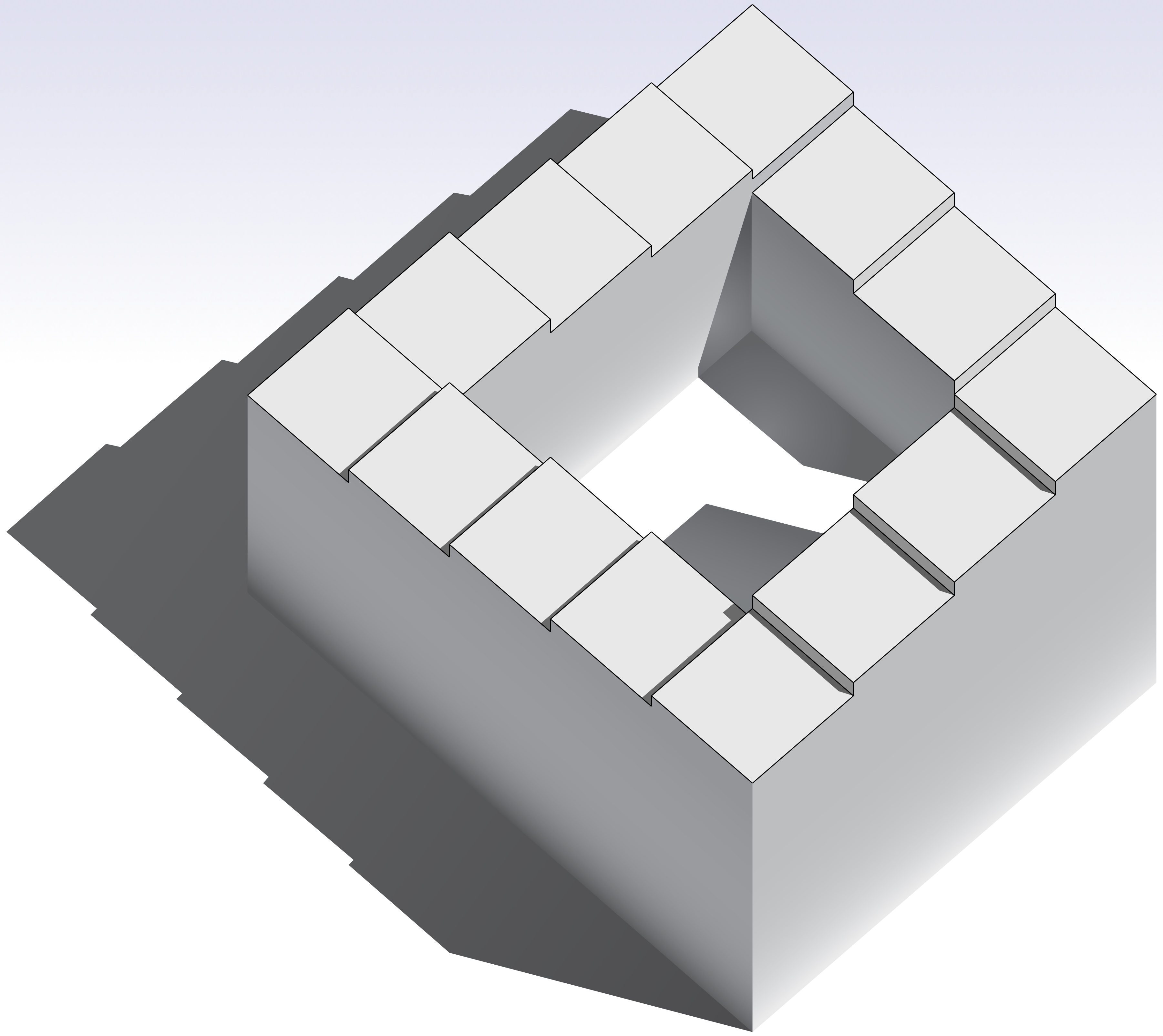
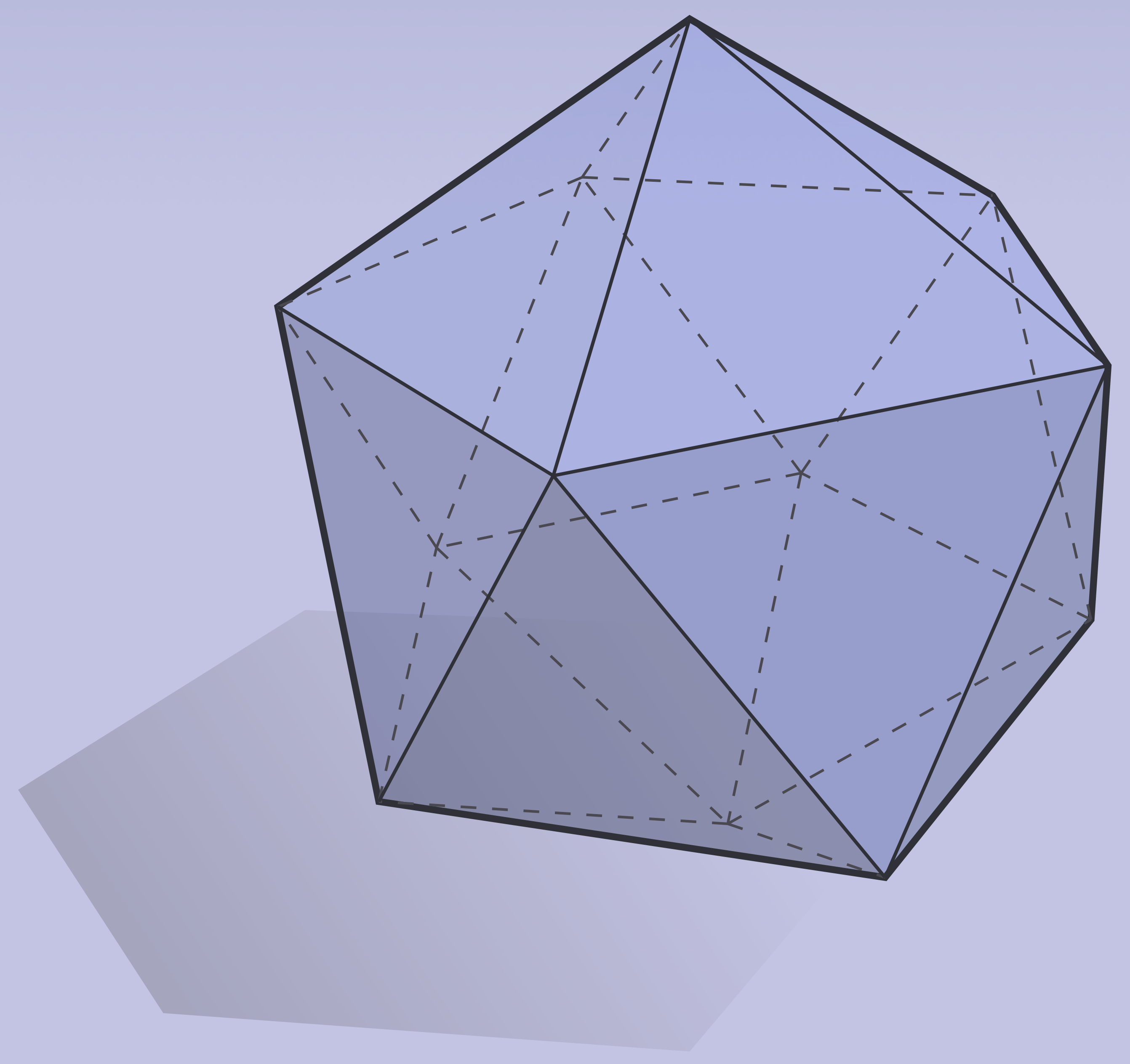


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

LECTURE 8:
SURFACES



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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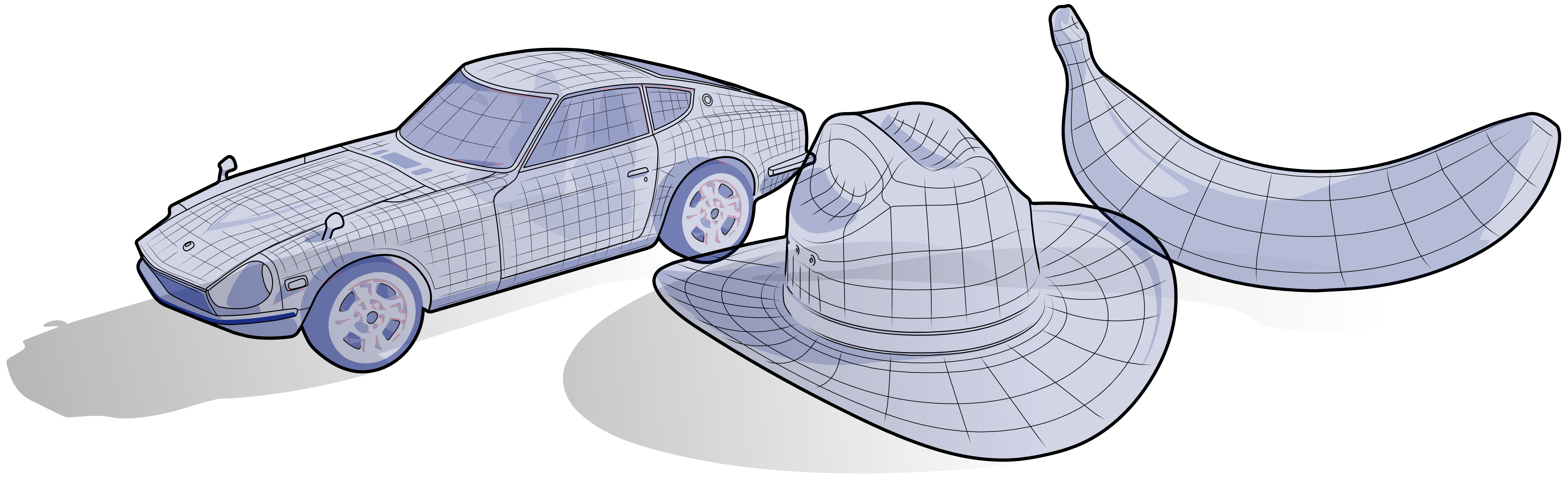
From Curves to Surfaces

- **Previously:** saw how to talk about 1D curves (both smooth and discrete)
- **Today:** will study 2D curved surfaces (both smooth and discrete)
  - Some concepts remain the same (*e.g.*, differential); others need to be generalized (*e.g.*, curvature)
  - Still use exterior calculus as our *lingua franca*

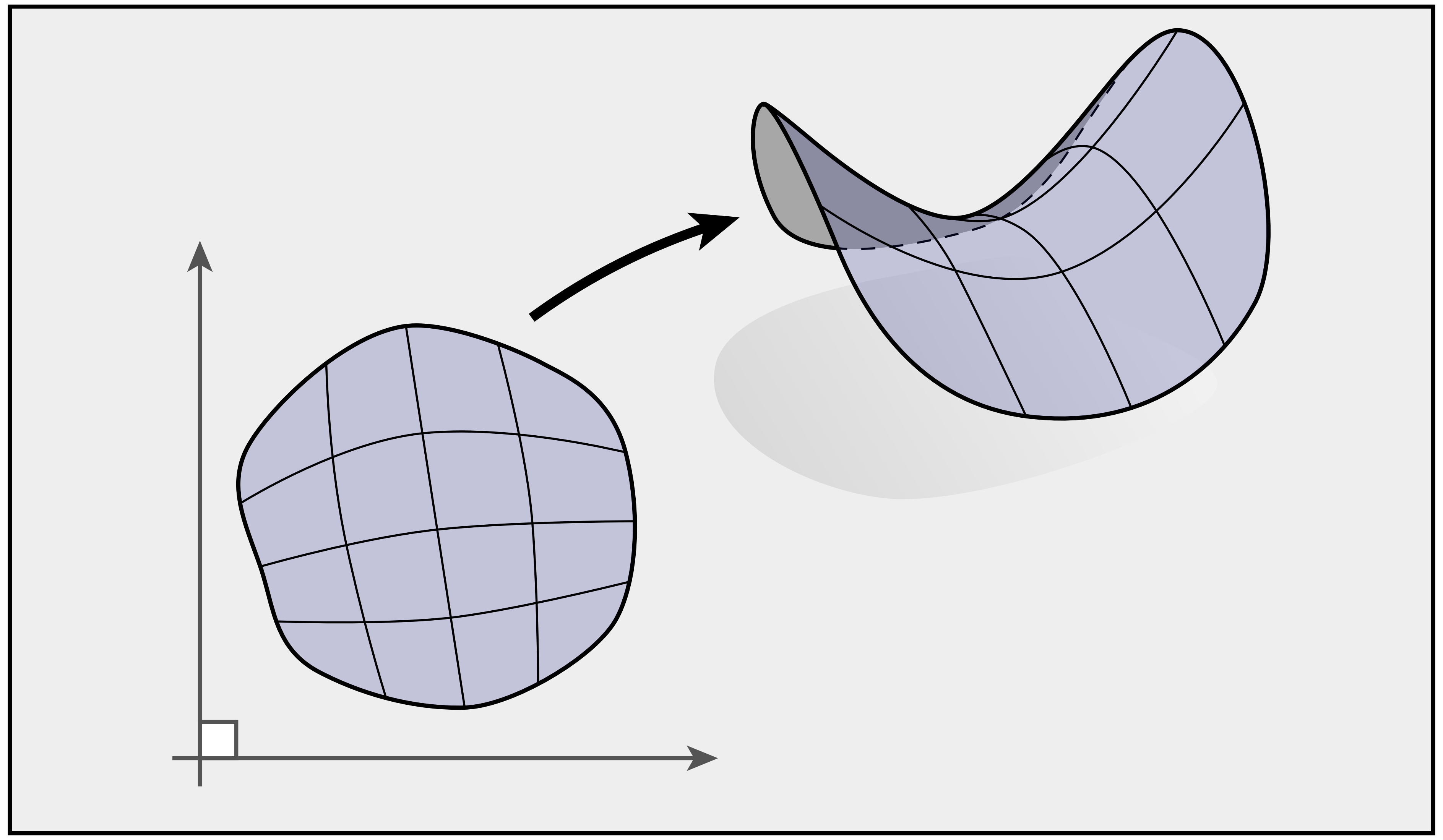
(Curves)

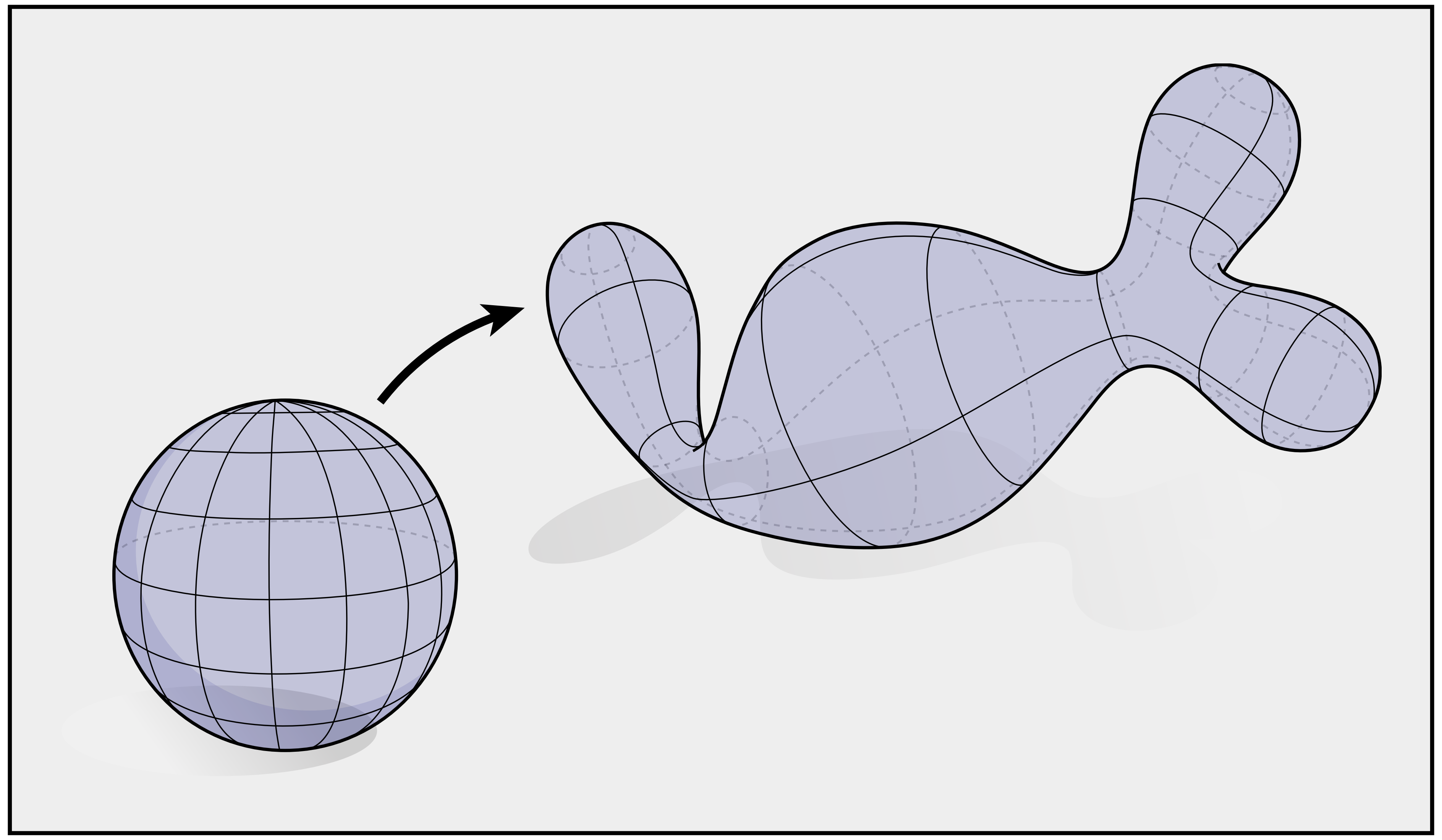


(Surfaces)

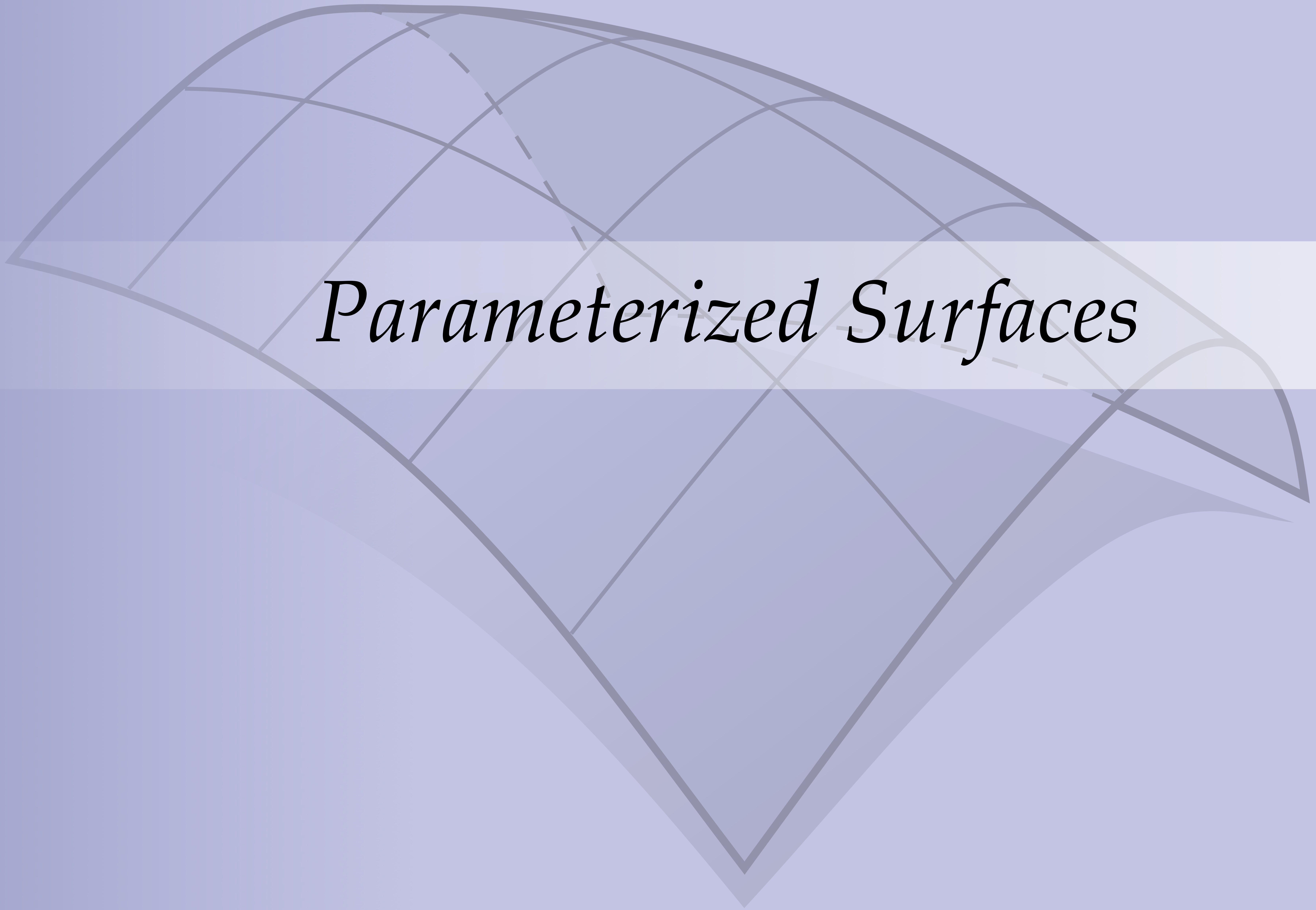
Surfaces—Local vs. Global View

- So far, we've only studied exterior calculus in R^n
- Will therefore be easiest to think of surfaces expressed in terms of patches of the plane (**local picture**)
- Later, when we study topology & smooth manifolds, we'll be able to more easily think about "whole surfaces" all at once (**global picture**)
- Global picture is *much* better model for **discrete** surfaces (meshes)…





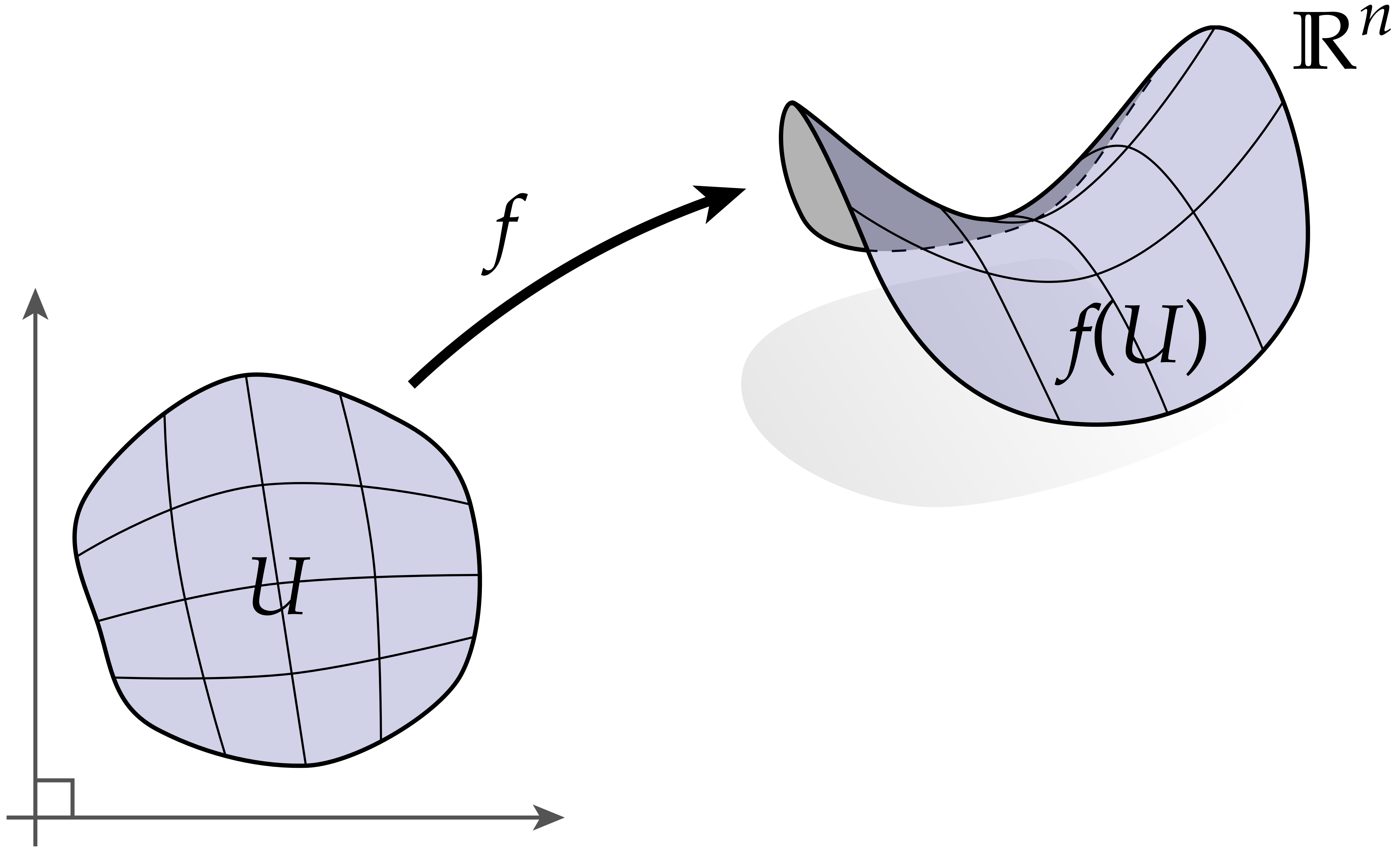
Parameterized Surfaces



Parameterized Surface

A **parameterized surface** is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$:

$$f : U \to \mathbb{R}^n$$



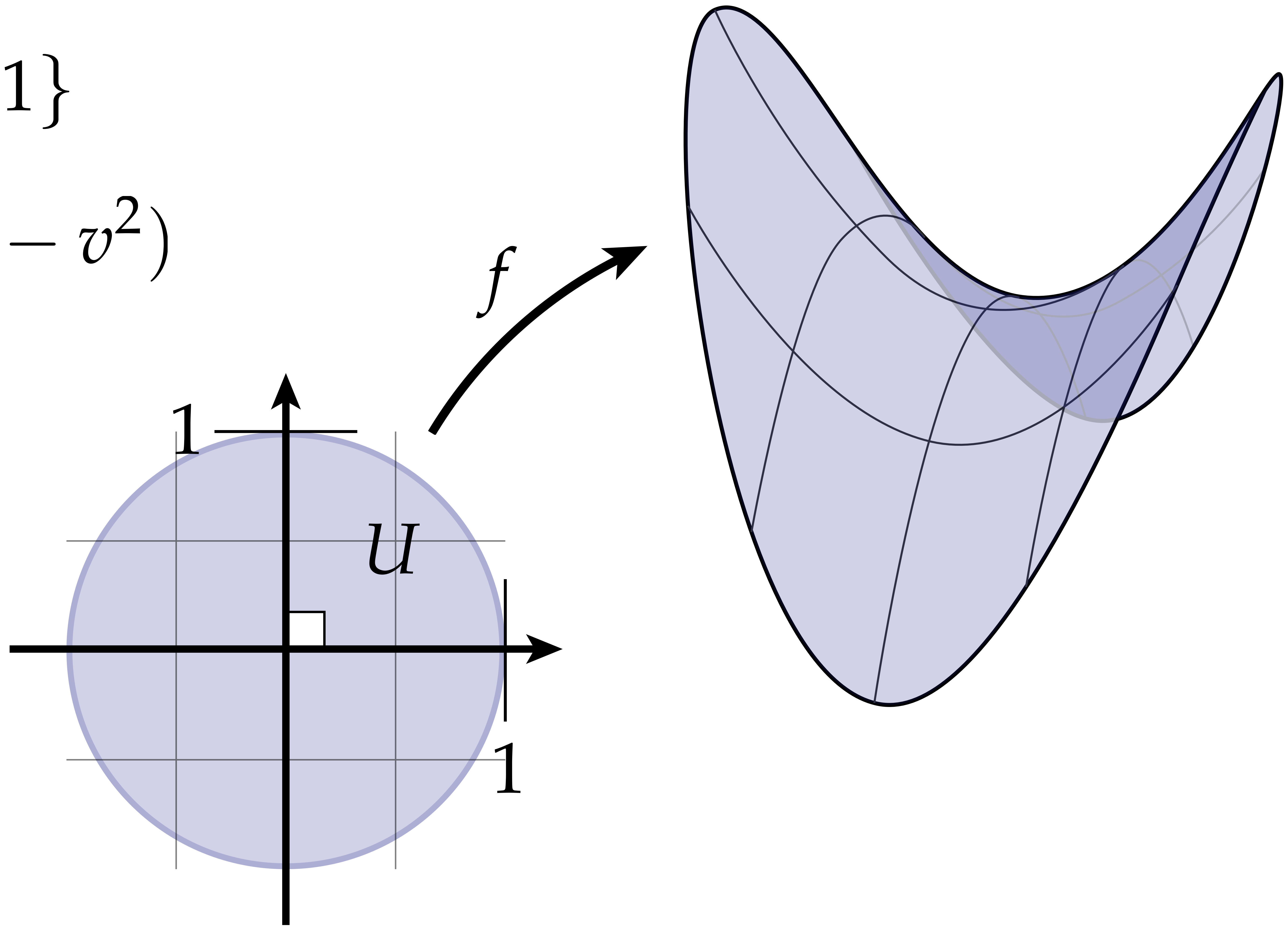
The set of points $f(U)$ is called the **image** of the parameterization.

Parameterized Surface—Example

- As an example, we can express a *saddle* as a parameterized surface:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u,v,u^2 - v^2)$$



Reparameterization

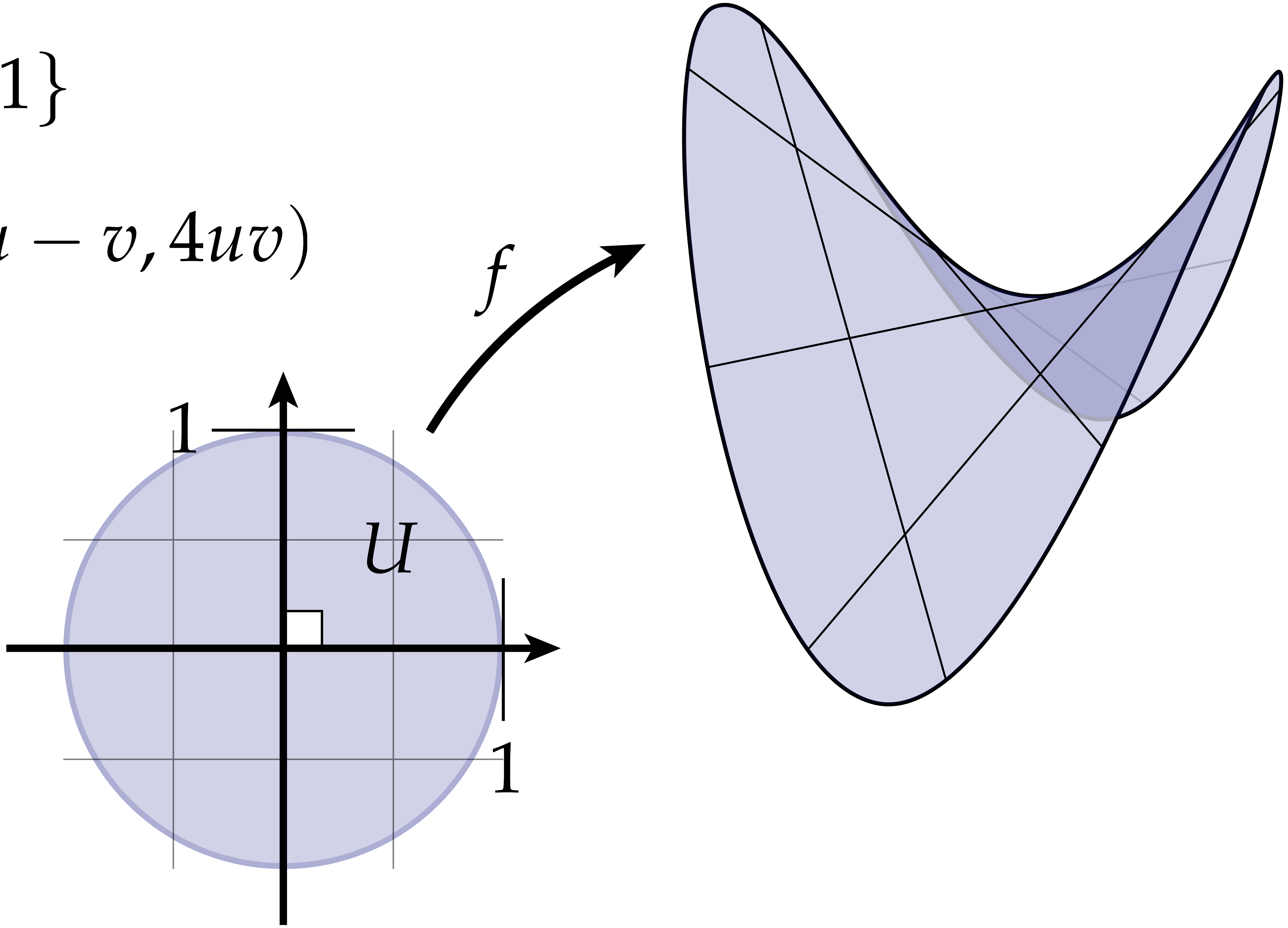
- Many different parameterized surfaces can have the same image:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u+v, u-v, 4uv)$$

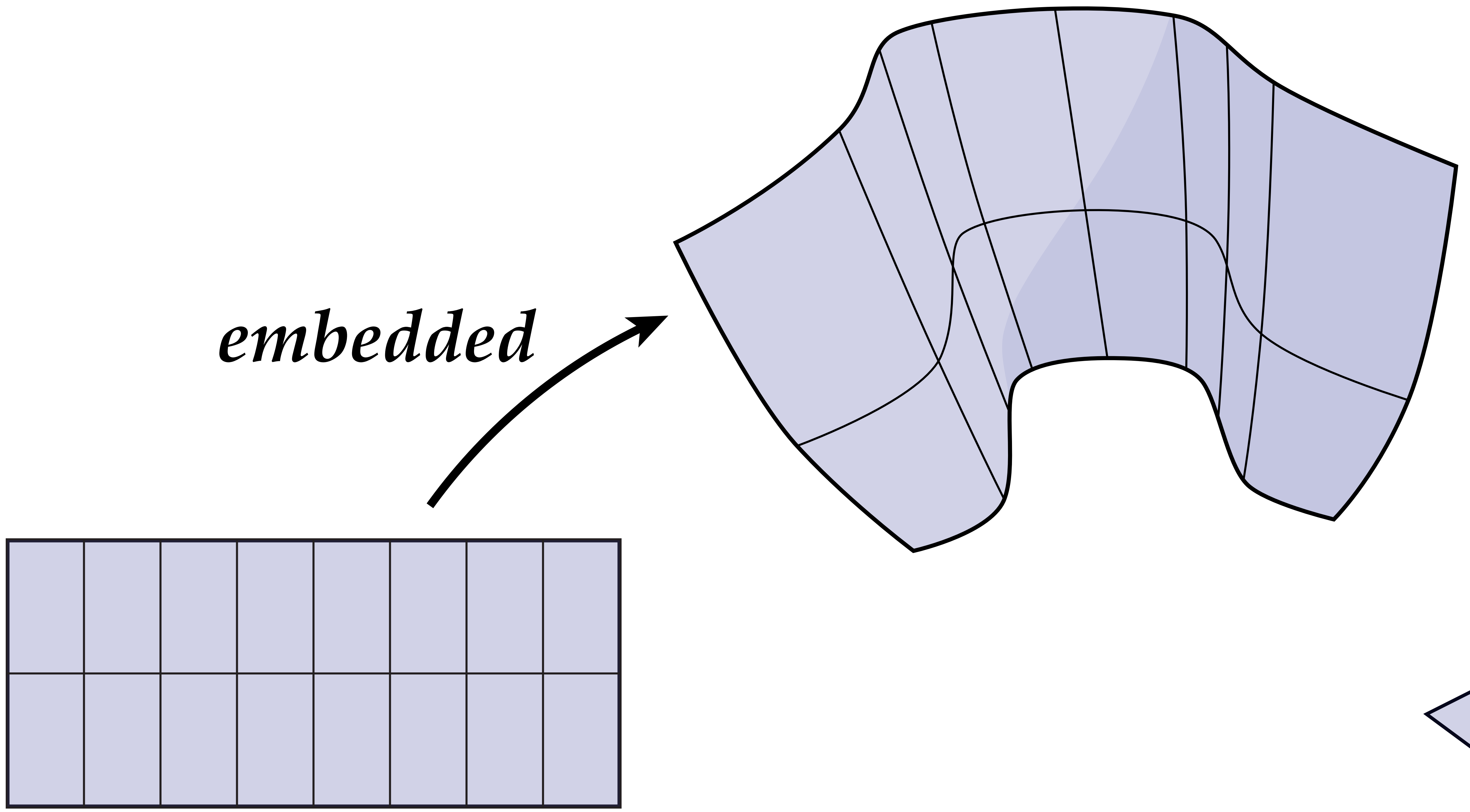
This *"reparameterization symmetry"* can be a major challenge in applications—*e.g.*, trying to decide if two parameterized surfaces (or meshes) describe the same shape.

