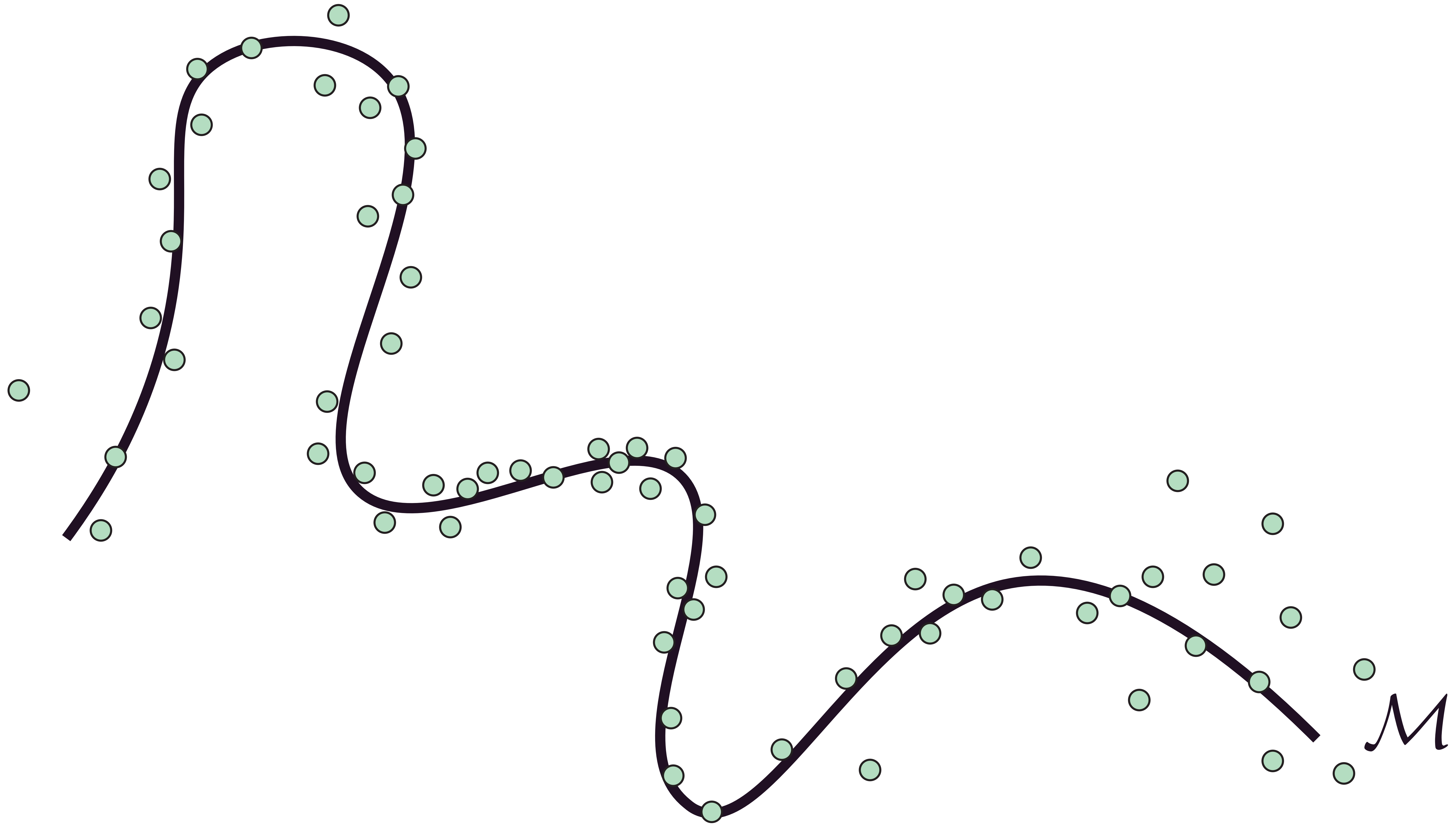

Curvature Estimation of High Dimensional Submanifolds Based on Noisy Observations

