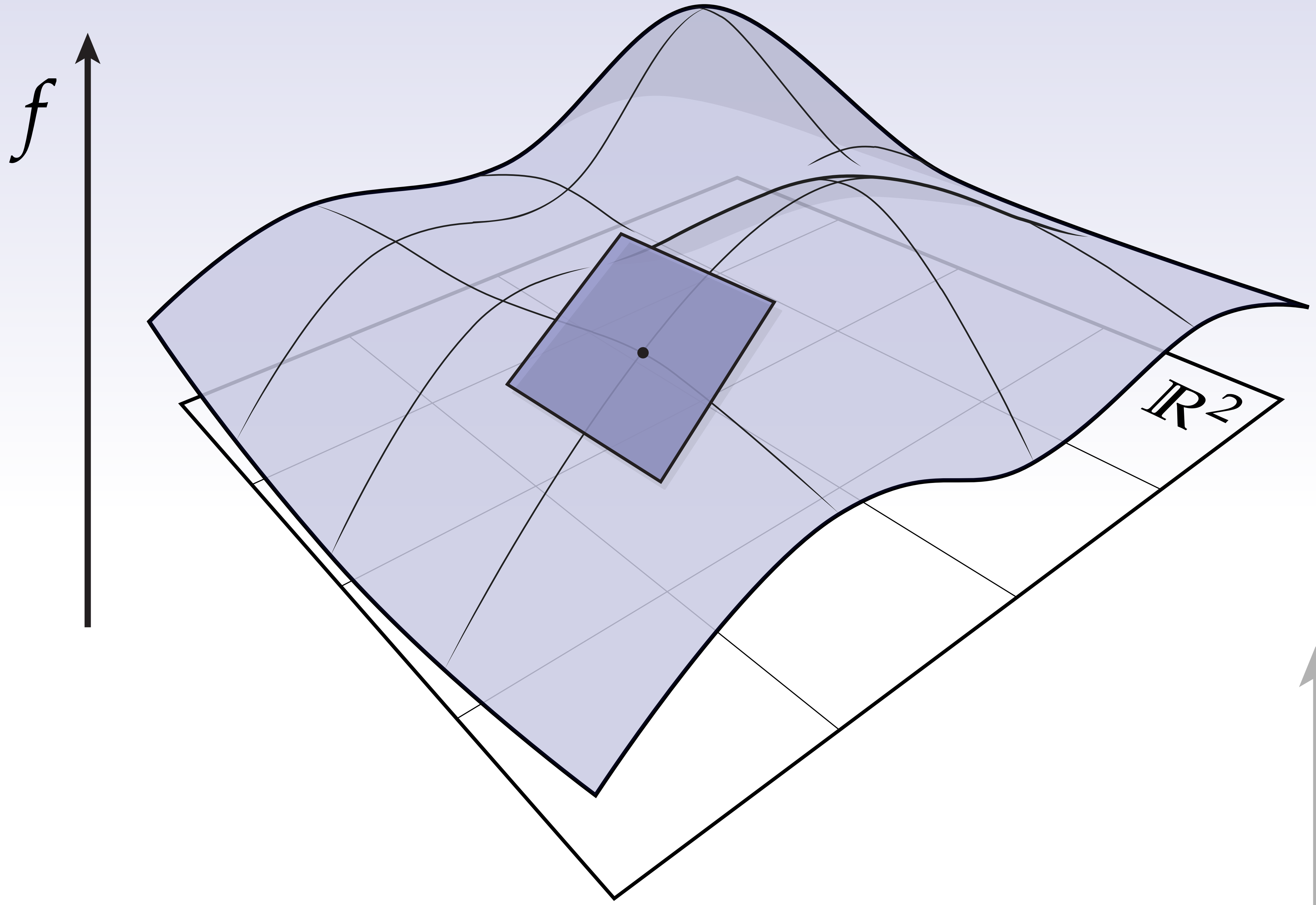
- **Goal:** integrate differential forms over meshes to get *discrete exterior calculus (DEC)*

Exterior Derivative

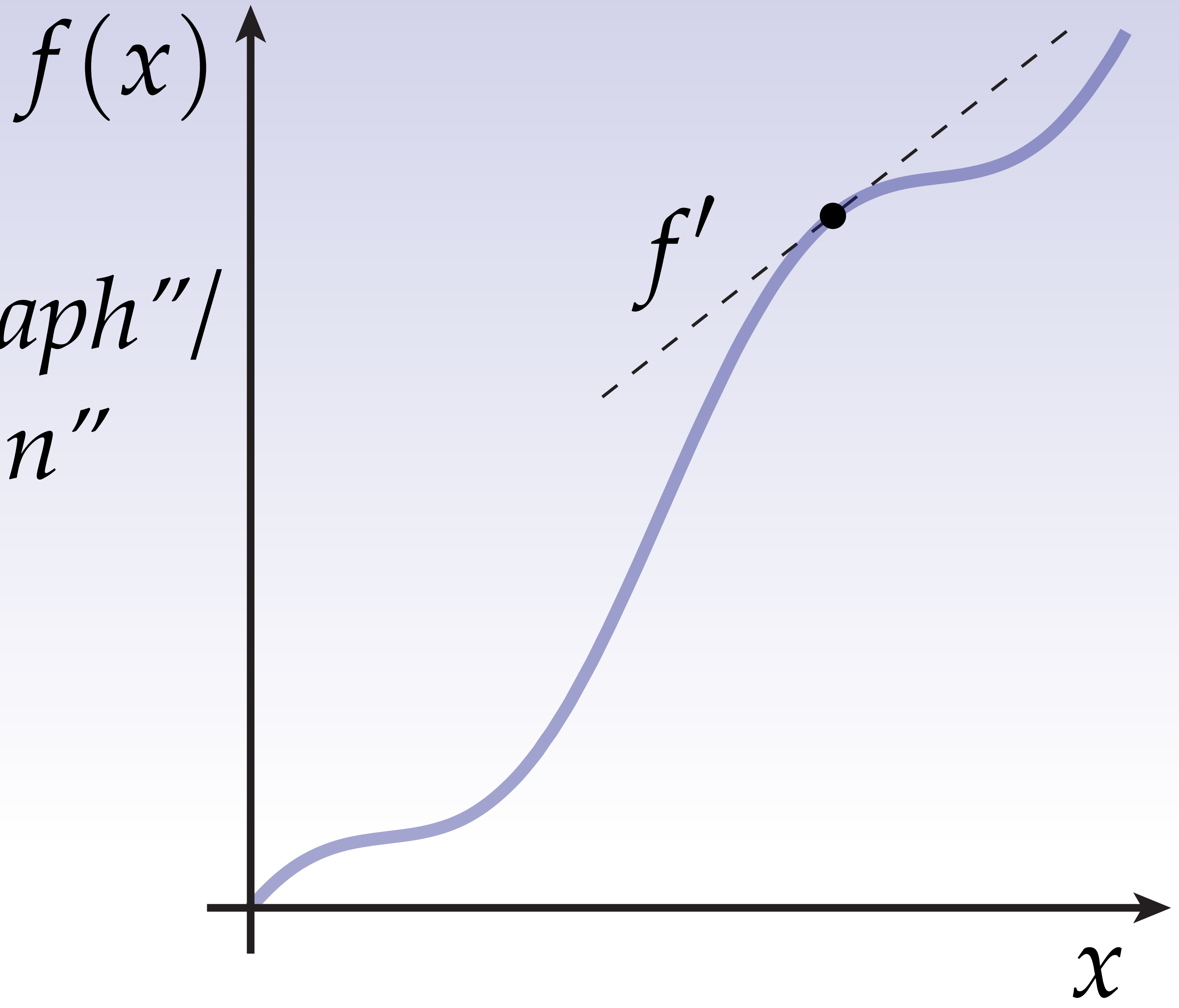


Derivative—Many Interpretations…

"best linear approximation"

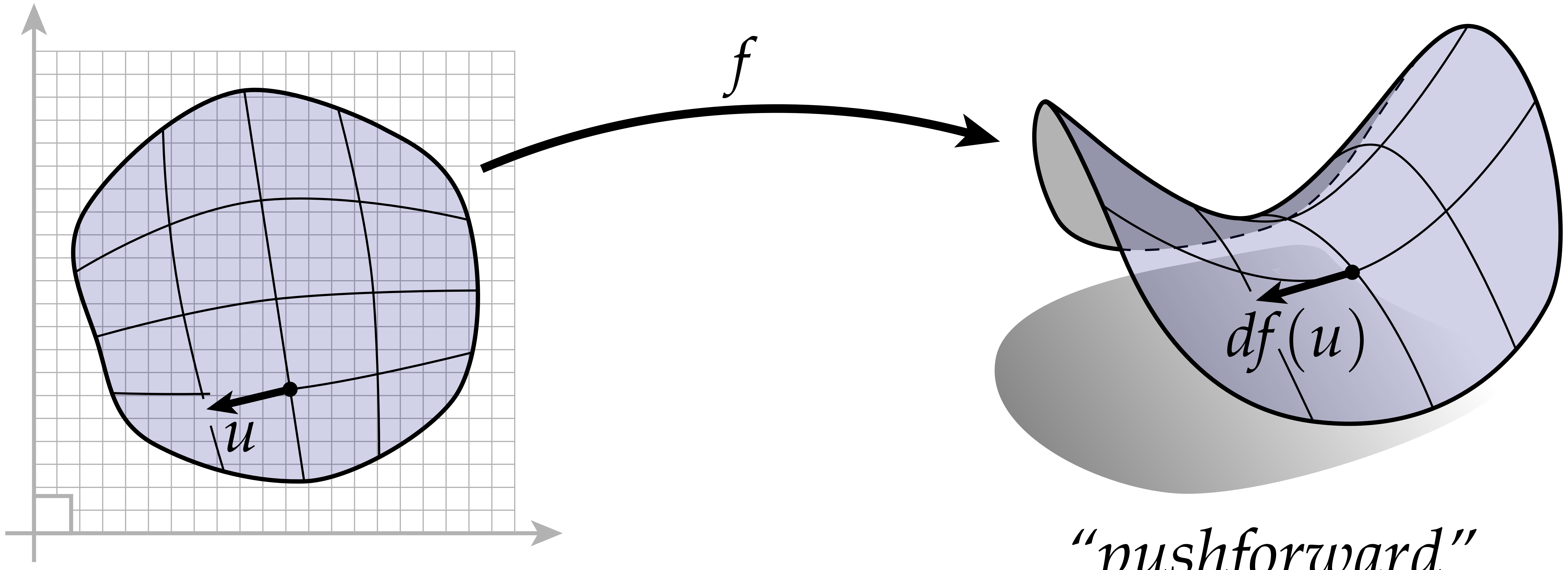


"slope of the graph"/
"rise over run"



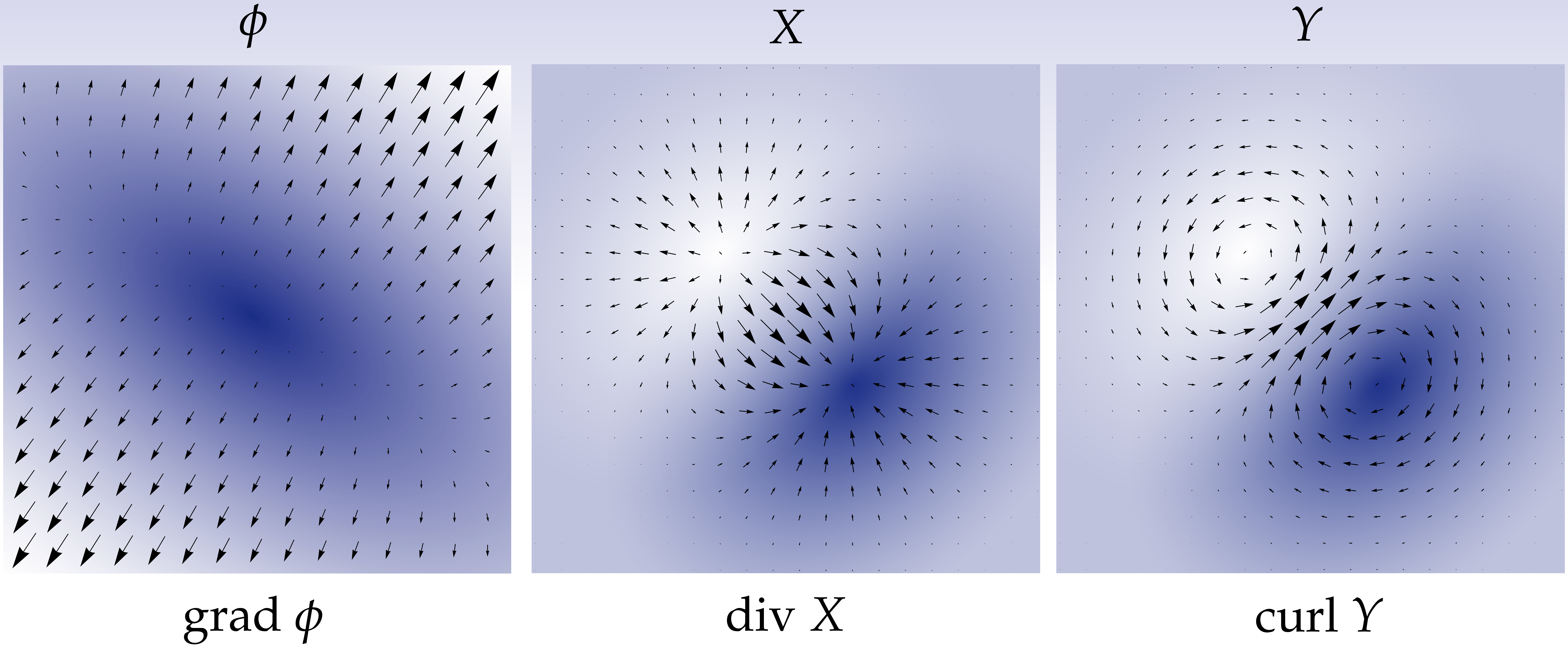
$$f'(x) := \lim_{\varepsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\varepsilon}$$

"difference in the limit"



"pushforward"

Vector Derivatives — Visualized



Review—Vector Derivatives in Coordinates

How do we express grad, div, and curl in coordinates?

Consider a scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$ and a vector field

$$X = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

where $u, v, w : \mathbb{R}^n \to \mathbb{R}$ are coordinate functions that vary over the domain, and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are the standard basis vector fields.

grad

$$\nabla\phi = \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}$$

div

$$\nabla \cdot X = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

curl

$$\nabla \times X = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial}{\partial z}$$

Exterior Derivative

(Ω^k — space of all differential k-forms)

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Where do these rules come from?
(What's the *geometric* motivation?)

Exterior Derivative — Differential

Review: Directional Derivative

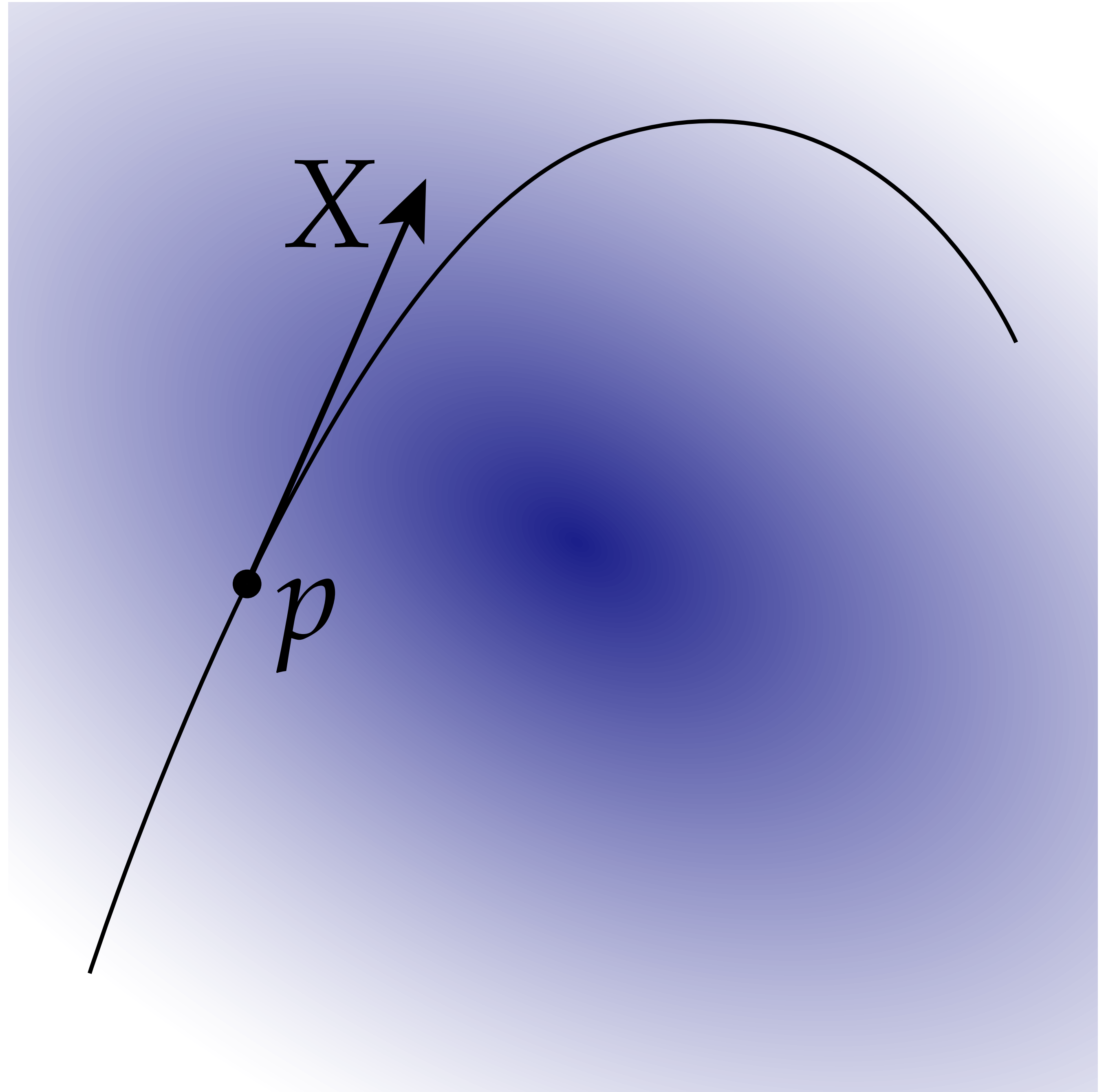
- The *directional derivative* of a scalar function at a point p with respect to a vector X is the rate at which that function increases as we walk away from p with velocity X.
- More precisely:

$$D_X\phi\Big|_p := \lim_{\varepsilon\to 0}\frac{\phi(p+\varepsilon X)-\phi(p)}{\varepsilon}$$

- Alternatively, suppose that X is a *vector field*, rather than just a vector at a single point. Then we can write just:

$$D_X\phi$$

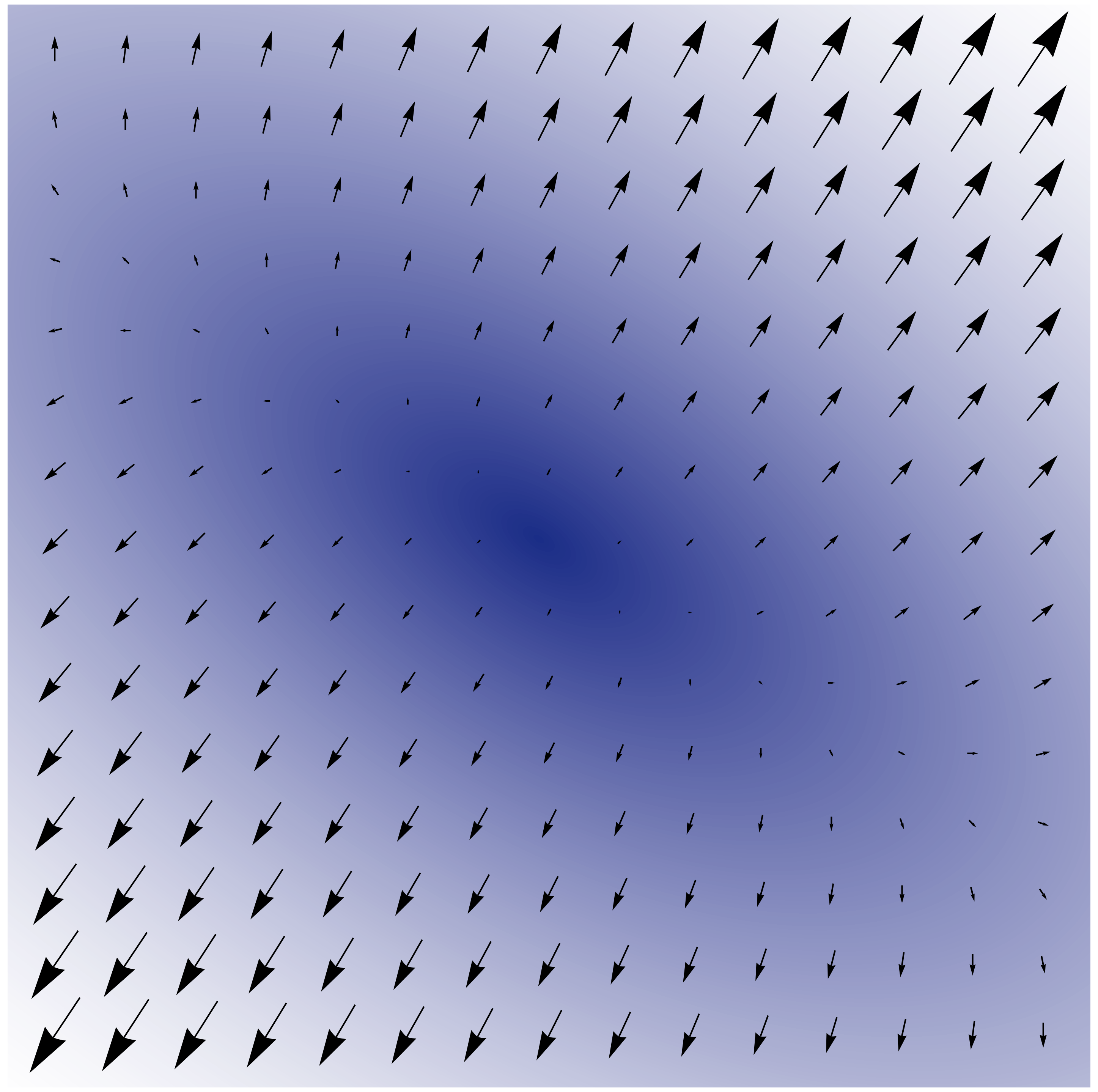
- The result is a *scalar function*, whose value at each point p is the directional derivative along the vector $X(p)$.



$$\phi : \mathbb{R}^2 \to \mathbb{R}$$

Review: Gradient

Let $\phi : \mathbb{R}^n \to \mathbb{R}$. What is the *gradient* of ϕ?



Geometric intuition. "Uphill direction."

Coordinate approach. In Euclidean $\mathbb{R}^n$, list of partials:

$$\nabla\phi = \frac{\partial\phi}{\partial x^1}\frac{\partial}{\partial x^1} + \cdots + \frac{\partial\phi}{\partial x^n}\frac{\partial}{\partial x^n} = \left[\begin{array}{ccc} \frac{\partial\phi}{\partial x^1} & \cdots & \frac{\partial\phi}{\partial x^n} \end{array}\right]^\mathsf{T}$$

Coordinate-free approach. $\langle \nabla\phi, X \rangle = D_X(\phi)$ for all X.

I.e., at each point the gradient is the unique vector* such that taking the inner product $\langle \cdot , \cdot \rangle$ with a given vector X yields the directional derivative along X.

*Assuming it exists! I.e., assuming the function is *differentiable*.

