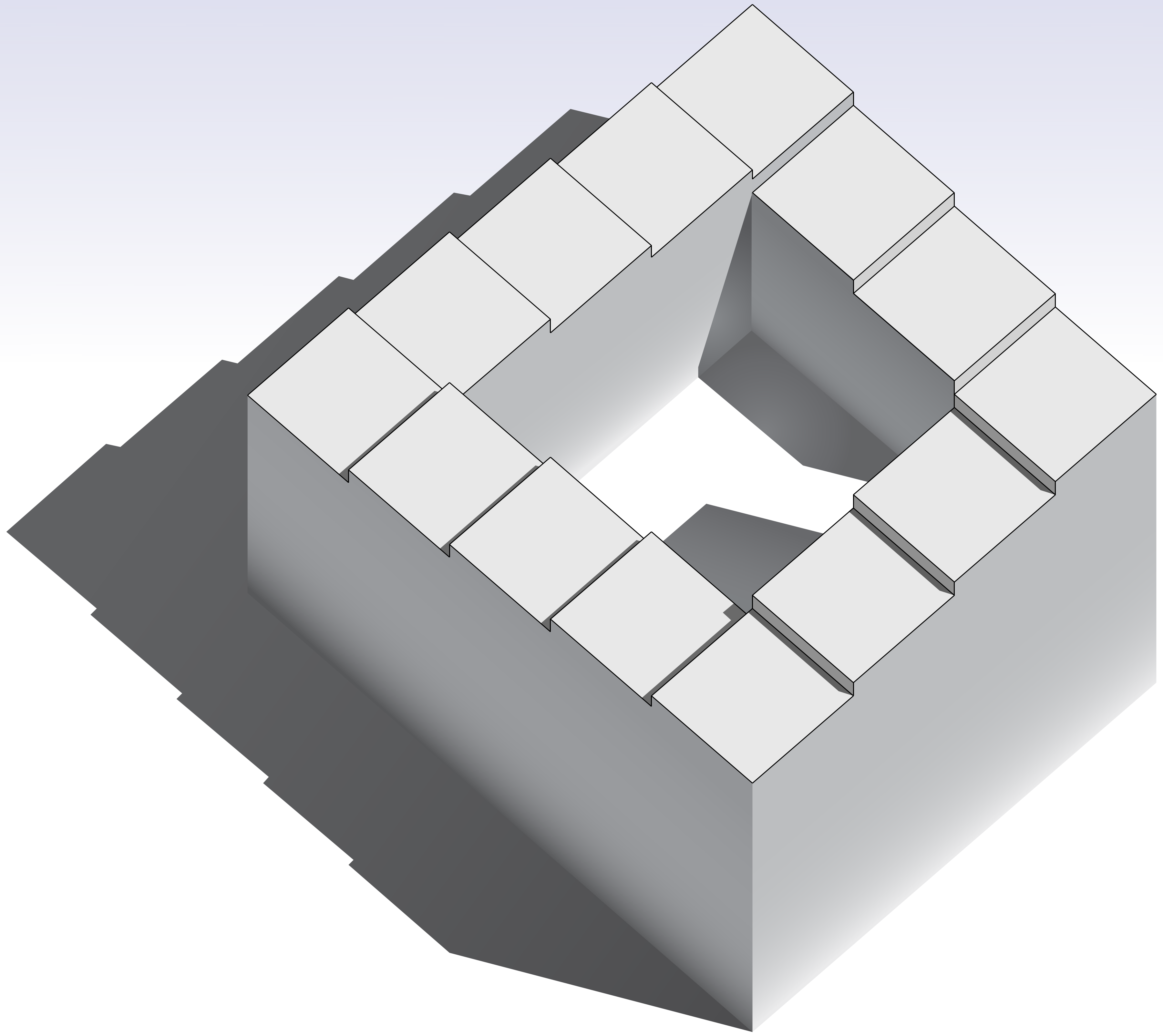
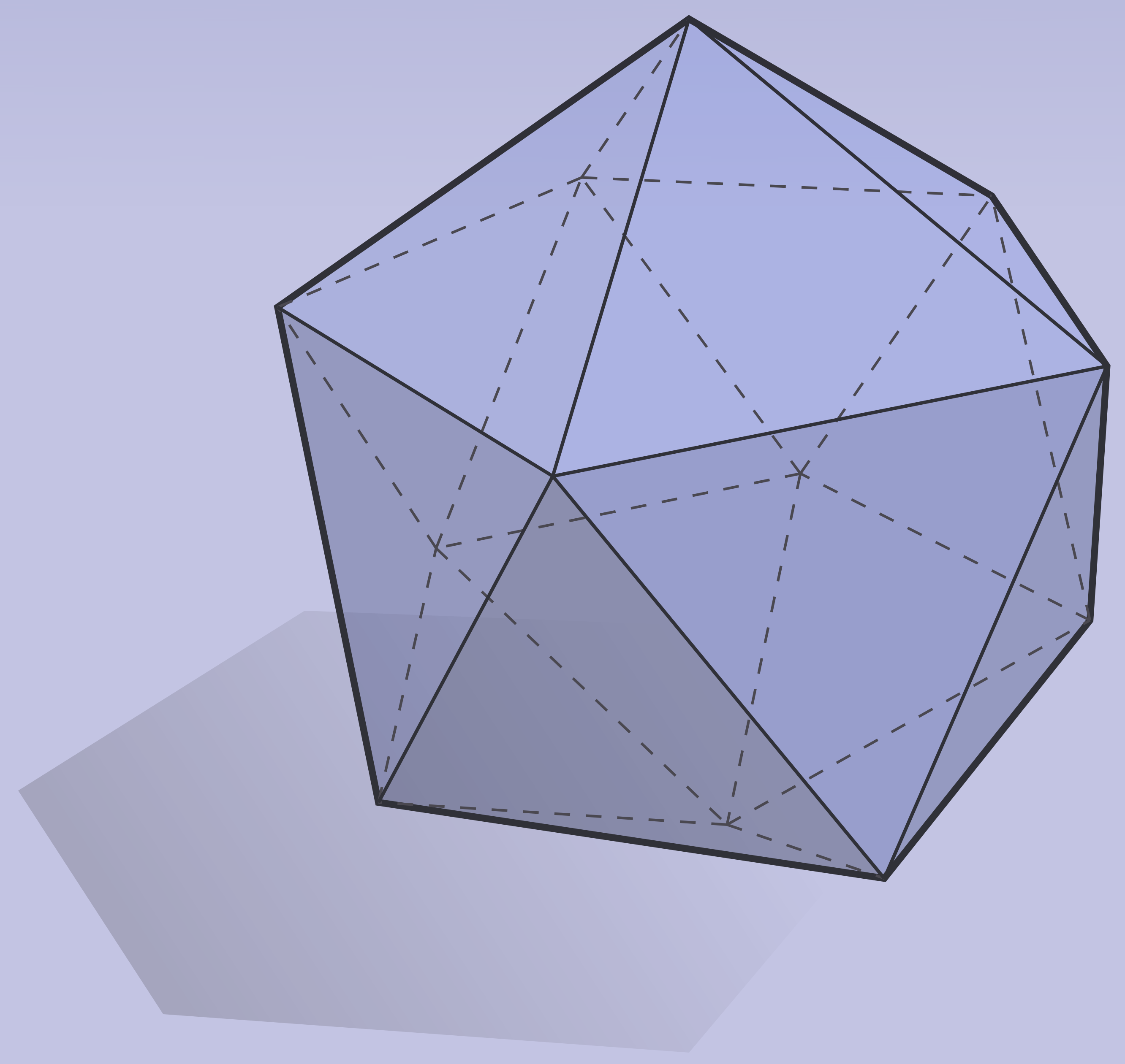


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-869(J) • Spring 2016

PART II:
DIFFERENTIABLE STRUCTURE



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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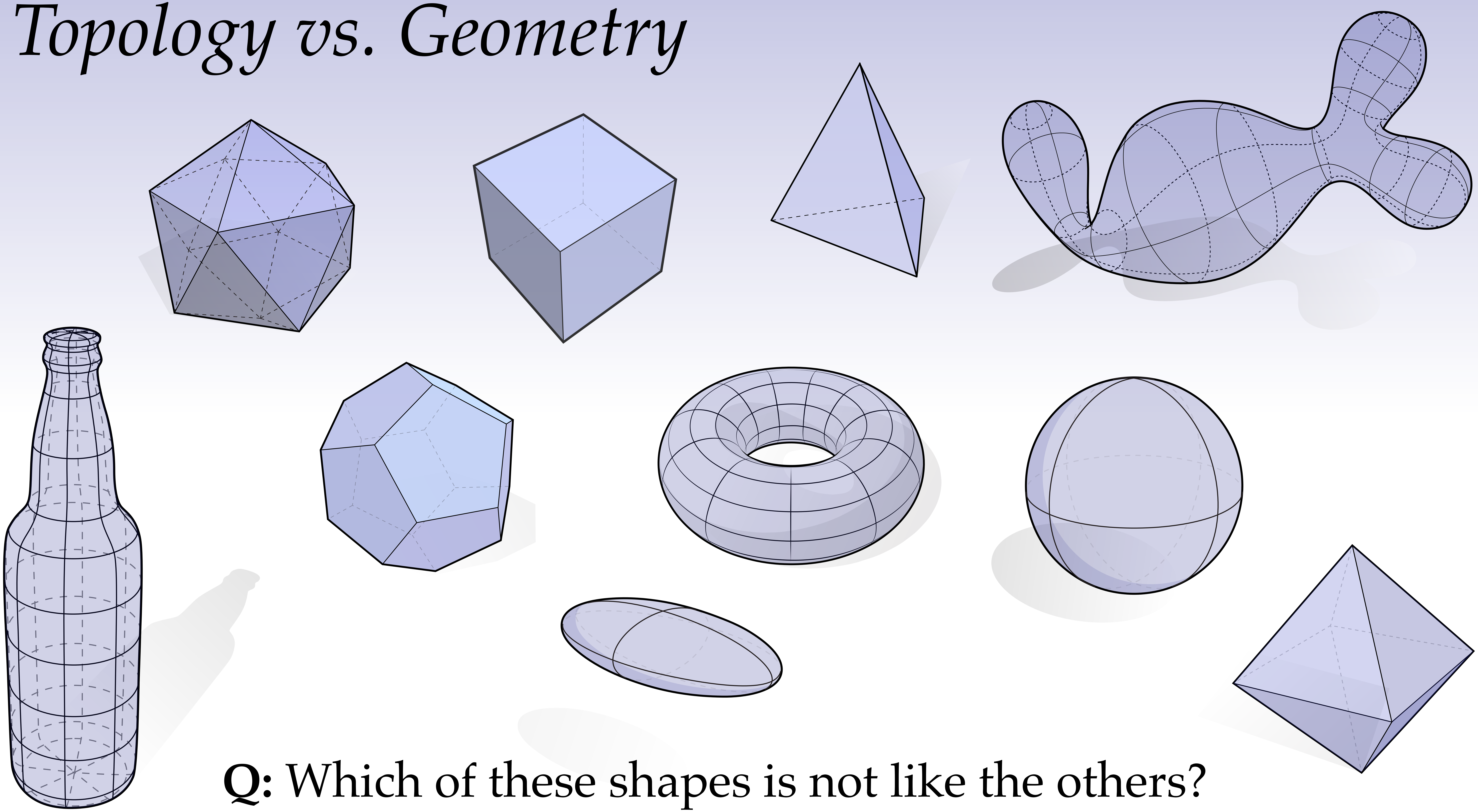
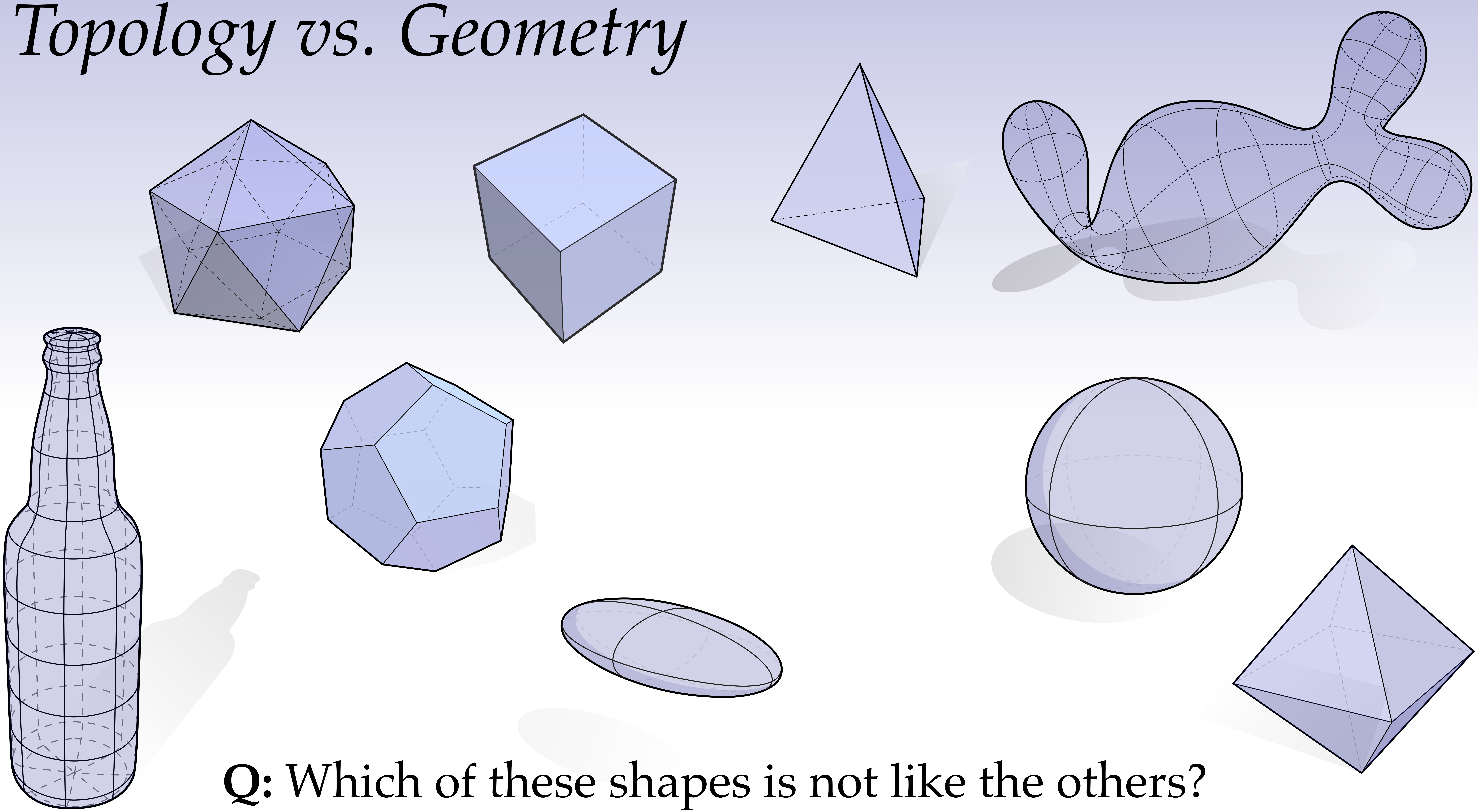
Topology vs. Geometry



Q: Which of these shapes is not like the others?

Topology vs. Geometry



Q: Which of these shapes is not like the others?

Topological vs. Geometric Structures

SET

∪

TOPOLOGICAL SPACE

∪

TOPOLOGICAL MANIFOLD

"topology"

∪

SMOOTH MANIFOLD

"differential topology"

∪

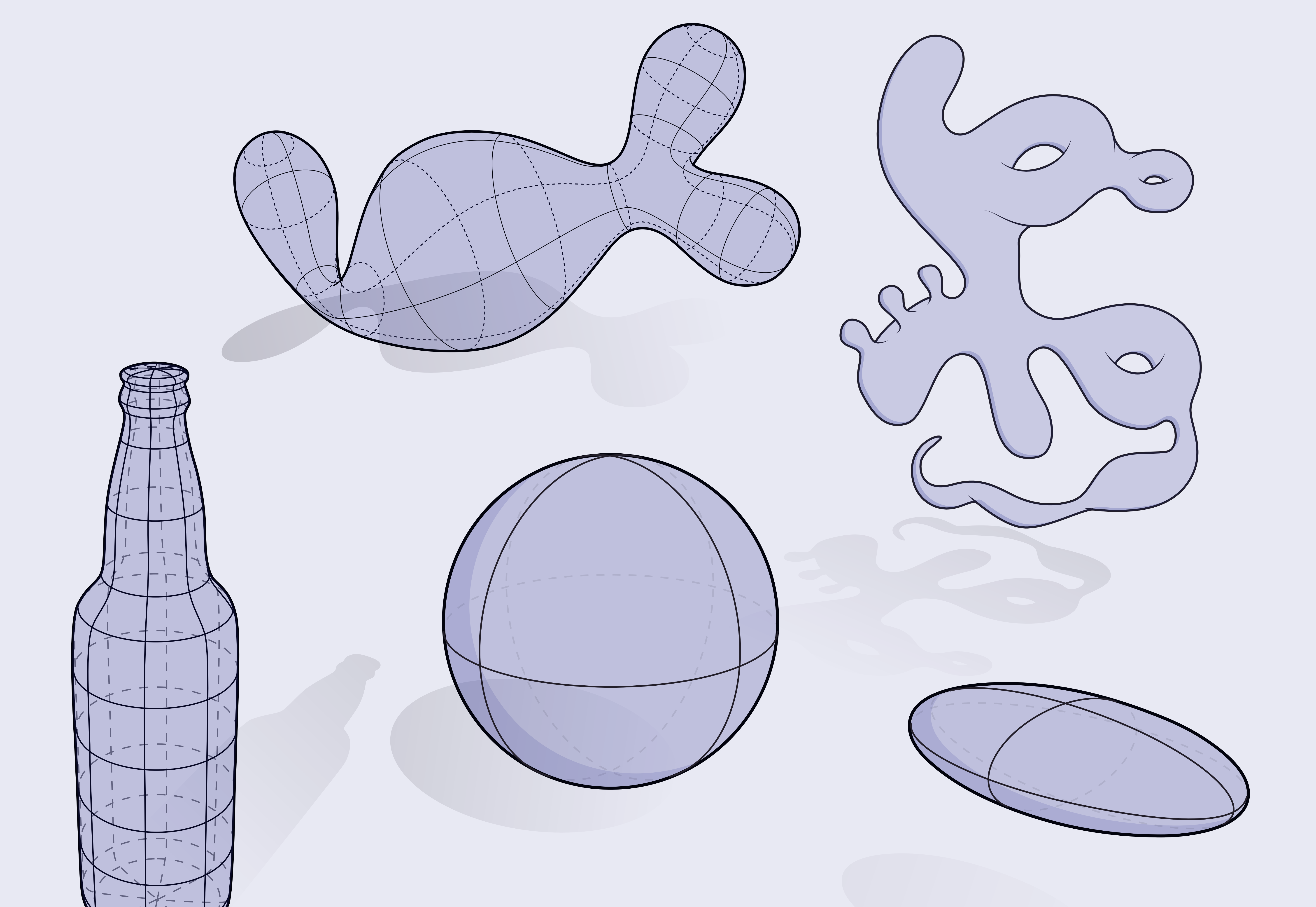
COMPLEX MANIFOLD

∪

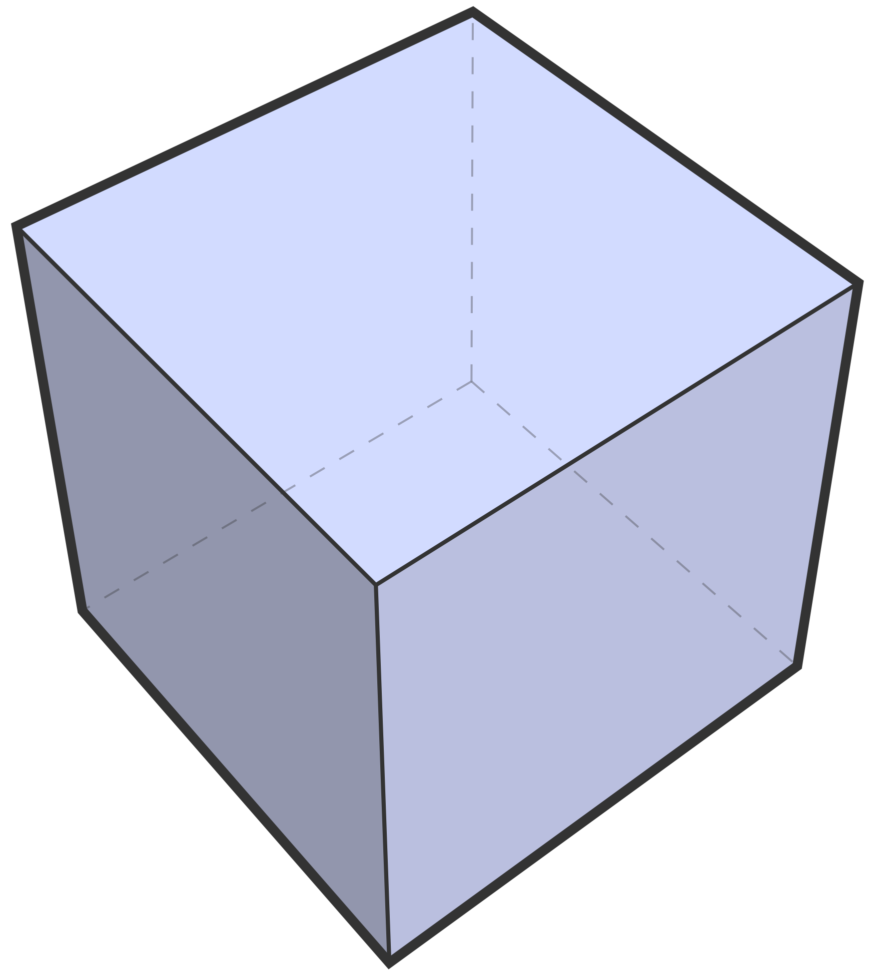
RIEMANNIAN MANIFOLD

"geometry"

Smooth Structure—Visualized



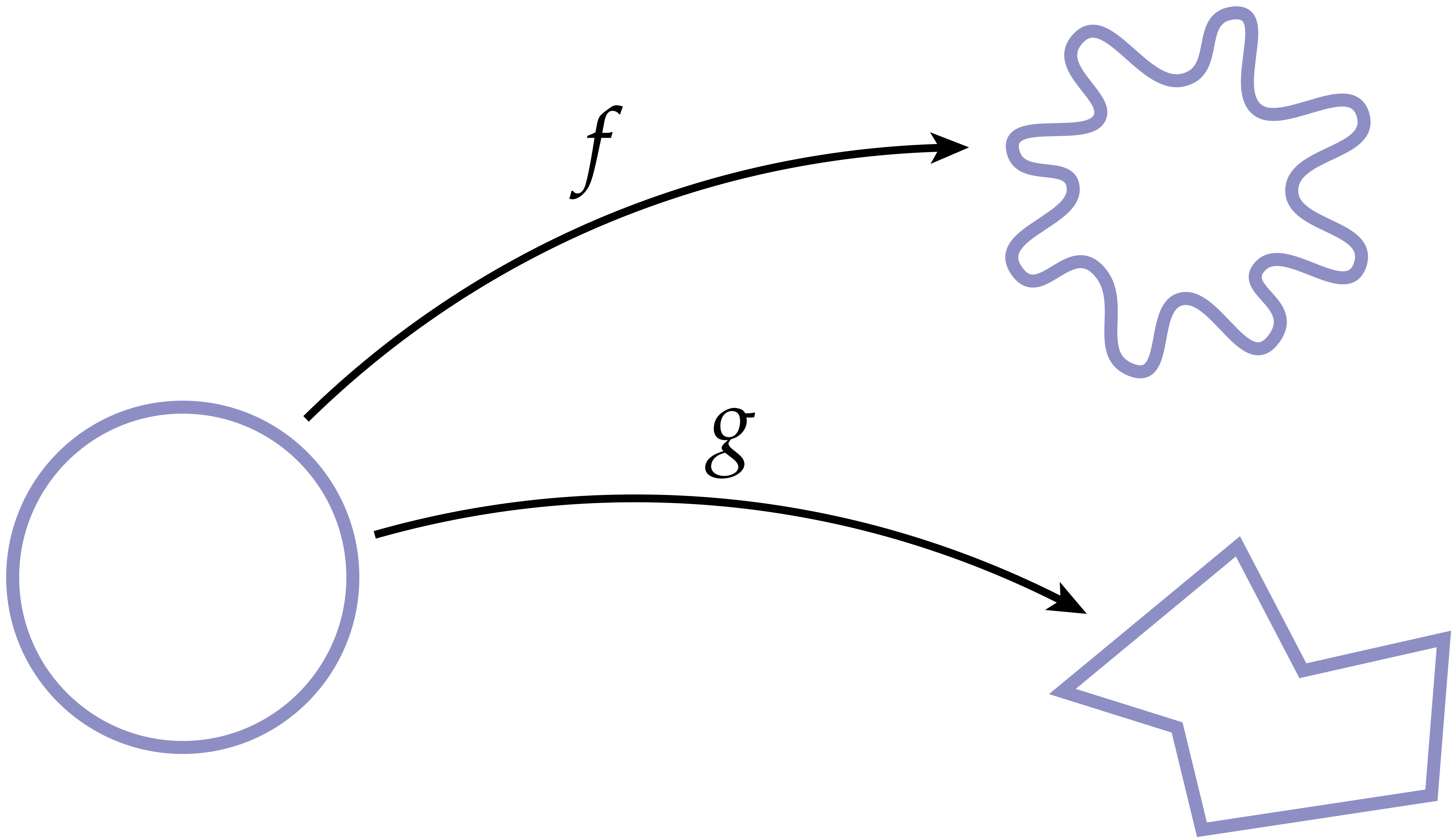
"same"



"different"

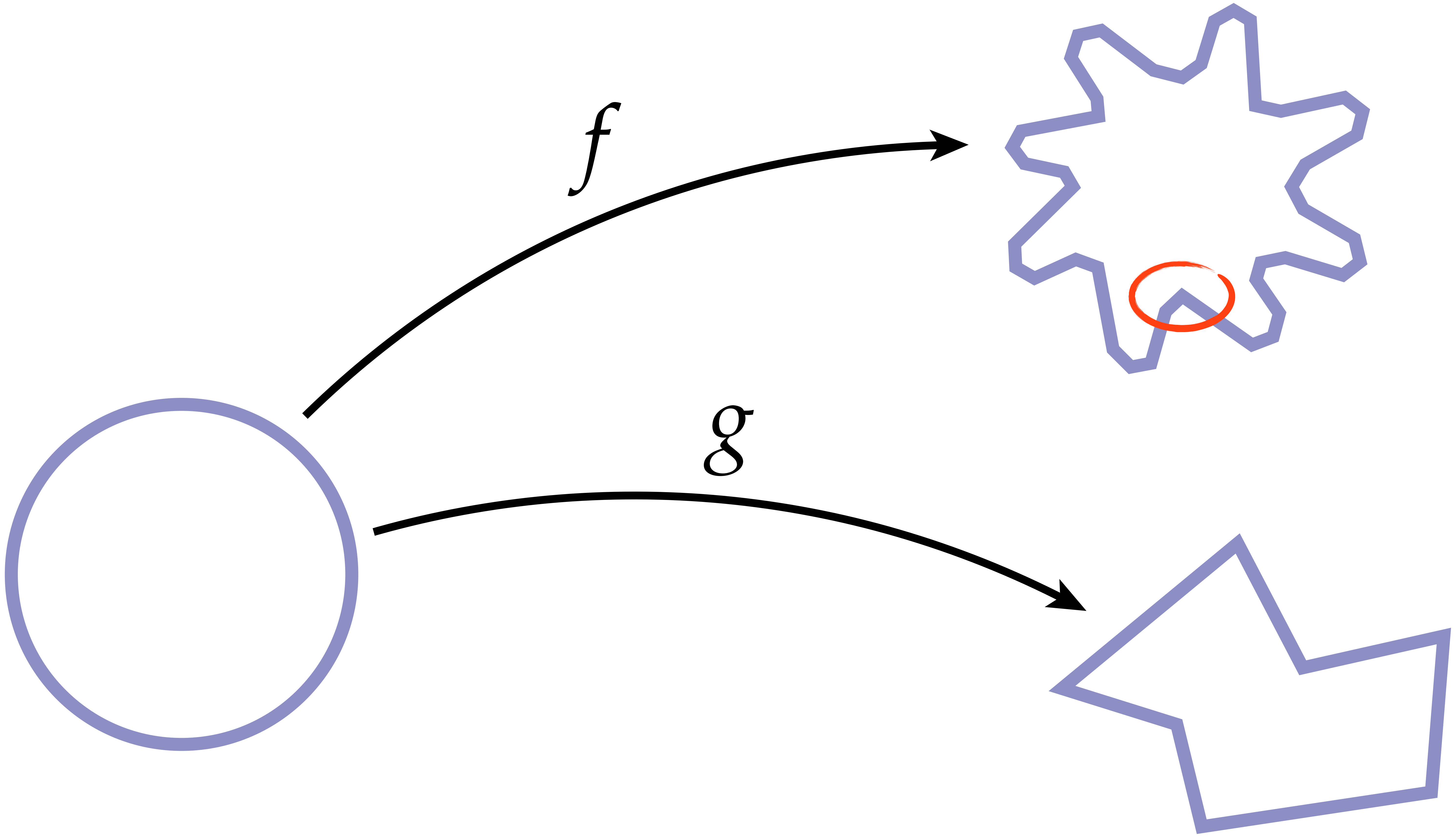
Differentiable Maps — Visualized

Which one is differentiable?