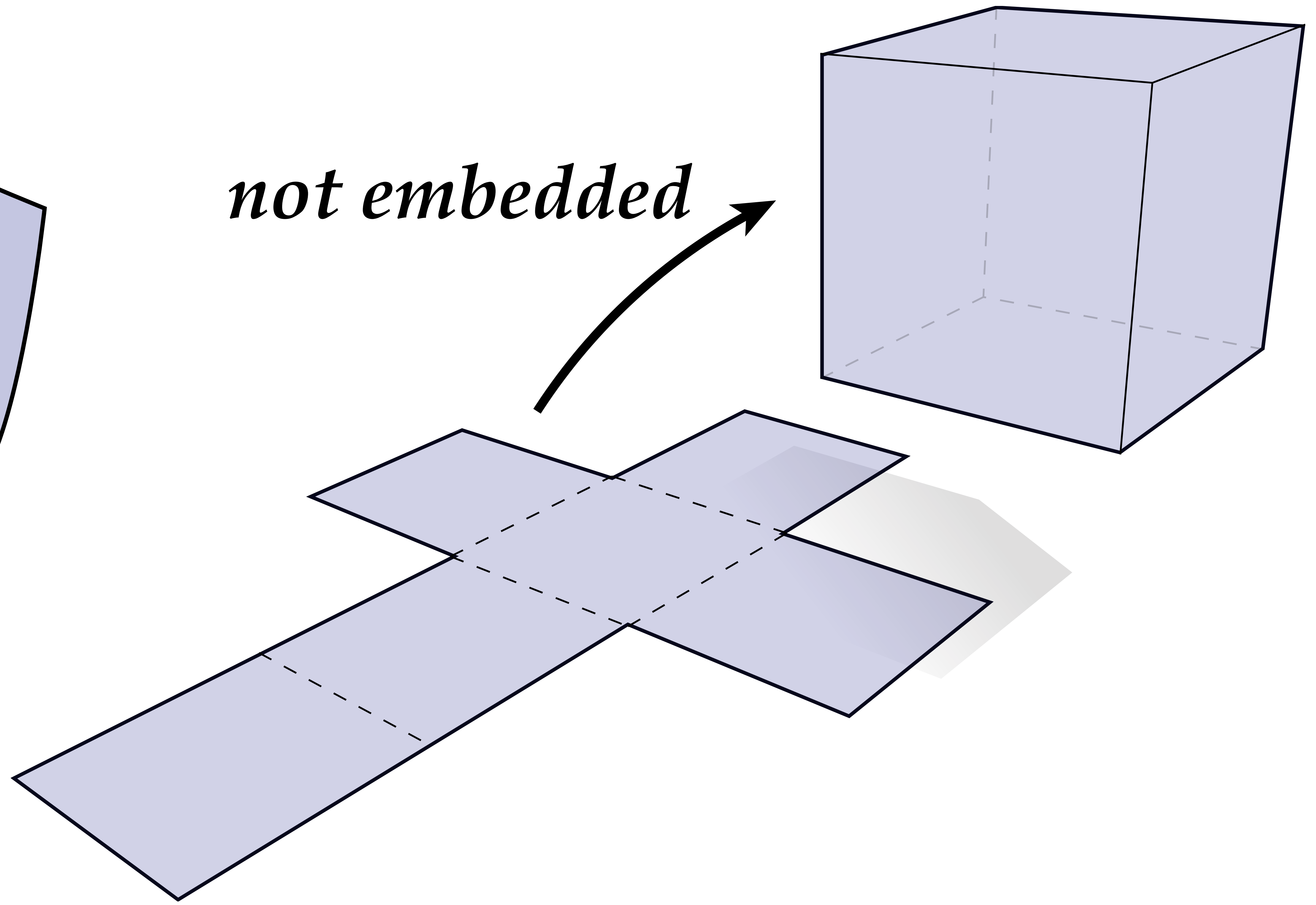
Analogy: graph isomorphism



Embedded Surface

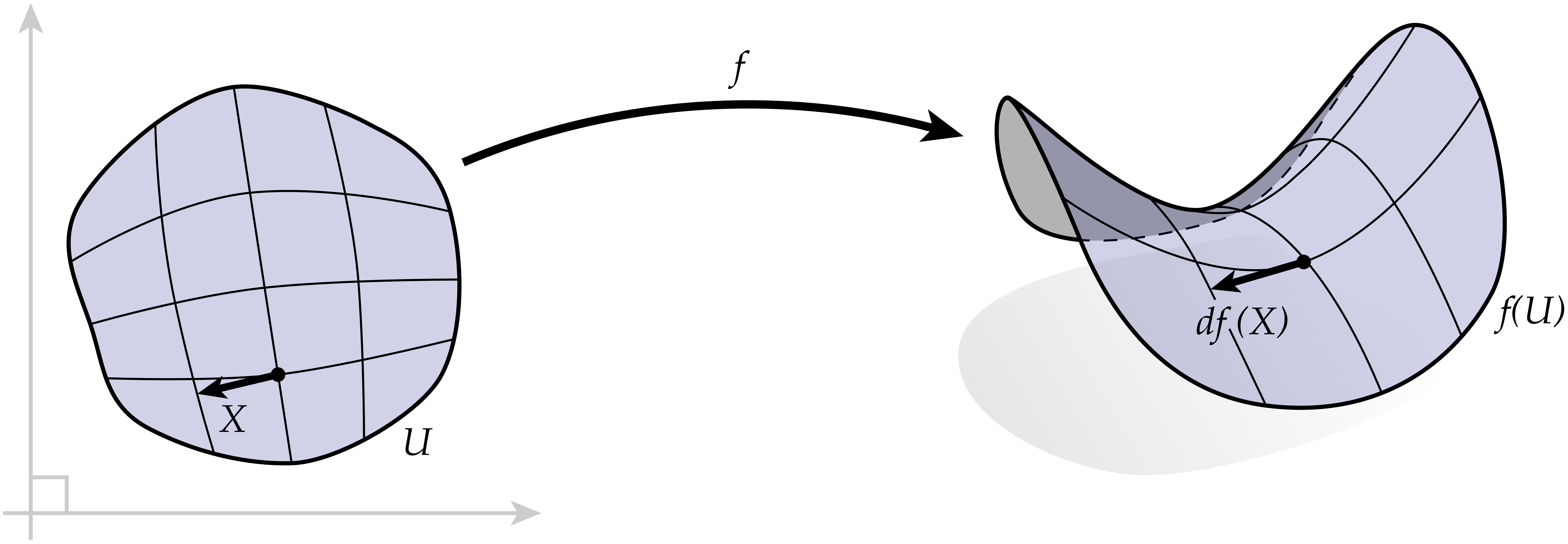
- Roughly speaking, an **embedded** surface does not self-intersect
- More precisely, a parameterized surface is an embedding if it is a continuous injective map, and has a continuous inverse on its image





Differential of a Surface

Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get mapped to vectors in space:



We say that df *"pushes forward"* vectors X into R^n, yielding vectors $df(X)$

Differential in Coordinates

In coordinates, the differential is simply the exterior derivative:

$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u,v,u^2 - v^2)$

$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv =$

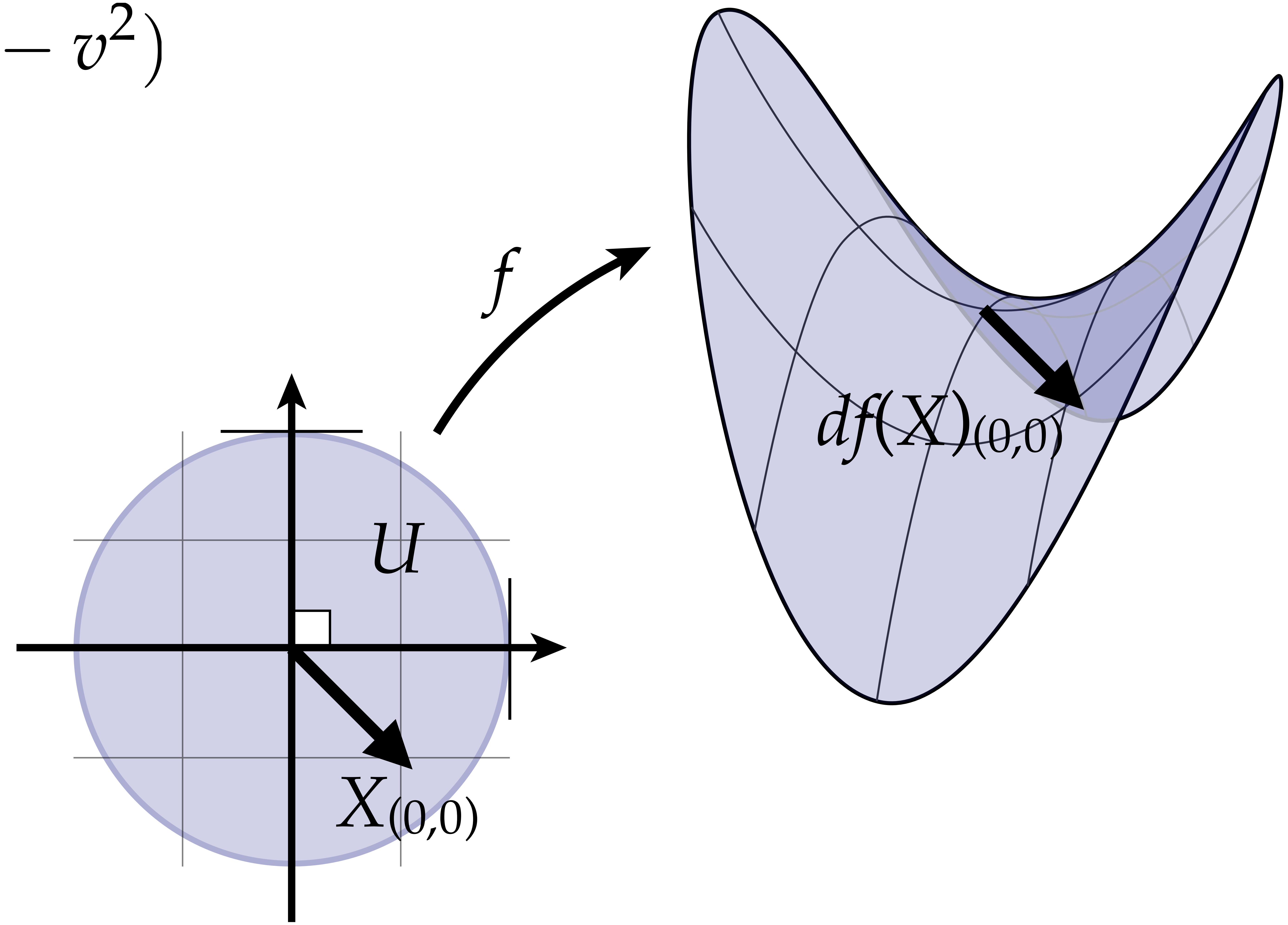
$(1,0,2u)du + (0,1,-2v)dv$

<u>Pushforward of a vector field:</u>

$X := \frac{3}{4}(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$

$df(X) = \frac{3}{4}(1,-1,2(u+v))$

E.g., at u=v=0: $(\frac{3}{4}, -\frac{3}{4}, 0)$



Differential—Matrix Representation (Jacobian)

Definition. Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$, and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. Then the *Jacobian* of f is the matrix

$$J_f := \begin{bmatrix} \partial f^1/\partial x^1 & \cdots & \partial f^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial f^m/\partial x^1 & \cdots & \partial f^m/\partial x^n \end{bmatrix},$$

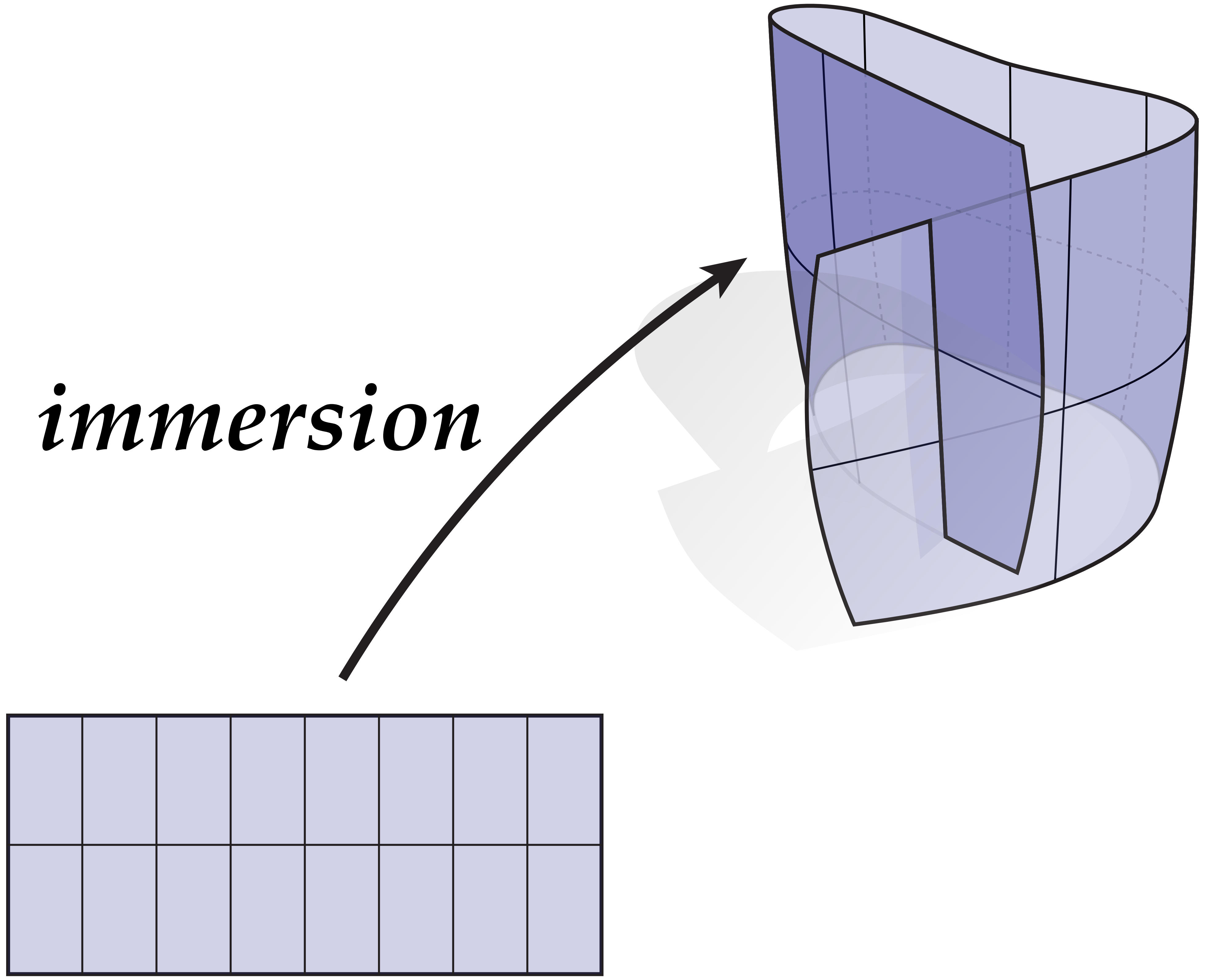
where $f^1, \ldots, f^m$ are the components of f w.r.t. some coordinate system on $\mathbb{R}^m$. This matrix represents the differential in the sense that $df(X) = J_f X$.

(In solid mechanics, also known as the *deformation gradient*.)

Note: does not generalize to infinite dimensions! (E.g., maps between functions.)

Immersed Surface

- A parameterized surface f is an *immersion* if its differential is nondegenerate, *i.e.*, if $df(X) = 0$ if and only if $X = 0$.



Intuition: no region of the surface gets "pinched"

Immersion — Example

Consider the standard parameterization of the sphere:

$$f(u,v) := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

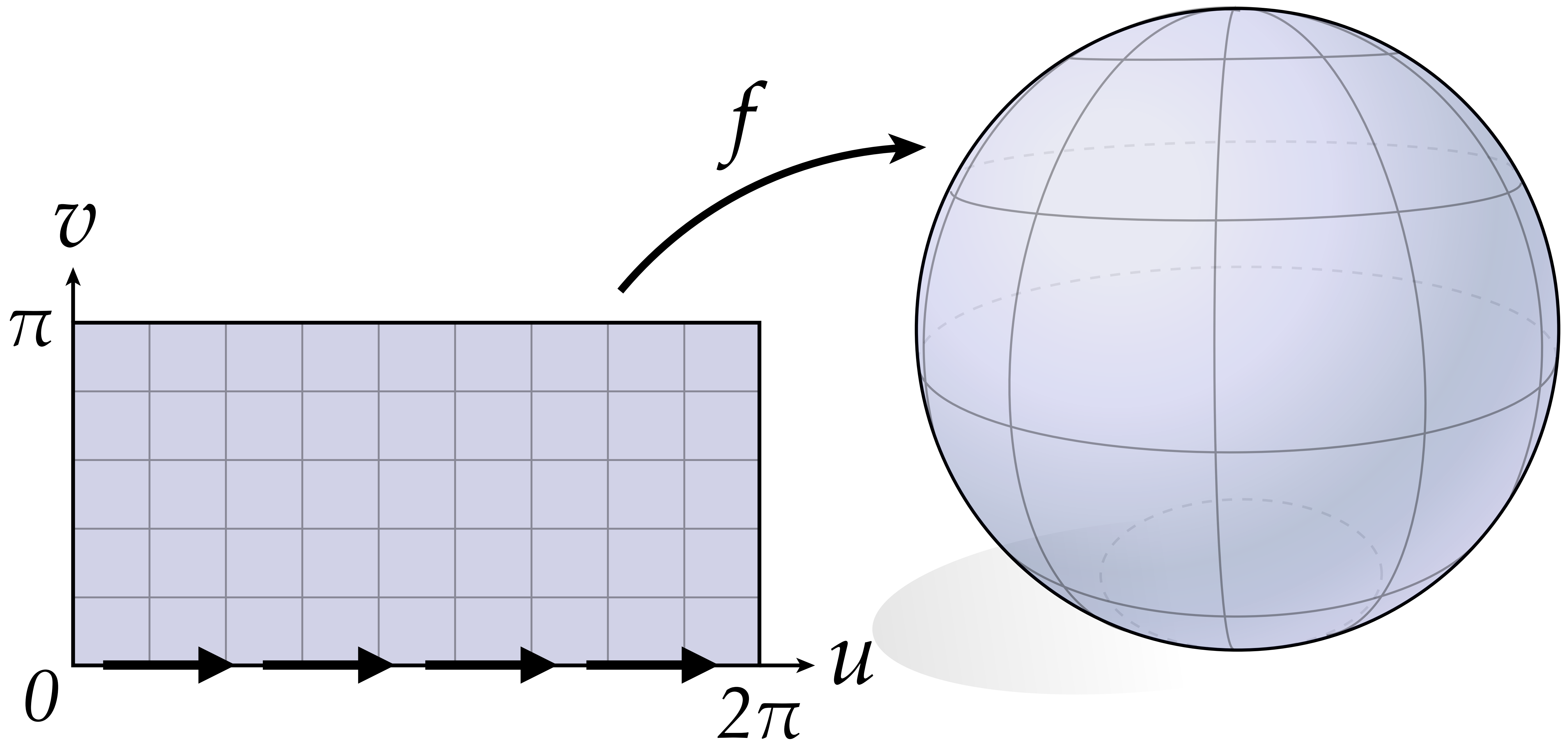
$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = \begin{array}{lccccl} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &) & du\, + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &) & dv \end{array}$$

Q: Is f an immersion?

A: No: when $v = 0$ we get

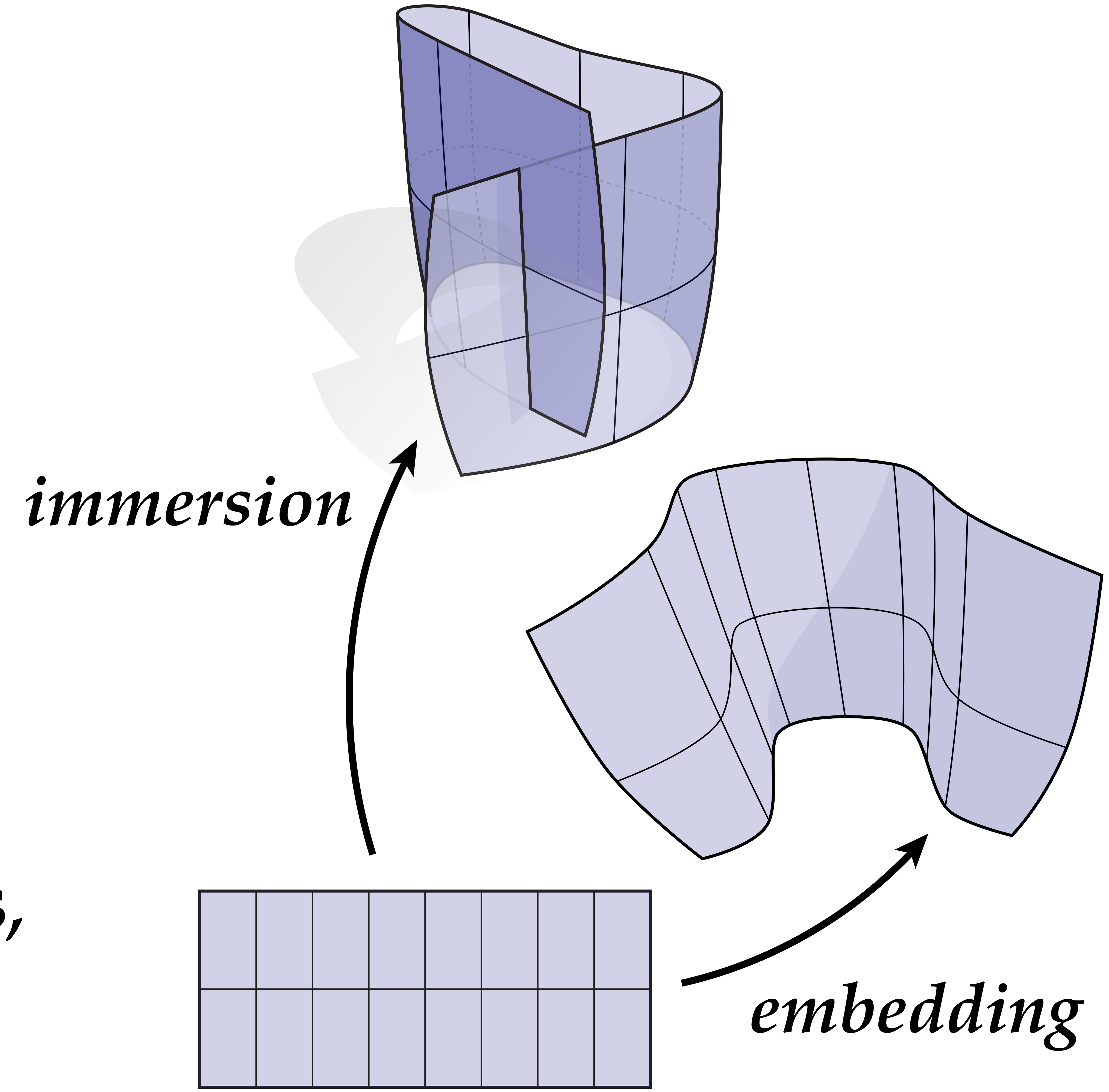
$$\begin{array}{lccccl} (& 0, & 0, & 0 &) & du\, + \\ (& \cos(u), & \sin(u), & -\sin(v) &) & dv \end{array}$$

Nonzero tangents mapped to zero!



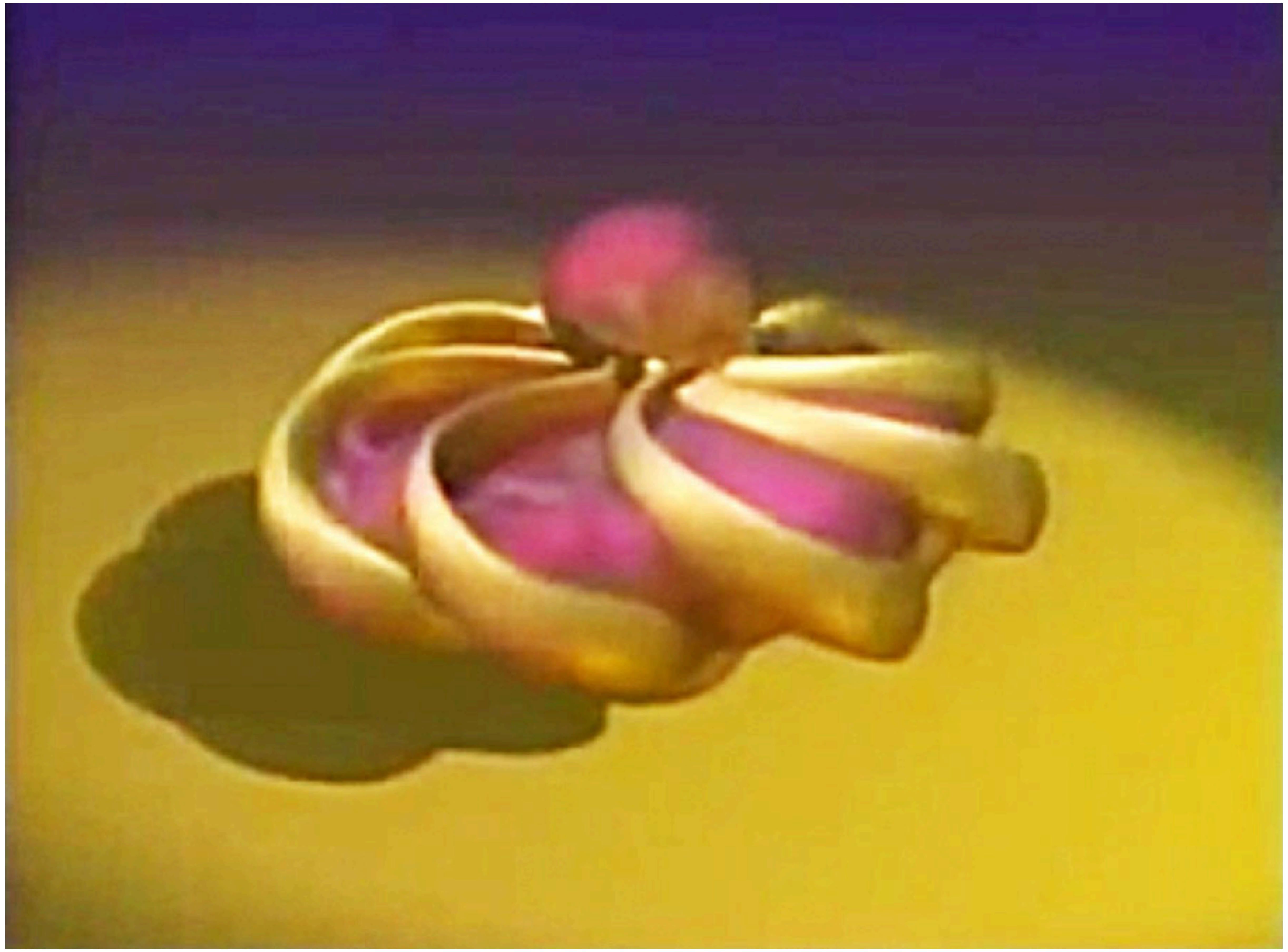
Immersion vs. Embedding

- In practice, ensuring that a surface is globally embedded can be challenging
- Immersions are typically "nice enough" to define local quantities like tangents, normals, metric, etc.
- Immersions are also a natural model for the way we typically think about meshes: most quantities (angles, lengths, etc.) are perfectly well-defined, even if there happen to be self-intersections



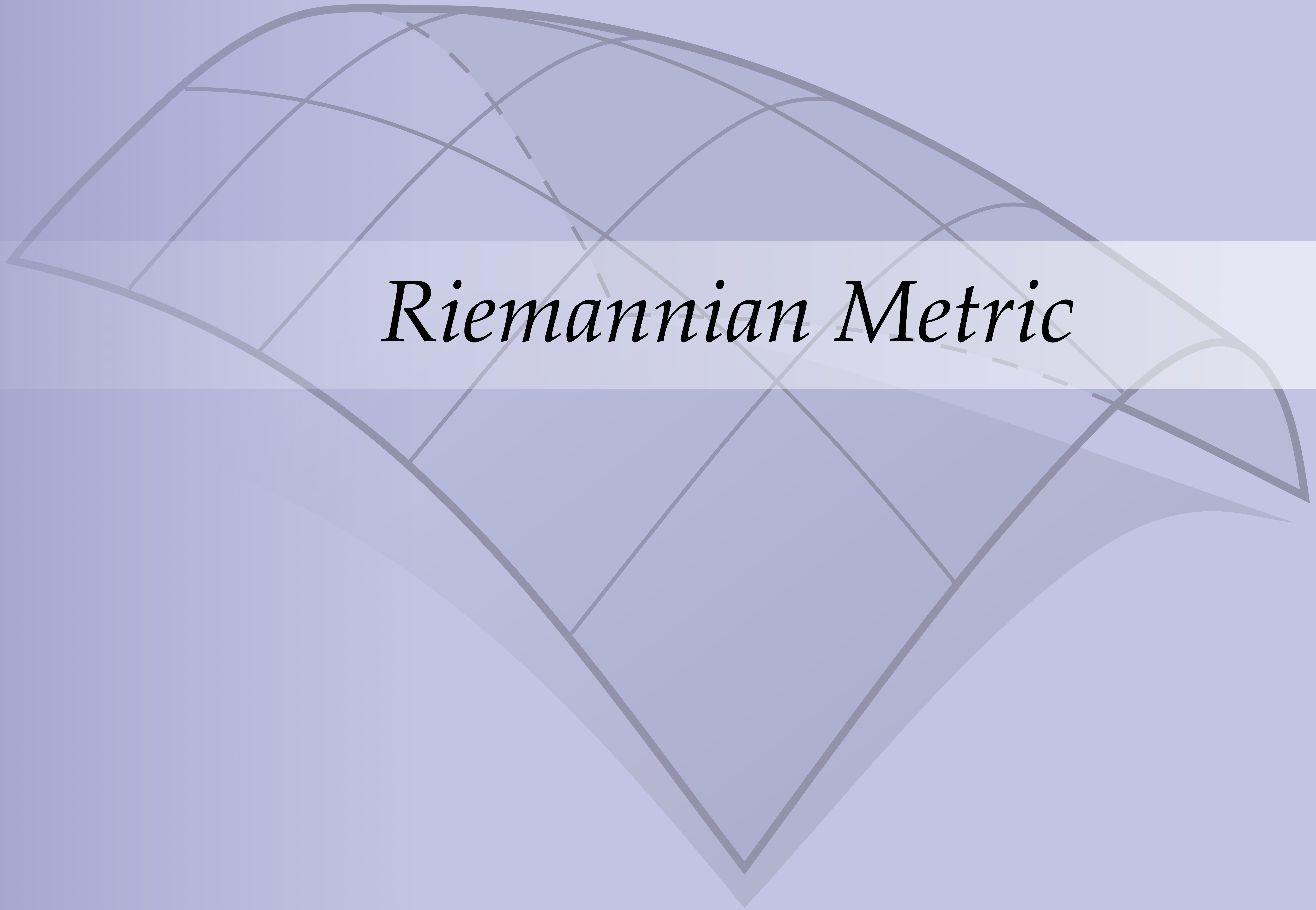
Sphere Eversion

Turning a Sphere Inside-Out (1994)