Differential of a Function

- Recall that differential 0-forms are just ordinary scalar functions
- Change in a scalar function can be measured via the *differential*
- Two ways to define differential:

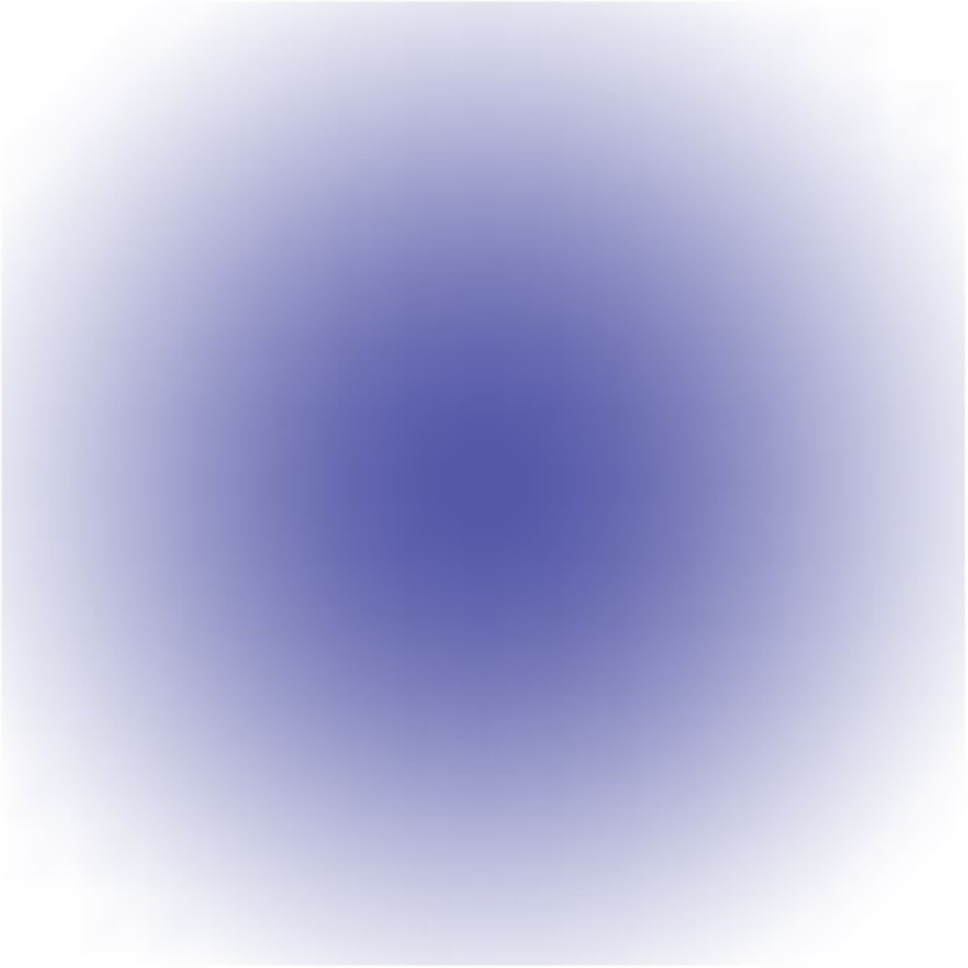
1. As unique 1-form such that applying to any vector field gives directional derivative along those directions:

$$d\phi(X) = D_X\phi$$

2. In coordinates:

$$d\phi(X) := \frac{\partial\phi}{\partial x^1}dx^1 + \cdots + \frac{\partial\phi}{\partial x^n}dx^n$$

…but wait, isn't this just the same as the *gradient?*



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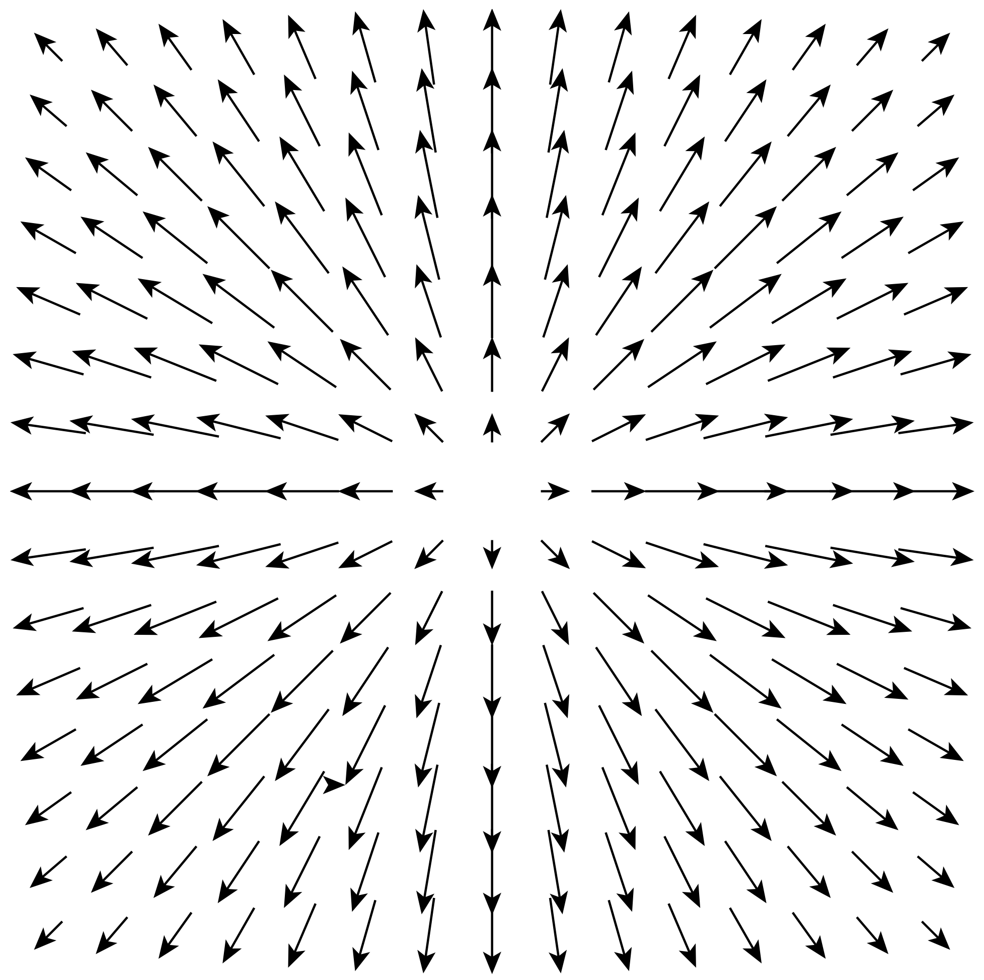


$d\phi$

Gradient vs. Differential

- Superficially, gradient and differential look quite similar (but not identical!):

$$\langle \nabla\phi, X\rangle = D_X\phi \qquad\qquad d\phi(X) = D_X\phi$$

- Especially in R^n:

$$\nabla\phi = \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} + \cdots + \frac{\partial}{\partial\phi x^n}\frac{\partial}{\partial x^n} \qquad\qquad d\phi = \frac{\partial\phi}{\partial x^1}dx^1 + \cdots + \frac{\partial\phi}{\partial x^n}dx^n$$

- So what's the difference?
  - For one thing, one is a *vector field*; the other is a *differential 1-form*
  - More importantly, gradient depends on *inner product*; differential doesn't

$$(df)^\sharp = \nabla\phi \iff \boxed{d\phi(\,\cdot\,) = \langle\nabla\phi,\,\cdot\,\rangle} \iff (\nabla\phi)^\flat = df$$

Makes a *big* difference when it comes to curved geometry, numerical optimization, …

Exterior Derivative — Product Rule

Exterior Derivative

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

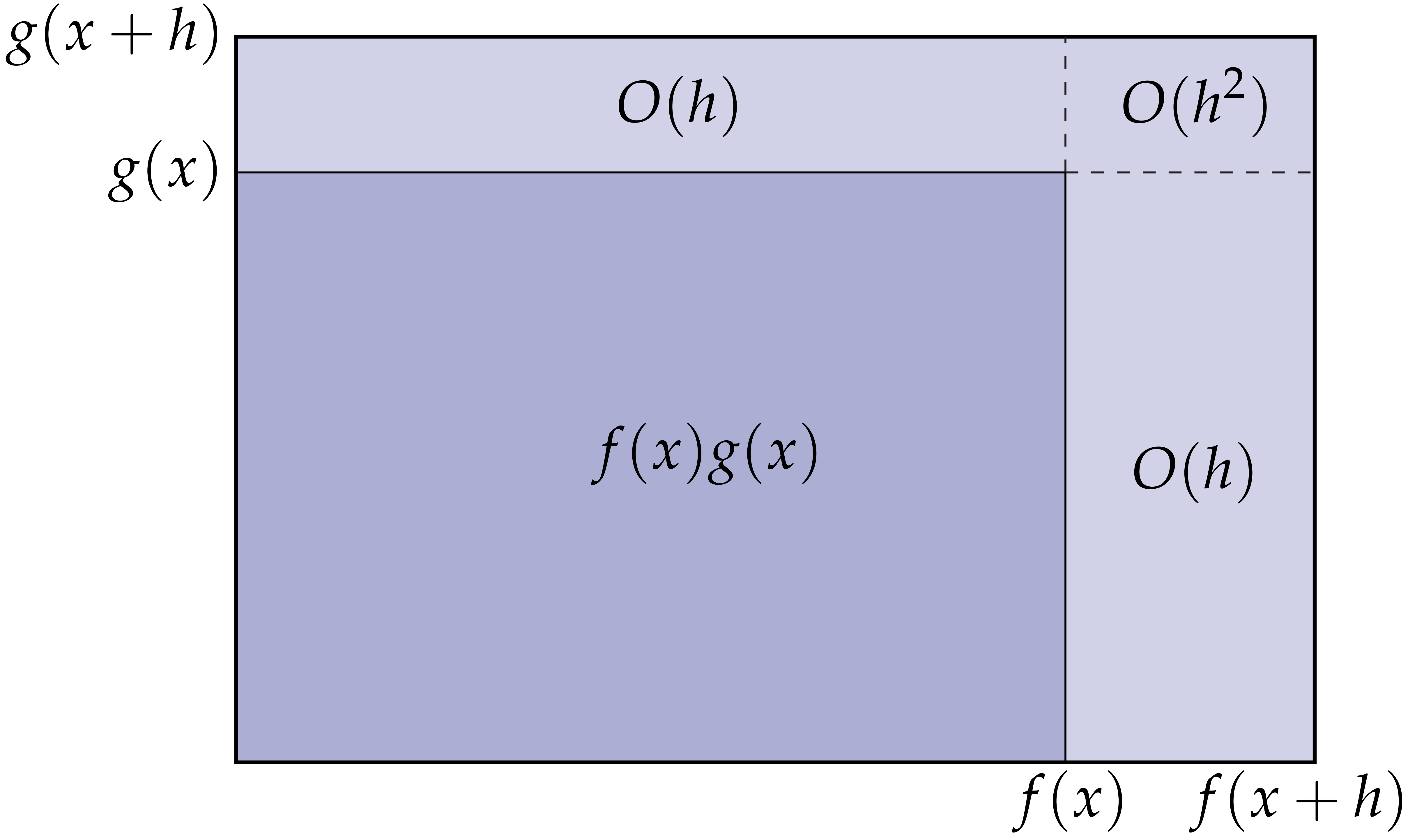
differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Product Rule—Derivative

Reminder: For any differentiable function $f : \mathbb{R} \to \mathbb{R}$, $(fg)' = f'g + fg'$.

Q: Why? What's the *geometric* interpretation?

$$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

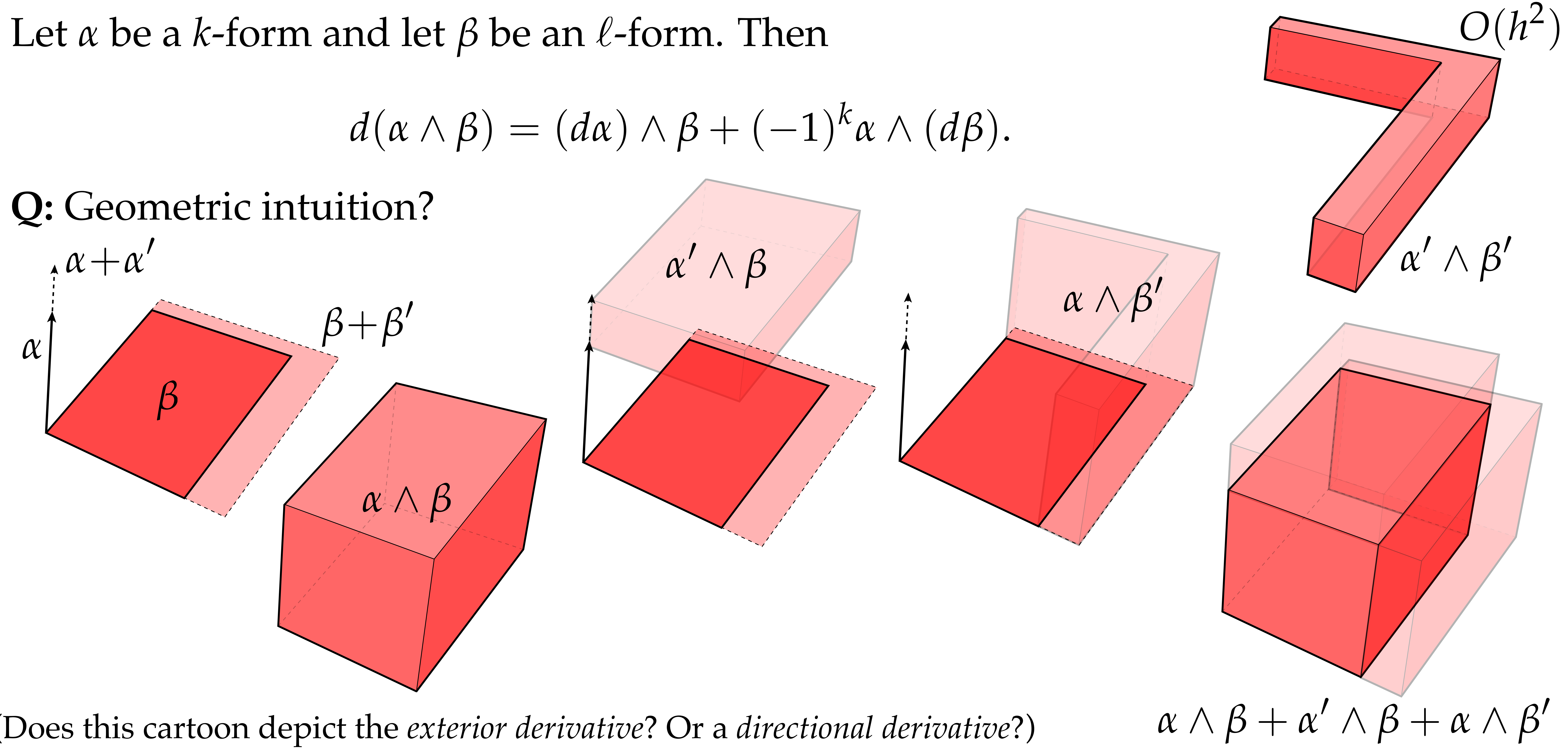


Product Rule—Exterior Derivative

Let α be a k-form and let β be an ℓ-form. Then

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

Q: Geometric intuition?



(Does this cartoon depict the *exterior derivative*? Or a *directional derivative*?)

Product Rule—"Recursive Evaluation"

Example. Let $\alpha := u\,dx$, $\beta := v\,dy$, and $\gamma := w\,dz$ be differential 1-forms on $\mathbb{R}^n$, where $u, v, w : \mathbb{R}^n \to \mathbb{R}$ are 0-forms, *i.e.*, scalar functions. Also, let $\omega := \alpha \wedge \beta$. Then

$$d(\omega \wedge \gamma) = (d\omega) \wedge \gamma + (-1)^2 \omega \wedge (d\gamma).$$

We can then "recursively" evaluate derivatives that appear on the right-hand side:

$$\begin{aligned}
d\omega &= (d\alpha) \wedge \beta + (-1)^1 \alpha \wedge (d\beta), \\
d\alpha &= (du) \wedge dx + (-1)^0 u \overbrace{(ddx)}^{0}, \\
d\beta &= (dv) \wedge dy + (-1)^0 v \overbrace{(ddy)}^{0}, \\
d\gamma &= (dw) \wedge dz + (-1)^0 w \overbrace{(ddz)}^{0}.
\end{aligned}$$

Key idea: The "base case" is the 0-forms, *i.e.*, computing the final result boils down to taking the differential of ordinary scalar functions.

Exterior Derivative — Examples

Example. Let $\phi(x,y) := \frac{1}{2}e^{-(x^2+y^2)}$. Then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$

$$= -2\phi(xdx + ydy)$$

Example. Let $\alpha(x,y) = xdx + ydy$. Then $d\alpha =$

$$(\tfrac{\partial x}{\partial x}dx + \tfrac{\partial x}{\partial y}dy) \wedge dx + (\tfrac{\partial y}{\partial x}dx + \tfrac{\partial y}{\partial y}dy) \wedge dy$$

$$= dx \wedge dx + dy \wedge dy \ = 0 + 0 \ = 0.$$

Example. Again let $\alpha(x,y) = xdx + ydy$. Then $d \star \alpha = d(x \star dx + y \star dy)$

$$= d(xdy - ydx)$$

$$= dx \wedge dy - dy \wedge dx$$

$$= 2dx \wedge dy.$$

Exterior Derivative—Exactness

Exterior Derivative

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Why?

Review: Curl of Gradient

ϕ grad ϕ $\text{curl} \circ \text{grad}\, \phi$

Key idea: exterior derivative should capture a similar idea.

What Happens if $d \circ d = 0$*?*

Q: Consider a 1-form $\alpha = udx + vdy + wdz$, where the coefficients u, v, w are each scalar functions $\mathbb{R}^3 \to \mathbb{R}$. What is the exterior derivative $d\alpha$ in coordinates x, y, z?

A: $d\alpha = d(udx + vdy + wdz) = du \wedge dx + u\cancelto{0}{ddx} + dv \wedge dy + v\cancelto{0}{ddy} + dw \wedge dz + w\cancelto{0}{ddz}$

$$= \begin{array}{l} (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz) \wedge dx + \\ (\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz) \wedge dy + \\ (\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz) \wedge dz \end{array} = \begin{array}{l} (\frac{\partial u}{\partial x}\cancelto{0}{dx \wedge dx} + \frac{\partial u}{\partial y}dy \wedge dx + \frac{\partial u}{\partial z}dz \wedge dx) + \\ (\frac{\partial v}{\partial x}dx \wedge dy + \frac{\partial v}{\partial y}\cancelto{0}{dy \wedge dy} + \frac{\partial v}{\partial z}dz \wedge dy) + \\ (\frac{\partial w}{\partial x}dx \wedge dz + \frac{\partial w}{\partial y}dy \wedge dz + \frac{\partial w}{\partial z}\cancelto{0}{dz \wedge dz}) \end{array}$$

$$= -\frac{\partial u}{\partial y}dx \wedge dy + \frac{\partial u}{\partial z}dz \wedge dx + \frac{\partial v}{\partial x}dx \wedge dy - \frac{\partial v}{\partial z}dy \wedge dz - \frac{\partial w}{\partial x}dz \wedge dx + \frac{\partial w}{\partial y}dy \wedge dz$$

$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})dy \wedge dz + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz \wedge dx + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx \wedge dy.$$

Q: Does this operation remind you of anything (*perhaps from vector calculus*)?

Exterior Derivative and Curl

Suppose we have a *vector field*

$$X := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

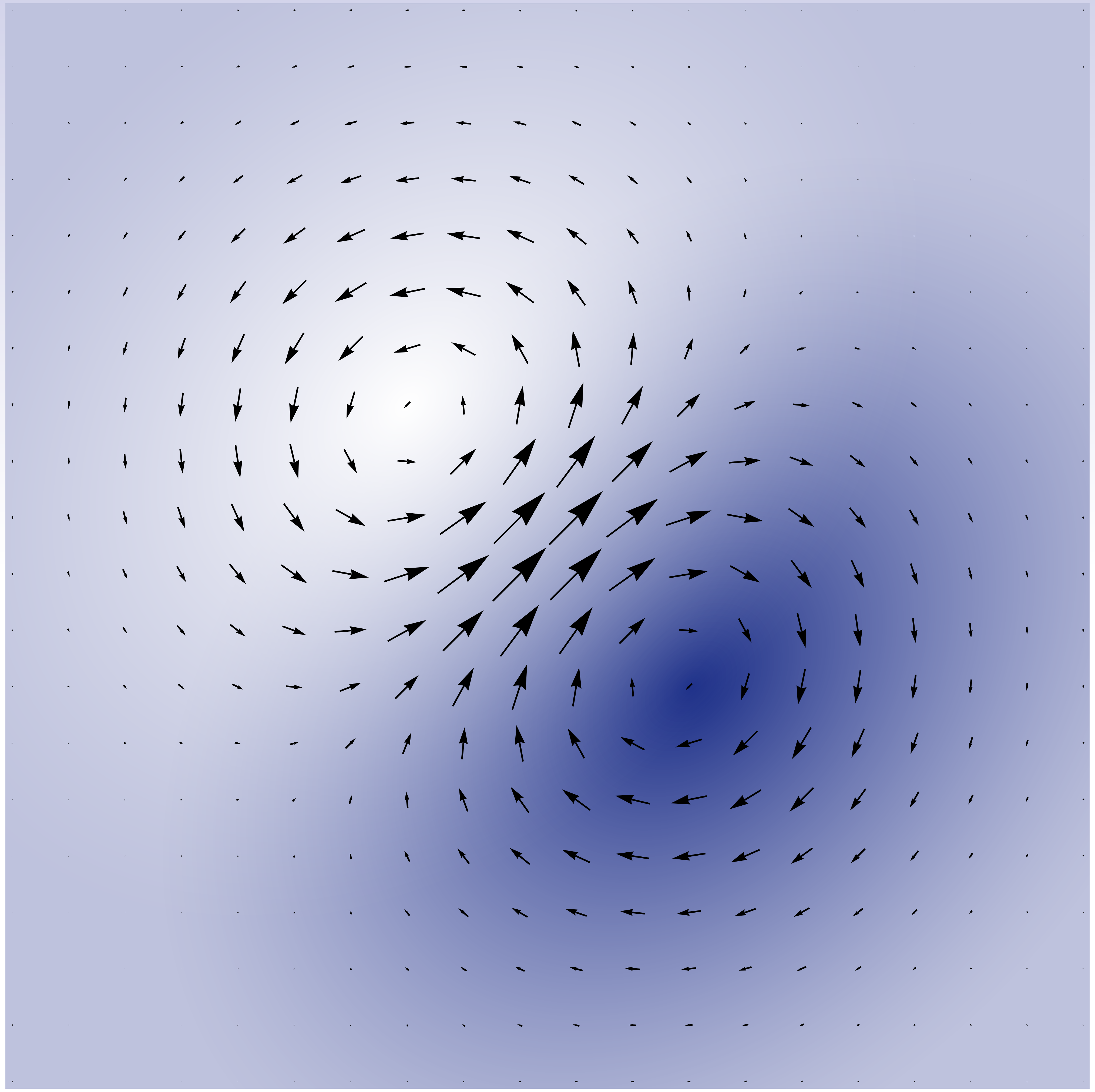
Its *curl* is then

$$\nabla \times X = \begin{array}{cccc} (\partial w/\partial y & - & \partial v/\partial z) & \frac{\partial}{\partial x} & + \\ (\partial u/\partial z & - & \partial w/\partial x) & \frac{\partial}{\partial y} & + \\ (\partial v/\partial x & - & \partial u/\partial y) & \frac{\partial}{\partial z} & \end{array}$$

Looks an awful lot like…

$$d\alpha = \begin{array}{cccc} (\partial w/\partial y & - & \partial v/\partial z) & dy \wedge dz & + \\ (\partial u/\partial z & - & \partial w/\partial x) & dz \wedge dx & + \\ (\partial v/\partial x & - & \partial u/\partial y) & dx \wedge dy & \end{array}$$

Especially if we then apply the *Hodge star.*



$$\nabla \times X \iff \star d\alpha$$

$$\nabla \times X = (\star d X^\flat)^\sharp$$

$d \circ d = 0$

ϕ $\quad$ $d\phi$ $\quad$ $dd\phi$

Intuition: in R^n, first d behaves just like gradient; second d behaves just like curl.

Exterior Derivative in 3D (1-forms)

Q: How about $d \star \alpha$? (Still for $\alpha = udx + vdy + wdz$.)

A:
$$\begin{aligned} d \star \alpha &= d(\star(udx + vdy + wdz)) \\ &= d(udy \wedge dz + vdz \wedge dx + wdx \wedge dy) \\ &= du \wedge dy \wedge dz + dv \wedge dz \wedge dx + dw \wedge dx \wedge dy \\ &= \tfrac{\partial u}{\partial x} dx \wedge dy \wedge dz + \tfrac{\partial v}{\partial y} dy \wedge dz \wedge dz + \tfrac{\partial w}{\partial z} dz \wedge dx \wedge dy \\ &= (\tfrac{\partial u}{\partial x} + \tfrac{\partial v}{\partial y} + \tfrac{\partial w}{\partial z}) dx \wedge dy \wedge dz \end{aligned}$$

Q: Does this operation remind you of anything (*perhaps from vector calculus*)?

Exterior Derivative and Divergence

Suppose we have a *vector field*

$$X := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

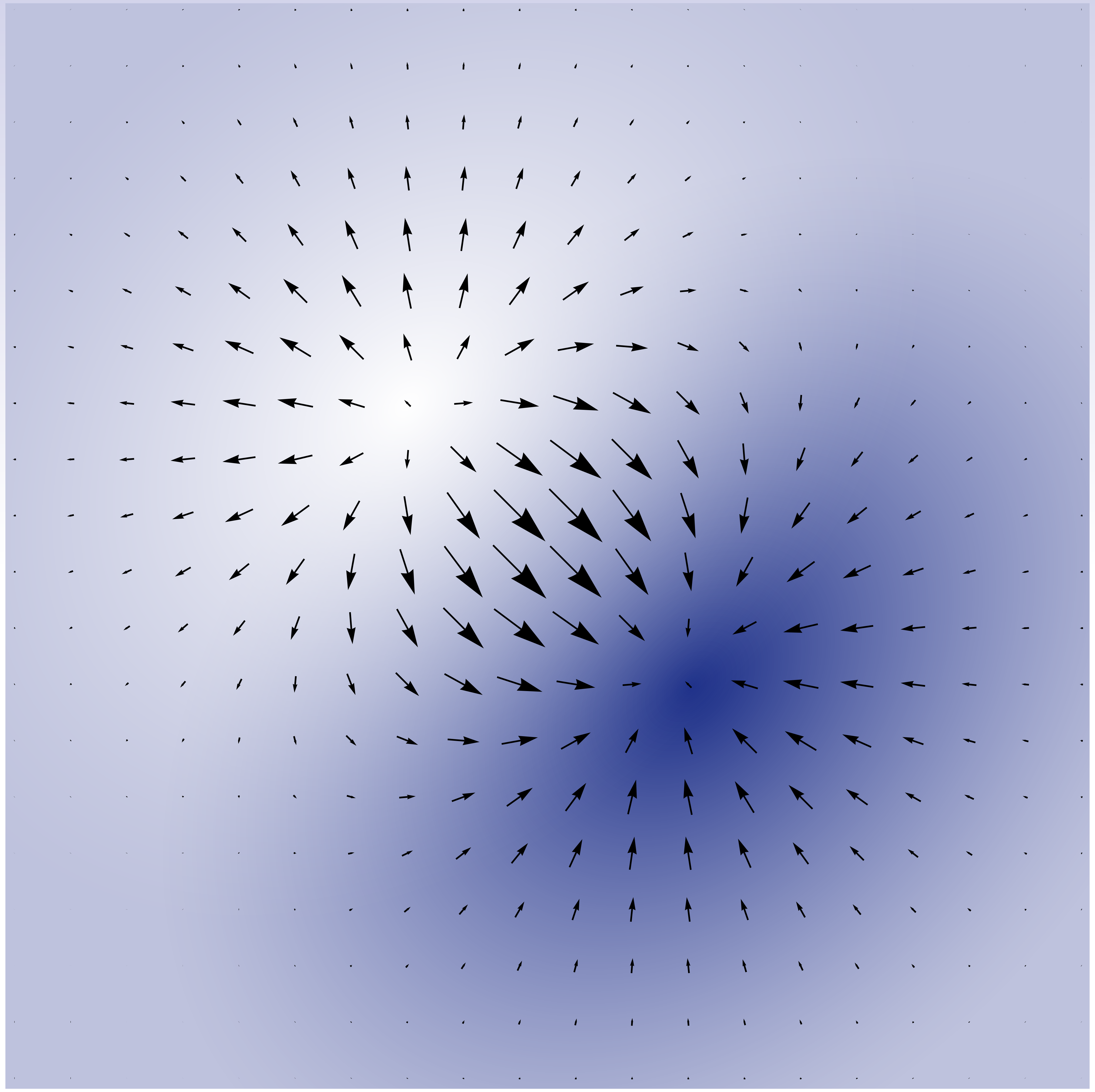
Its *divergence* is then

$$\nabla \cdot X = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial u} + \frac{\partial w}{\partial z}$$

Looks an awful lot like...