"Discrete" Differentiable Maps?

Is one "discretely" differentiable?



Derivative as Slope

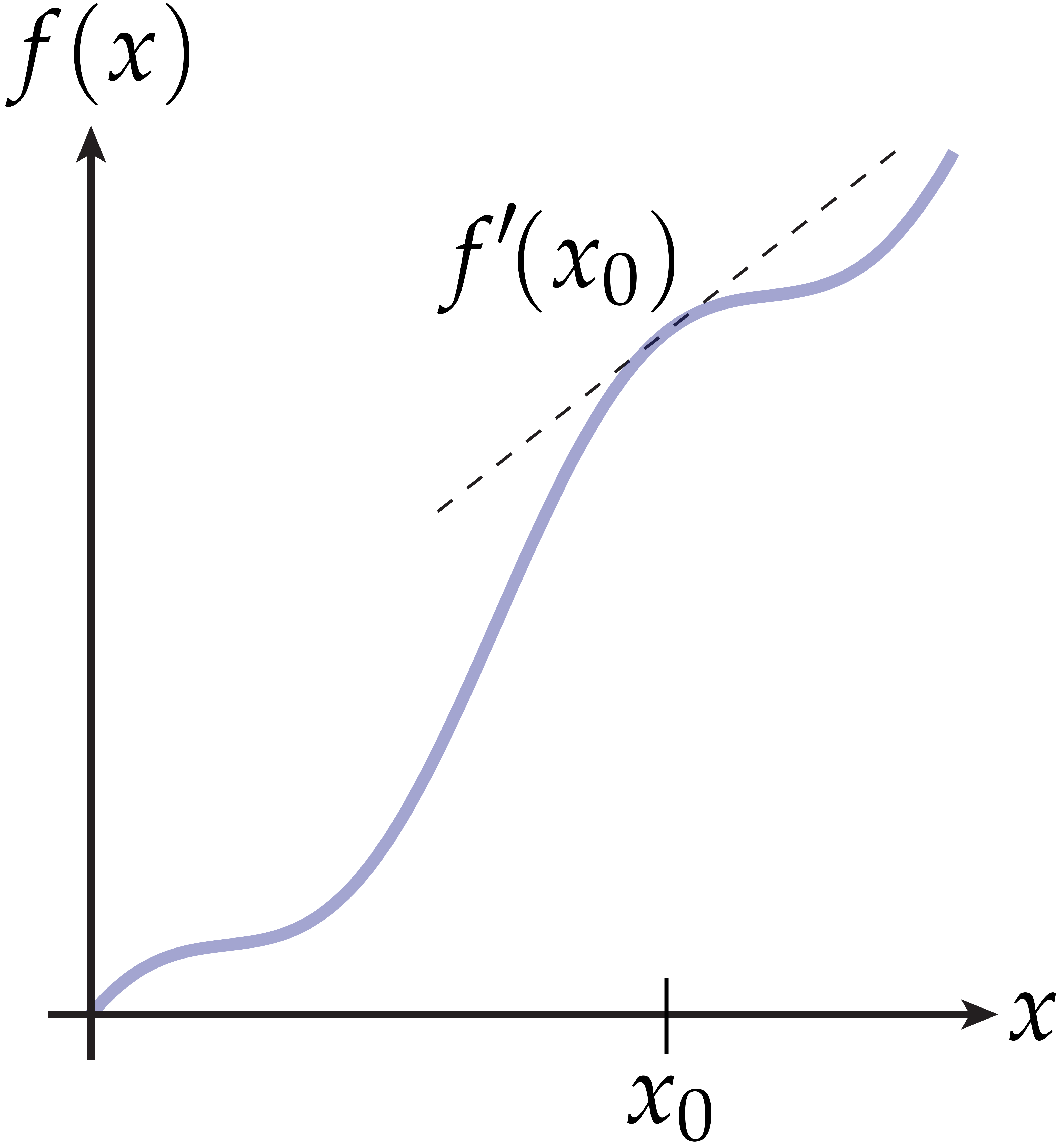
Definition. Consider a map $f : \mathbb{R} \to \mathbb{R}$. At each point $x_0 \in \mathbb{R}$, let

$$f^+(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon},$$

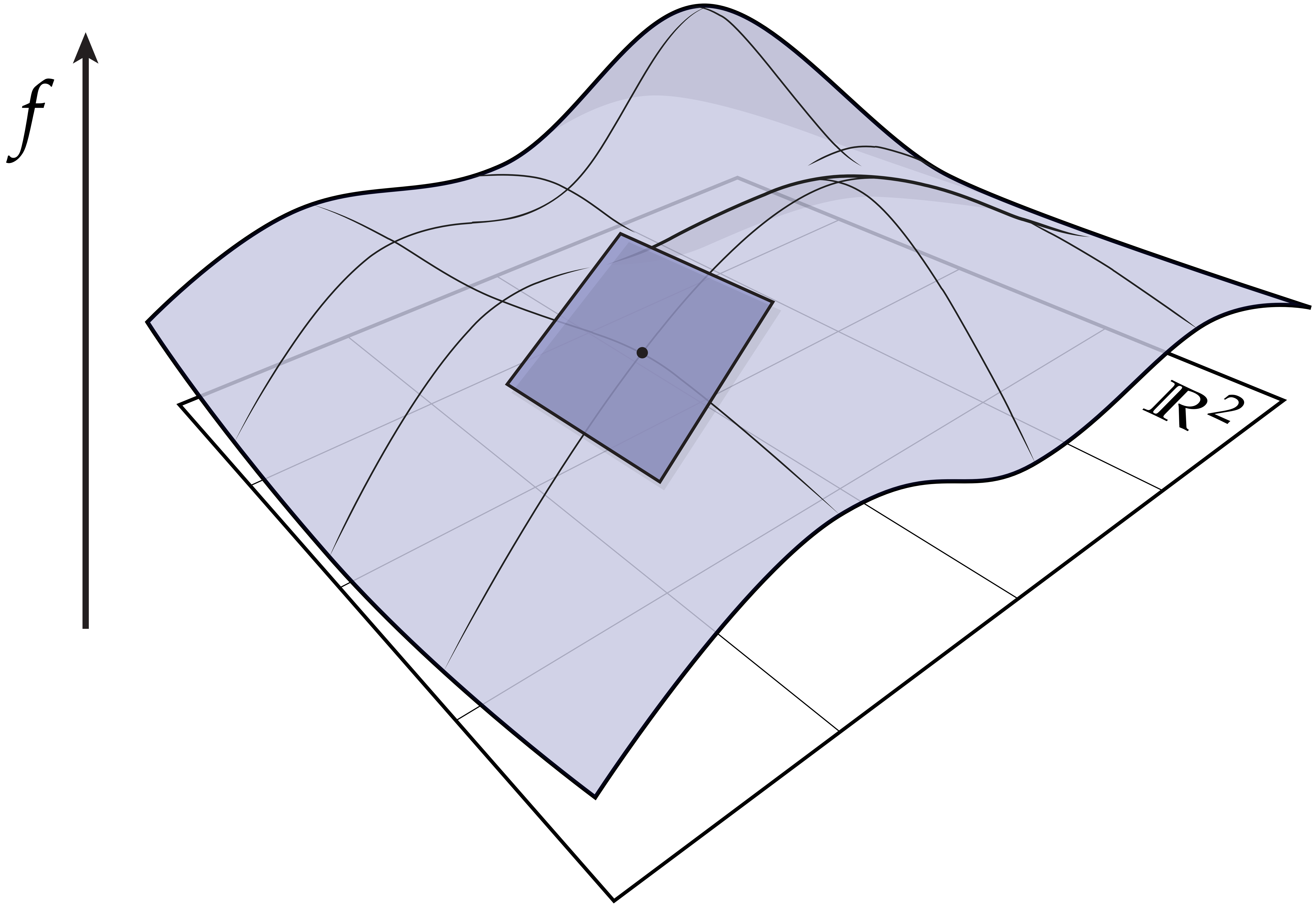
and likewise

$$f^-(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0) - f(x_0 - \varepsilon)}{\varepsilon}.$$

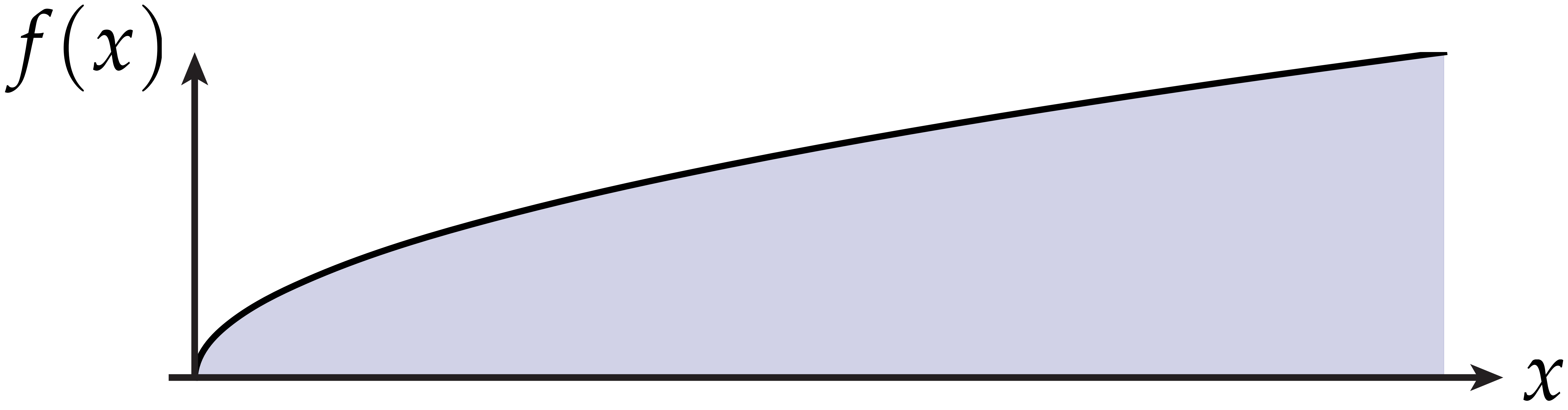
If $f^+ = f^-$ then f is *differentiable* and $f' := f^+ = f^-$ is its *derivative*.



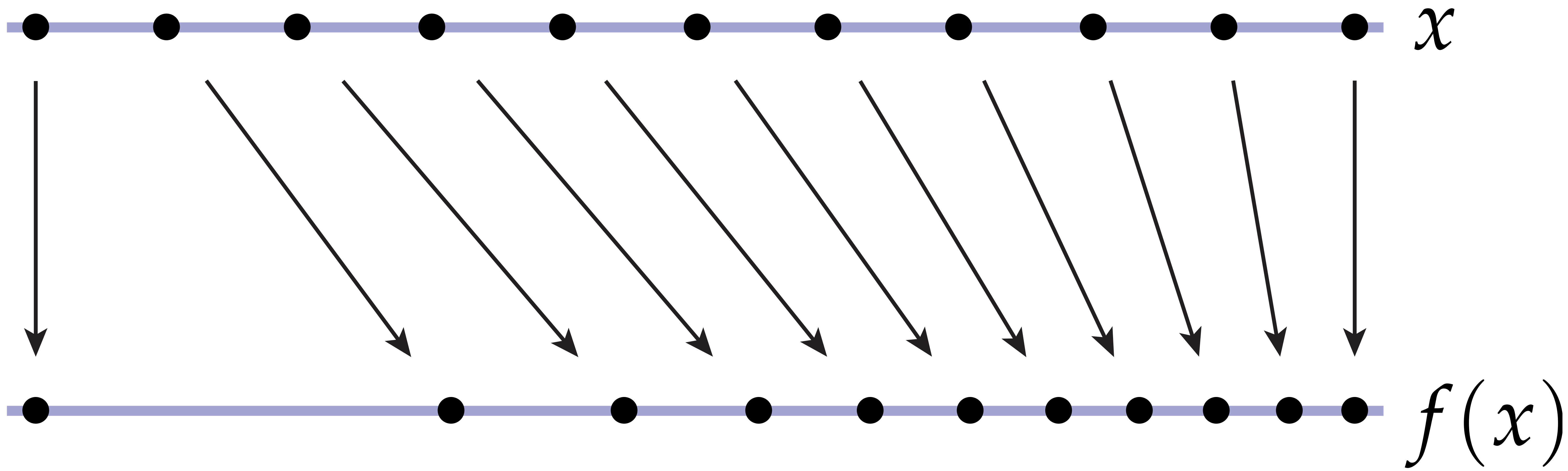
Derivative as Linear Approximation



Derivative as Stretching

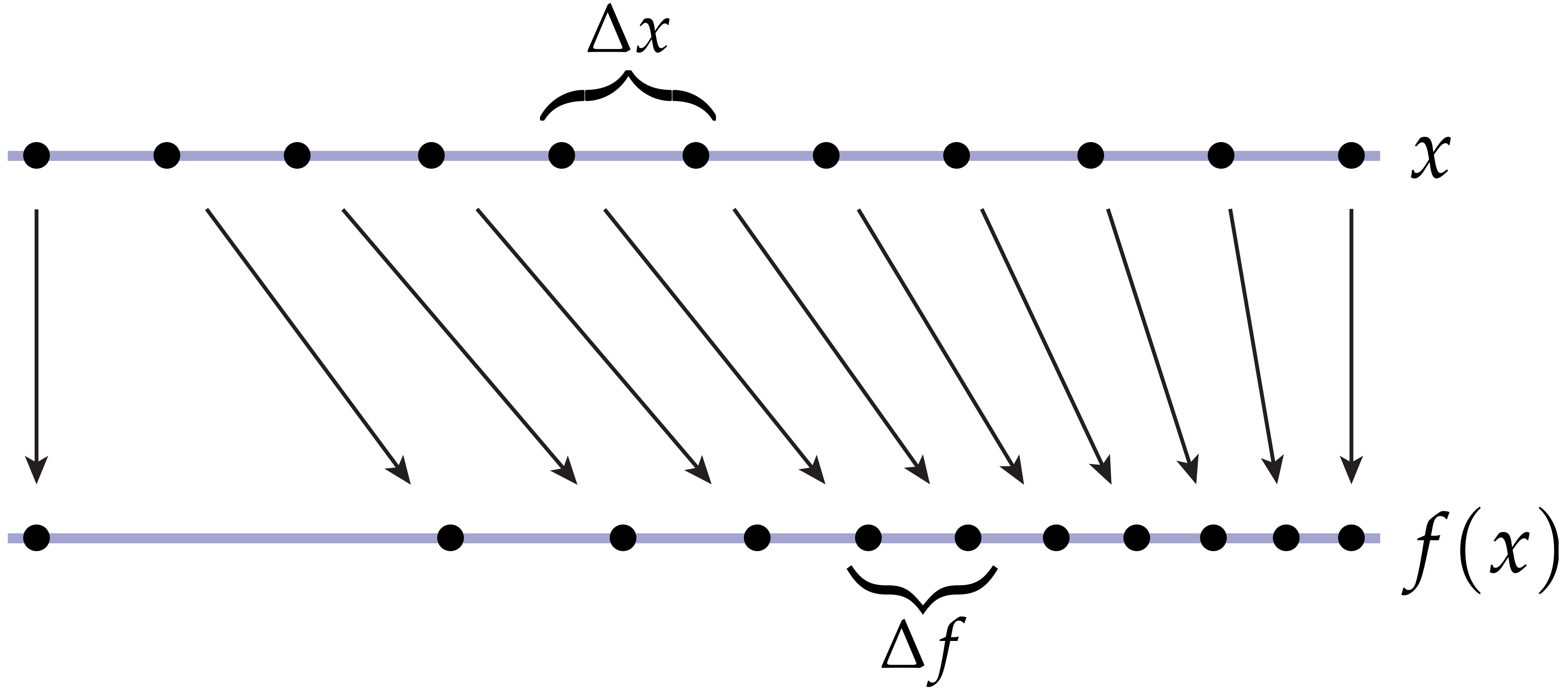


...same function, different picture...



Visual Calculus—Derivative as Stretching

Derivative is no longer the slope; it is now the amount by which the function locally gets squashed/stretched:

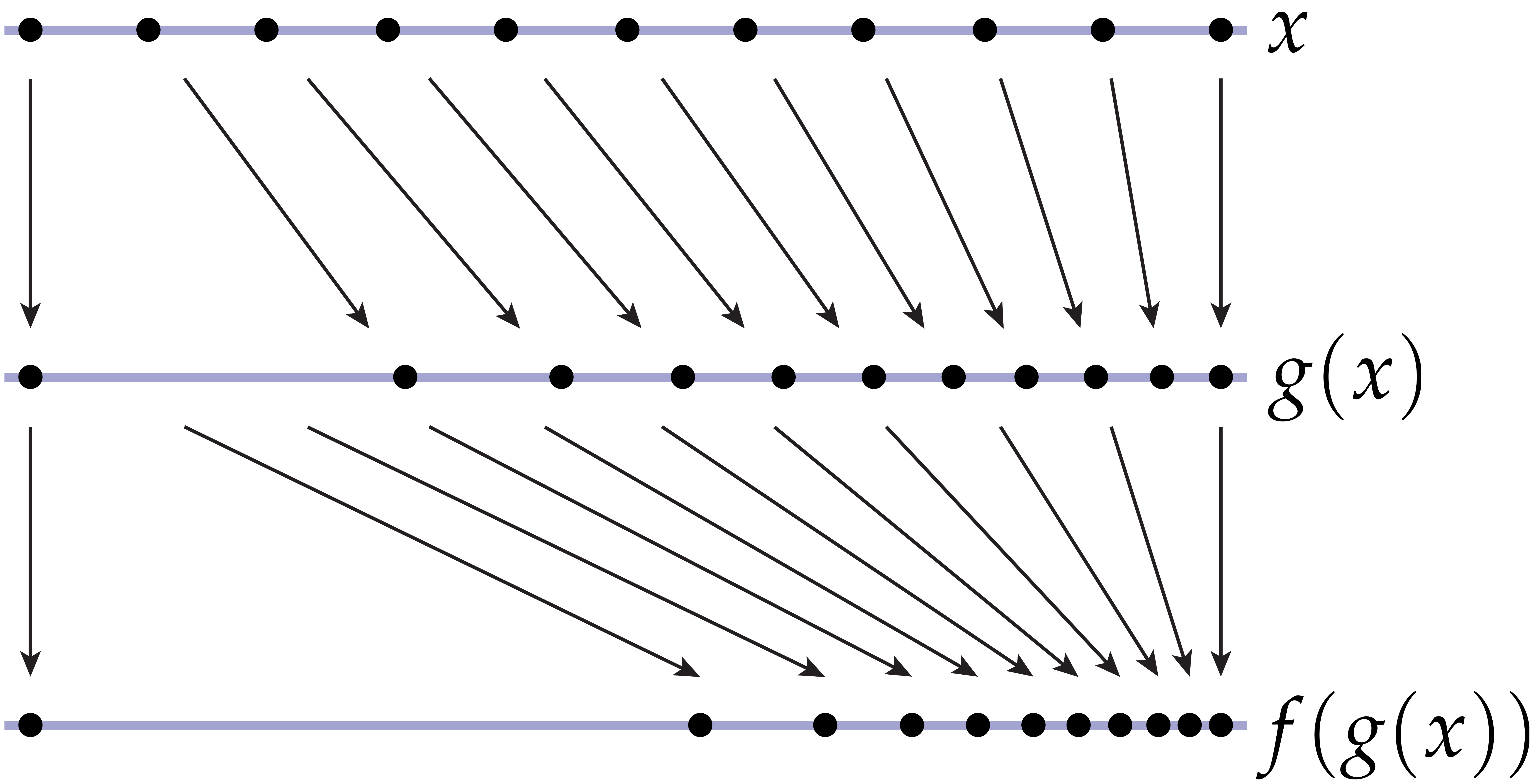


$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x}$$

Visual Calculus—The Chain Rule

Q: Why does $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$?

A:

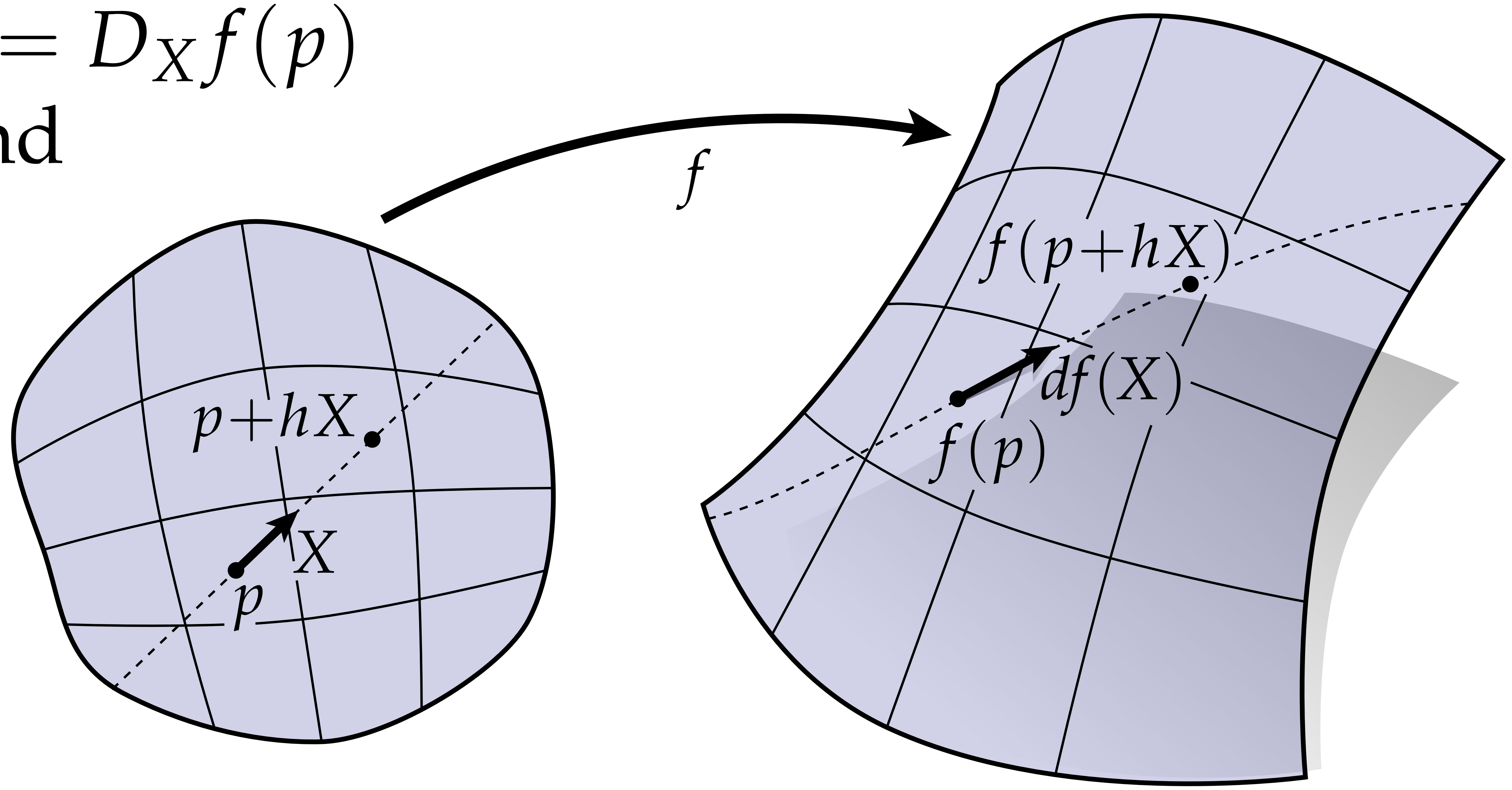


Differential

Definition. Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$. At any point $p \in \mathbb{R}^n$ and in any direction $X \in \mathbb{R}^n$, define the *directional derivative*

$$D_X f(p) := \lim_{\varepsilon \to 0} \frac{f(p + \varepsilon X) - f(p)}{\varepsilon}$$

for $\epsilon > 0$. Suppose that at each point $p \in \mathbb{R}^n$ there exists a **linear** map $df_p : \mathbb{R}^n \to \mathbb{R}^m$ such that $df_p(X) = D_X f(p)$ for all X. Then f is a *differentiable function*, and df is its *differential*.