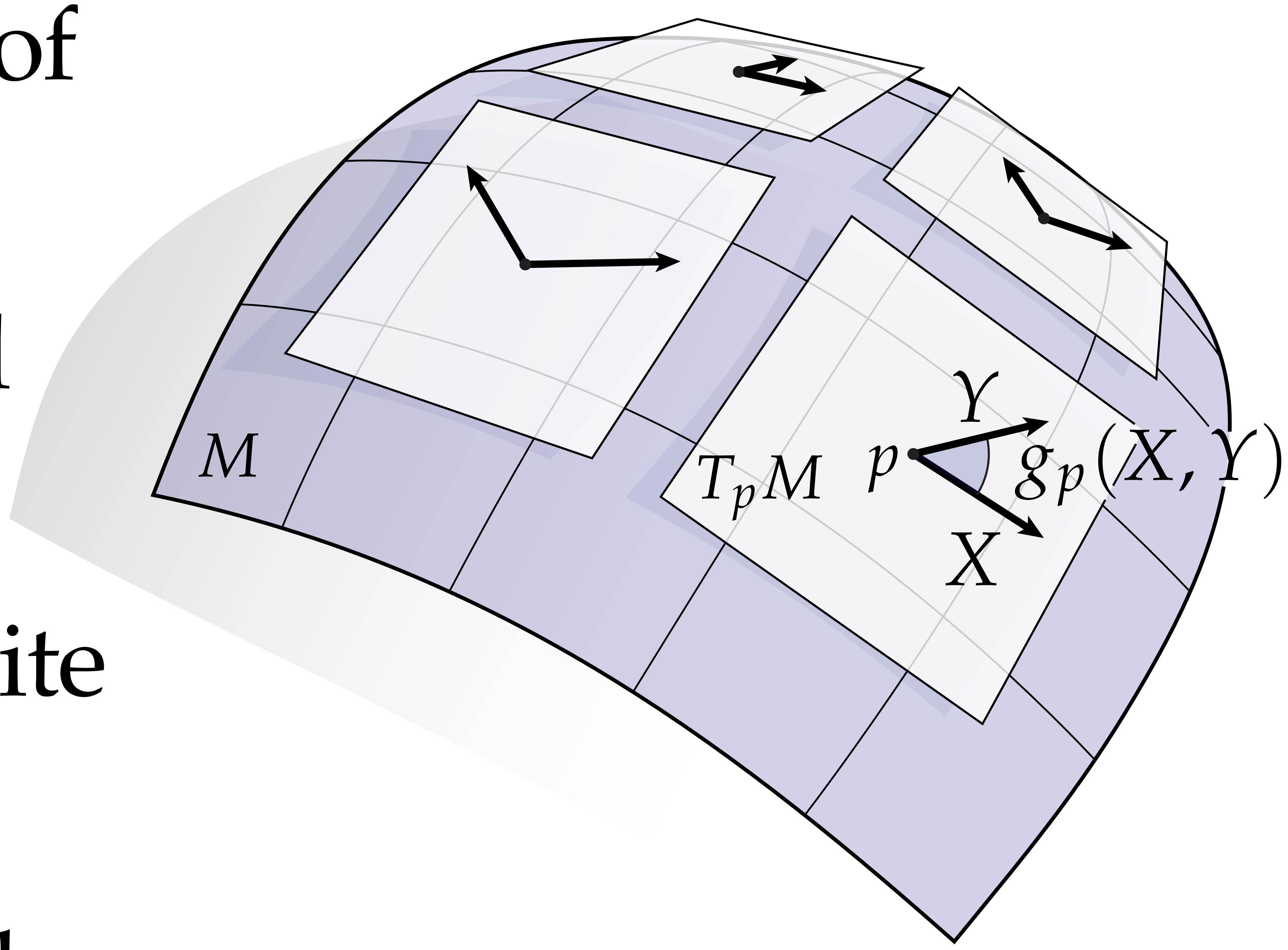
https://youtu.be/-6g3ZcmjJ7k

Riemannian Metric



Riemann Metric

- Many quantities on manifolds (curves, surfaces, *etc.*) ultimately boil down to measurements of *lengths* and *angles* of tangent vectors
- This information is encoded by the so-called *Riemannian metric**
- Abstractly: smoothly-varying positive-definite bilinear form
- For immersed surface, can (and will!) describe more concretely/geometrically

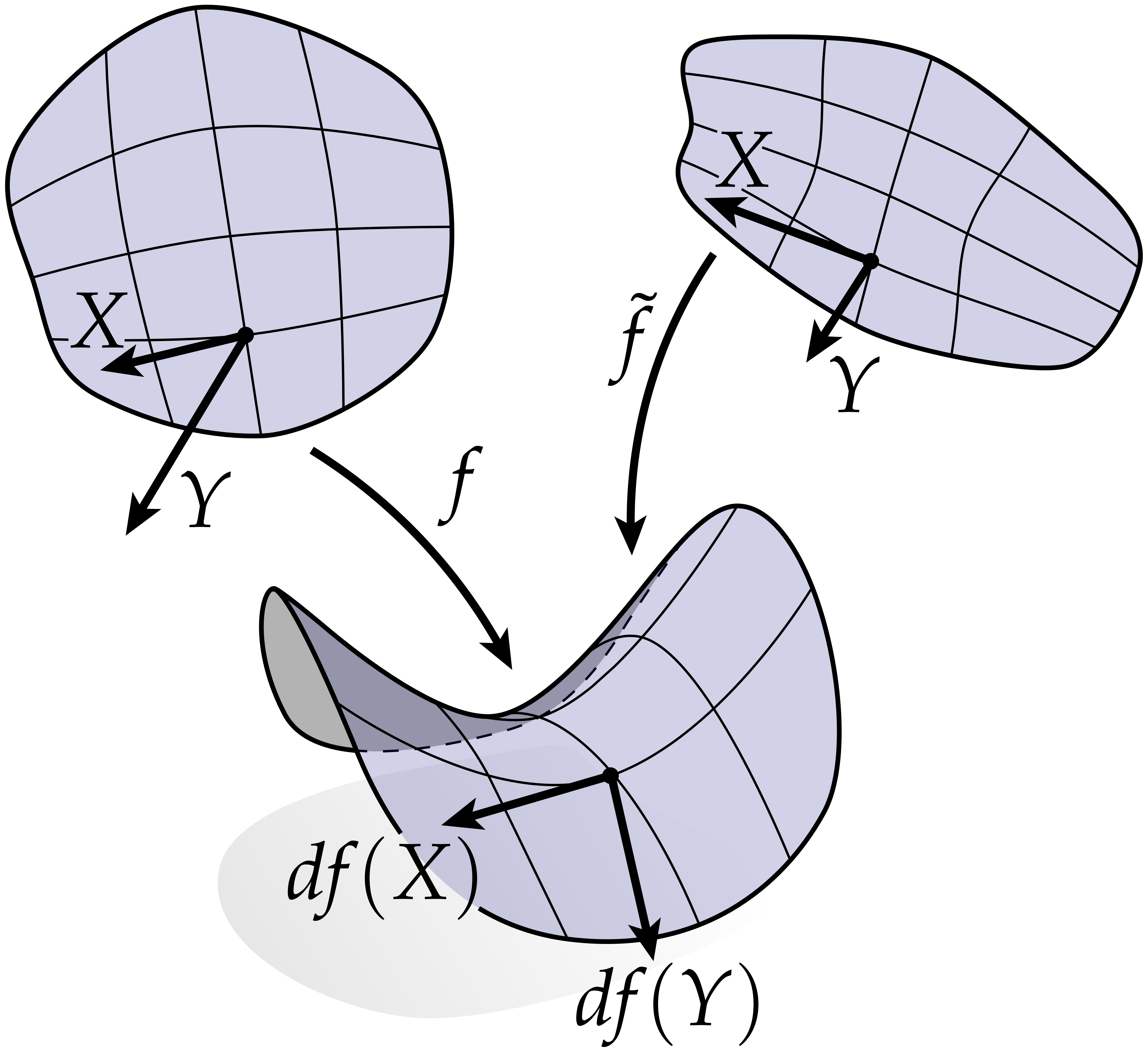


***Note:** *not* the same as a point-to-point distance metric $d(x,y)$

Metric Induced by an Immersion

- Given an immersed surface *f*, how should we measure inner product of vectors *X*, *Y* on its domain *U*?
- We should **not** use the usual inner product on the plane! (Why not?)
- Planar inner product tells us *nothing* about actual length & angle on the surface (and changes depending on choice of parameterization!)
- Instead, use **induced metric**

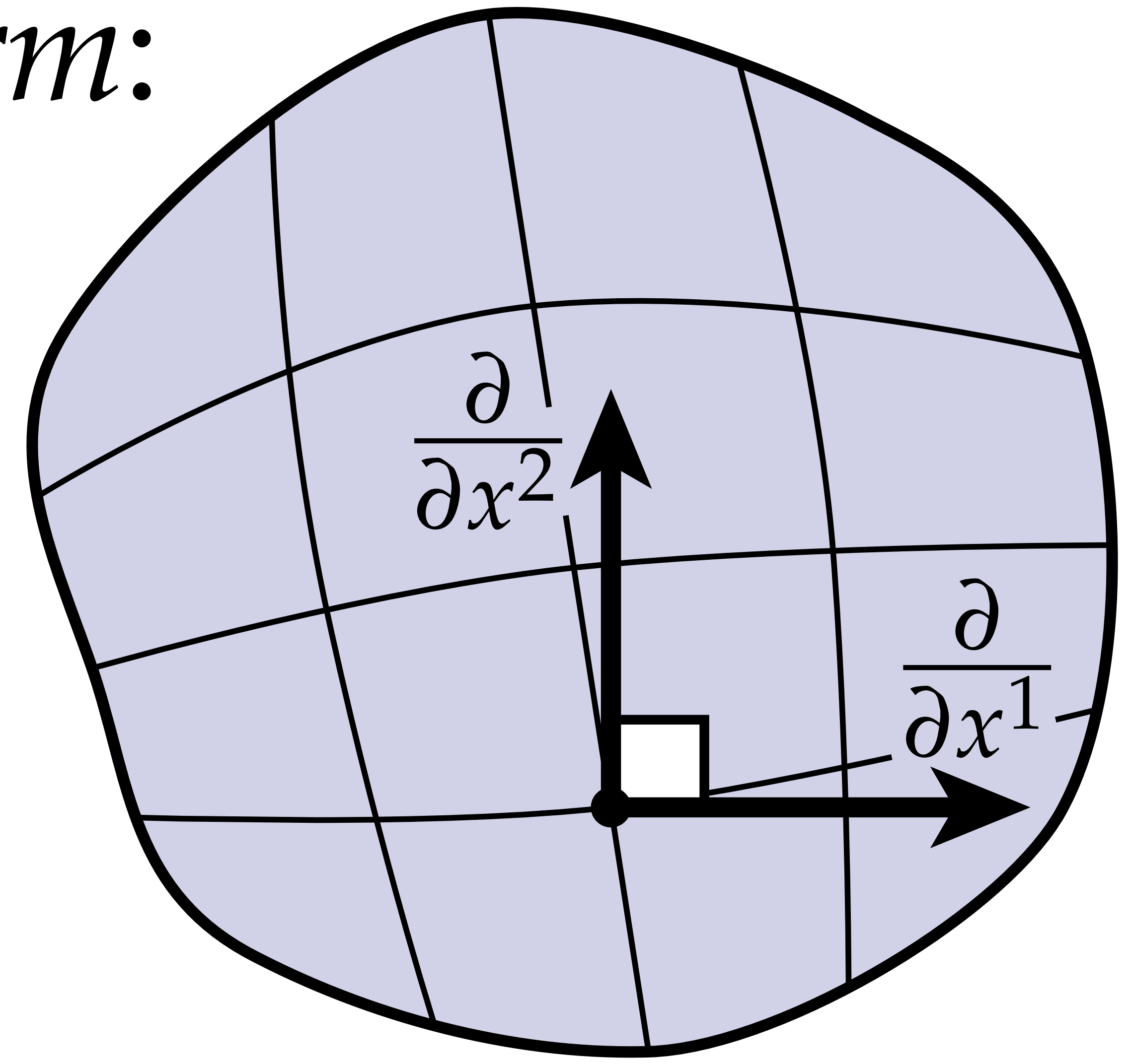
$$g(X,Y) := \langle df(X), df(Y) \rangle$$



Key idea: must account for "stretching"

Induced Metric—Matrix Representation

- Metric is a bilinear map from a pair of vectors to a scalar, which we can represent as a 2x2 matrix **I** called the *first fundamental form*:



$$g(X,Y) = X^T \mathbf{I} Y$$

$$\Rightarrow \mathbf{I}_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle df\left(\frac{\partial}{\partial x^i}\right), df\left(\frac{\partial}{\partial x^j}\right)\right\rangle$$

- Alternatively, can express first fundamental form via Jacobian:

$$g(X,Y) = \langle df(X), df(Y)\rangle = (J_f X)^\mathsf{T}(J_f Y) = X^\mathsf{T}(J_f^\mathsf{T} J_f)Y$$

$$\Rightarrow \mathbf{I} = J_f^\mathsf{T} J_f$$

Induced Metric—Example

Can use the differential to obtain the induced metric:

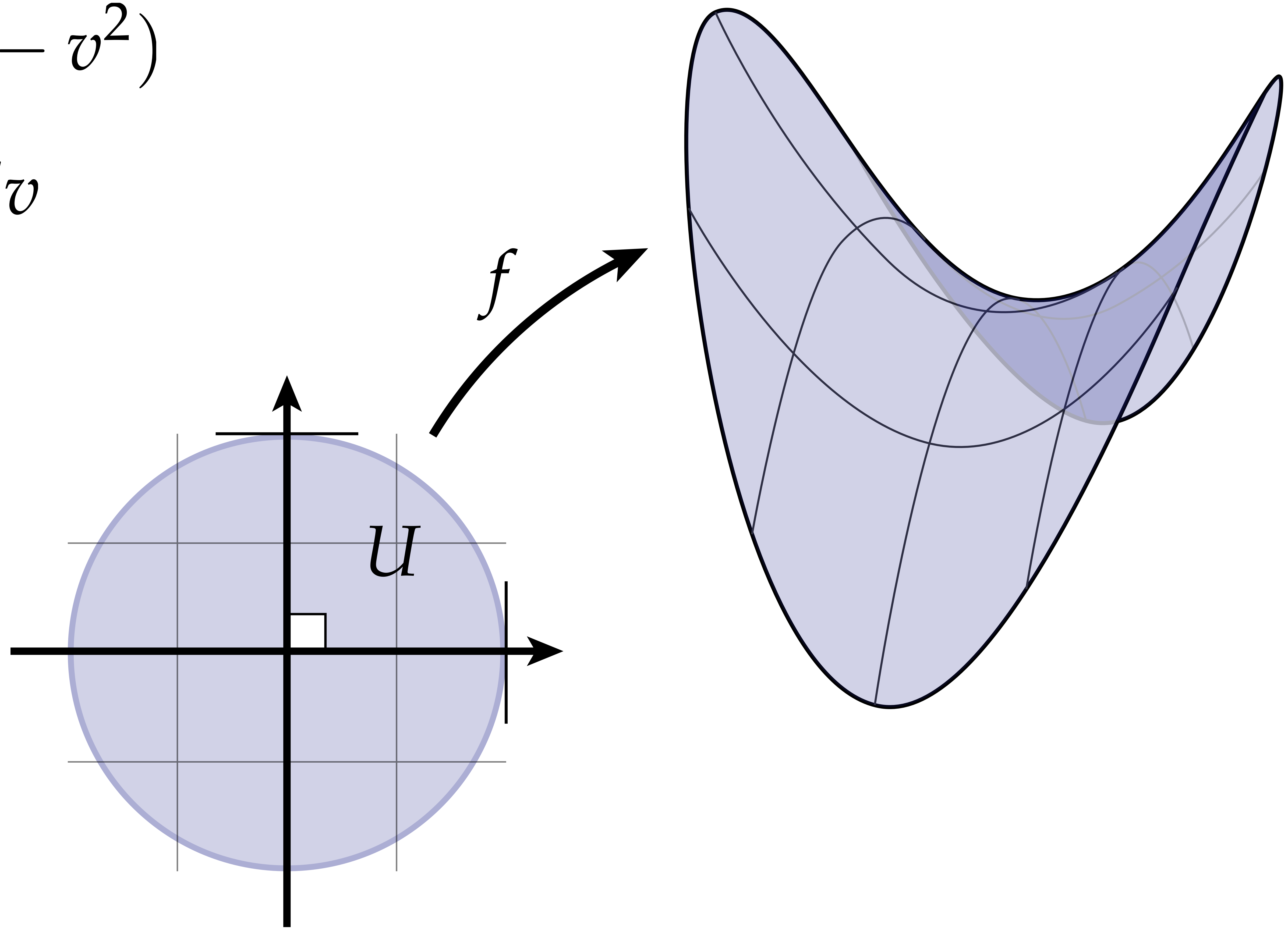
$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u,v,u^2 - v^2)$$

$$df = (1,0,2u)du + (0,1,-2v)dv$$

$$J_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

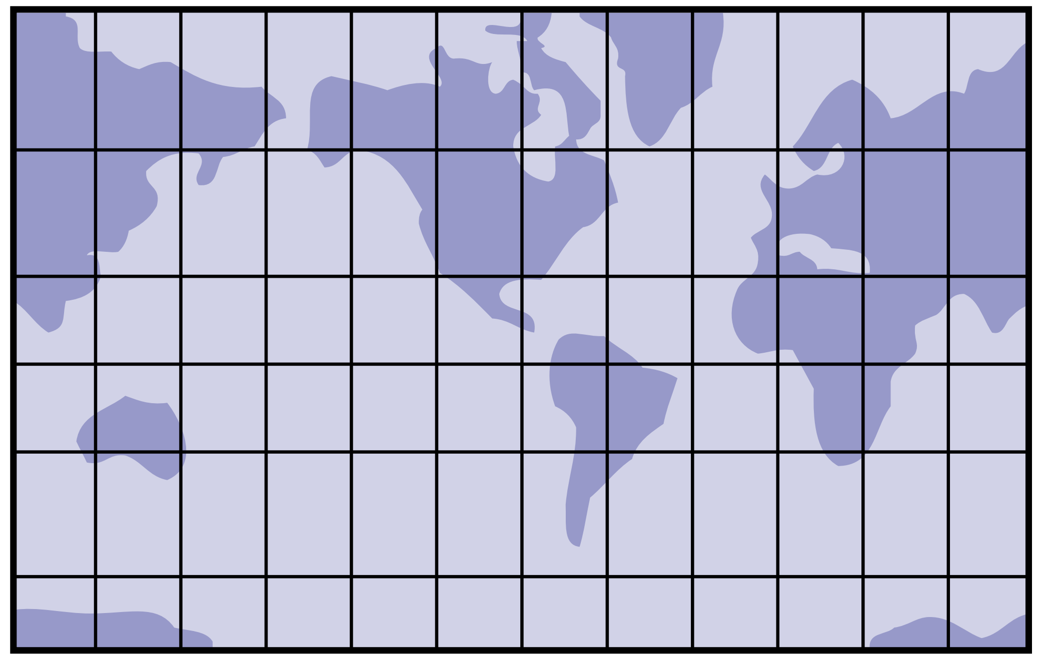
$$\mathbf{I} = J_f^\mathsf{T} J_f$$

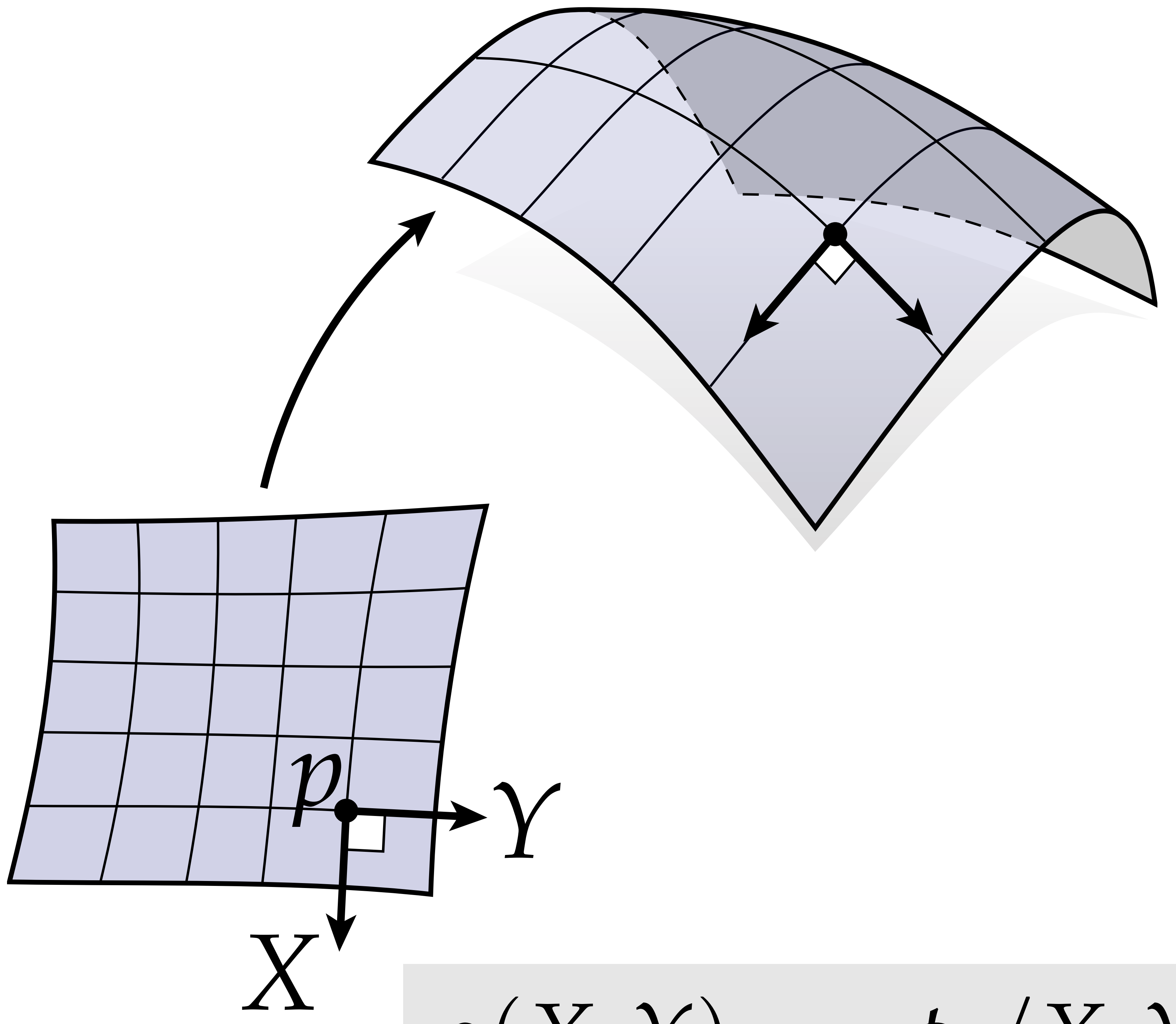
$$= \begin{bmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{bmatrix}$$



Conformal Coordinates

- As we've just seen, there can be a complicated relationship between length & angle on the domain (2D) and the image (3D)
- For curves, we simplified life by using an *arc-length* or *isometric* parameterization: lengths on domain are identical to lengths along curve
- For surfaces, usually not possible to preserve all *lengths* (e.g., globe). Remarkably, however, can always preserve *angles* (**conformal**)
- Equivalently, a parameterized surface is *conformal* if at each point the induced metric is simply a positive rescaling of the 2D Euclidean metric



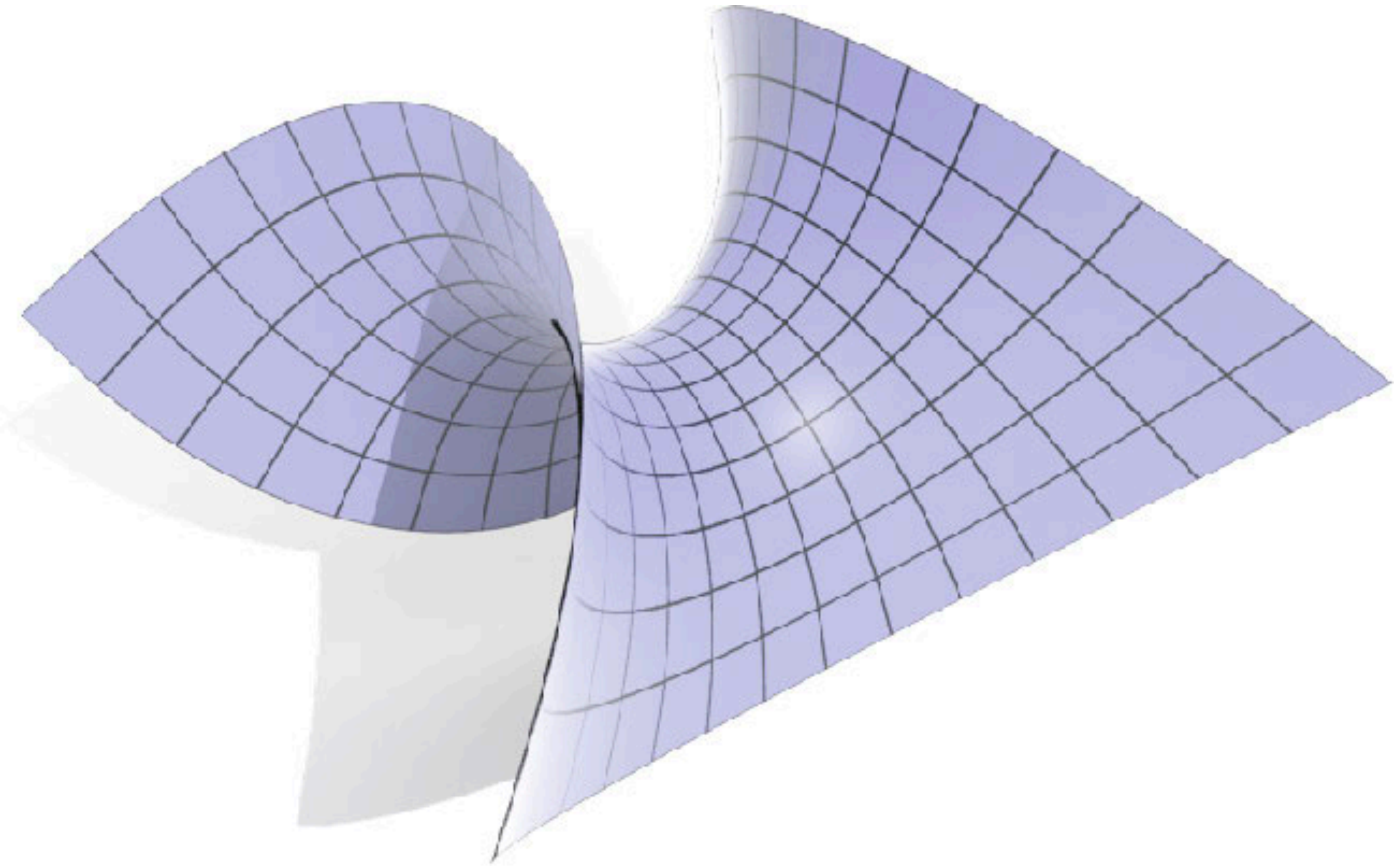


$$g(X,Y)_p = \phi_p \langle X, Y \rangle$$

Example (Enneper Surface)

Consider the surface

$$f(u,v) := \begin{bmatrix} uv^2 + u - \frac{1}{3}u^3 \\ \frac{1}{3}v\left(v^2 - 3u^2 - 3\right) \\ (u-v)(u+v) \end{bmatrix}$$



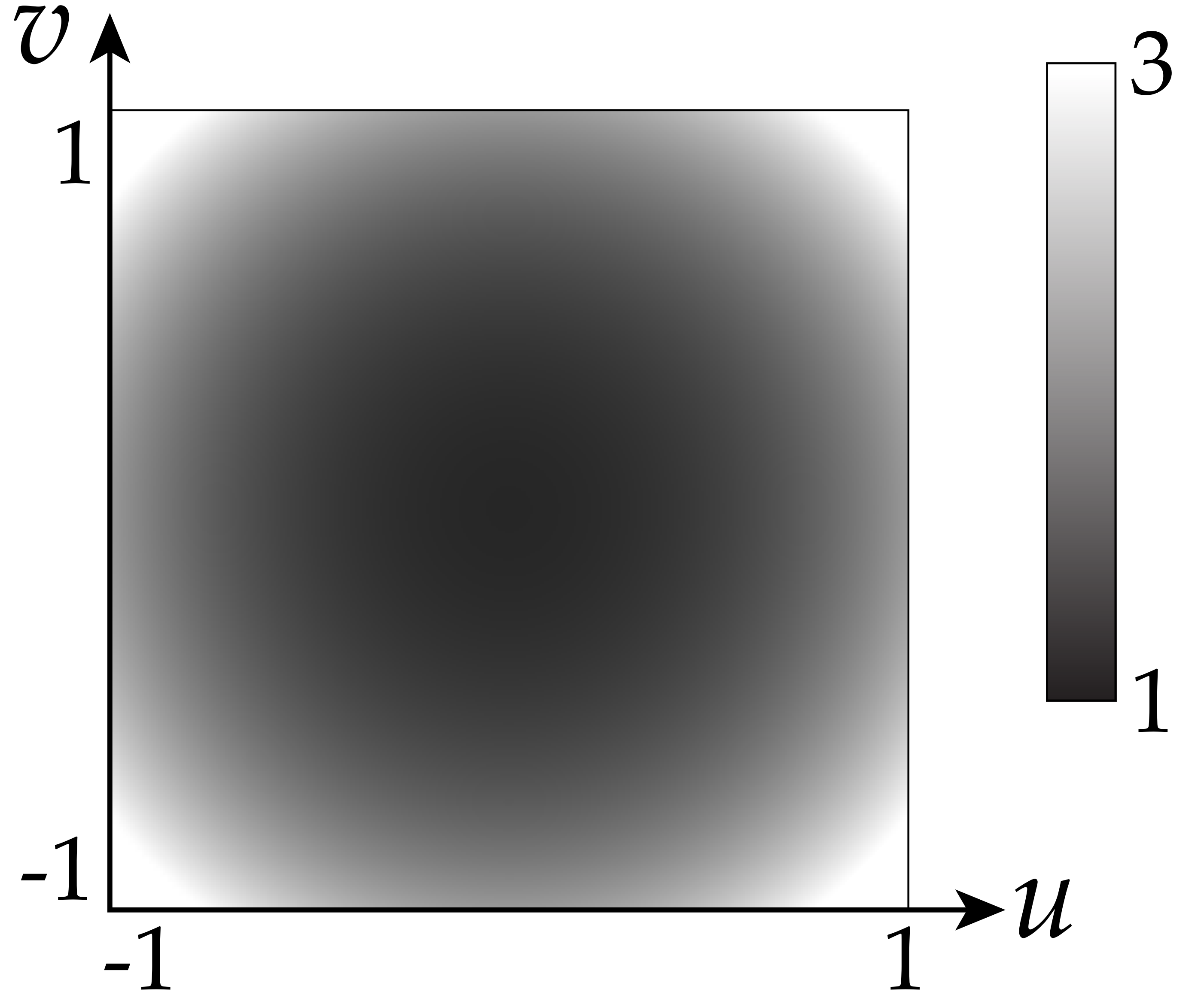
Its Jacobian matrix is

$$J_f = \begin{bmatrix} -u^2 + v^2 + 1 & 2uv \\ -2uv & -u^2 + v^2 - 1 \\ 2u & -2v \end{bmatrix}$$

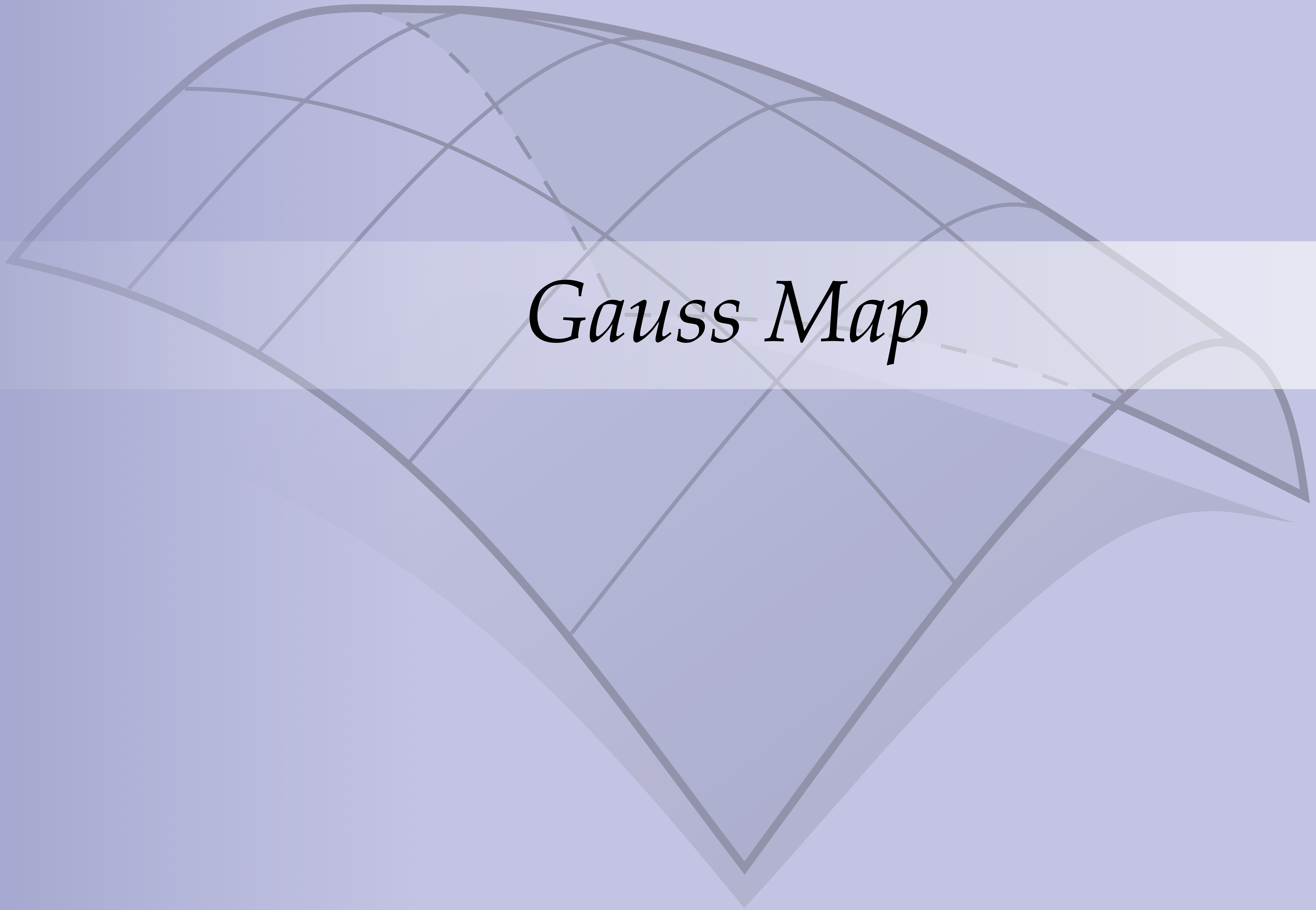
Its metric then works out to be just a scalar function times the usual metric of the Euclidean plane:

$$\mathbf{I} = J_f^T J_f = \left(u^2 + v^2 + 1\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This function is called the *conformal scale factor*.



