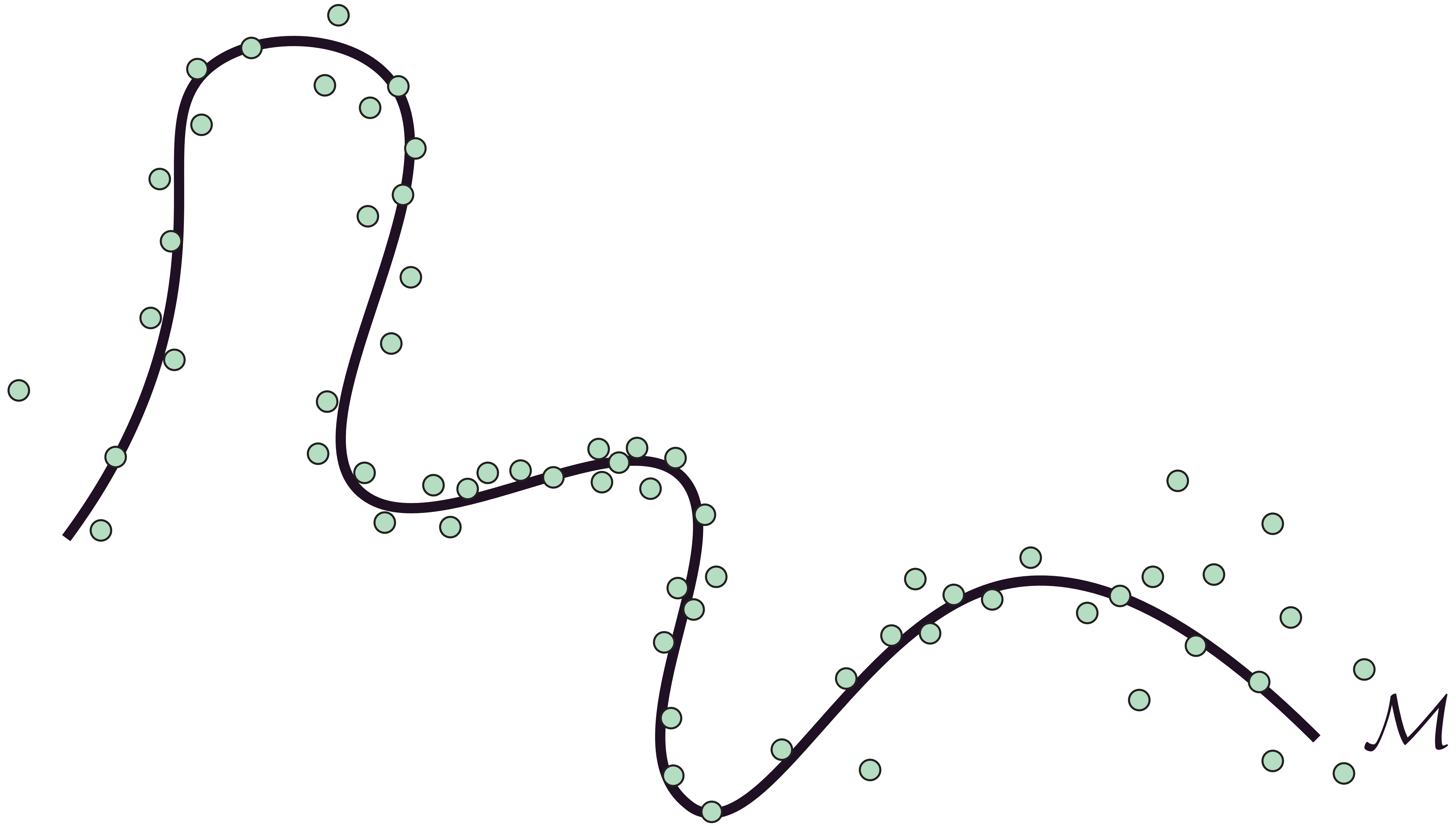


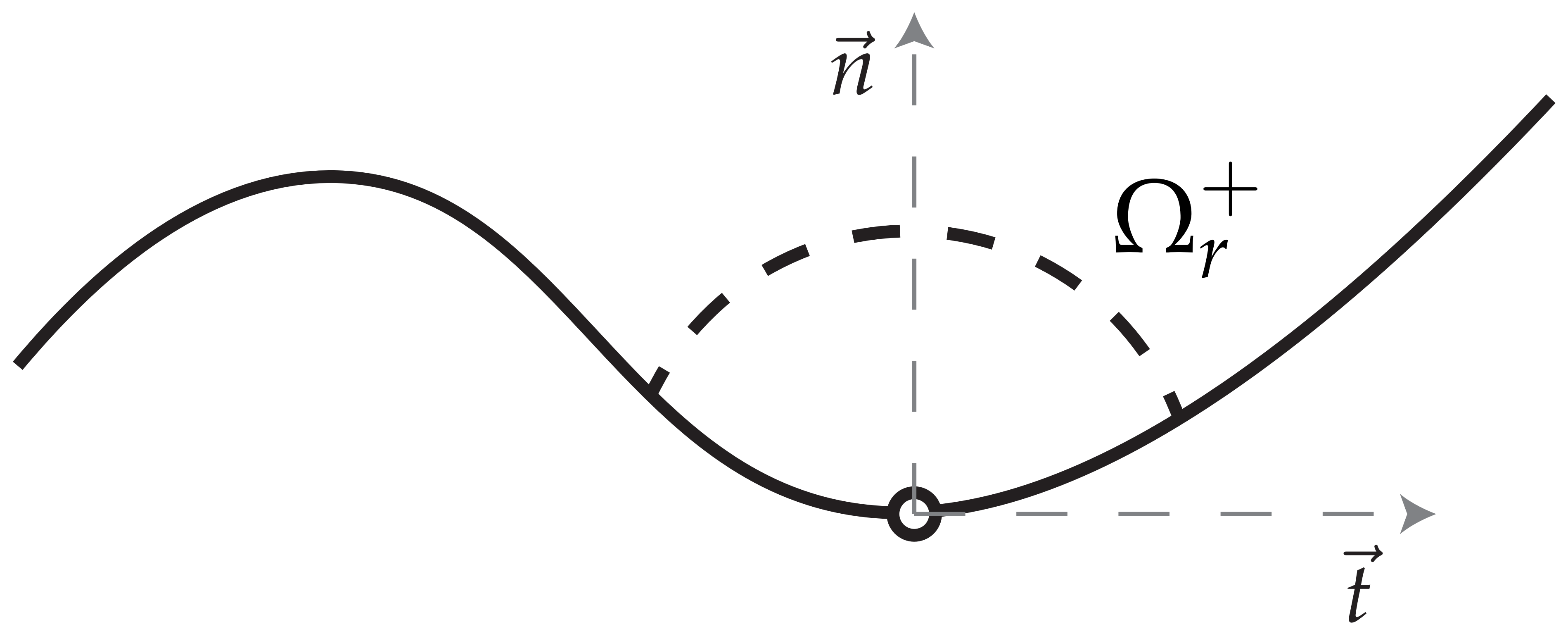
Curvature Estimation of High Dimensional Submanifolds Based on Noisy Observations: Some Results

