15-869J Course Project: On the Conformal Maps of Triangle Linkages

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Abstract

This writeup studies the nature of conformal maps, particularly in connection with discrete differential geometry. The discrete model we focus on is the triangular linkage geometry introduced in Konakovic, et.al., [KCD$^+$16]. Abstractly, these linkages are equilateral triangles such that pairs of triangles meet at vertices and the triangles are connected in cycles of length six. In practice, such surfaces can be manufactured from flat "auxetic" (opening) materials with slits cut in them, providing many more degrees a freedom than ordinary developable (no-cut) surfaces. We present an overview of discretization of conformal geometry, both in the traditional Lagrangian element model as well as in the Crouzeix-Raviart element model. We describe how this theory connects to the geometry of triangular linkages, laying a foundation of discrete differential geometry for these structures. Furthermore, we propose a working definition of discrete conformal maps on triangular linkages, and prove some implications.

1 Introduction: Conformal Geometry in the Smooth Setting

The over-arching theme of this project is the study of conformal maps on two-dimensional surfaces, particularly in the discrete setting. Intuitively, conformal maps keep angles on the surface unchanged but may change relative scaling in the process. Classic examples of conformal maps include the Mercator and stereographic projections of the surface of the Earth onto the plane (see Figure 1).

In the smooth setting, conformal maps have been studied extensively for over a century, and have found numerous applications. In fluid mechanics, conformal maps have been used to reconstruct two-dimensional flows based on boundary conditions [MZ83]. In geometric optics, conformal maps can be used to describe a change of coordinates in four-dimensional space-time [Bat09]. Furthermore, conformal maps are important in the study of general relatively and cosmology [FGN99].

There are many ways to formally define conformal maps. We present a handful of these ways to motivate the discretizations explored in subsequent sections. From the perspective of complex analysis (e.g., [Nee98, Cra15]), a conformal map is one which preserves angles between tangent vectors. If a two-dimensional surface $M \subseteq \mathbb{R}^3$ has two tangent vectors v_0 and v_1 at $p \in M$ which make an angle of θ, then a conformal map $f : M \to \mathbb{C}$ will send v_0 and v_1 to a new pair of tangent vectors v_0' and v_1' at $f(p)$ make the same angle of θ. Furthermore, the angles still have the same orientation. This can be elegantly captured by the Cauchy-Riemann equation (from e.g., [Cra15])

$$idf(v) = df(\mathcal{J}v), \tag{1}$$

e where $df(v)$ is the directional derivative of f in the direction v at the point p, and $\mathcal{J}$ is the 90° counter-clockwise rotation of M on the surface of M. If the orientation reverses everywhere, we say that the map f is *anti-conformal*. See [Nee98] for a brief introduction to how conformal maps

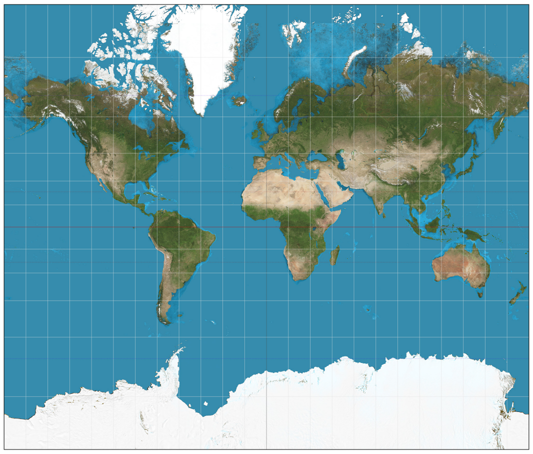 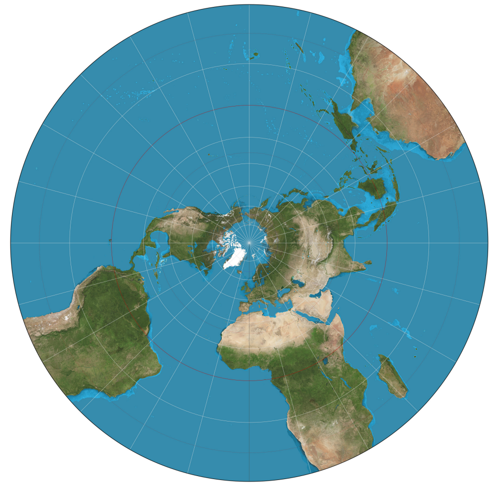

Figure 1: The Mercator projection (left) and the stereographic projection (right) of the Earth are an examples of conformal map from the sphere to the plane. Note that the shapes of continents are (in small regions) similar to those on a globe, but the relative sizes are distorted. Image urls: `https://upload.wikimedia.org/wikipedia/commons/f/f4/Mercator_projection_SW.jpg` and `https://upload.wikimedia.org/wikipedia/commons/a/a6/Stereographic_projection_SW.JPG`. Attribution: "By Strebe (Own work) [CC BY-SA 3.0 (`http://creativecommons.org/licenses/by-sa/3.0`)], via Wikimedia Commons"

connect to complex analysis. Maps to surfaces embedded in $\mathbb{R}^3$ instead of the complex plane can be described in an analogous way using *quaternions* (see, e.g., [CPS11]).

We can also view conformal maps from the perspective of metrics. If we have a manifold M which comes with a metric g, we may define a conformal map entirely in terms of g. More precisely, a map $f : M \to N$ between n-dimensional manifolds M and N with metrics g and $\hat{g}$ is conformal if and only if there exists a scalar $\phi : M \to \mathbb{R}$ such that

$$g_p(u, v) = e^{2\phi(p)} \hat{g}_{f(p)}(df(u), df(v)), \text{ for all } p \in M \text{ and tangent vectors } u, v. \tag{2}$$

See, e.g. [BPS15]. Intuitively, this says that a conformal map re-scales different regions of the geometry of M but otherwise does preserves the structure.

A third definition of conformal map is defined in terms of the *conformal energy* of a manifold. From the course lecture notes [Cra15], the conformal energy $D_C(f)$ of a map $f : M \to \mathbb{C}$ is defined to be to what extent (1) fails to be true. The formula for this is often written as (see p. 92)

$$D_C(f) = \frac{1}{2} \langle\langle \Delta f, f \rangle\rangle_M - \mathcal{A}(f),$$

where $\langle\langle \cdot, \cdot \rangle\rangle_M$ is the inner product operator on M, Δ is the Laplace-Beltrami operator, and $\mathcal{A}(f)$ is the area of the image of M with respect to f . The first term of the sum is often known as the *Dirichlet energy* of f. We have that f is conformal, if $D_C(f)$ is minimized, but we also need to specify that f is bounded away from the 0 function (see pages 93-94).