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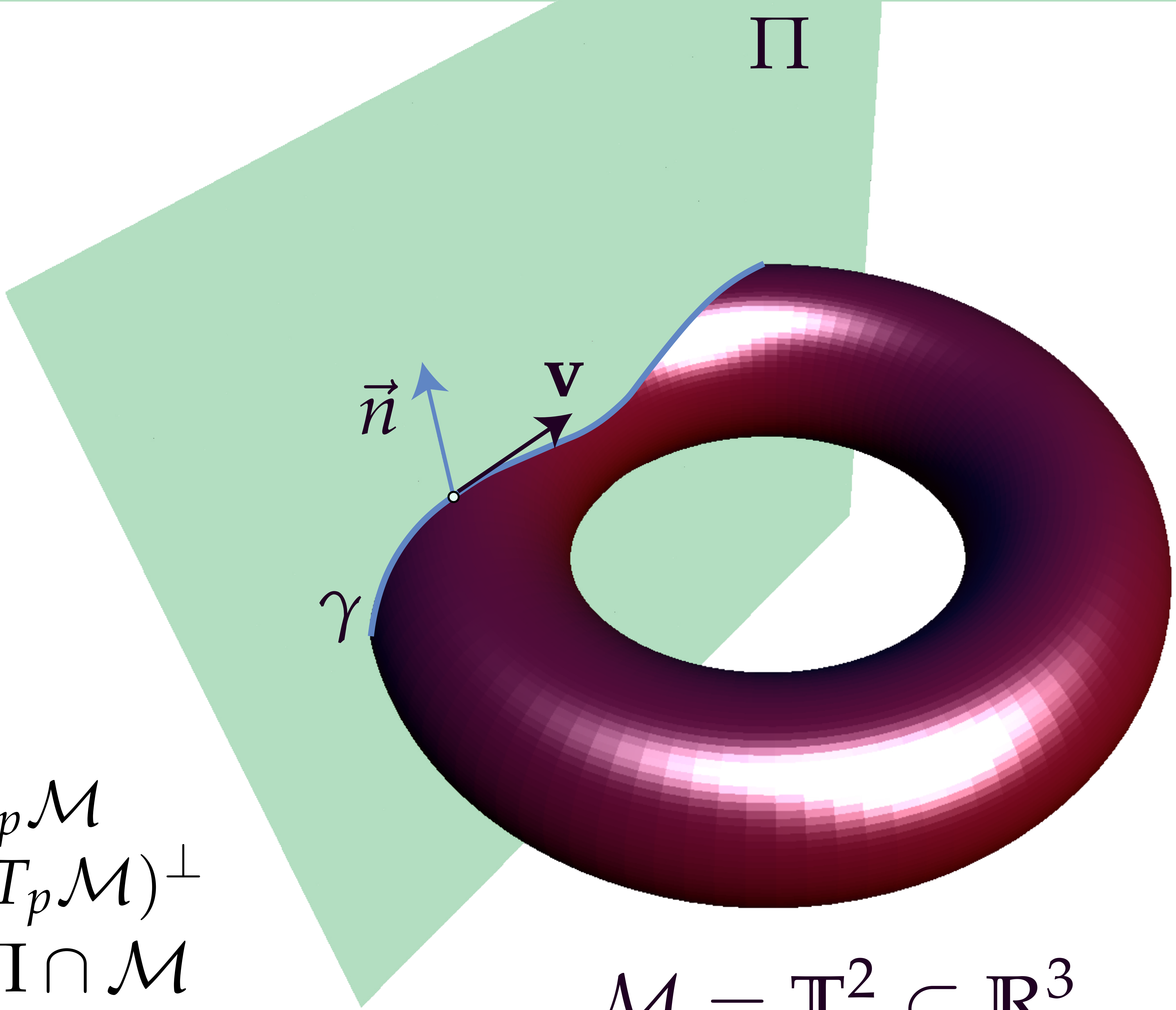
Principal Curvatures of Surfaces

$\kappa(\mathbf{v})$ is the curvature of γ

$$\kappa_1 = \max_{\mathbf{v} \in T_p\mathcal{M}} \kappa(\mathbf{v})$$

$$\kappa_2 = \min_{\mathbf{v} \in T_p\mathcal{M}} \kappa(\mathbf{v})$$



$\mathbf{v} \in T_p\mathcal{M}$

$\vec{n} \in (T_p\mathcal{M})^\perp$

$\gamma = \Pi \cap \mathcal{M}$

$\mathcal{M} = \mathbb{T}^2 \subset \mathbb{R}^3$

Principal Curvatures of Hypersurfaces

The second fundamental form:

$$\mathbb{I}_p : T_p\mathcal{M} \to (T_p\mathcal{M})^\perp$$

$$\mathbb{I}_p\left(v^i\partial_i\right) = \sum_{i,j=1}^{m}\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}(0)\right)^\perp v^i v^j$$