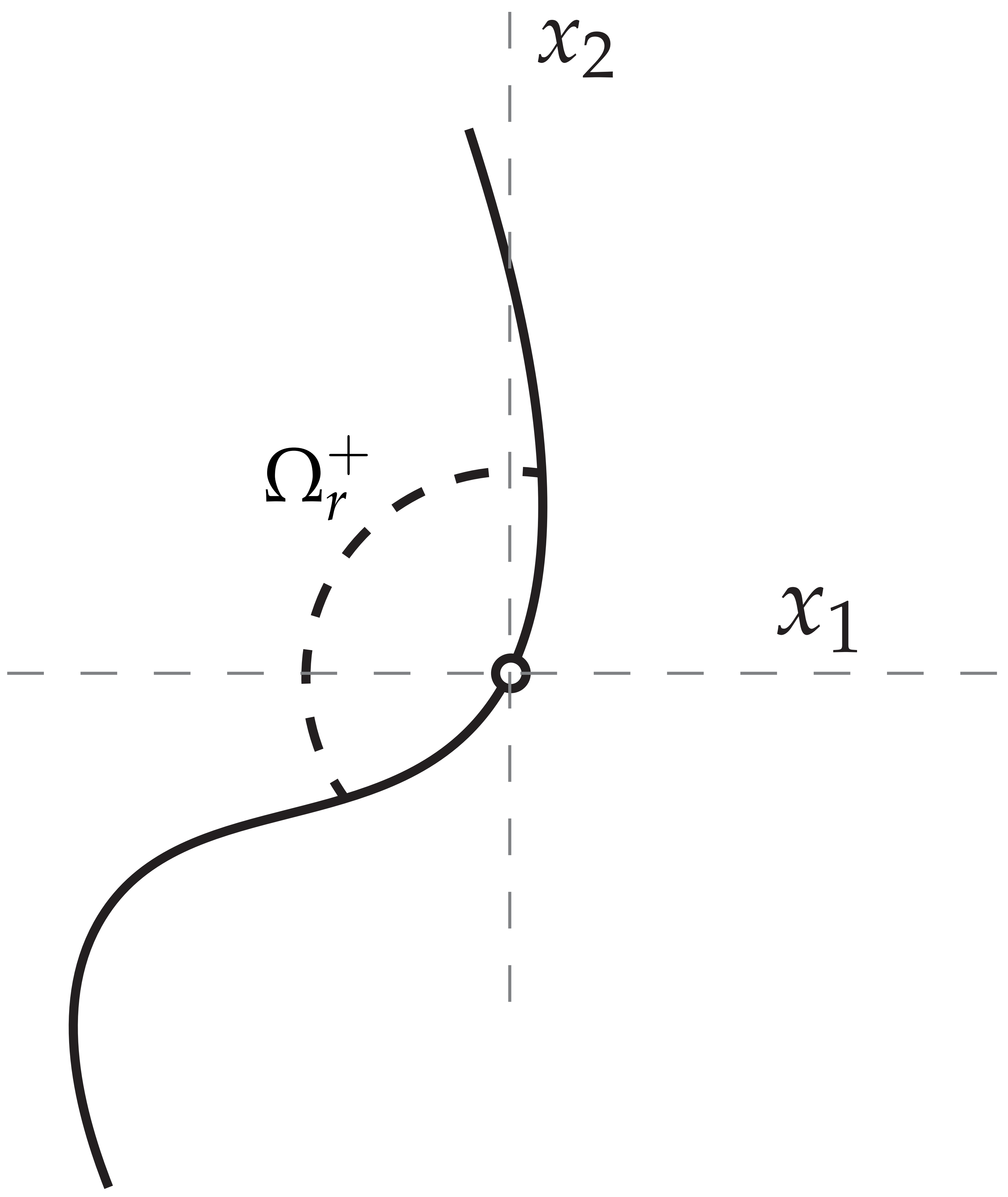
Yousuf M. Soliman

Quick Review of Integral Invariants and Curvature Estimation

$$I(f) = \int_{\Omega_r^+} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

$$f(x) = \frac{1}{2}\sum_{i=1}^{m} \kappa_i x_i^2 + \mathcal{O}(\|x\|^3)$$





Quick Review of Integral Invariants and Curvature Estimation

Rather than just computing $I(1)$, we can get better estimates of the curvature by computing the covariance matrix:

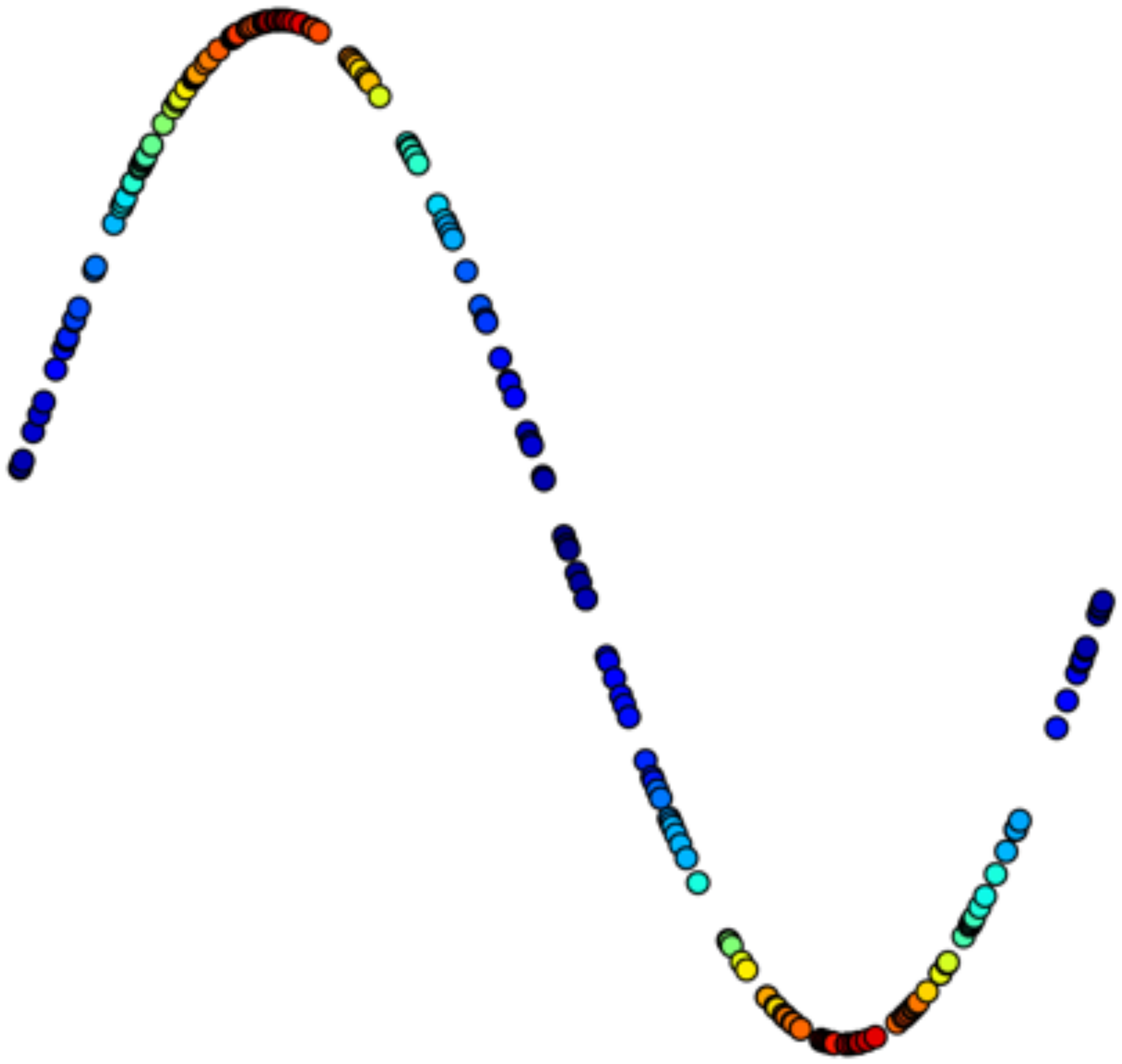
$$\Sigma(\Omega_r^+) = \int_{\Omega_r^+} (\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^\top \, \mathrm{d}\mathbf{x}$$

$$= \begin{pmatrix} I(x^2) & I(xy) \\ I(xy) & I(y^2) \end{pmatrix} - \frac{1}{\mathrm{vol}(\Omega_r^+)} \begin{pmatrix} I(x)^2 & I(x)I(y) \\ I(x)I(y) & I(y)^2 \end{pmatrix}$$

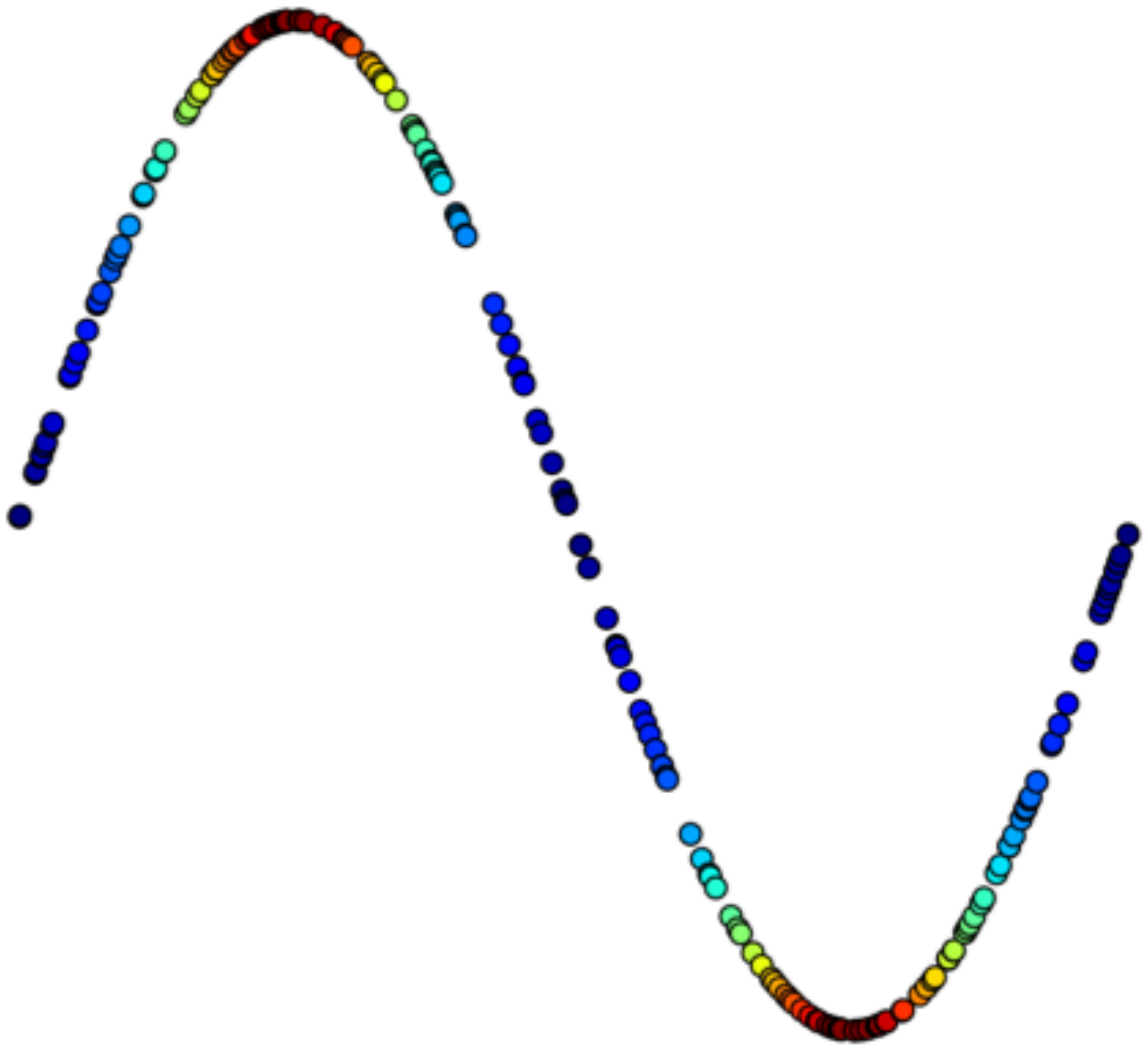
Furthermore, eigenanalysis of the covariance matrix reveals the principal curvatures