With these different definitions of conformal maps in the smooth setting, we explore methods of discretizing the these notions.

2 Discrete Conformal Geometry: Theory and Applications

In the past couple of decades, much work has been done to discretize the theory of conformal maps in ways which are amenable to computation. In the section, we survey many of these works, most of which pertain to the traditional 'triangle mesh' model of discrete differential geometry.

The first approaches reduced finding discrete conformal maps to optimization problems. One approach, called "Least Squares Conformal Maps" is due to Levy, et.al., [LPRM02] and similarly discovered by Desbrun, et.al., [DMA02]. They use the Cauchy-Riemann equation as a guide by attempting to minimize the squared error of a discrete version of the equation within the local neighborhood of each vertex. Another approach due to Sheffer, et.al., [SdS01], finds a mapping of the whole mesh into the plane such that the angles of the triangles of the original mesh are distorted to a minimum. This algorithm was later improved upon in a collaboration by Sheffer, Levy, et.al., [SLMB05].

Later on, models of conformal maps drew more richly from the theory of Discrete Differential Geometry. One such model of conformal maps on triangle meshes was developed by Kharevych, et.al., [KSS06] based on the circumcircles of the faces of the mesh. In particular, they define a map to be conformal if the angle between circles (when flattened out) is preserved. This most closely generalization the angle-preservation definition of a conformal map in the smooth setting. This was one of the first approaches for discretizing conformal maps which resulted in a large number of degrees of freedom (e.g. one could set the entire boundary) yet also having efficient algorithms for applications.

In another model due to Springborn, et.al., [SSP08], conformal maps are discretized at the vertex level. Recall that if a mesh has a vertex set V and an edge set E, a *discrete metric* is a function $\ell : E \to \mathbb{R}^+$ such that is satisfies the triangle inequality on the faces. Two different metrics ℓ and $\hat{\ell}$ are then conformally equivalent if there is a function $\phi : V \to \mathbb{R}$ such that

$$\ell(u, v) = e^{\phi(u)+\phi(v)}\hat{\ell}(u, v), \text{ for all } (u, v) \in E. \tag{3}$$

([BPS15] cites [Luo04] as the original source.) Note that this equation is nearly identical to definition (2) of conformal maps.

Another example of a discrete conformal map is Discrete Ricci Flow (e.g., [GY08]).

The above results mostly assume that the triangular meshes we are constructing are "C^0" that they don't have any breaks in them. Other work, such as [KMB$^+$09], has shown how to study discrete differential geometry on surfaces with cuts and aberrations using *harmonic* functions, which are less specific than conformal maps.

Other work, such as Crane, et.al.,[CPS11] applies an adaptation of these discrete conformal maps, particularly the work of [LPRM02], to allow for dynamic conformal perturbations of triangle meshes, although they also make their own theoretical contributions. They also introduce other applications such as efficiently computing a *conformal flow* from a surface to the plane.

Further theoretical work by Bobenko, et.al., [BPS15] shows the connections of these discrete conformal maps to hyperbolic geometry as well as rigorously exploring the the theory of the model studied by Springborn, et.al.

In all of the above papers, the discretized objects consisted of "C^0" triangle meshes. That is, the faces are linearly interpolated between vertices. The following works [Pol00, War06] consider models of discrete differential geometry with relatively more freedom, where the only constraint on the faces is that they are continuous at the edge midpoints. See the right side of Figure 4 for an example. We soon delve more deeply into this model.

Wardetzky [War06] shows how geometry can be discretized both using Lagrangian elements (vertex-based elements to traditional triangle meshes) as well as Crouzeix-Raviart elements (edge-based elements). The latter elements have been used in numerous applications in PDEs (e.g., [HL03]).

In the subsequent sections, we explore a relatively new model of discrete differential geometry, known as *triangular linkages*, which have strong connections with conformal maps.

3 Defining the Object: Triangular Linkages

The primary focus of the implementation phase of the project is to study a relatively unexplored model of discrete geometry known as *triangular linkages*, introduced by Konakovic, et.al.,[KCD$^+$16]. Their main motivation for studying such linkages is due to their ease of manufacturing as well as being an *auxetic materials* (materials which expand in all directions when being stretched). Such materials have been previously studied by material scientists in connection with foams (e.g. [Lak87]).

What is a triangular linkage? Informally, it is a collection of triangles connected to each other at vertices, in contrast with edges in the classical model). Otherwise, the triangles are free to move about with their remaining degrees of freedom. See Figure 2 for an example of what a triangular linkage looks like. The triangular linkages studied in Konakovic, et.al., [KCD$^+$16], consist of equilateral triangles which are etched out of a 2-dimensional sheet and then subsequently manipulated into a 3-dimensional shape. As noted in the article, this pattern is purposely chosen so that the scaling of the triangular linkage can be uniform. They note that this mode of manufacturing allows for a much greater variety of surfaces to be produced from 2-dimensional materials compared to that of folding or origami-based methods (known as "developable" surfaces). They further suggested some potential applications such as in the manufacturing of clothing and light fixtures.

The authors of [KCD$^+$16] noted many qualitative properties about the triangular linkages. In particular, the authors suggest that the movement of these linkages have the behavior suggesting that of a conformal map. One reason they give is that the number of degrees of freedom is correlated with the number of triangles on the boundary (see Appendix A of [KCD$^+$16]). A more intuitive reason is that the linkages such the faces in Figure 2 have a 'smooth-scaling' look to them that is characteristic of a conformal map. One caveat they due give is that these linkages could in no way express *all* conformal maps because they triangles can only open up so much (see the left-hand side of Figure 2).

To assist in discussing the formal geometric properties of these meshes, we now present formal definitions from which we will develop a geometric theory.

Definition 1. A *triangular linkage* $\mathcal{T} = (V, T)$ consists a set $V \subseteq \mathbb{R}^3$ of finitely many *vertices* along with an *incidence structure* $T \subseteq V^3$ which are triples of distinct vertices of V. For each $(u, v, w) \in T$, we say that the convex hull of (u, v, w) is a *face* of $\mathcal{T}$. We assume that for all $v \in S$, there is at least one and at most two $t \in T$ for which v is a vertex of t. We furthermore assume that any two distinct $t_1, t_2 \in T$ overlap in at most one vertex.