$\mathcal{M} \subset \mathbb{R}^n$

$\dim \mathcal{M} = m$

$(U \subset \mathbb{R}^m, \varphi)$ is a local chart around p

$\varphi(0) = p$

∂_i - basis for $T_p\mathcal{M}$

In codimension 1:

$$\mathbb{I}(\mathbf{v}) = (\mathbb{I}(\mathbf{v}) \cdot \vec{n}) \cdot \vec{n}$$

$$\mathbb{I}(v^i\partial_i) \cdot \vec{n} = \sum_{i,j=1}^{m}\left(\Phi_{ij}v^i v^j\right)$$

The principal curvatures are the eigenvalues of Φ
The principal directions are the eigenvectors of Φ

$\Phi \in \mathbb{R}^{m\times m}$

Local Representation of Submanifolds

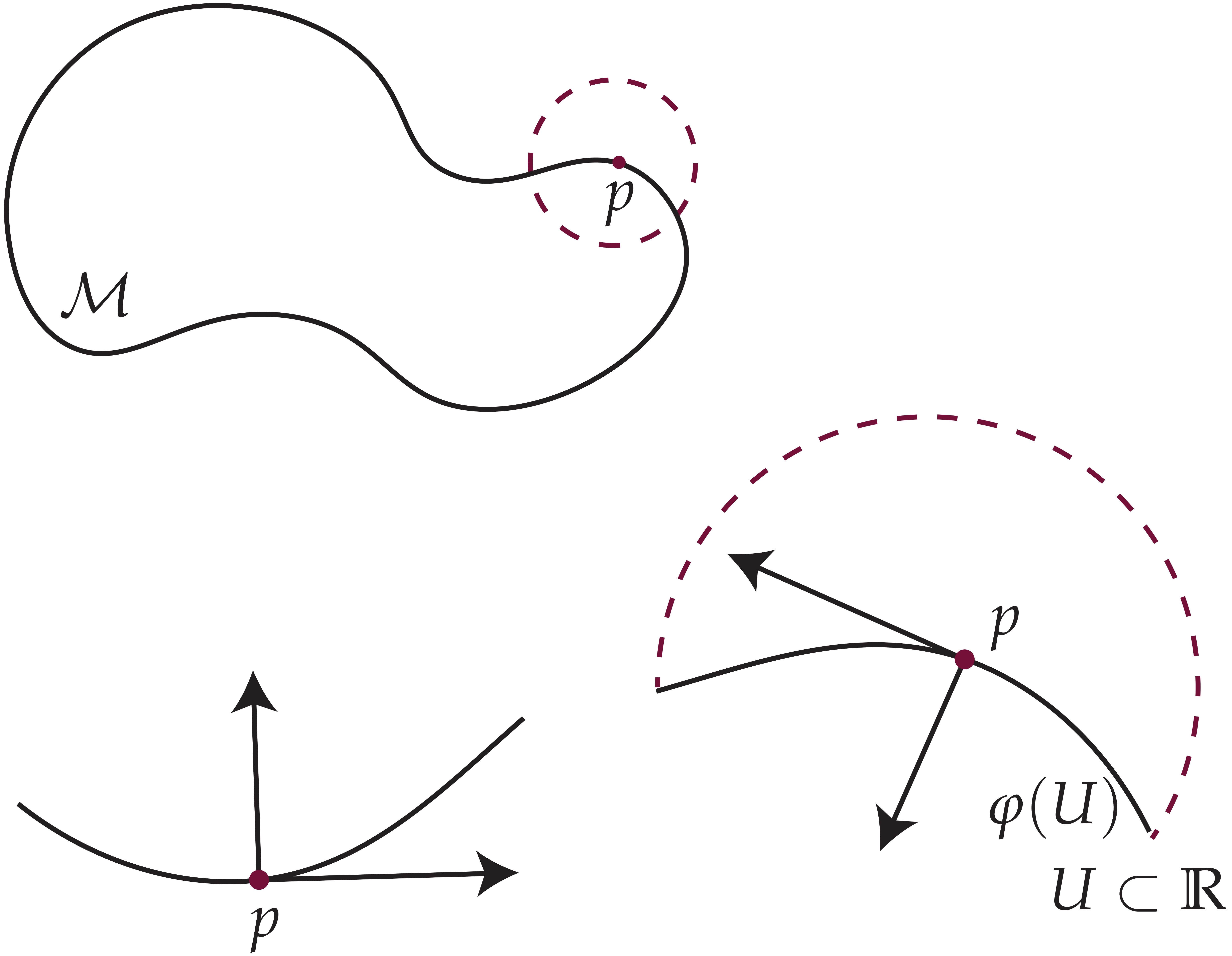
Note that the entries of $\mathrm{I\!I}_p$ are the second partial derivatives $\left(\Phi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0) \cdot \vec{n}\right)$, so we have that they coincide with the Hessian. So we can express the normal change of a hypersurface locally using a Taylor series:

$$
\begin{aligned}
f(x_1, \ldots, x_m) &= \frac{1}{2} \mathrm{I\!I}_p(x_1, \ldots, x_m) + \mathcal{O}(\|x\|^3) \\
&= \frac{1}{2} \sum_{i=1}^{m} \kappa_i x_i^2 + \mathcal{O}(\|x\|^3)
\end{aligned}
$$

More generally, we have that

$$
(\varphi(x+h) - \varphi(x))^{\perp} = \frac{1}{2} \mathrm{I\!I}_p(h) + \mathcal{O}(\|h\|^3)
$$

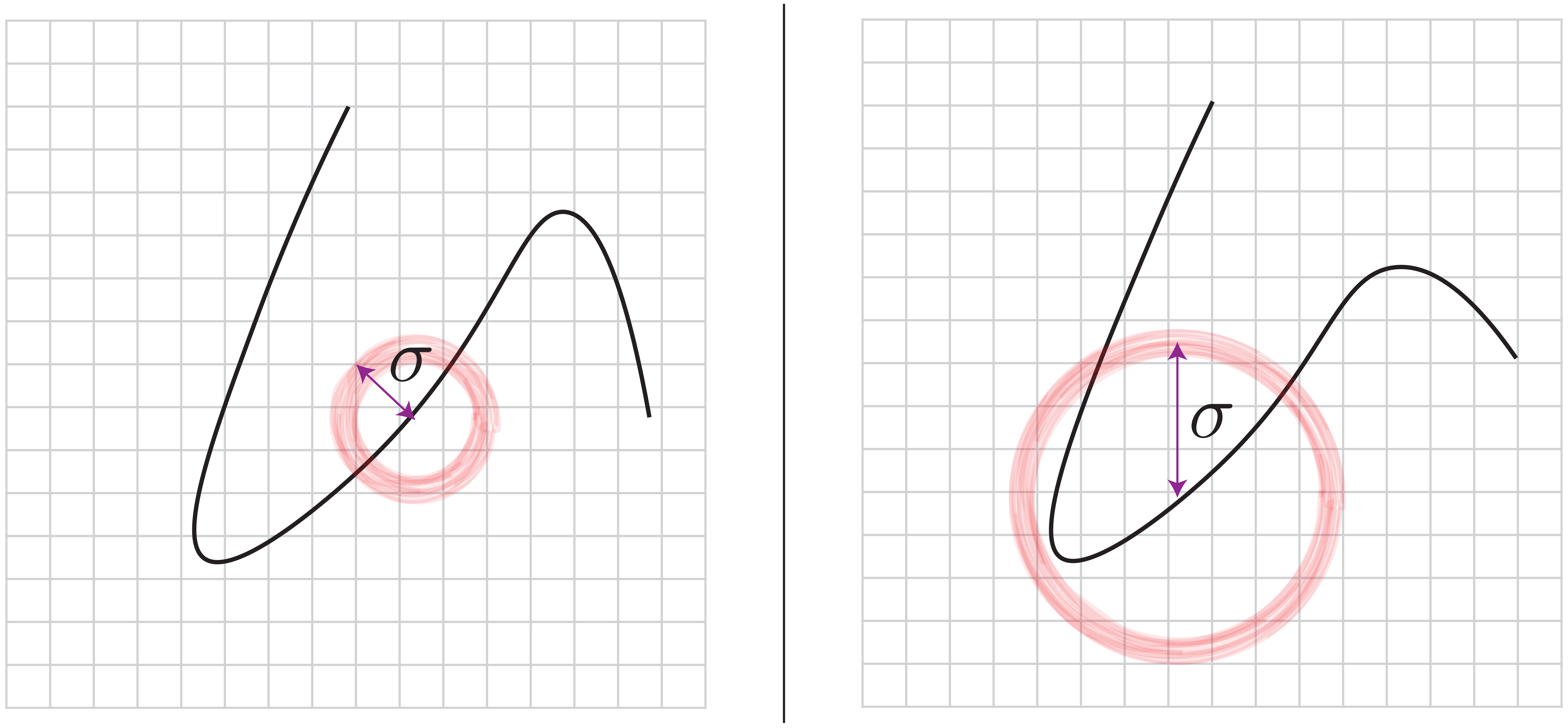