Initial Results: $(y = \sin x)$

$n = 200$



Estimation Using Integral Invariants

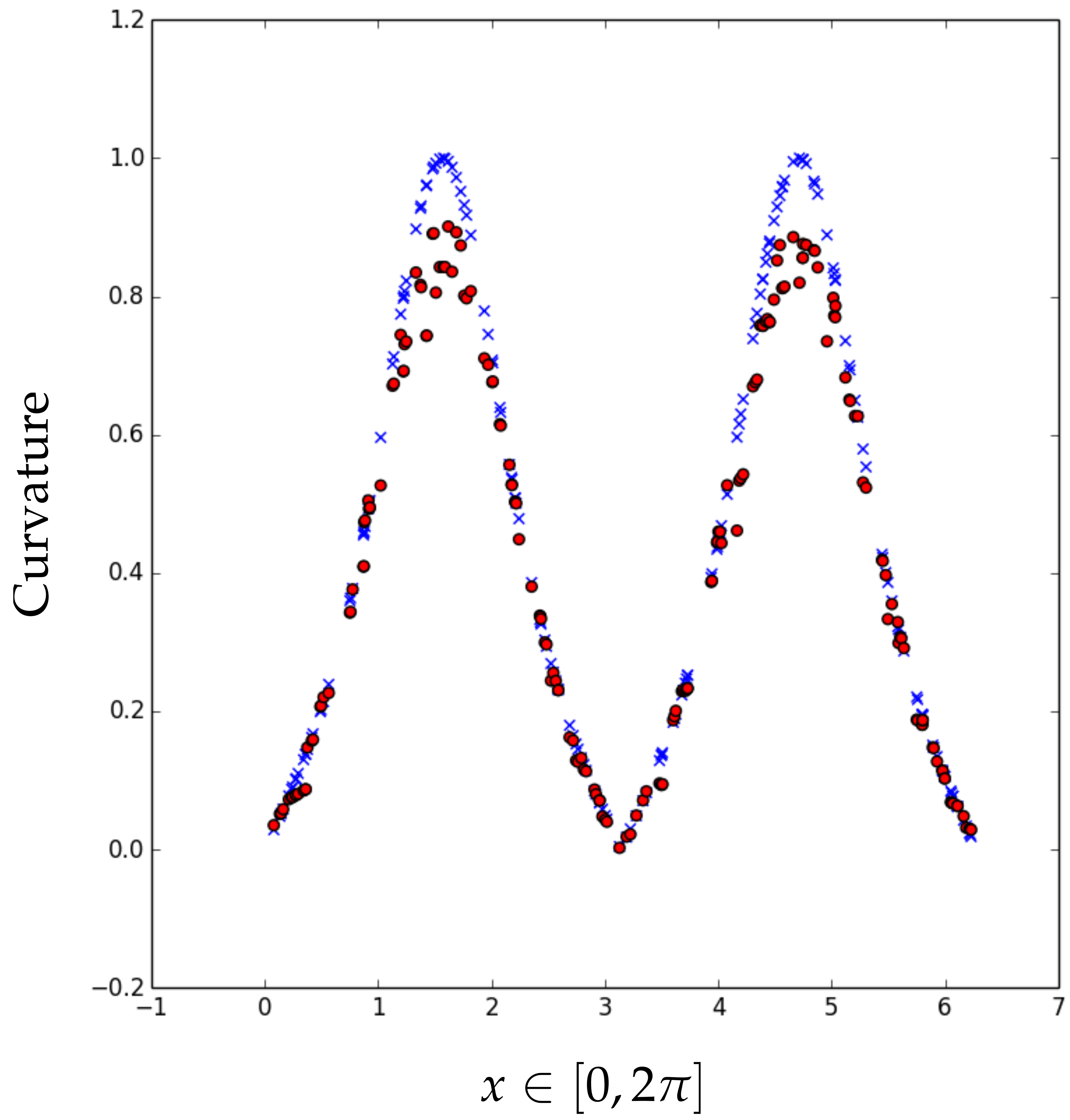


True Curvature

Initial Results: $(y = \sin x)$

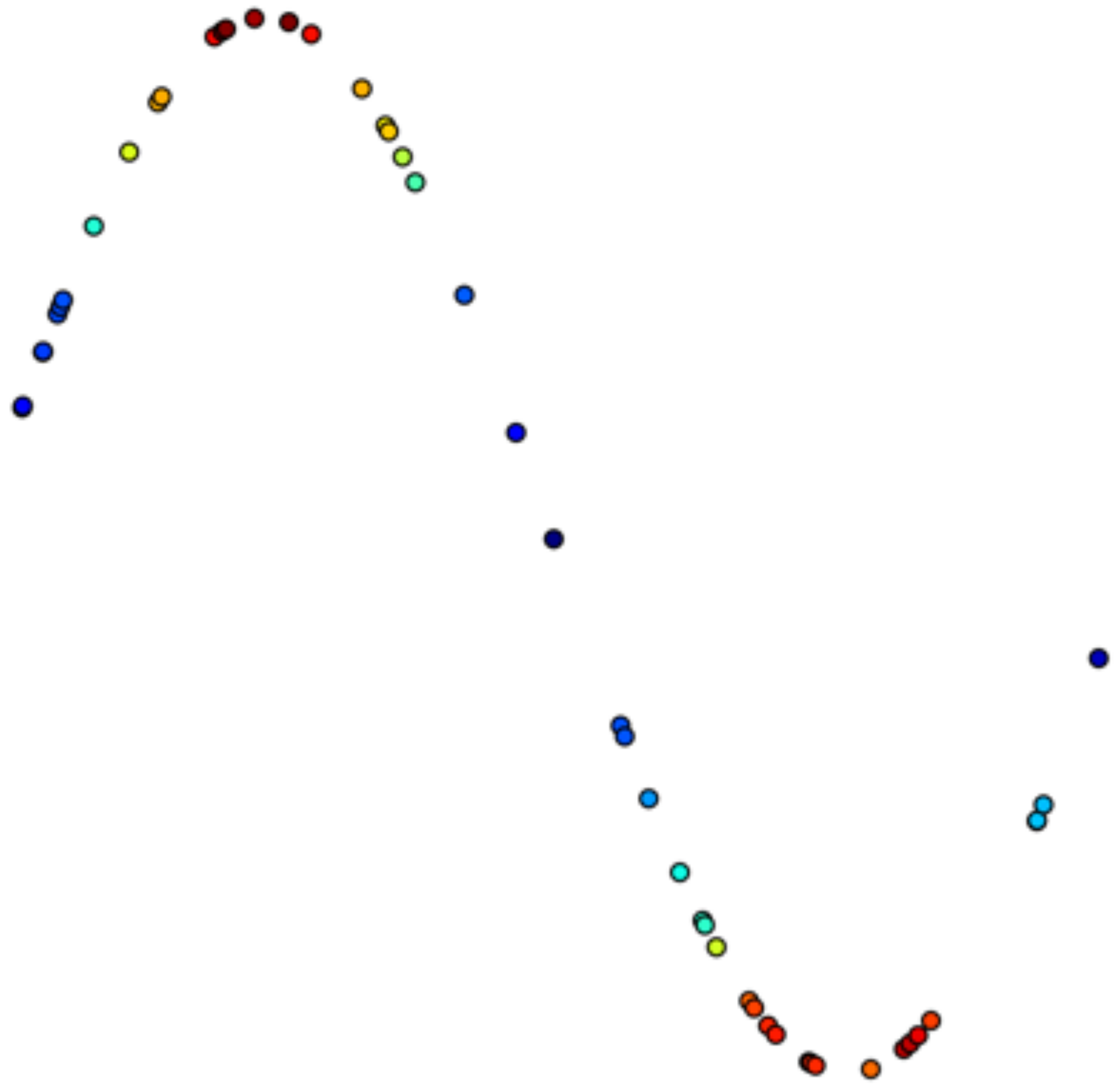
mse ≈ 0.00148

average percent error: $\approx 6\%$