Note that typically we would like to have the vertices of each face be non-colinear. We say that a triangular linkage $\mathcal{T}$ is *uniform* if every face of $\mathcal{T}$ is an equilateral triangle . Our discussion mostly focuses on uniform $\mathcal{T}$. To more easily discuss each triangles position in space, we define the concepts of the *center* and *normal* of each triangle.

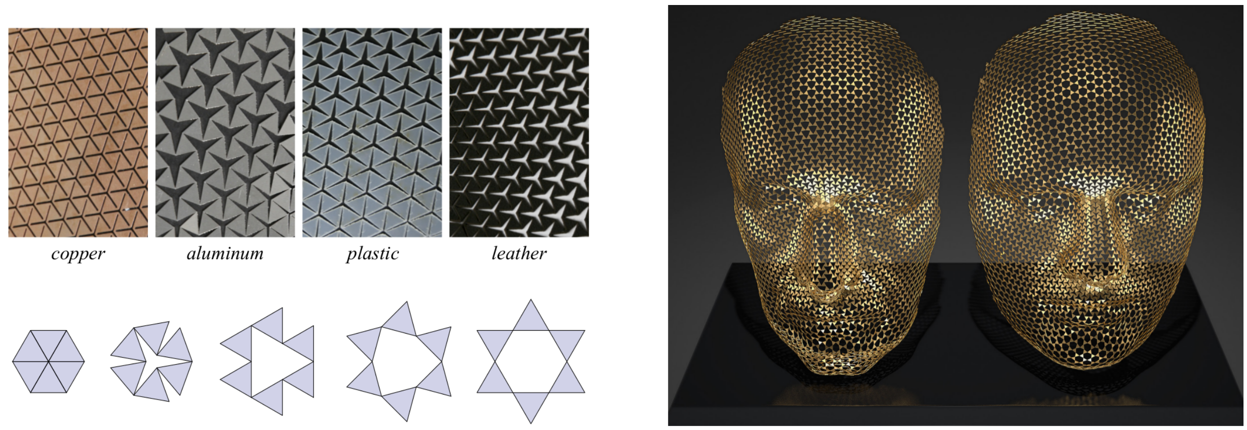

Figure 2: Left: Manufactured triangle linkages and a schematic of the linkages opening up. Right: An elaborate example of a triangular linkage. Images from [KCD$^+$16] with permission.

Definition 2. Let (u, v, w) be the vertices of a face t of a triangle linkage $\mathcal{T}$. The *center* of the face is the circumcenter of u, v, w in the plane through (u, v, w). The *normal* of the face is the unique unit vector $n(t)$ which which is normal to the plane through (u, v, w) and the path $u \to v \to w \to u$ goes around $n(t)$ in the counter-clockwise orientation.

The above definition loses one key property of triangular linkages which is that they typically arise from 2-dimensional surfaces. Throughout our discussion, we will often think of the *dual graph* of $\mathcal{T}$.

Definition 3. The *dual graph* $G_{\mathcal{T}}$ of $\mathcal{T} = (S, T)$ is an undirected graph on vertex set T such that two elements of T are connected by an edge if and only if that share a vertex in $\mathcal{T}$.

Throughout our discussion, we assume that $G_{\mathcal{T}}$ is *planar* that it can be embedded in the plane such that no two edges cross. Note that $G_{\mathcal{T}}$ is planar whenever the $\mathcal{T}$ arose from triangles cut out of the plane. Each face of $G_{\mathcal{T}}$ corresponds to a collection of triangles of T, we call such a collection a *void*.

This leads to our main question to be investigated in the implementation phase of this project.

Question 1. *Can we develop a theory of discrete differential geometry on conformal maps? More specifically, can we explain the conformal structure of these triangular linkages as noted by [KCD$^+$16] in a matter which is harmonious with other theories of discrete differential geometry?*

4 Initial Attempt: Mesh Identification

One naive approach to understanding the triangular linkages is to bootstrap the methods of traditional simplicial-complex model of discrete differential geometry by finding an 'auxiliary mesh' M which tracks the behavior of $\mathcal{T}$ as it is manipulated. By a triangle mesh, we are referring to a C^0 manifold constructed from Lagrangian elements. In particular, we would like for the graph formed by the vertices and edges of M to be the same as the dual graph $G_{\mathcal{T}}$. One way to do this would be as follows, for each triangle t of $\mathcal{T}$, we would like there to be a corresponding triangle t' in M. We can't force t and t' to be similar triangles, of else the mesh M would be too rigid. Instead, we would like to stipulate that the normal for t' is in the same direction as the normal for t. The vertices of M would then correspond to voids between the triangles (e.g. Figure 2). The following lemma shows that such a correspondence is impossible.

Lemma 1. *Given an arbitrary triangular linkage $\mathcal{T}$, there may not exist a C^0 triangular mesh M with the same normals.*

Sketch of proof. It suffices to find a local obstruction to this fact. That is, we only need to find a subset of $\mathcal{T}$ which fails to have a corresponding partial mesh. To do this, we will use a degrees of freedom argument, that $\mathcal{T}$ has more freedoms than M. Define a *cycle* of triangles $t_1, \ldots, t_\ell$ of $\mathcal{T}$ to be a collection of triangles such that consecutive triangles (including t_1 and t_ℓ) share a vertex, and that these triangles wrap around a single void of $\mathcal{T}$. Let $t'_1, \ldots, t'_\ell$ be the other triangle which is connected to each of these triangles. Let $\mathbf{n}_1, \ldots, \mathbf{n}_\ell$ and $\mathbf{n}'_1, \ldots, \mathbf{n}'_\ell$ be the normals for $t_1, \ldots, t_\ell$ and $t'_1, \ldots, t'_\ell$, respectively. Assume there there exists a mesh M with vertex v corresponding to the void around $t_1, \ldots, t_\ell$. Let $v_1, \ldots, v_\ell$ be the vertices of M connected to v such that triangle vv_iv_{i+1} has $\mathbf{n}_i$ as a normal for all $i = 1, \ldots, \ell$ (where $v_{\ell+1} = v_1$). Then, we can infer a lot about how M sits in $\mathbb{R}^3$. In particular, we know that the vector vv_{i+1} must be in the direction of $\mathbf{n}_i \times \mathbf{n}_{i+1}$ since the edge is in planes normal to $\mathbf{n}_i$ and $\mathbf{n}_{i+1}$. We may assume that we have chosen $\mathcal{T}$ so that these cross products are never 0.