Linear Maps

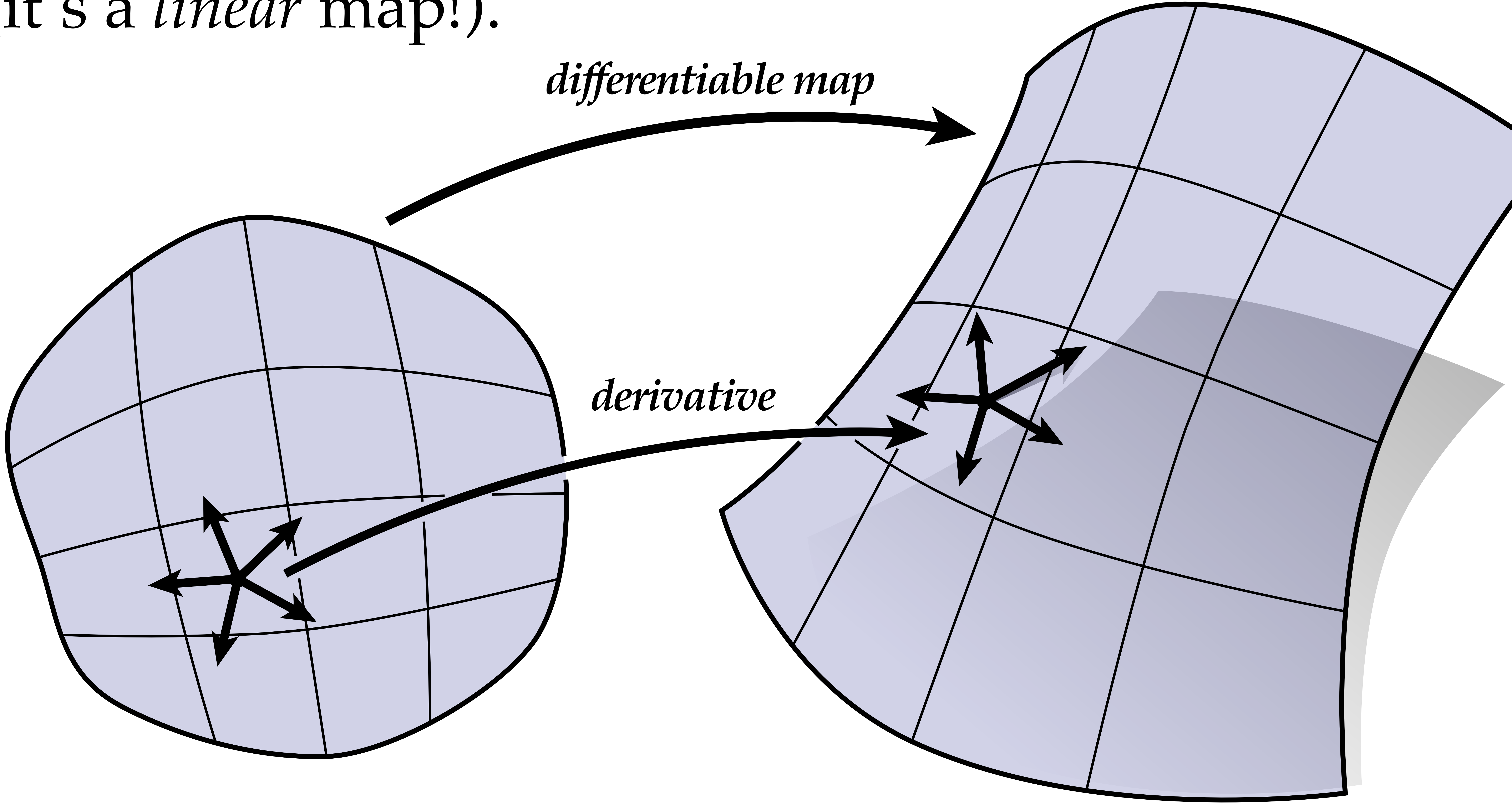
- We said linear algebra is about "vector spaces and maps between them."
- Now that we know what a vector space is, what is a linear map?
- Formally: $f : V_1 \to V_2$ is a *linear map* if

$$f(ax + by) = af(x) + bf(y) \;\forall x, y \in V_1,\; a, b \in \mathbb{R}$$

- (Notice that definition makes sense only when we know how to add, scale, etc., vectors, i.e., when we have a *vector space structure* on our set.)
- **Q:** What does it mean geometrically?
- **A:** Vectors get mapped to vectors; lines (through origin) mapped to lines

Derivative as Linear Map

Key idea: derivative can always be visualized as a map from vectors to vectors (it's a *linear* map!).



Differential in Coordinates (Jacobian)

Definition. Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$, and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. Then the *Jacobian* of f is the matrix

$$J_f := \begin{bmatrix} \partial f^1/\partial x^1 & \cdots & \partial f^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial f^m/\partial x^1 & \cdots & \partial f^m/\partial x^n \end{bmatrix},$$

where $f^1, \ldots, f^m$ are the components of f w.r.t. some coordinate system on $\mathbb{R}^m$. This matrix represents the differential in the sense that $df(X) = J_f X$.

Note: does not generalize to infinite dimensions! (E.g., maps between functions.)

Differential of a Graph

$$x \stackrel{\phi}{\longmapsto} (x, f(x))$$

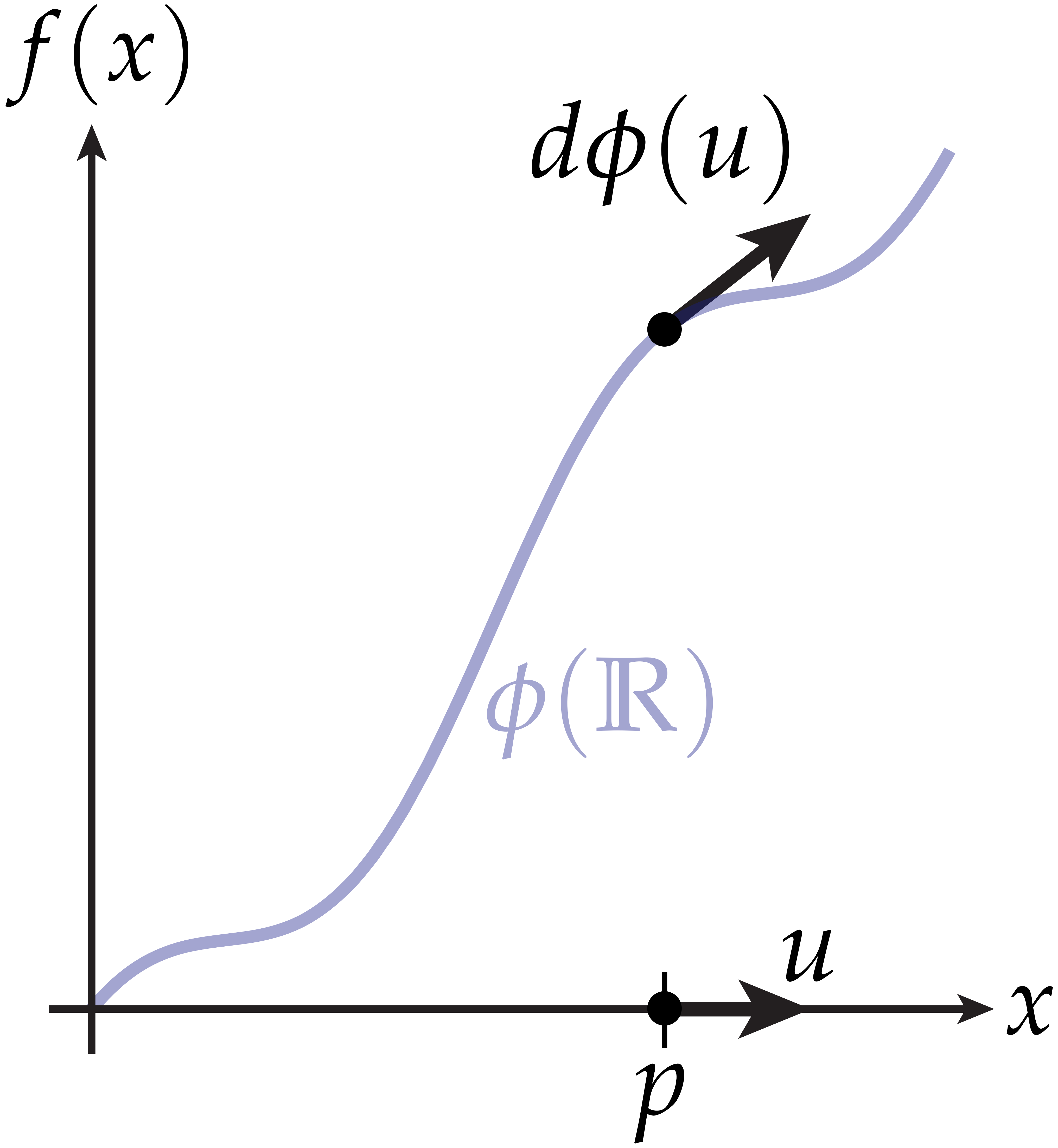
$$d\phi(u) = (u, uf')$$

$$J_\phi = \begin{bmatrix} 1 \\ f' \end{bmatrix}$$

unit

$$L_a^b = \int_a^b |d\phi(u)|\, dx$$

$$= \int_a^b \sqrt{1 + (f')^2}\, dx$$



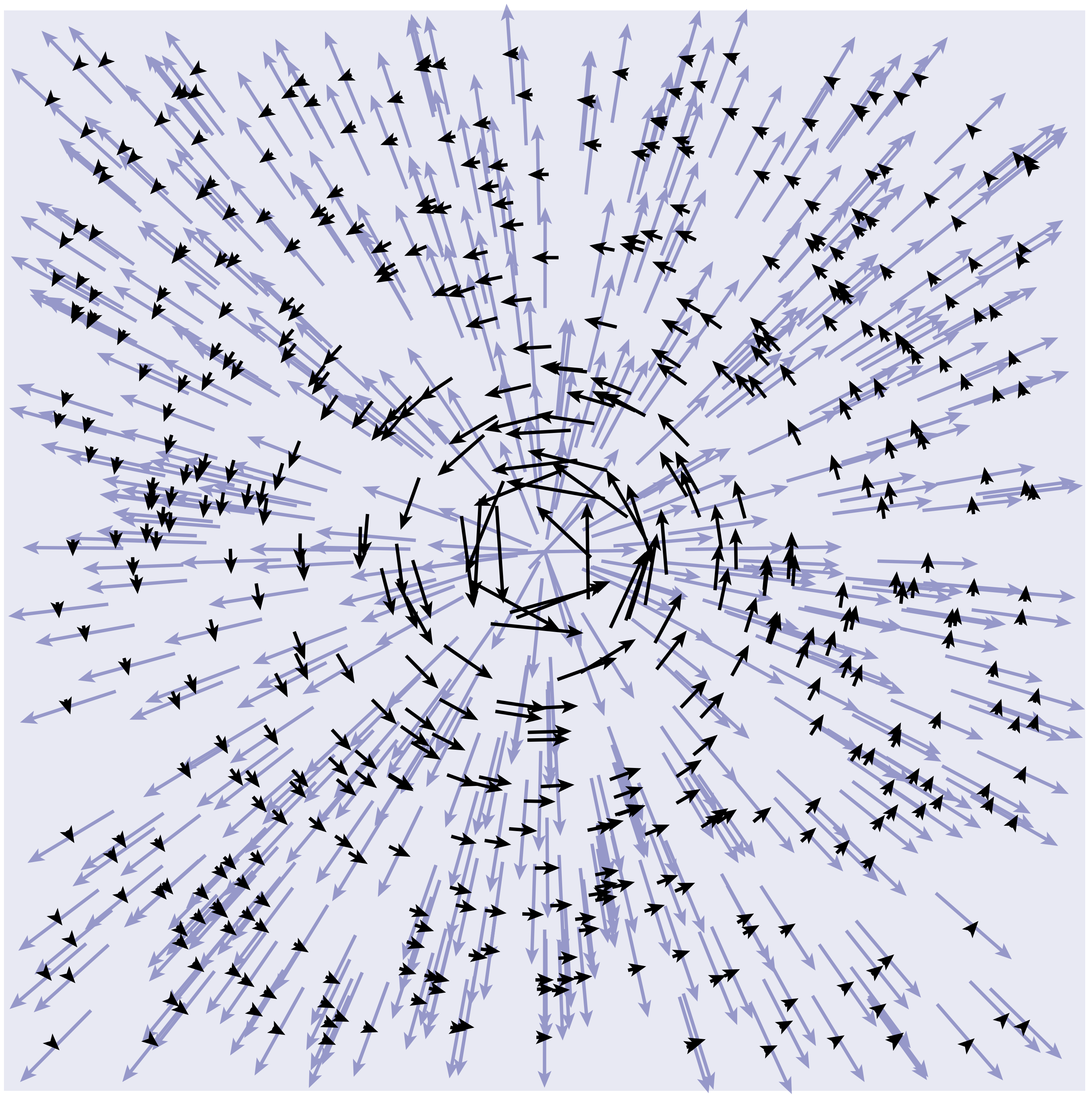
Differential — Example

Example. Consider the map

$$\underbrace{(x, y)}_{=:X} \overset{f}{\mapsto} (r, \theta),$$

taking rectangular coordinates to polar coordinates. The corresponding differential is

$$df(X) = (X/r, X^{\perp}/r^2).$$

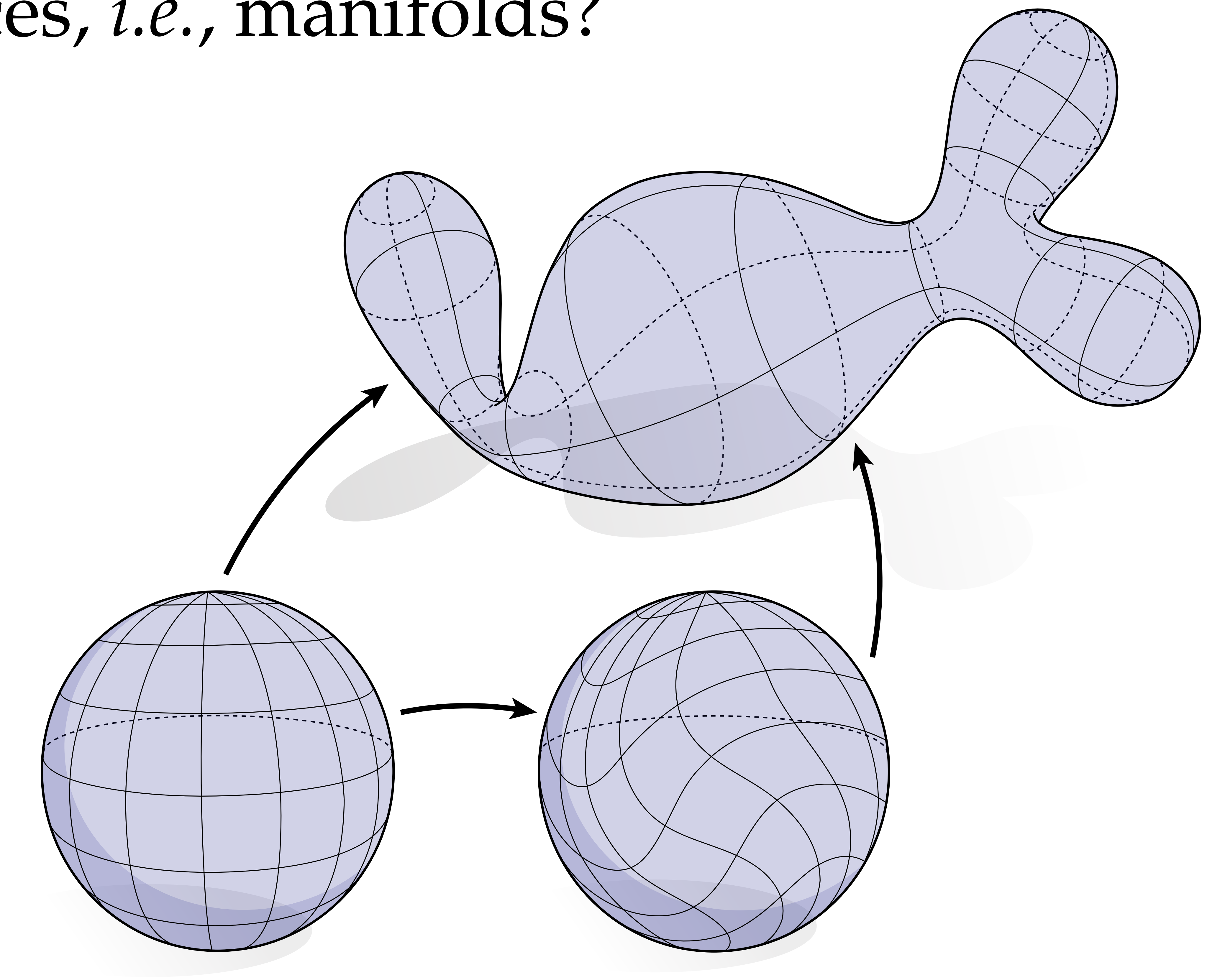


df

(Could also write out Jacobian... but why? What will we learn by doing this?)

Differentiable Maps Between Manifolds?

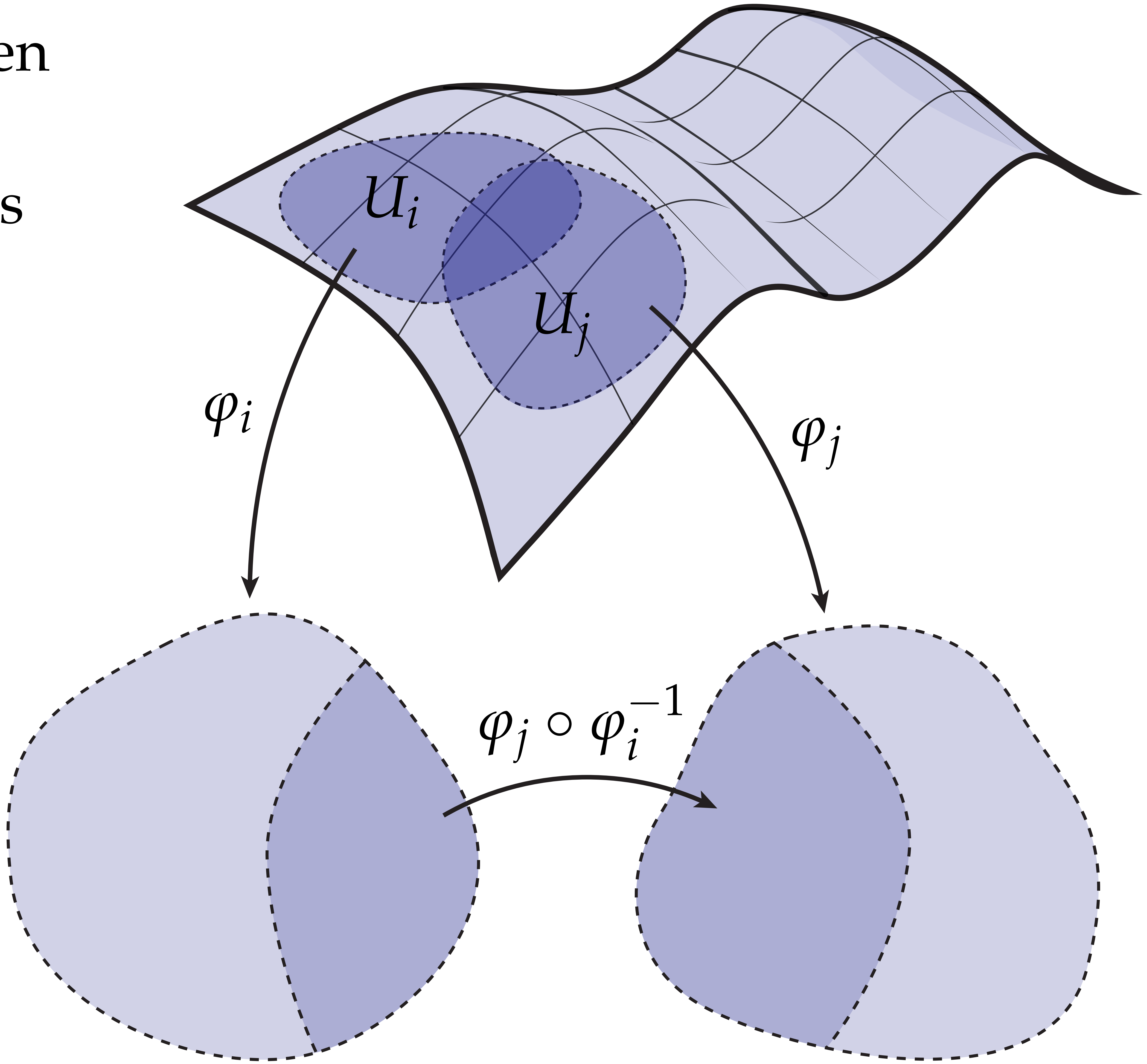
Ok, so we know how to talk about differentiation on $\mathbb{R}^n$. What about more general spaces, *i.e.*, manifolds?



Key idea in *differential* geometry: leverage $\mathbb{R}^n$ to talk about derivatives on manifolds. (*"Calculus on manifolds."*)

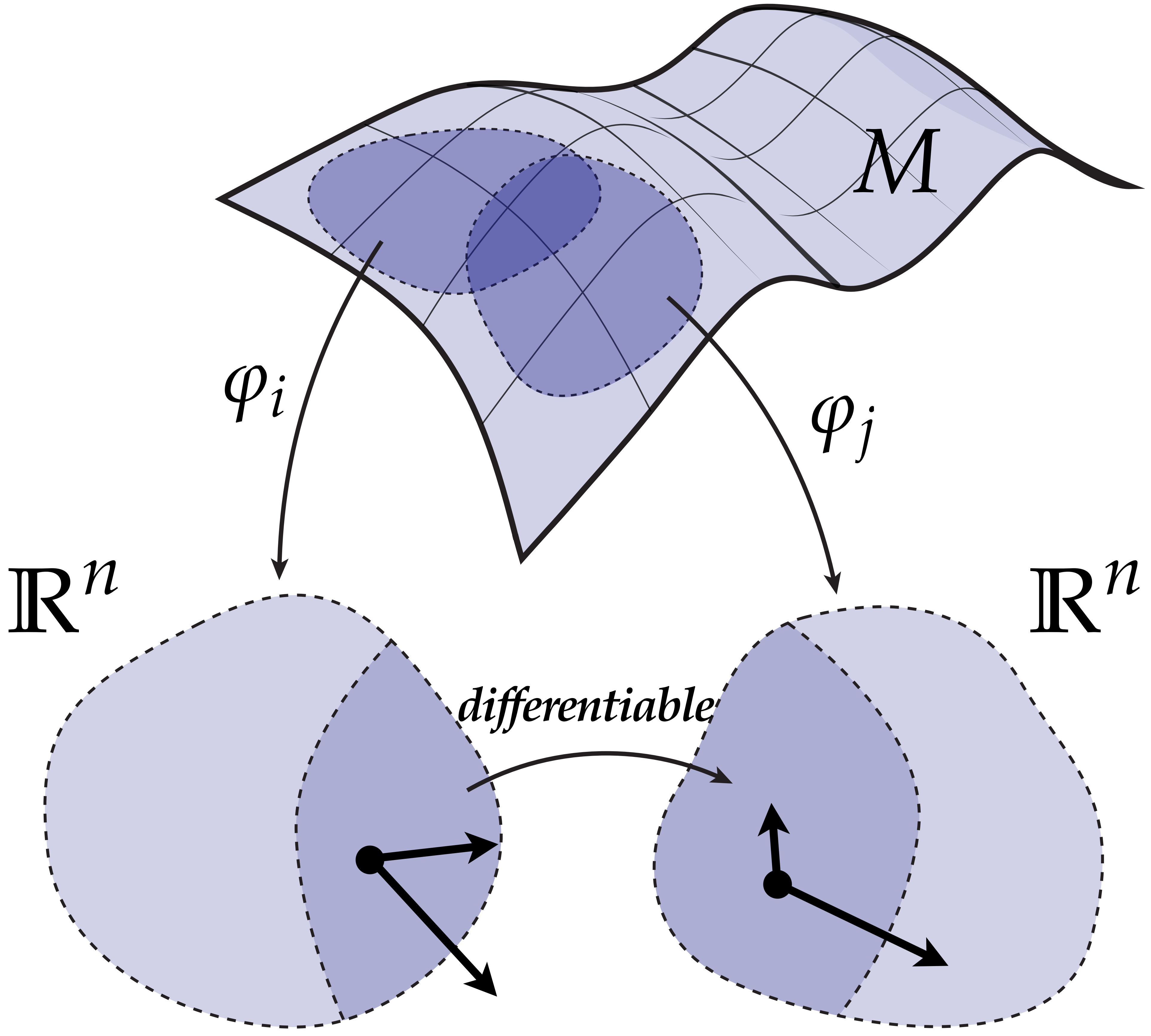
Reminder—Compatibility of Charts

Key idea: if some object or quantity (open sets, lengths, angles, etc.) are consistent across overlaps "below," then it provides a unique and meaningful definition "above."



Differentiable Manifold

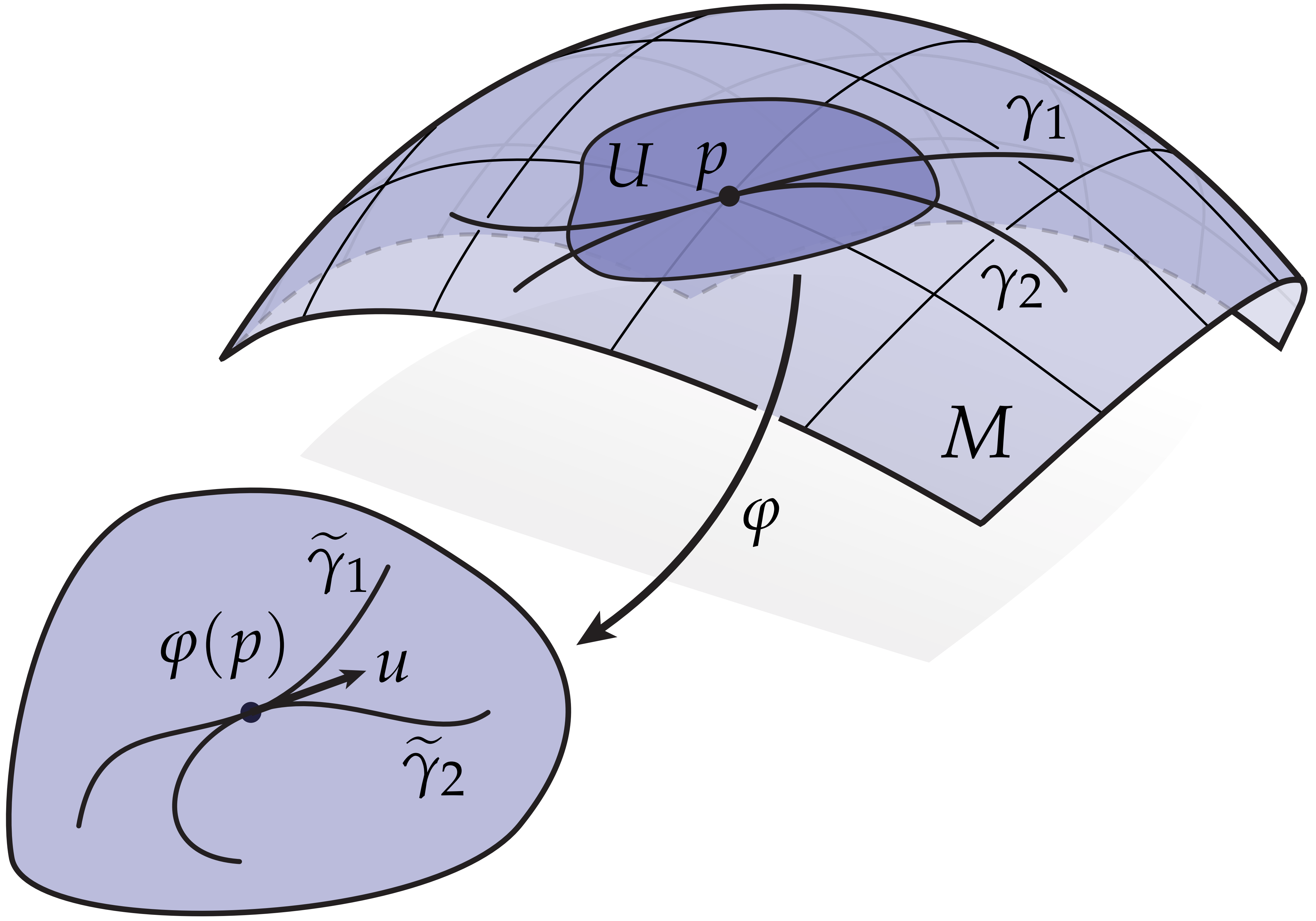
Definition. Let M be a topological manifold with an atlas of charts $\varphi_i : M \supset U_i \to \mathbb{R}^n$. If the overlap maps $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}$ are k-times differentiable, then M is a C^k *differentiable manifold*; if they are infinitely many times differentiable, then M is a C^∞ or *smooth manifold*.