$$d \star \alpha = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx \wedge dy \wedge dz$$

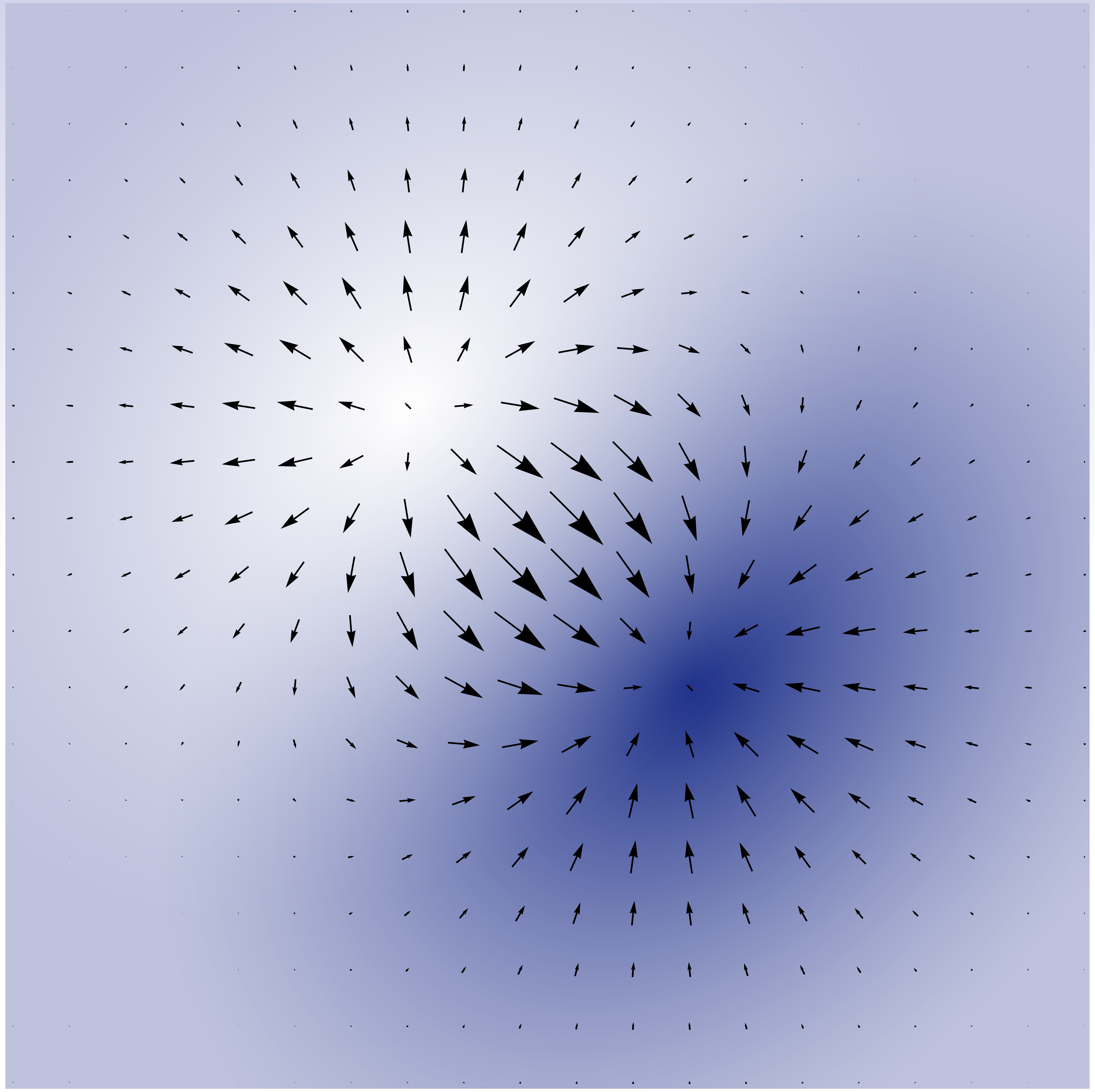
Especially if we then apply the *Hodge star.*



$$\nabla \cdot X \quad \Longleftrightarrow \quad \star d \star \alpha$$

$$\nabla \cdot X = \star d \star X^\flat$$

Exterior Derivative - Divergence

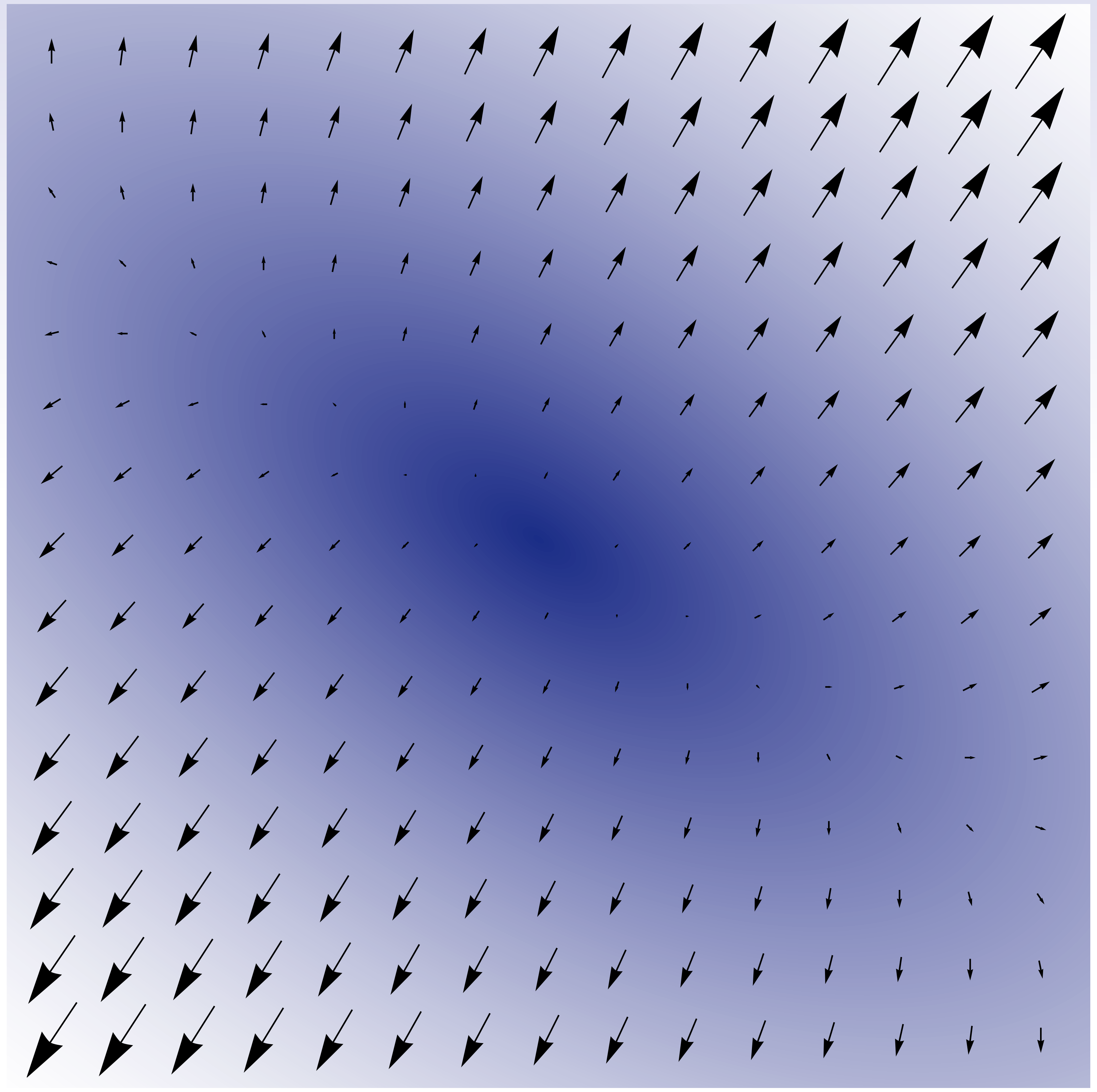


$$\nabla \cdot X = \star d(\star X^\flat)$$

(*codifferential:* $\delta := \star d \star$)

Exterior vs. Vector Derivatives — Summary

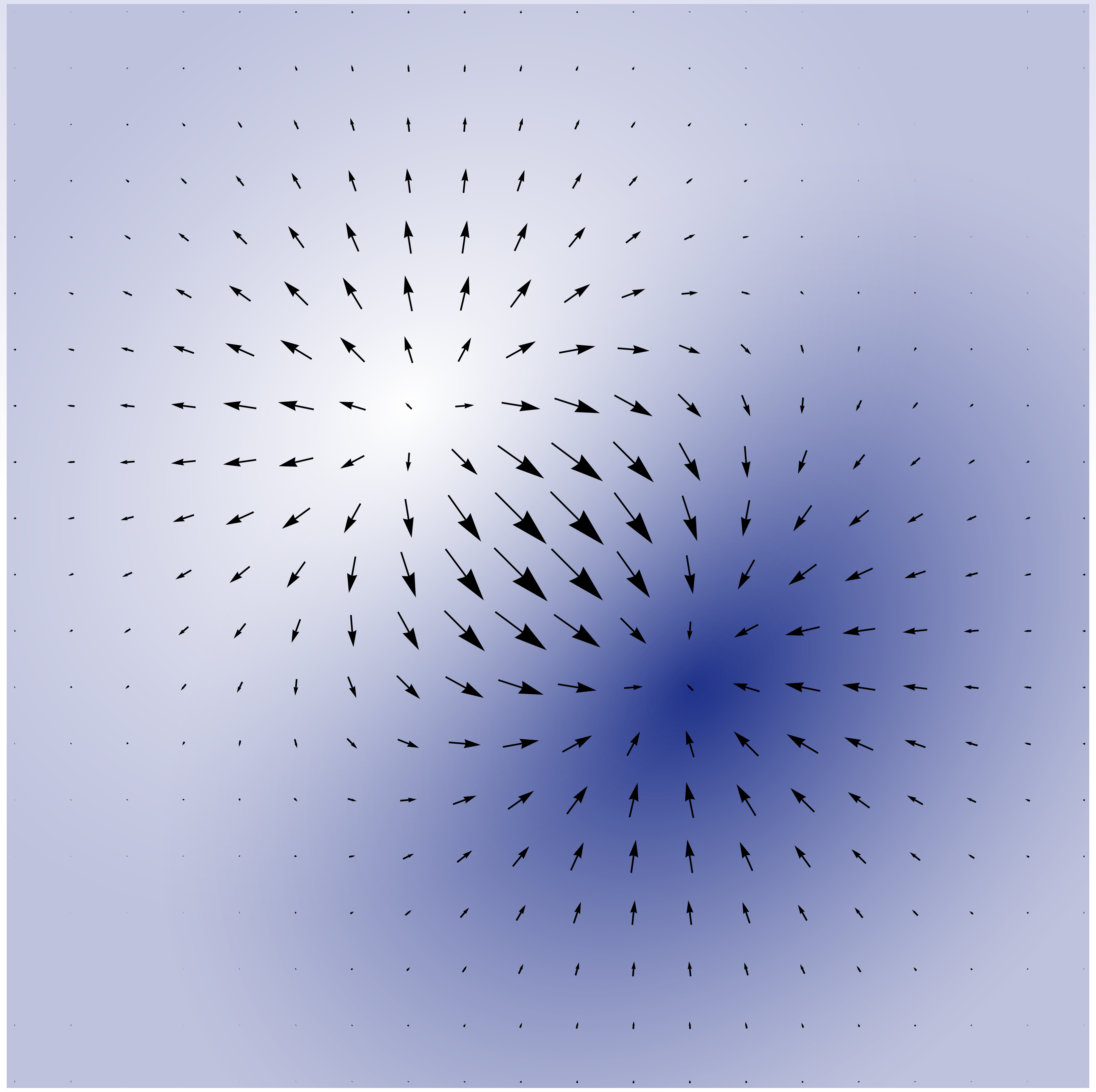
ϕ



grad ϕ

$(d\phi)^\sharp$

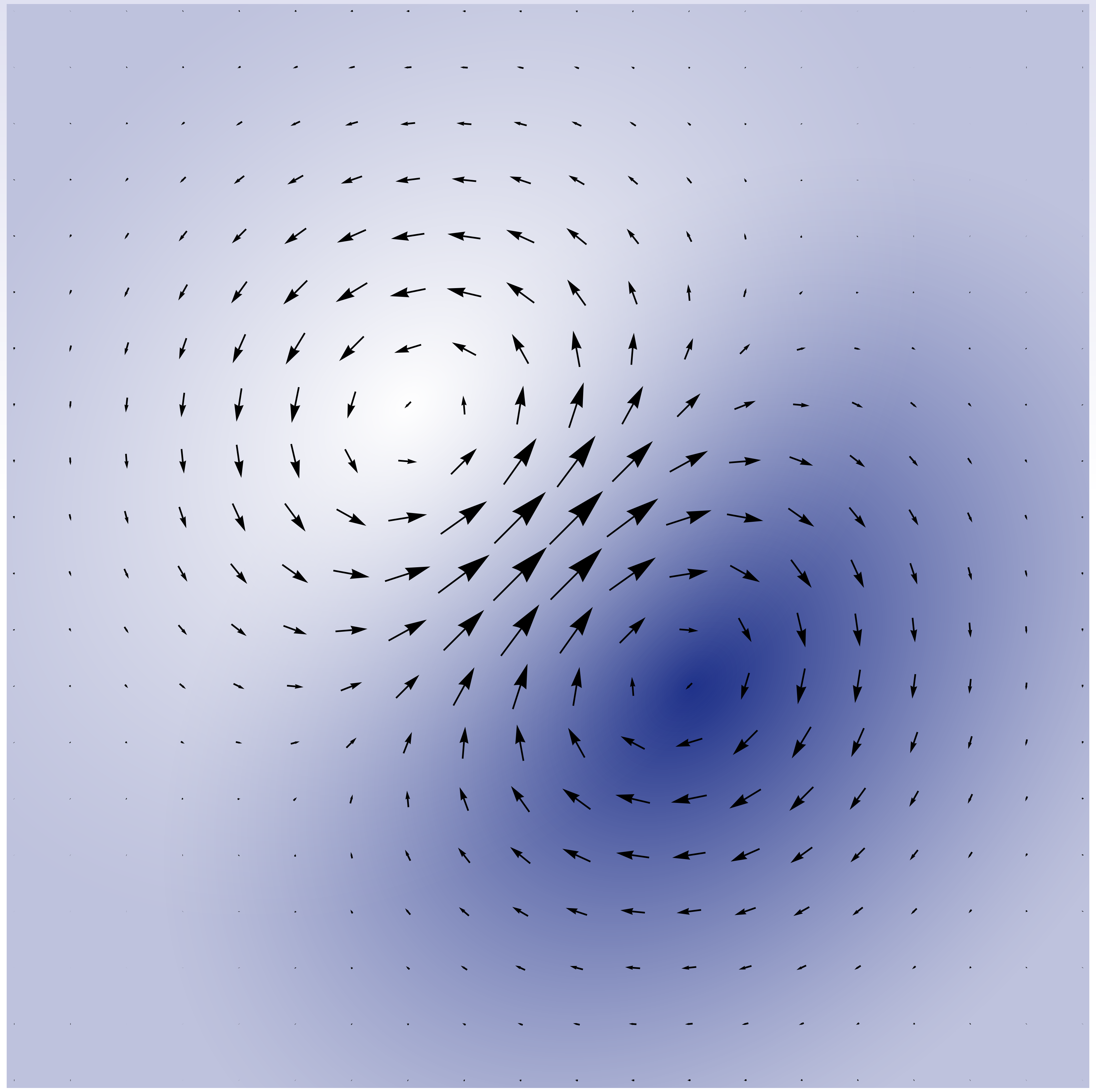
X



div X

$\star d(\star X^\flat)$

Y



curl Y

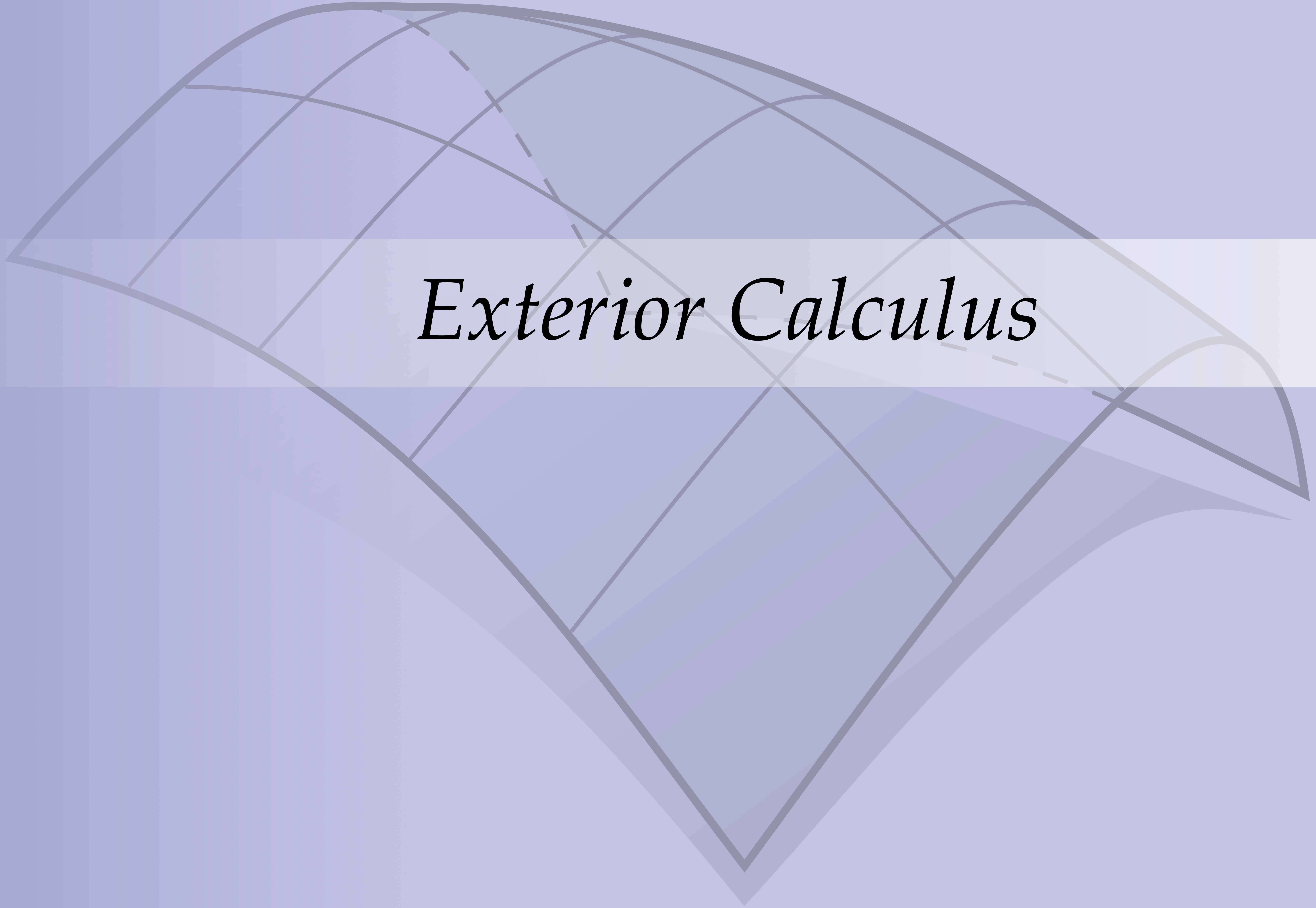
$(\star(dX^\flat))^\sharp$

Exterior Derivative

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

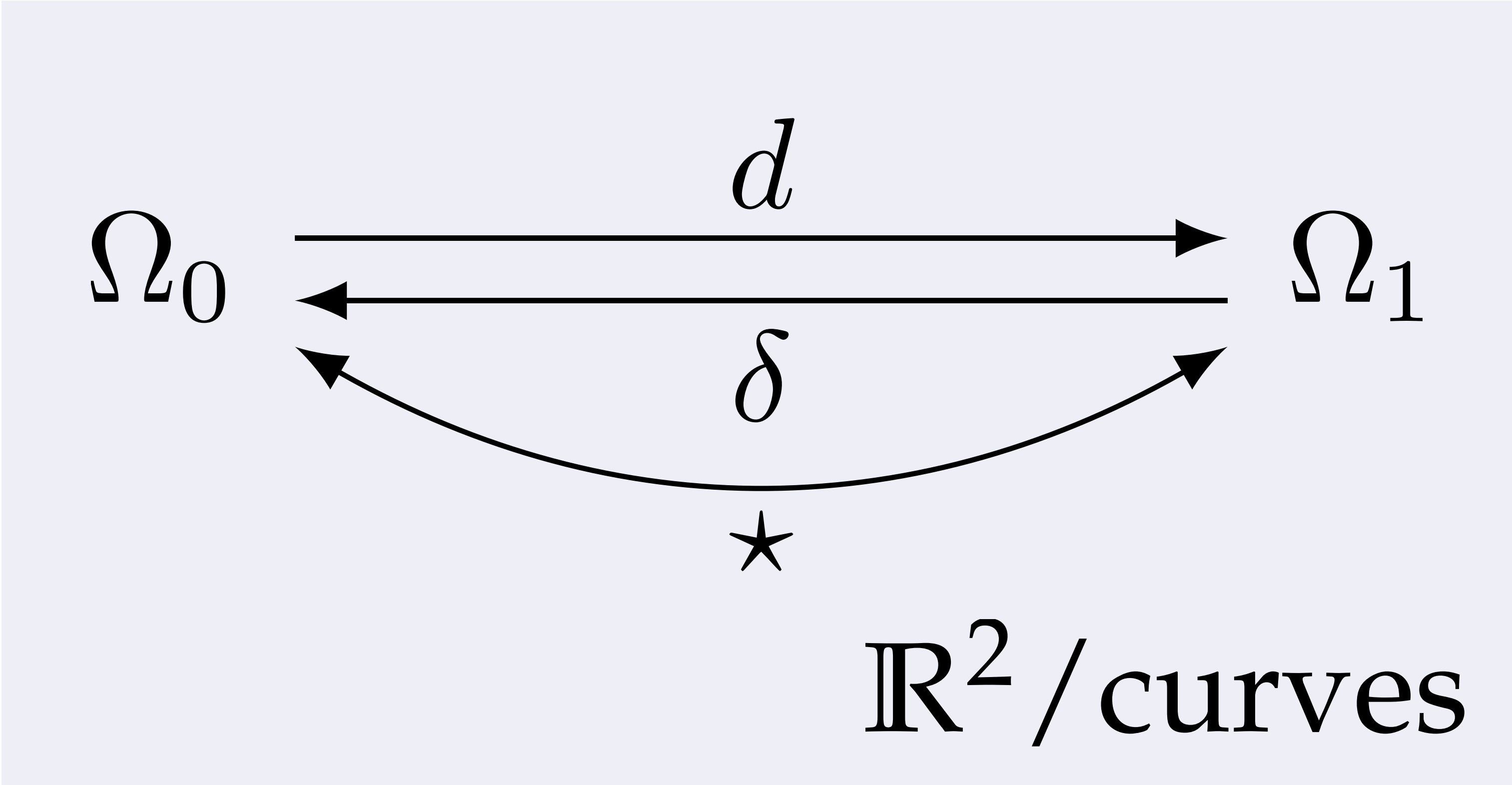
differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

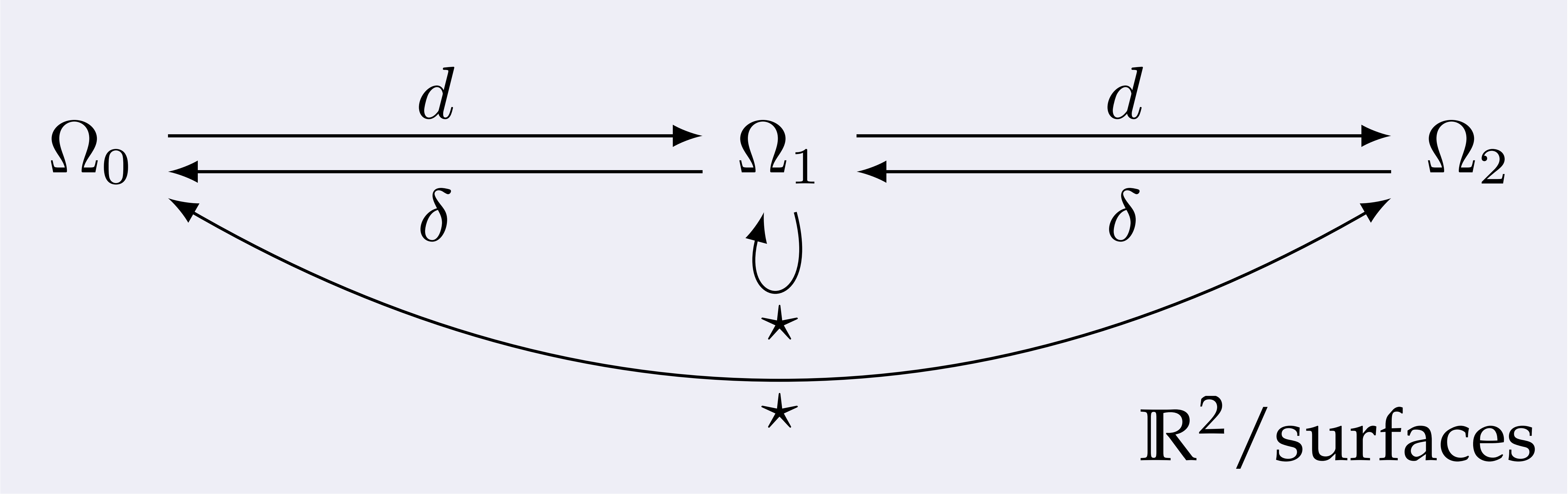
Exterior Calculus

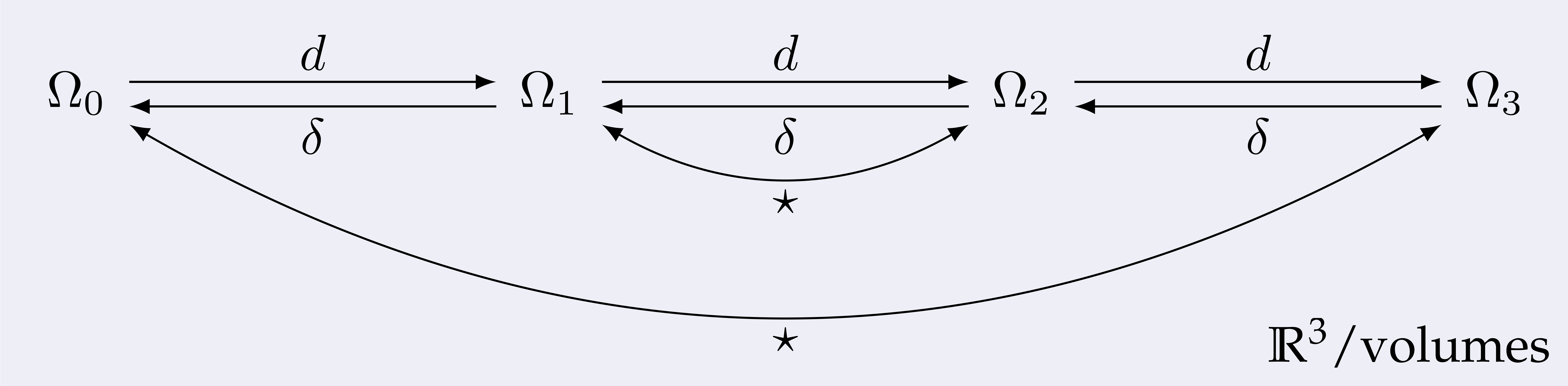


Exterior Calculus—Diagram View

- Taking a step back, we can draw many of the operators seen so far as diagrams:







Ω_k—differential k-forms

Laplacian

- Can now compose operators to get other operators
- E.g., *Laplacian* from vector calculus:

$$\Delta := \text{div} \circ \text{grad}$$

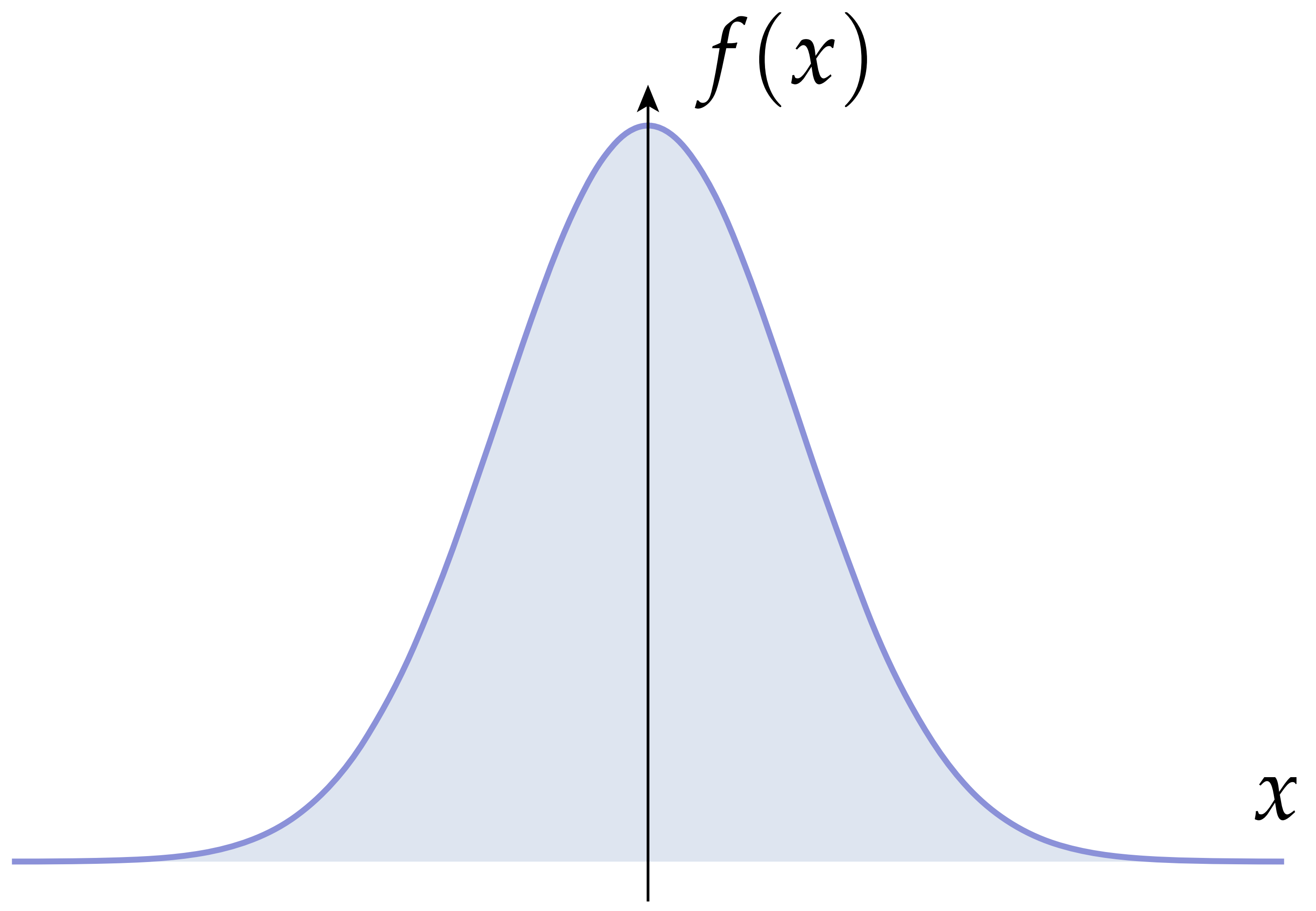
- Can express exact same operator via exterior calculus:

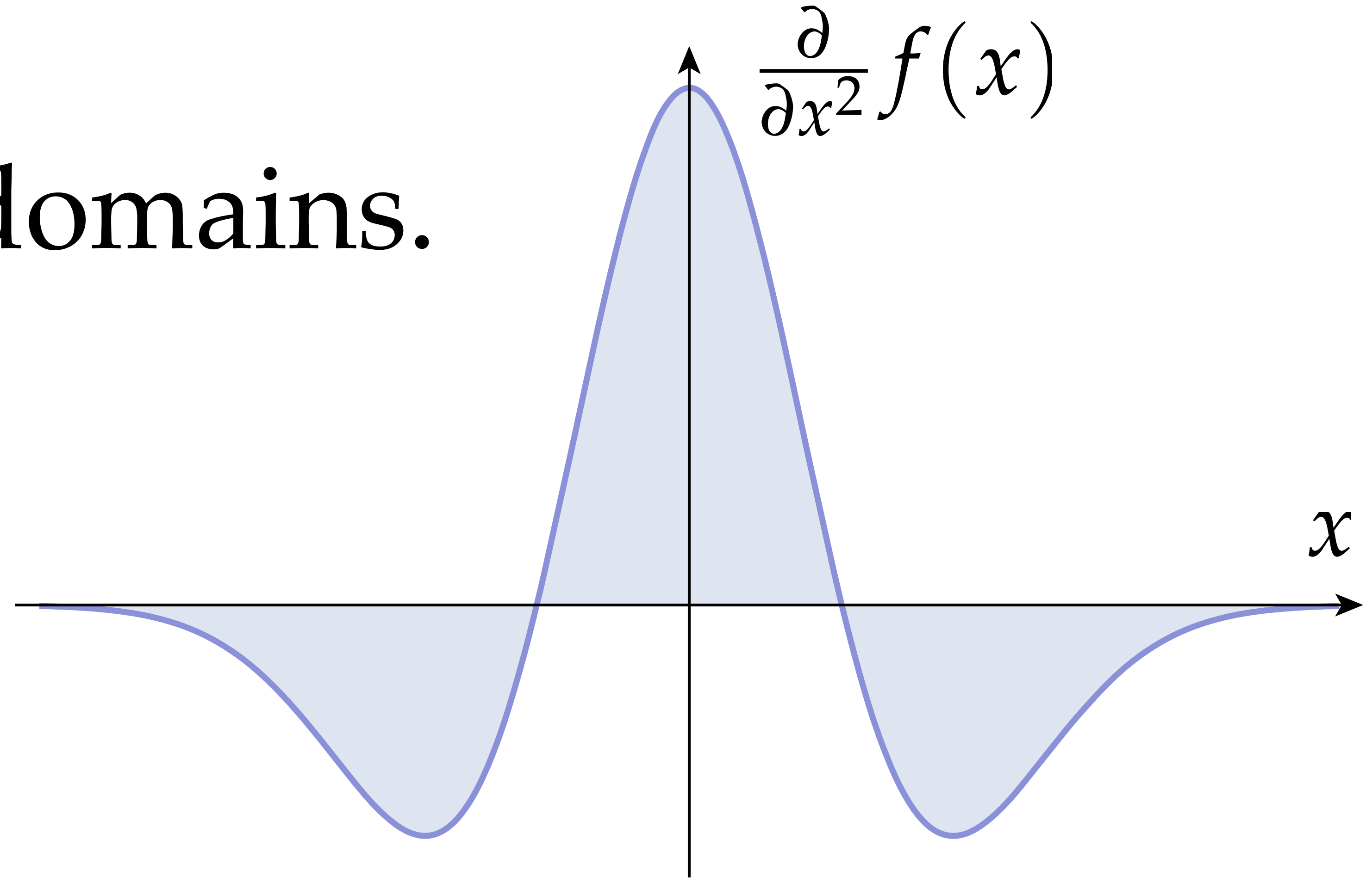
$$\Delta = \star\, d \star d$$

- …except that this expression easily generalizes to curved domains.
- Can also generalize to k-forms:

$$\Delta := \star d \star d + d \star d \star$$

- Will have **much** more to say about the Laplacian later on!

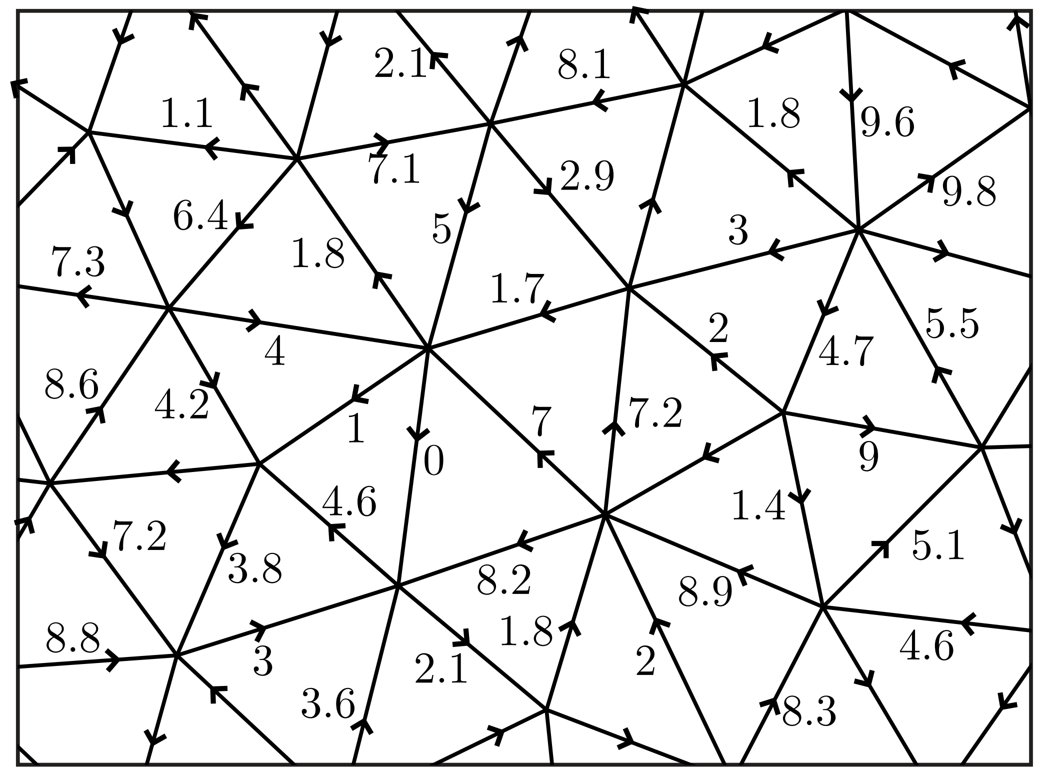


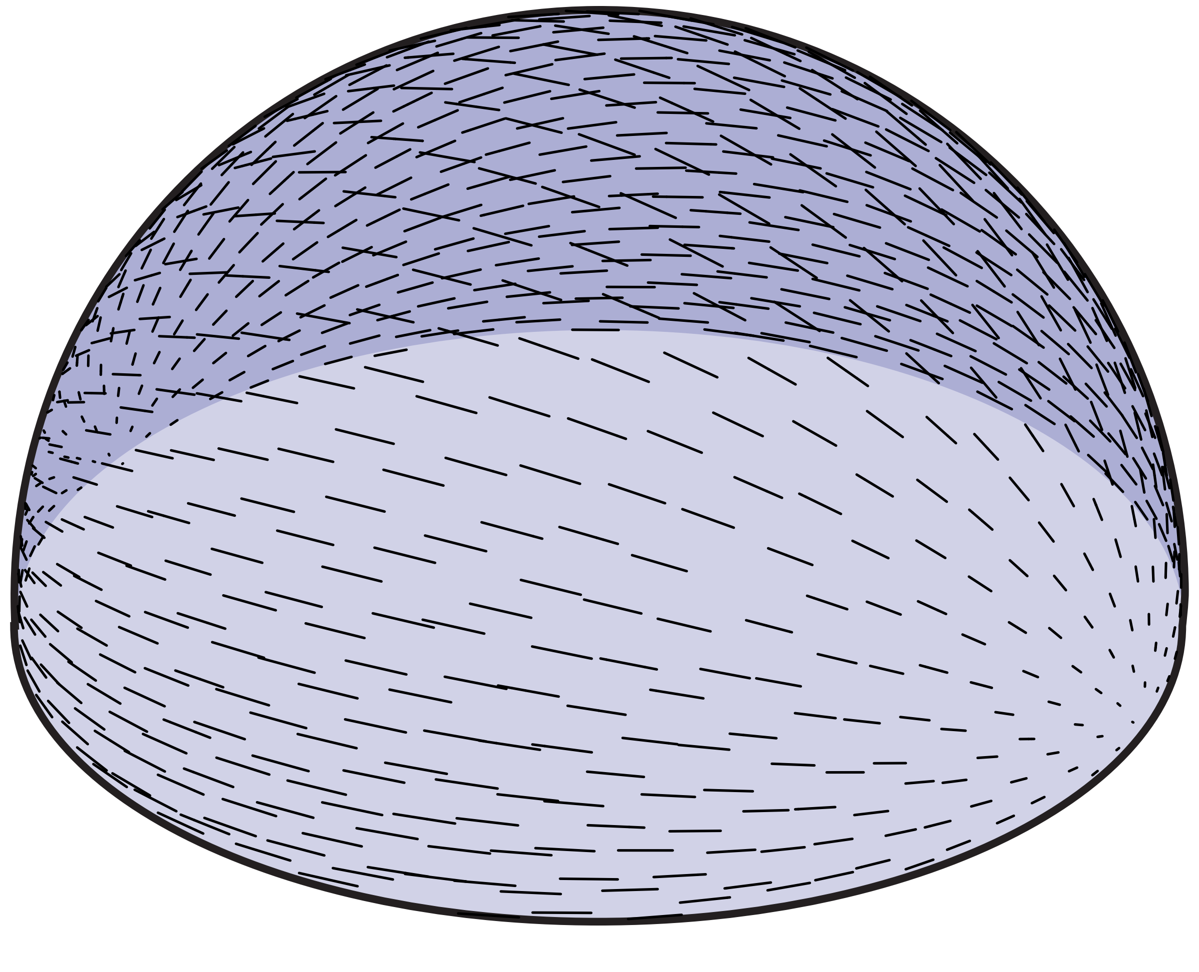


Preview: Exterior Calculus Beyond R^n

- Why study these two very similar viewpoints? (*I.e.*, **vector** vs. **exterior** calculus)
  - Hard to measure change in *volumes* using basic vector calculus
  - Duality clarifies the distinction between different concepts/quantities
  - **Topology**: notion of differentiation that does not require metric (e.g., *cohomology*)
  - **Geometry**: clear language for calculus on *curved* domains (Riemannian manifolds)
  - **Physics**: clear distinction between physical quantities (e.g., *velocity* vs. *momentum*)
  - **Computer Science**: *Leads directly to discretization/computation!*

[DEMO]

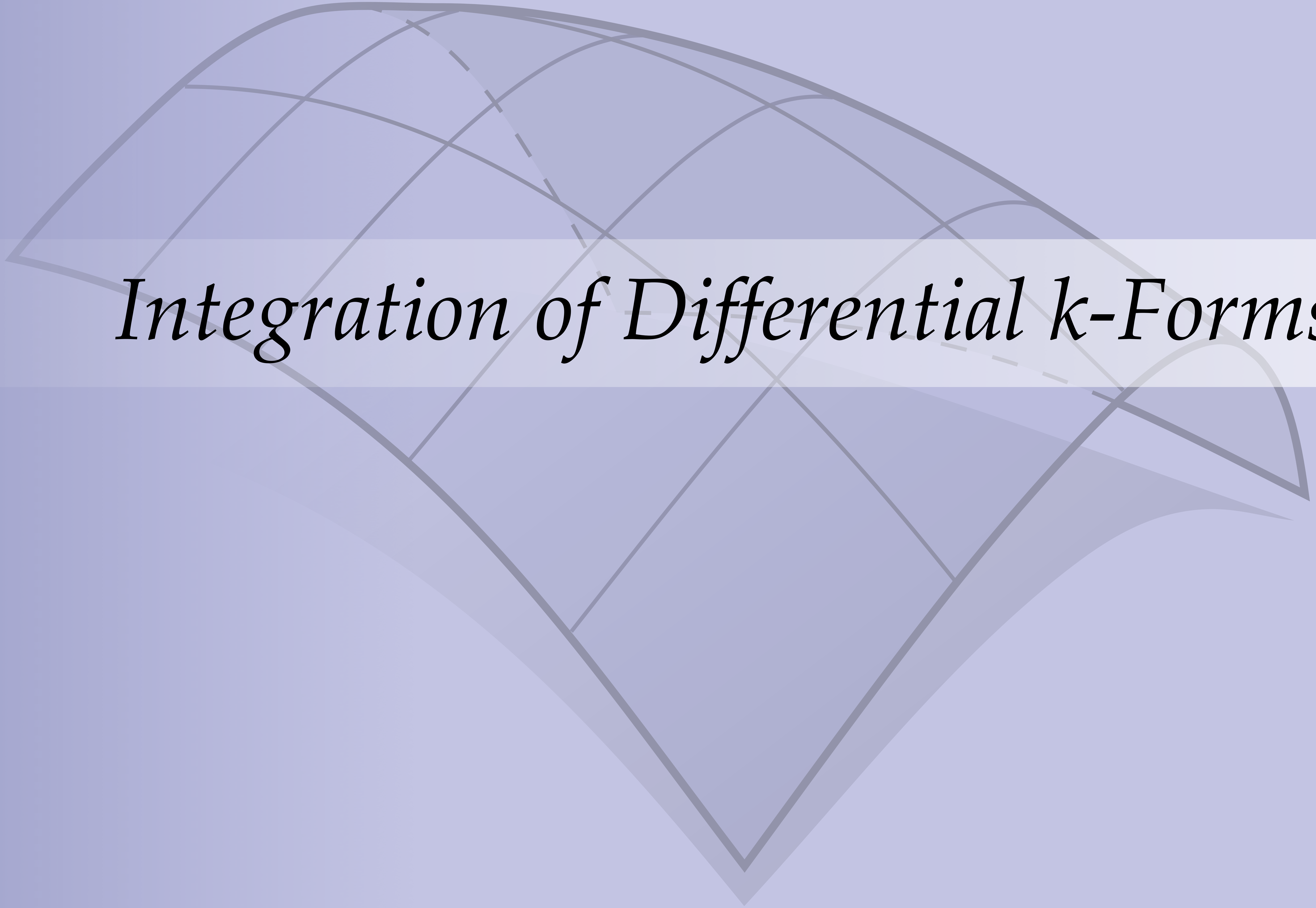




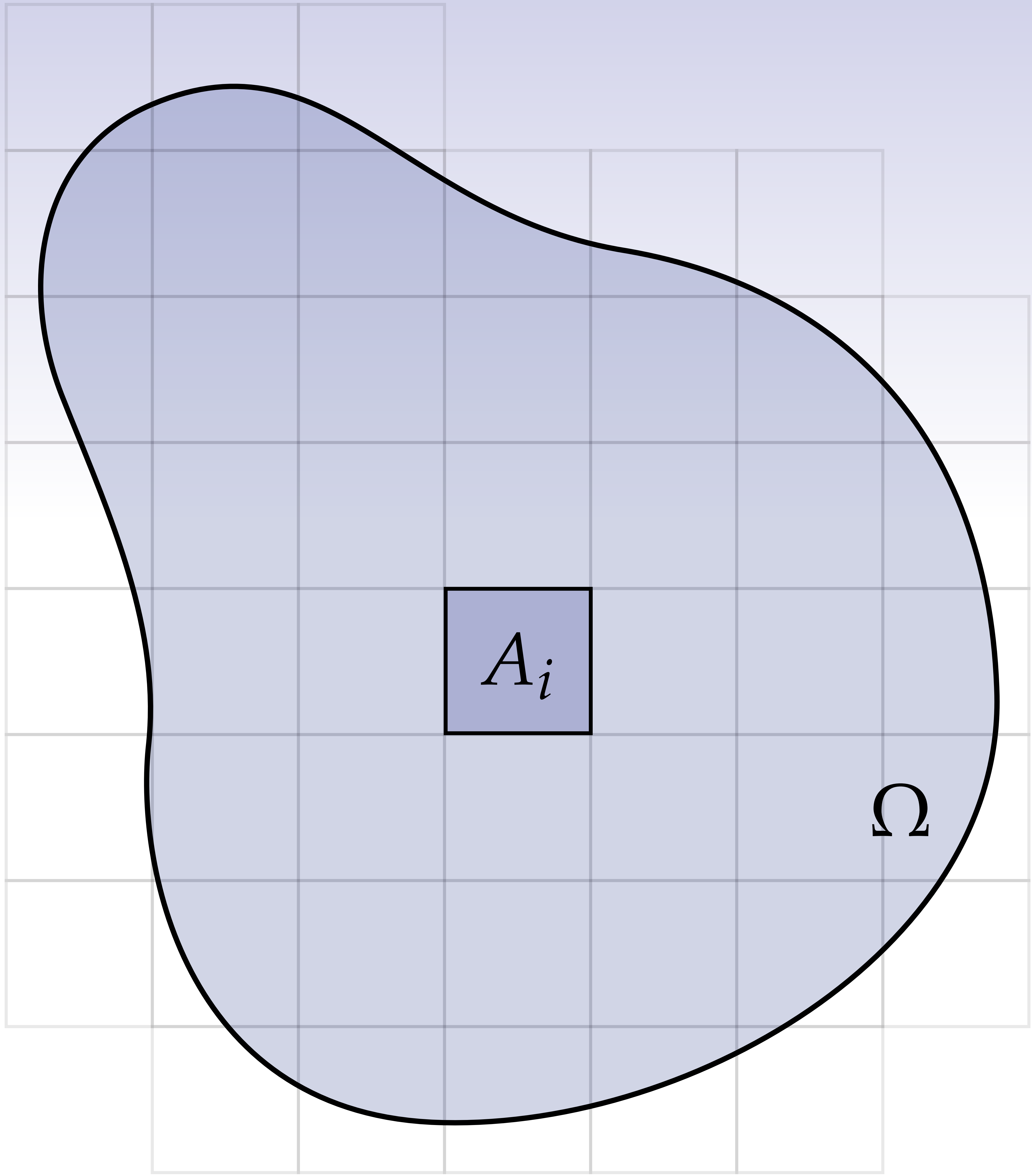
Exterior Derivative - Summary

- Exterior derivative d used to differentiate k-forms
  - 0-form: “gradient”
  - 1-form: “curl”
  - 2-form: “divergence” (codifferential δ)
  - and more...
- Natural product rule
- ***d* of *d* is zero**
  - Analogy: curl of gradient
  - More general picture (soon!) via *Stokes' theorem*