Likewise, we know that the vector v_iv_{i+1} is in the direction of $\mathbf{n}_i \times \mathbf{n}'_i$ for all i. Since none of these constraints on M fix the translation or scaling of M, we may assume without loss of generality that v is the origin in $\mathbb{R}^3$ and that vv_1 has length 1. Then, since we know the directions v_1v_2 and vv_2, we have forces the location of v_2 (unless there is a degeneracy). Likewise, we know the locations of all of $v_1, \ldots, v_\ell$, and in deducing these we did *not* need to know the value of $\mathbf{n}'_\ell$. But, we still have the constraint, that the vector $v_\ell v_1$ is in the direction of $\mathbf{n}_\ell \times \mathbf{n}'_\ell$. Since M has run out of degrees of freedom, we can perturbate $\mathbf{n}'_\ell$ without affecting any of $\mathbf{n}_1, \ldots, \mathbf{n}_\ell$ and $\mathbf{n}'_1, \ldots, \mathbf{n}'_{\ell-1}$ so that $v_\ell v_1$ is not in the same direction as $\mathbf{n}_\ell \times \mathbf{n}'_\ell$. Thus, our auxiliary mesh M does not exist in general. $\square$

Although the above proof shows that M does not exist in general, we can still adapt the normal-based definition of Gaussian curvature [Cra15] to obtain a notion of Gaussian curvature in our setting. If v is a vertex of M and $n_1, \ldots, n_\ell$ are the normals to the faces with vertex v ordered in counterclockwise order, then the Gaussian curvature of vertex v is the signed area of the spherical polygon formed by connecting $n_1, \ldots, n_\ell$ with geodesics on the unit sphere. Likewise, we can define the Gaussian curvature of a void of $\mathcal{T}$ in the same way (see Figure 4). Since these normals partition the unit sphere, it is intuitively clear that the Gauss-Bonnet theorem holds if $\mathcal{T}$ lacks boundary.

5 Application of Work of Polthier and Wardetzky

A more sophisticated attack on this question is to integrate this geometry with the discrete conformal methods of [Pol00] and [War06]. As previously discussed, that context concerns the nature of triangles connected at edge midpoints. To connect that with our geometry, we can imagine quadrupling the area of each of our triangles by reflecting the triangle across each of its edges (see Figure 4). That way, the triangles will be connected at the midpoints but will overall have the same geometry. In particular, the angles of the larger triangles are identical to the angles of the smaller triangles (in the uniform case they are all equilateral).

As mentioned previously, one line of attack of coming up with a theory of discrete differential geometry for triangle linkages is using the methods of Polthier [Pol00] and Wardetzky [War06].

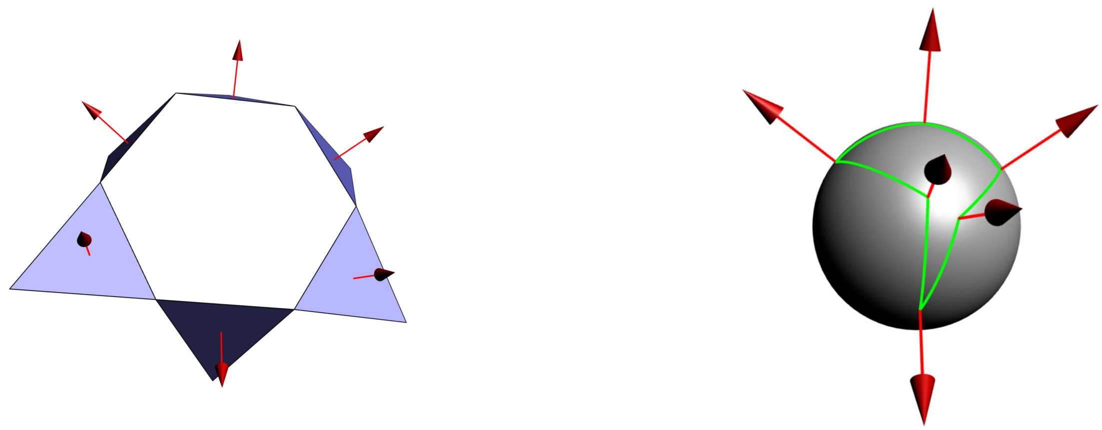

Figure 3: On the left is a void of a triangular linkage with normals. On the right is the normals placed on a unit sphere. The area of the region bounded by the green arcs is what we define to be the Gaussian curvature of the void.

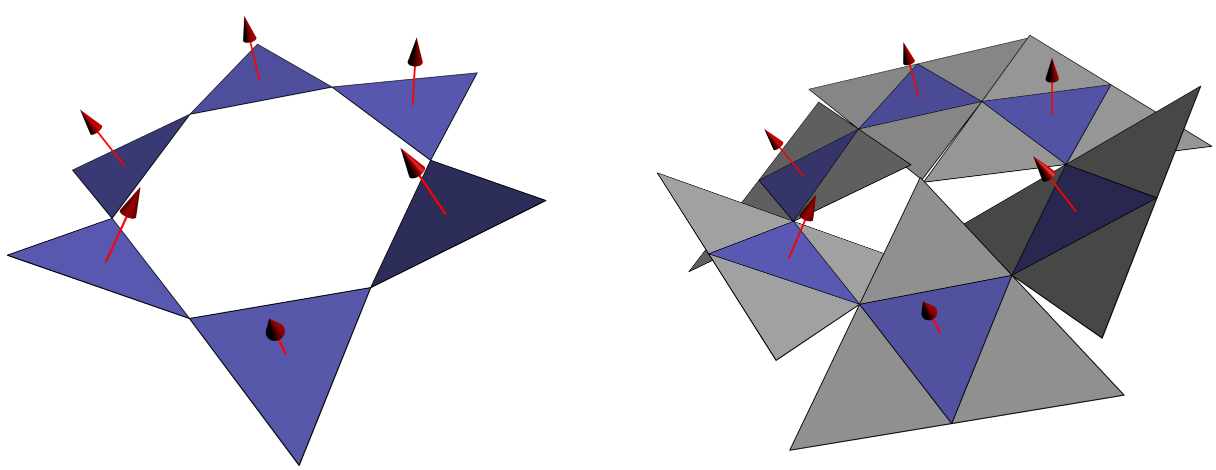

Figure 4: A method of transforming the triangular linkages into a midpoint-connected triangular mesh by reflecting each triangle through each of its edges. Normals and centers are added for reference.