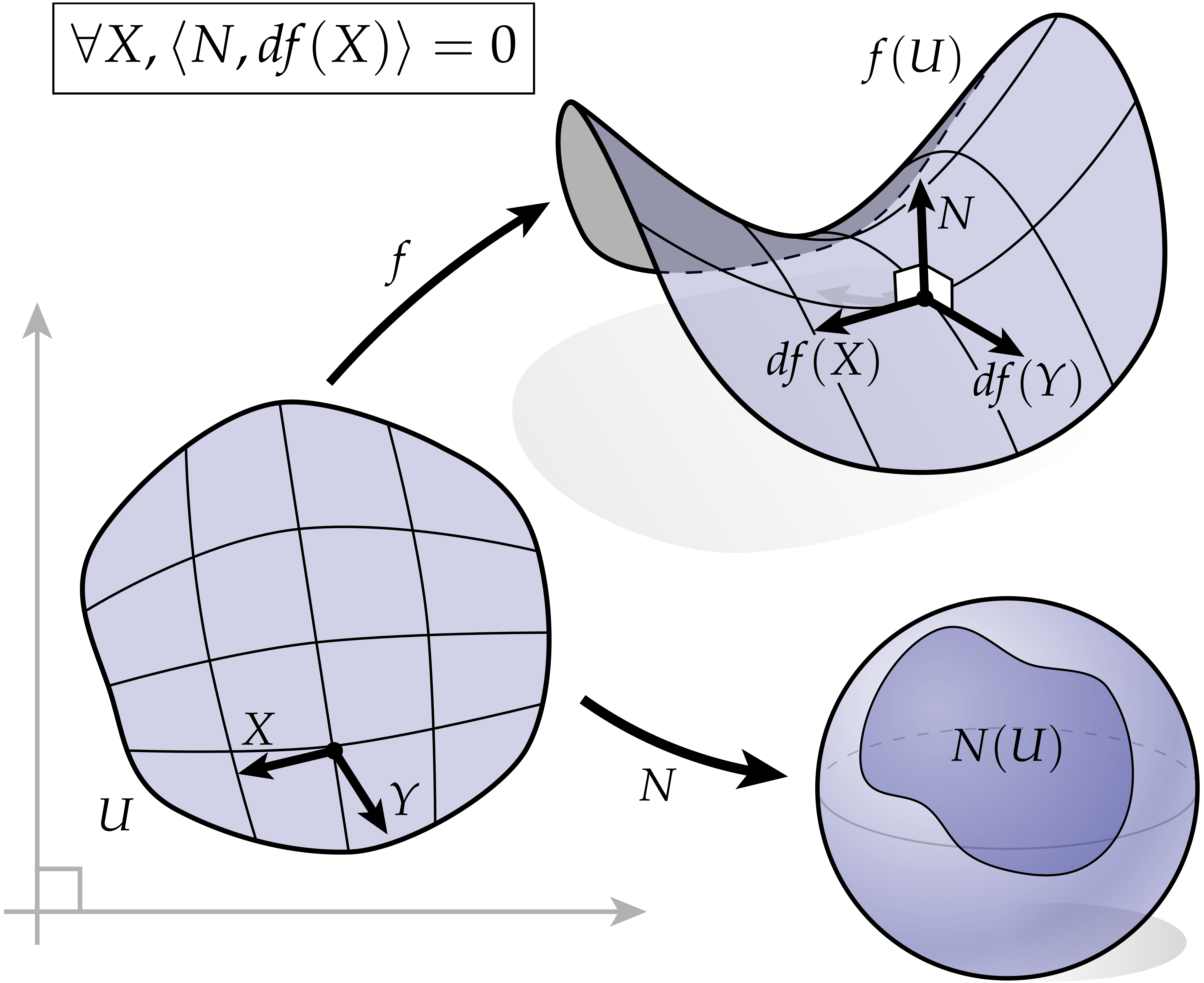
Gauss Map



Gauss Map

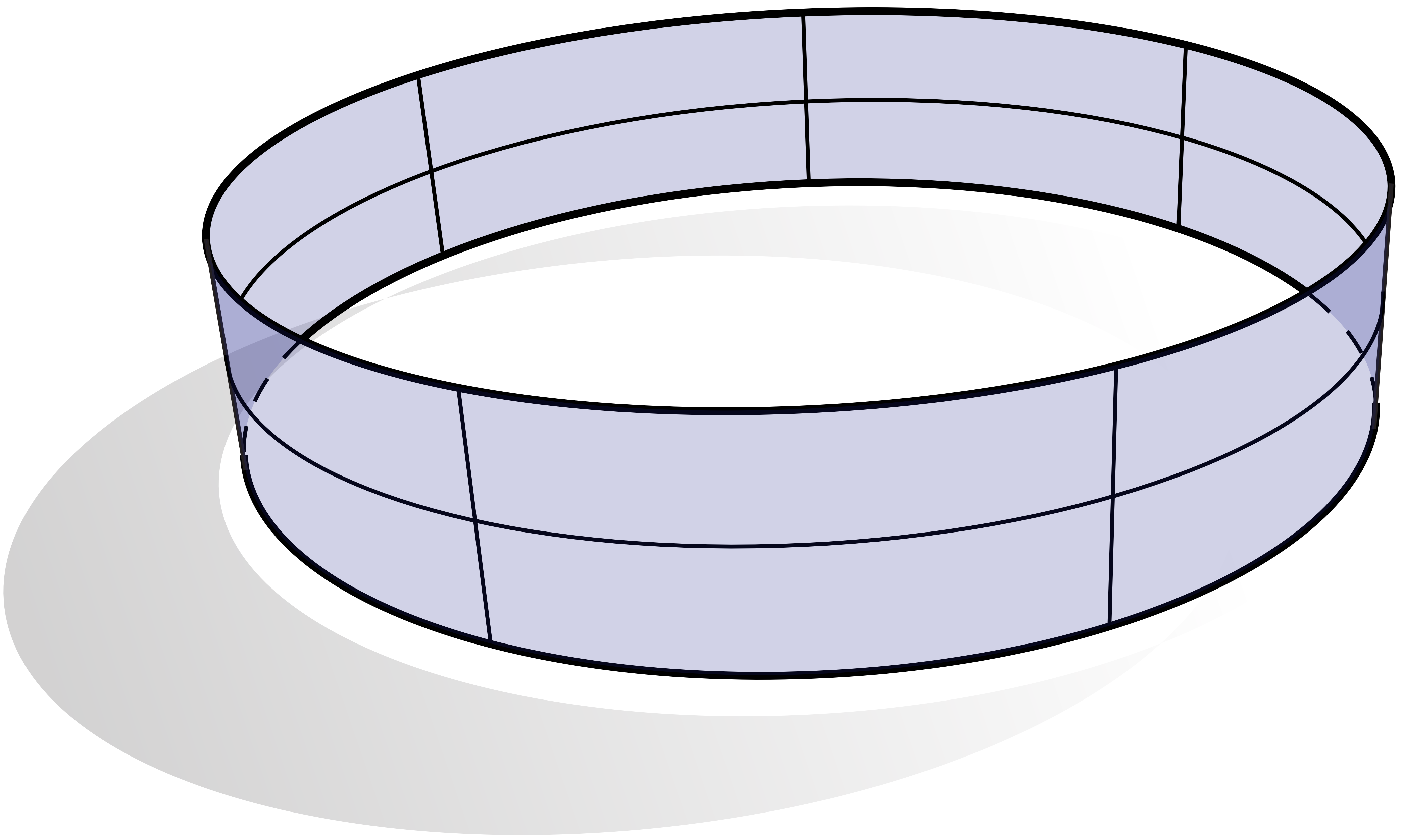
- A vector is **normal** to a surface if it is orthogonal to all tangent vectors
- **Q:** Is there a *unique* normal at a given point?
- **A:** No! Can have different magnitudes/directions.
- The **Gauss map** is a *continuous* map taking each point on the surface to a *unit* normal vector
- Can visualize Gauss map as a map from the surface to the unit sphere

$$\forall X, \langle N, df(X) \rangle = 0$$

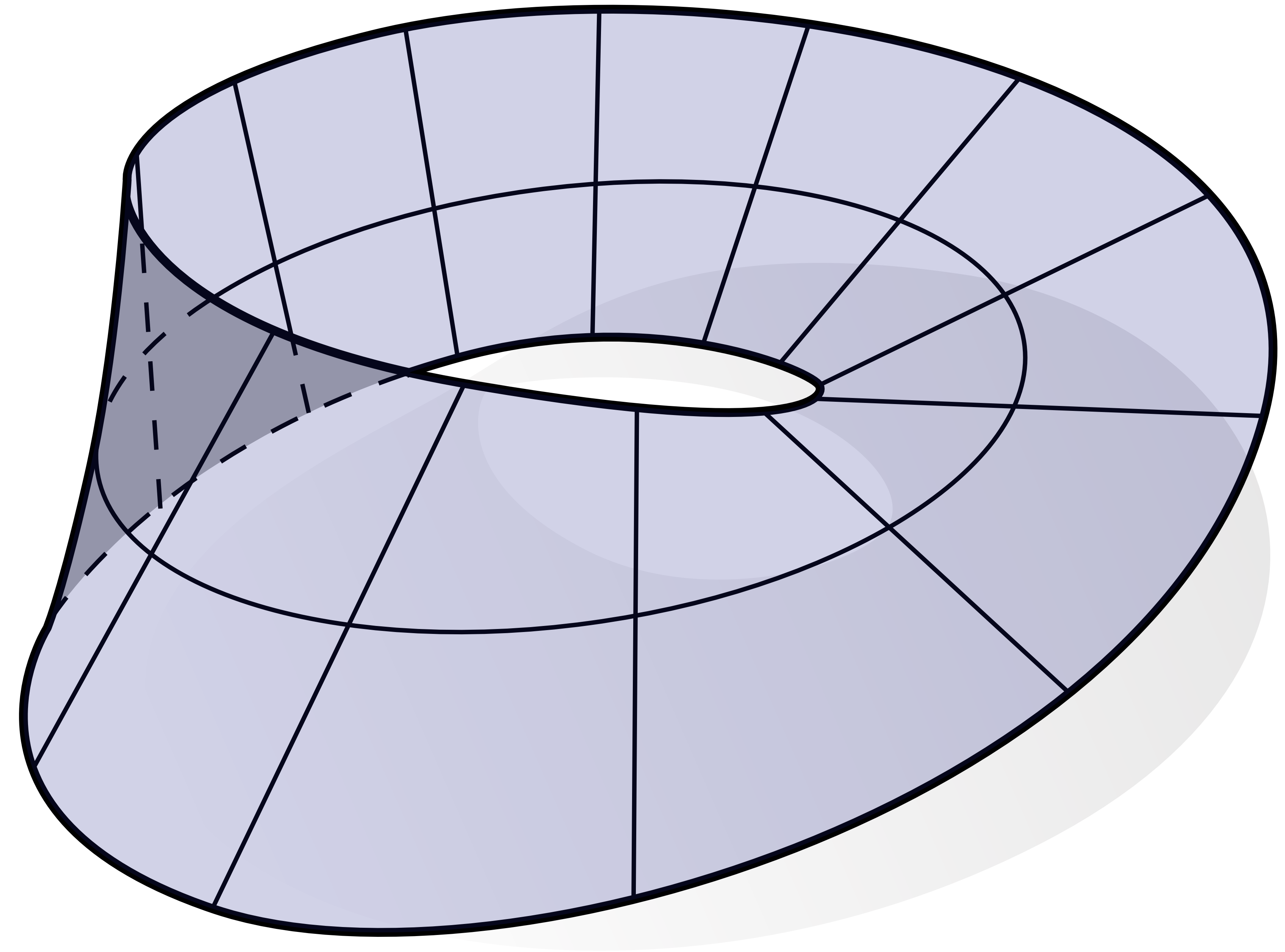


Orientability

Not every surface admits a Gauss map (globally):



orientable



nonorientable

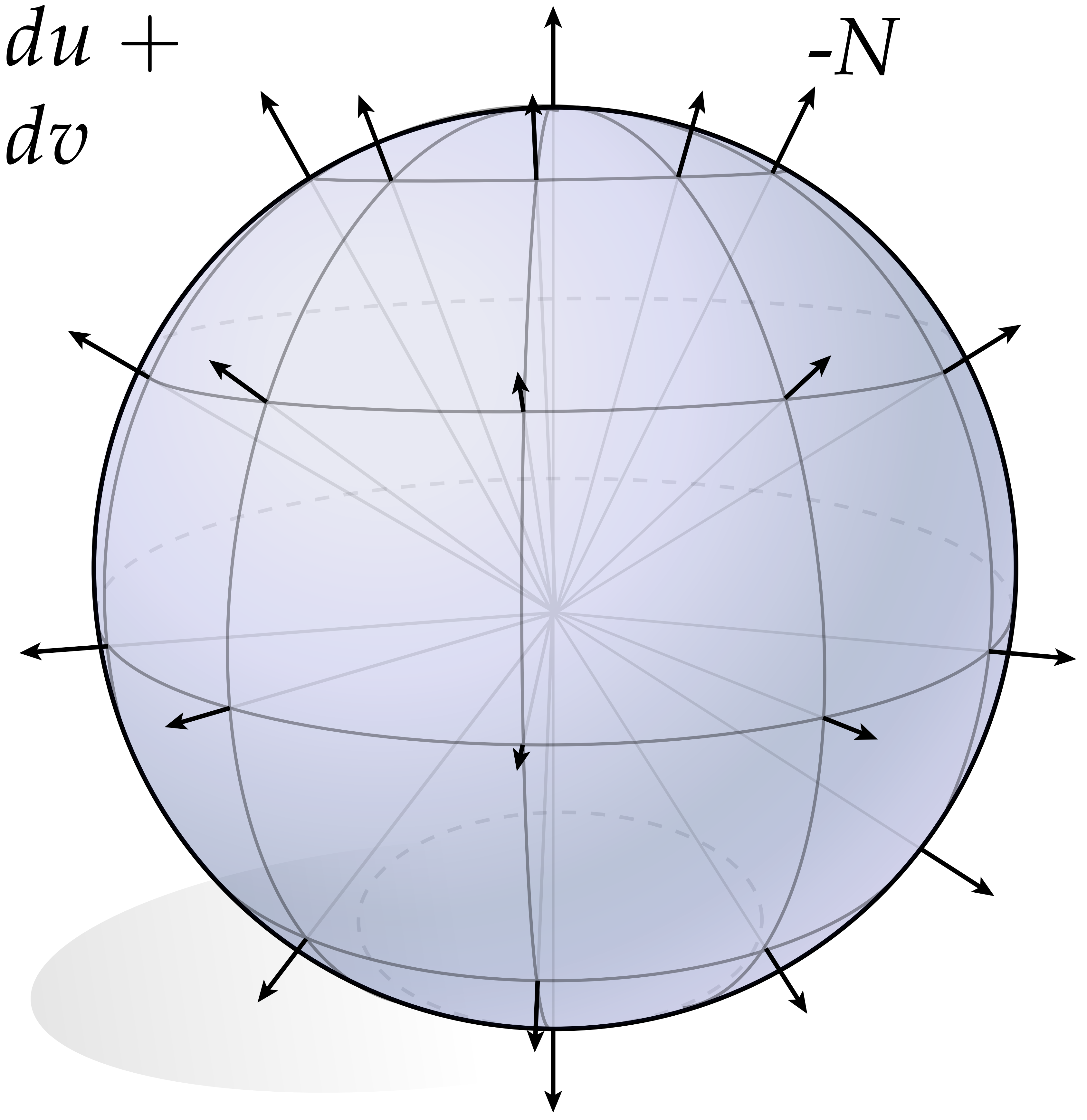
Gauss Map—Example

Can obtain unit normal by taking the cross product of two tangents*:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{array}{lllll} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &)\ du\ + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &)\ dv \end{array}$$

$$df(\tfrac{\partial}{\partial u}) \times df(\tfrac{\partial}{\partial v}) = \begin{bmatrix} -\cos(u)\sin^2(v) \\ -\sin(u)\sin^2(v) \\ -\cos(v)\sin(v) \end{bmatrix}$$



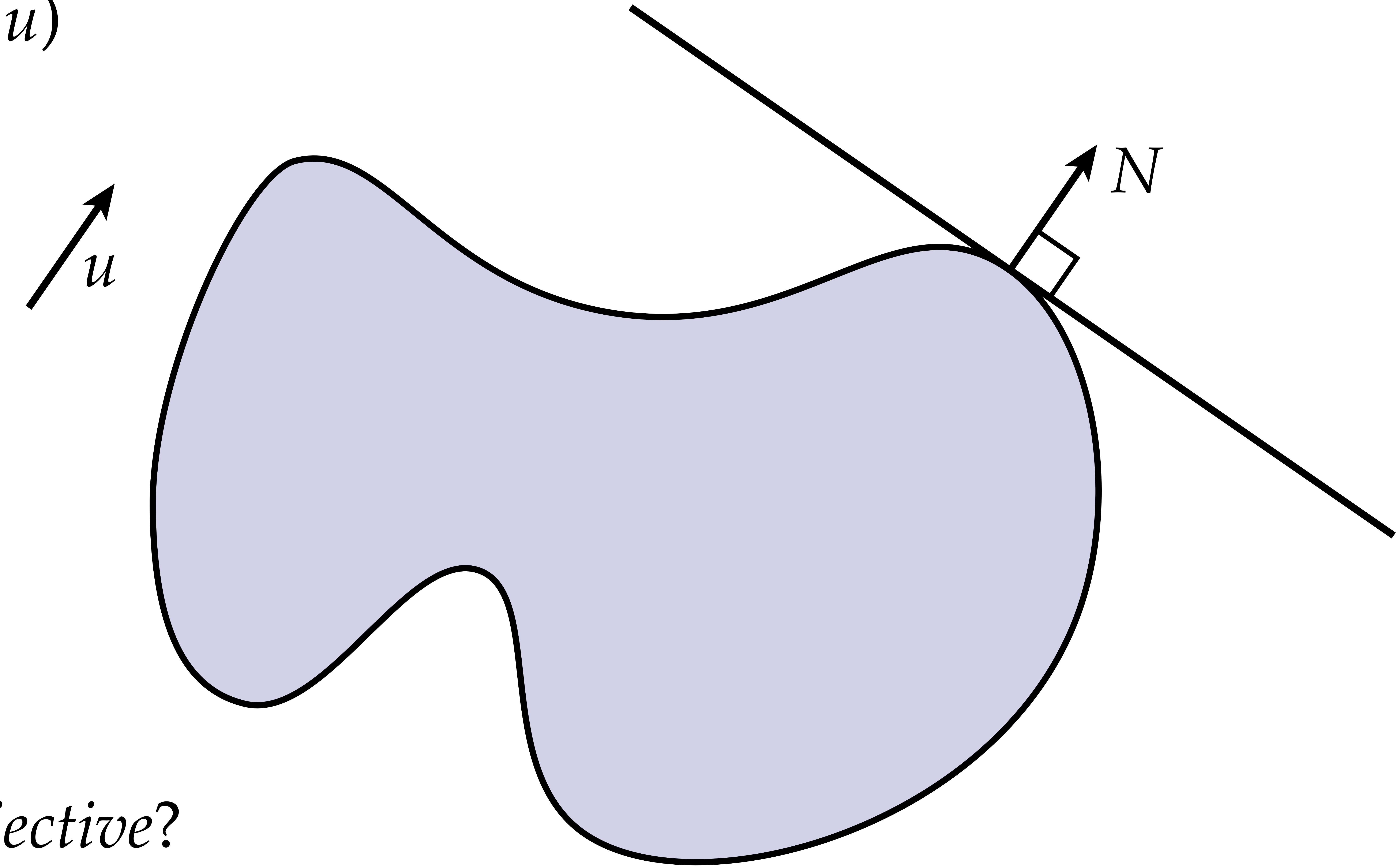
To get *unit* normal, divide by length. In this case, can just notice we have a constant multiple of the sphere itself:

$$\Rightarrow N = -f$$

*Must not be parallel!

Surjectivity of Gauss Map

- Given a unit vector u, can we always find some point on a surface that has this normal? ($N = u$)
- Yes! **Proof** (Hilbert):



Q: Is the Gauss map *injective*?

Vector Area

- Given a little patch of surface Ω, what's the "average normal"?
- Can simply integrate normal over the patch, divide by area:

$$\frac{1}{\text{area}(\Omega)} \int_\Omega N \, dA$$

- Integrand $N\,dA$ is called the **vector area**. (Vector-valued 2-form)
- Can be easily expressed via exterior calculus*:

$$\begin{aligned} df \wedge df(X,Y) = {} & df(X) \times df(Y) - df(Y) \times df(X) = \\ & 2df(X) \times df(Y) = \\ & 2NdA(X,Y) \end{aligned}$$

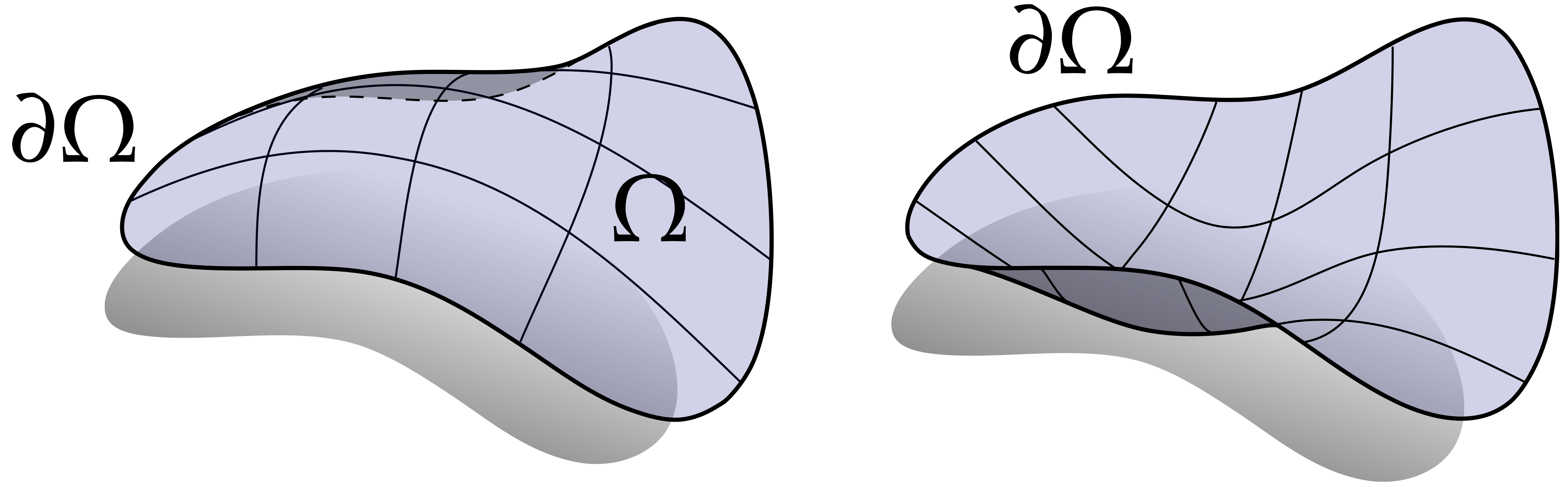
$$\Longrightarrow \boxed{\mathcal{A} = \tfrac{1}{2} df \wedge df}$$

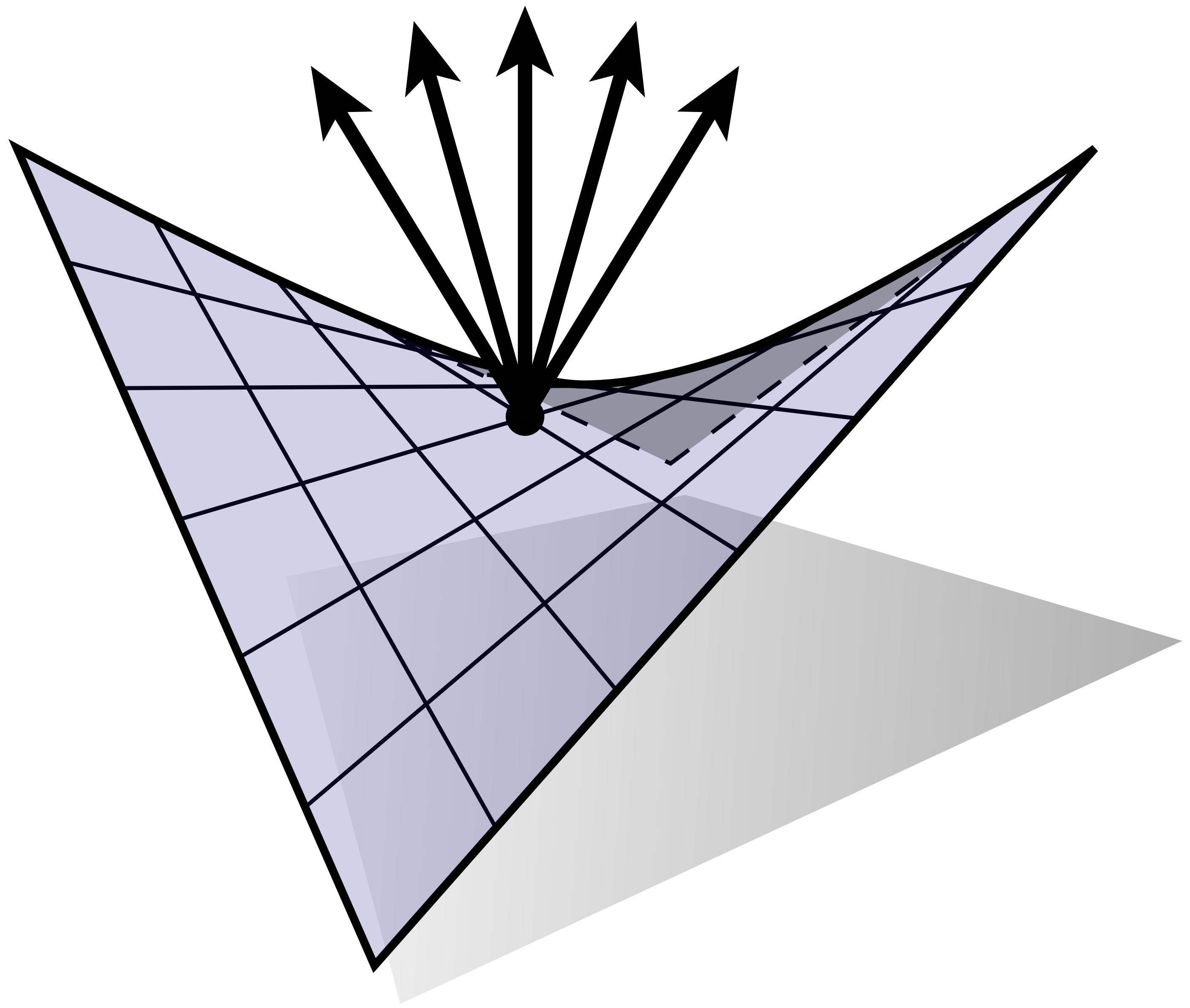
Vector Area, continued

- By expressing vector area this way, we make an interesting observation:

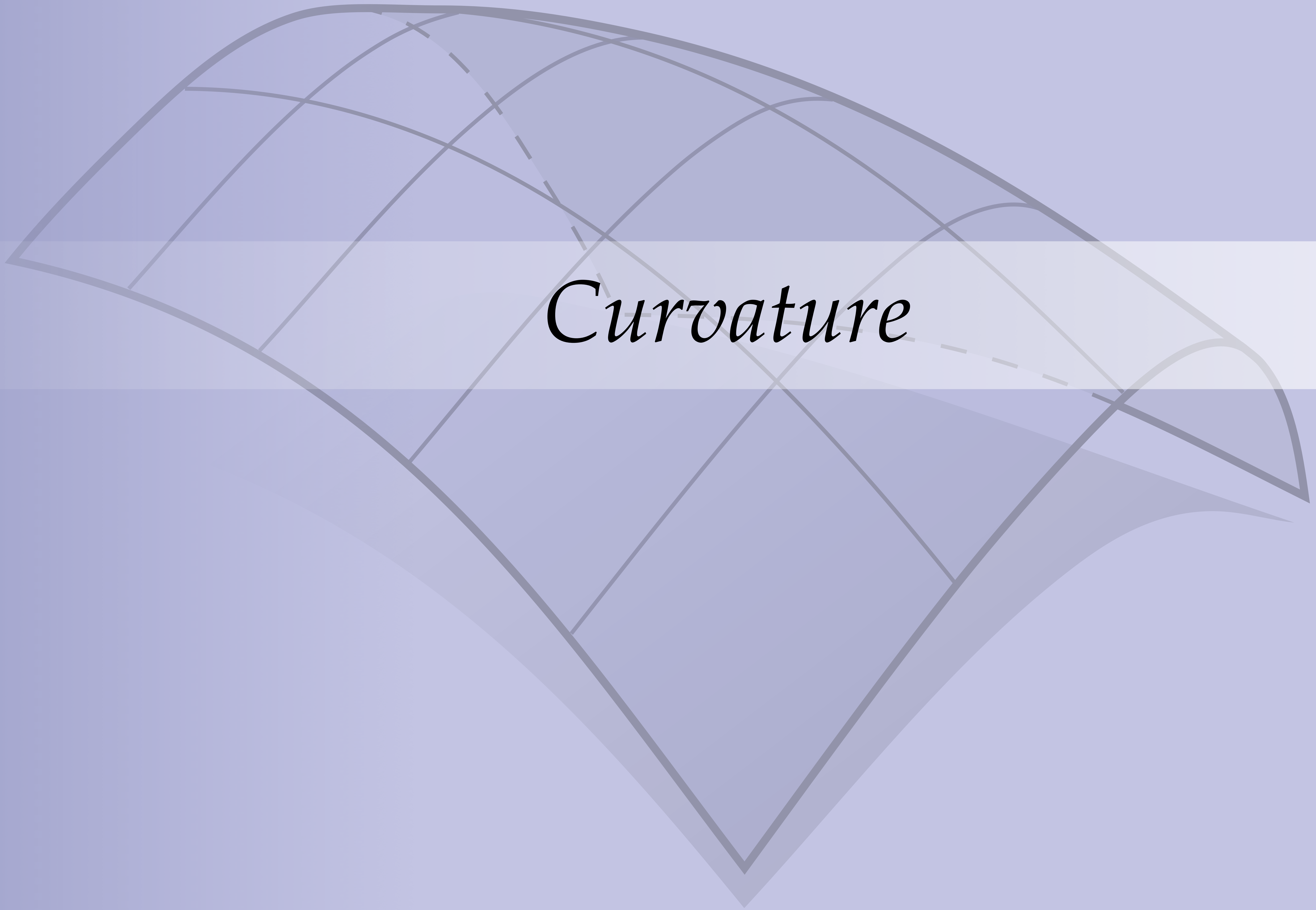
$$2\int_\Omega N\, dA = \int_\Omega df \wedge df = \int_\Omega d(f df) = \int_{\partial\Omega} f df = \int_{\partial\Omega} f(s) \times df(T(s))\, ds$$

- Hence, vector area is the same for any two patches w/ same boundary
- Can define "normal" given **only** boundary (*e.g.*, nonplanar polygon)
- **Corollary:** *integral of normal vanishes for any closed surface*



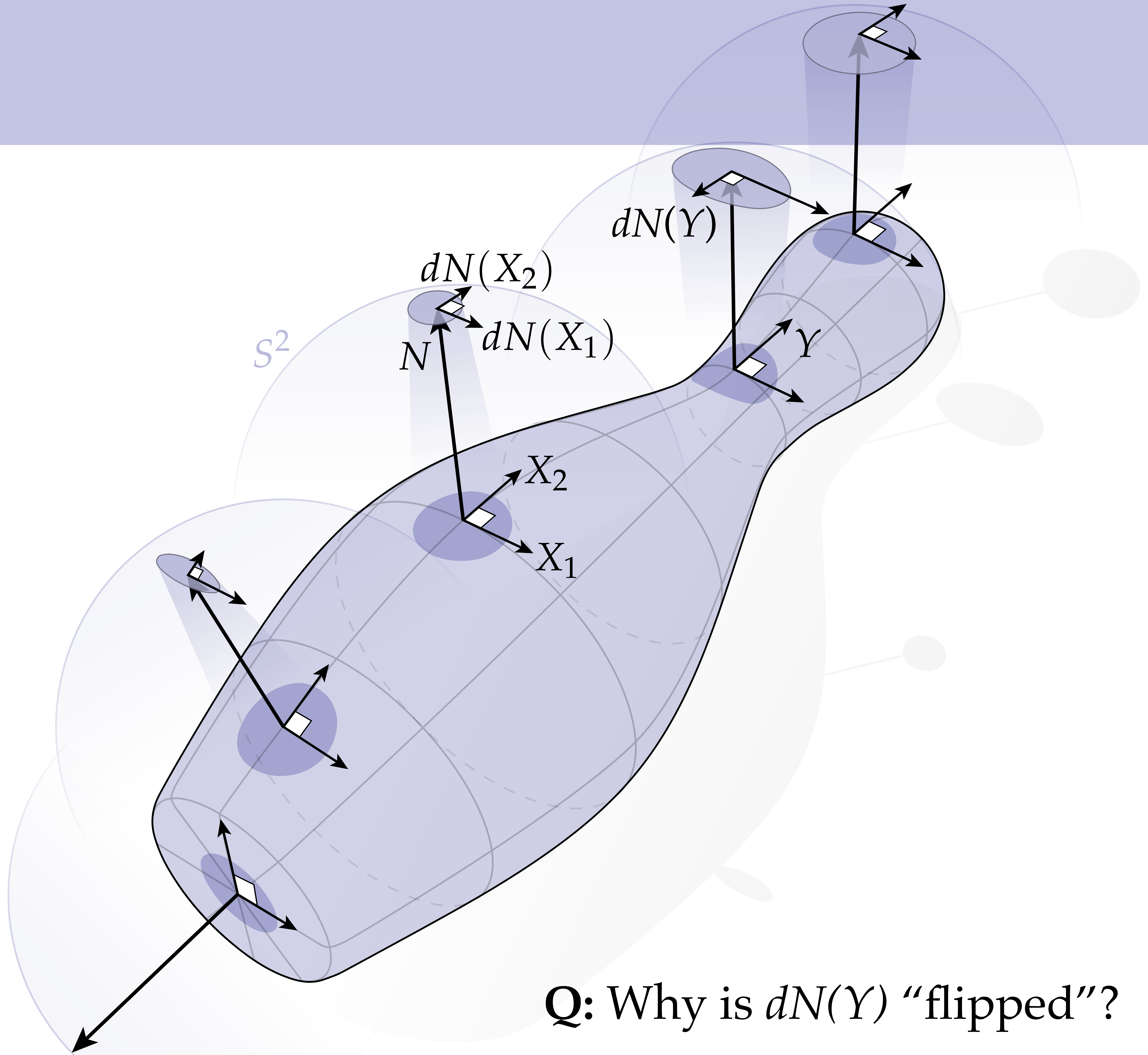


Curvature



Weingarten Map

- The **Weingarten map** dN is the differential of the Gauss map N
- At each point, tells us the change in the normal vector along any given direction X
- Since change in *unit* normal cannot have any component in the normal direction, $dN(X)$ is always tangent to the surface
- Can also think of it as a vector tangent to the unit sphere S^2



Q: Why is $dN(Y)$ “flipped”?