Local Representation of Submanifolds



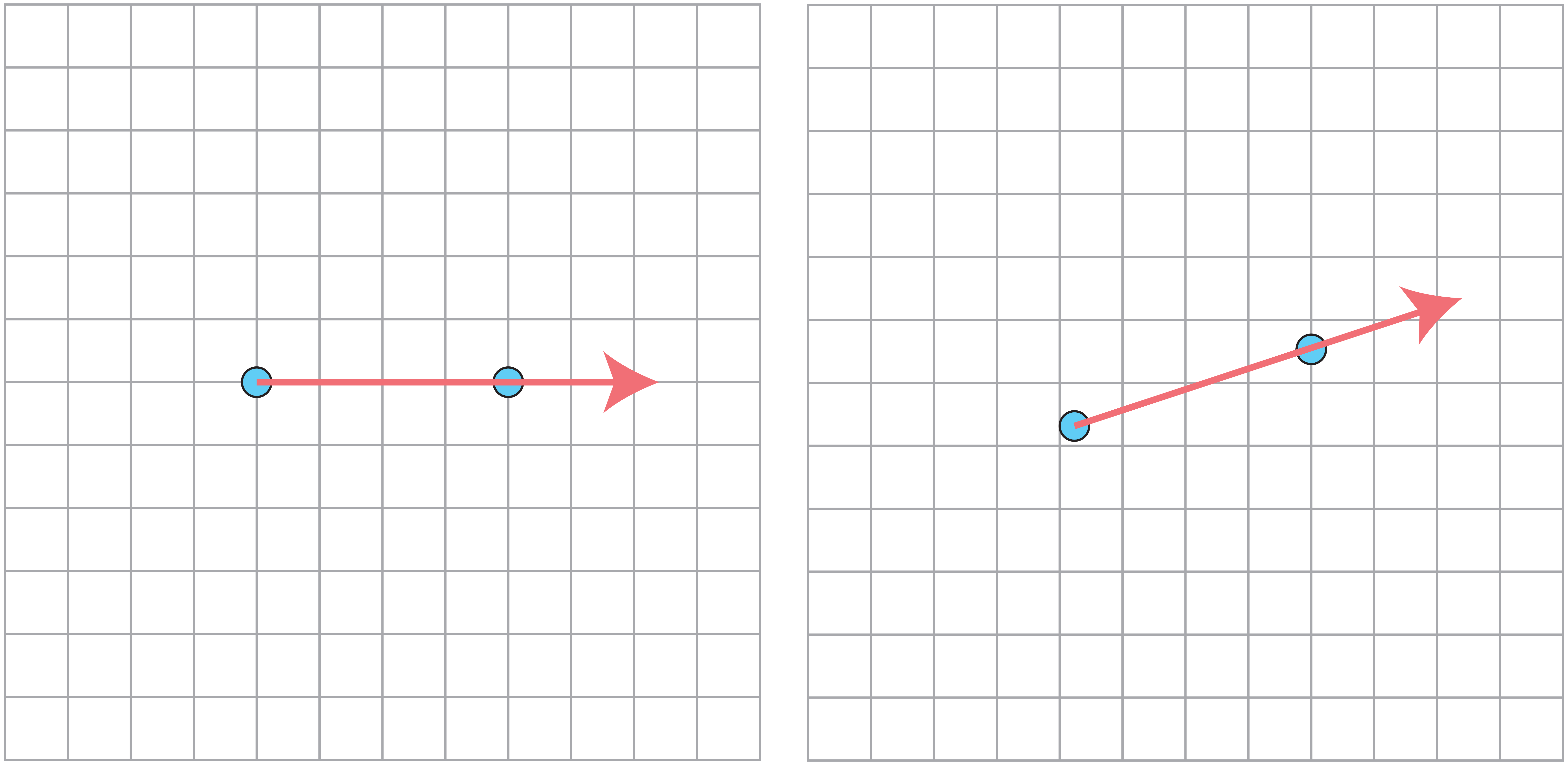
Curvature as a Shape Descriptor

- Local Features:
  - Summarize the manifold by a subset of significant points
  - Construct informed triangulations of neighborhoods
  - A denoising algorithm
- Global Features:
  - Principal Curves
  - Data Segmentation
  - Registration and Comparison
- Manifold Learning and Estimation