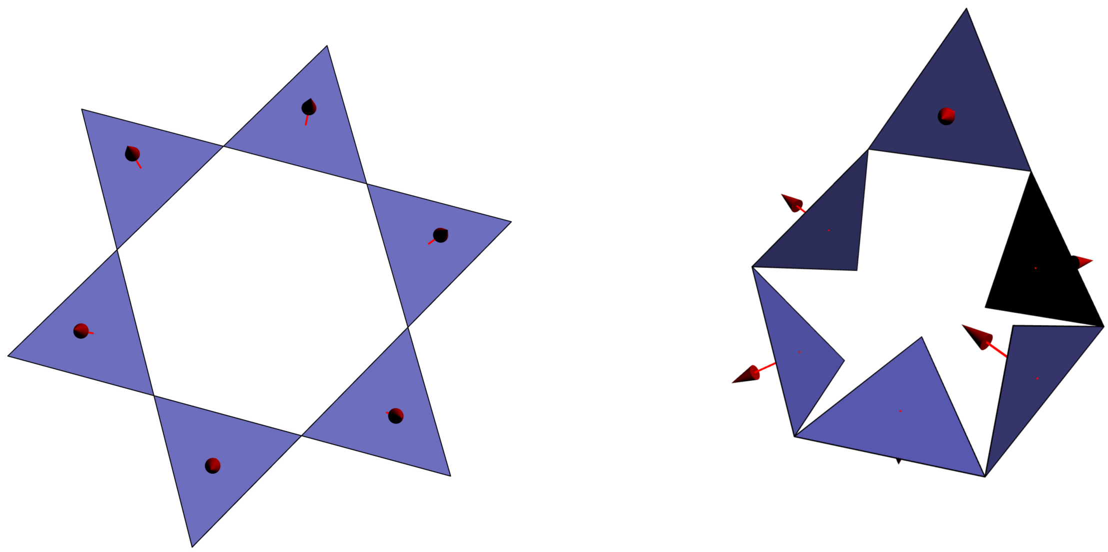

Figure 5: On the left is an example of a triangular linkage which should be consider to have large-area and thus conformal. On the right, an example of a triangular linkage which should be considered to have smaller area and thus non-conformal.

Recall from the lecture notes [Cra15] how the complex plane can be discretized using Lagrangian elements. In that particular case, the discretization of the complex plane was the lattice $\mathbb{Z}[\omega]$, where $\omega = \frac{1+i\sqrt{3}}{2}$ is a sixth root of unity. Then, for any $z \in Z[\omega]$, we have the neighbors $w + \omega^0, w + \omega^1, \ldots, w + \omega^5$. Then, any discrete map $\hat{f} : \mathbb{Z}[\omega] \to V$, where V is some vector space, can be extended to C^0 map $\mathbb{C} \to V$ as follows

$$f(z) = a\hat{f}(z_1) + b\hat{f}(z_2) + c\hat{f}(z_3)$$

where (a, b, c) are the barycentric coordinates of the lattice triangle (z_1, z_2, z_3) which z is in. The contribution of each vertex $z \in \mathbb{Z}[\omega]$ is known as a *Lagrange element*. Note that if $V = \mathbb{R}^3$, then f is an embedding of a (infinite) triangle mesh.

Polthier and Wardetzky use an alternative basis known as *Crouzeix-Raviart elements* for discretization. Let $m(\mathbb{Z}[\omega])$ be the set of midpoints of adjacent lattice points. Our discretized function is then $\hat{f} : m(\mathbb{Z}[\omega]) \to V$. We can extend this to a function $f : \mathbb{C} \to V$ as follows.

$$f(z) = \left(\frac{1}{2} - a\right)\hat{f}\left(\frac{z_2+z_3}{2}\right) + \left(\frac{1}{2} - b\right)\hat{f}\left(\frac{z_1+z_3}{2}\right) + \left(\frac{1}{2} - c\right)\hat{f}\left(\frac{z_1+z_2}{2}\right),$$

where (a, b, c) are the barycentric coordinates of the *lattice* triangle (z_1, z_2, z_3) and thus $(z_1 + z_2)/2, (z_1 + z_3)/2$, and $(z_2 + z_3)/2$ are the midpoints. We need to be careful defining this function along the edges of the lattice. Note that f is continuous at the points of $m(\mathbb{Z}[\omega])$ as well as the interiors of the lattice triangles, but is not necessarily continuous elsewhere on the edges. Relaxing this constriction yields many degrees of freedom. We can canonically choose which triangle each boundary vertex is a part of to avoid difficulty.

5.1 Connection to Triangular Linkages

A triangular linkage can be described a map $\hat{f} : E \to \mathbb{R}^3$, where $E \subseteq m(\mathbb{Z}[\omega])$ represent the 'vertices' where two triangles meet in the triangular linkage. We say that the map $\hat{f}$ is an *embedding* if for any $w, z \in E$ which are adjacent; that is $\|w - z\|_2 = 1/2$, then $\|\hat{f}(w) - \hat{f}(z)\|_2 = 1$. We further constrain that the extension f is injective within the convex hull of a triple of three adjacent points (i.e. a face).

The faces of the embedding are then $(\hat{f}(z_1), \hat{f}(z_2), \hat{f}(z_3))$, where $z_1, z_2, z_3 \in E$ form an equilateral triangle of side length $1/2$. We say that $z \in E$ is a *boundary* vertex if it is adjacent to at most one triangle. Note that we can identify *voids* of the triangular linkages with points $z \in \mathbb{Z}[\omega]$ such that $z + \omega^k/2 \in E$ for $k \in \{0, 1, \ldots, 5\}$. This is now our formal definition of void.