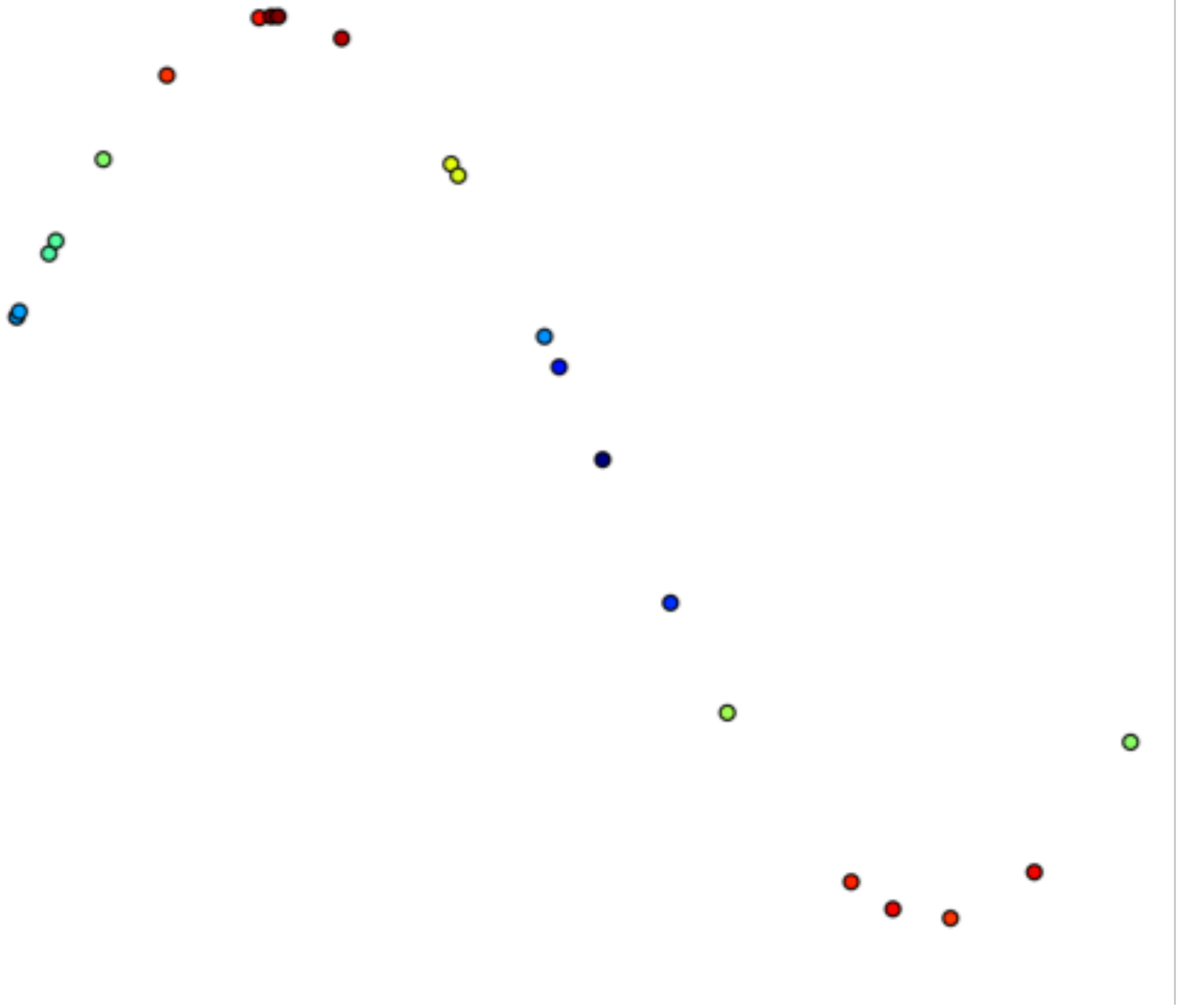
Effects of Sparsity

$n = 50$



$\approx$ 16% error

$n = 25$



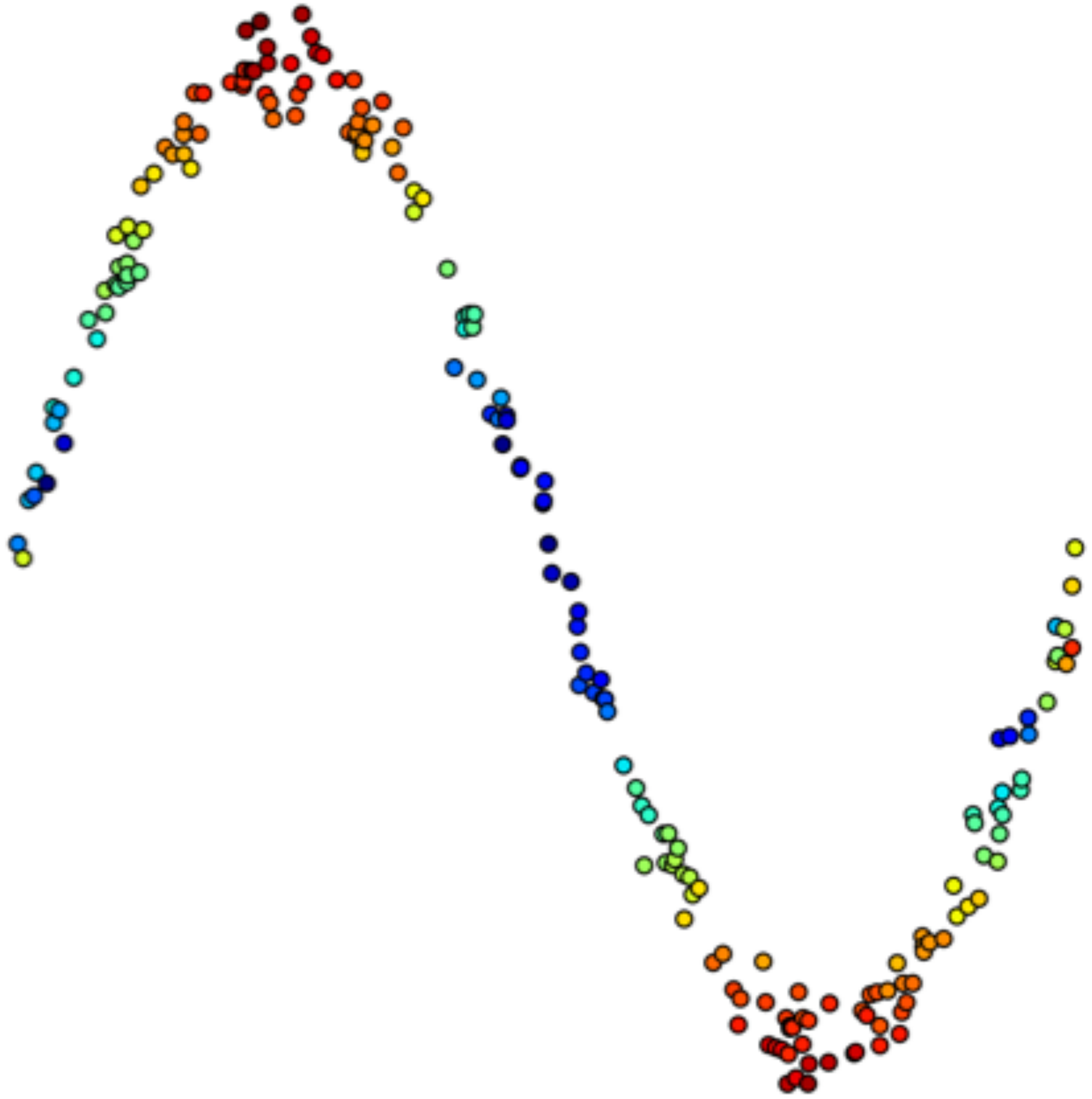
$\approx$ 27% error

Effects of Noise

$y_i = \sin x_i + \varepsilon_i$

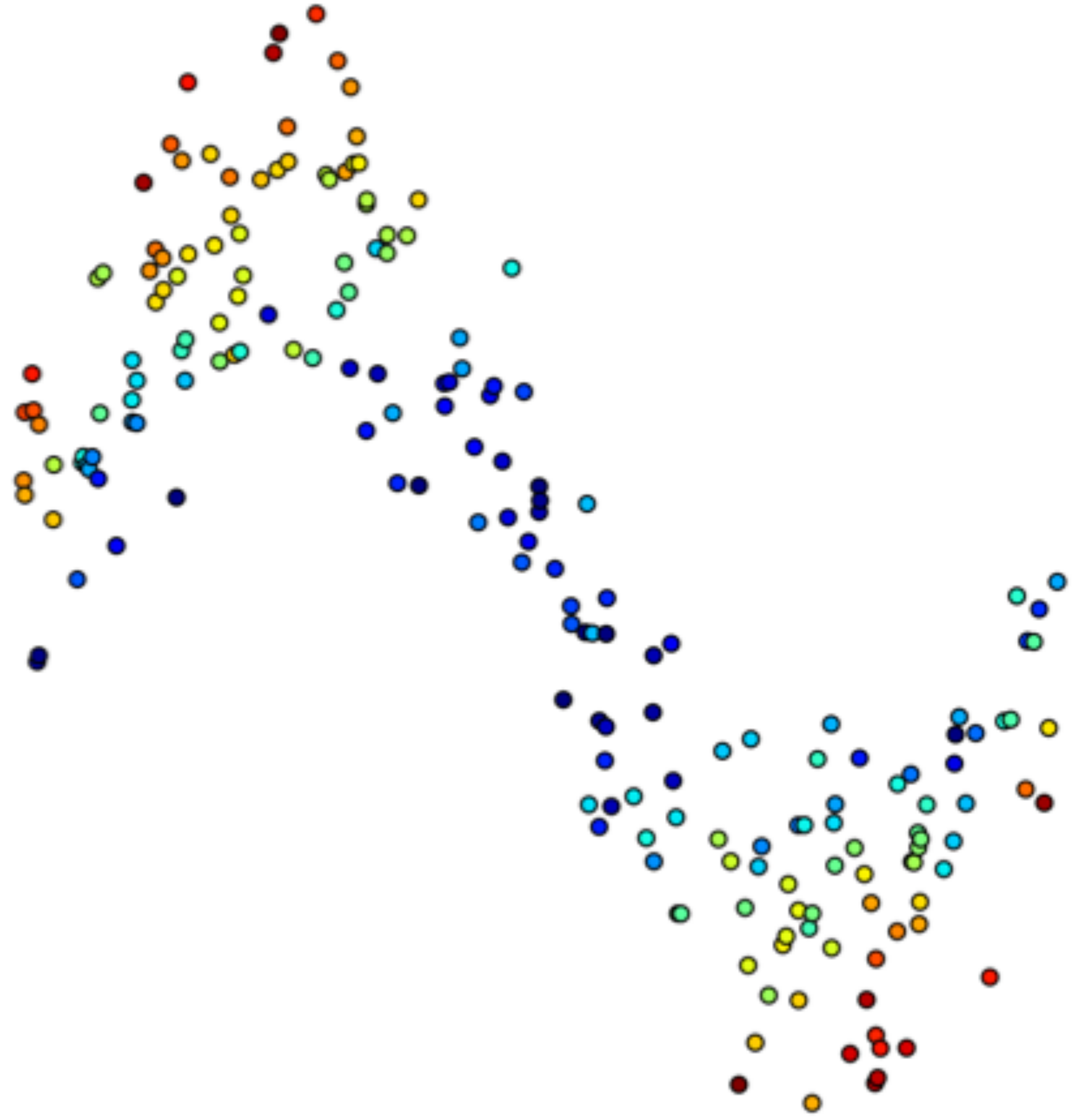
$n = 200$

$\varepsilon_i \sim \mathcal{N}(0, 0.05)$