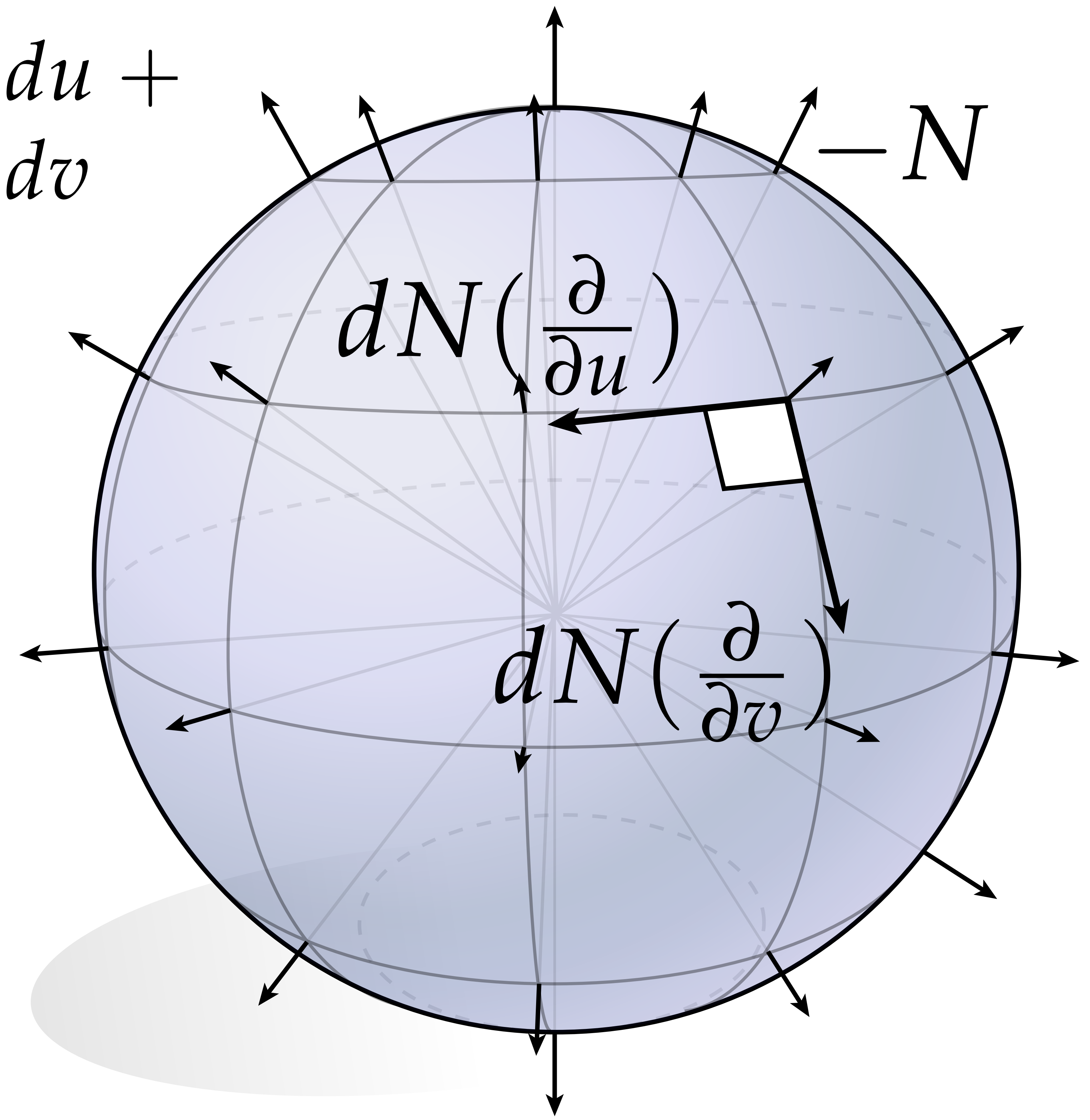
Weingarten Map—Example

- Recall that for the sphere, $N = -f$. Hence, Weingarten map dN is just $-df$:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{array}{l} (\;\; -\sin(u)\sin(v), \;\; \cos(u)\sin(v), \;\; 0 \;\;)\; du + \\ (\;\; \cos(u)\cos(v), \;\; \cos(v)\sin(u), \;\; -\sin(v) \;\;)\; dv \end{array}$$

$$dN = \begin{array}{l} (\;\; \sin(u)\sin(v), \; -\cos(u)\sin(v), \;\; 0 \;\;)\; du \\ (-\cos(u)\cos(v), \; -\cos(v)\sin(u), \; \sin(v) \;\;)\; dv \end{array}$$



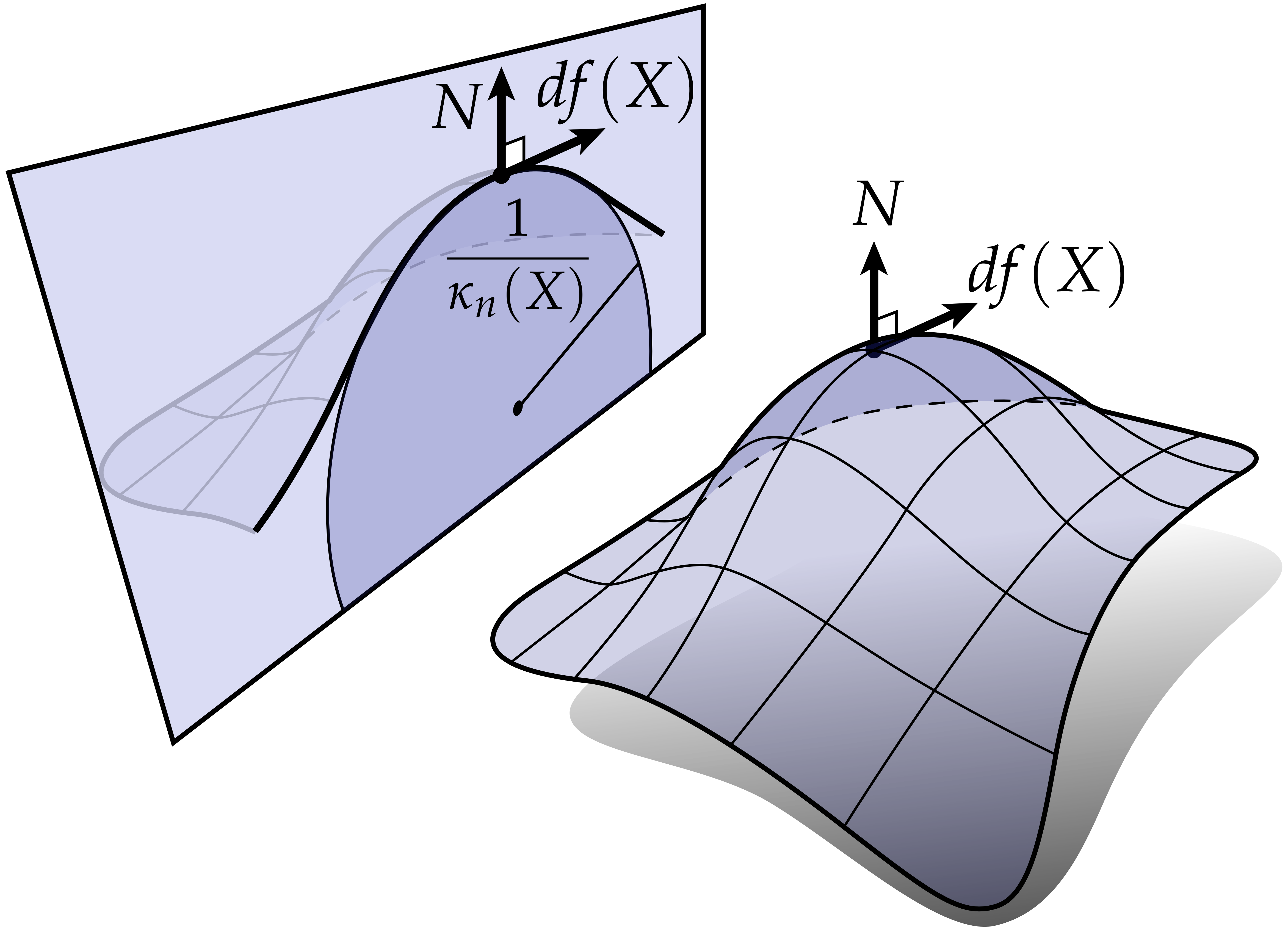
Key idea: computing the Weingarten map is no different from computing the differential of a surface.

Normal Curvature

- For curves, curvature was the rate of change of the *tangent*; for immersed surfaces, we'll instead consider how quickly the *normal* is changing.*
- In particular, **normal curvature** is rate at which normal is bending along a given tangent direction:

$$\kappa_N(X) := \frac{\langle df(X), dN(X)\rangle}{|df(X)|^2}$$

- Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve



*For plane curves, what would happen if we instead considered change in N?

Normal Curvature—Example

Consider a parameterized cylinder:

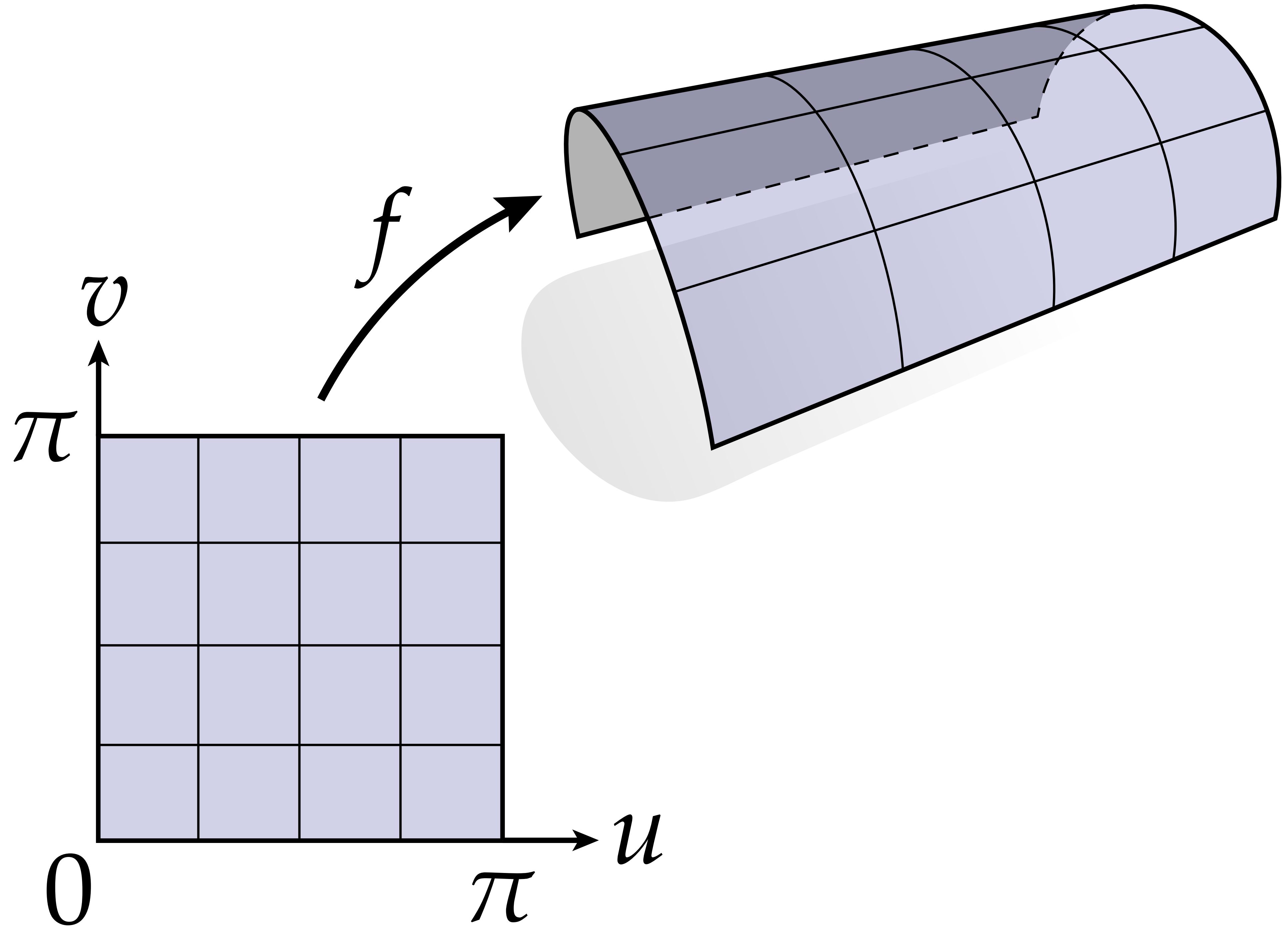
$$f(u,v) := (\cos(u), \sin(u), v)$$

$$df = (-\sin(u), \cos(u), 0)du + (0,0,1)dv$$

$$N = (-\sin(u), \cos(u), 0) \times (0,0,1)$$

$$= (\cos(u), \sin(u), 0)$$

$$dN = (-\sin(u), \cos(u), 0)du$$



$$\kappa_N(\tfrac{\partial}{\partial u}) = \frac{\langle df(\frac{\partial}{\partial u}), dN(\frac{\partial}{\partial u})\rangle}{|df(\frac{\partial}{\partial u})|^2} = \frac{(-\sin(u),\cos(u),0)\cdot(-\sin(u),\cos(u),0)}{|(-\sin(u),\cos(u),0)|^2} = 1$$

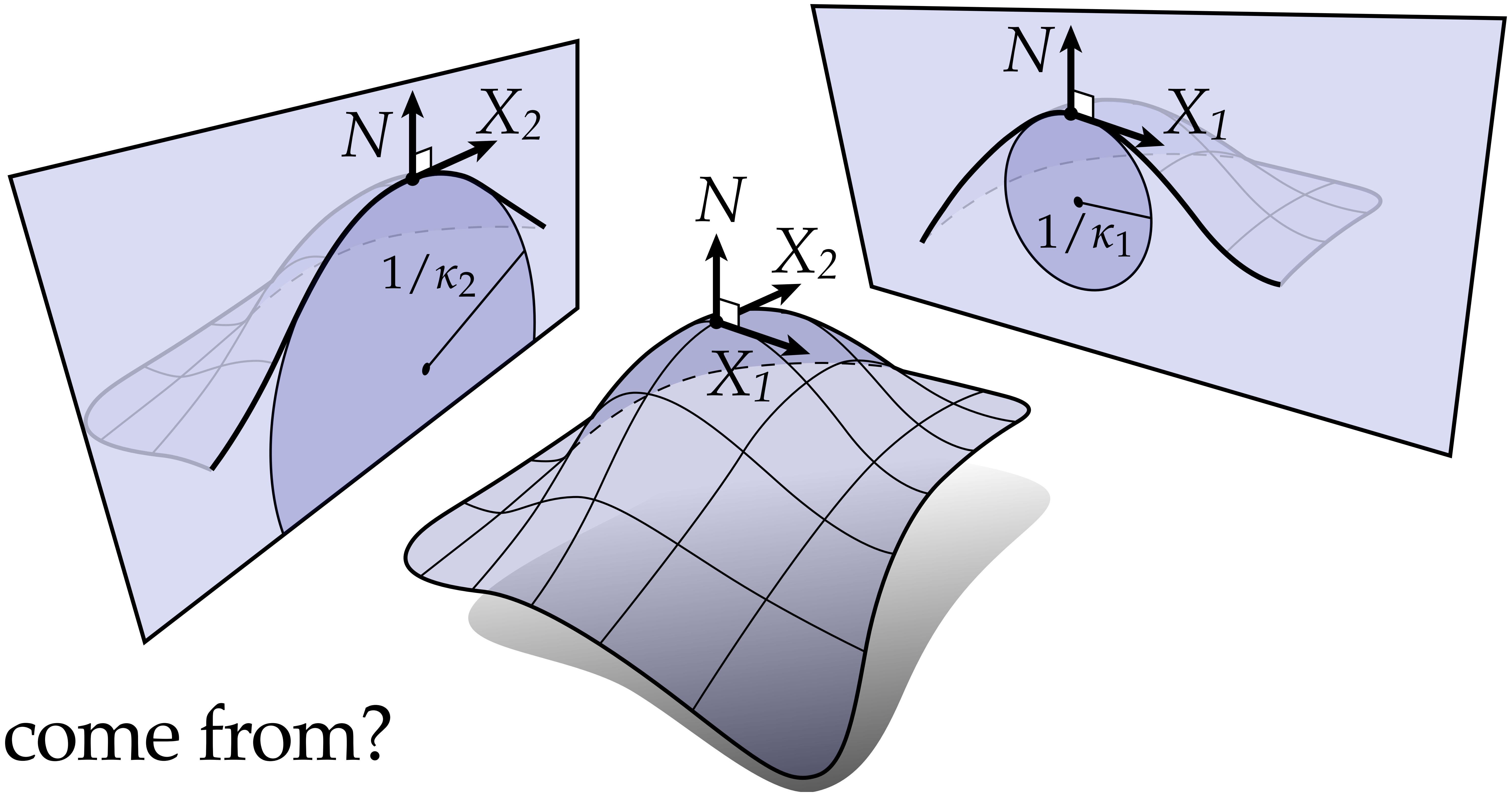
$$\kappa_N(\tfrac{\partial}{\partial v}) = \cdots = 0$$

Q: Does this result make sense geometrically?

Principal Curvature

- Among all directions X, there are two **principal directions** X_1, X_2 where normal curvature has minimum/maximum value (respectively)
- Corresponding normal curvatures are the **principal curvatures**
- Two critical facts*:

1. $g(X_1, X_2) = 0$
2. $dN(X_i) = \kappa_i df(X_i)$



Where do these relationships come from?

Shape Operator

- The change in the normal N is always *tangent* to the surface
- Must therefore be some linear map S from tangent vectors to tangent vectors, called the **shape operator**, such that

$$df(SX) = dN(X)$$

- Principal directions are the *eigenvectors* of S
- Principal curvatures are *eigenvalues* of S
- **Note:** S is not a symmetric matrix! Hence, eigenvectors are not orthogonal in R^2; only orthogonal with respect to induced metric g.

Shape Operator — Example

Consider a nonstandard parameterization of the cylinder (*sheared* along z):

$$f(u,v) := (\cos(u), \sin(u), u+v) \qquad df = (-\sin(u), \cos(u), 1)du + (0,0,1)dv$$

$$N = (\cos(u), \sin(u), 0) \qquad dN = (-\sin(u), \cos(u), 0)du$$

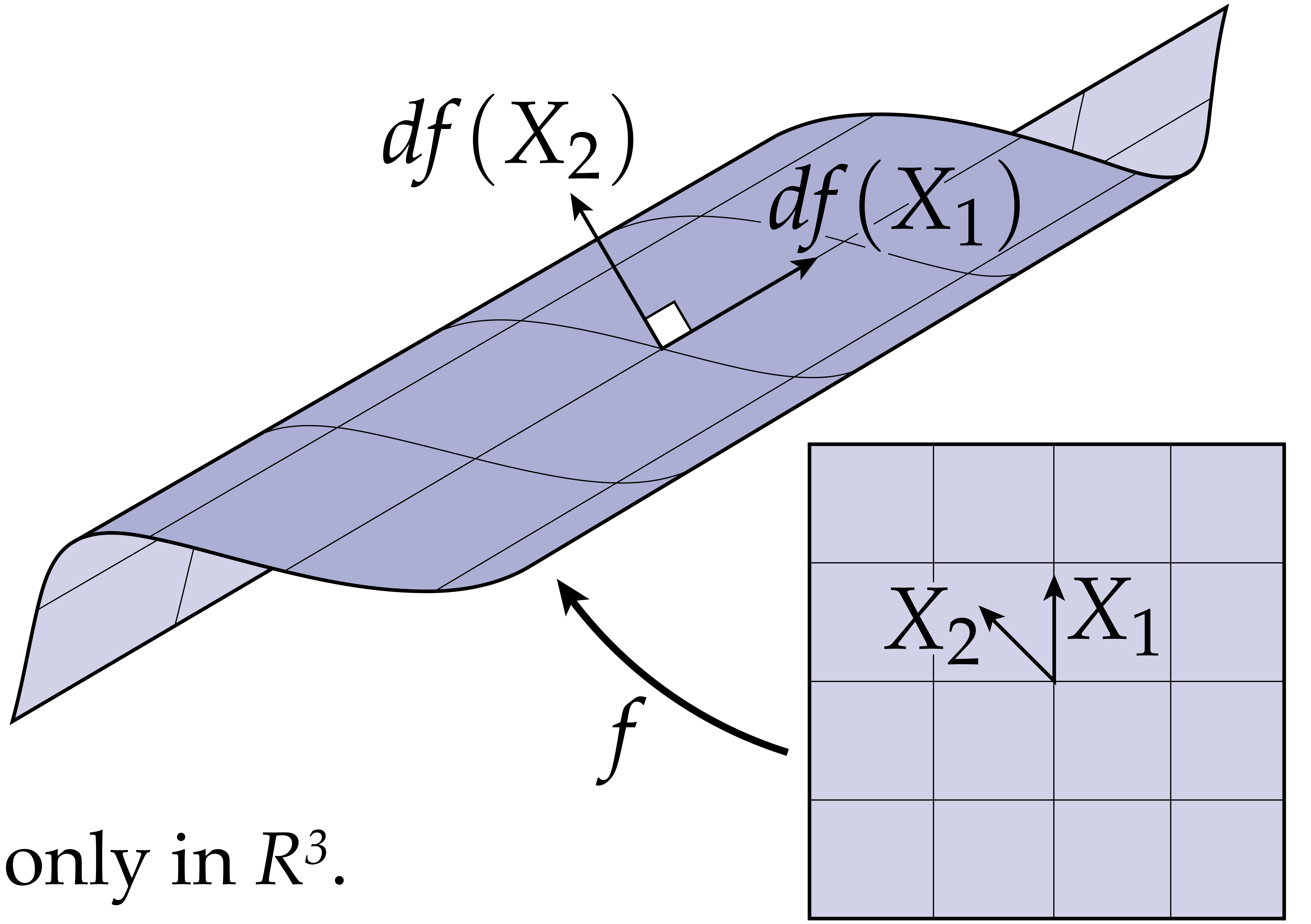
$$\boxed{df \circ S = dN}$$

$$\begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$df(X_1) = (0,0,1) \qquad \kappa_1 = 0$$

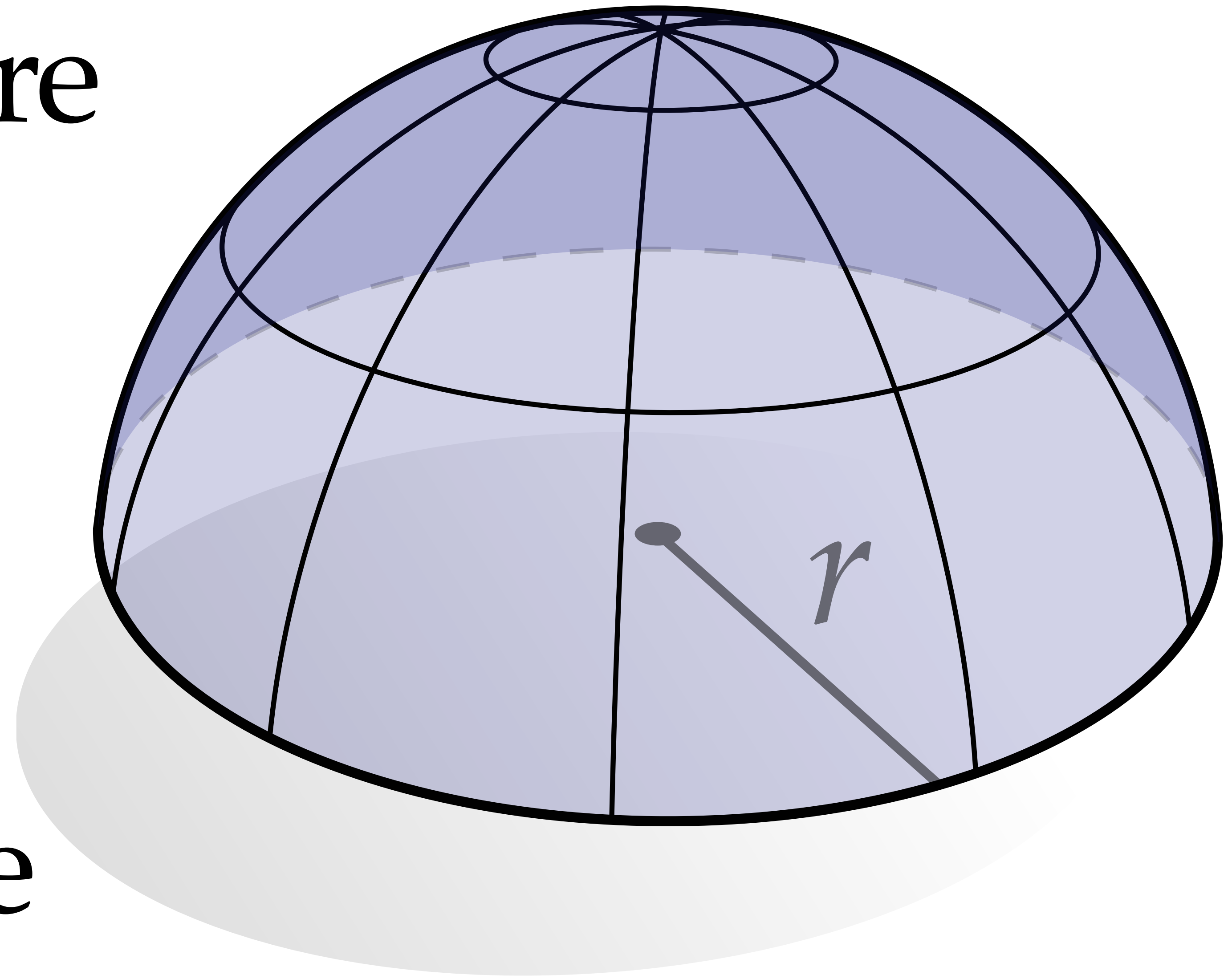
$$df(X_2) = (\sin(u), -\cos(u), 0) \qquad \kappa_2 = 1$$



Key observation: principal directions orthogonal only in R^3.