5.2 Laplacian and Dirichlet Energy

Wardetzky [War06] developed the notion of the Laplace-Beltrami operator for these surfaces. Recall that if we have a Lagrangian mesh, $\hat{f}_L : (V \subseteq \mathbb{Z}[\omega]) \to \mathbb{R}^3$, then the Laplacian [War06, Cra15] is

$$\Delta \hat{f}_L(z) = \frac{1}{2} \sum_{k=0}^{5} (\cot \alpha_k + \cot \beta_k)(\hat{f}_L(z + \omega^k) - \hat{f}_L(z)),$$

where α_k and β_k are the two angles opposite the segment from $\hat{f}_L(z)$ to $\hat{f}_L(z + \omega^k)$. Now consider our Crouzeix-Raviart mesh $\hat{f}_{CR} : (E \subseteq m(\mathbb{Z}[\omega])) \to \mathbb{R}^3$. Wardetzky chooses a Laplacian of

$$\Delta \hat{f}_{CR}(w) = 2 \sum_{w', \|w' - w\| = 1/2} \cot \alpha_{w,w'} (\hat{f}_{CR}(w') - \hat{f}_{CR}(w)).$$

Assuming that $w \in E$ is not on the boundary of the triangular linkage, this sum is taken over 4 vertices of the triangles connecting w. The $\alpha_{w,w'}$ is the angle opposite edge from $f(w)$ to $f(w')$ which will always be $60°$ in our case.

Likewise, Polthier [Pol00] (and perhaps also Wardetzky) defines a notion of the *Dirichlet energy* of one of these discrete maps which is important in determining if a map is harmonic. His definition is

$$E_D(\hat{f}_{CR}) = \sum_{v \in E} \left(\cot \alpha_v \|\hat{f}_{CR}(v_1) - \hat{f}_{CR}(v_2)\|_2^2 + \cot \beta_v \|\hat{f}_{CR}(v_{-1}) - \hat{f}_{CR}(v_{-2})\|_2^2 \right),$$

where v_1, v_2 and v_{-1}, v_{-2} are the vertices on the two faces incident with v, and α_v and β_v are the angles v makes with these two pairs of vertices. In the case of embedding a triangular linkage, all of these values are constant, so our embeddings already have 'minimized' Dirichlet energy.

Wardetzky [War06] also defines other operators such as the curl and divergence in these settings, but the ones presented seem the most applicable to what we seek to study.

Note that in our triangular linkage construction all of these quantities, the distances and the angles, remain constant. Thus, physical transformations of the triangular linkage can be described as Dirichlet-energy preserving maps! As previously discussed, low Dirichlet energy is not quite the required condition for a conformal map. The proper constraint is that we seek to minimize is the conformal energy (see Equation (3))which is the *difference* between the Dirichlet energy and the area of the resulting surface (see [Cra15]). Since the Dirichlet energy is constant, the conformal maps correspond to triangular linkages which are the most 'spread out.' Note that the non-degeneracy constraints are not of concern since the faces of our triangular mesh are rigid and cannot collapse to a point.

Of course, this leads to a new problem, what does area mean in this context? Intuitively our definition of area should say that shapes like those in Figure 2 and the left-hand-side of Figure 5 have large area and thus are 'near-conformal' while excluding those in the right-hand-side of

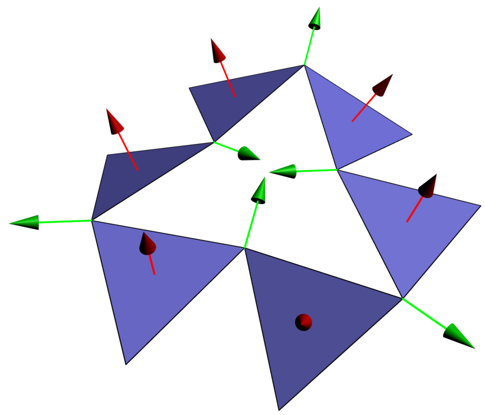

Figure 6: The green vector depict the (negated) mean curvature normals of the triangle mesh. The red vectors are the face normals.

Figure 5. In Section 6, we answer this question by proposing a model of discrete conformal maps on triangular linkages.

5.3 Mean Curvature Normals

Recall from Homework 4 how the planar Laplacian relates to mean curvature.

$$\Delta f = 2HN,$$

where $f : \mathbb{C} \to \mathbb{R}^3$ is our embedding. In the case where f was constructed from the Lagrange basis, we used this identity yield both a normal vector as well as a mean curvature value (up to sign). Wardetzky notes that using the Crouzeix-Raviart elements, we may use the discretized Laplacian to compute mean curvature normals at each edge of our mesh.

Although this gives a definition of mean curvature normals, in practice they do not seem to be the optimal definition as they are extremely sensitive to the local geometry. See Figure 5.3.

5.4 Connecting Back to Lagrangian Elements

As discussed with the mean curvature normals, there are unsettling discontinuities when building the discrete operators off of these Crouzeix-Raviart elements. One workaround suggested by Wardetzky (See Lemma 2.4.1) is by deriving a Lagrangian map $\hat{f}_L : \mathbb{Z}[\omega] \to V$ from the Crouzeix-Raviart map $\hat{f}_{CR} : m(\mathbb{Z}[\omega]) \to V$ via the transformation

$$\hat{f}_L(z) = \frac{1}{2} \sum_{k=0}^{5} \hat{f}_{CR}(z + \omega^k/2).$$

We could apply this to the mean curvature normals to potential get more sensible-looking normals.

6 Conformal Scale Factors

By consulting the literature, we have found a few discrete operators for triangular linkages, and have applied some desirable properties to obtain other operators (such as edge normals and mean curvature). These constructions only give partial insight the main motivation of this project which is to understand how the 'natural' triangle linkage positions relate to conformal maps. In this section, we directly attack the question of conformal behavior of triangular linkages by using elementary geometry.

6.1 Defining Triangular Conformal Maps

Looking closely at the elegant triangular linkages in Konakovic, et.al., [KCD$^+$16], one may notice that the six vertices surrounding each void bunch in groups of three, an 'inward' cluster of three vertices and an 'outward' cluster. Furthermore, each of these clusters appears to form an equilateral triangle. That behavior is our motivation of a definition of conformal maps on triangular linkages.

Definition 4. Consider $E \subseteq m(\mathbb{Z}[\omega])$, and let $\hat{f} : E \to \mathbb{R}^3$ be an embedding. We say that the embedding is *conformal* if for any $z_0, z_1, z_2 \in E$ there exists $w \in \mathbb{Z}[\omega]$ such that $z_i k = w + \omega^{2k}$ for $k \in \{0, 1, 2\}$, then $(\hat{f}(z_0), \hat{f}(z_1), \hat{f}(z_2))$ is an equilateral triangle.

Unpacking the formal definition, we have that a conformal map preserves one of the two equilateral triangles in each void, and the triangle we pick is consistent between voids so that each vertex is part of at most one constrained equilateral triangle. We call the triangle we picked the void's equilateral triangle. This also yields a canonical 1-to-3 injective map from voids to vertices. Note that this choice of definition allows for much freedom in the geometry.