Integration of Differential k-Forms

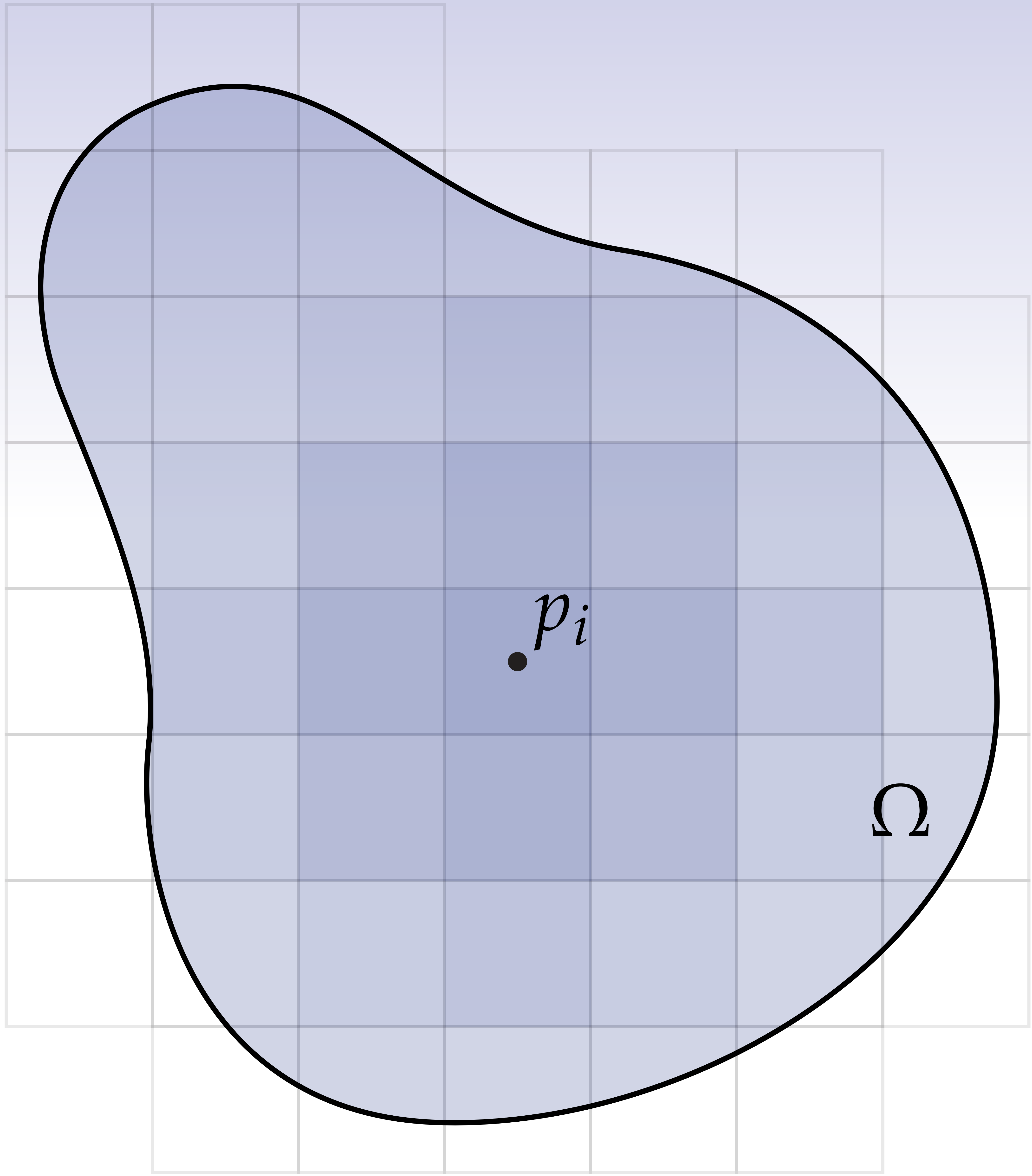


Review—Integration of Area



$$\sum_i A_i \implies \int_\Omega dA$$

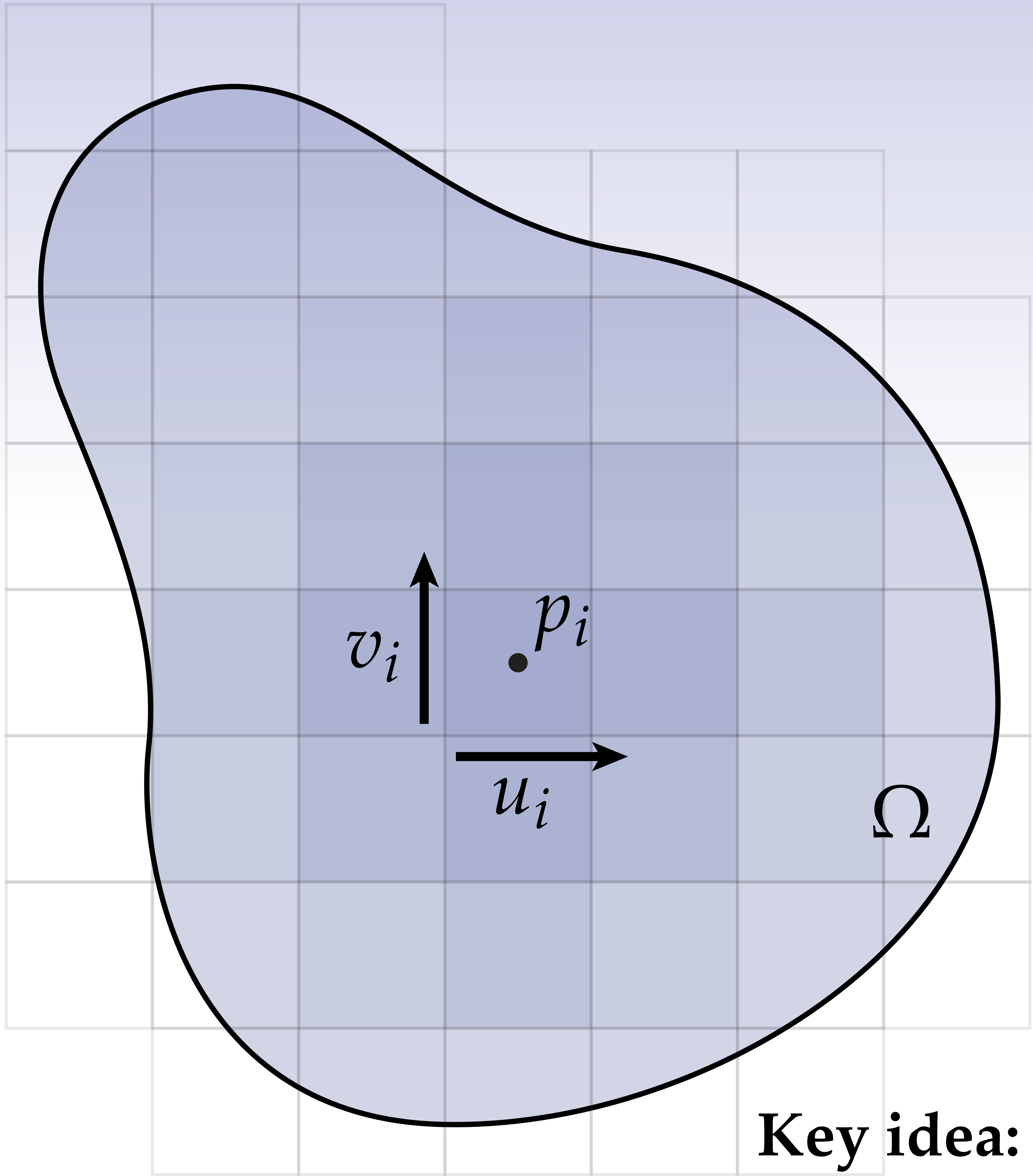
Review—Integration of Scalar Functions



$$\phi : \Omega \to \mathbb{R}$$

$$\sum_i A_i \phi(p_i) \implies \int_\Omega \phi \, dA$$

Integration of a 2-Form



ω — differential 2-form on Ω

$$\sum_i \omega_{p_i}(u_i, v_i) \implies \int_\Omega \omega$$

Key idea: integration *always* involves differential forms!

Integration of Differential 2-forms—Example

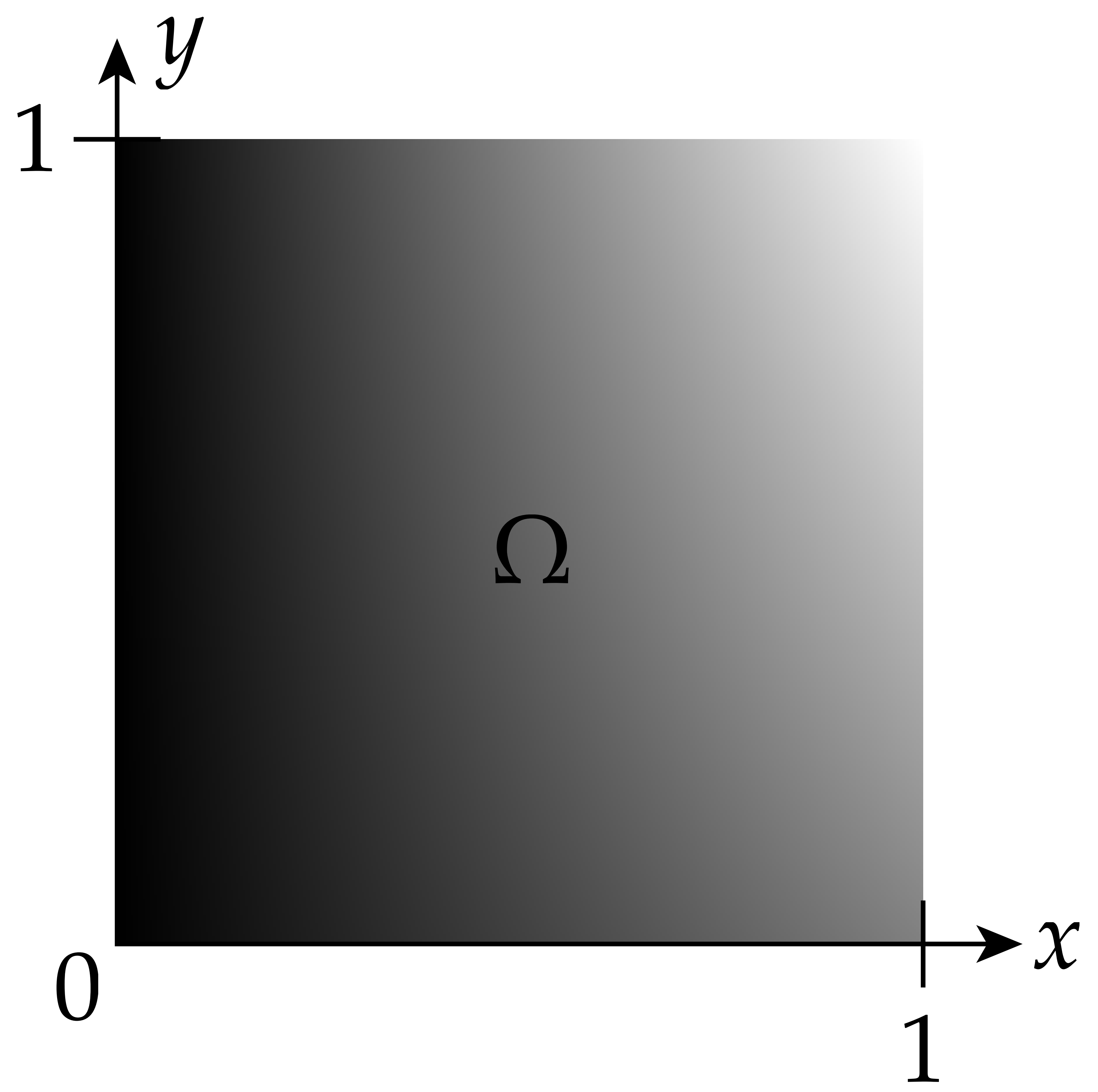
- Consider a differential 2-form on the unit square in the plane:

$$\omega := (x + xy)dx \wedge dy$$

$$\int_\Omega \omega = \int_\Omega (x + xy)dx \wedge dy$$

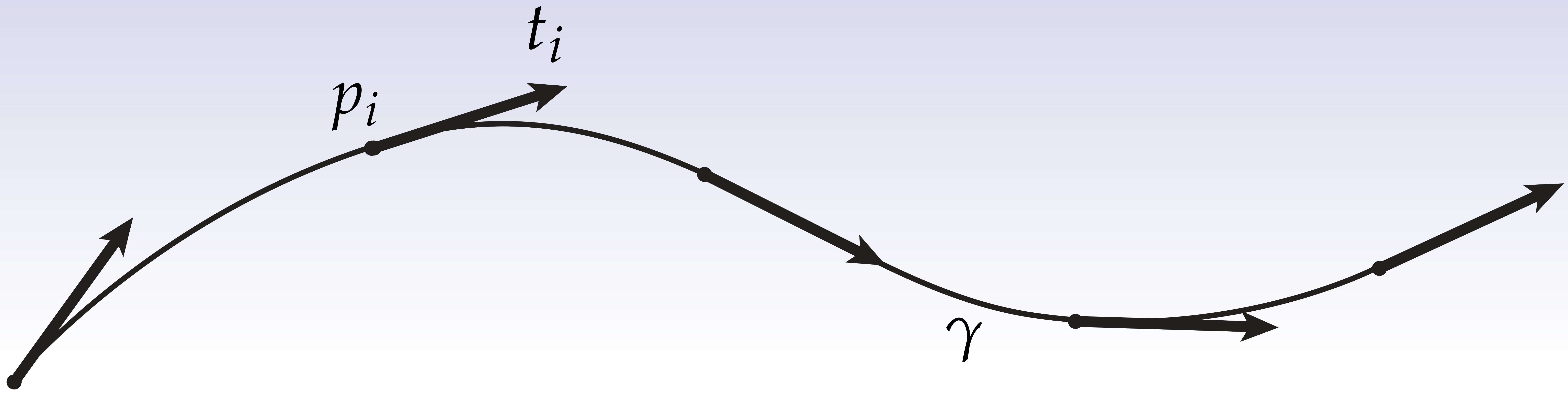
$$= \int_0^1 \int_0^1 (x + xy)dx \wedge dy$$

$$= \cdots = \frac{3}{4}$$



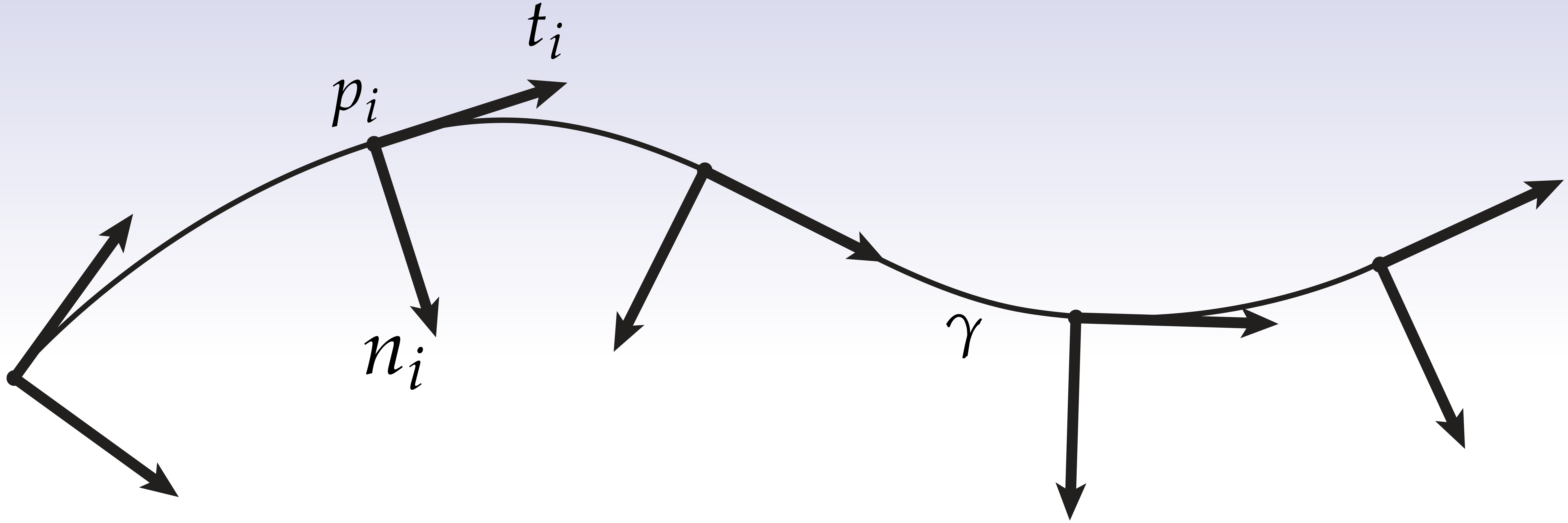
- In this case, no different from usual "double integration" of a scalar function.

Integration on Curves



$$\int_\gamma \alpha \approx \sum_i \alpha_{p_i}(t_i)$$

Integration on Curves



$$\int_\gamma \star\alpha \approx \sum_i \star\alpha_{p_i}(t_i) = \sum_i \alpha_{p_i}(n_i)$$

Integration on Curves — Example

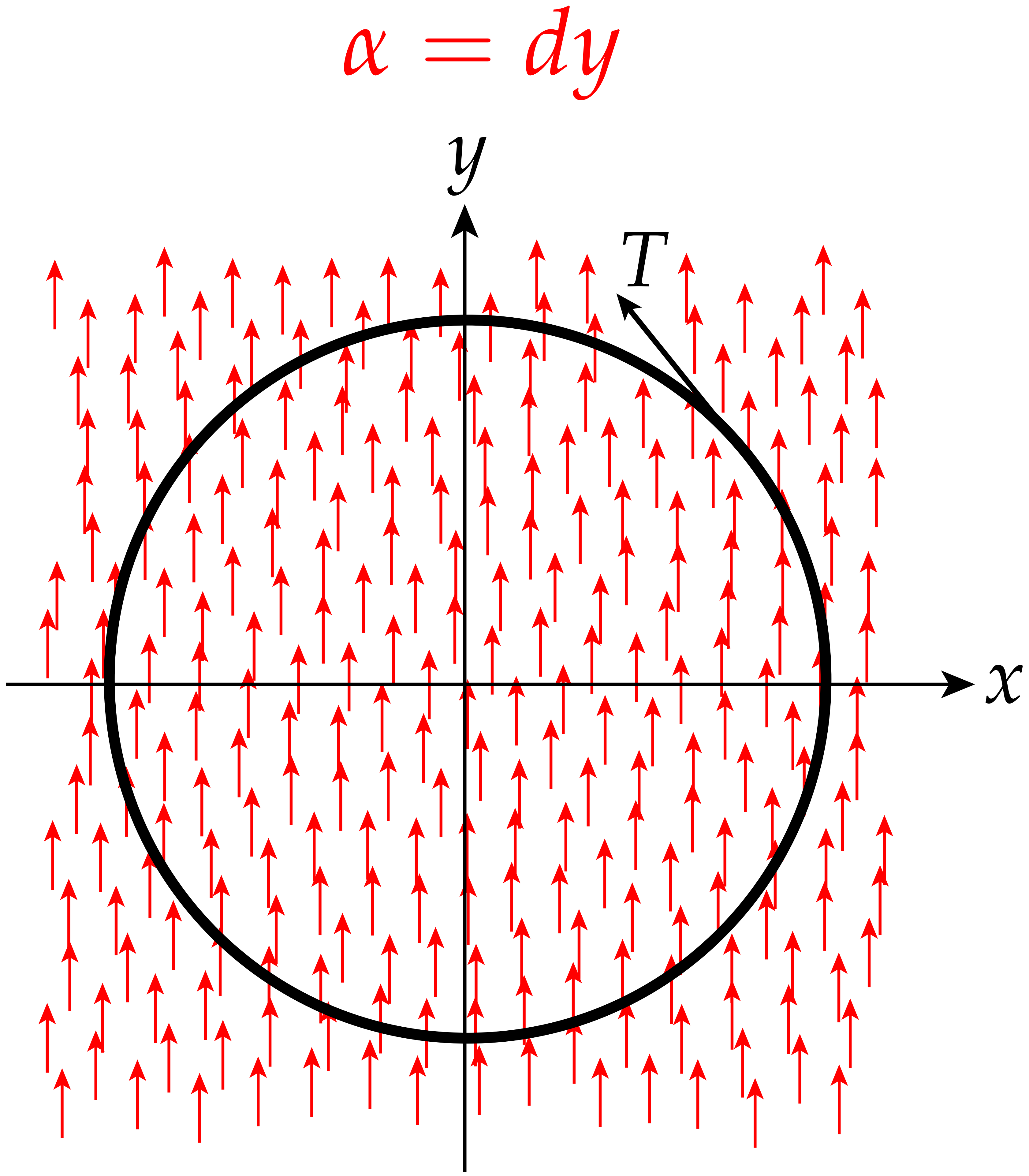
- Now consider a 1-form in the plane, which we will integrate over the unit circle:

$$\int_{S^1} \alpha = \int_0^{2\pi} \alpha_{\gamma(s)}(T(s))\, ds =$$

$$\int_0^{2\pi} \alpha_{\gamma(s)}(\cos(s)\tfrac{\partial}{\partial x} + \sin(s)\tfrac{\partial}{\partial y})\, ds =$$

$$\int_0^{2\pi} dy(\cos(s)\tfrac{\partial}{\partial x} + \sin(s)\tfrac{\partial}{\partial y})\, ds =$$

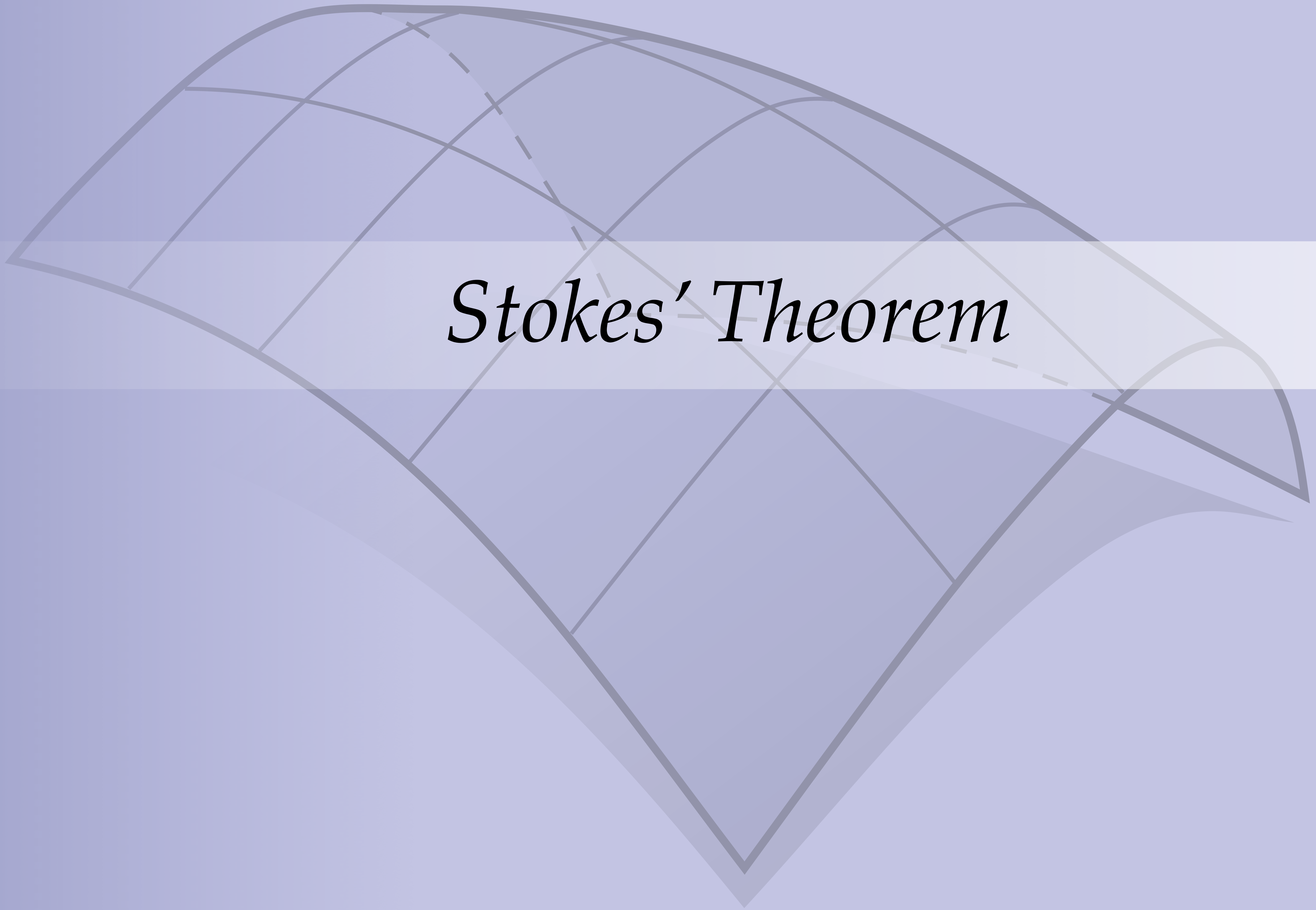
$$\int_0^{2\pi} \sin(s)\, ds = 0$$



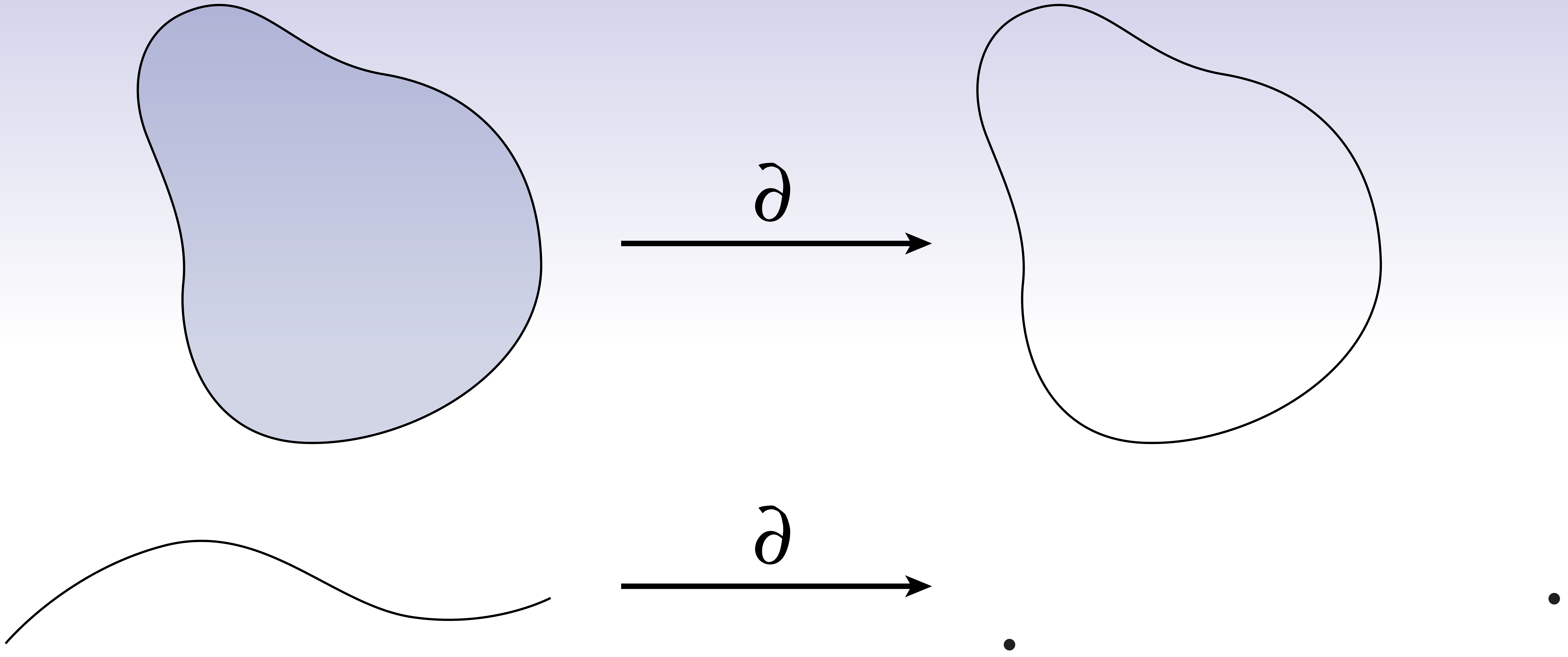
$\gamma : [0, 2\pi) \to \mathbb{R}^2; s \mapsto (\cos(s), \sin(s))$

(Why does this result make sense geometrically?)