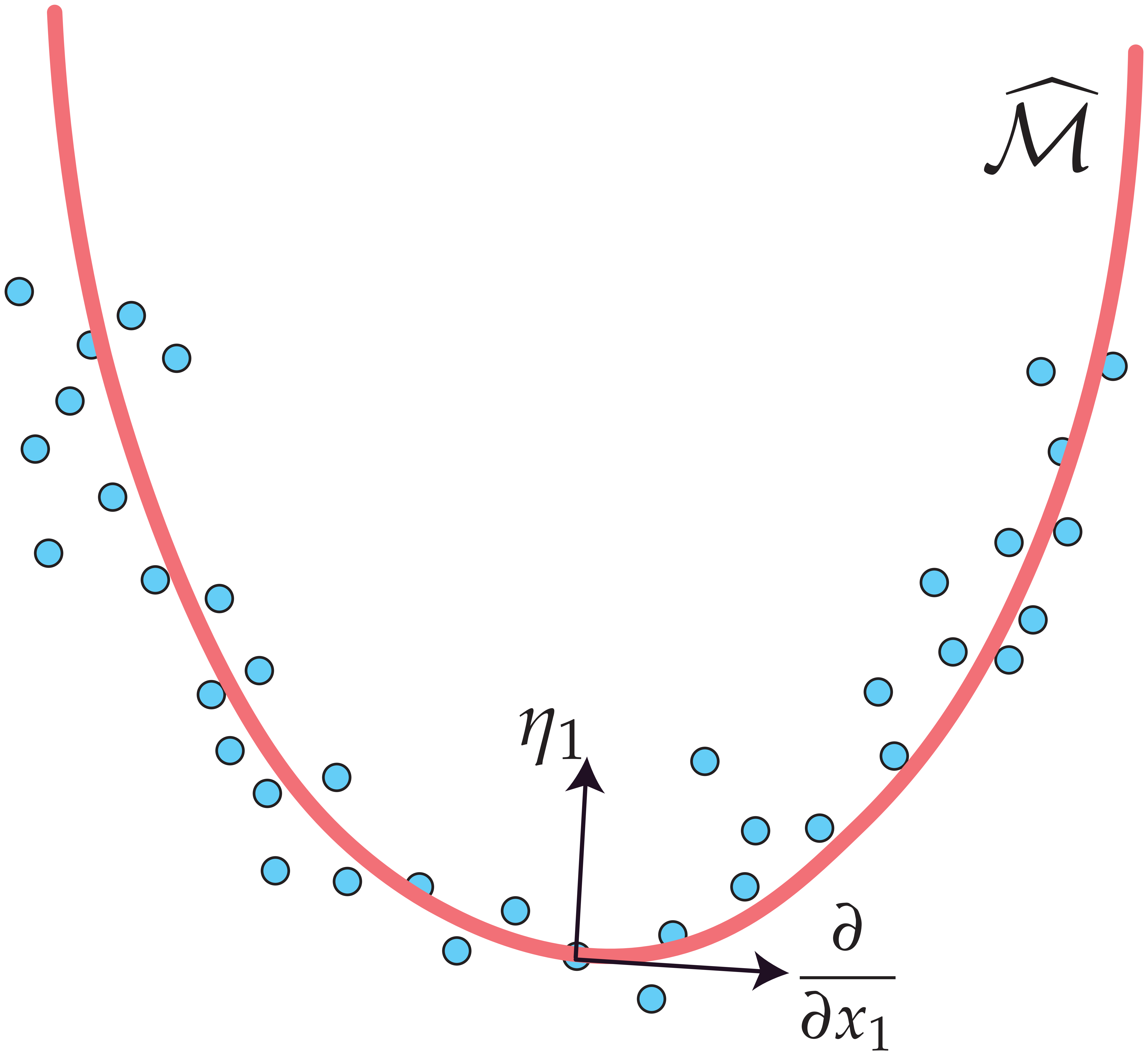
The Effects of Noise on Geometric Inference