Key idea: M now has (tangent) vectors

Tangent Vectors

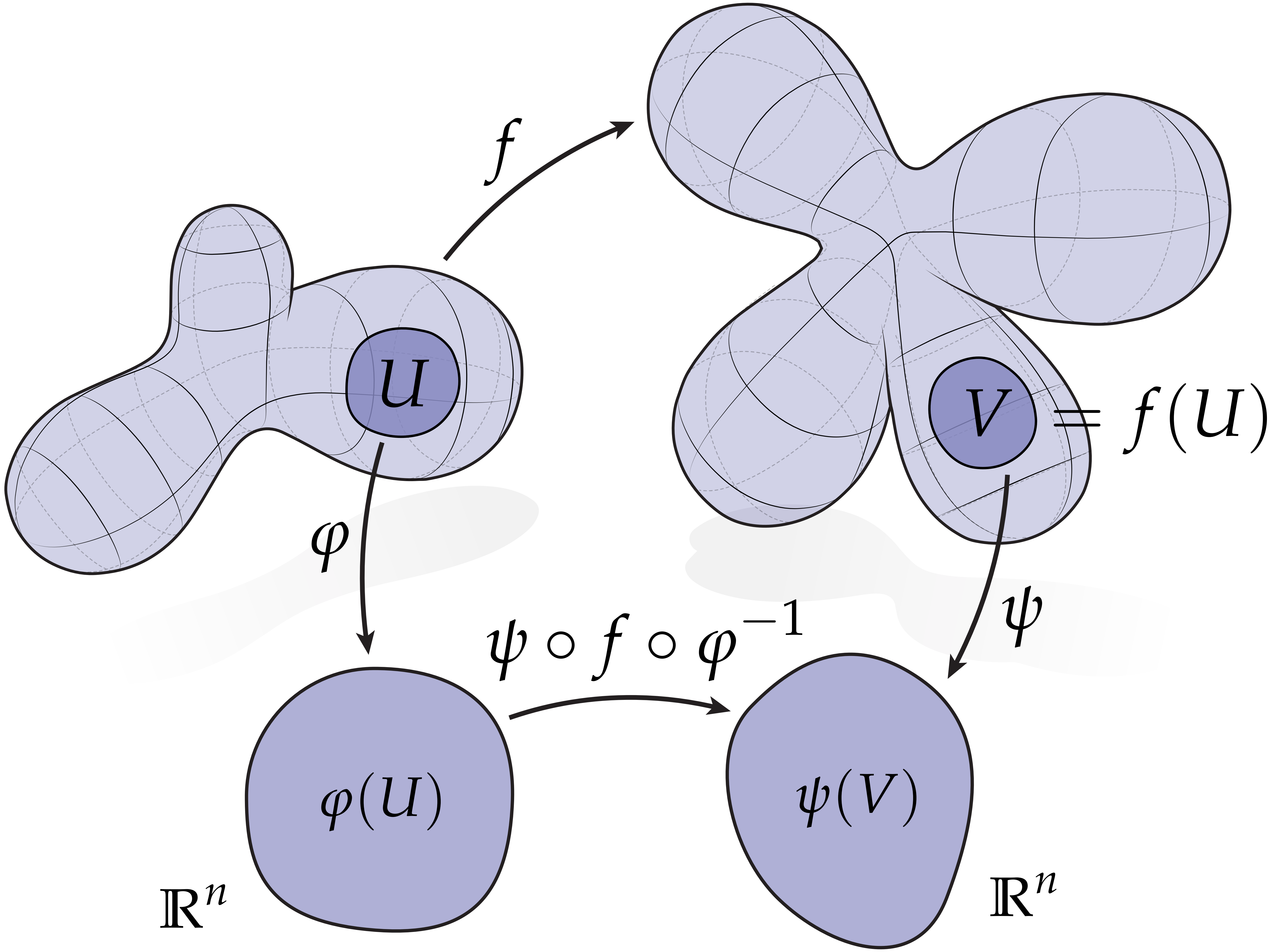
Definition. Let M be a differentiable manifold; a *curve through* $p \in M$ is a map $\gamma : \mathbb{R} \to \mathbb{M}$ such that $\gamma(0) = p$. For any chart $\varphi : U \to \mathbb{R}^n$ on M, let $\tilde{\gamma}$ denote the Euclidean realization of a curve γ, *i.e.*, $\tilde{\gamma} := \varphi \circ \gamma|_{\gamma^{-1}(\gamma(\mathbb{R}) \cap U)}$. Let two curves γ_1, γ_2 be equivalent at p if and only if their Euclidean velocity is the same, *i.e.*, if $\tilde{\gamma}_1'(0) = \tilde{\gamma}_2'(0) = u$ for some fixed vector u. A *tangent vector* at p is then an equivalence class of curves. The *tangent space* at p, denoted T_pM is the collection of all curves, together with the usual vector space operations in $\mathbb{R}^n$. The *tangent bundle* is the collection of all tangent spaces T_pM, $p \in M$.



Q: Why did we demand a *differentiable* manifold for this definition?

Differentiable Maps and Diffeomorphisms

Definition. Let $f : M \to N$ be a continuous map between differentiable manifolds M and N, let $\{\varphi_i : U_i \to \mathbb{R}^n\}$ be an atlas on M, and let $\{\psi_i : V_i \to \mathbb{R}^n\}$ be an atlas on N where $V_i := f(U_i)$. If the compositions $\psi_i \circ f \circ \varphi_i^{-1}$ are differentiable (smooth), then f is a *differentiable (smooth) map*. If $f : M \to N$ is a differentiable bijection with differentiable inverse, then it is a *diffeomorphism*, and we say that M and N have the same *differentiable (smooth) structure*.

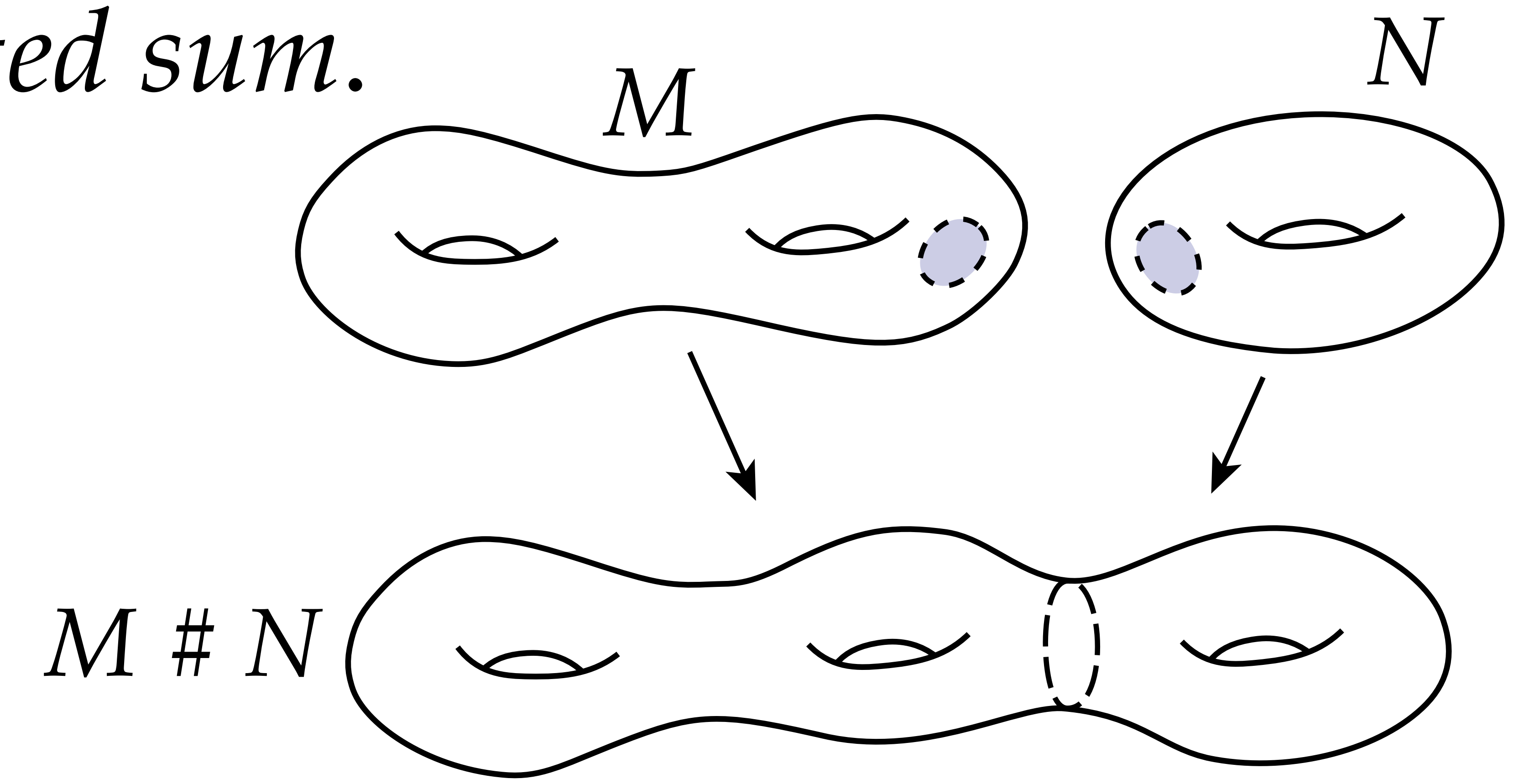


Topological vs. Smooth Structure

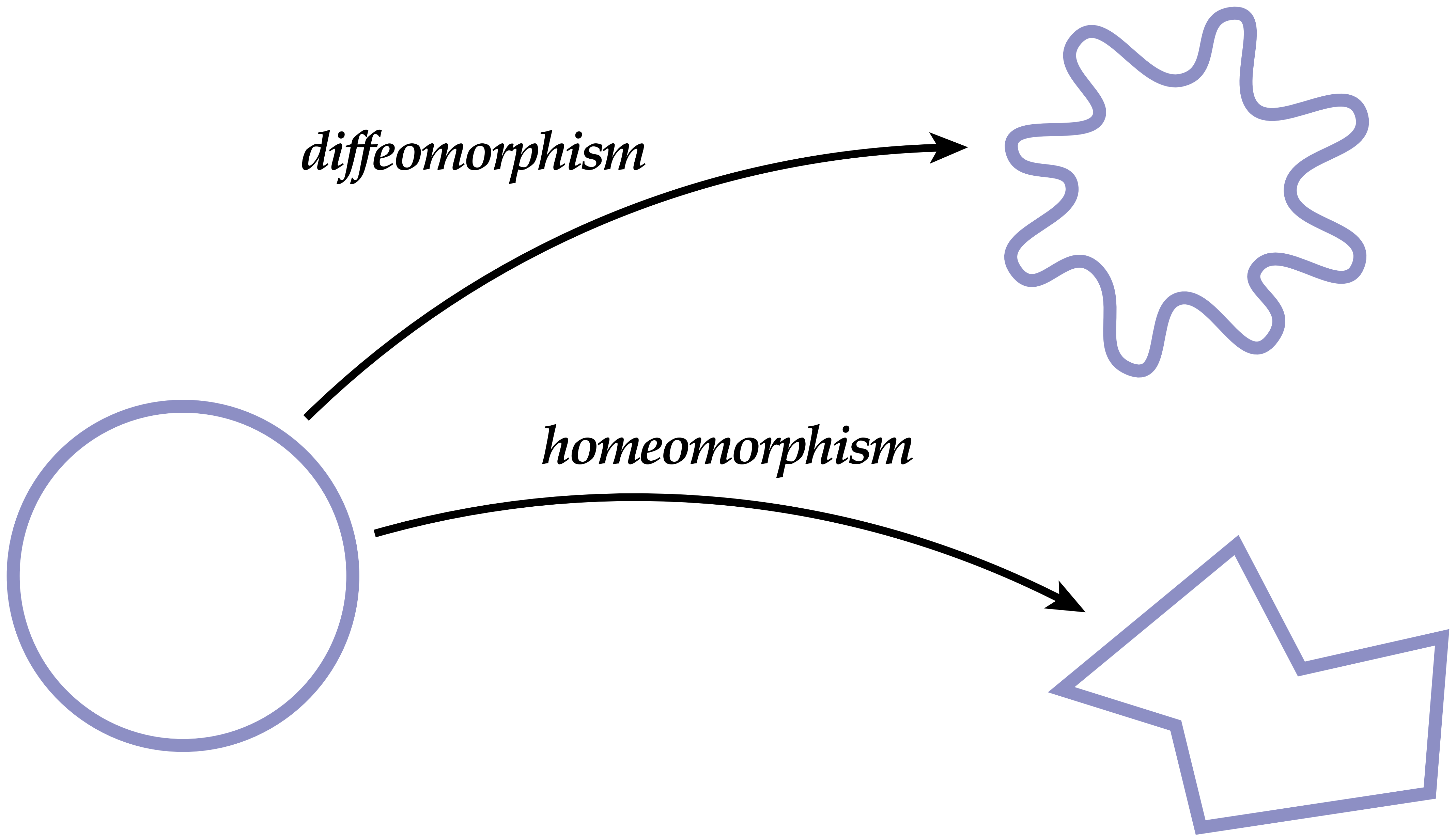
Fact. In dimension $n \leq 3$, there is only a single unique smooth structure that can be put on a given topological n-manifold, *i.e.*, if two smooth manifolds are homeomorphic, then they are automatically diffeomorphic. (Hence, one does not typically bother to specify the smooth structure.)

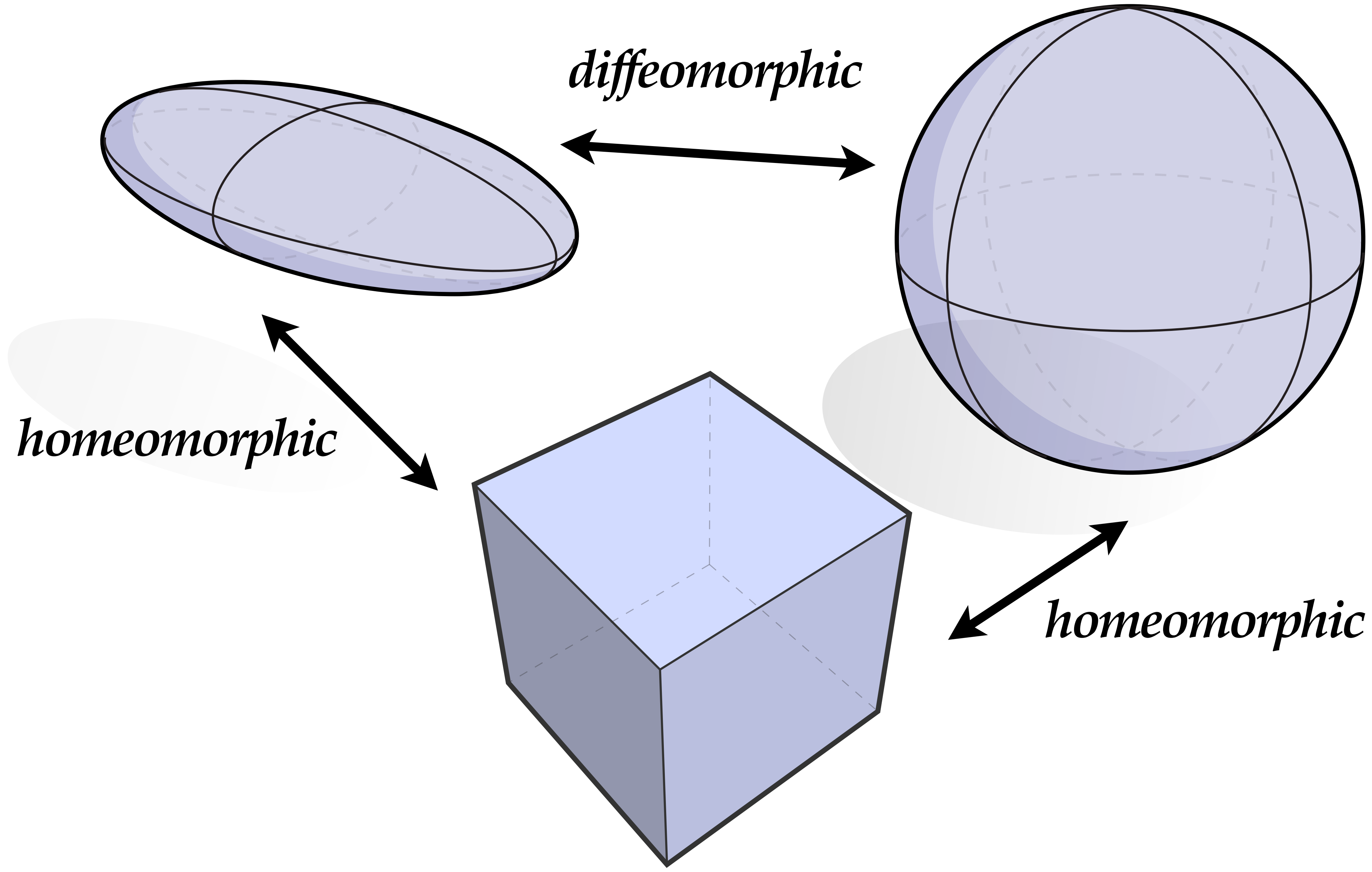
Fun Fact. There is a unique differentiable structure compatible with the Euclidean topology on $\mathbb{R}^n$, except in the case $n = 4$ where there are infinitely many "exotic" differentiable structures.

Fun Fact. The 7-dimensional sphere admits 28 distinct differentiable structures, which form an abelian monoid with respect to the *connected sum*.

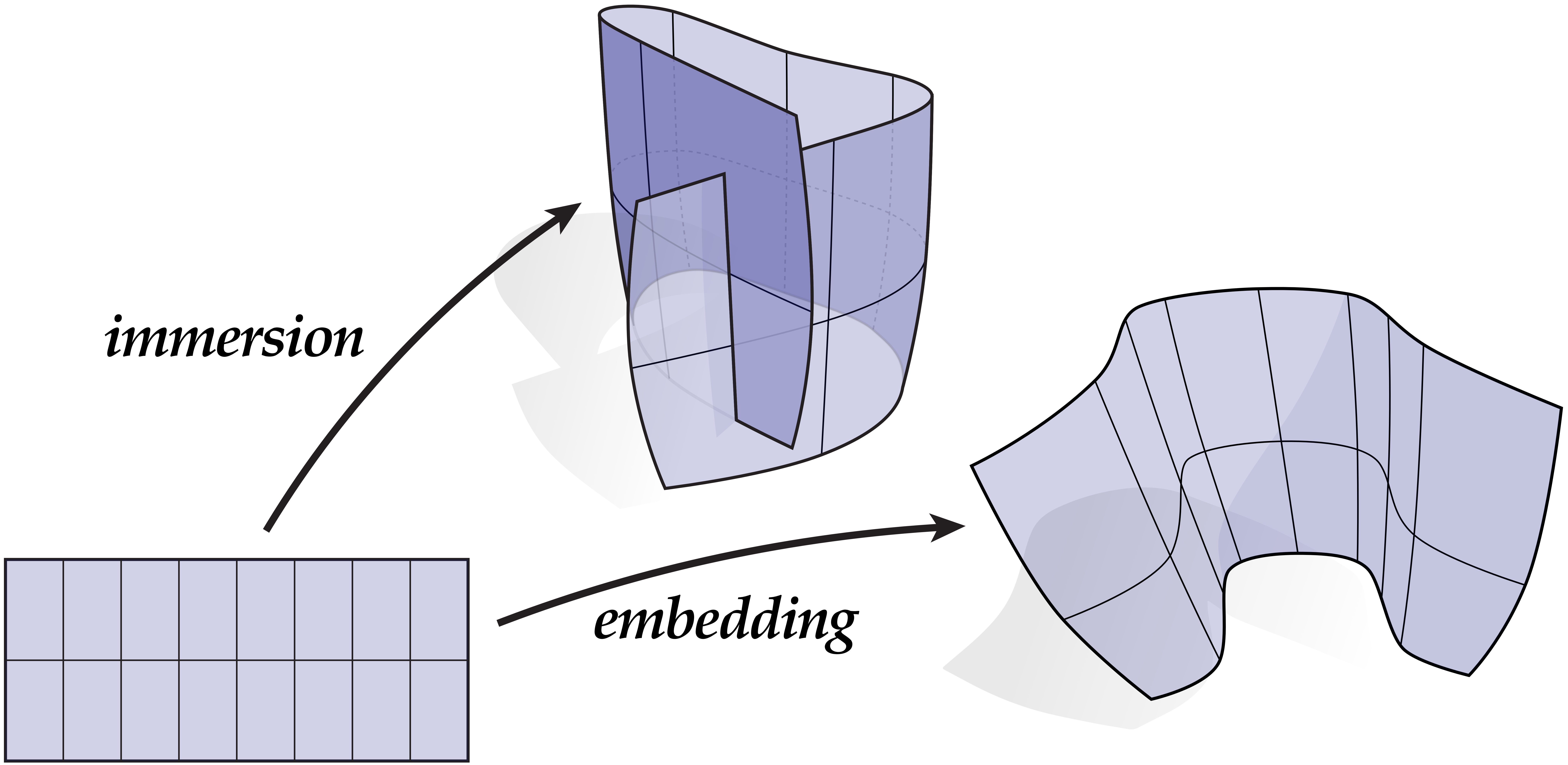


Homeomorphism vs. Diffeomorphism



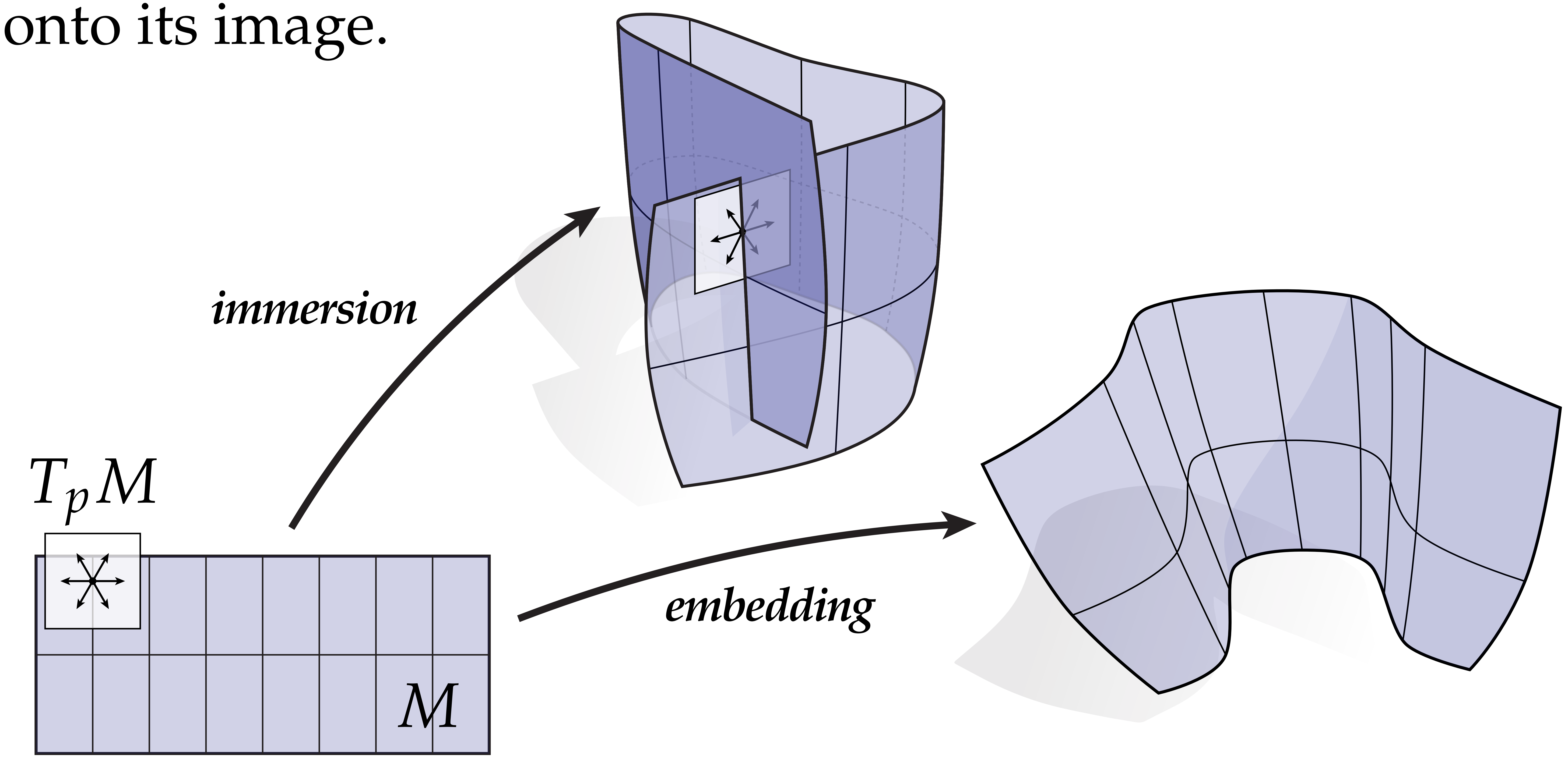


Immersion vs. Embedding — Visualized



Immersion vs. Embedding

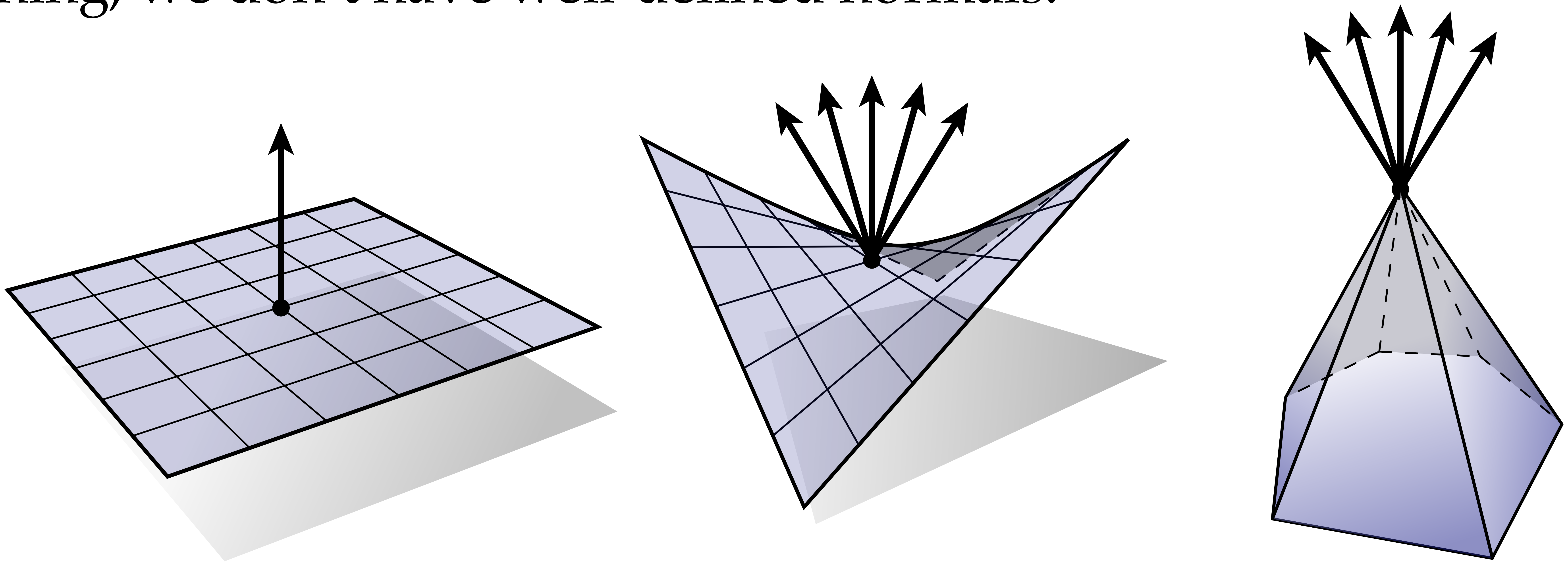
Definition. A differentiable map $f : M \to N$ between differentiable manifolds is an *immersion* if its differential $df_p : T_pM \to T_pN$ is nondegenerate at each point $p \in M$, *i.e.*, if $df_p(X) = 0$ if and only if $X = 0$. In other words, if the differential is an injective linear map between tangent spaces. An immersion is an *embedding* if it is also a homeomorphism onto its image.