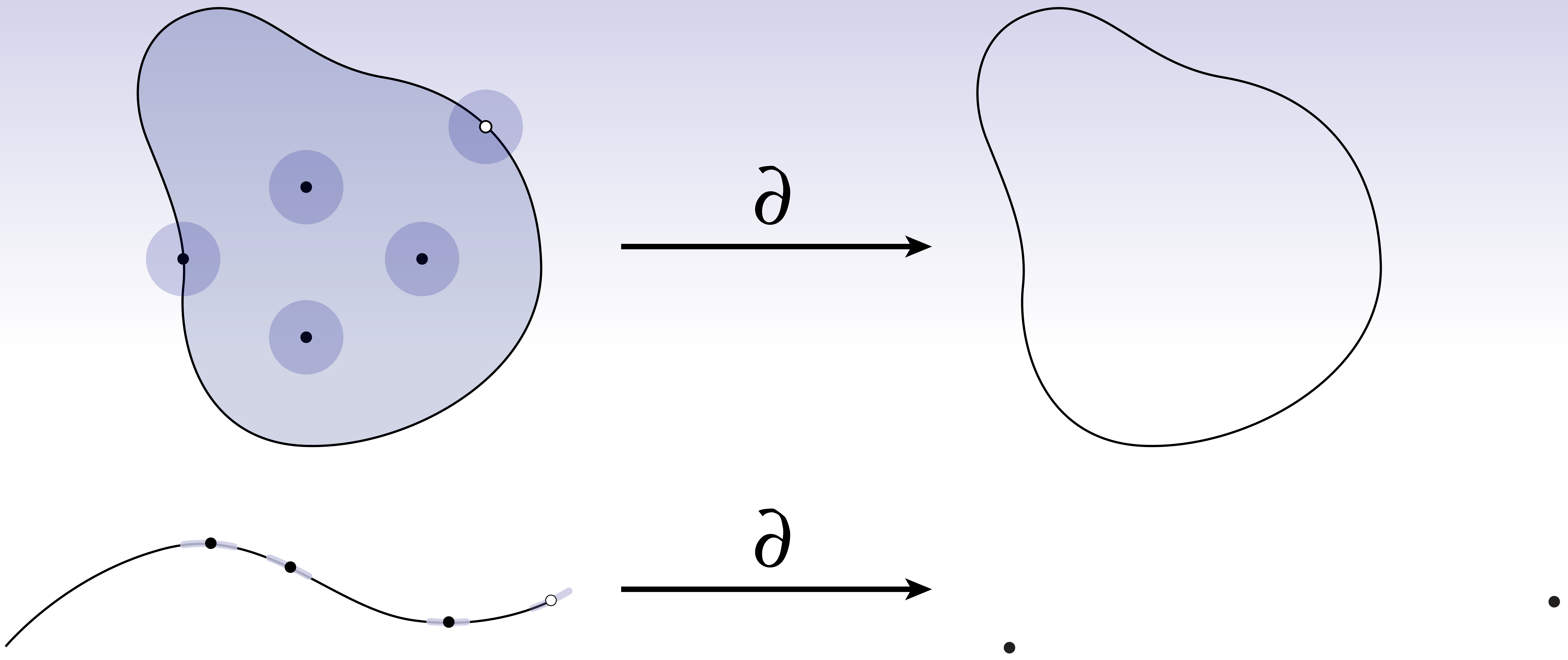
Stokes' Theorem



Boundary



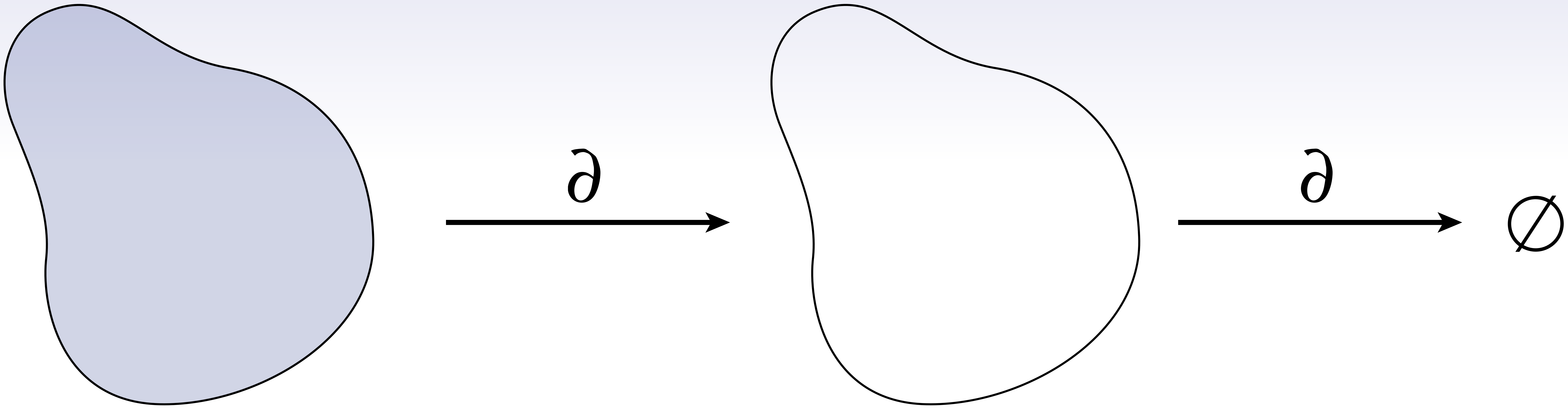
Boundary



Basic idea: for an n-dimensional set, the boundary points are those not contained in any n-ball strictly inside the domain.

Boundary of a Boundary

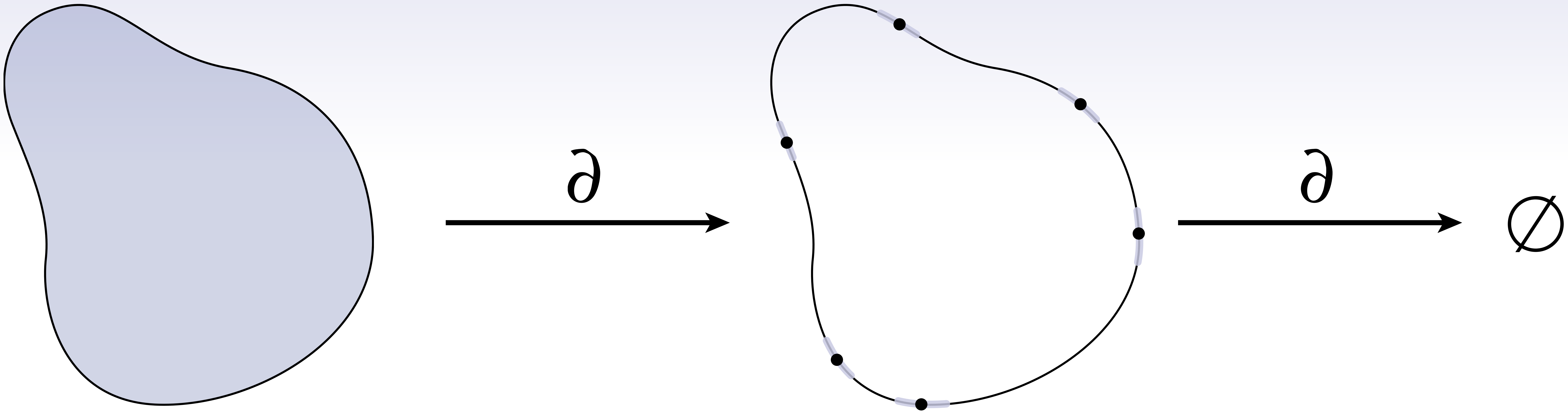
Q: Which points are in the boundary of the boundary?



A: No points! Boundary of a boundary is always *empty*.

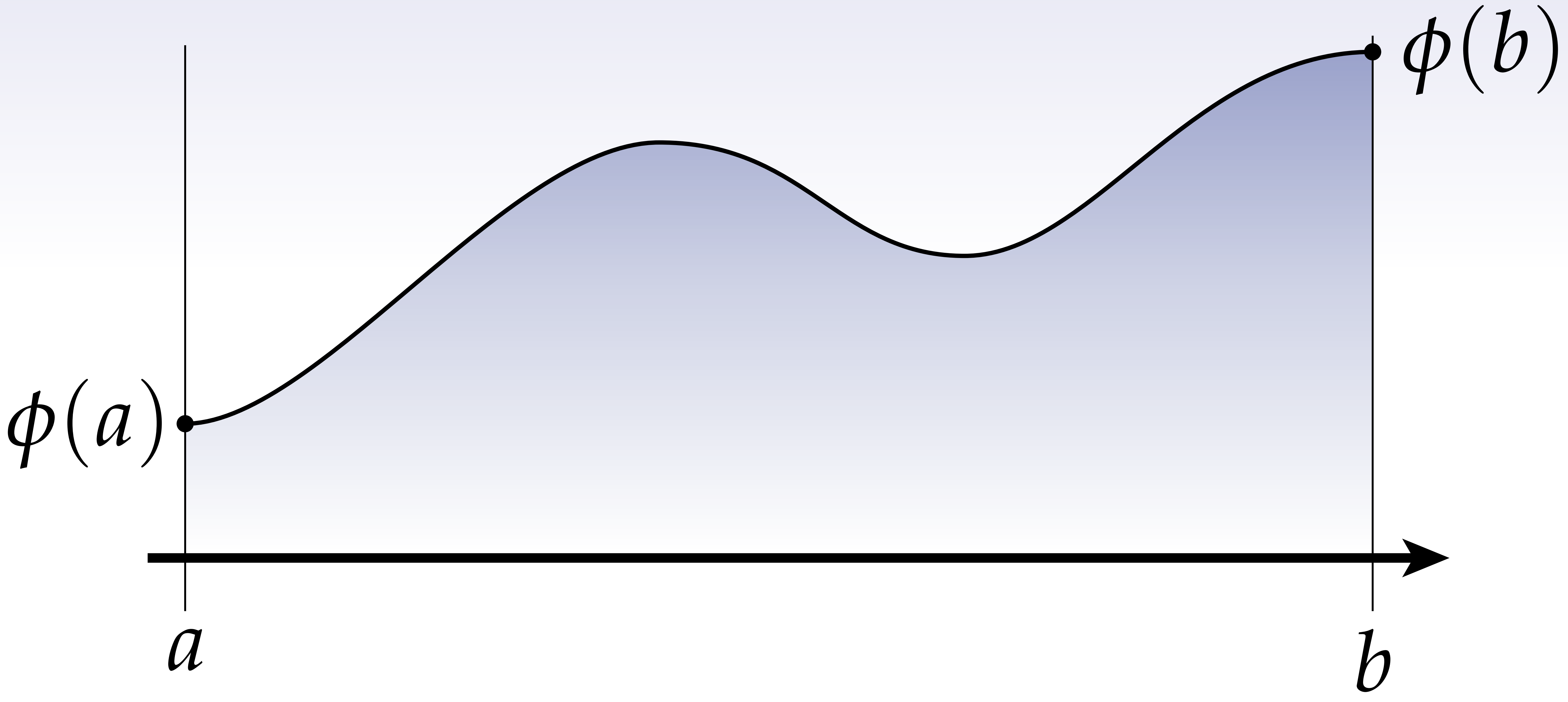
Boundary of a Boundary

Q: Which points are in the boundary of the boundary?



A: No points! Boundary of a boundary is always *empty*.

Review: Fundamental Theorem of Calculus



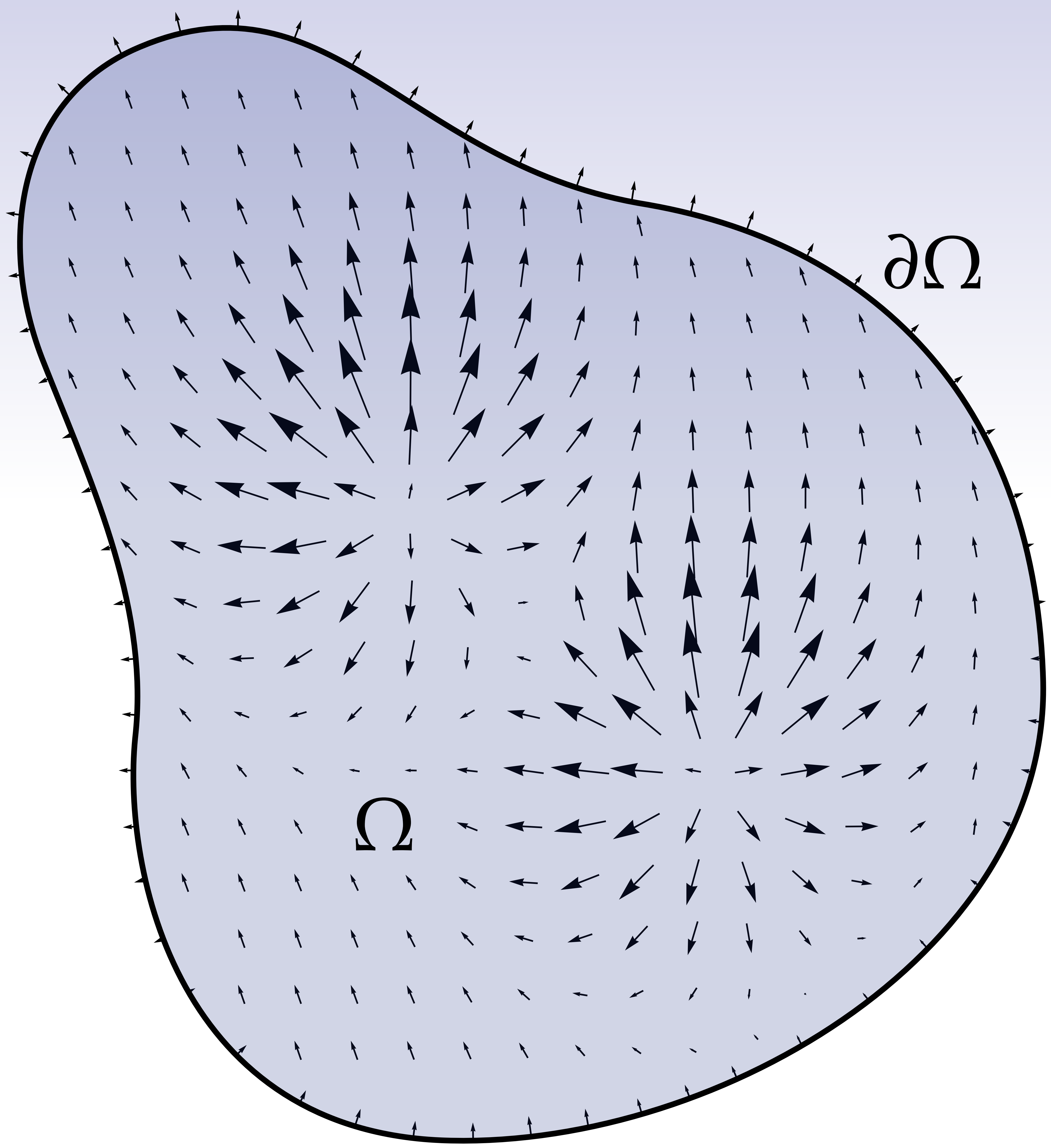
$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a)$$

Stokes' Theorem

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

Analogy: fundamental theorem of calculus

Example: Divergence Theorem



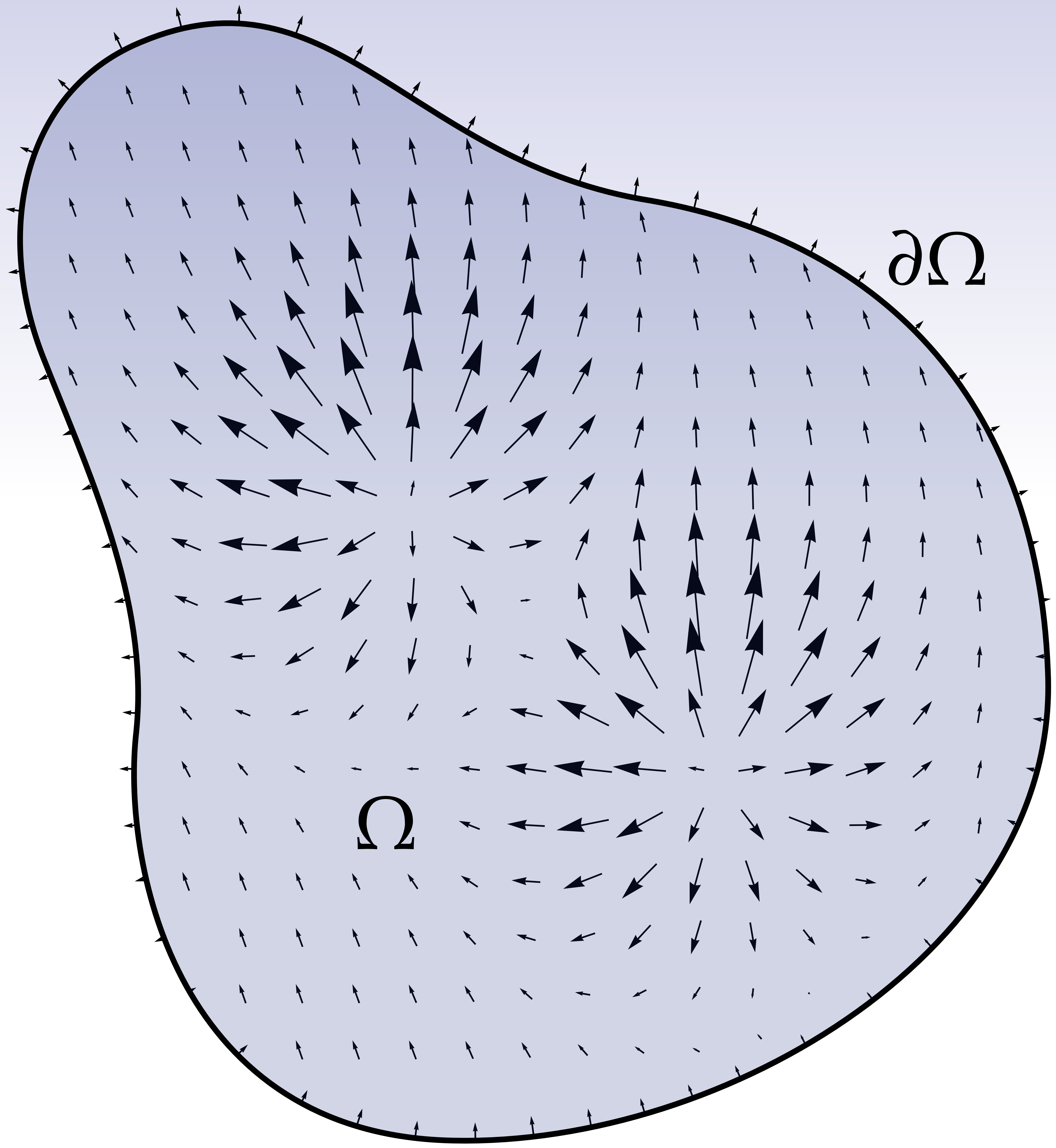
$$\int_\Omega \nabla \cdot X \, dA = \int_{\partial\Omega} n \cdot X \, d\ell$$

$$\Longleftrightarrow$$

$$\int_\Omega d \star \alpha = \int_{\partial\Omega} \star\alpha$$

What goes in, must come out!

Stokes' Theorem

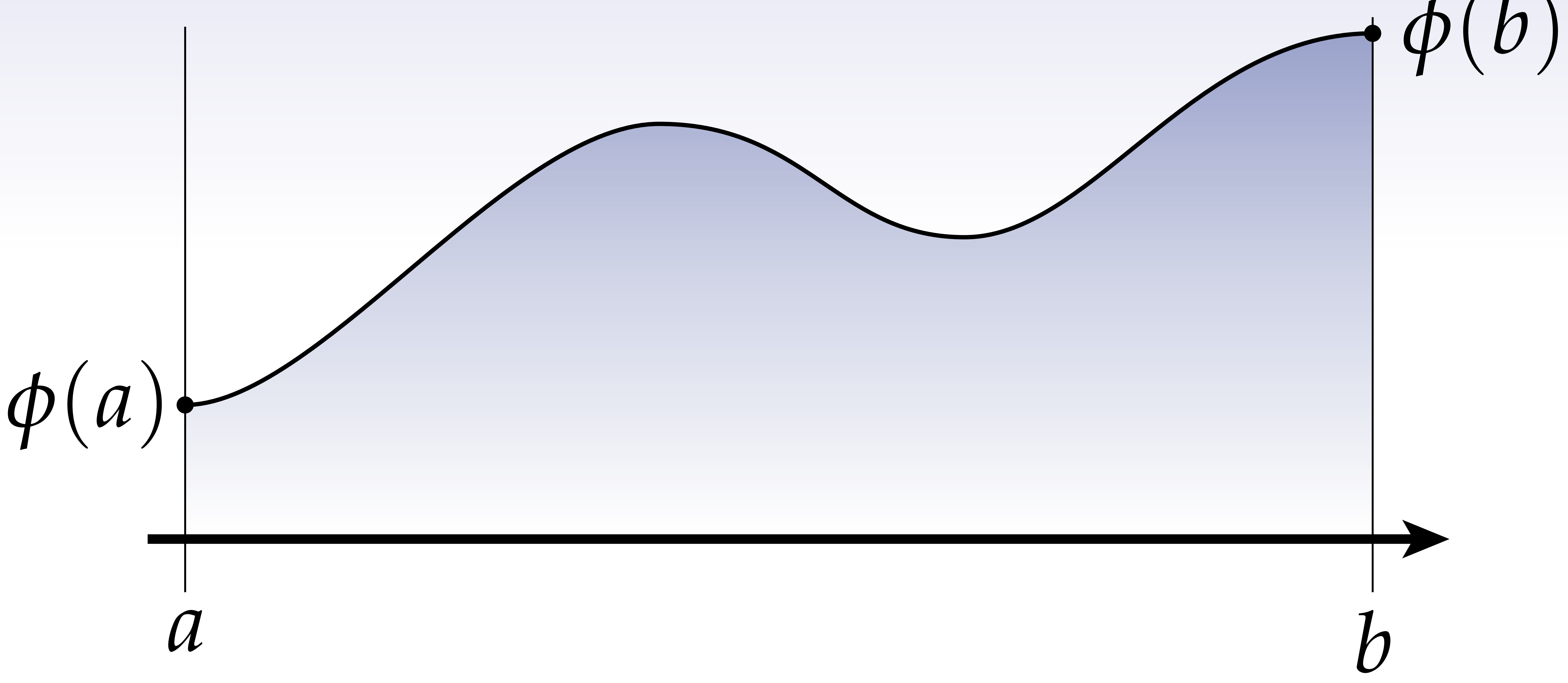


$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

"The change we see on the outside is purely a function of the change within."

—Zen koan

Fundamental Theorem of Calculus & Stokes'

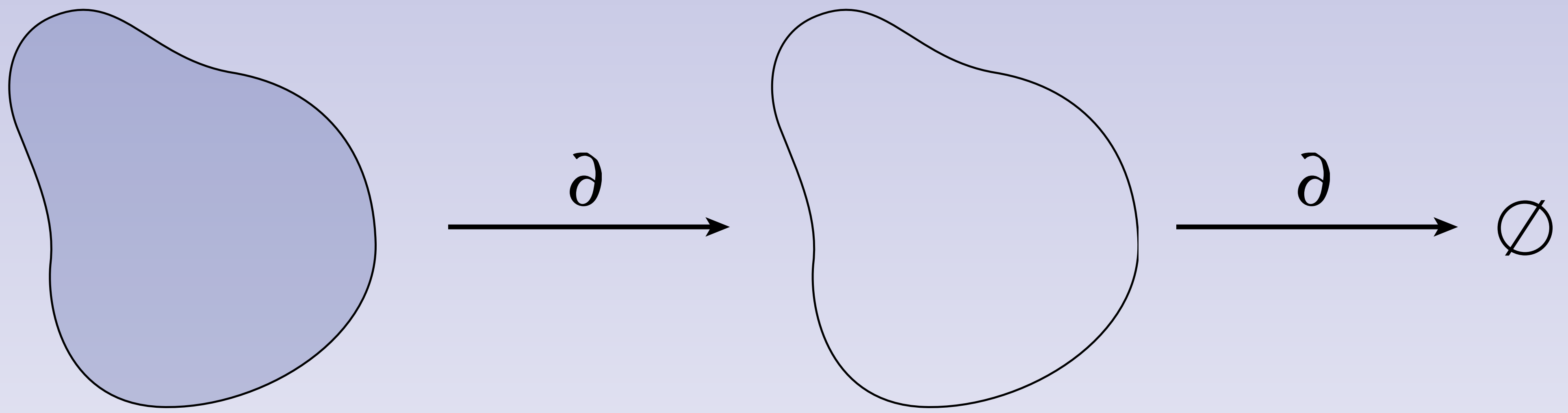


$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a)$$

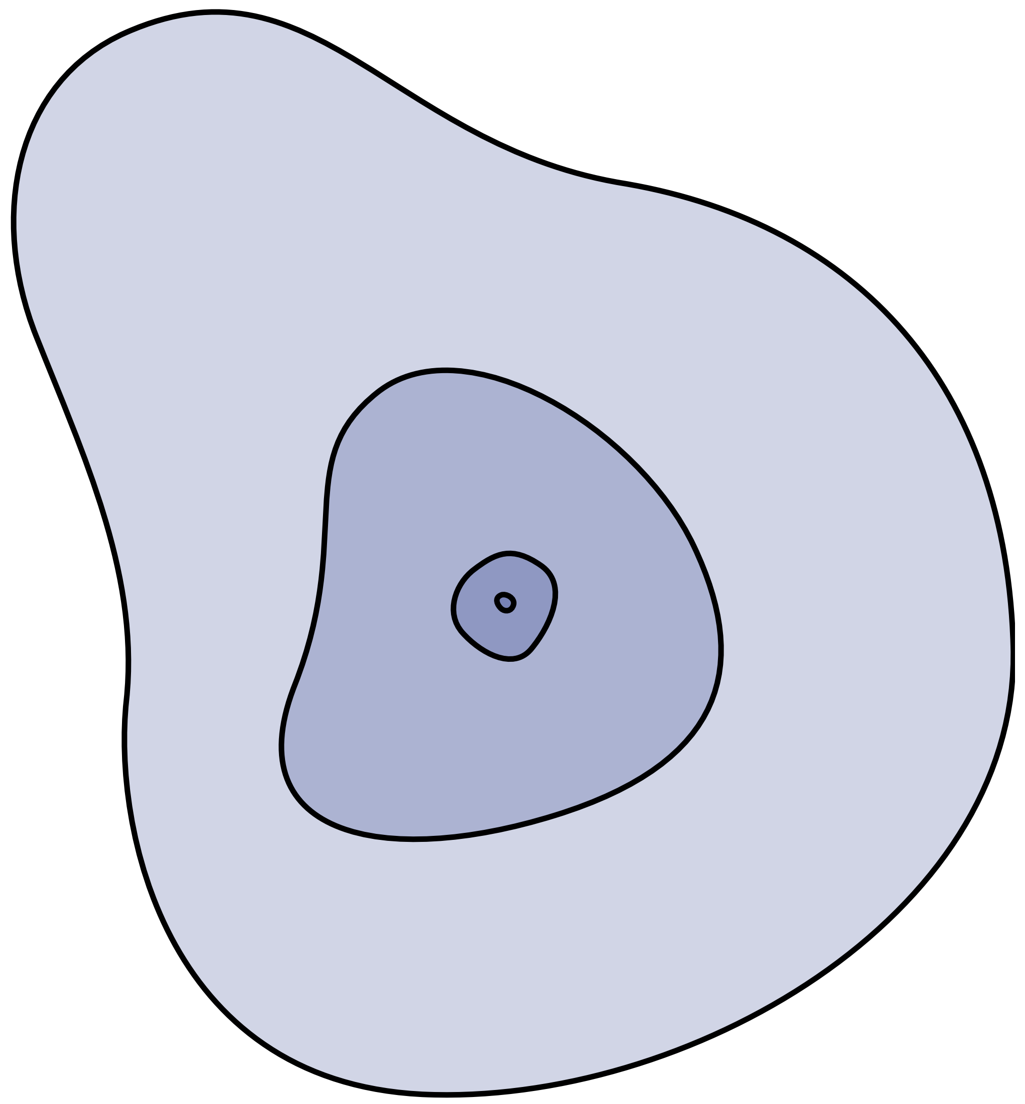
$$\Longleftrightarrow$$

$$\int_{[a,b]} d\phi = \int_{\partial [a,b]} \phi$$

Why is $d \circ d = 0$*?*



$$\int_{\Omega} dd\phi = \int_{\partial\Omega} d\phi = \int_{\underbrace{\partial\partial\Omega}_{\emptyset}} \phi = 0$$



...for *any* Ω (no matter how small!)