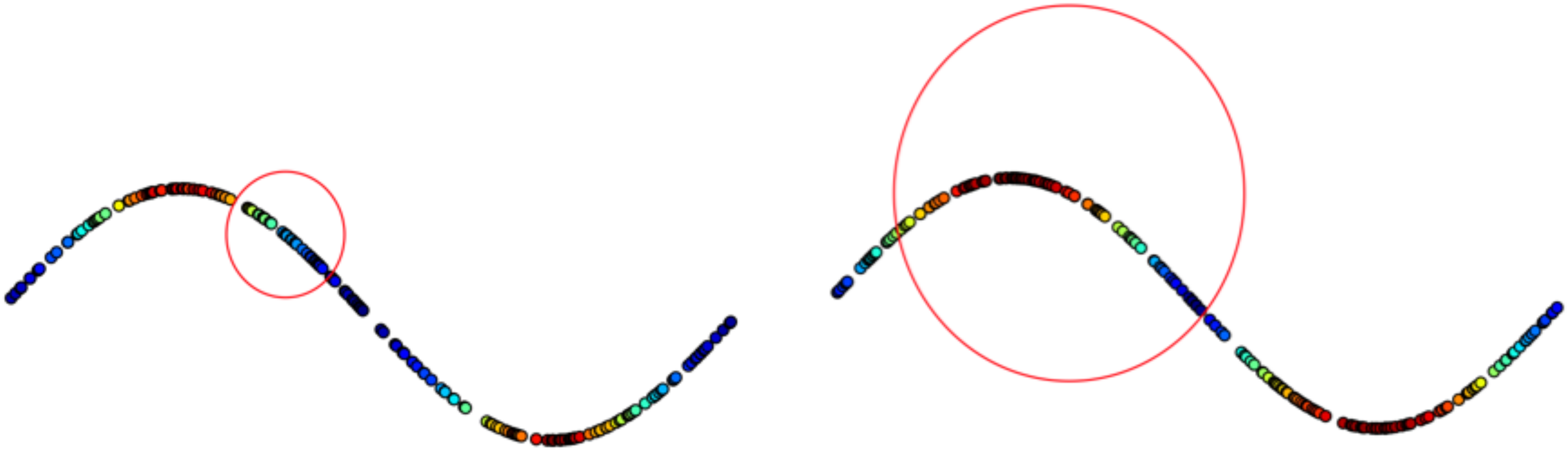
$\approx 52\%$ error

$\varepsilon_i \sim \mathcal{N}(0, 0.25)$



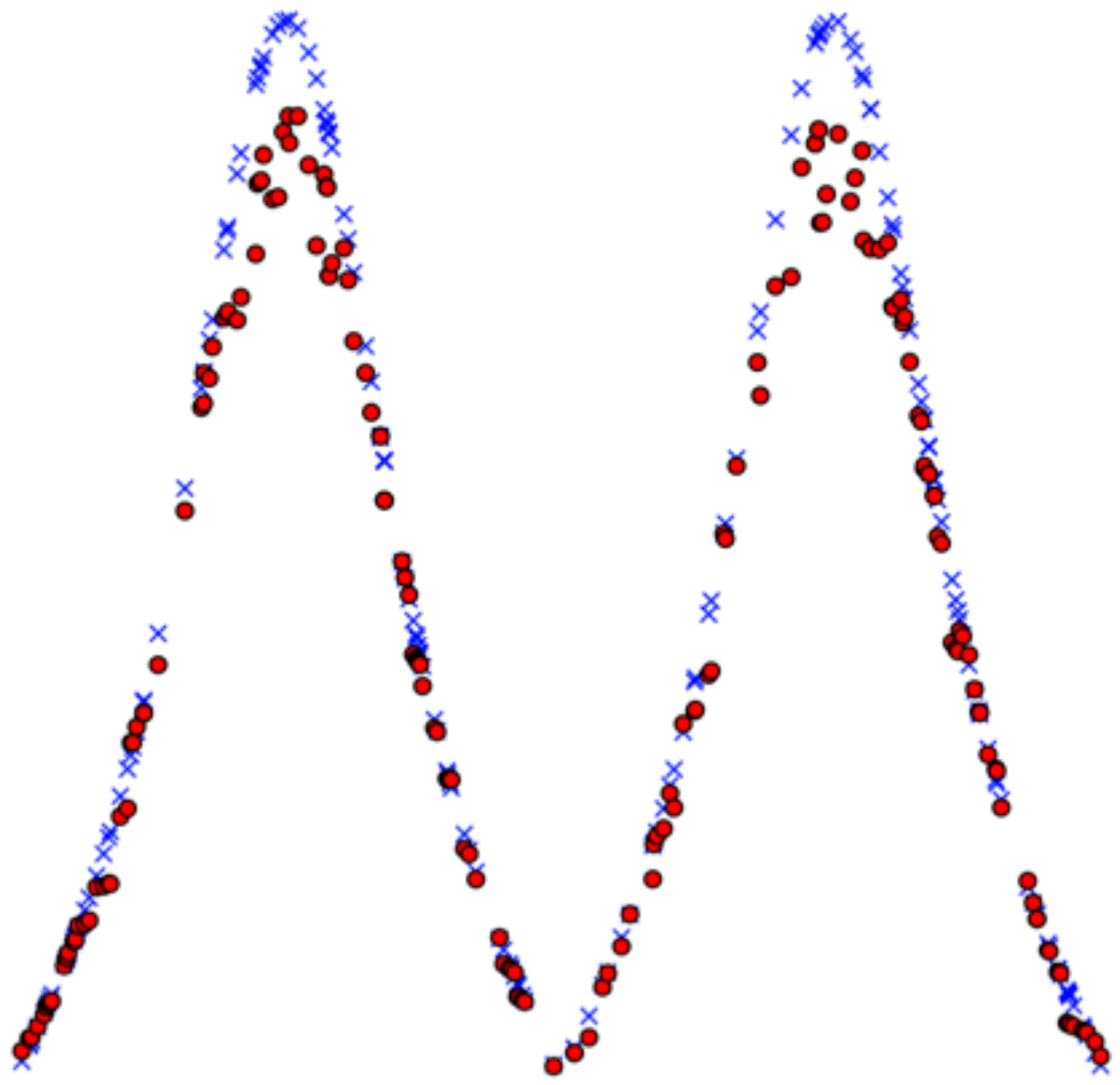
$\approx 112\%$ error

Effects of Smoothing

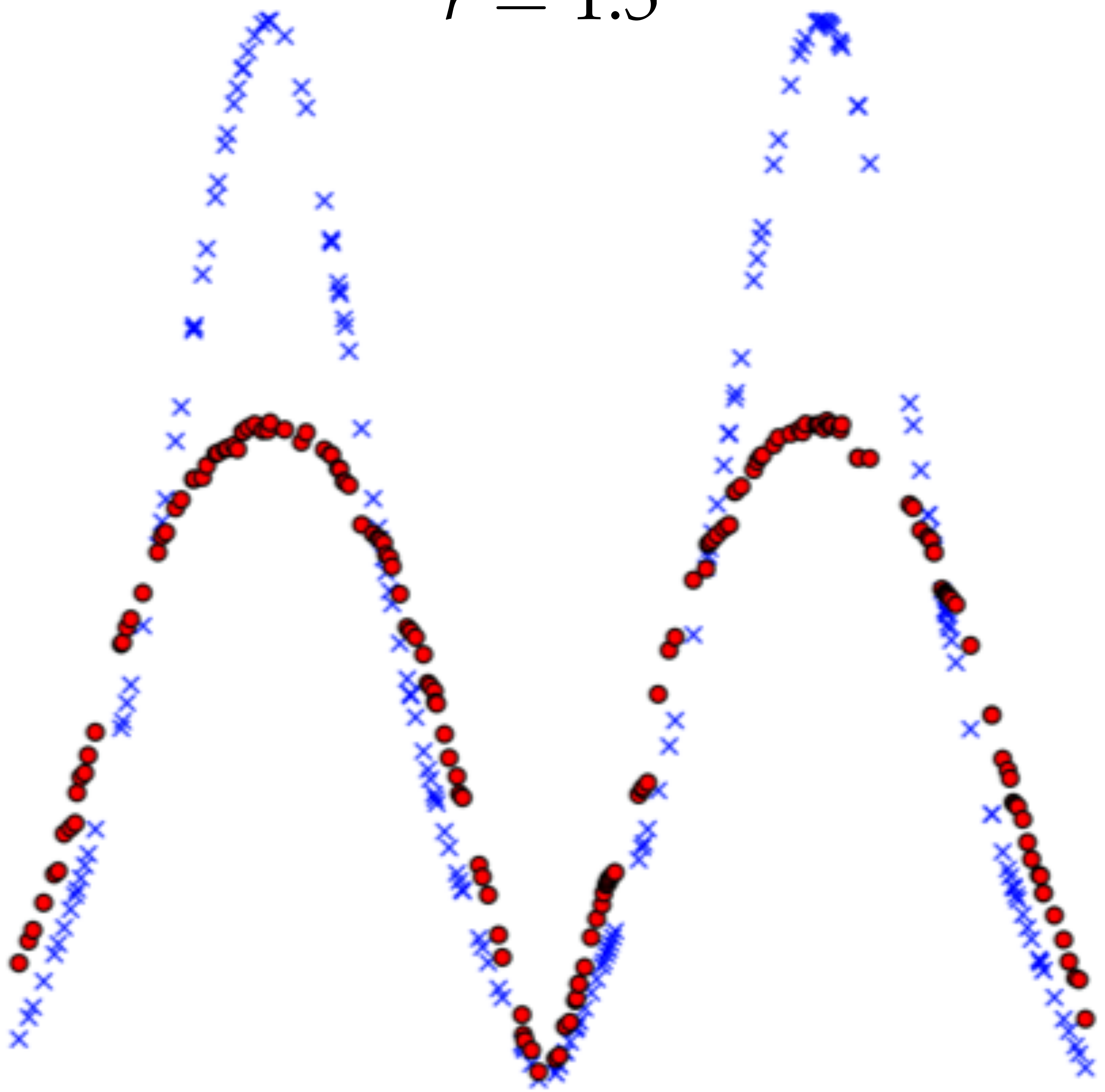


Effects of Smoothing

$r = 0.5$