Discrete Tangent Vectors?

How do we define tangent vectors for a discrete manifold?

For one thing, we don't have well-defined normals:

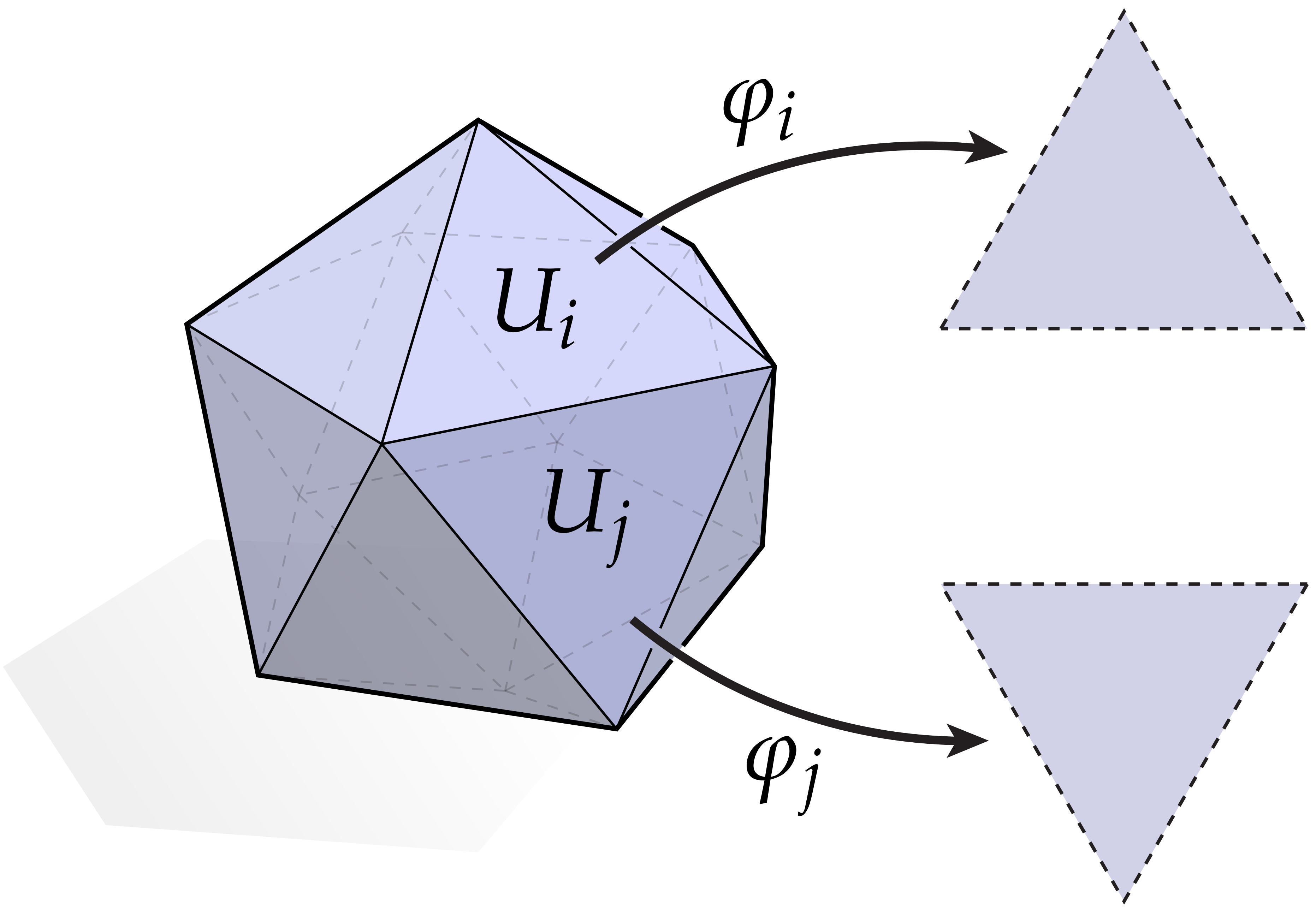


…In fact, we don't know *anything* about how the manifold sits in space!

(So far, our "discrete manifold" is just a gluing together of abstract simplices.)

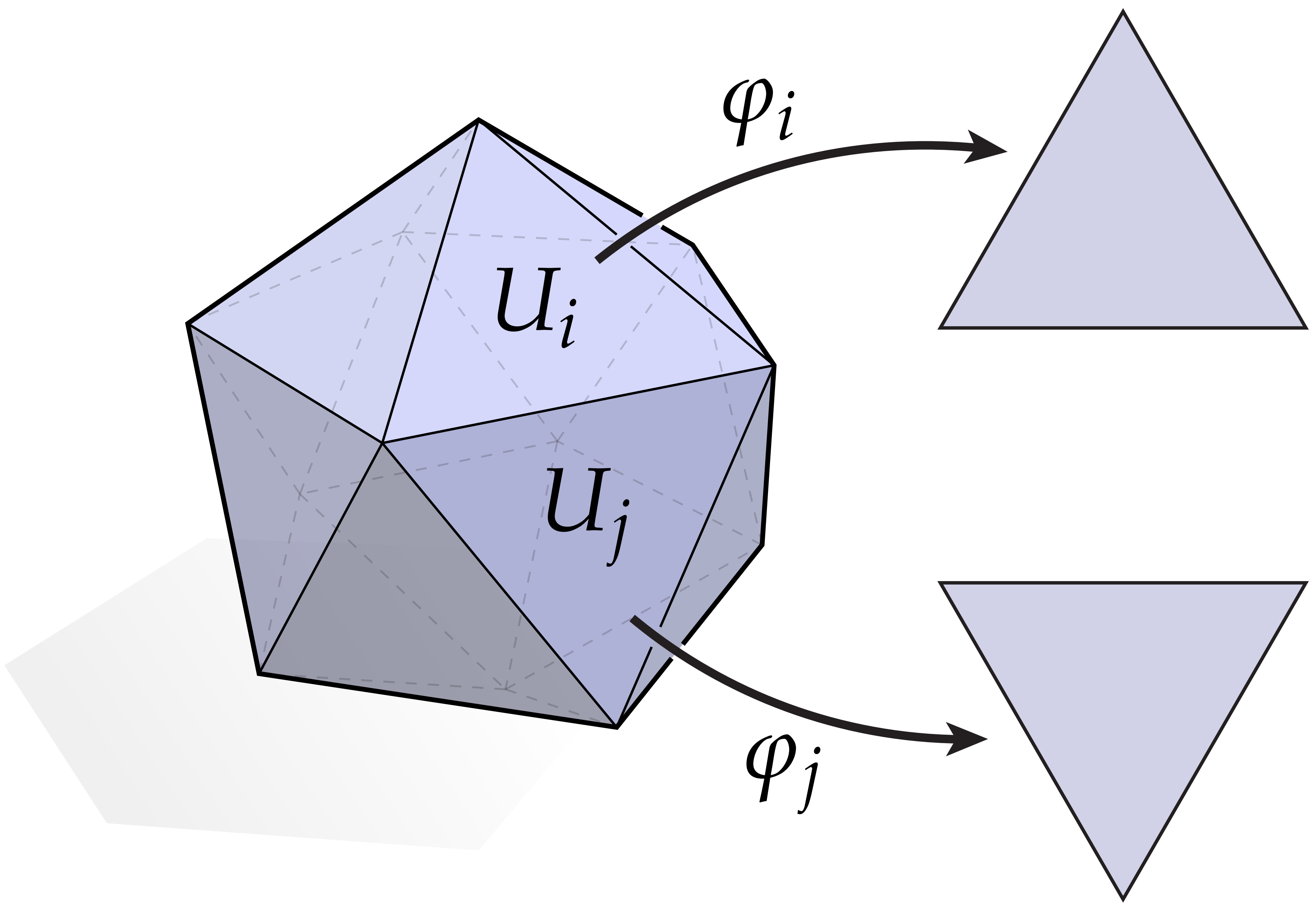
Atlas on a Discrete Manifold...?

Natural idea: use *simplices* as charts (w/ mapping to standard simplex):



Open simplices?

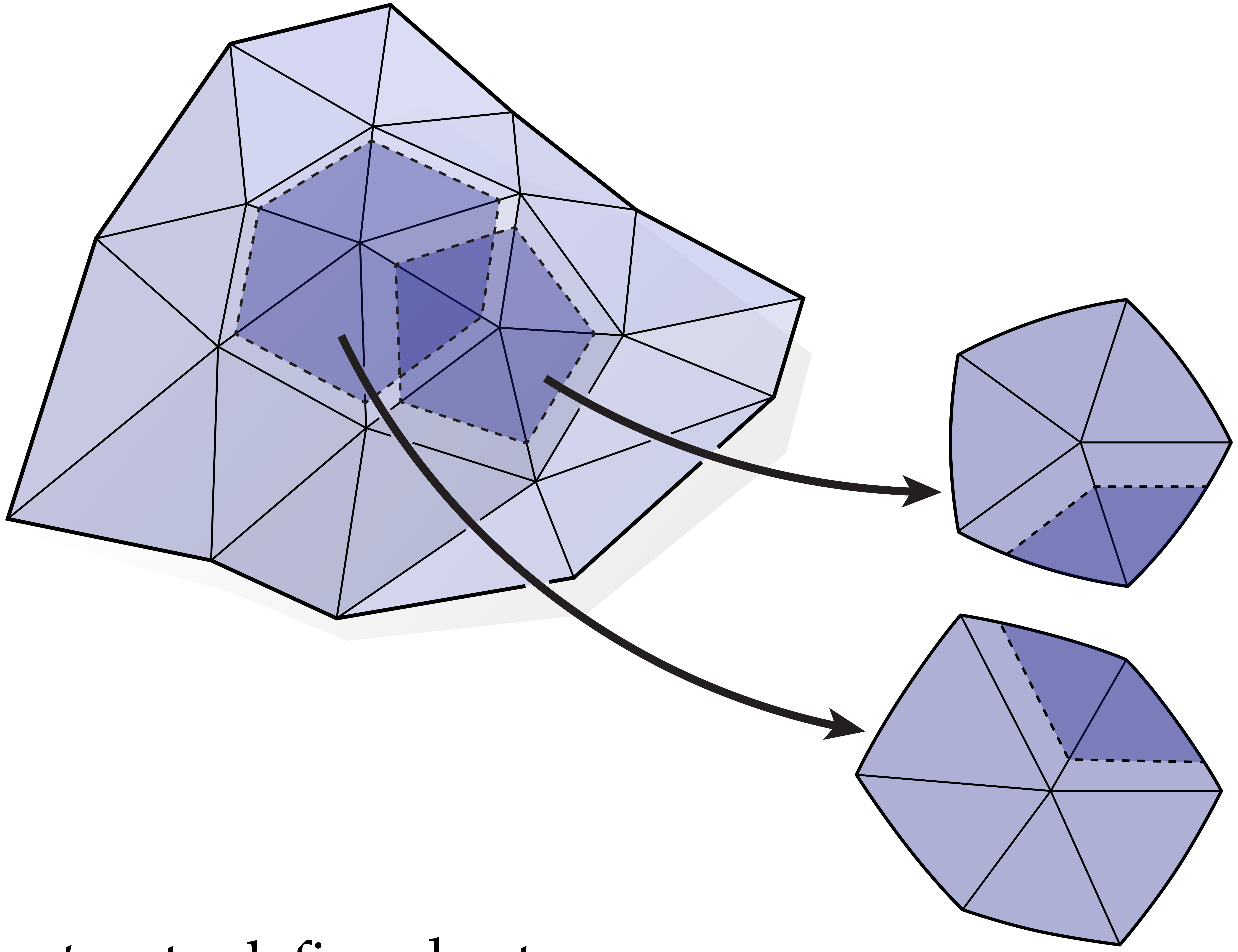
Problem: overlap is *empty!*
No way to compare quantities.



Closed simplices?

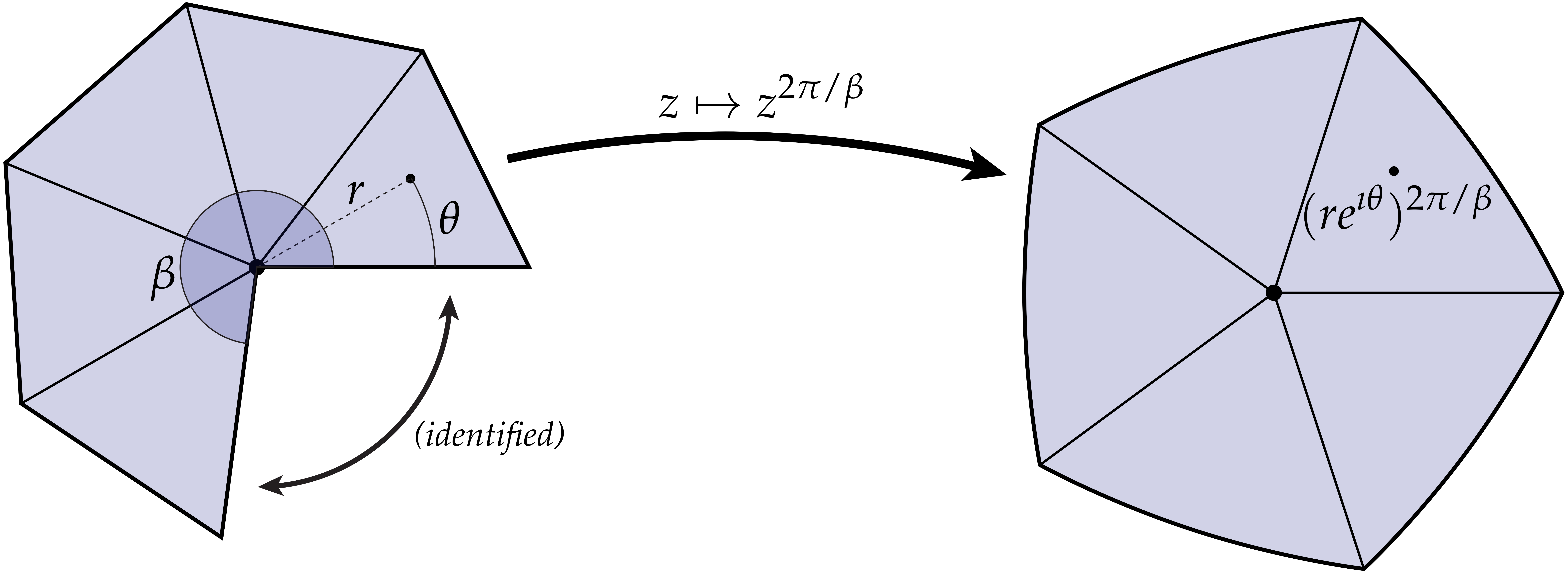
Problem: overlap is *trivial!*
…Still no way to compare quantities.

Simplicial Atlas



Key idea: use *vertex stars* to define charts.

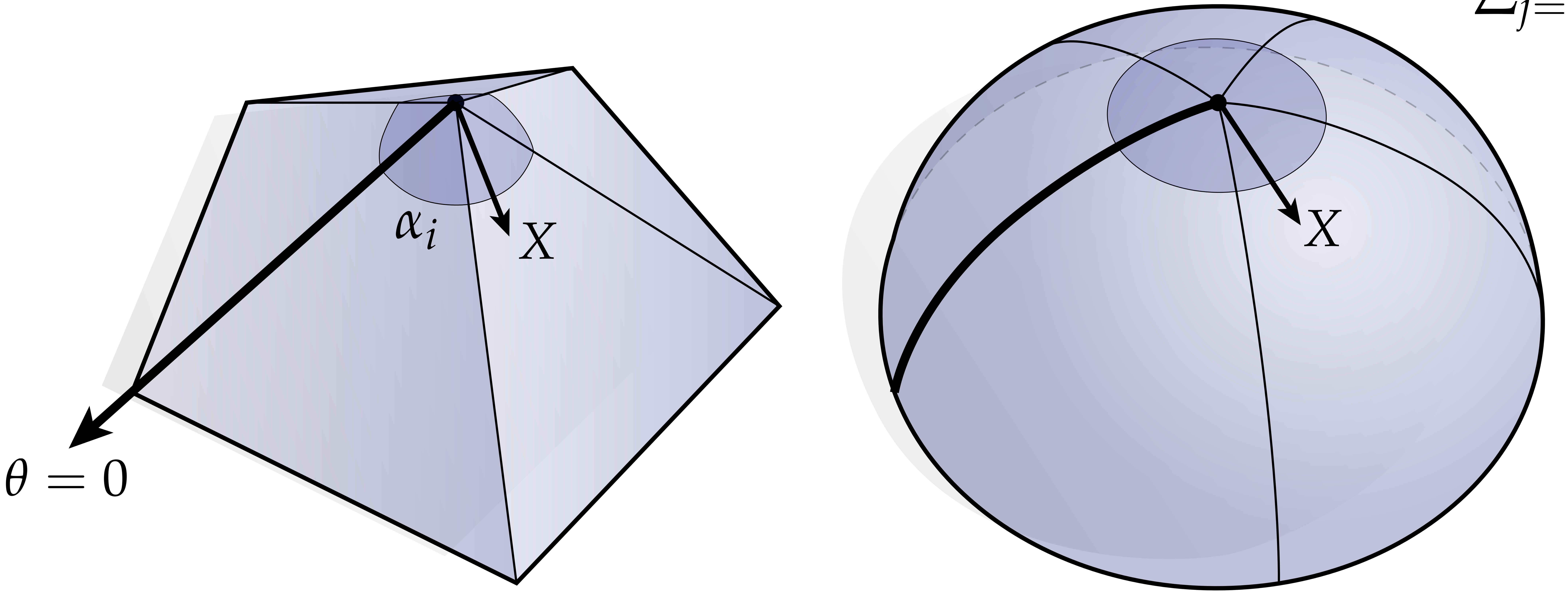
Parameterization of Simplicial Charts



Bijective? Continuous? Homeomorphism? *Diffeomorphism?*

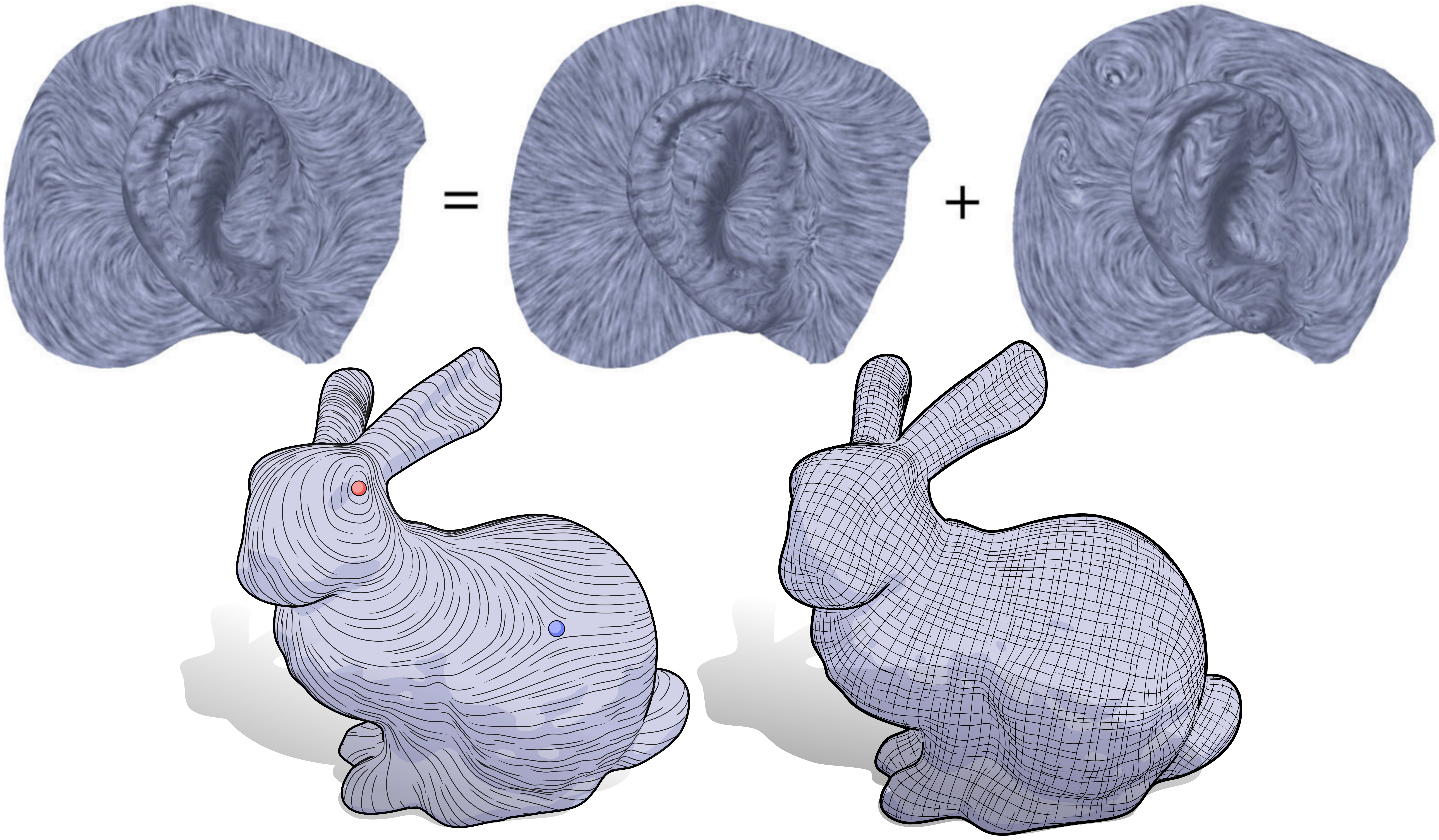
Coordinates on Discrete Tangent Spaces

$$\tilde{\alpha}_i := \frac{2\pi}{\sum_{j=1}^{n} \alpha_j} \alpha_i$$



(Generalization to 3D?)