Umbilic Points

- Points where principal curvatures are equal are called **umbilic points**
- Principal *directions* are not uniquely determined here
- What happens to the shape operator S?
  - May still have full rank!
  - Just have repeated eigenvalues, 2-dim. eigenspace

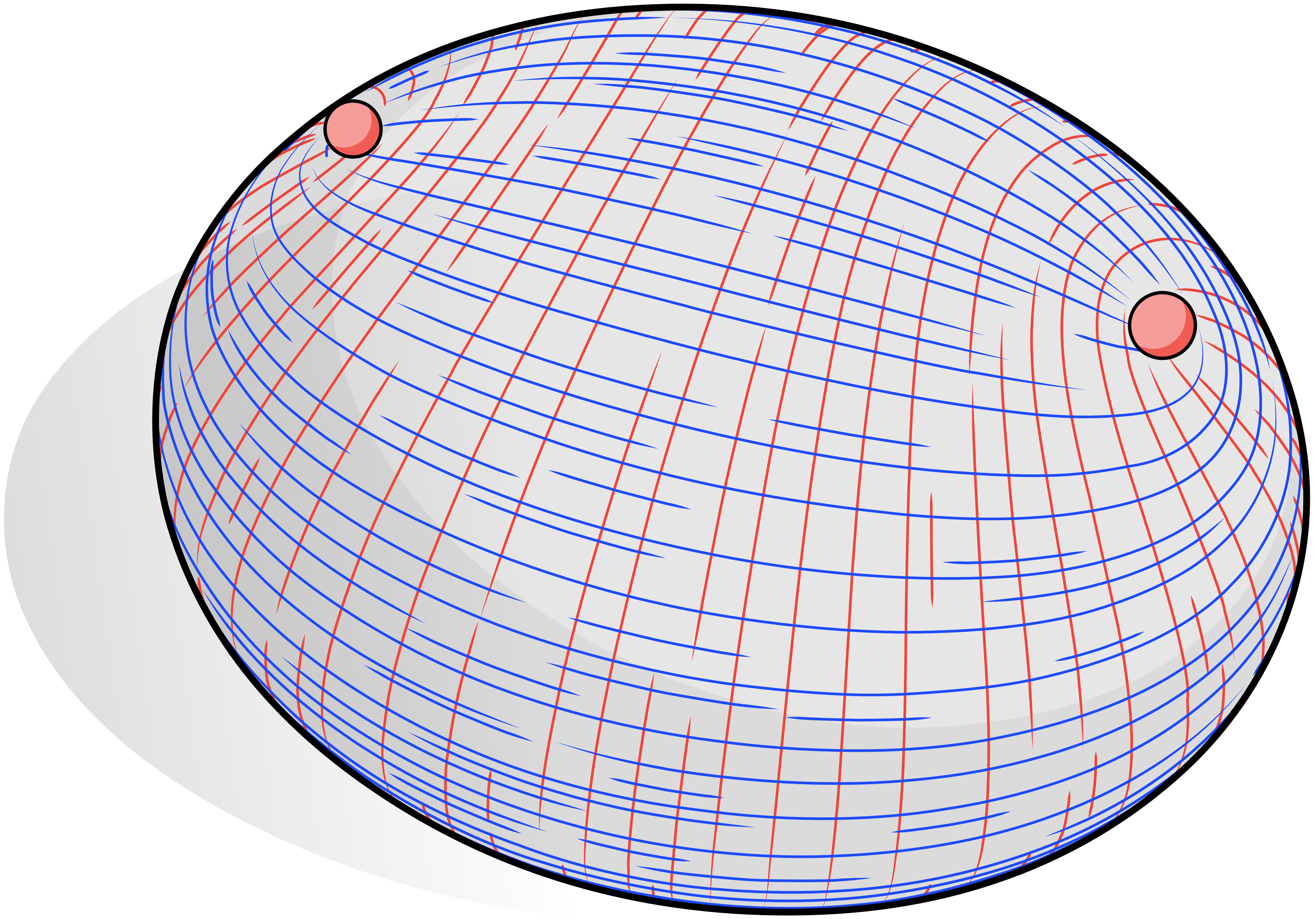


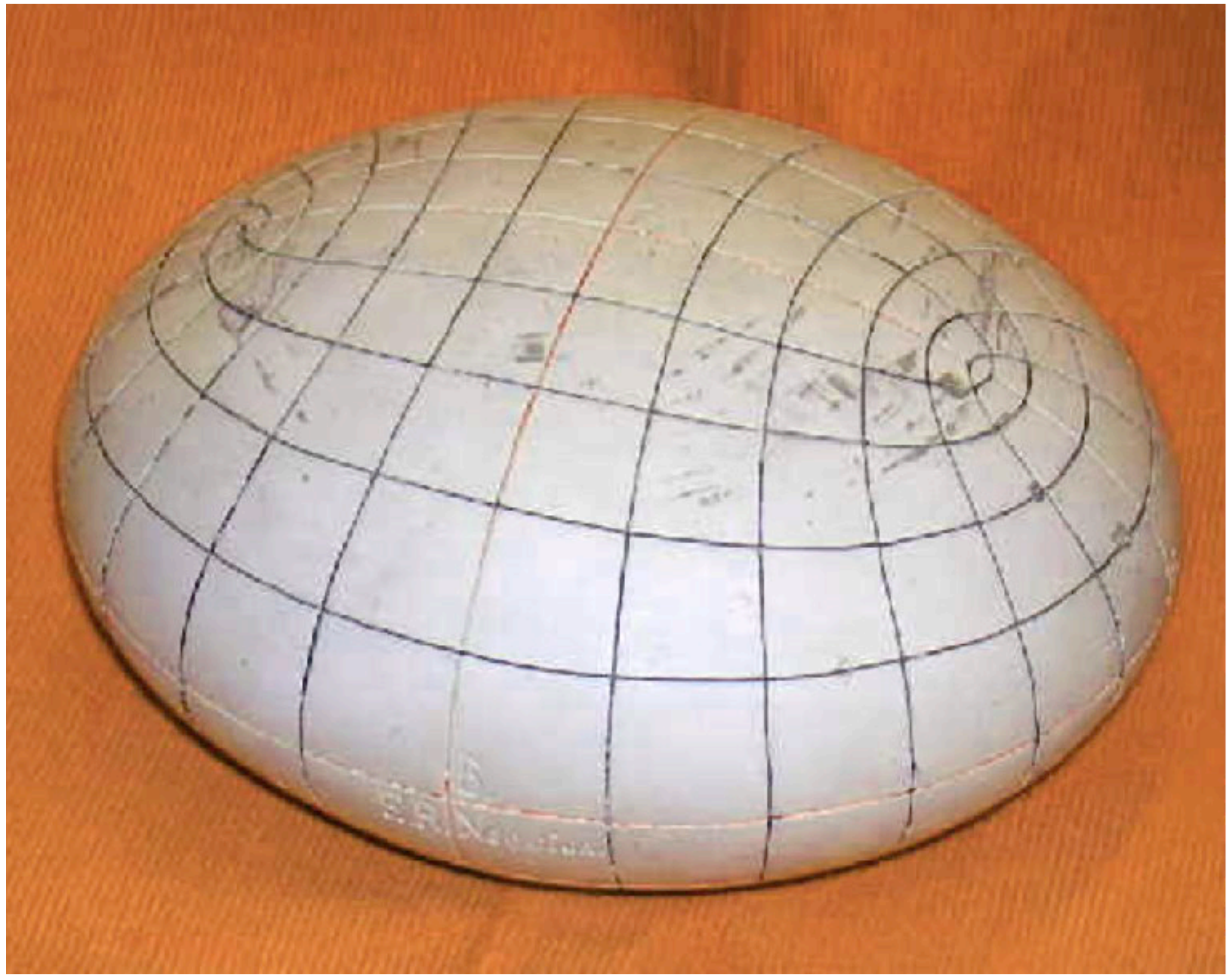
$$S = \begin{bmatrix} 1/r & 0 \\ 0 & 1/r \end{bmatrix} \qquad \kappa_1 = \kappa_2 = \frac{1}{r} \qquad \forall X,\ SX = \tfrac{1}{r}X$$

Could still of course choose (arbitrarily) an orthonormal pair $X_1, X_2 \ldots$

Principal Curvature Nets

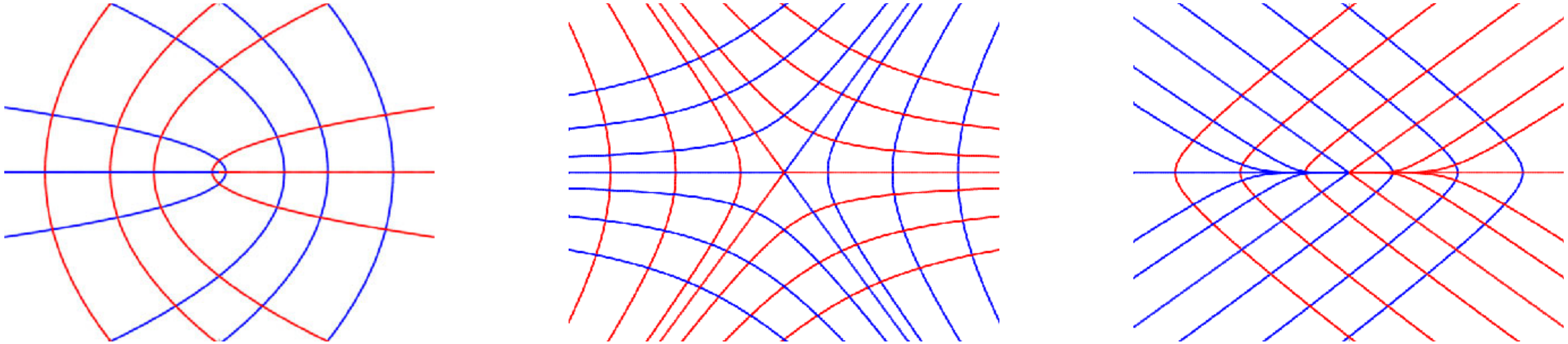
- Walking along principal direction field yields **principal curvature lines**
- Collection of all such lines is called the **principal curvature network**





Topological Invariance of Umbilic Count

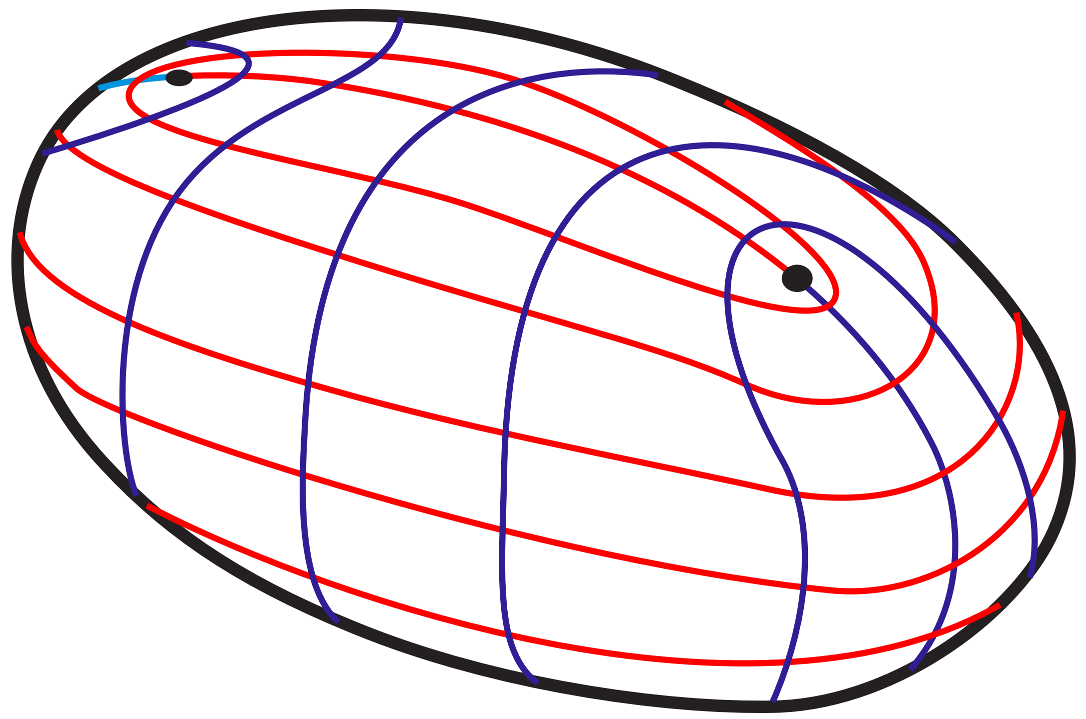
Can classify regions around umbilics into three types based on behavior of principal network: *lemon*, *star*, and *monstar*



lemon (k_1) **star** (k_2) **monstar** (k_3)

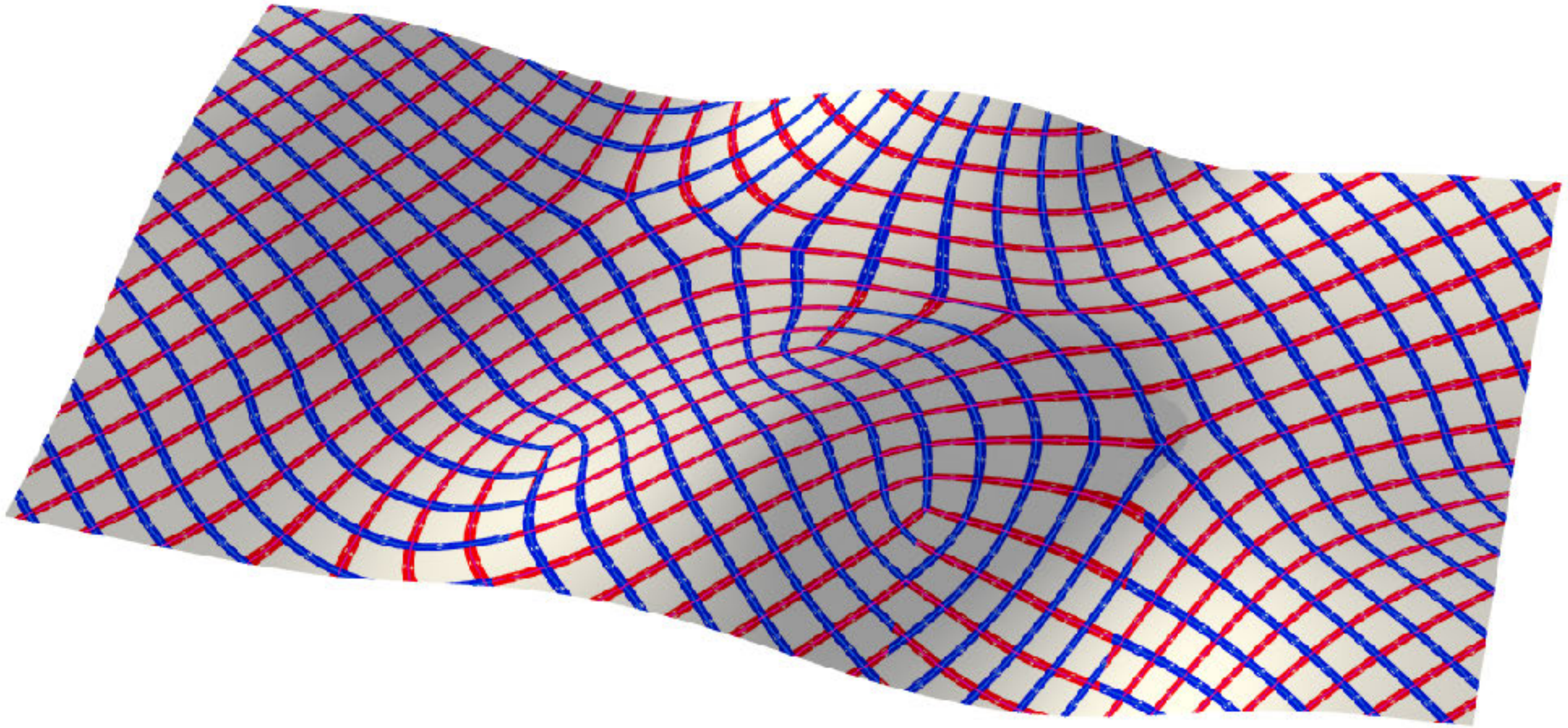
Fact. If k_1, k_2, k_3 are number of umbilics of each type, then

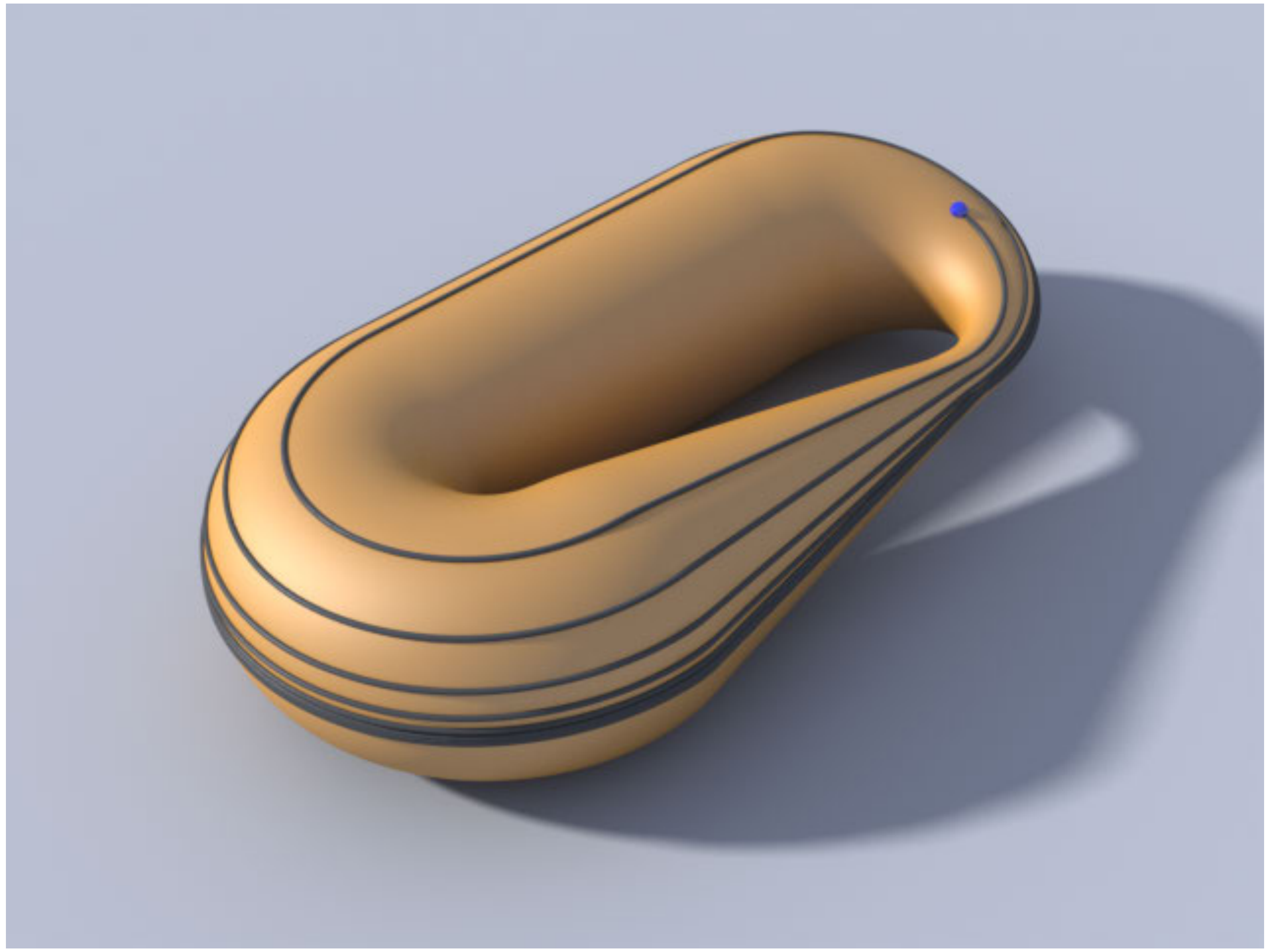
$$\boxed{\kappa_1 - \kappa_2 + \kappa_3 = 2\chi}$$



Separatrices and Spirals

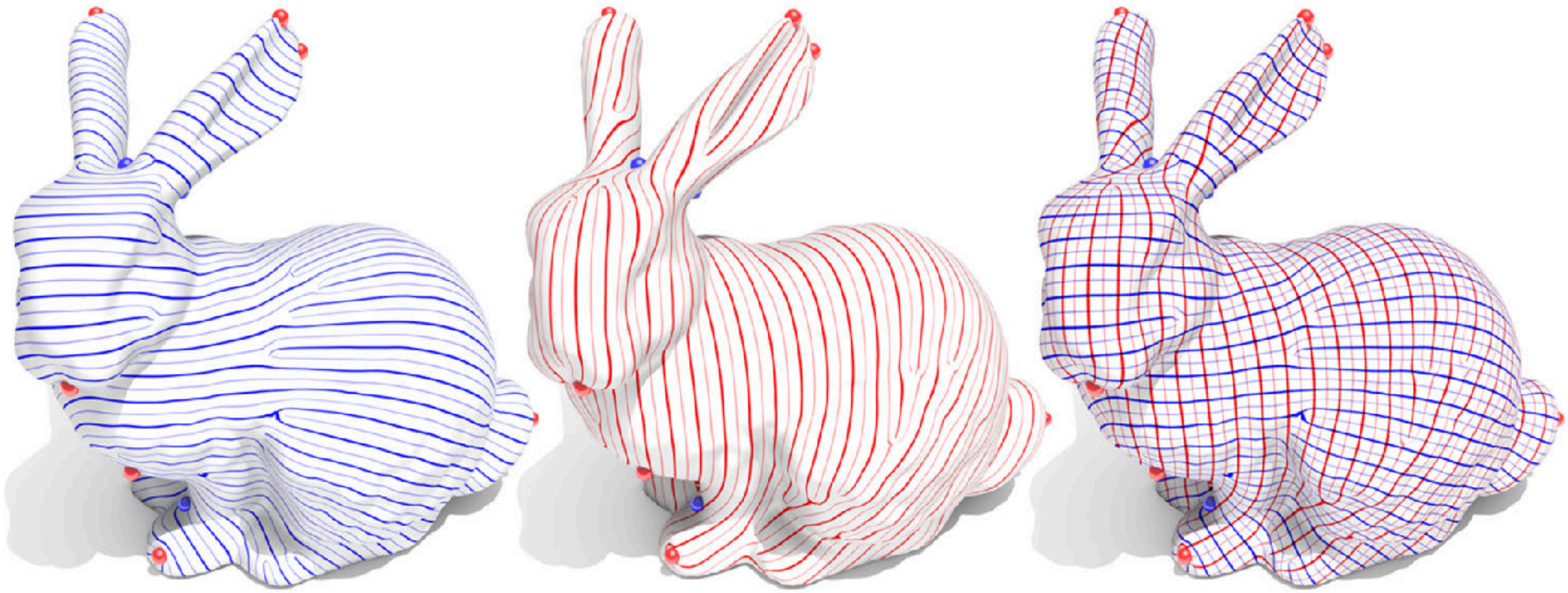
- If we walk along a principal curvature line, where do we end up?
- Sometimes, a curvature line terminates at an umbilic point in both directions; these so-called **separatrices** (can) split network into regular patches.
- Other times, we make a closed loop. More often, however, behavior is *not* so nice!





Application—Quad Remeshing

- Recent approach to meshing: construct net *roughly* aligned with principal curvature—but with separatrices & loops, not spirals.



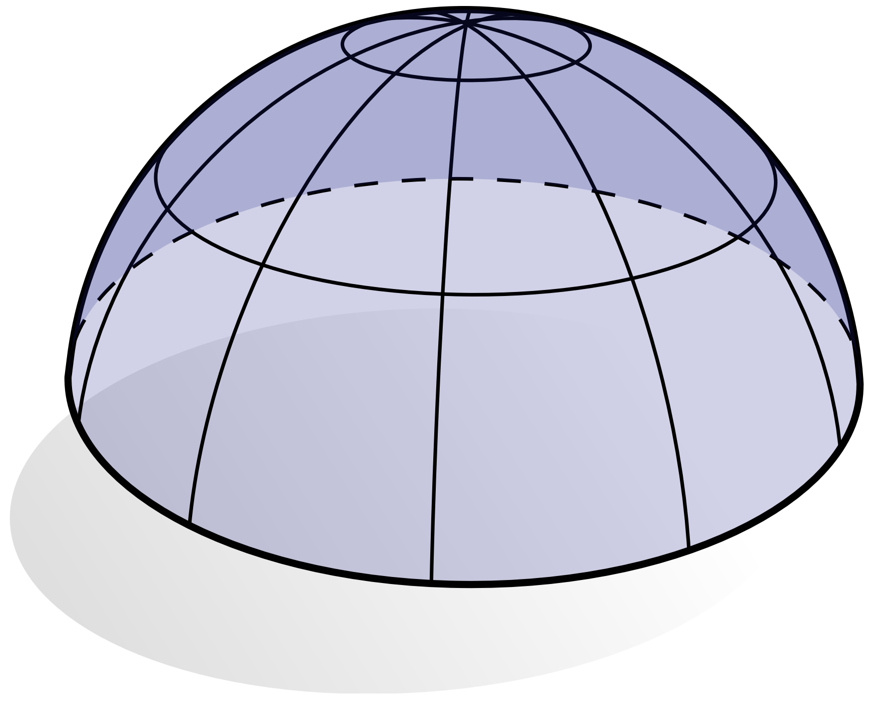
from Knöppel, Crane, Pinkall, Schröder, *"Stripe Patterns on Surfaces"*

Gaussian and Mean Curvature

Gaussian and *mean* curvature also fully describe local bending:

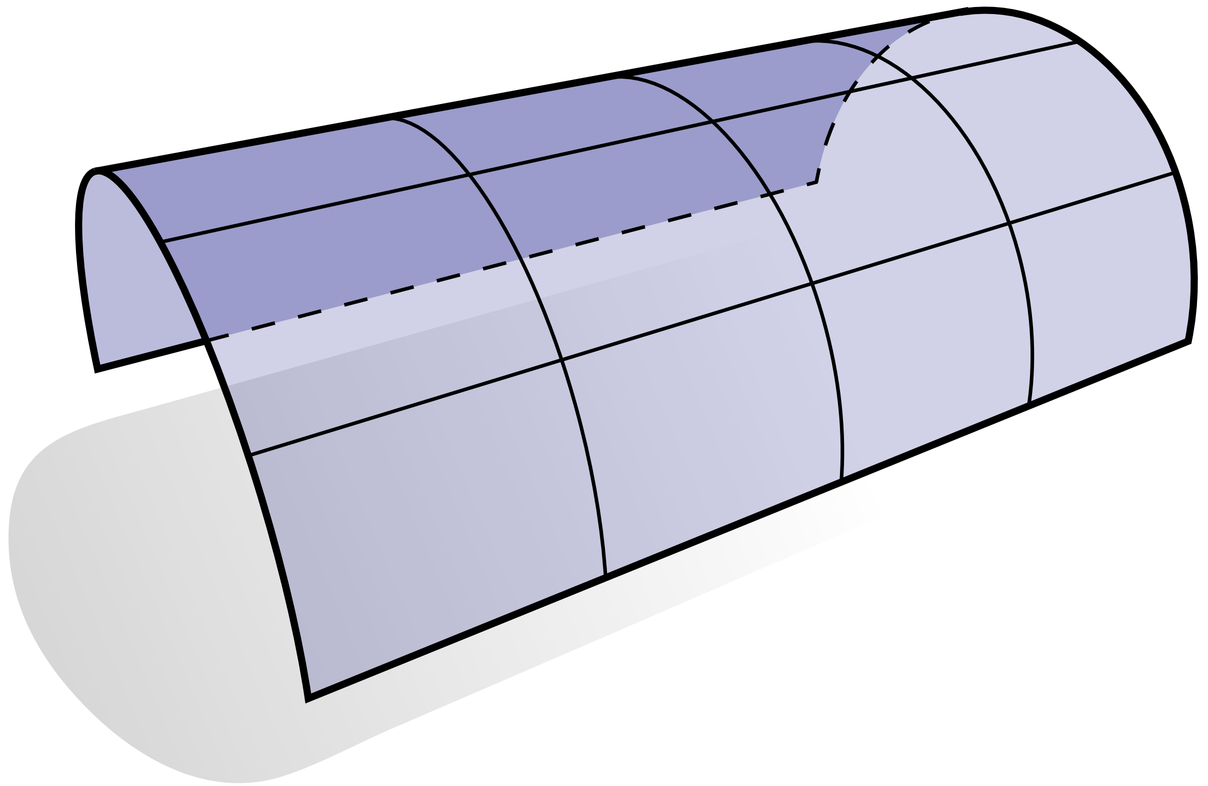
Gaussian $K := \kappa_1 \kappa_2$

mean* $H := \frac{1}{2}(\kappa_1 + \kappa_2)$



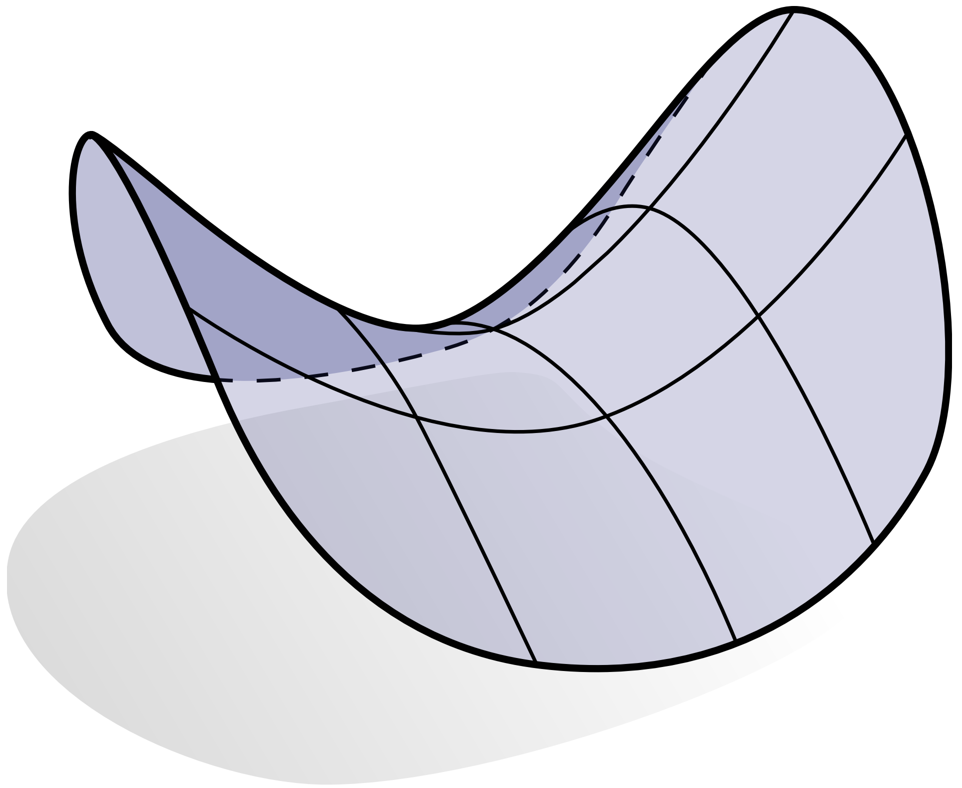
$K > 0$

$H \neq 0$



"developable" $K = 0$

$H \neq 0$



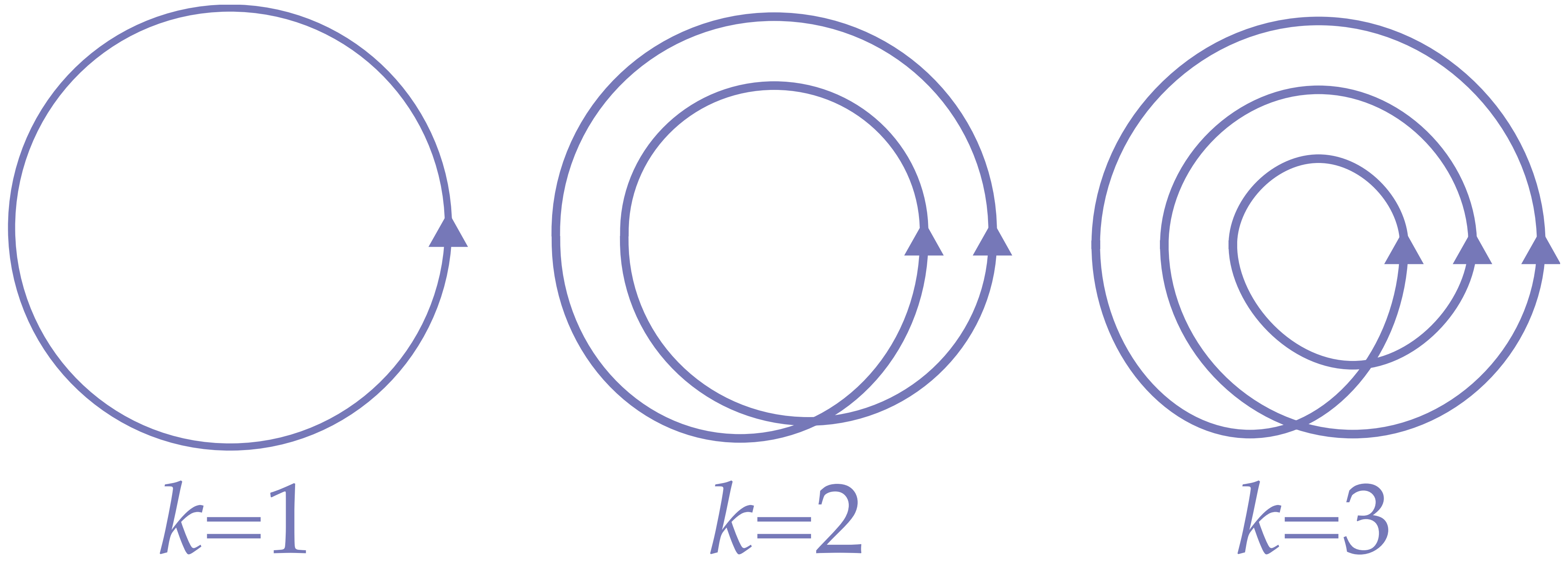
$K < 0$

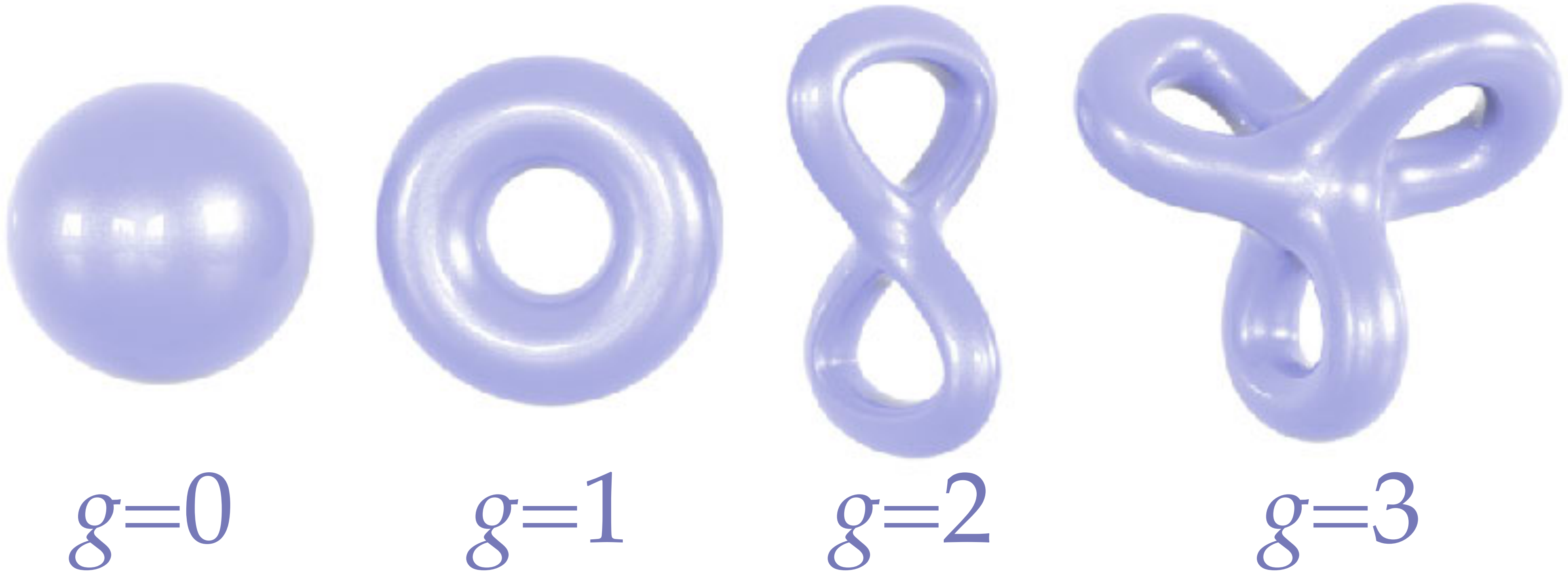
"minimal" $H = 0$

***Warning:** another common convention is to omit the factor of $1/2$

Gauss-Bonnet Theorem

- Recall that the total curvature of a closed plane curve was always equal to 2π times *turning number k*
- **Q:** Can we make an analogous statement about surfaces?
- **A:** Yes! Gauss-Bonnet theorem says total Gaussian curvature is always 2π times *Euler characteristic* χ
- Euler characteristic can be expressed in terms of the *genus* (number of "handles")





<u>Curves</u>

$$\int_0^L \kappa\, ds = 2\pi k$$

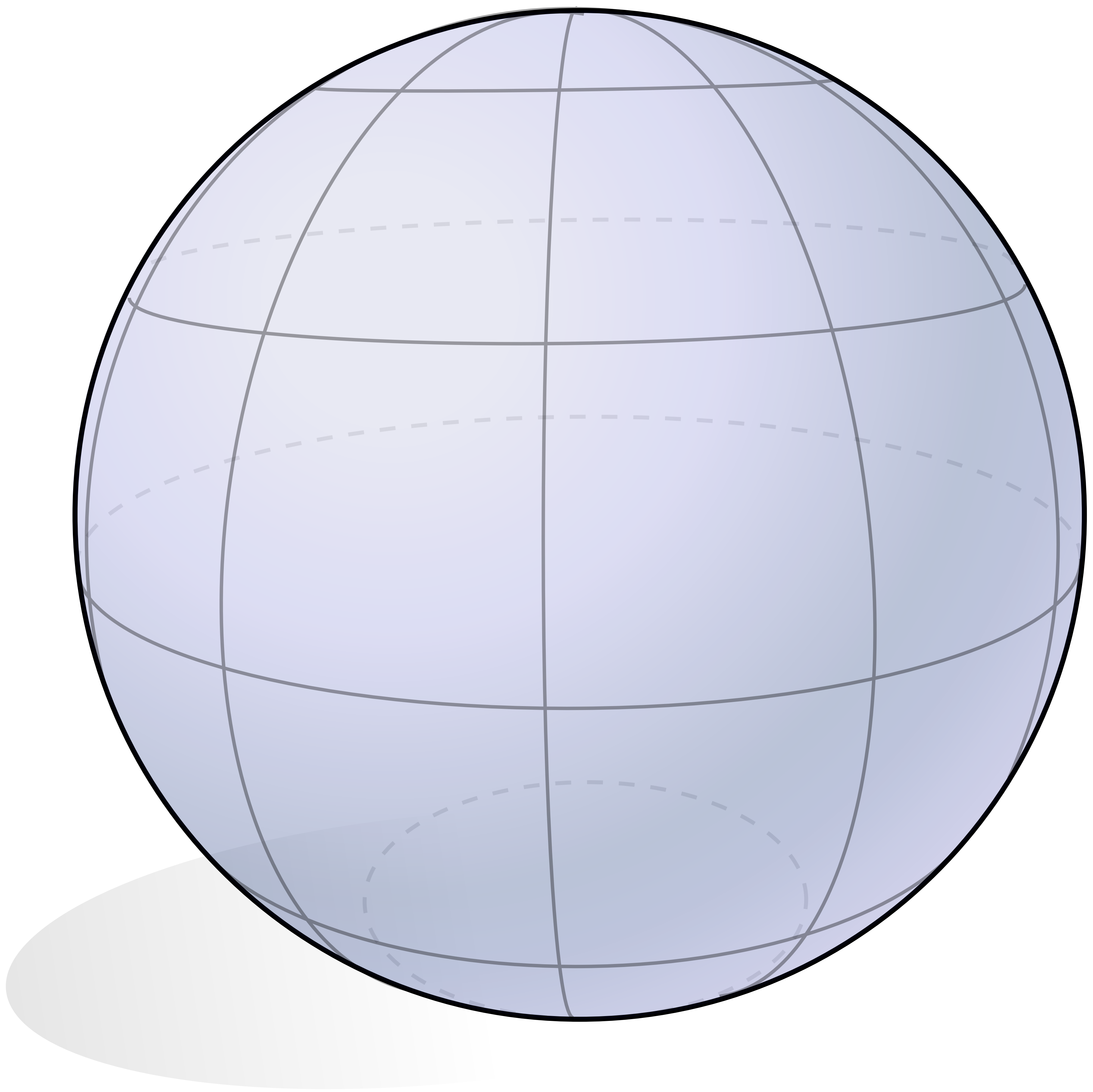
<u>Surfaces</u>

$$\int_M K\, dA = 2\pi\chi$$

Total Mean Curvature?