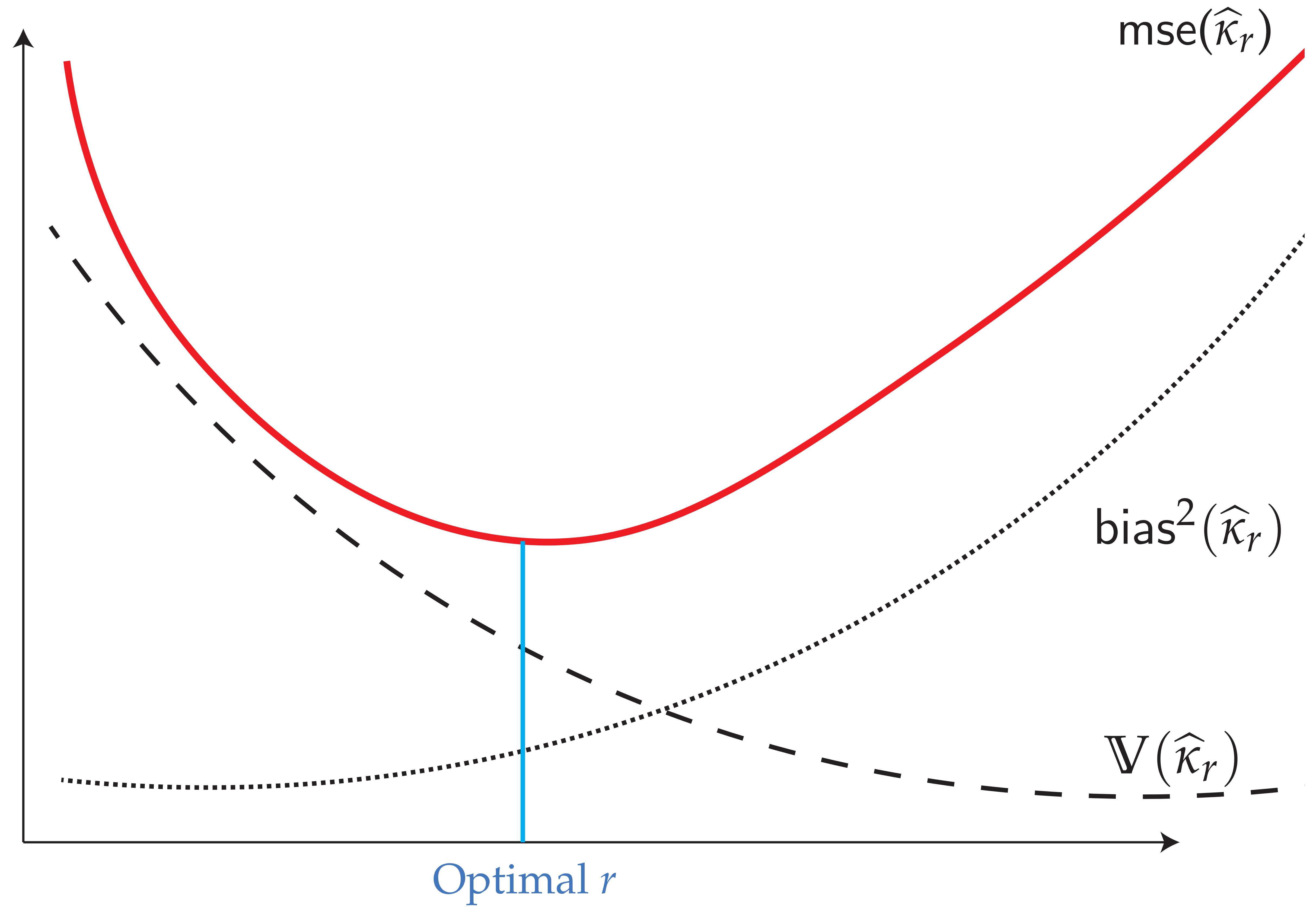
$r = 1.5$



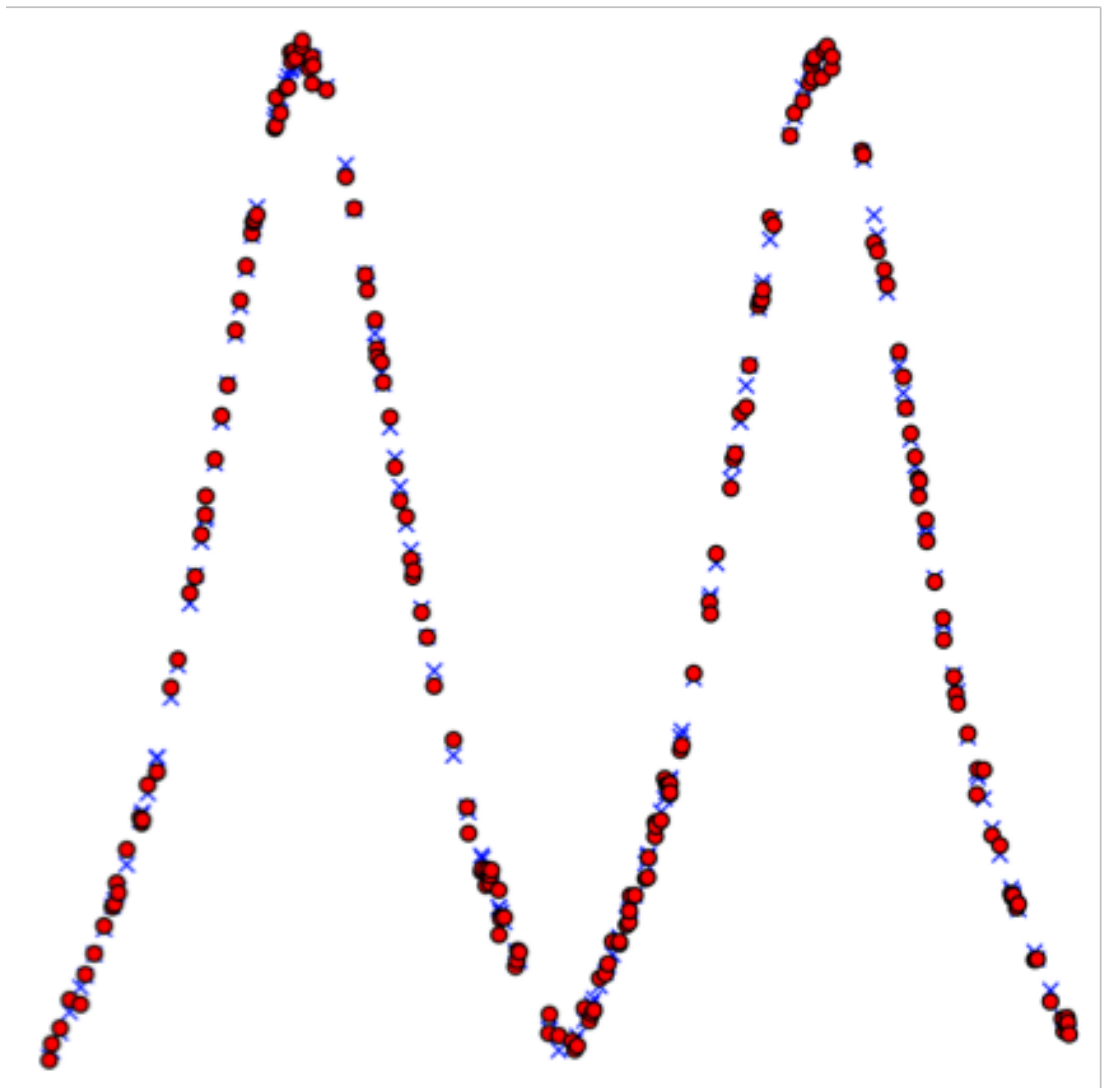
Choosing the Right Scale Using Nonparametric Bootstrap

- Compute an initial estimate $\widehat{\kappa}_r$
- Draw n samples with replacement from the empirical distribution