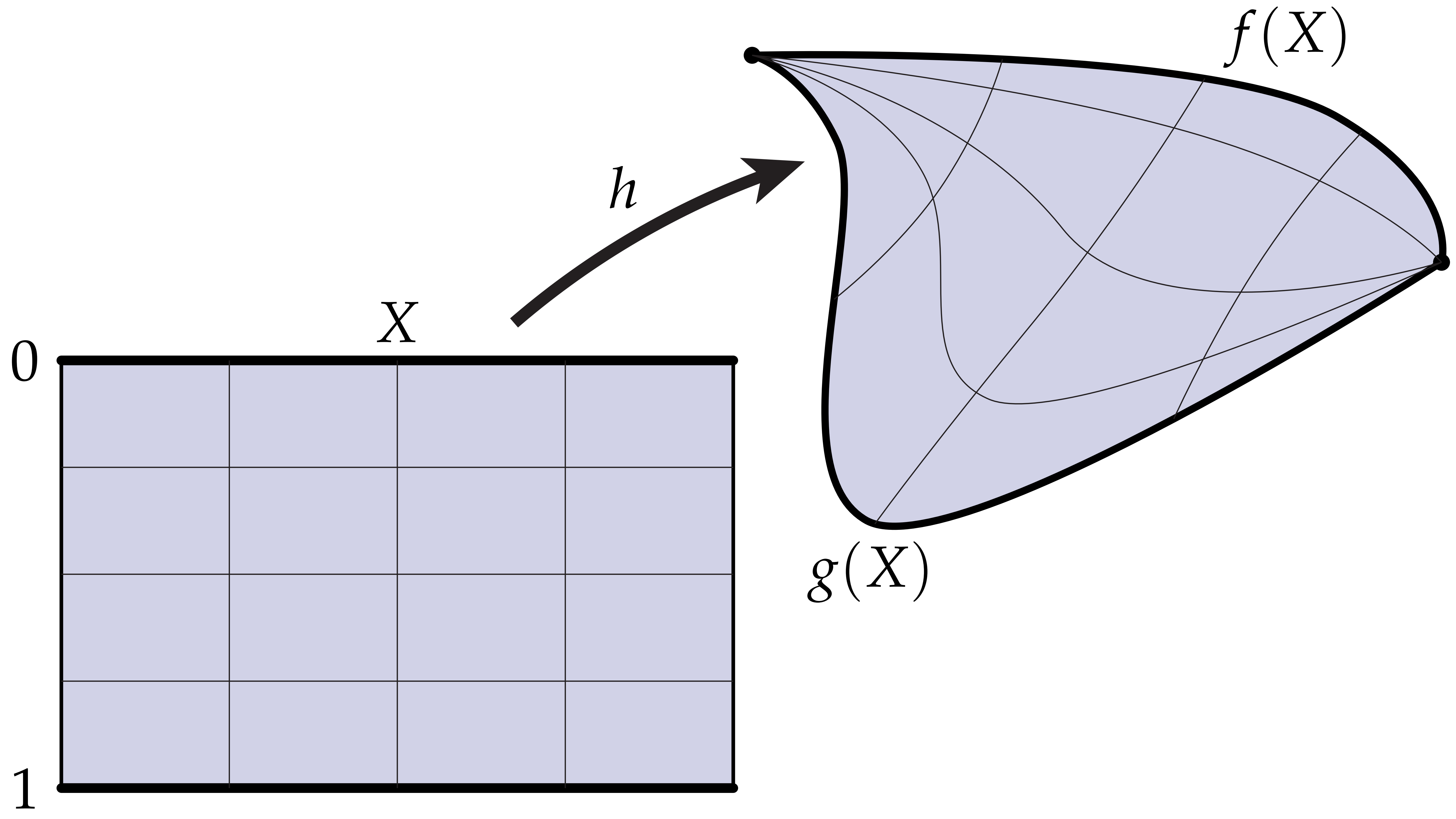
Tangent Vector Field Processing—Preview

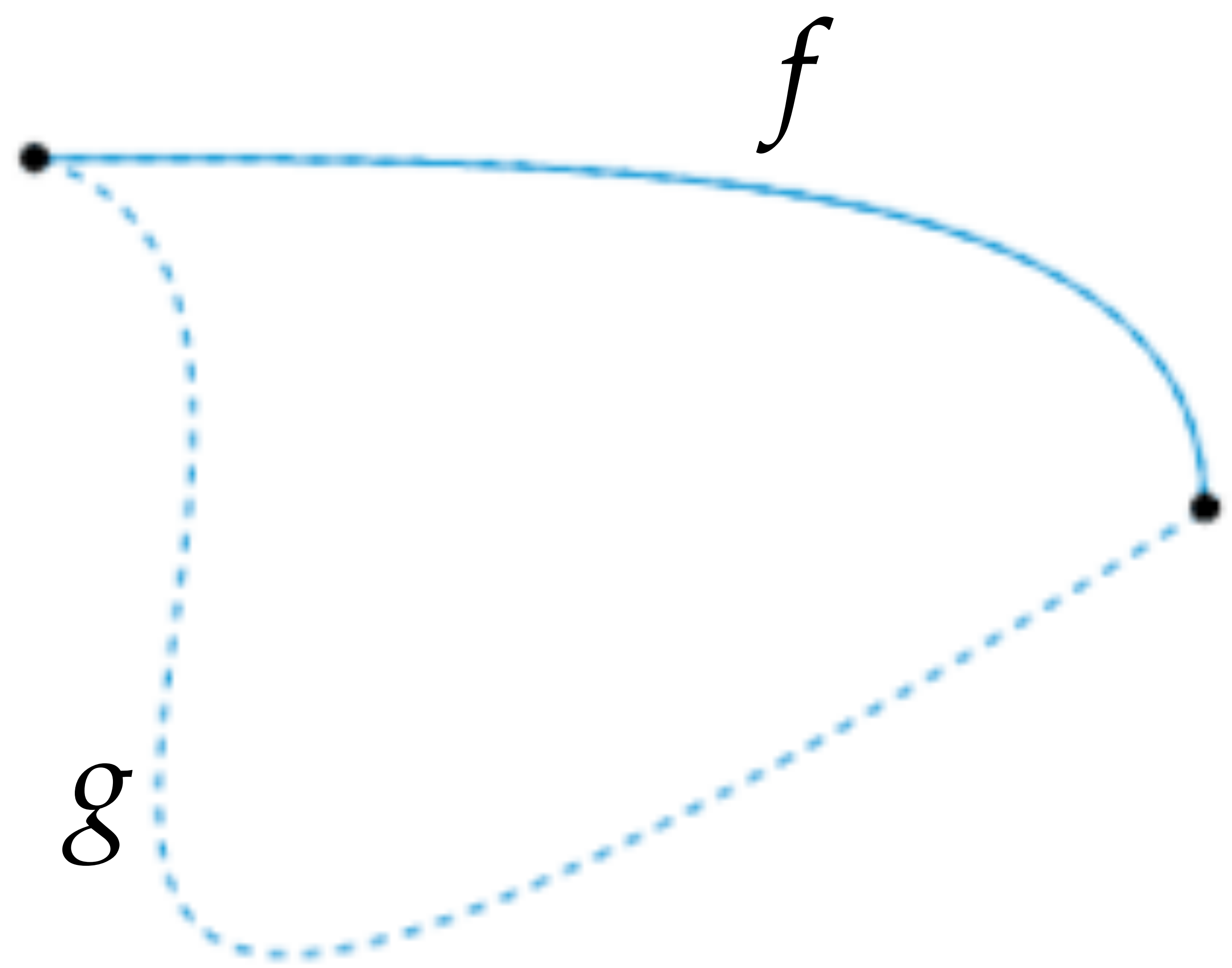


Tangent Vector Field Processing — Preview

Homotopy and Isotopy

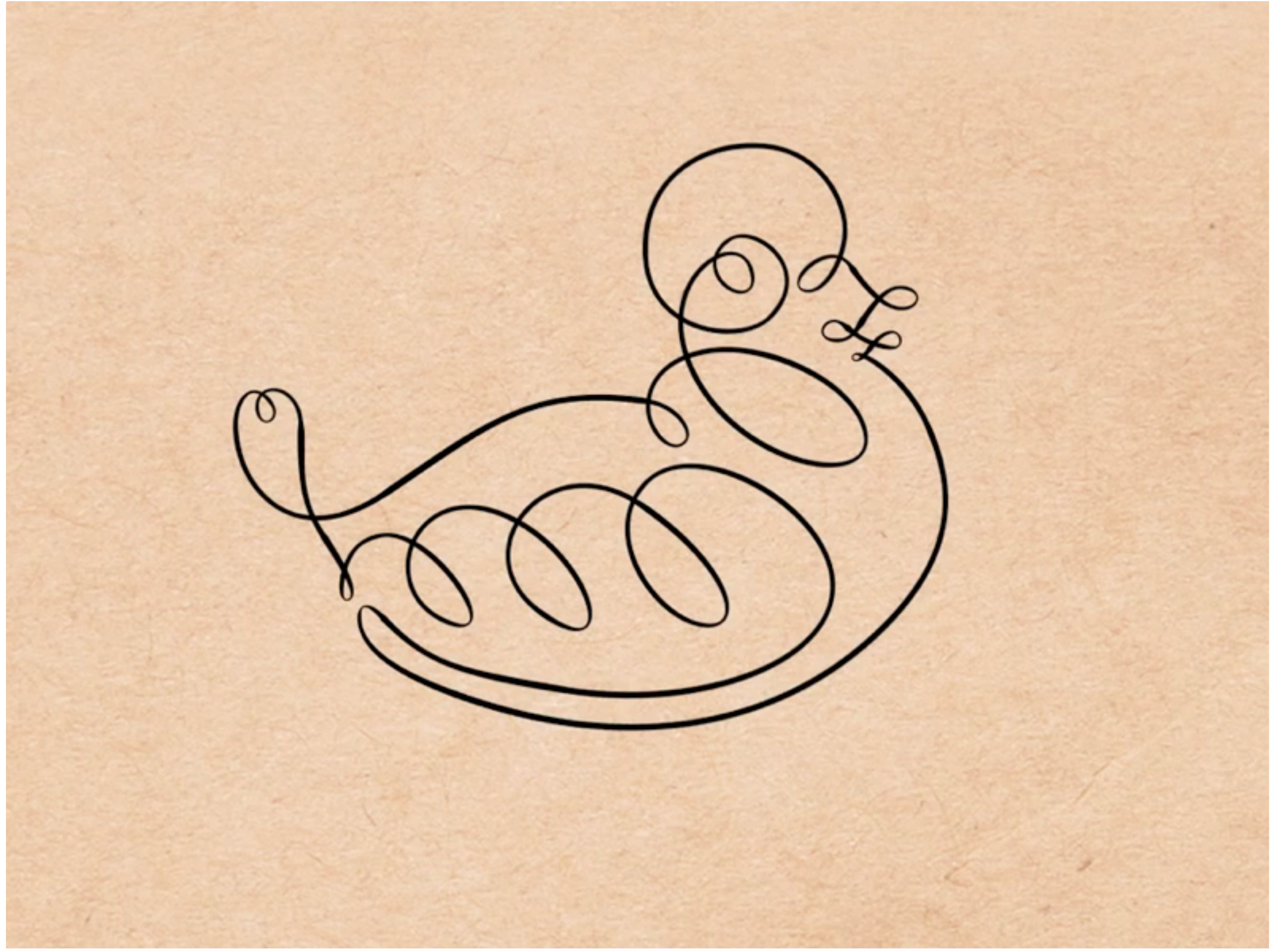
Definition. Let X, Y be topological spaces. Two maps $f, g : X \to Y$ are *homotopic* if there exists a continuous map $h : X \times [0, 1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. If each map $h(., t)$ is a homeomorphism onto its image, then h is an *isotopy*.





Regular Homotopy and Isotopy

Definition. Let M, N be differentiable manifolds. A *regular homotopy* between two immersions $f, g : M \to N$ is a continuous family of immersions $h(t)$, $t \in [0,1]$ such that $h(0) = f$ and $h(1) = g$. A *regular isotopy* is a homotopy by differentiable embeddings.



Crane et al, *"Robust Fairing via Conformal Curvature Flow"*

Winding Number

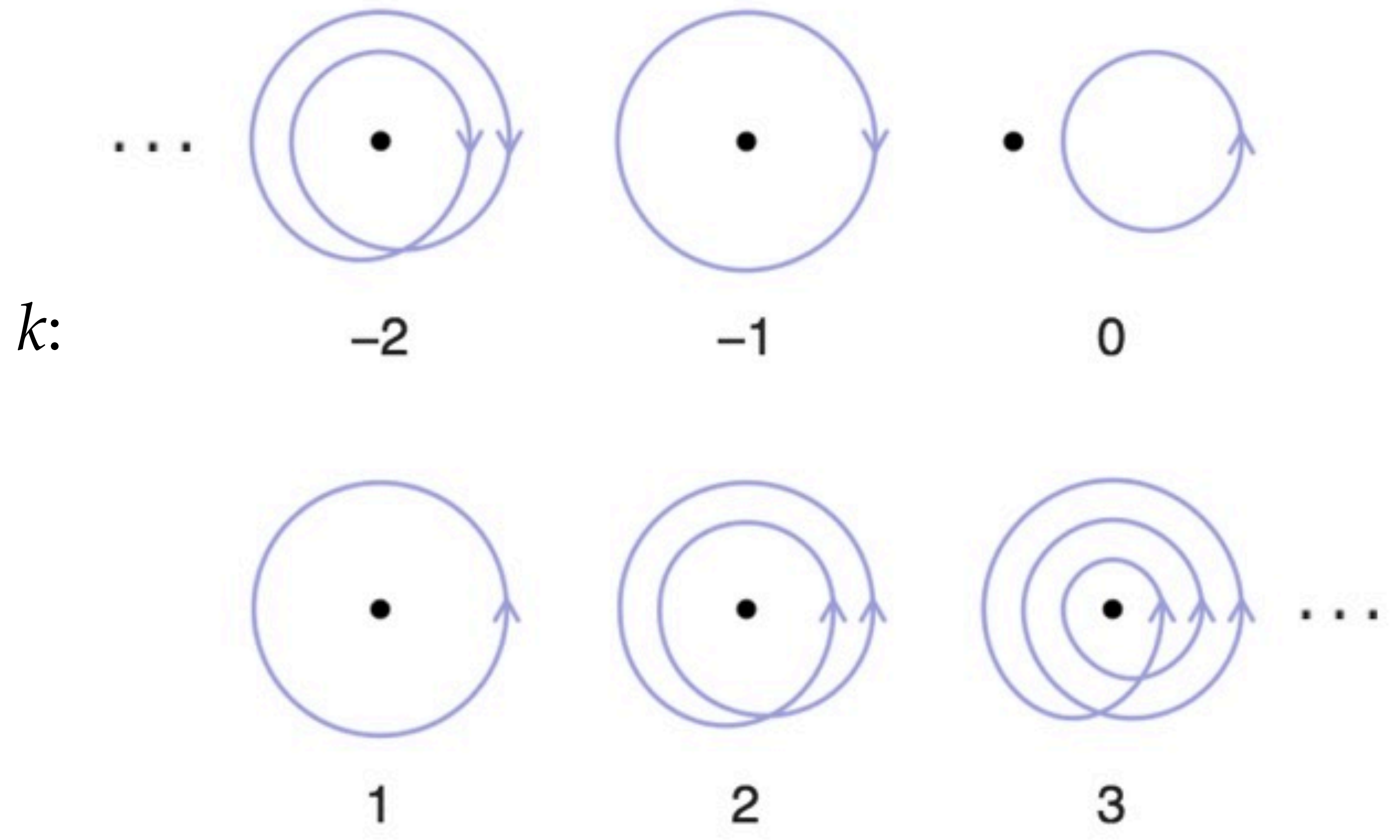
Definition. For any integer $k \in \mathbb{Z}$ we can define a k-fold covering of the circle

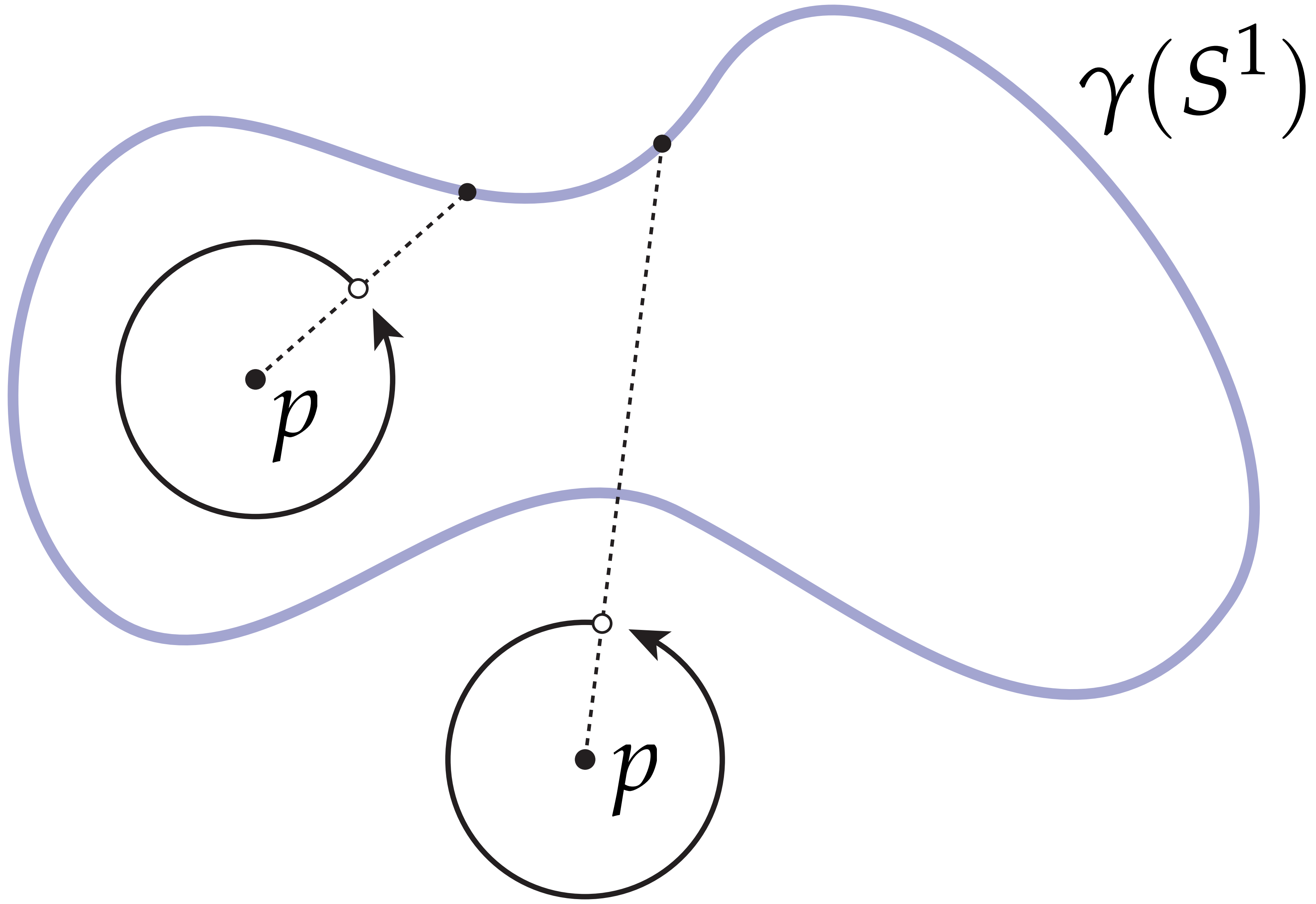
$$\eta_k : [0, 2\pi) \to S^1; \;\; s \mapsto (\cos(ks), \sin(ks)).$$

Let $\gamma : S^1 \to \mathbb{R}^2$ be a continuous map, and let $p \in \mathbb{R}^2$ be any point not on $\gamma(S^1)$. Then

$$\phi := \frac{\gamma - p}{|\gamma - p|}$$

defines a map from the circle to itself. If ϕ is homotopic to η_k, then we say that k is the *winding number* of γ.





Generalized Winding Number

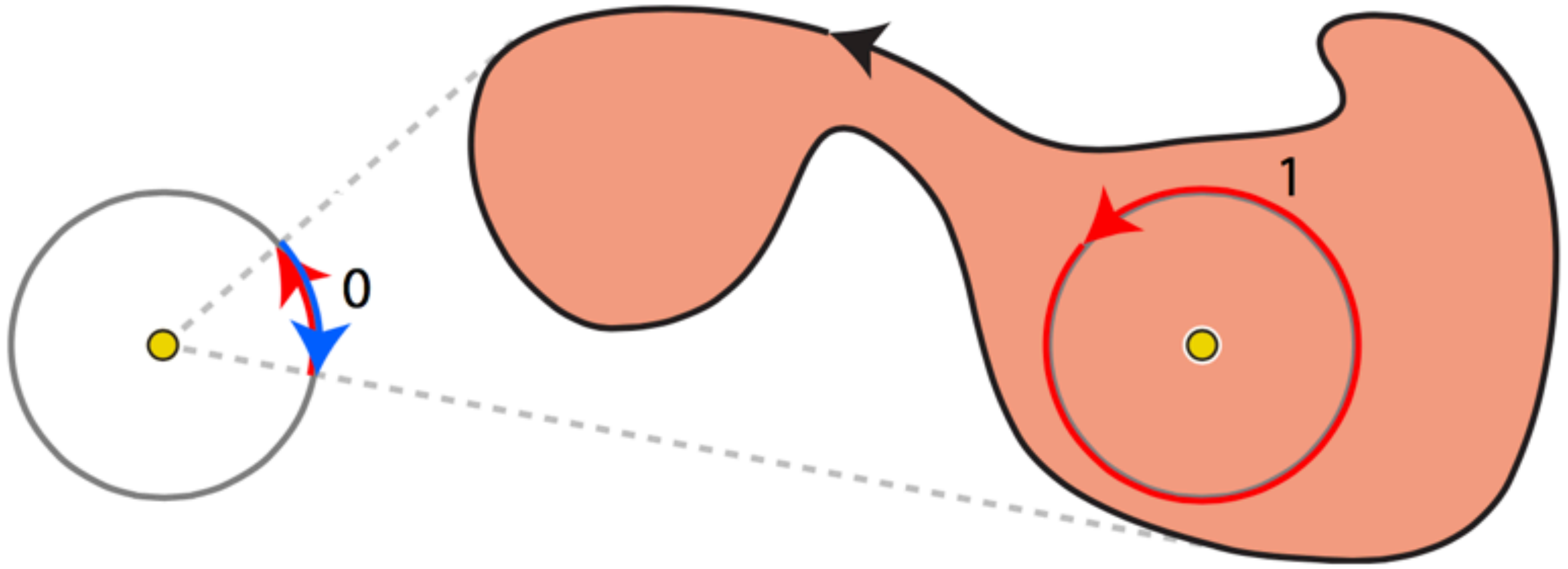
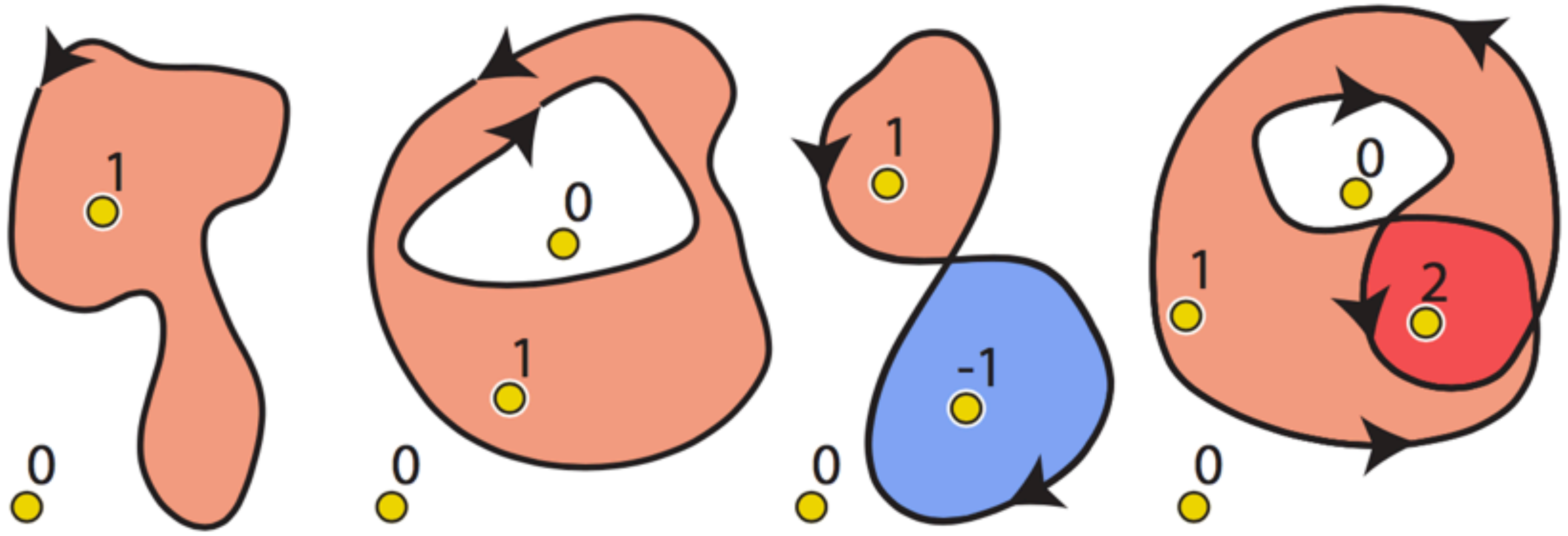
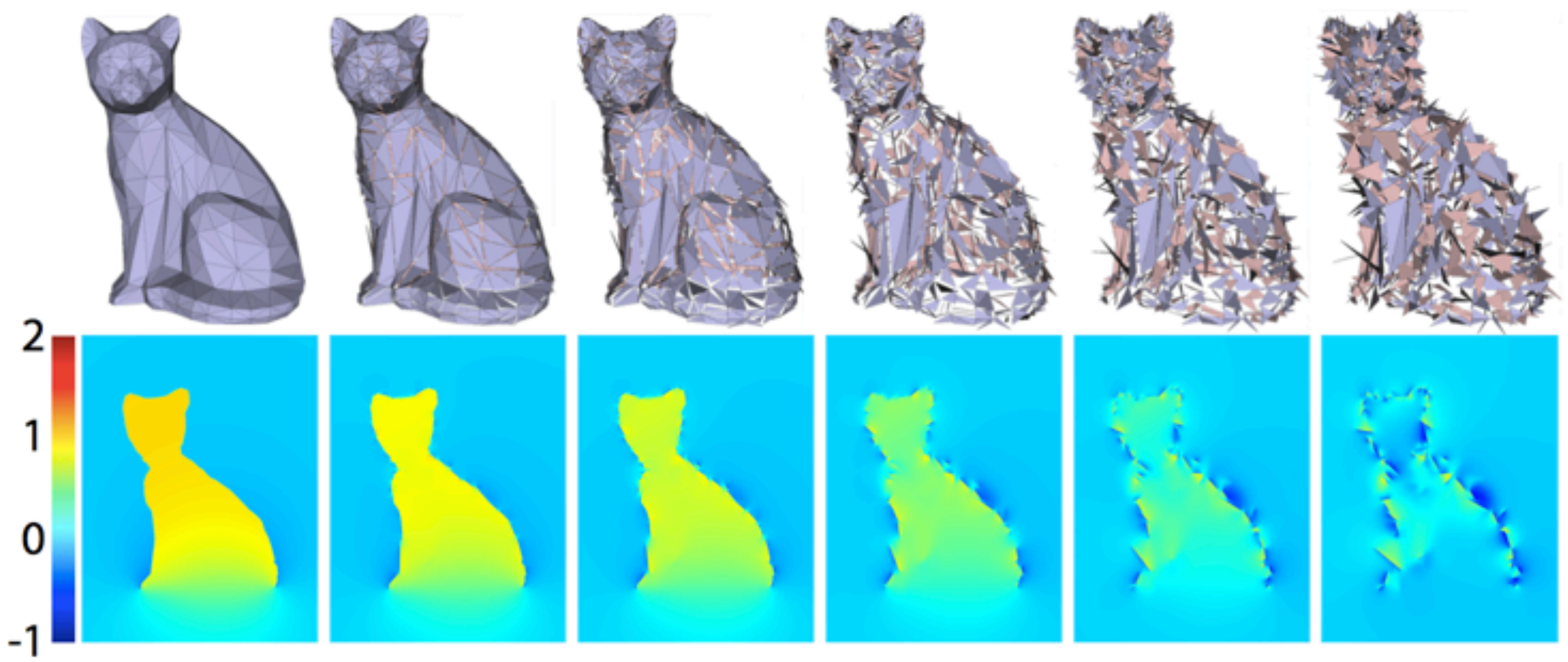


Figure 4: *Winding number is the signed length of the projection of a curve onto a circle at a given a point divided by* 2π*. Outside the curve, the projection cancels itself out. Inside, it measures one.*





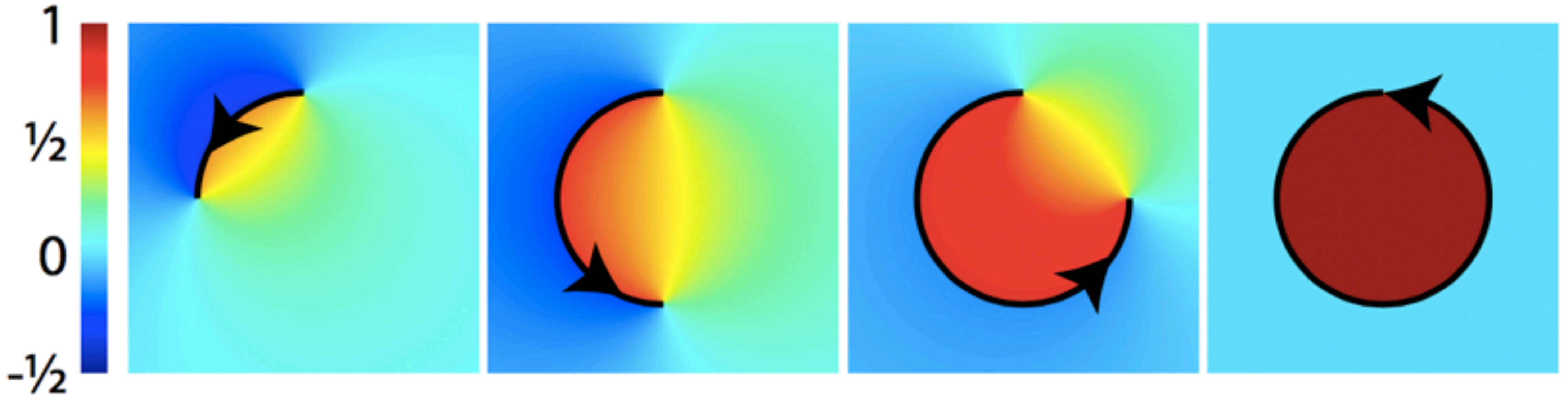


Figure 6: *Left to right: winding number field with respect to an open, partial circle converging to a closed circle. Note the* ± 1 *jump discontinuity across the curve. Otherwise the function is harmonic: smooth with minimal oscillation.*