Theorem (Minkowski): for a regular closed embedded surface,

$$\int_M H \, dA \geq \sqrt{4\pi A}$$



Q: When do we get equality?

A: For a sphere.

Curvature of a Curve in a Surface

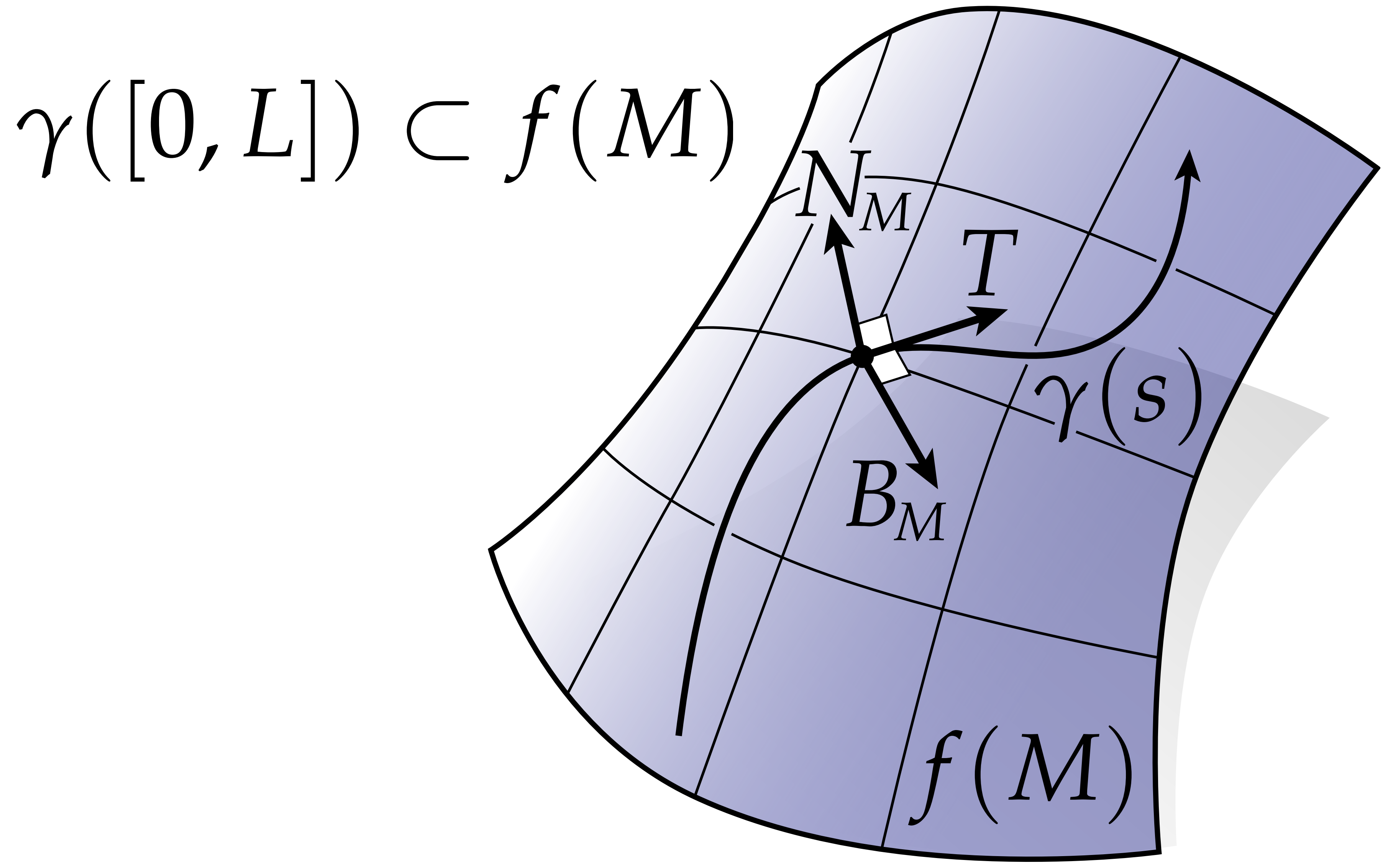
- Earlier, broke the "bending" of a space curve into curvature (κ) and torsion (τ)
- For a curve *in a surface*, can instead break into *normal* and *geodesic* curvature:

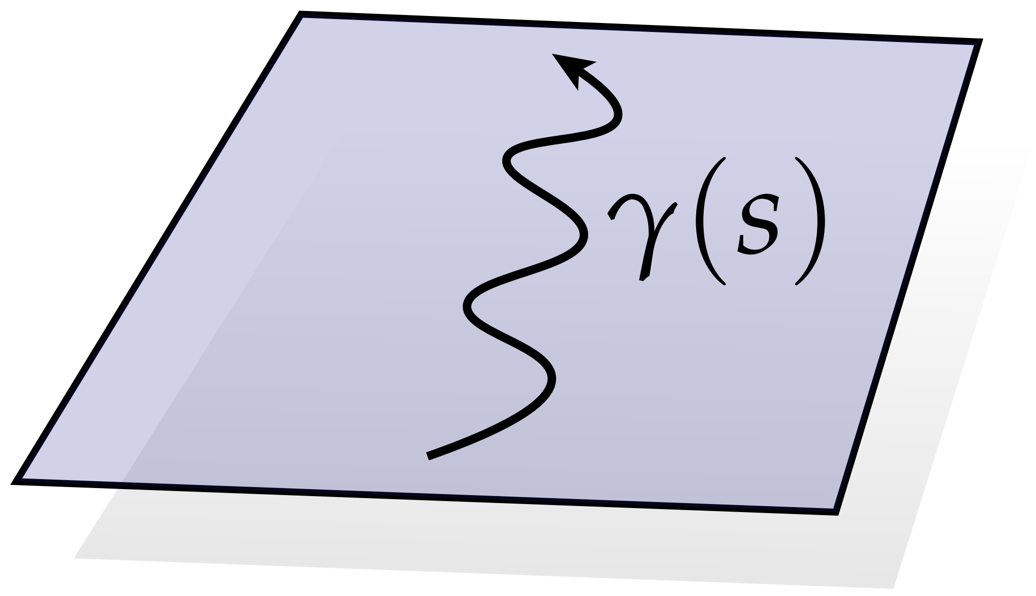
$$\kappa_n := \langle N_M, \tfrac{d}{ds}T \rangle$$

$$\kappa_g := \langle B_M, \tfrac{d}{ds}T \rangle$$

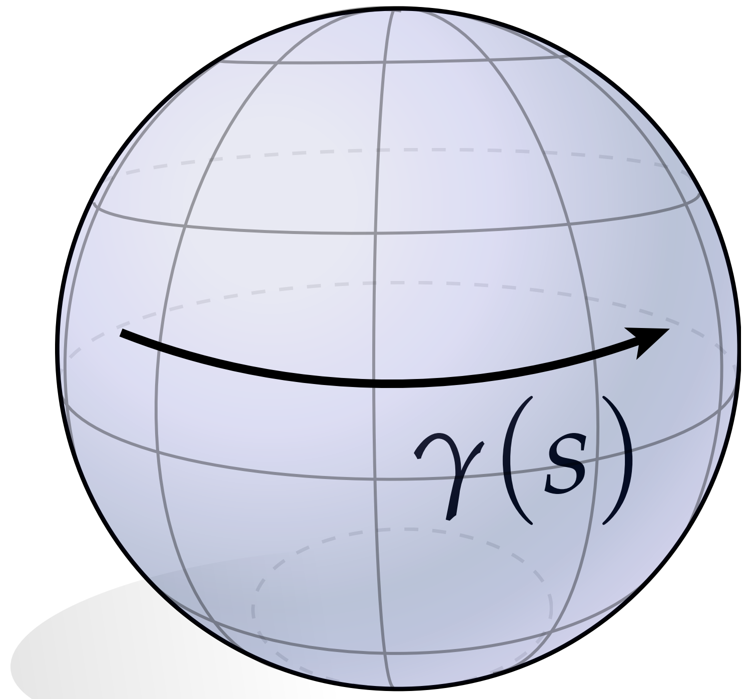
- T is still tangent of the curve; but unlike the Frenet frame, N_M is the normal of the surface and $B_M := T \times B_M$

Q: Why no third curvature $\langle T_M, \tfrac{d}{ds}T \rangle$?





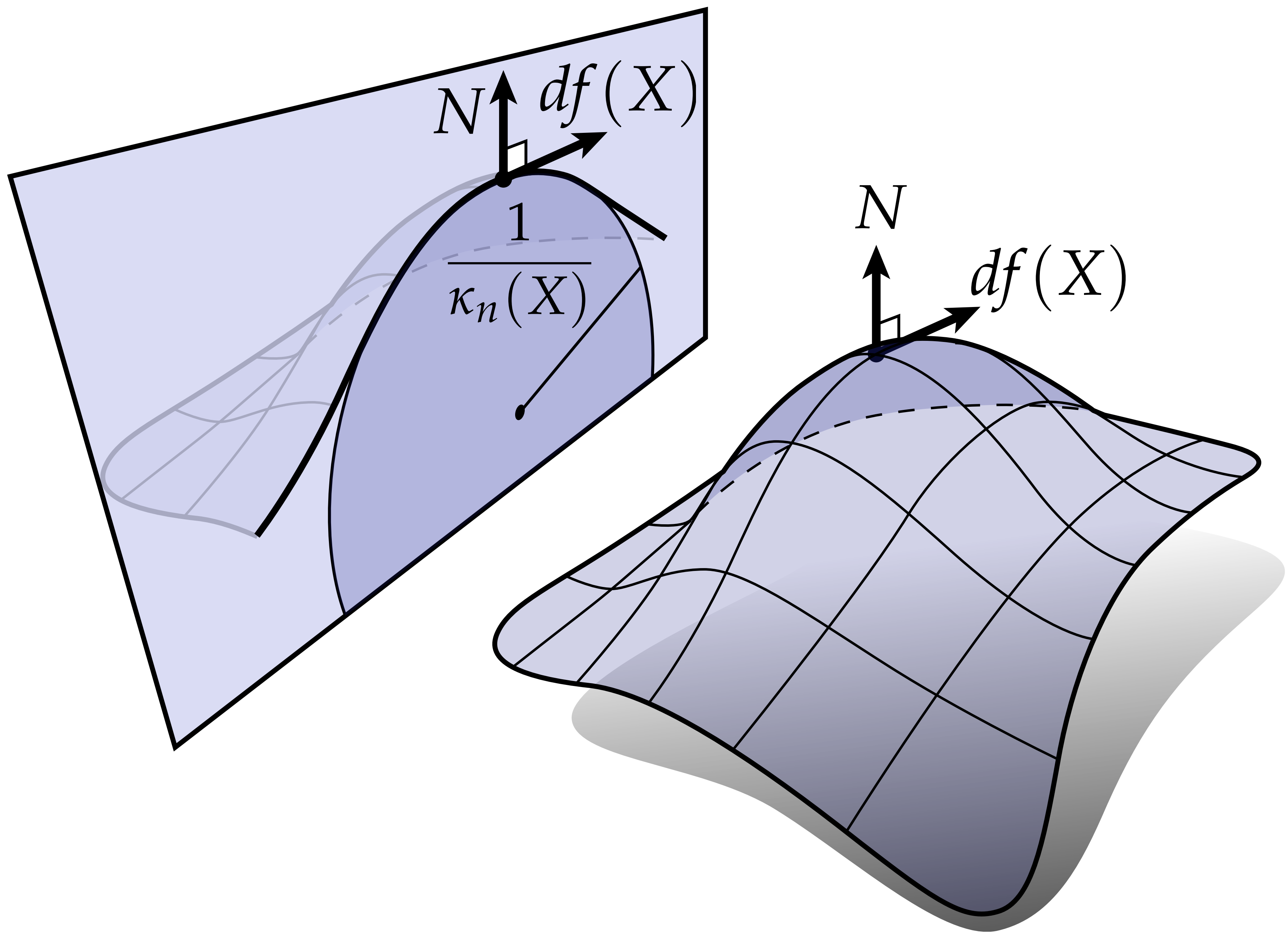
large κ_g;
small κ_n



large κ_n;
small κ_g

Second Fundamental Form

- Second fundamental form is closely related to principal curvature
- Can also be viewed as change in *first* fundamental form under motion in normal direction
- Why “fundamental?”First & second fundamental forms play role in important theorem…



$$\mathbf{II}(X,Y) := \langle dN(X), df(Y) \rangle$$

$$\kappa_N(X) := \frac{df(X), dN(X)}{|df(X)|^2} = \frac{\mathbf{II}(X,X)}{\mathbf{I}(X,X)}$$

Fundamental Theorem of Surfaces

- **Fact.** Two surfaces in R^3 are congruent if and only if they have the same first and second fundamental forms
  - …However, not every pair of bilinear forms **I**, **II** on a domain U describes a valid surface—must satisfy the **Gauss Codazzi** equations
- Analogous to fundamental theorem of plane curves: determined up to rigid motion by curvature
  - …However, for *closed* curves not every curvature function is valid (*e.g.*, must integrate to $2k\pi$)

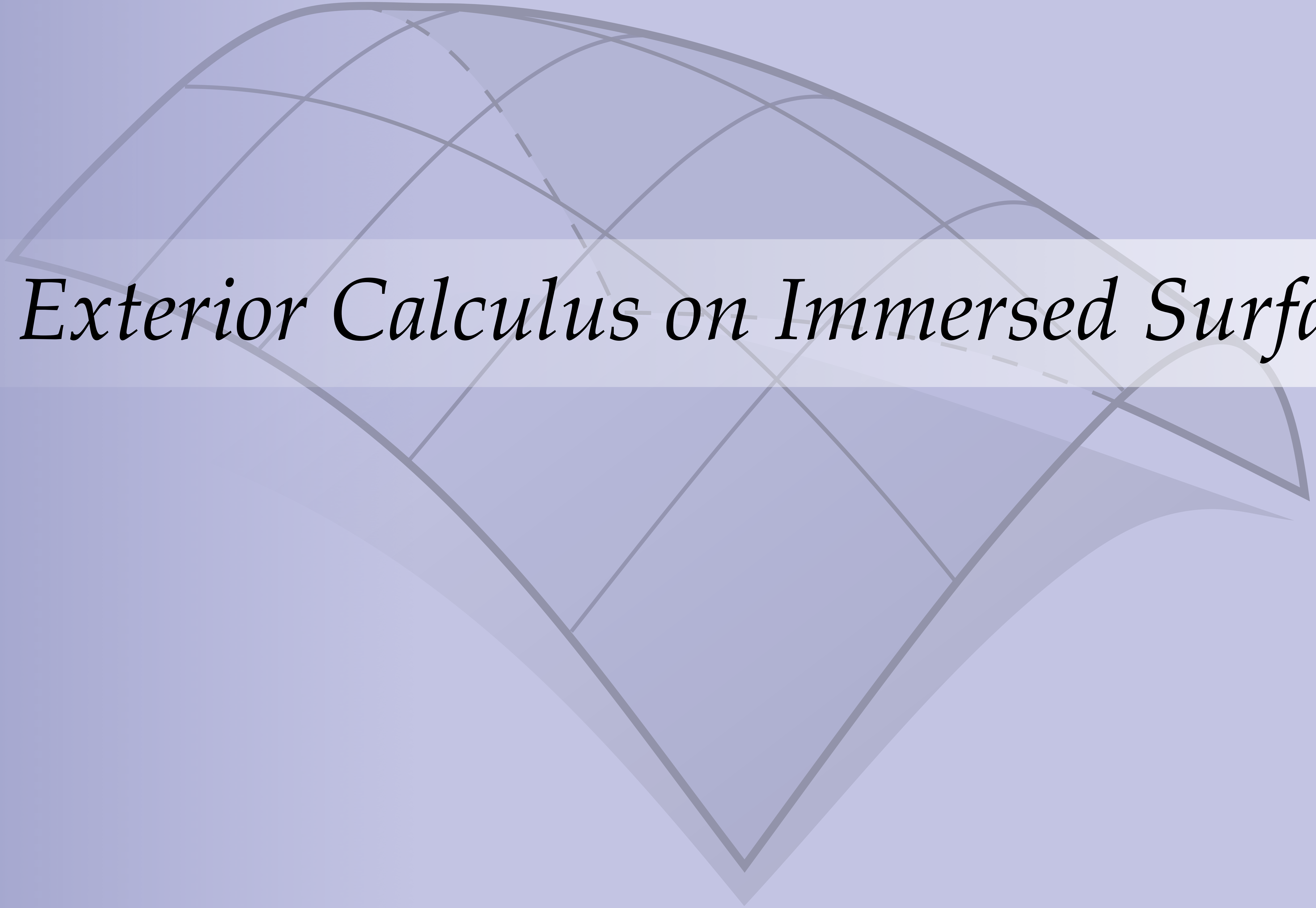
Other Descriptions of Surfaces?

- Classic question in differential geometry:

> *"What data is sufficient to completely determine a surface in space?"*

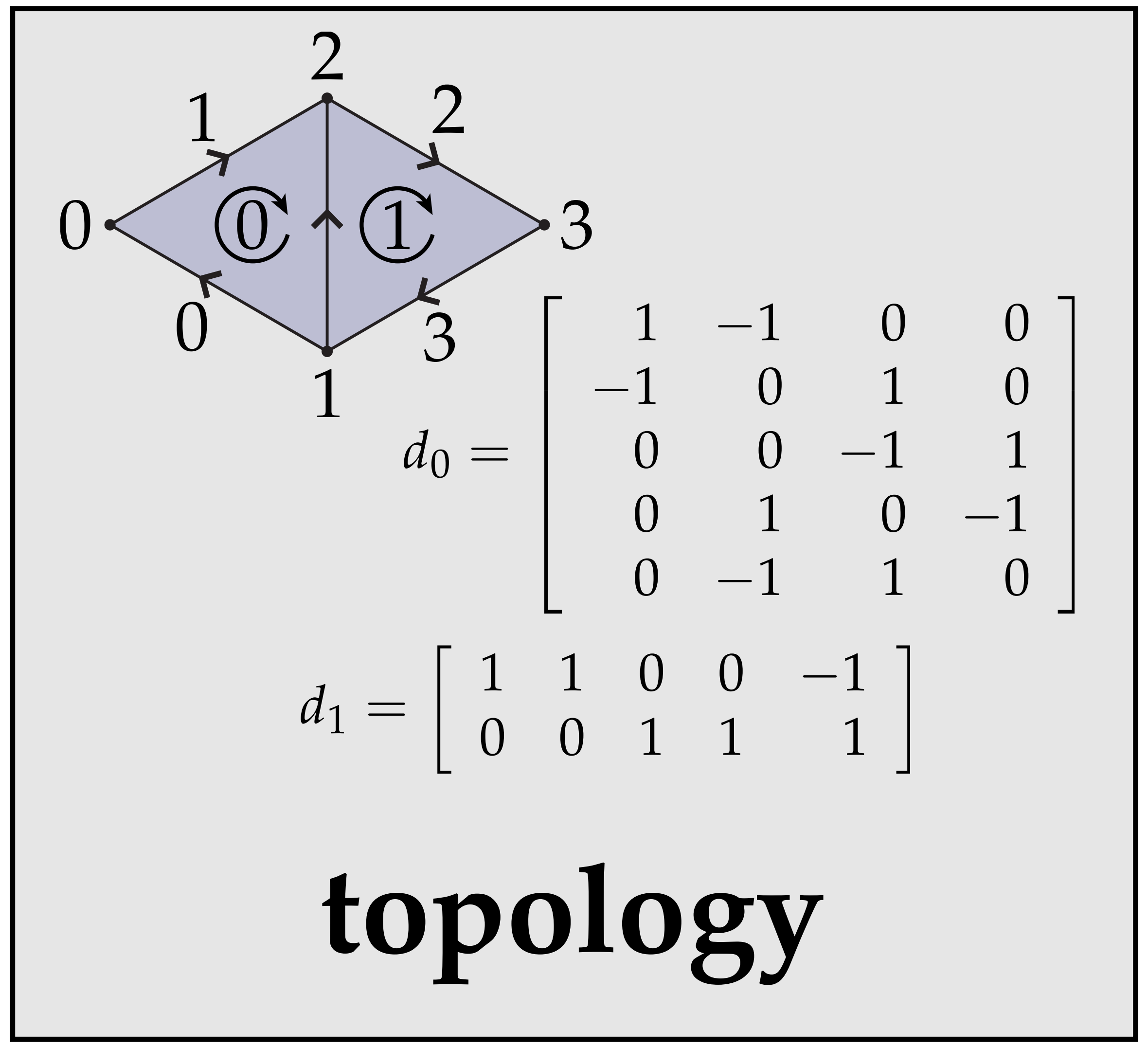
- Many possibilities…
  - First & second fundamental form (Gauss-Codazzi)
  - Mean curvature and metric (up to "Bonnet pairs")
  - Convex surfaces: metric alone is enough (Alexandrov/Pogorolev)
  - Gauss curvature essentially determines metric (Kazdan-Warner)
- …in general, still a surprisingly murky question!

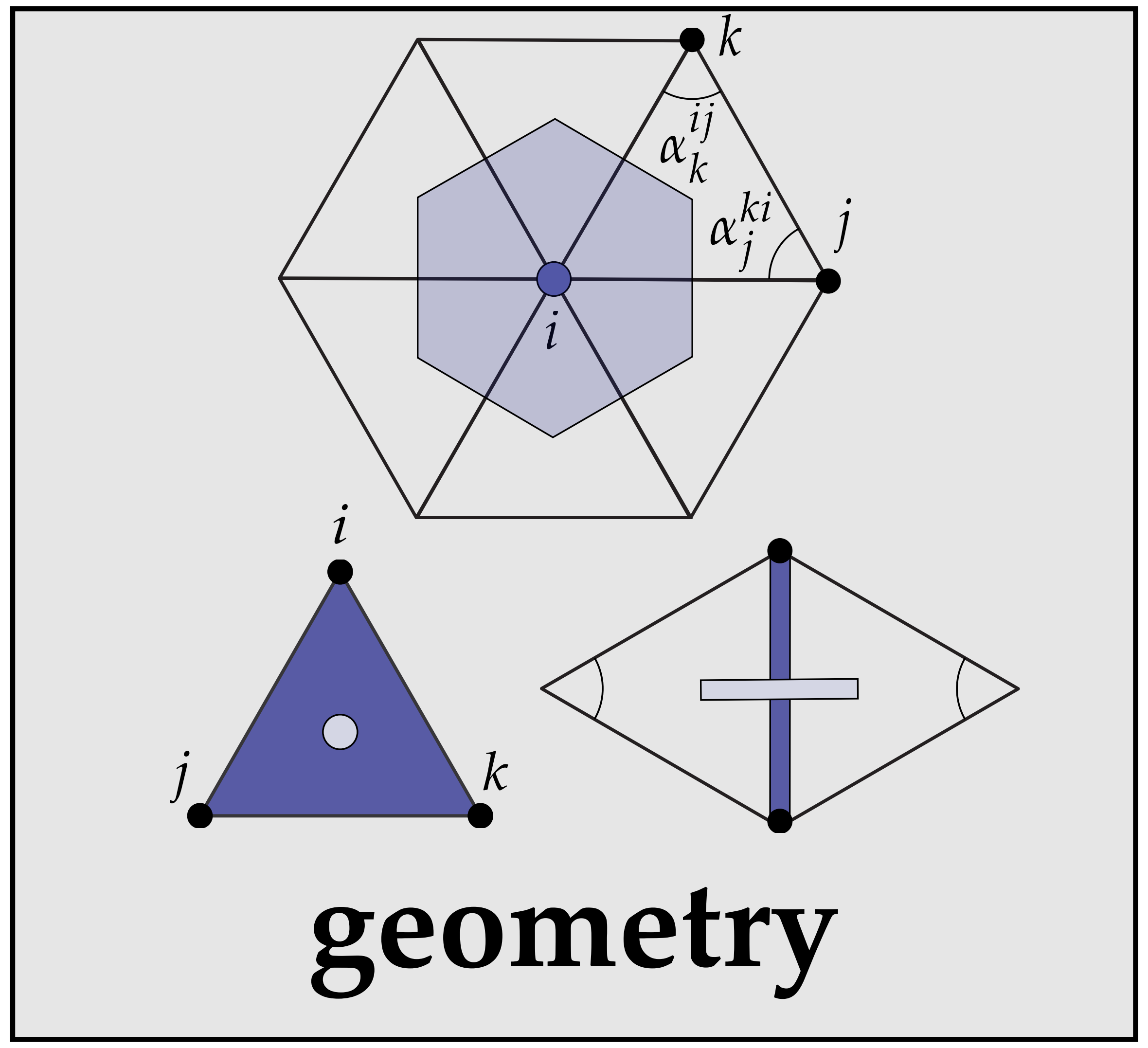
Exterior Calculus on Immersed Surfaces



Exterior Calculus on Curved Domains

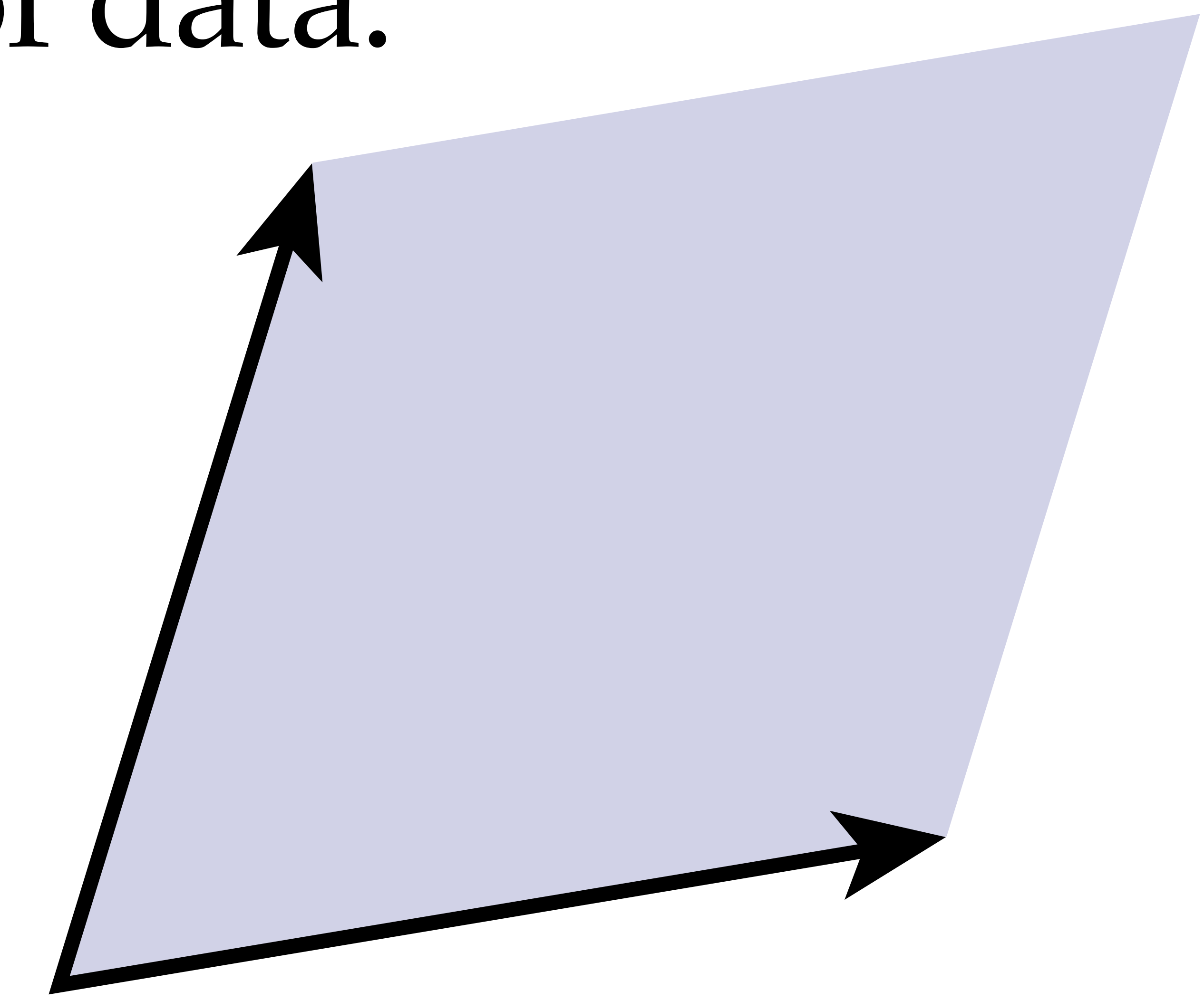
- Initial study of differential forms was in **flat** Euclidean R^n
- How do we do exterior calculus on **curved** spaces?
- Recall that operators nicely "split up" topology & geometry:
  - **(topology)** wedge product (^), exterior derivative (d)
  - **(geometry)** Hodge star (★)
- For instance, discrete d uses only mesh connectivity **(topology)**; discrete ★ involves only ratios of volumes **(geometry)**
- Therefore, to get exterior calculus to work with curved spaces, we just need to figure out what the Hodge star looks like!
- Traditionally taught from abstract **intrinsic** point of view; we'll start with the concrete **extrinsic** picture (which fewer people know… but is more directly relevant for real applications!)