- Compute $\widehat{\kappa}_r^*$ with this new sample
- Repeat B times
- Compute the estimated MSE of $\widehat{\kappa}_r$
- Choose the scale r that minimizes the MSE of $\widehat{\kappa}_r$

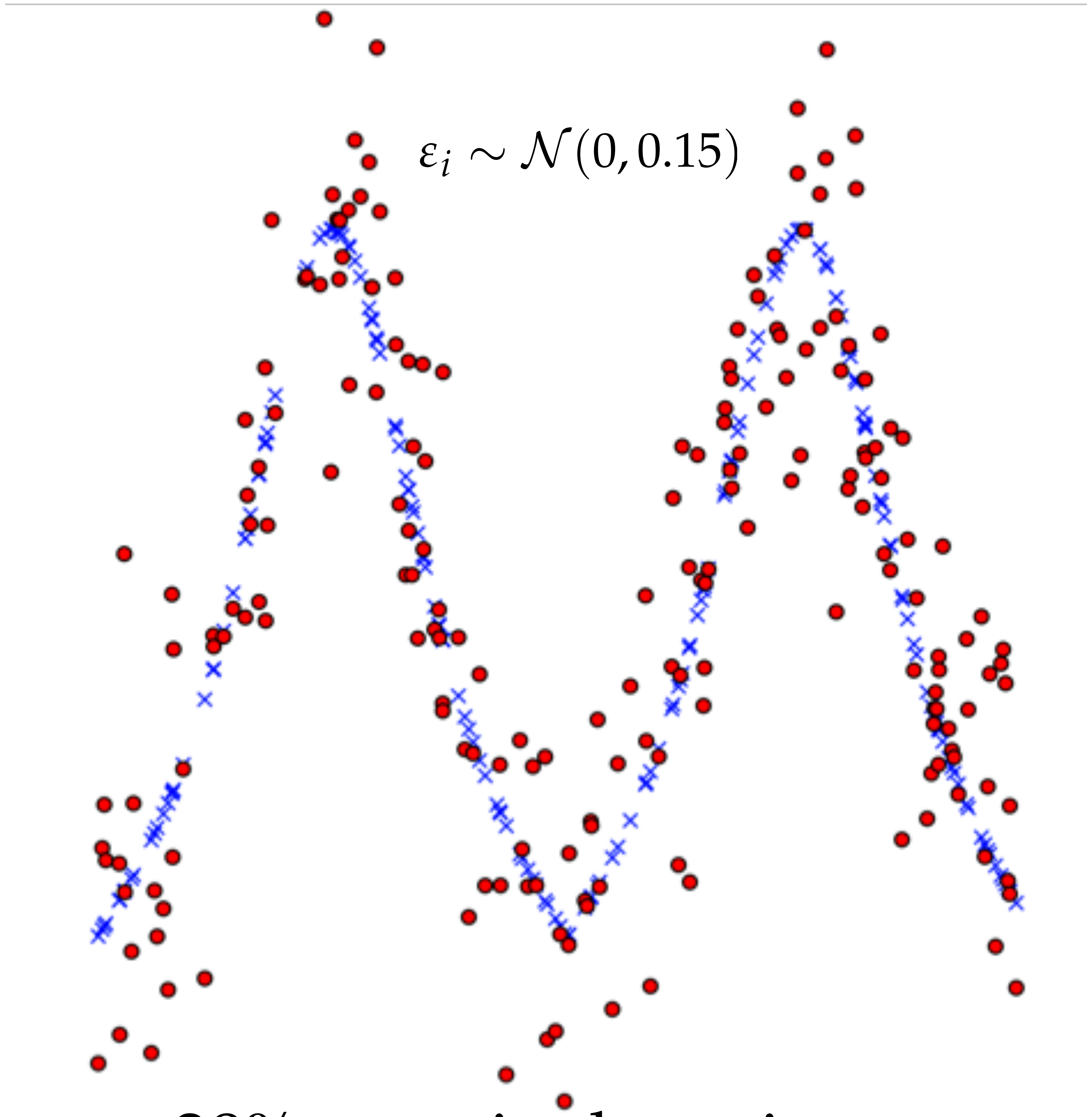
$$\mathsf{mse} = \mathbb{E}(\widehat{\kappa}_r - \kappa)^2$$