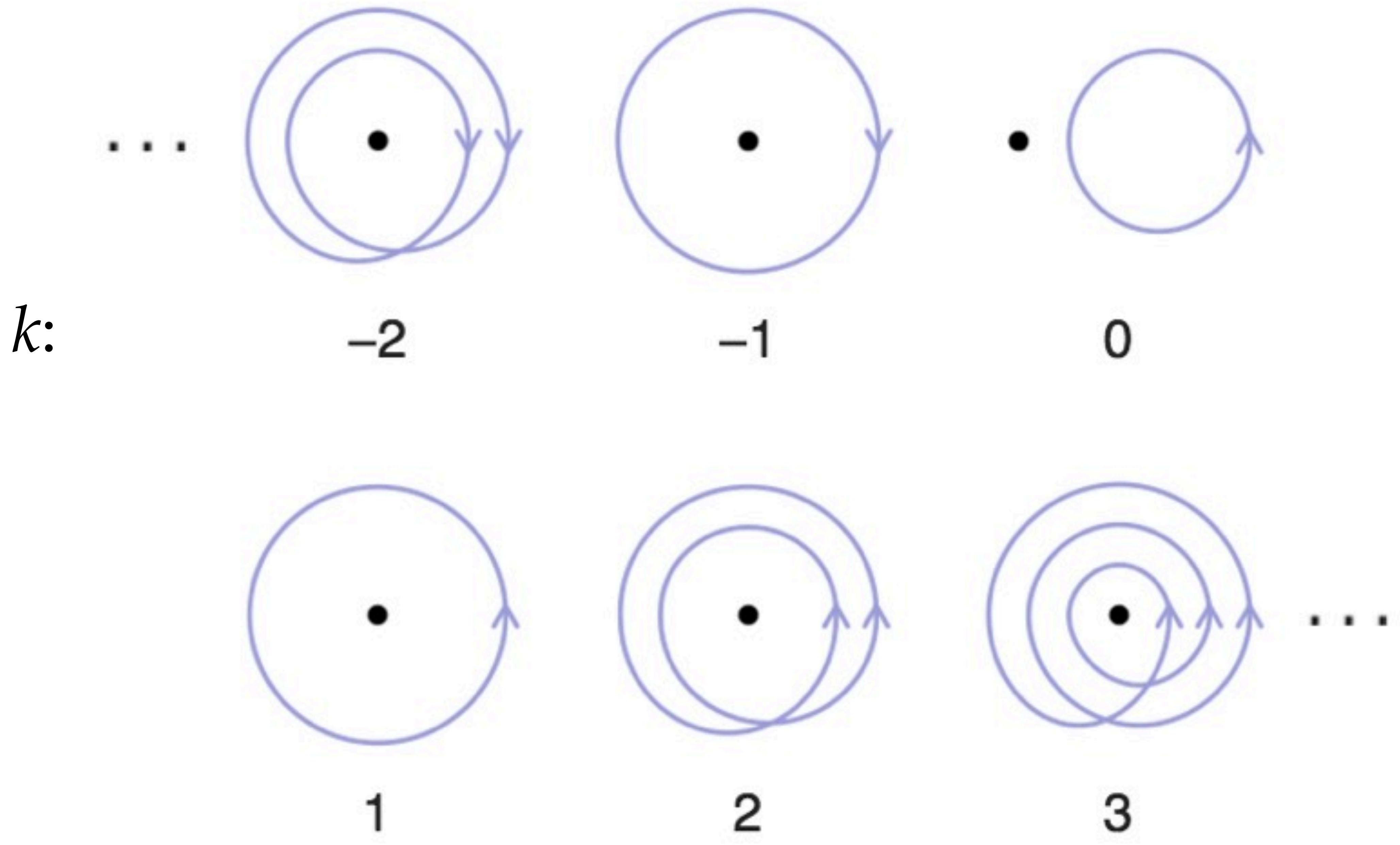
Jacobson et al, *"Robust Inside-Outside Segmentation using Generalized Winding Numbers"*

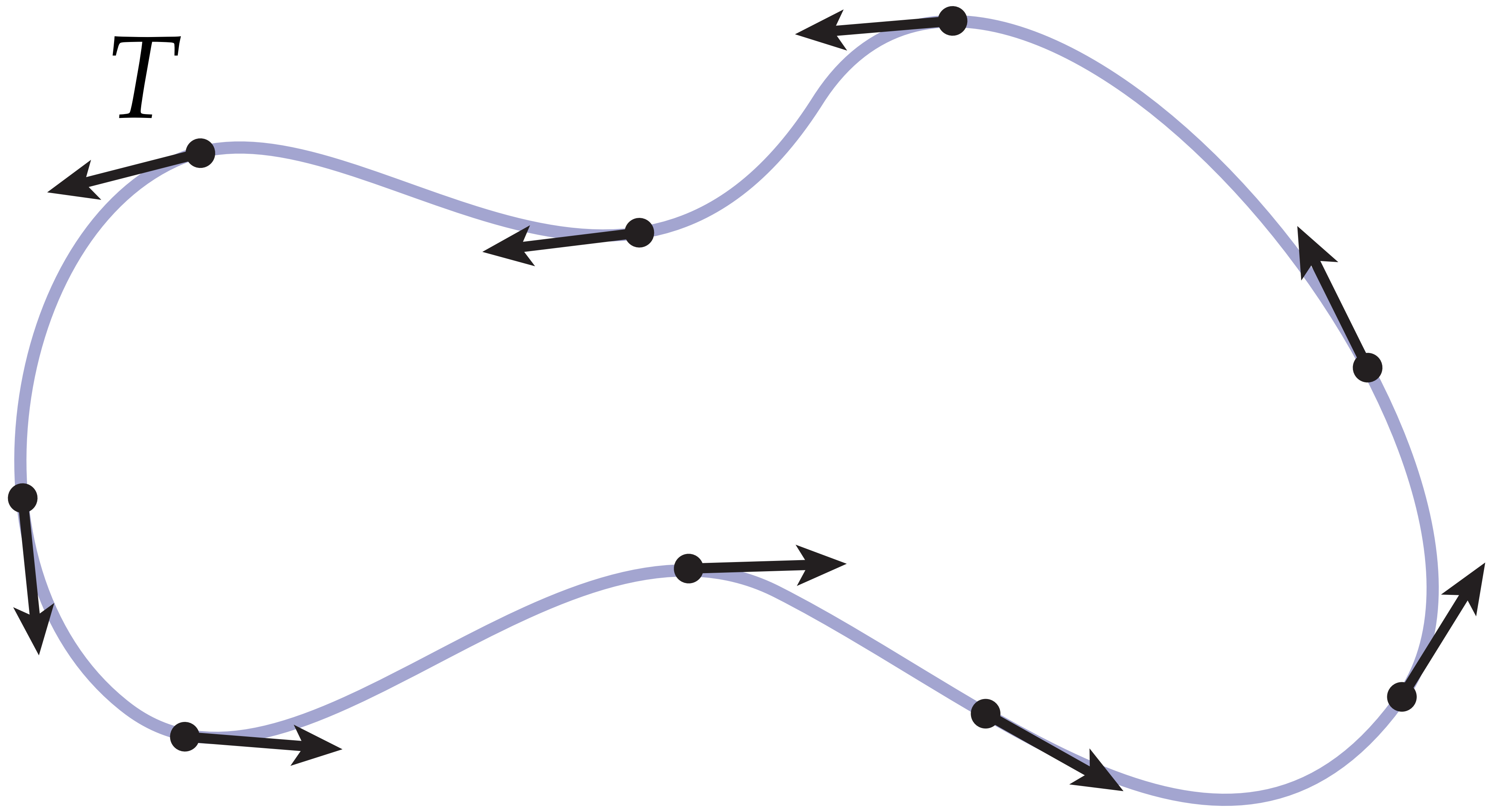
Turning Number

Definition. Let $\gamma : S^1 \to \mathbb{R}^2$ be an immersion, and let

$$T := \gamma' / |\gamma'|$$

be its unit tangent field; note that T is a map from S^1 to S^1. If T is homotopic to η_k, then k is the *turning number* of γ.

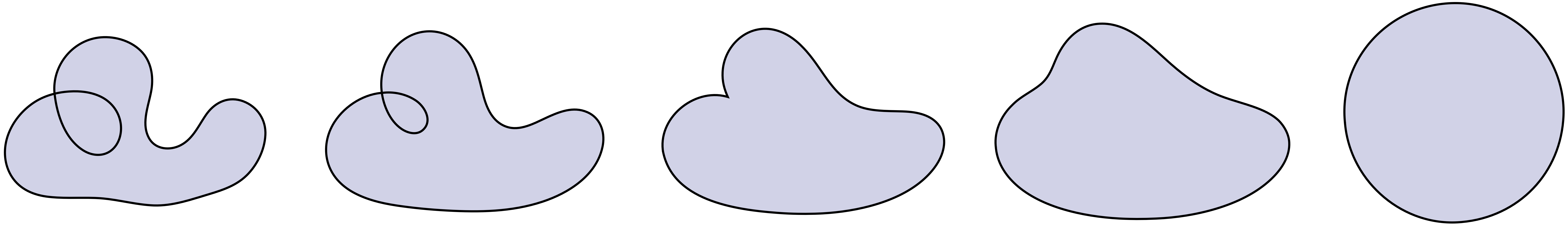




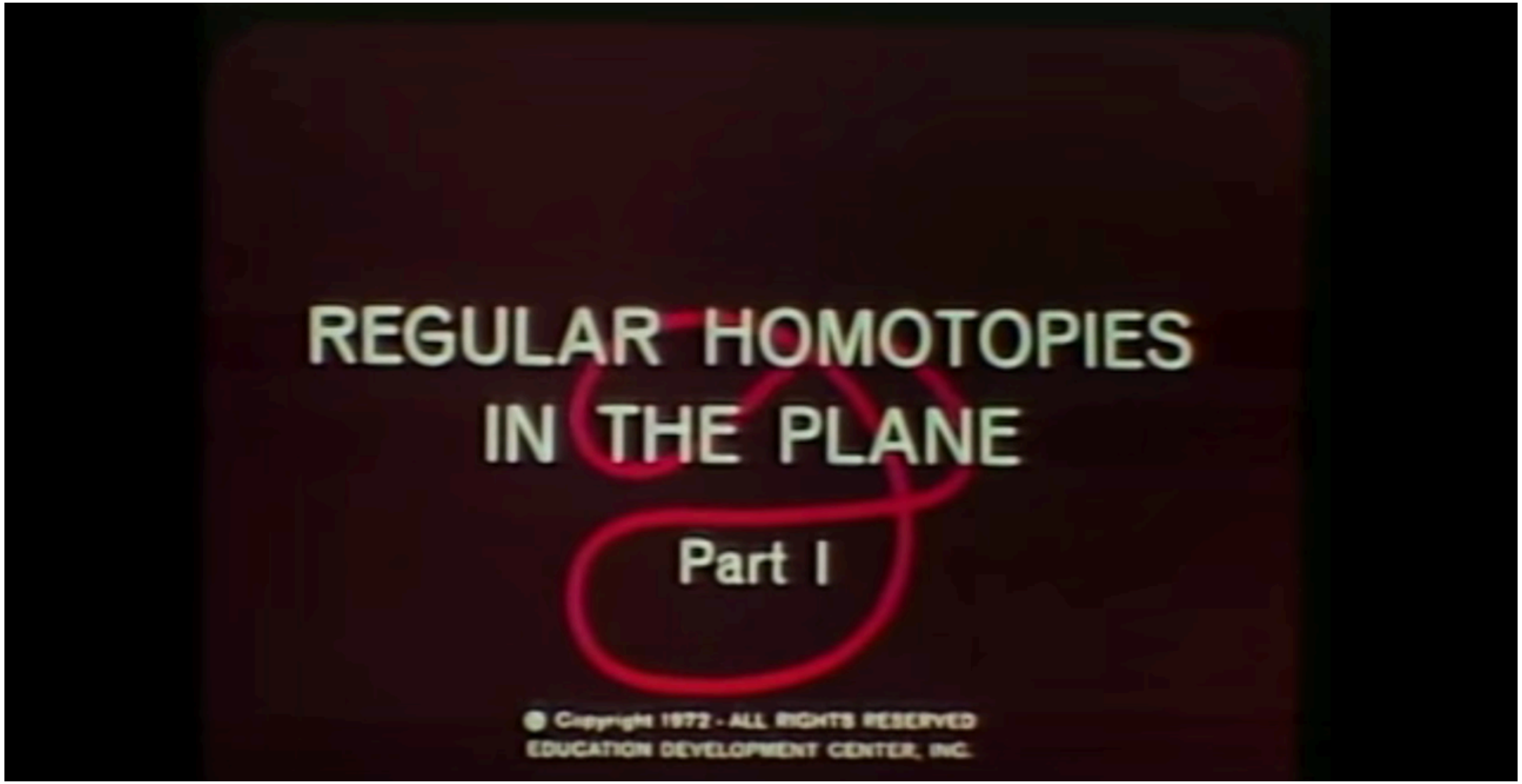
Whitney-Graustein

Theorem (Whitney-Graustein). Two regularly homotopic curves have the same *turning number*.

Corollary. In the plane, you can't turn the circle inside-out, *i.e.*, there is no *eversion* of the circle.



Regular Homotopies in the Plane



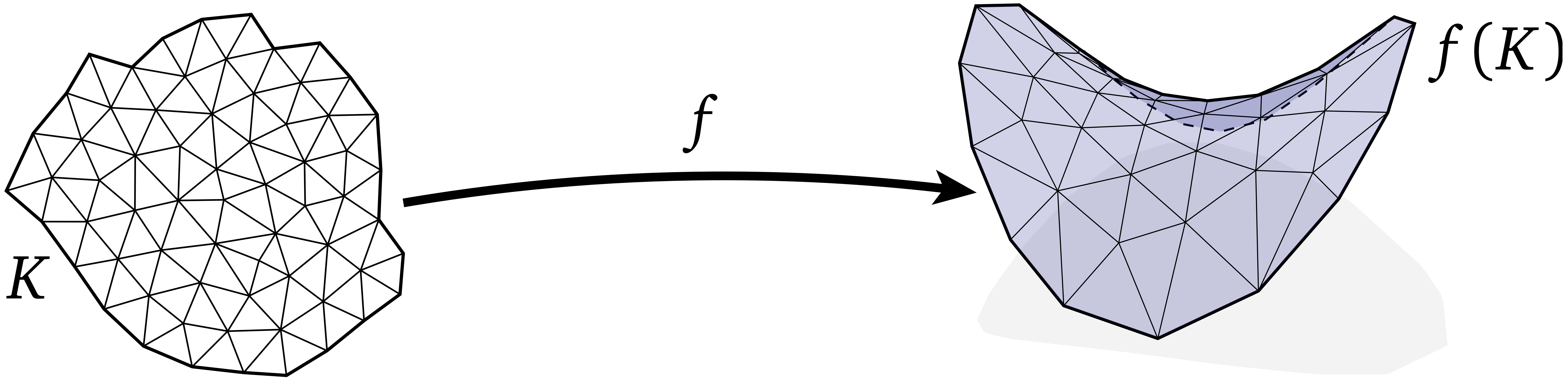
https://youtu.be/m7k8fxaAC40 https://youtu.be/fKFH3c7b57s

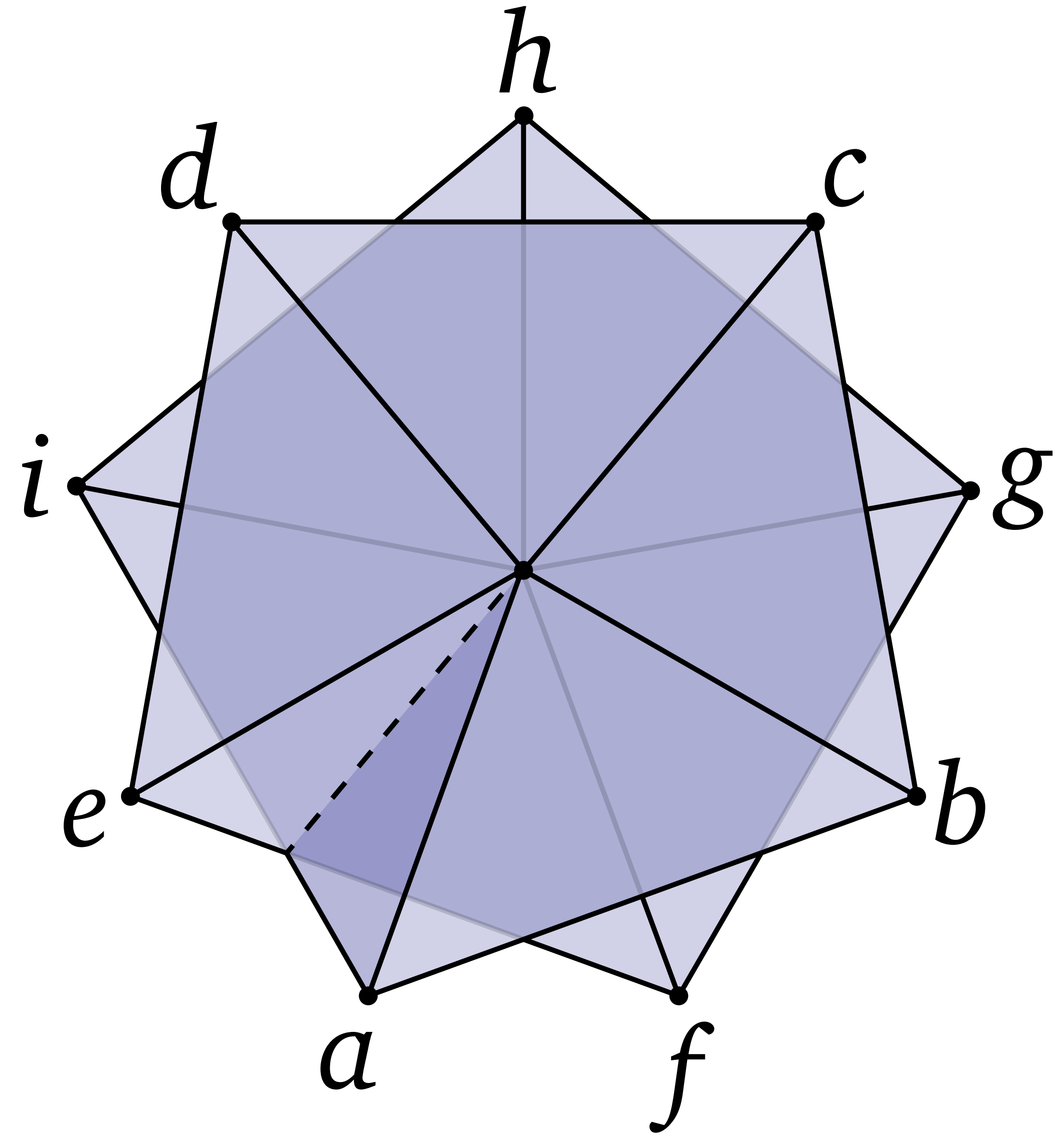
https://youtu.be/mY-V0TSMVCY https://youtu.be/olVQt_qx-bw

Sphere Eversion

Discrete Immersion

Definition. A simplicial map f is a *discrete immersion* if it is locally injective, *i.e.*, if around every point p there exists a small neighborhood U such that $f|_U$ is injective. Equivalently, f is a discrete immersion if every vertex star $\mathrm{St}(v)$ is mapped bijectively onto its image. A *discrete immersion* is a globally injective discrete immersion that is homeomorphic onto its image.





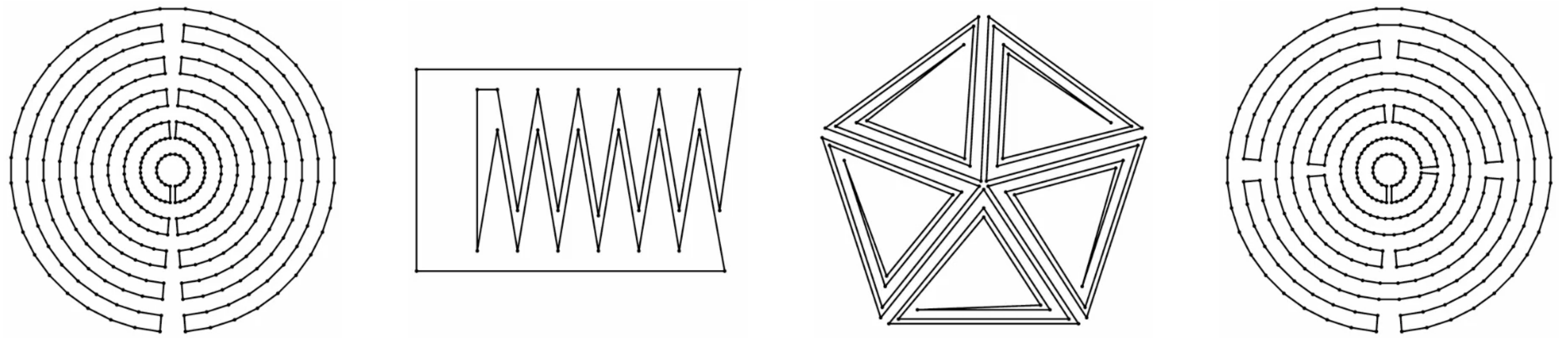
Note: "no degenerate simplices" is **NOT** the same as "locally injective!"

Discrete Embedding

What do you think?

Discrete Regular Homotopy / Isotopy

Definition. A family of simplicial maps f_t is a *discrete regular homotopy* if each map in the family is a discrete immersion; it is a *discrete regular isotopy* if each map is a discrete embedding.



Cantarella et al, *"An Energy-Driven Approach to Linkage Unfolding"*

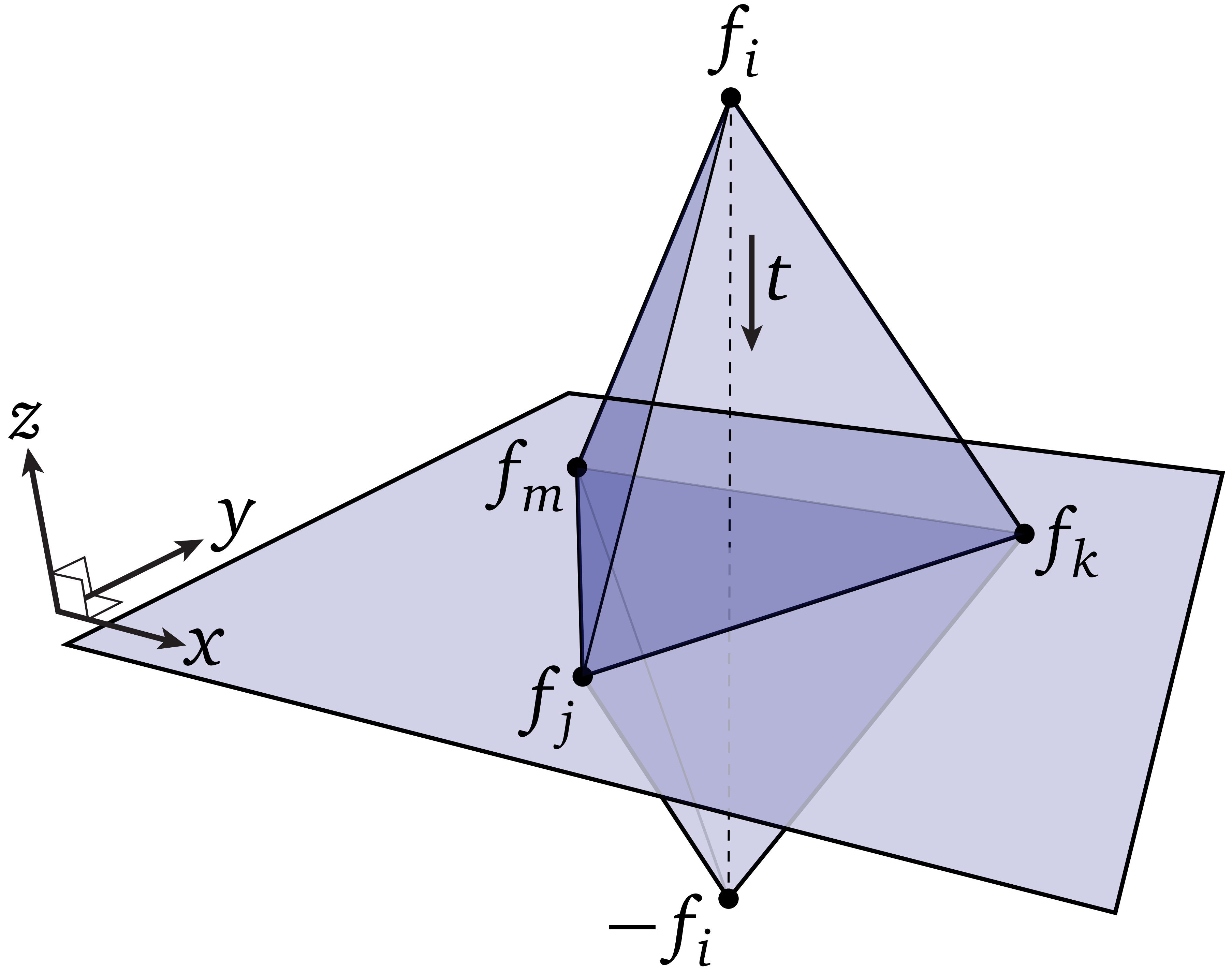
Discrete Sphere Eversion

A *discrete regular homotopy* is a homotopy by discrete immersions.

Can you *always* turn a simplicial sphere inside-out?

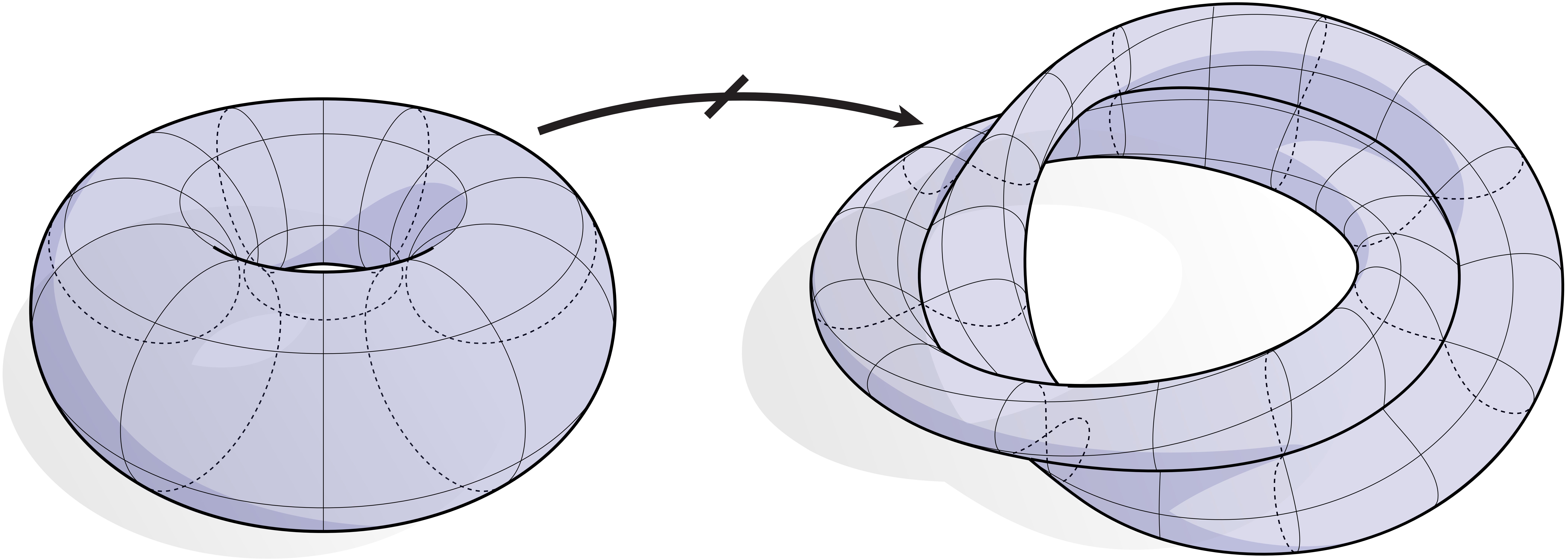
Can you turn a simplicial sphere inside-out?