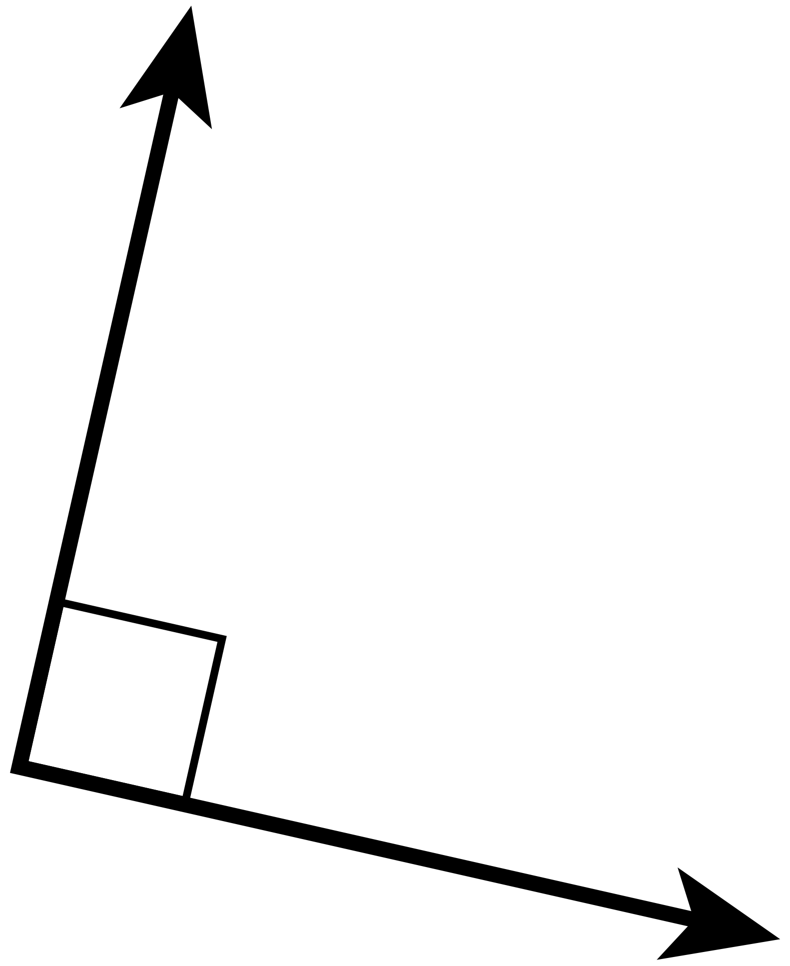




Exterior Calculus on Immersed Surfaces

- For surface immersed in 3D, just need two pieces of data:
  - **Area form**—*"how big is a given region?"*
    - lets us define Hodge star on 0/2-forms
    - can express via cross product in R^3
  - **Complex structure**—*"how do we rotate by 90°?"*
    - lets us define Hodge star on 1-forms
    - can express via cross product w/ surface normal
- All of this data also determined by induced metric

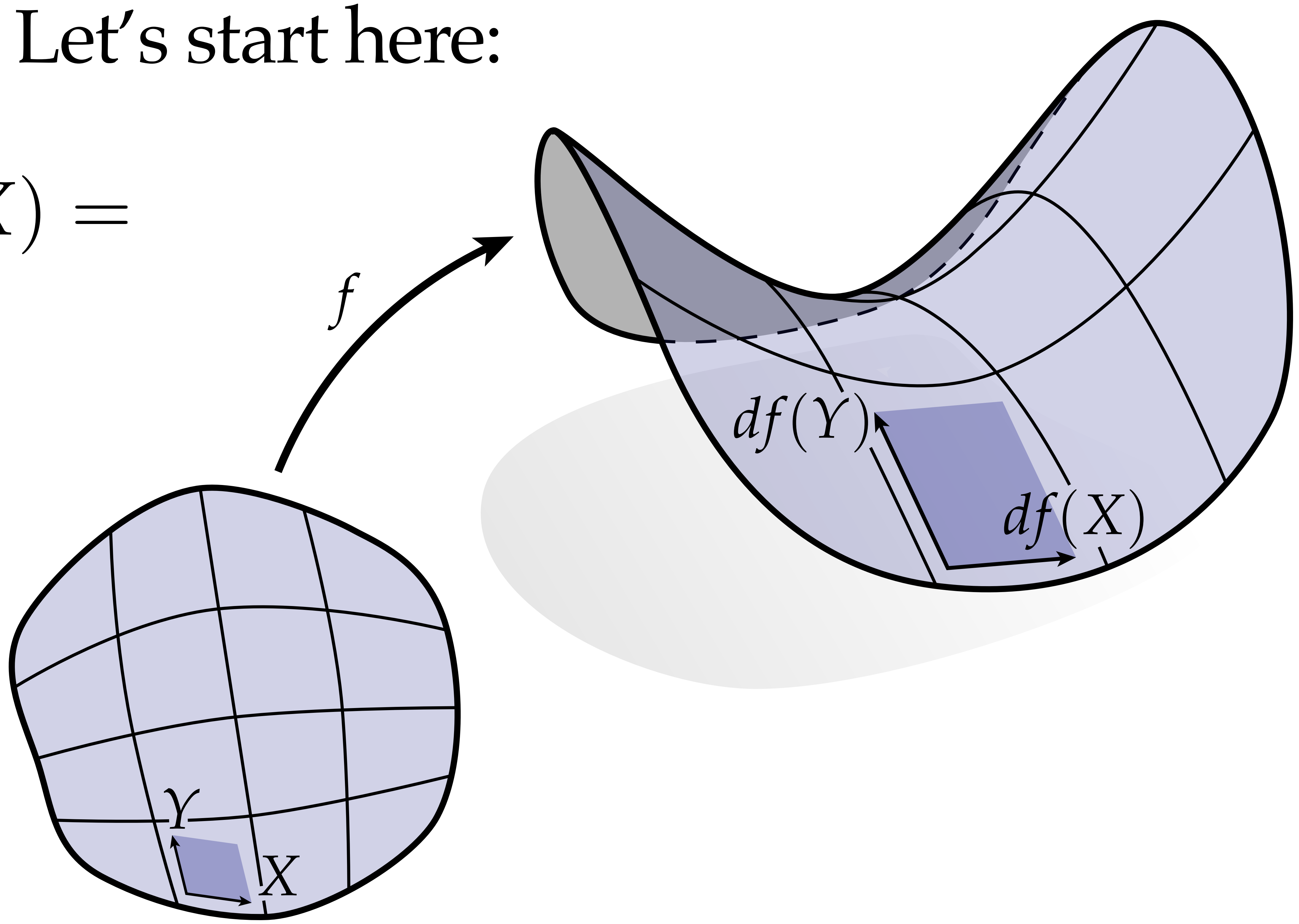




Induced Area 2-Form

- What signed area should we associate with a pair of vectors X, Y on the domain?
- Not just their cross product! Need to account for "stretching" caused by immersion f
- What's the signed area of the stretched vector? Let's start here:

$$df \wedge df(X,Y) = df(X) \times df(Y) - df(Y) \times df(X) = 2df(X) \times df(Y)$$



Since $df(X)$ and $df(Y)$ are *tangent*, we get

$$df \wedge df(X,Y) = 2NdA(X,Y)$$

where dA is the area 2-form on $f(M)$. Hence,

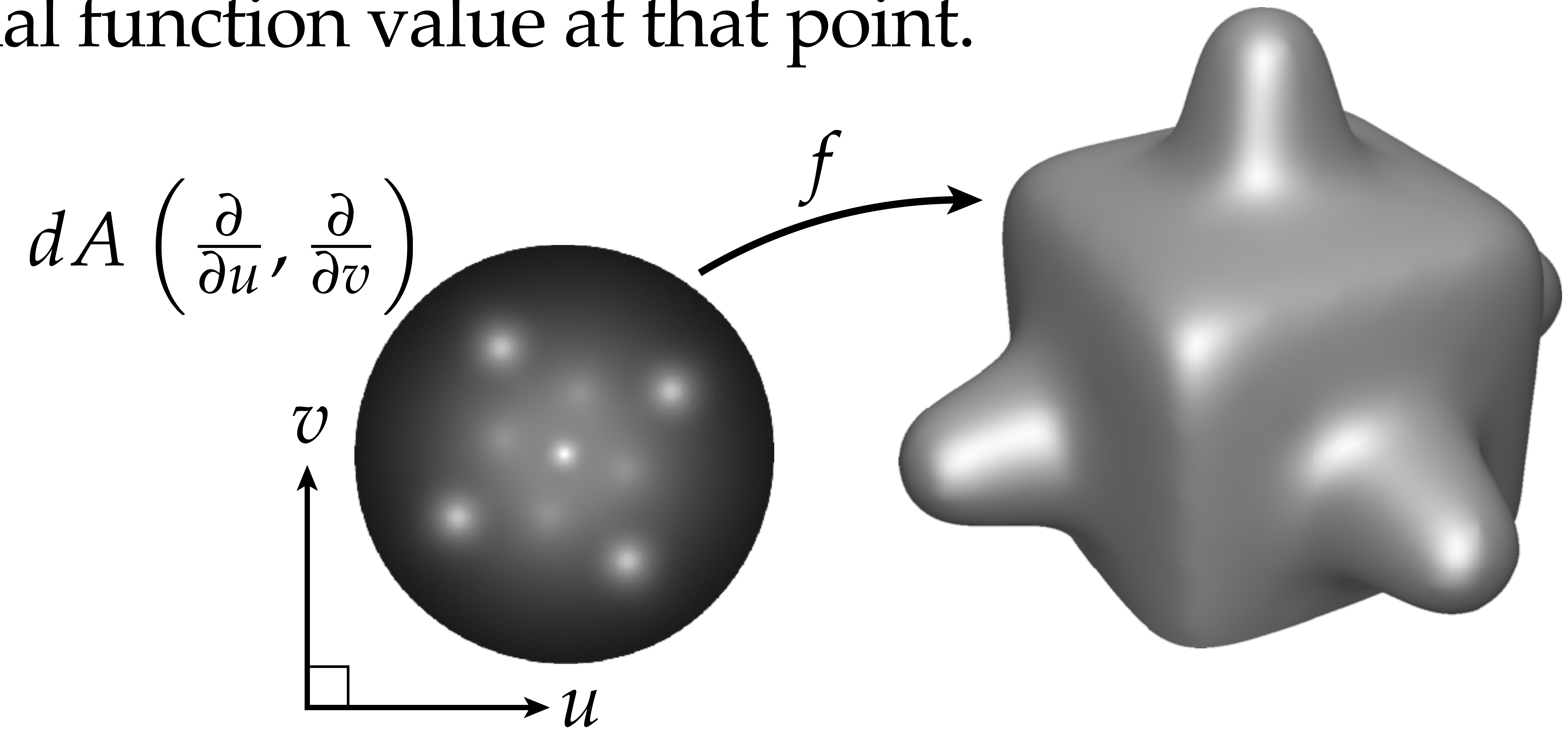
$$\boxed{dA = \frac{1}{2}\langle N, df \wedge df \rangle}$$

Induced Hodge Star on 0-Forms

- Given the area 2-form dA, can easily define Hodge star on 0-forms:

$$\phi \overset{\star}{\longmapsto} \phi\, dA$$

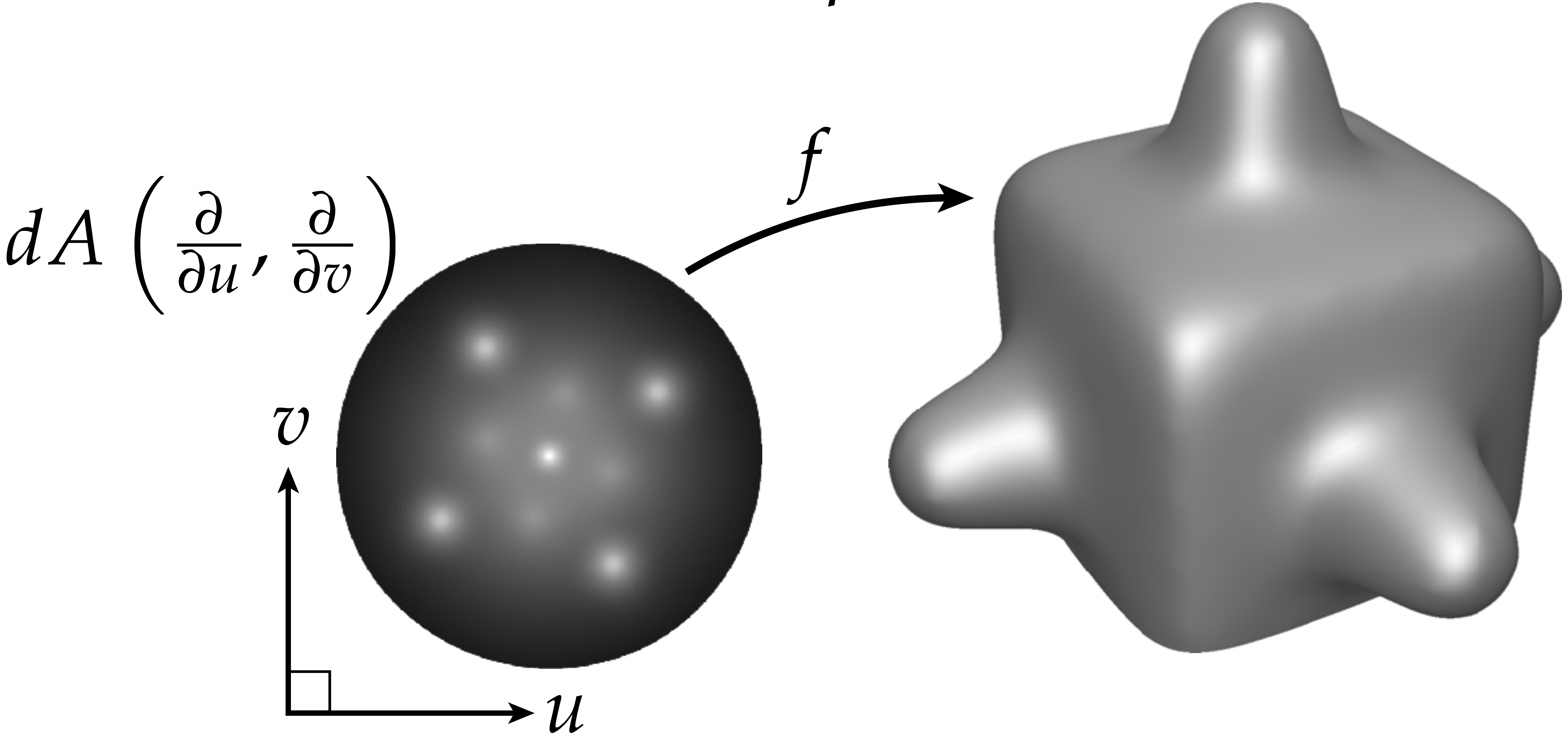
- **Meaning?** Applying this new 2-form to a unit area *on the surface* yields the original function value at that point.



Induced Hodge Star on 2-Forms

- To get the 2-form Hodge star, we just go the other way
- Suppose ω is a 2-form on $f(M)$. Then its Hodge dual is the unique 0-form ϕ such that

$$\omega = \phi \, dA$$



Complex Structure

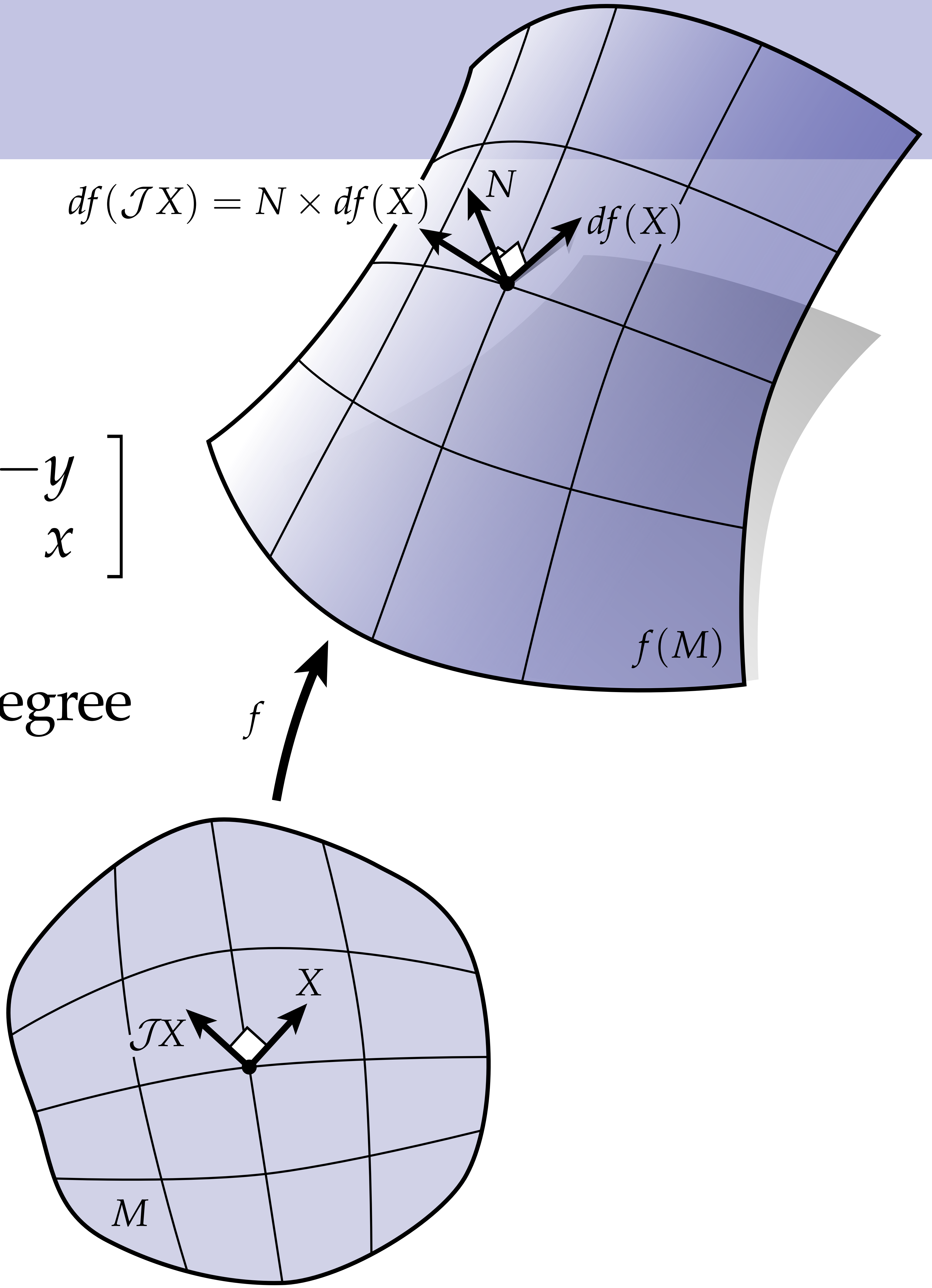
- The *complex structure** tells us how to rotate by 90°
- In R^2, we just replace (x,y) with $(-y,x)$:

$$\mathcal{J}_{\mathbb{R}^2} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \mathcal{J}_{\mathbb{R}^2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

- For a surface immersed in R^3, we can express a 90-degree rotation via a cross product with the unit normal N:

$$df(\mathcal{J}_f X) := N \times df(X)$$

- This relationship uniquely determines $\mathcal{J}_f$
- An immersion is conformal if and only if $\mathcal{J}_f = \mathcal{J}_{\mathbb{R}^2}$



*Sometimes called *linear complex structure*; same thing for surfaces.

Complex Structure in Coordinates

- Suppose we want to explicitly compute the linear complex structure*
- Similar strategy to shape operator: solve a matrix equation for $\mathcal{J}$

$$\widehat{N} := \begin{bmatrix} 0 & -N_z & N_y \\ N_z & 0 & -N_x \\ -N_y & N_x & 0 \end{bmatrix}$$

cross product w/ normal

$(N \times u = \widehat{N}u)$

$$A := \begin{bmatrix} \partial f_x/\partial u & \partial f_x/\partial v \\ \partial f_y/\partial u & \partial f_y/\partial v \\ \partial f_z/\partial u & \partial f_z/\partial v \end{bmatrix}$$

Jacobian

$$J := \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

complex structure

$$\boxed{df(\mathcal{J}X) = N \times df(X)} \implies \boxed{J = A^T \widehat{N} A}$$

*__Note:__ not something you do much in practice, but may help make definition feel more concrete…

Induced Hodge Star on 1-Forms

- Recall that for a 1-form α in the plane, applying $\star\alpha$ to a vector X is the same as applying α to a 90-degree rotation of X:

$$\star_{\mathbb{R}^2}\alpha(X) = \alpha(\mathcal{J}_{\mathbb{R}^2}X)$$

- For 1-forms on an immersed surface f, we instead want to apply a 90-degree rotation with respect to the surface itself:

$$\star_f\alpha(X) = \alpha(\mathcal{J}_f X)$$

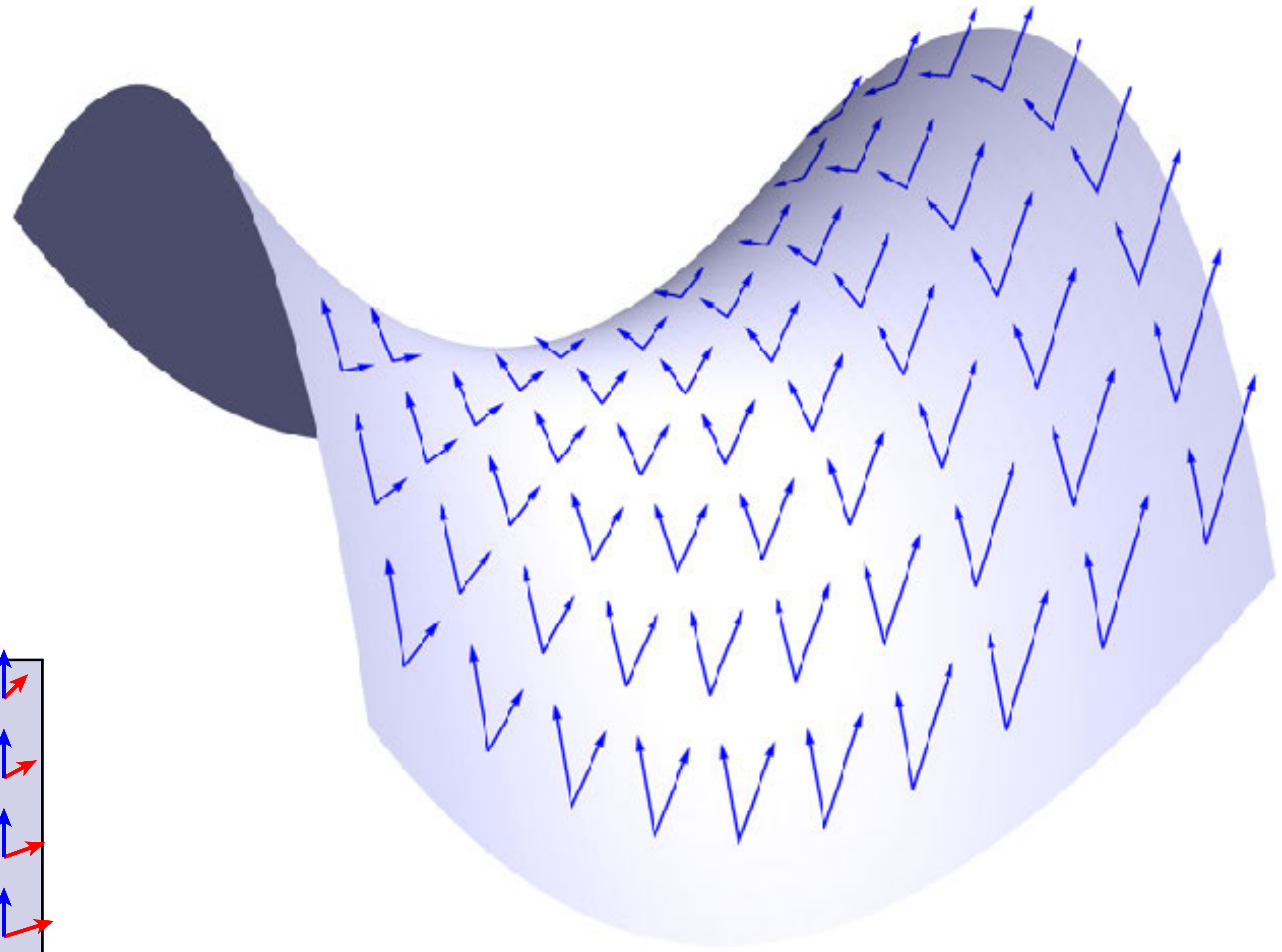
- At this point we have everything we need to do calculus on curved surfaces: 0-, 1-, and 2-form Hodge star. (Will see more general/abstract/intrinsic definitions for n-manifolds later on.)

Sharp and Flat on a Surface

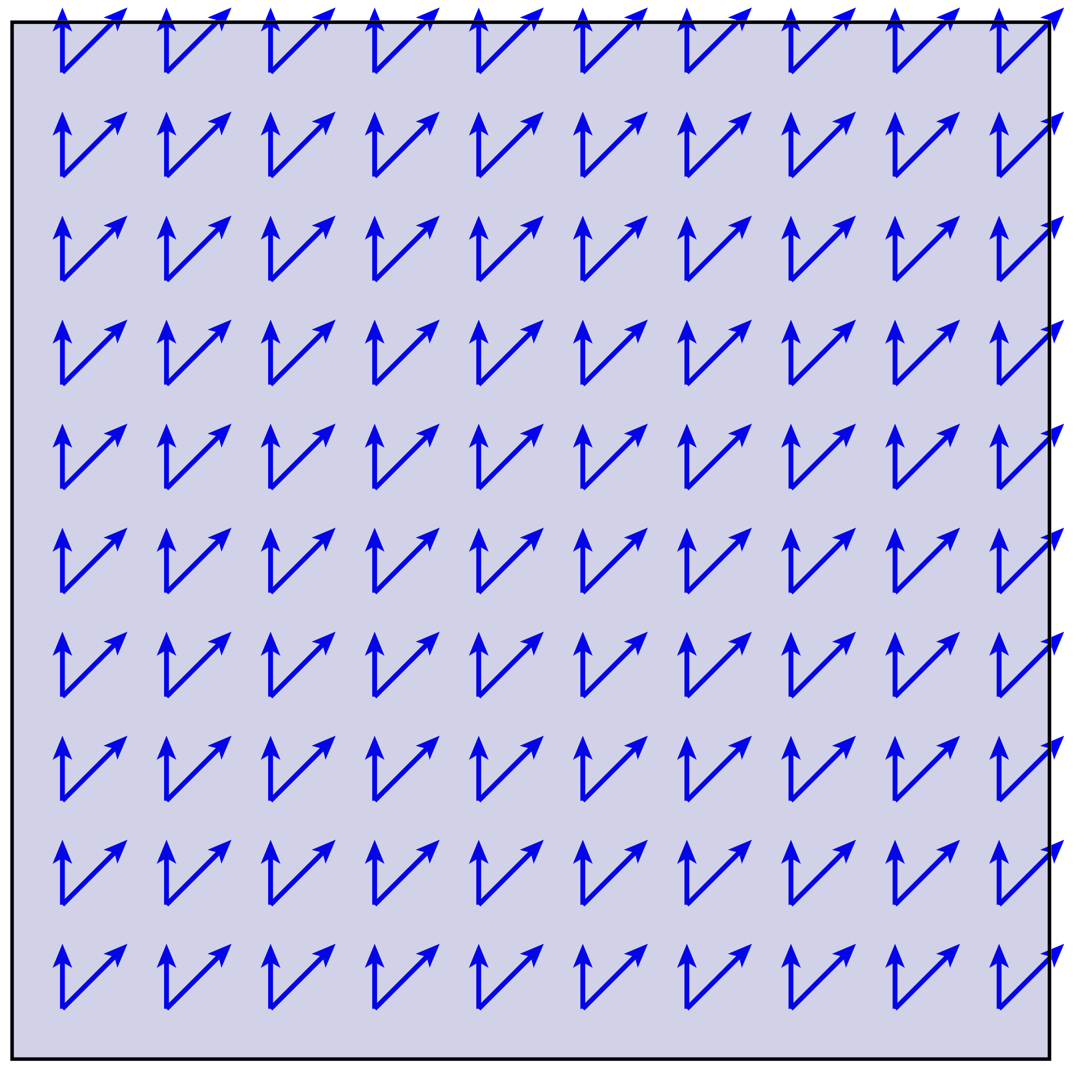
- Can use induced metric to translate between vector fields and 1-forms:

$$X^\flat(Y) := g(X,Y) \qquad\qquad g(\alpha^\sharp, Y) := \alpha(Y)$$

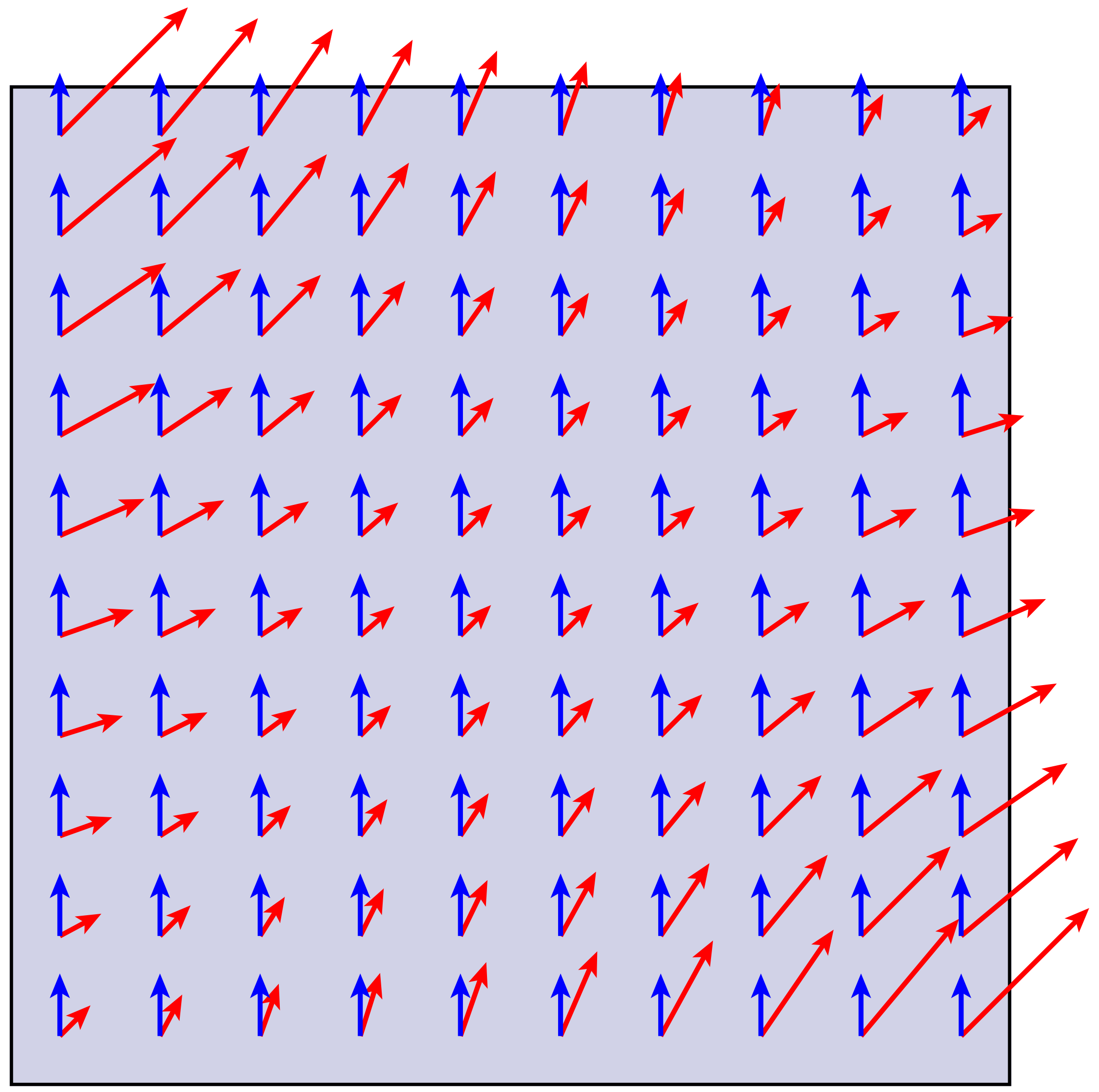
- No longer just a trivial "transpose" (as in Euclidean R^n)
- E.g., flat correctly encodes inner product on surface



$df(X)\cdot df(Y)$

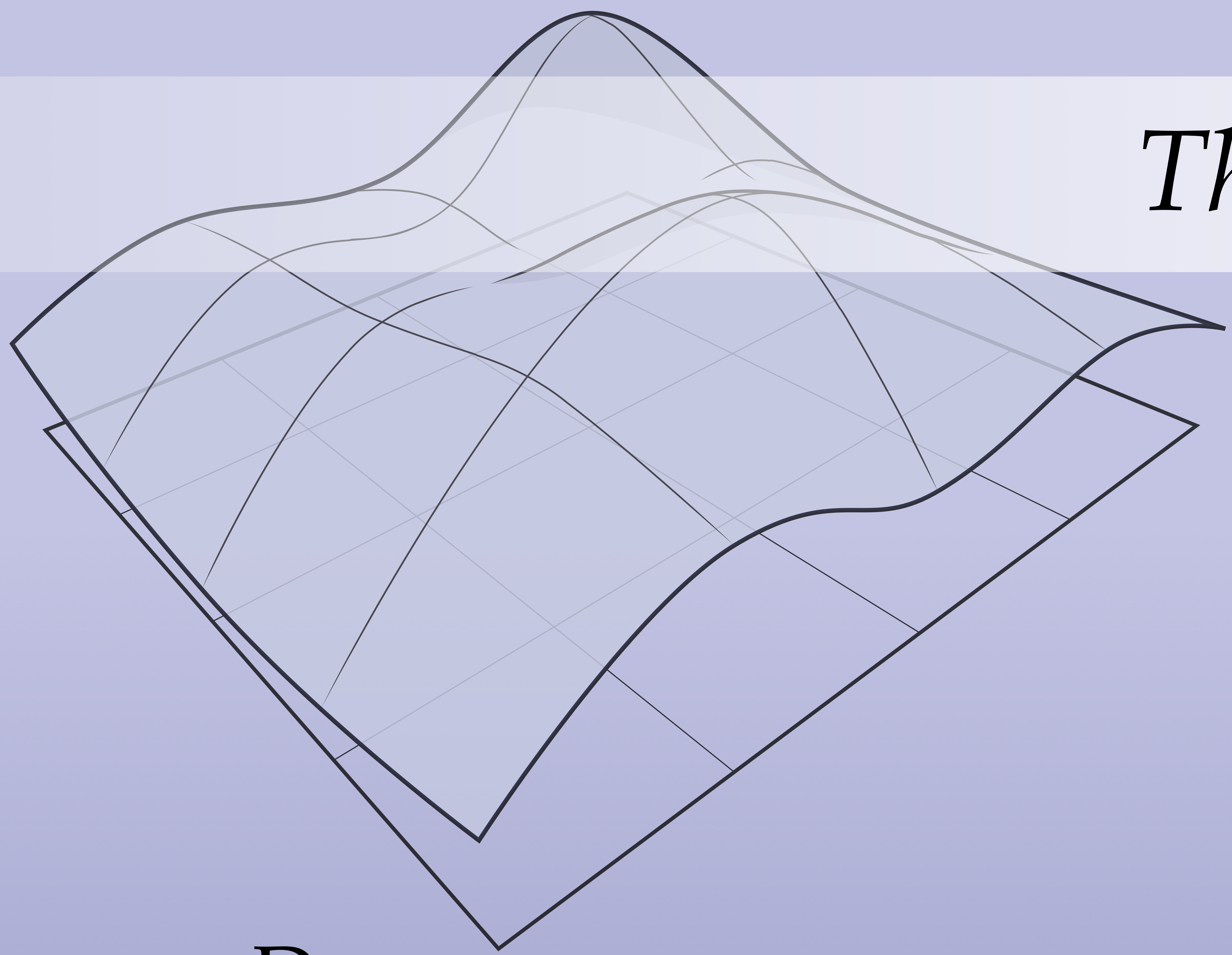


$X \cdot Y \neq df(X)\cdot df(Y)$



$X^\flat(Y) = df(X)\cdot df(Y)$

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017