Choosing the Right Scale Using Nonparametric Bootstrap

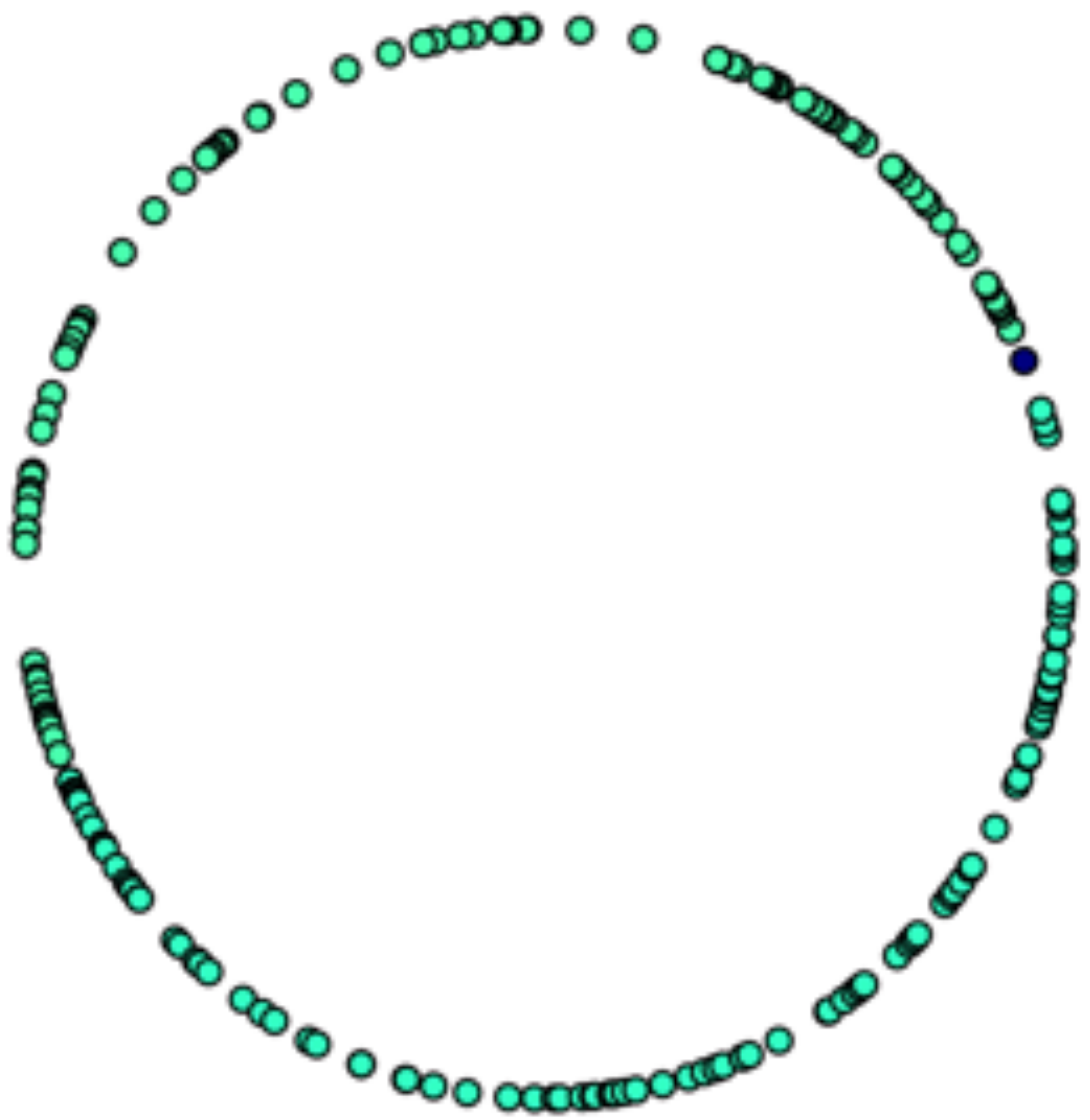


$\approx 0.01\%$ error in the noiseless case

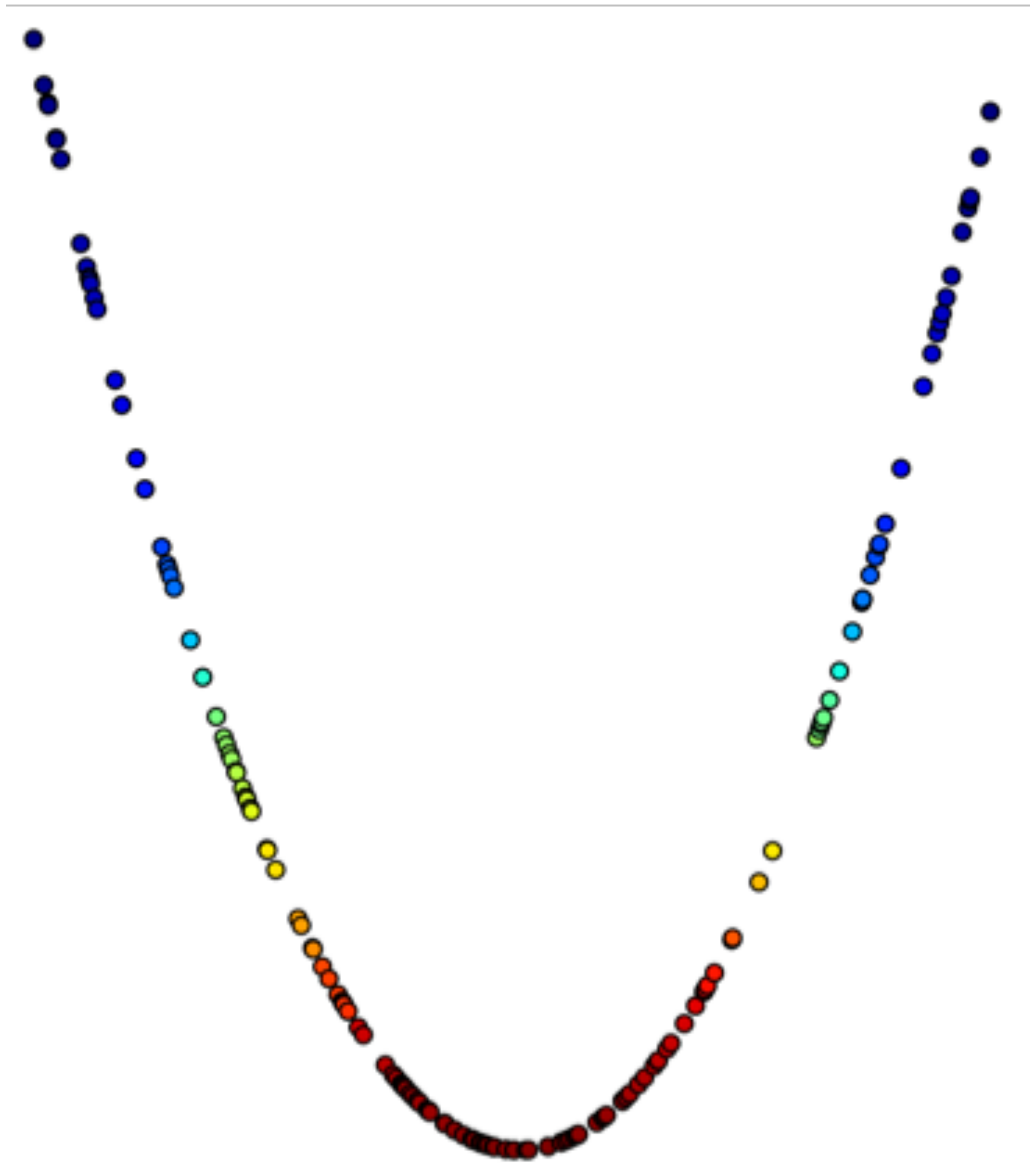


$\approx 28\%$ error in the noisy case

Some Other Functions

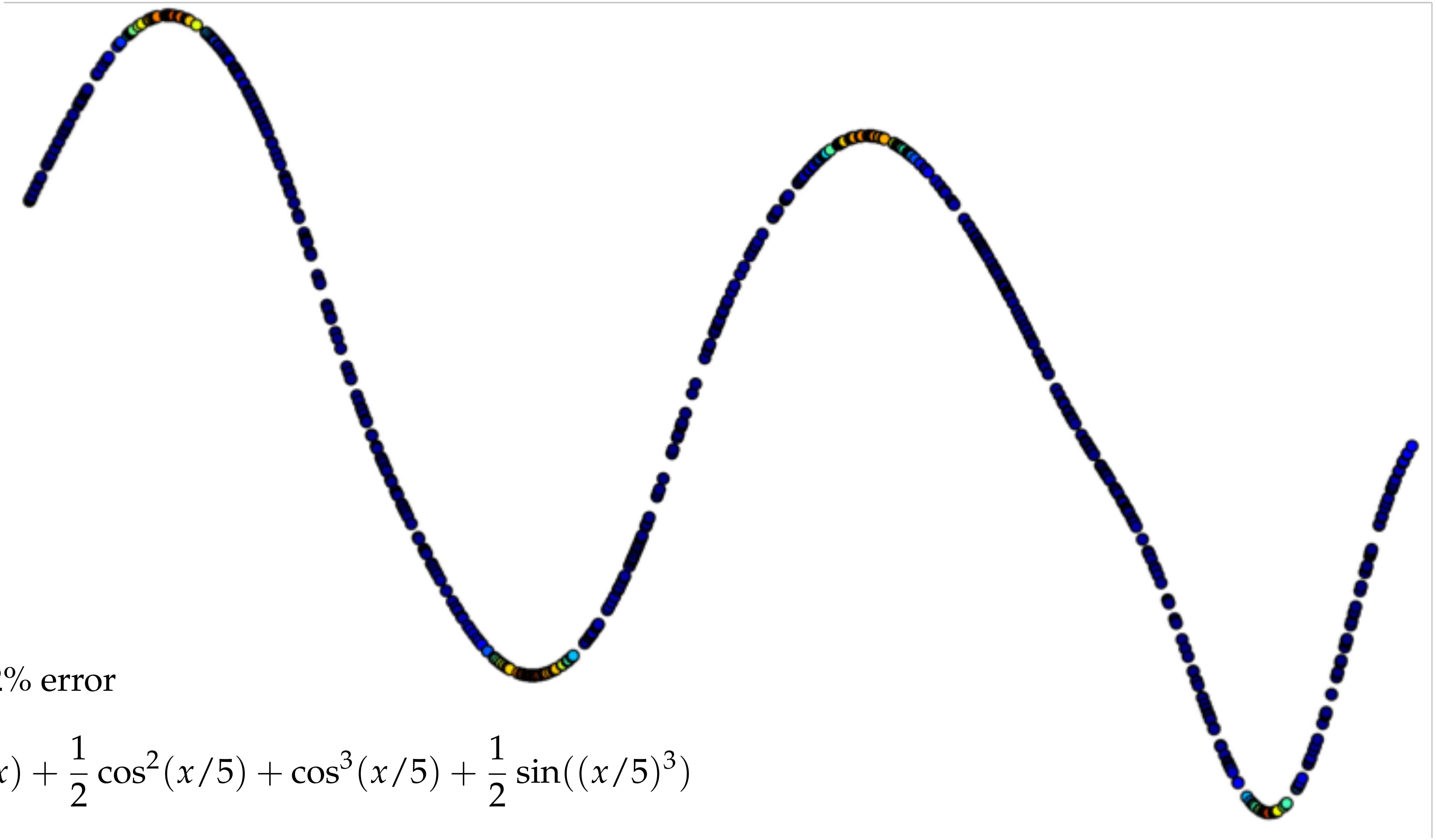


$\mathbb{S}^1$



$f(x) = x^2$

Some Other Functions



$\approx 2\%$ error

$$y = 5\sin(x) + \frac{1}{2}\cos^2(x/5) + \cos^3(x/5) + \frac{1}{2}\sin((x/5)^3)$$

Problems

- Didn't have time for full theoretical analysis
- I'm having trouble with computing numerical integrals bounded by a set of data points in arbitrary dimensions
- Didn't code other techniques to compare them
- Edge Cases