Denner, *"Polyhedral Eversions of the Sphere"*



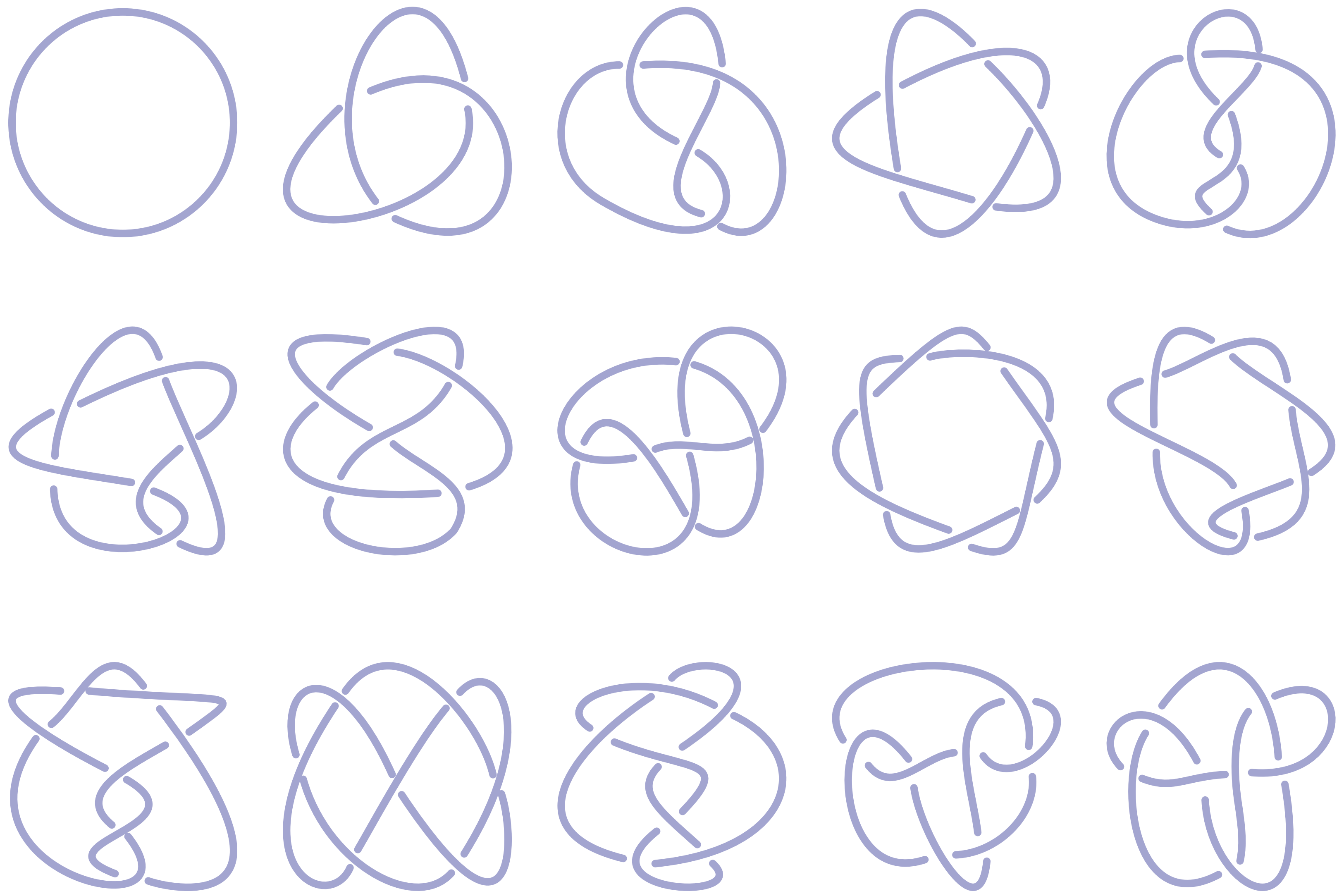
Regular Homotopy Classes of Surfaces

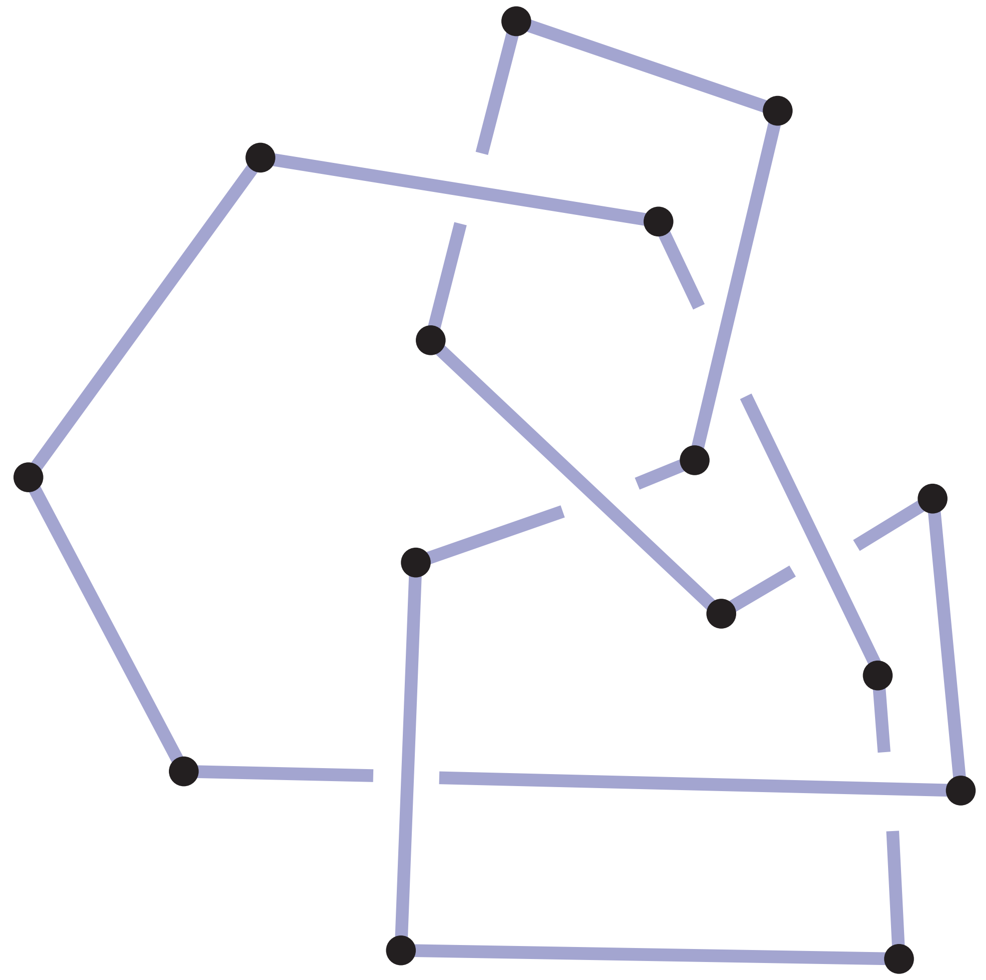
Theorem (Pinkall)**.** For a surface of genus g, there are 2^{2g} regular homotopy classes of immersions into $\mathbb{R}^3$.

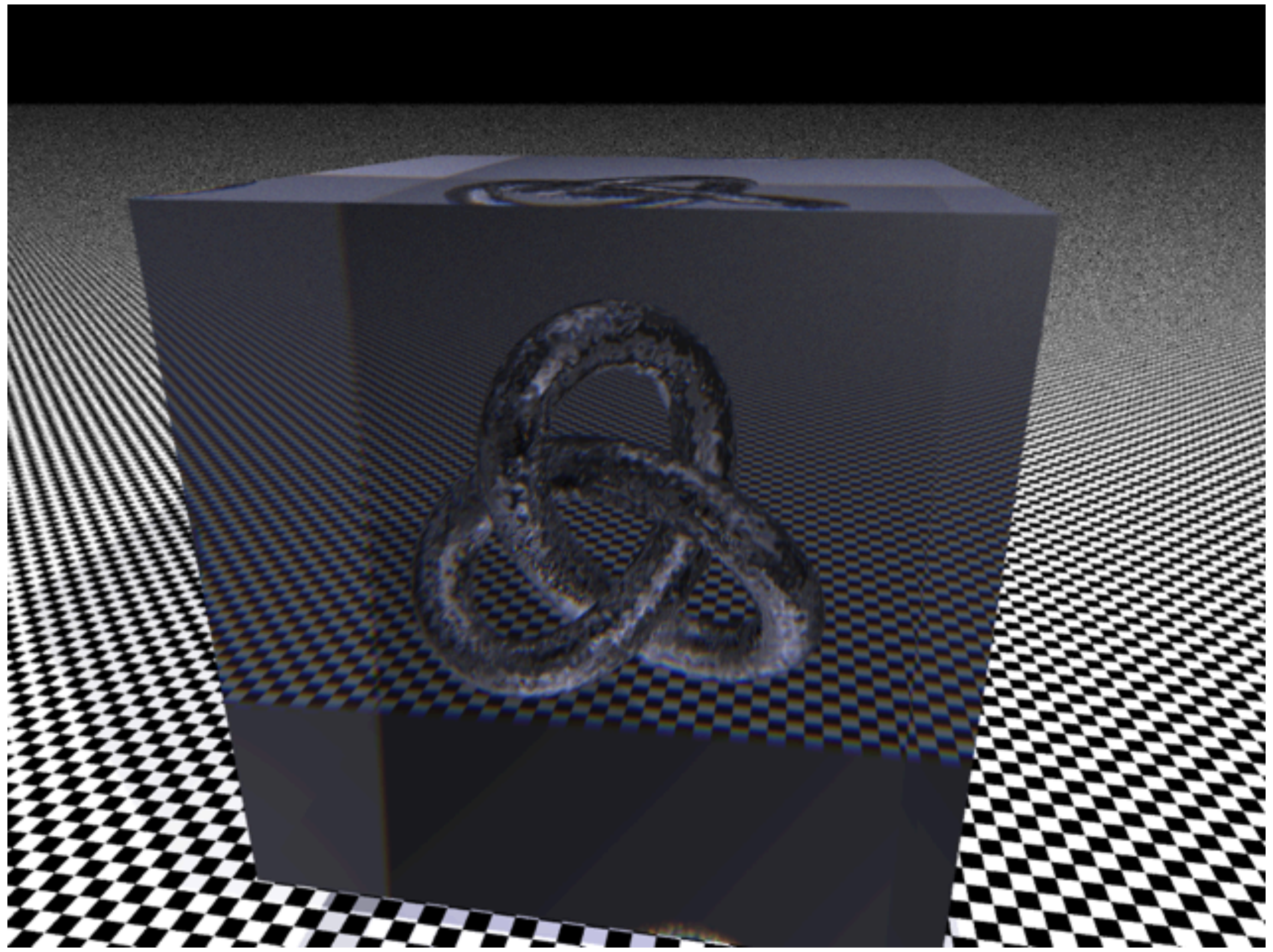


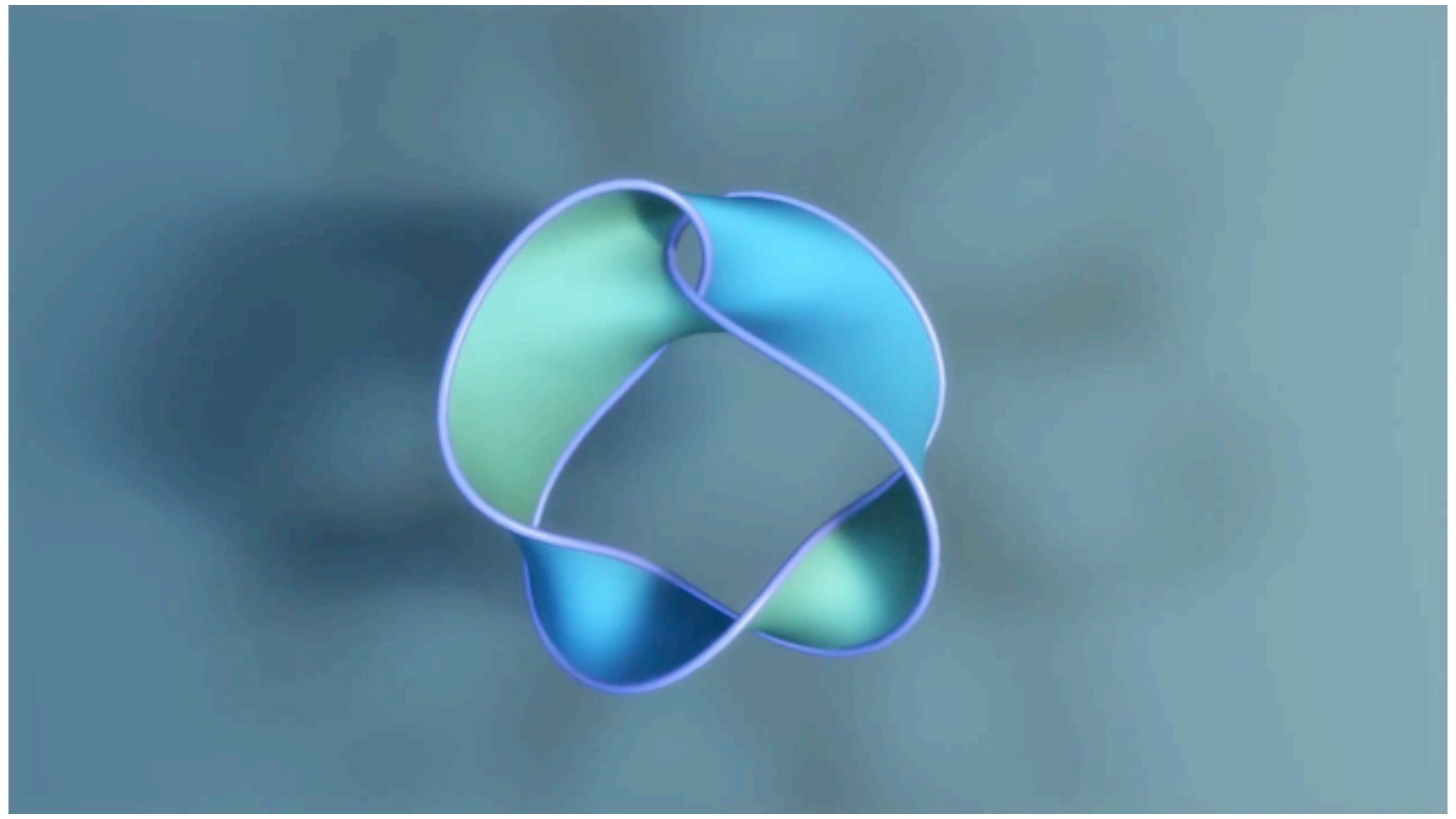
(Discrete theorem...?)

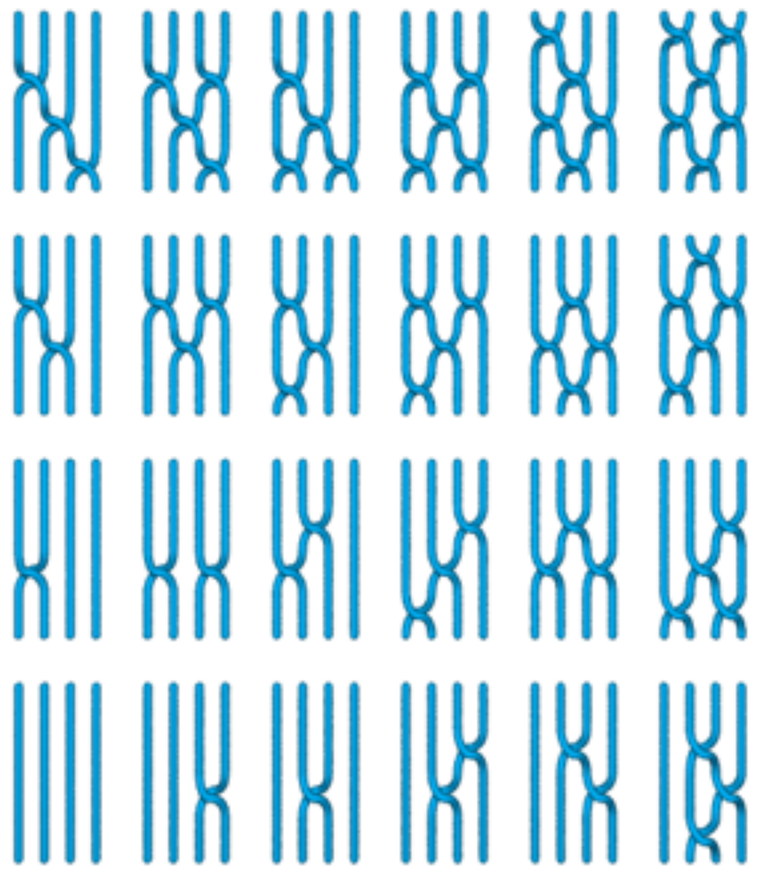
Geometric Topology











Mustafa Hajij / SeifertView

Computational Geometric Topology

What we know: the complexity of $\mathrm{EMBED}_{k \to d}$

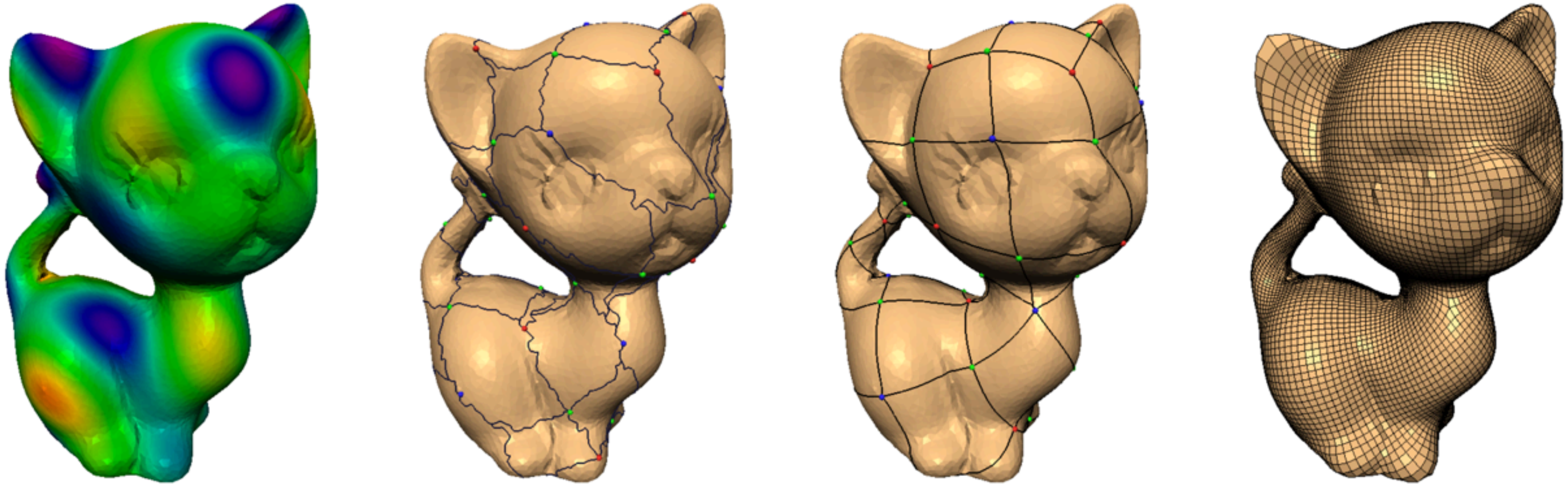
						d							
k	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P												
2	P	D	NPh										
3		D	NPh	NPh	P								
4			NPh	und	NPh	NPh	P						
5				und	und	NPh	NPh	P	P				
6					und	und	NPh	NPh	NPh	P	P		
7						und	und	NPh	NPh	NPh	P	P	P

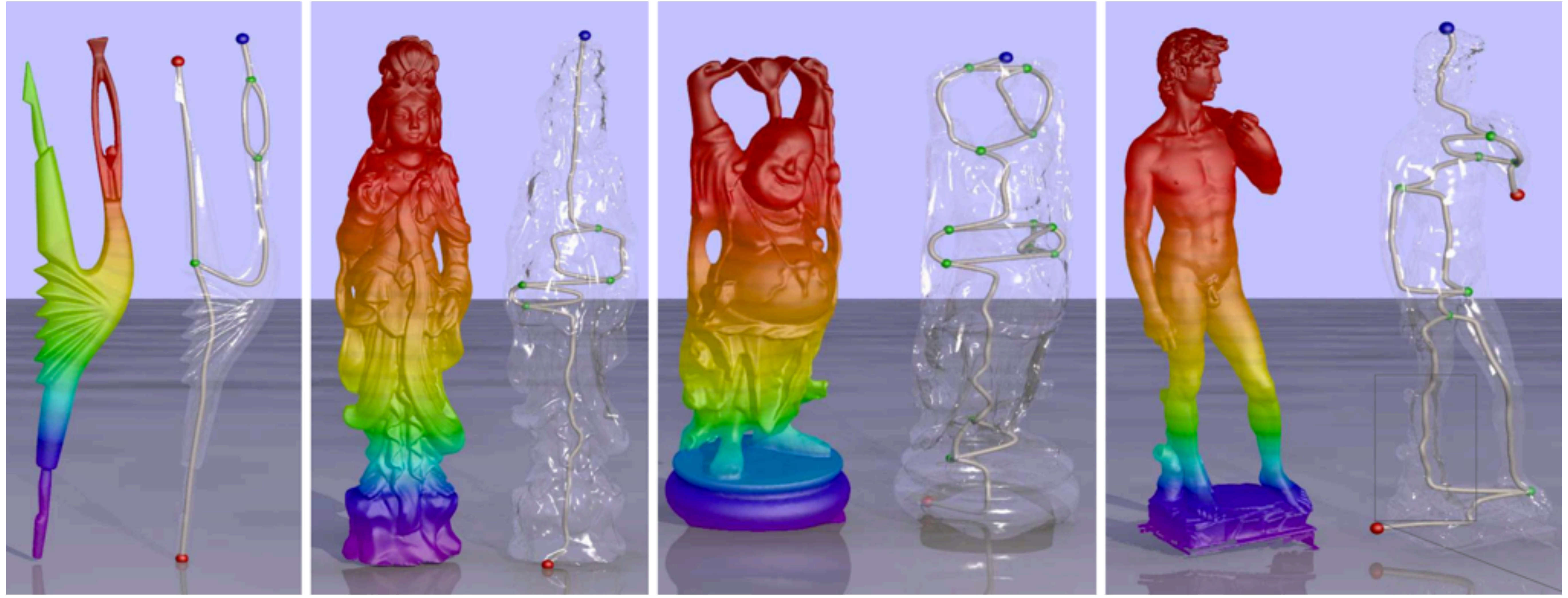
und = algorithmically undecidable [Matoušek, Tancer, W.]
NPh = NP-hard [Matoušek, Tancer, W.]
D = algorithmically decidable [Matoušek, Sedgwick, Tancer, W.]
P = polynomial-time solvable; new results based on algorithmic homotopy classification of (equivariant) maps [Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, W.]

[Courtesy Uli Wagner]

Computational Topology in Geometry Processing

Garland et al, *"Spectral Surface Quadrangulation"*





Pascucci et al, *"Robust On-line Computation of Reeb Graphs"*

Summary—Differentiable Manifolds

CONTINUOUS	DISCRETE
derivative	difference
differentiable manifold	*simplicial manifold*
differentiable map	*simplicial map*
tangent space	rescaled angles
immersion	locally injective simplicial map
…	…

Fact. For $n \leq 3$, smooth structure is uniquely determined by topology.

Fact. There are often more regular homotopy classes in the discrete case than in the continuous setting.

*Hence, **not everything** is captured by discretization!*

Topological vs. Geometric Structures

SET

$\cup$

TOPOLOGICAL SPACE

$\cup$

TOPOLOGICAL MANIFOLD

$\cup$

SMOOTH MANIFOLD

$\cup$

COMPLEX MANIFOLD

$\cup$

RIEMANNIAN MANIFOLD

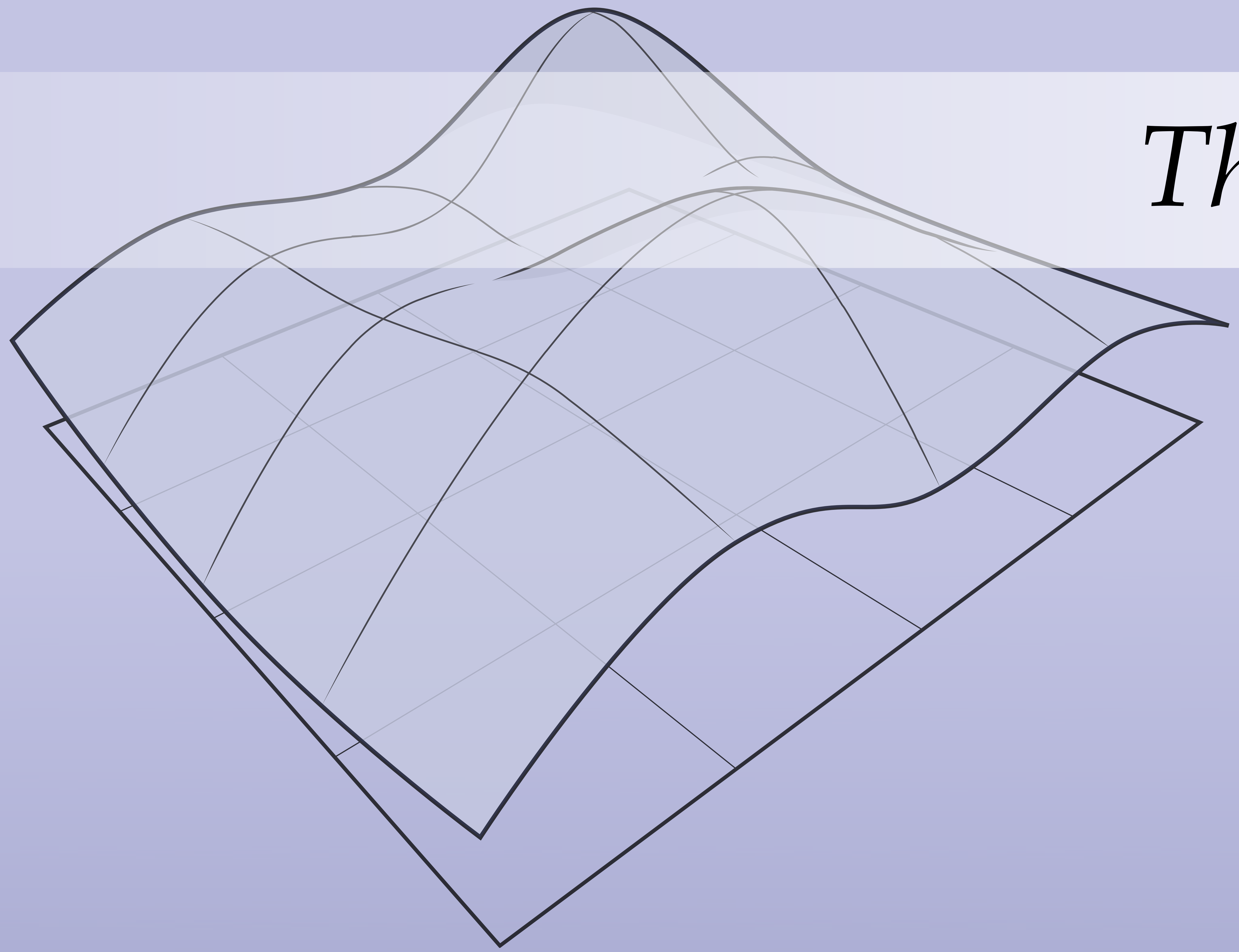
"floppier" ↑

"more rigid" ↓

"topology"

"differential topology"

"geometry"



Thanks!

DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-869(J) • Spring 2016