Why is $d \circ d = 0$?

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

"behaves like gradient for functions"

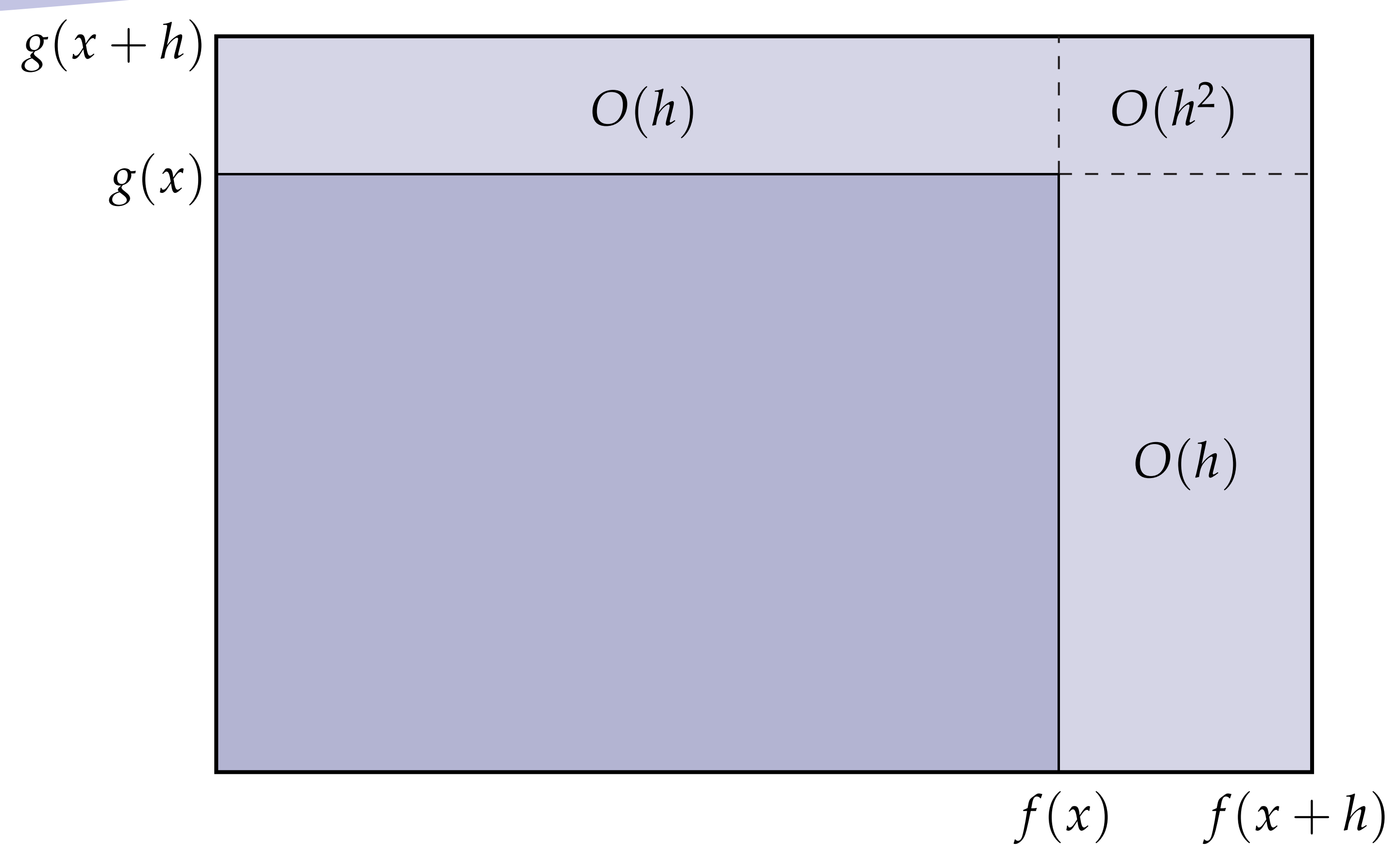
differential $d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \cdots + \frac{\partial \phi}{\partial x^n} dx^n$

product rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta(-1)^k \alpha \wedge d\beta$

~~**exactness** $d \circ d = 0$~~

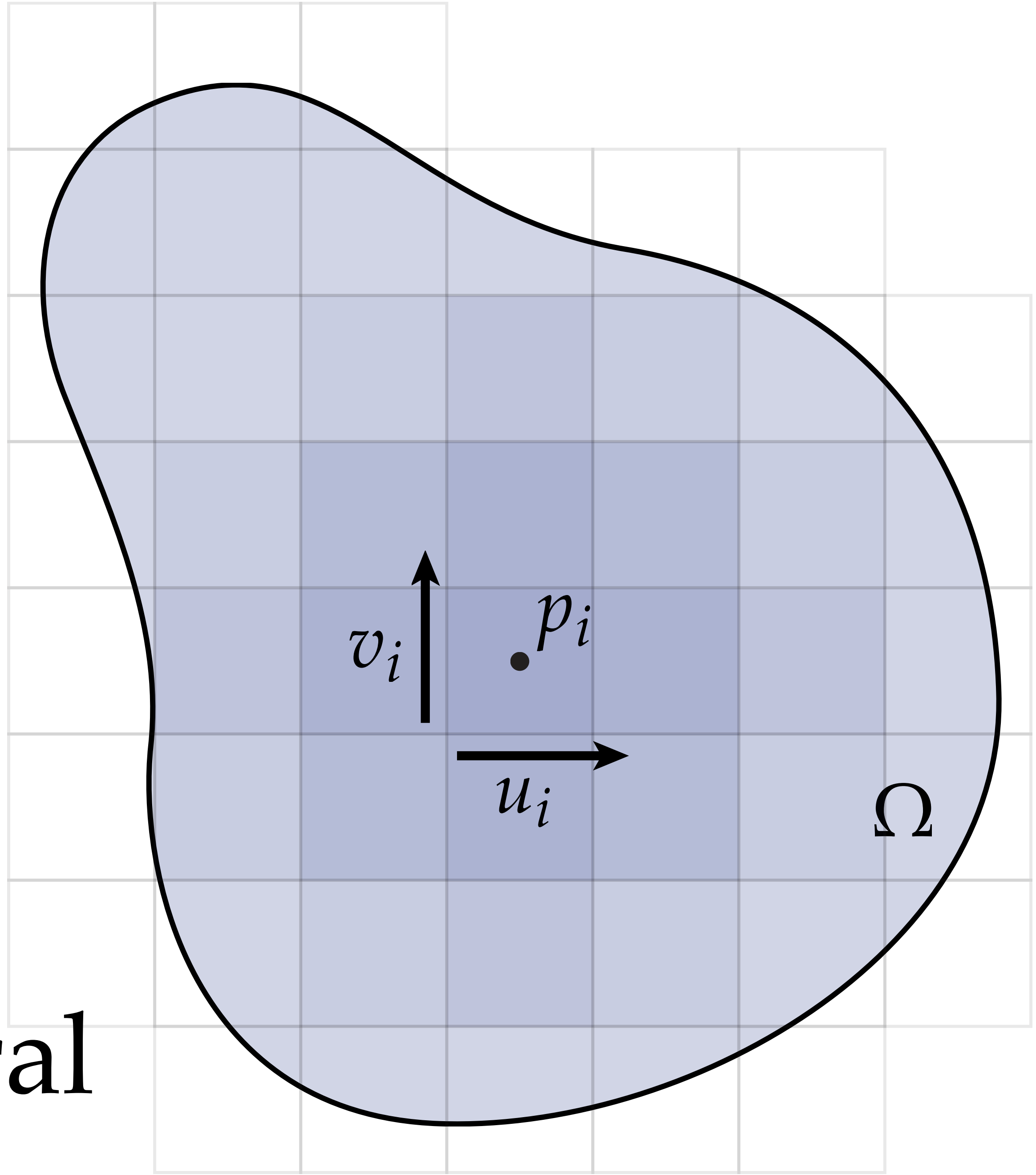
Stokes' theorem $\int_\Omega d\alpha = \int_{\partial\Omega} \alpha$

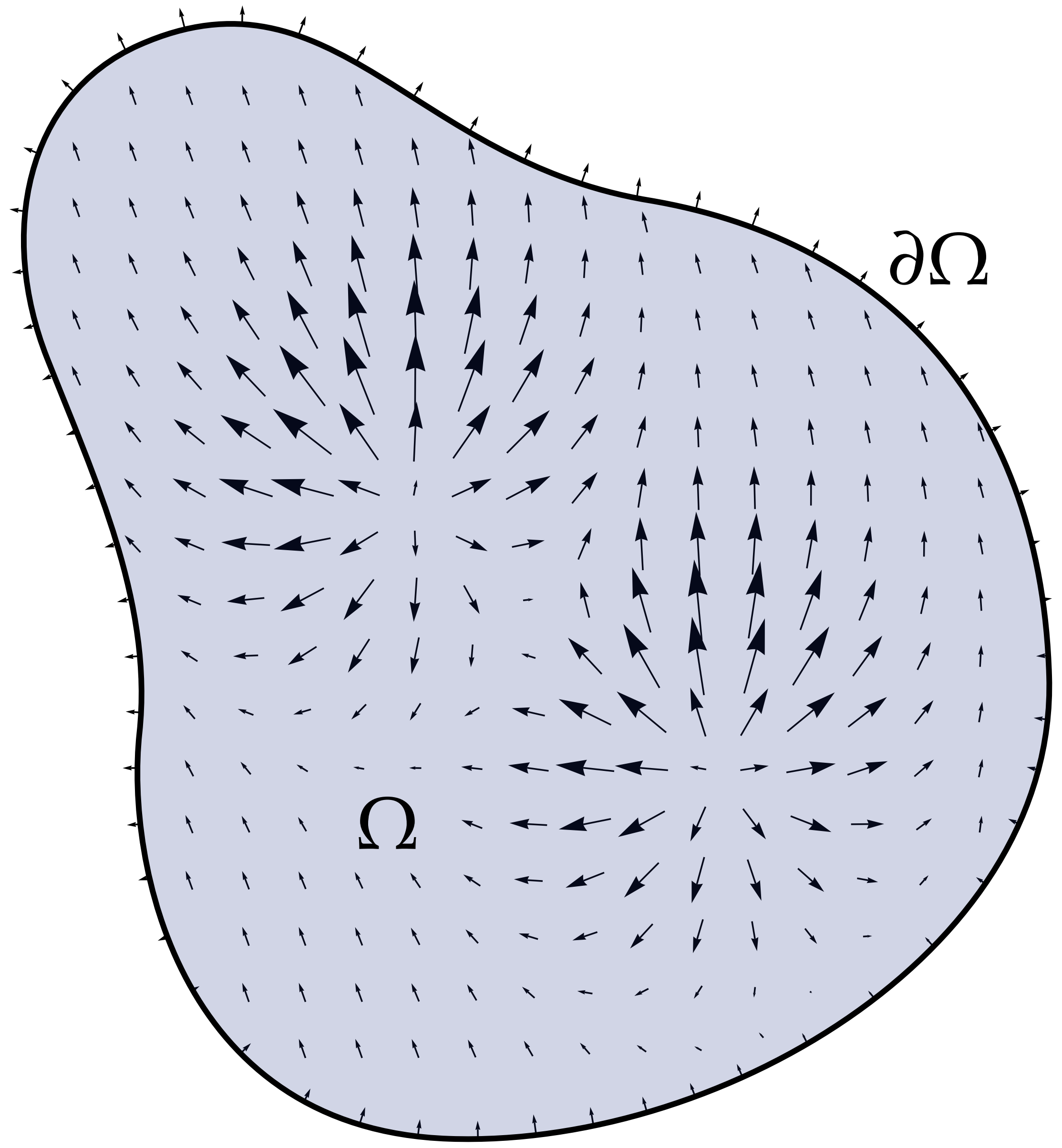
what goes in, must come out!



Integration & Stokes' Theorem - Summary

- Integration
  - break domain into small pieces
  - measure each piece with *k*-form
- Stokes' theorem
  - convert region integral to boundary integral
  - super useful—lets us "skip" a derivative
  - special cases: divergence theorem, F.T.C., *many more!*
  - will use *over and over* again in DEC/geometry processing





$$\int_M d\alpha = \int_{\partial M} \alpha$$

Inner Product on Differential k-Forms

Inner Product—Review

- Recall that a *vector space V* is any collection of "arrows" that can be added, scaled, …
- **Q:** What's an *inner product* on a vector space?
- **A:** Loosely speaking, a way to talk about lengths, angles, etc., in a vector space
- More formally, a symmetric positive-definite bilinear map:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

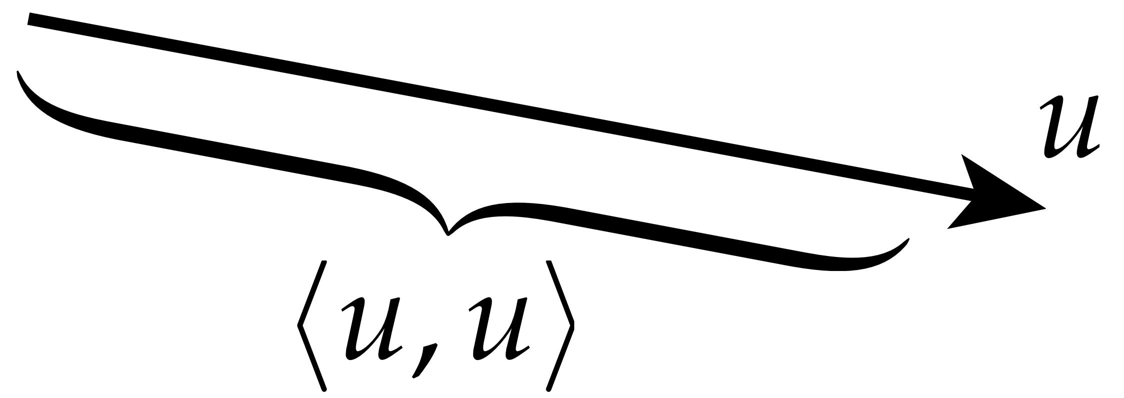
$$\langle u, v \rangle = \langle v, u \rangle$$

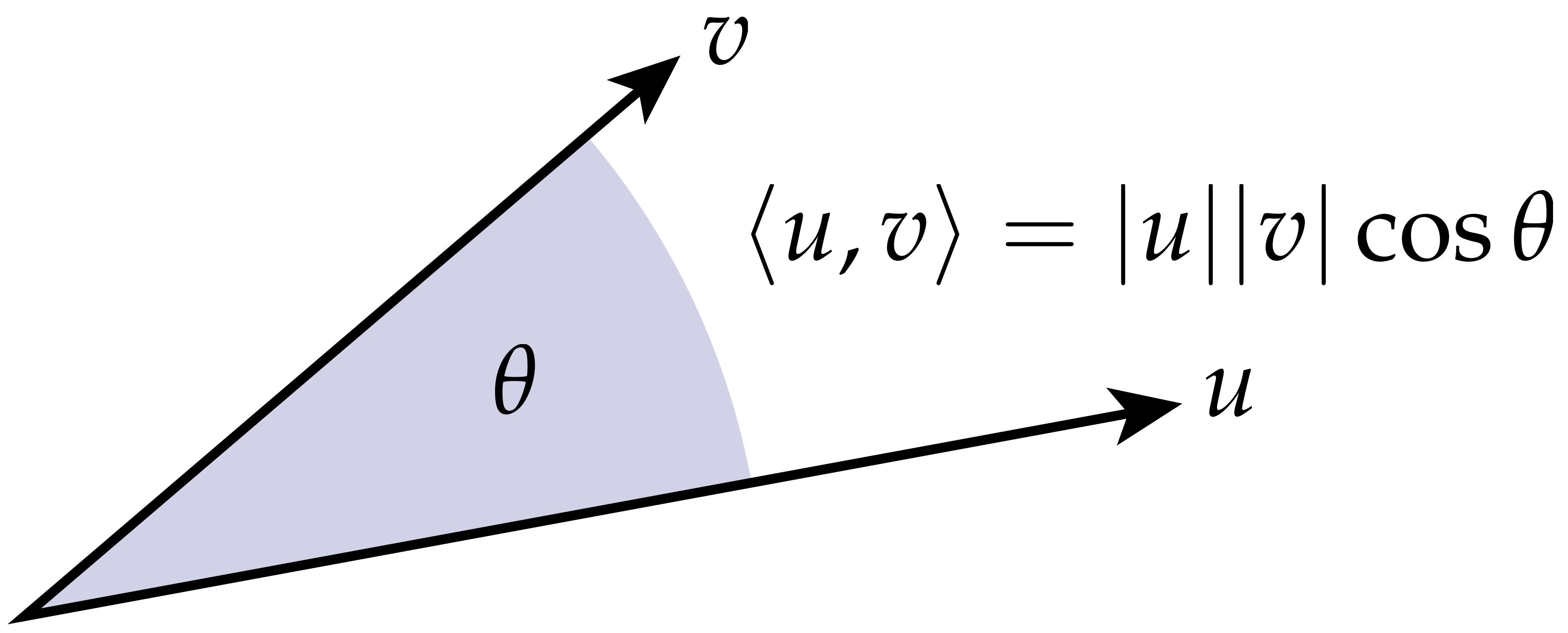
$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle au, v \rangle = a\langle u, v \rangle$$

$$\langle u, u \rangle \geq 0; \; \langle u, u \rangle = 0 \iff u = 0$$

for all vectors *u*,*v*,*w* in *V* and scalars *a*.





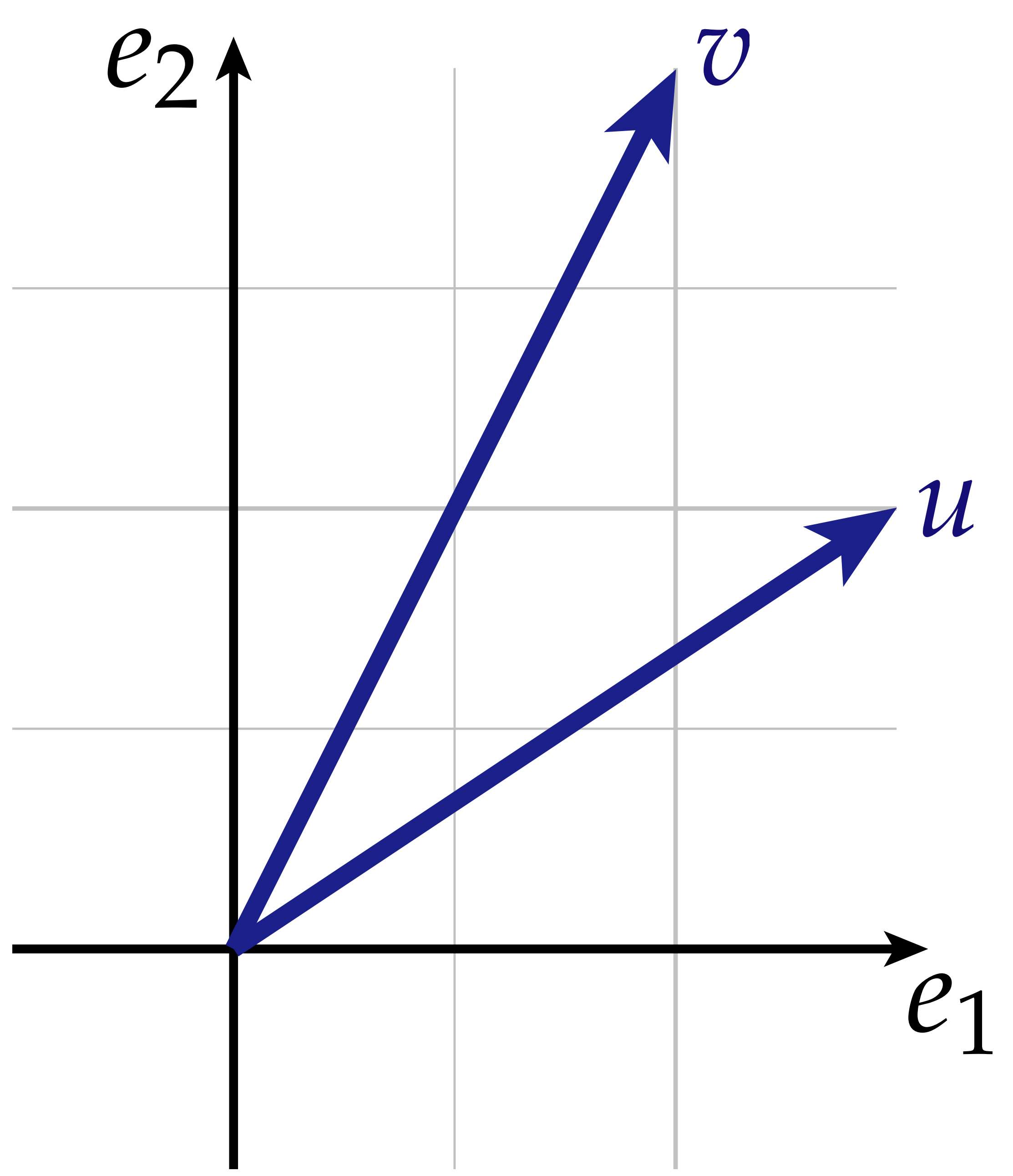
(Geometric interpretation of these rules?)

Euclidean Inner Product—Review

- Most basic inner product: inner product of two vectors in Euclidean R^n
- Just sum up the product of components:

$$u = u^1 e_1 + \cdots + u^n e_n$$
$$v = v^1 e_1 + \cdots + v^n e_n$$

$$\langle u, v \rangle := \sum_{i=1}^{n} u^i v^i$$



<u>Example.</u>

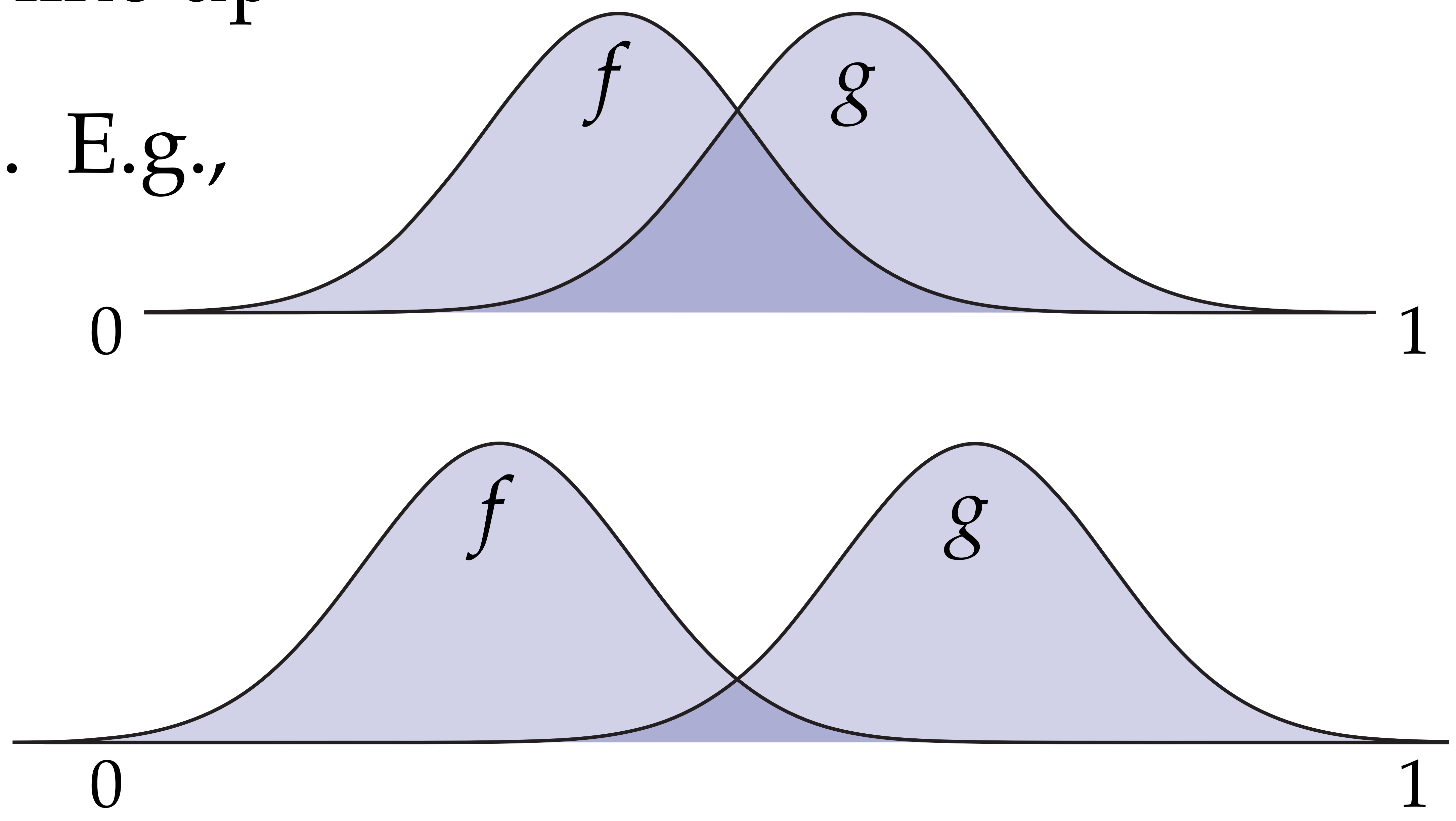
$$u = 3e_1 + 2e_2$$
$$v = 2e_1 + 4e_2$$
$$\langle u, v \rangle = 3 \cdot 2 + 2 \cdot 4 = 14$$

(Does this operation satisfy all the requirements of an inner product?)