Lemma 2. *The number of real degrees of freedom of conformal triangular map is at least the sum of the number of voids and the number of boundary vertices.*

Proof. Let E be the number of vertices in the triangular linkage (i.e. Crouzeix-Raviart edges), let V be the number of voids, and let B be the number of vertices on the boundary, and let F be the number of faces. Since each void is adjacent to six non-boundary vertices and each vertex is adjacent to at most two voids, we have the identity $V \leq \frac{1}{3}(E - B)$. Each face is adjacent to three vertices, and each vertex is adjacent to at most two faces, so $F \leq (2/3)E$.

Next we compute the number of real degrees of freedom in the triangular linkage. Each vertex has 3 real degrees of freedom a priori, but each face removes 3 real degrees of freedom (to specify the edge lengths), and each each void stipulates another 2 real degrees of constraints to enforce its equilateral triangle. Thus, there are at least

$$3E - 3F - 2V \geq 3E - 2E - 2V = E - 2V \geq 3V + B - 2V = V + B,$$

degrees of freedom. □

Note that if we would have enforces that both triangles of each void would need to be equilateral, then we would have $B - V$ degrees of freedom, which would often be negative, and thus force a rigid configuration. One can directly prove this without a degrees of freedom argument, but this is omitted. Note that this degrees of freedom argument does not rigorously ensure that all the constraints are consistent, but the examining the triangular linkages from [KCD$^+$16] should be convincing evidence that we made the right decision.

6.2 Scale factors

To give another motivating reason for why this choice of conformal maps is the 'correct' choice, we now derive a what the scale factor should be of our conformal maps.

Let's restrict our attention to the case that our conformal embedding $\hat{f}_{CR} : m(\mathbb{Z}[\omega]) \to \mathbb{R}^3$ maps to the plane where the z coordinate is 0. Although we will not rigorously show it, such conformal embeddings are extremely rigid. Once you fix the equilateral triangle of a void, you also essentially fix the six faces surrounding that void. With a little work, these fixed triangles "propagate" to fix neighboring triangles. Thus, it appears that the only such embeddings are the uniform scalings of the plane depicted in [KCD$^+$16]. Because of this rigidity, we can unambiguously define the scale factor of each void.

Definition 5. Let $\hat{f}_{CR} : m(\mathbb{Z}[\omega]) \to \mathbb{R}^3$ be a *planar* embedding. Let $z \in \mathbb{Z}[\omega]$ be a void. Define the scale factor $\phi_f(z)$ to be the sum of twice the area of a unit equilateral triangle and the area of the (possibly non-convex) hexagon with $\hat{f}_{CR}(z + \omega^k)$, $k \in \{0, 1, \dots, 5\}$, as vertices.

The reason we add twice the area of a unit equilateral triangle is that each void is adjacent to two six faces, but each face is adjacent to at most 3 voids, so the scale factor of a void should account for a third of the area of the surrounding faces, which is 2 units equilateral triangles, or $\frac{\sqrt{3}}{2}$. Note that the 'units' of are scale factor are squared units.

Lemma 3. *Let $\hat{f}_{CR} : m(\mathbb{Z}[\omega]) \to \mathbb{R}^3$ be a* planar *conformal embedding in which both triangles of each void are equilateral. Let $z \in \mathbb{Z}[\omega]$ be a void. Then*

$$\phi_f(z) = 2(A_1 + A_2) - \sqrt{3},$$

where A_1 is the area of the triangle with vertices $\hat{f}_{CR}(z + 1), \hat{f}_{CR}(z + \omega^2), \hat{f}_{CR}(z + \omega^4)$, and A_2 is the area of the triangle with vertices $\hat{f}_{CR}(z + \omega), \hat{f}_{CR}(z + \omega^3), \hat{f}_{CR}(z + \omega^5)$.

Proof. For notational simplicity, let $V_k = \hat{f}_{CR}(z+\omega^k)$. Let $\alpha = \angle V_0V_1V_2$. Since $V_0V_2 = V_2V_4 = V_4V_0$ and $V_iV_{i+1} = 1$ for all i, we have that $V_0V_1V_2 \cong V_2V_3V_4 \cong V_4V_5V_0$ by SSS. Thus, $\alpha = \angle V_2V_3V_4 = \angle V_4V_5V_0$. Thus,

$$\begin{aligned}
\angle V_1V_2V_3 &= \angle V_1V_2V_0 + \angle V_2V_0V_4 + \angle V_4V_2V_3 \\
&= \frac{180^\circ - \alpha}{2} + 60^\circ + \frac{180^\circ - \alpha}{2} \\
&= 240^\circ - \alpha,
\end{aligned}$$

where we are considering the internal angle of the hexagon. Thus, by Law of Cosines

$$\begin{aligned}
(V_0V_2)^2 &= (V_0V_1)^2 + (V_1V_2)^2 - 2(V_0V_1)(V_1V_2)\cos\alpha = 2 - 2\cos\alpha \\
(V_1V_3)^2 &= (V_1V_2)^2 + (V_2V_3)^2 - 2(V_1V_2)(V_2V_3)\cos(240^\circ - \alpha) = 2 + \cos\alpha + \sqrt{3}\sin\alpha.
\end{aligned}$$

Since $V_0V_2V_4$ and $V_1V_3V_5$ are equilateral, we have that

$$\begin{aligned}
A_1 + A_2 &= \frac{\sqrt{3}}{4}((V_0V_2)^2 + (V_1V_3)^2) \\
&= \frac{\sqrt{3}}{4}(4 - \cos\alpha + \sqrt{3}\sin\alpha) \\
&= \sqrt{3} - \frac{\sqrt{3}}{4}\cos\alpha + \frac{3}{4}\sin\alpha.
\end{aligned}$$

Now observe that since the area of a unit equilateral triangle is $\sqrt{3}4$

$$
\begin{aligned}
\phi_f(z) &= A(V_0V_1V_2) + A(V_2V_3V_4) + A(V_4V_5V_0) + A(V_0V_2V_4) + \frac{\sqrt{3}}{2} \\
&= 3A(V_0V_1V_2) + \frac{\sqrt{3}}{4}(2 - 2\cos\alpha) + \frac{\sqrt{3}}{2} \\
&= \frac{3}{2}\sin\alpha + \frac{\sqrt{3}}{4}(2 - 2\cos\alpha) + \frac{\sqrt{3}}{2} \text{ (sine area formula)} \\
&= \sqrt{3} - \frac{\sqrt{3}}{2}\cos\alpha + \frac{3}{2}\sin\alpha \\
&= 2(A_1 + A_2) - \sqrt{3},
\end{aligned}
$$

as desired. □

With this elegant alternative definition, we adopt it as our definition of scale factor for a general conformal embedding. In fact, since the definition does not even require the void's triangles to be equilateral, this definition holds for general embeddings. Looking at the images of [KCD$^+$16], the change in scale factor is fairly continuous throughout the objects, which they mention is a key sign that the mappings are conformal.

Now that we have a notion of scale factor, we can relate it to other quantities such as the Gaussian curvature (see e.g., [KCD$^+$16]).

$$K = \frac{\Delta(\log\phi)}{2\phi}$$

Most likely, this does not correspond to the normal-based definition of curvature defined previously, but it would be an interesting experiment to see how the two compare. Furthermore, it would be nice to have a 'complete' theory of discrete differential geometry for at least this special family of triangular linkages.

7 Future Directions and Applications

There are many potential directions of further exploration.

- Most of the above discussion has deal with the *extrinsic* geometry of triangle linkages. Can we develop a compatible theory of *intrinsic* geometry? In particular, what is the natural discrete metric on these triangle linkages?
- Earlier we proposed an extrinsic definition of discrete Gaussian curvature. Can we find other natural definitions of Gaussian curvature which can be proven to be equivalent? In particular, can we find a compatible *intrinsic* definition of Gaussian curvature for such a surface?
- Can we build a theory of discrete exterior calculus on these surfaces? What would be natural notions of 0-, 1-, and 2-forms? What should the exterior derivative and Hodge star be?
- Would it be possible to unify the traditional and triangle linkage models of differential geometry? Although the naive attempt presented in Section 4 failed, there may be more subtle ways to create a correspondence.

- Can we find efficient algorithms pertaining to triangular linkages? In particular, it would be desirable to find an efficient algorithm for constructing a triangular linkage approximating a given surface.

By developing such an alternate theory of discrete differential geometry, we hope to lay the groundwork for future computationally-driven applications of (e.g. novel application of 'real-time' conformal maps). From an algorithmic perspective, our proposed definition of conformal maps should be amenable to computation. For example, approximating a Lagrangian mesh with a triangular mesh could be done as some sort of an optimization problem. Furthermore, it would be desirable to develop some motion-planning algorithm which starts with cut sheet metal and describes how to deform it into the prescribe linkage.

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References

[Bat09] Harry Bateman. The conformal transformations of a space of four dimensions and their applications to geometrical optics. *Proceedings of the London Mathematical Society*, 2(1):70–89, 1909.

[BPS15] Alexander I Bobenko, Ulrich Pinkall, and Boris A Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. *Geometry & Topology*, 19(4):2155–2215, 2015.

[CPS11] Keenan Crane, Ulrich Pinkall, and Peter Schröder. Spin transformations of discrete surfaces. *ACM Trans. Graph.*, 30(4):104:1–104:10, July 2011.

[Cra15] Keenan Crane. *Discrete Differental Geometry: An Applied Introduction*. 2015.

[DMA02] Mathieu Desbrun, Mark Meyer, and Pierre Alliez. Intrinsic parameterizations of surface meshes. *Computer Graphics Forum*, 21(3):209–218, 2002.

[FGN99] V Faraoni, E Gunzig, and P Nardone. Conformal transformations in classical gravitational theories and in cosmology. *Fundamentals of Cosmic Physics*, 20:121–175, 1999.

[GY08] Xianfeng David Gu and Shing-Tung Yau. *Computational Conformal Geometry*. Advanced Lectures in Mathematics. International Press, Somerville, Mass., USA, 2008.

[HL03] Peter Hansbo and Mats G Larson. Discontinuous galerkin and the crouzeix–raviart element: application to elasticity. *ESAIM: Mathematical Modelling and Numerical Analysis*, 37(01):63–72, 2003.

[KCD+16] Mina Konakovic, Keenan Crane, Bailin Deng, Sofien Bouaziz, Daniel Piker, and Mark Pauly. Beyond developable: Computational design and fabrication with auxetic materials. *ACM Trans. Graph.*, 35, 2016.

[KMB+09] Peter Kaufmann, Sebastian Martin, Mario Botsch, Eitan Grinspun, and Markus Gross. Enrichment textures for detailed cutting of shells. *ACM Trans. Graph.*, 28(3):50:1–50:10, July 2009.

[KSS06] Liliya Kharevych, Boris Springborn, and Peter Schröder. Discrete conformal mappings via circle patterns. *ACM Trans. Graph.*, 25(2):412–438, April 2006.

[Lak87] Roderic Lakes. Foam structures with a negative poisson's ratio. *Science*, 235(4792):1038–1040, 1987.

[LPRM02] Bruno Lévy, Sylvain Petitjean, Nicolas Ray, and Jérome Maillot. Least squares conformal maps for automatic texture atlas generation. *ACM Trans. Graph.*, 21(3):362–371, July 2002.

[Luo04] Feng Luo. Combinatorial yamabe flow on surfaces. *Communications in Contemporary Mathematics*, 6(05):765–780, 2004.

[MZ83] Ralph Menikoff and Charles Zemach. Rayleigh-taylor instability and the use of conformal maps for ideal fluid flow. *Journal of Computational Physics*, 51(1):28–64, 1983.

[Nee98] Tristan Needham. *Visual complex analysis*. Oxford University Press, 1998.

[Pol00] Konrad Polthier. Conjugate harmonic maps and minimal surfaces. 2000.

[SdS01] A. Sheffer and E. de Sturler. Parameterization of faceted surfaces for meshing using angle-based flattening. *Engineering with Computers*, 17(3):326–337, 2001.

[SLMB05] Alla Sheffer, Bruno Lévy, Maxim Mogilnitsky, and Alexander Bogomyakov. Abf++: Fast and robust angle based flattening. *ACM Trans. Graph.*, 24(2):311–330, April 2005.

[SSP08] Boris Springborn, Peter Schröder, and Ulrich Pinkall. Conformal equivalence of triangle meshes. *ACM Trans. Graph.*, 27(3):77:1–77:11, August 2008.

[War06] Max Wardetzky. *Discrete Differential Operators on Polyhedral Surfaces–Convergence and Approximation*. PhD thesis, Université Claude Bernard Lyon 1, 2006.