L^2 Inner Product of Functions / 0-forms

- Remember that in many situations, *functions* are also vectors
- What does it mean to measure the inner product between functions?
- Want some notion of how well two functions "line up"
- One idea: mimic what we did for vectors in R^n. E.g.,

$$f : [0,1] \to \mathbb{R}$$
$$g : [0,1] \to \mathbb{R}$$
$$\langle\langle f, g \rangle\rangle := \int_0^1 f(x)g(x)dx$$



- Called the L^2 inner product. (**Note:** *f* and *g* must each be square-integrable!)
- Does this capture notion of "lining up"? Does it obey rules of inner product?

Inner Product on k-Forms

Definition. Let $\alpha, \beta \in \Omega^k$ be any two differential k-forms. Their (L^2) inner product is defined as*

$$\langle\langle \alpha, \beta \rangle\rangle := \int_\Omega \star\alpha \wedge \beta$$

Q: What happens when k=0?

A: We just get the usual L^2 inner product on functions.

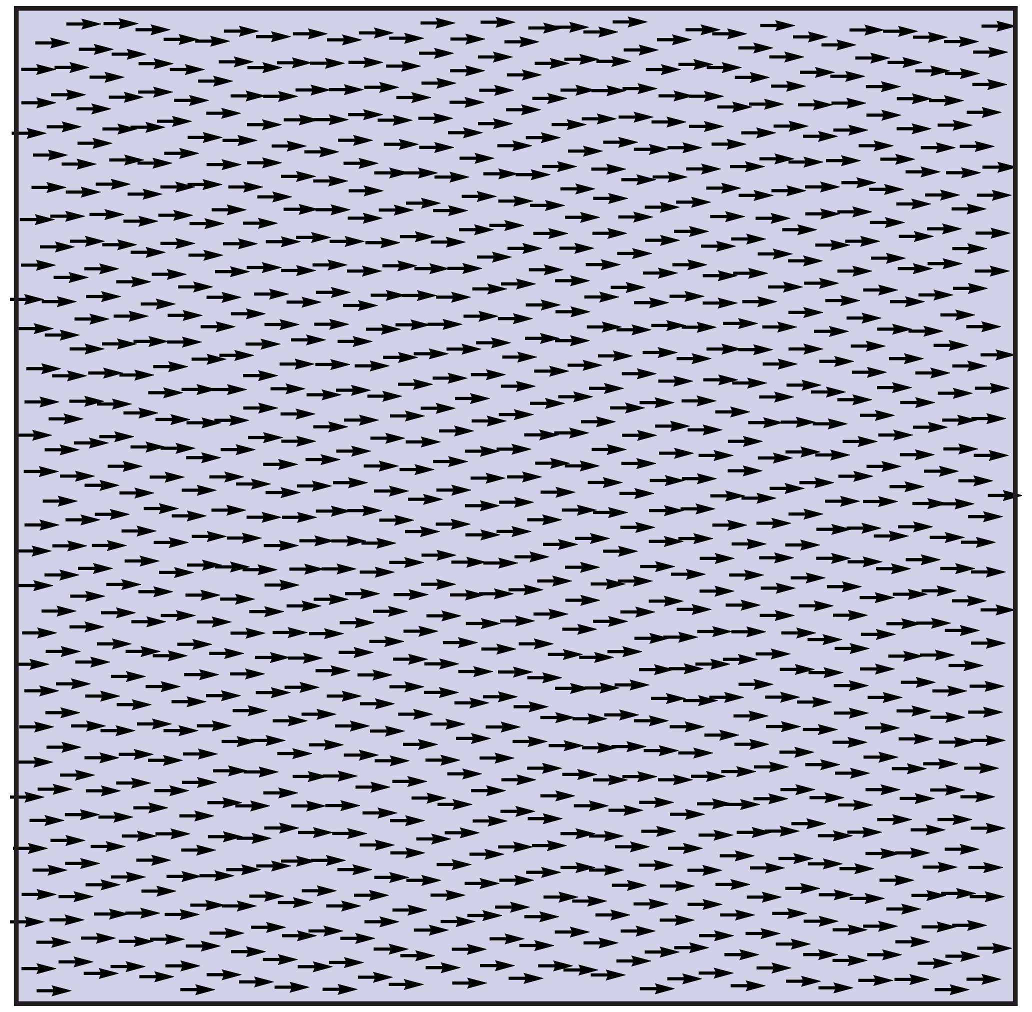
Q: What's the degree (k) of the integrand? Why is that important?

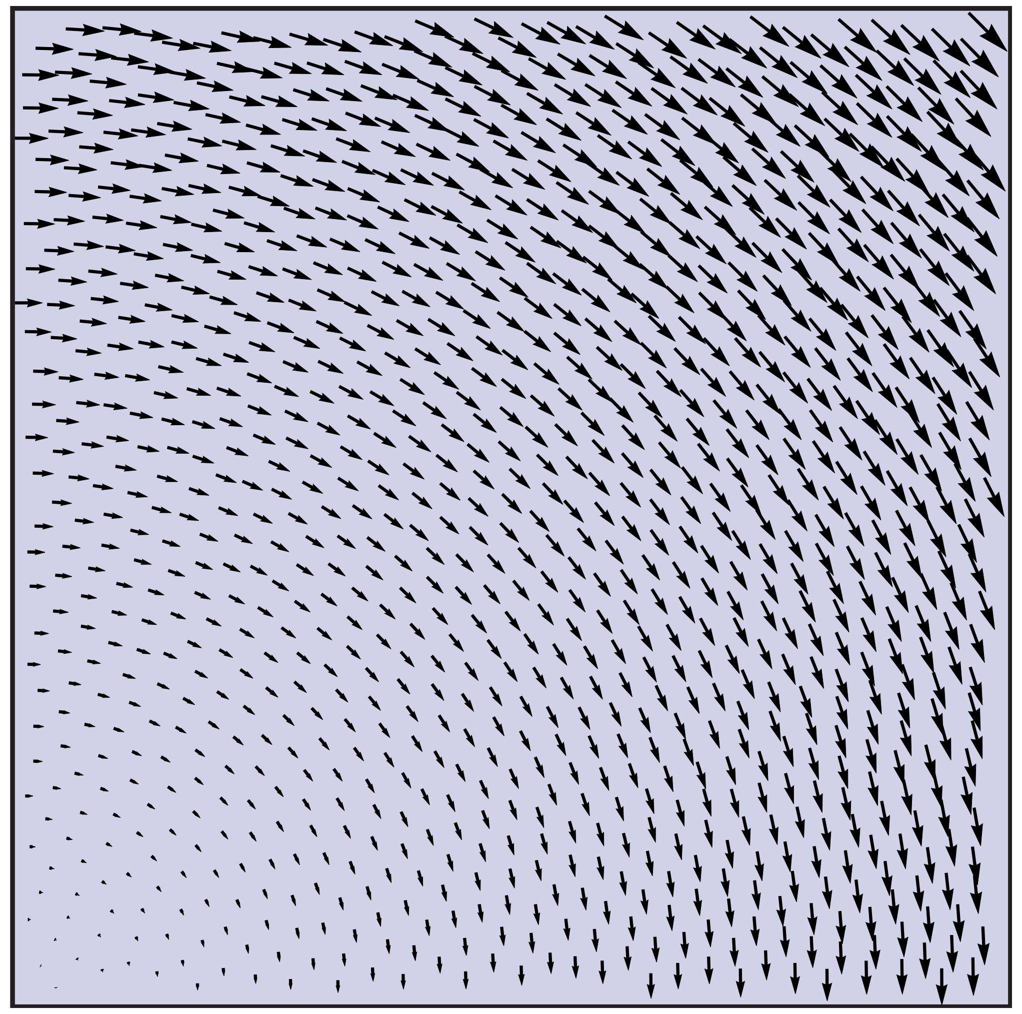
A: Integrand is always an n-form, which is the only thing we can integrate in n-D!

*Some authors define the integrand as $\alpha \wedge \star\beta$; our convention is consistent with the convention that in 2D the 1-form Hodge star is a *counter*-clockwise rotation.

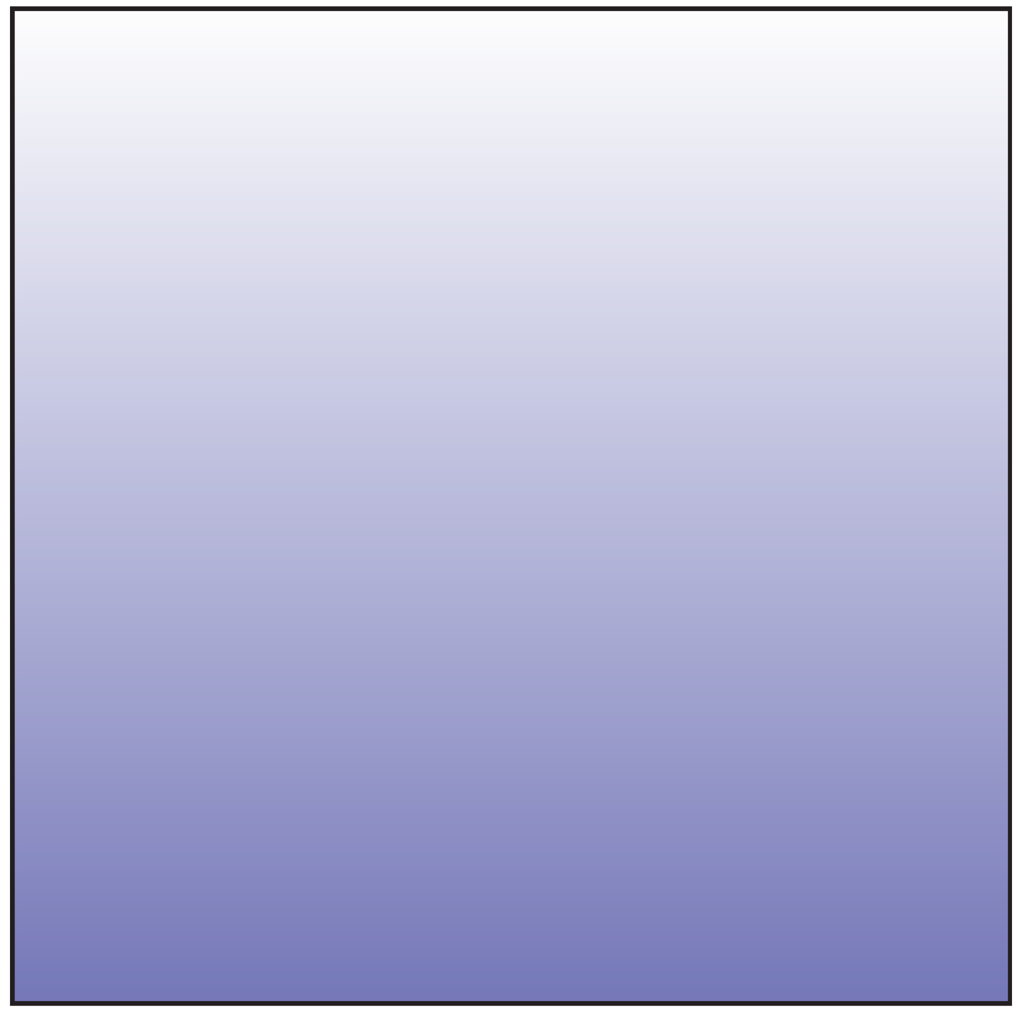
Inner Product of 1-Forms — Example

α





β



$\star\alpha \wedge \beta$

Example. Consider two 1-forms on the unit square $[0,1] \times [0,1]$ given by

$$\begin{aligned} \alpha &:= du, \\ \beta &:= v\,du - u\,dv. \end{aligned}$$

Their inner product is

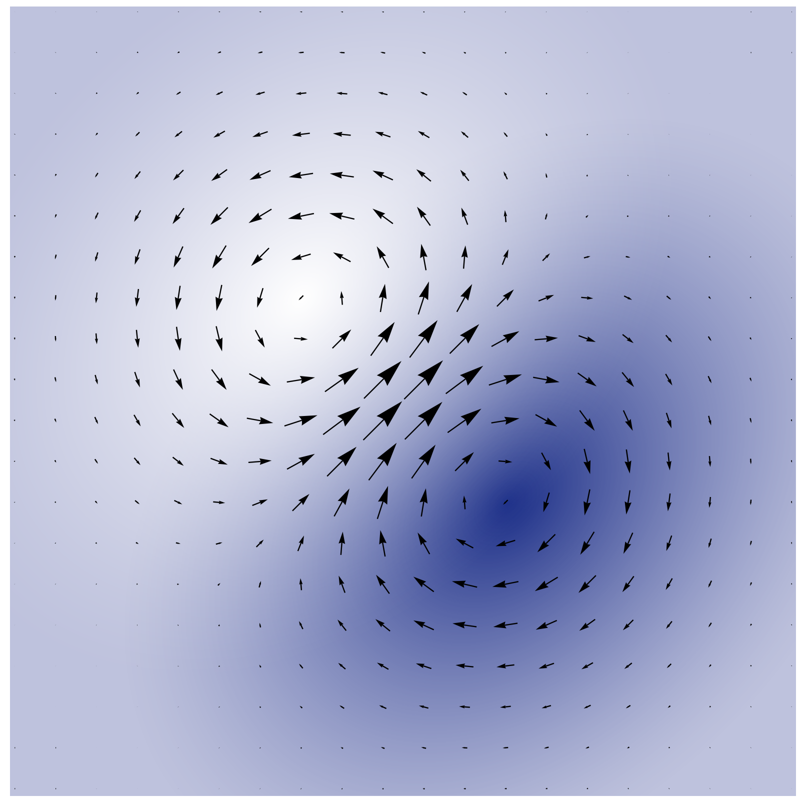
$$\langle\langle \alpha, \beta \rangle\rangle = \int_0^1 \int_0^1 (\star\alpha) \wedge \beta =$$

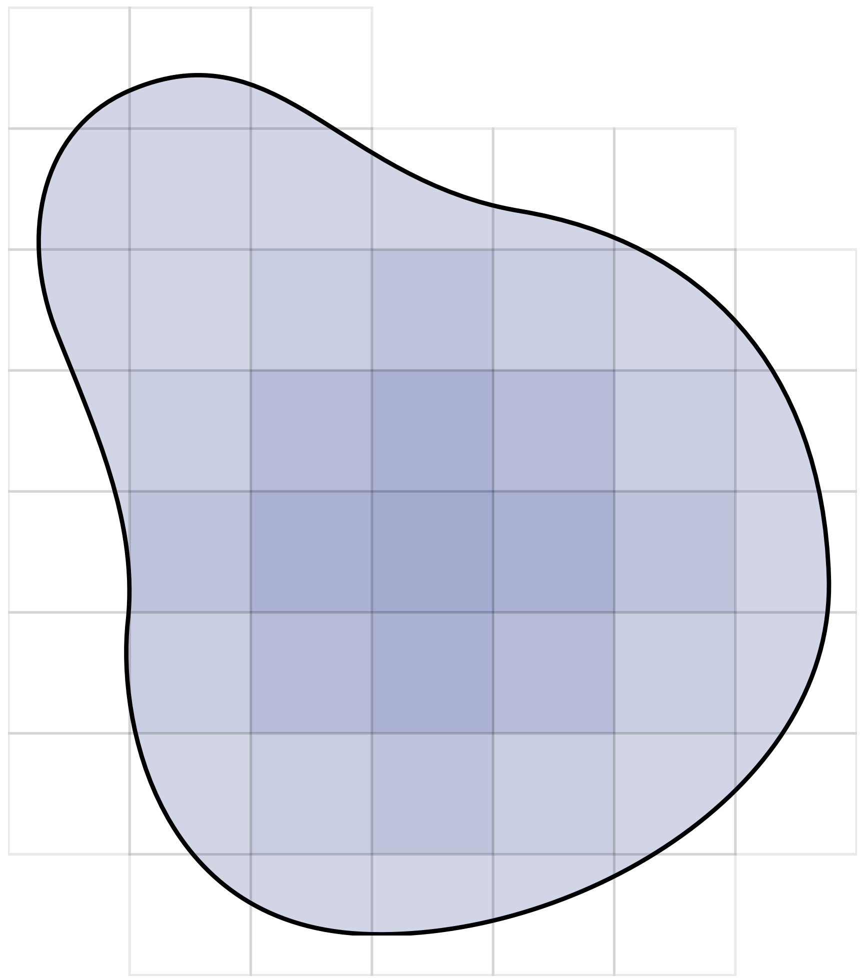
$$\int_0^1 \int_0^1 dv \wedge (v\,du - u\,dv) =$$

$$-\int_0^1 \int_0^1 v\,du \wedge dv = \frac{1}{2}.$$

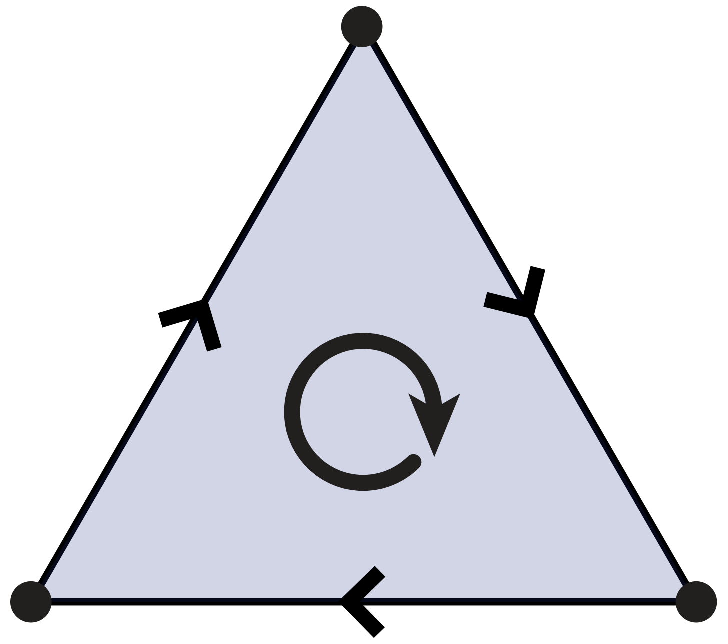
Summary

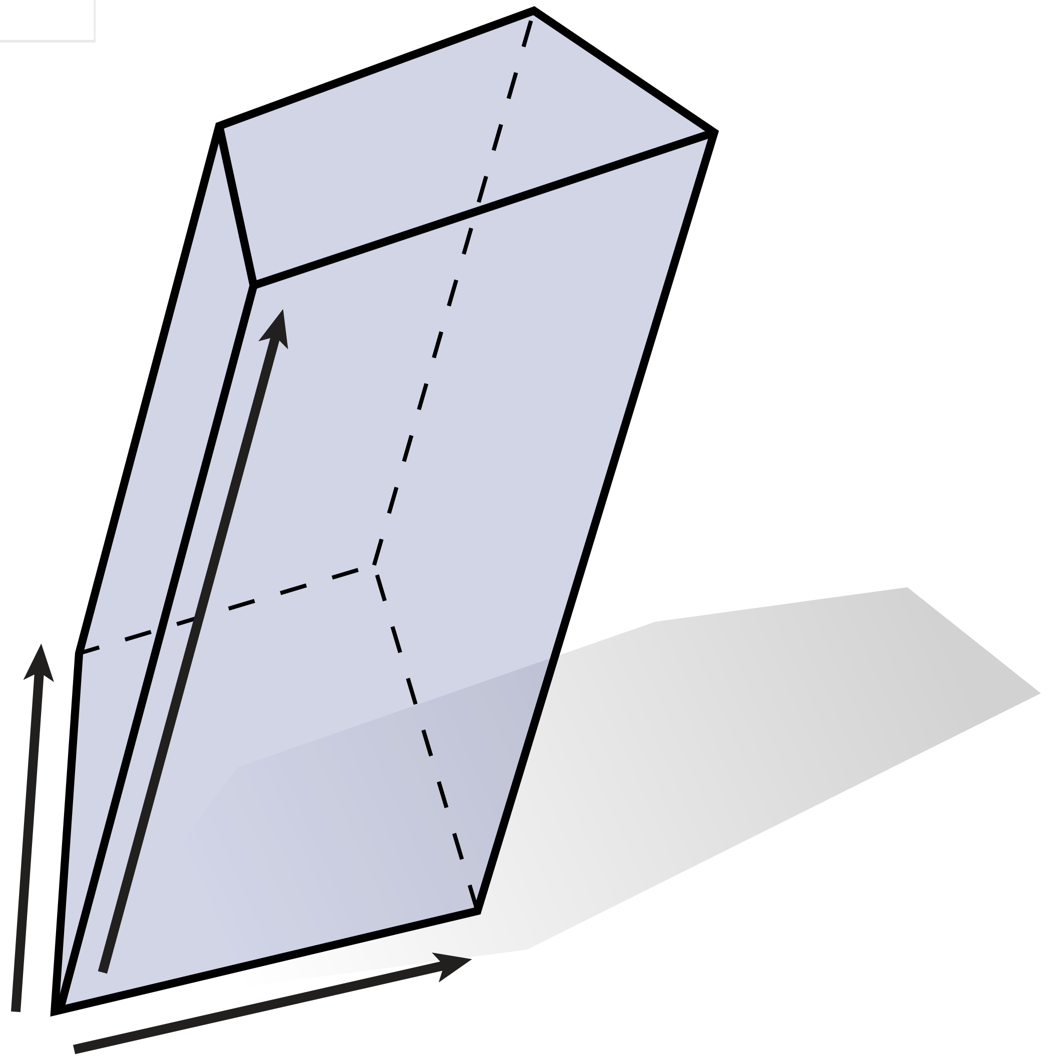
Exterior Calculus—Summary



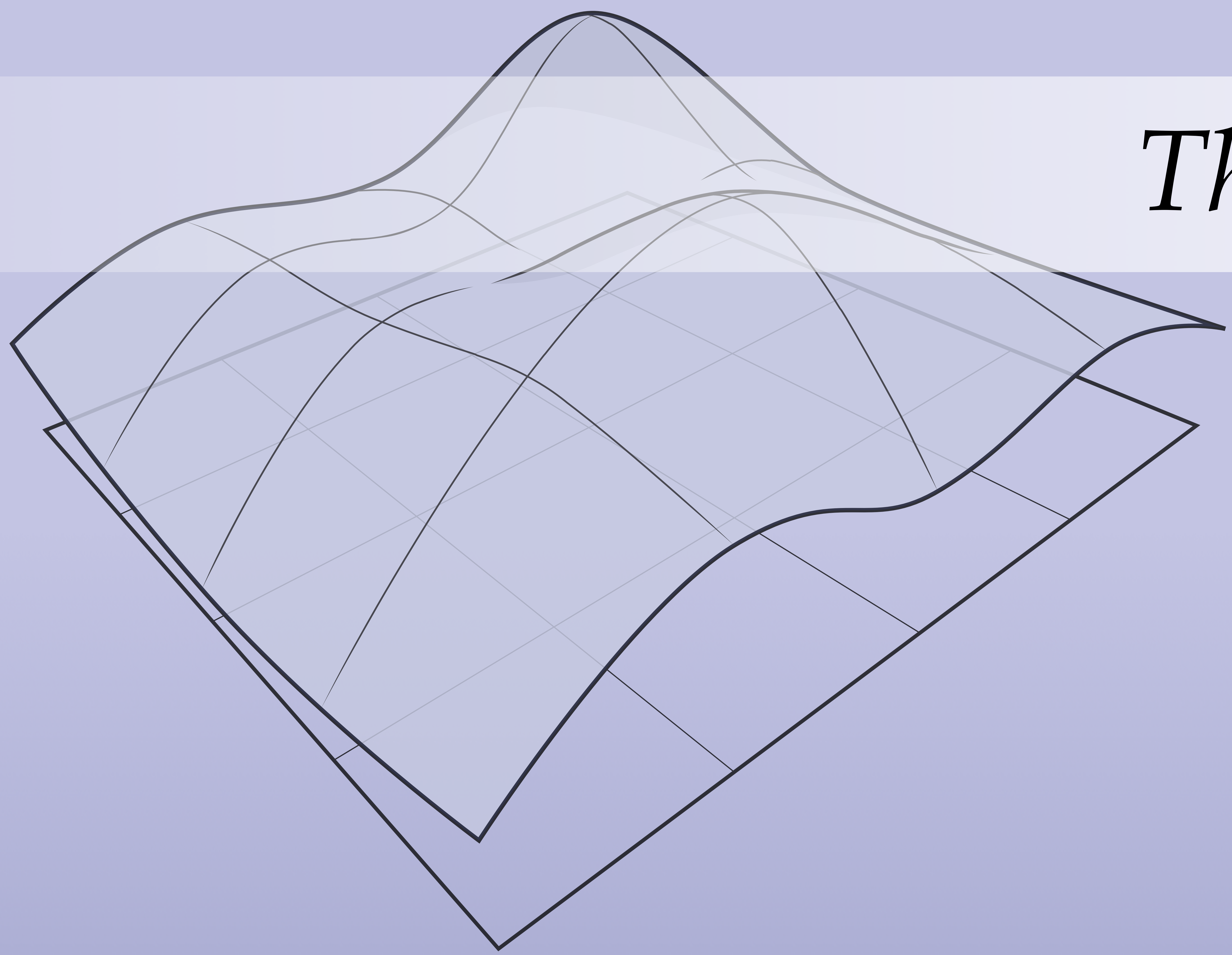


- **What we've seen so far:**
- *Exterior algebra*: language of volumes (*k*-vectors)
- *k-form:* measures a k-dimensional volume
- *Differential forms: k*-form at each point of space
- *Exterior calculus*: differentiate/integrate forms
- *Simplicial complex*: mesh made of vertices, edges, triangles…
- **Next up:**
  - Put all this machinery together
  - *Integrate* to get discrete exterior calculus (DEC)





Thanks!



Discrete Differential Geometry

an Applied Introduction