Differentiation is Not Robust to Noise



Curvature Estimation - Local Chart Estimation

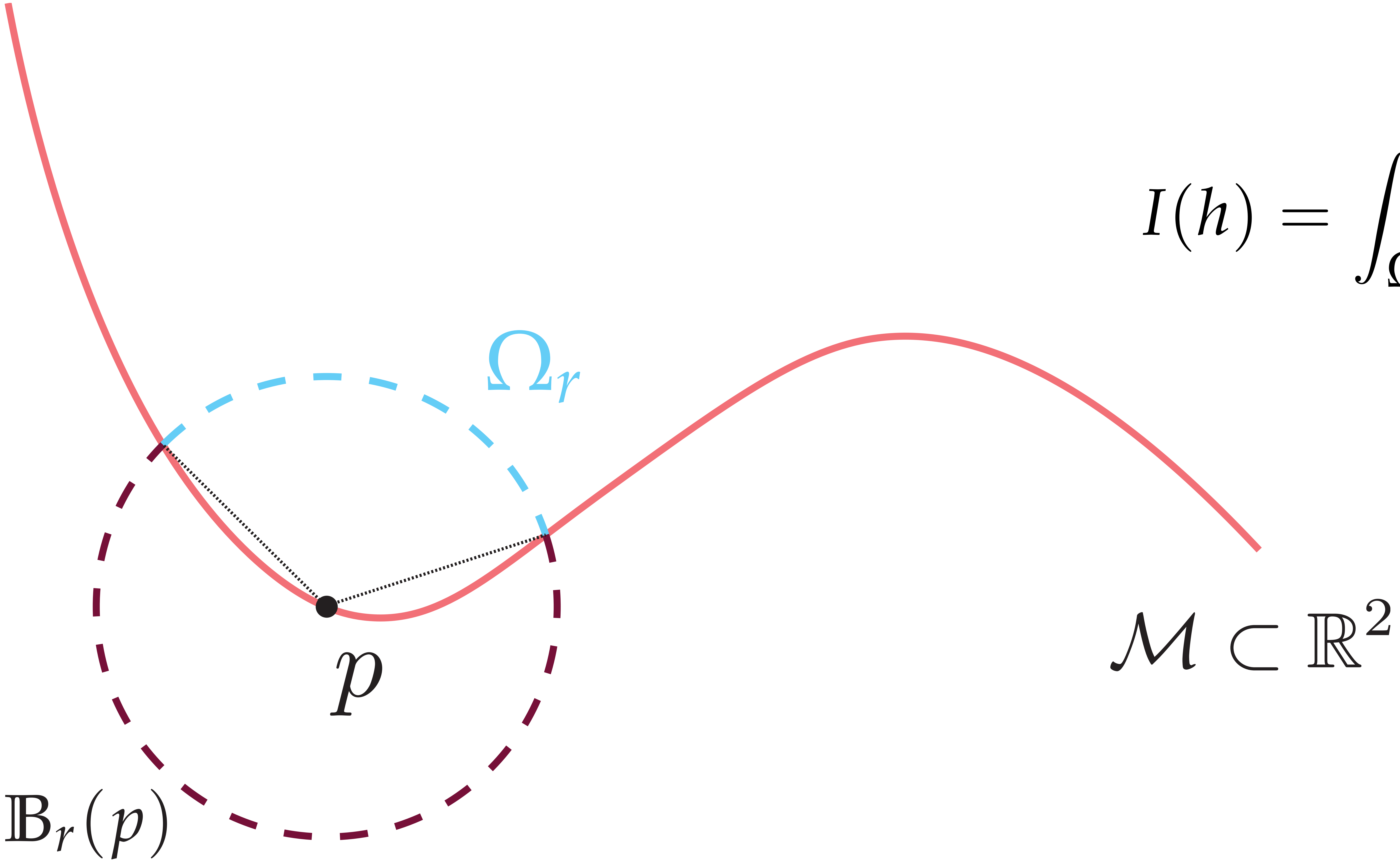


Curvature Estimation - Tensor Voting

- Directly estimate the second fundamental form
- First the neighboring points vote on the normal vector
- Next we compute the estimated curvature in the directions of the neighboring points
- Eigenvalues and Eigenvectors of $\mathbb{I}$ are the principal curvature and directions

Curvature Estimation - Integral Invariants

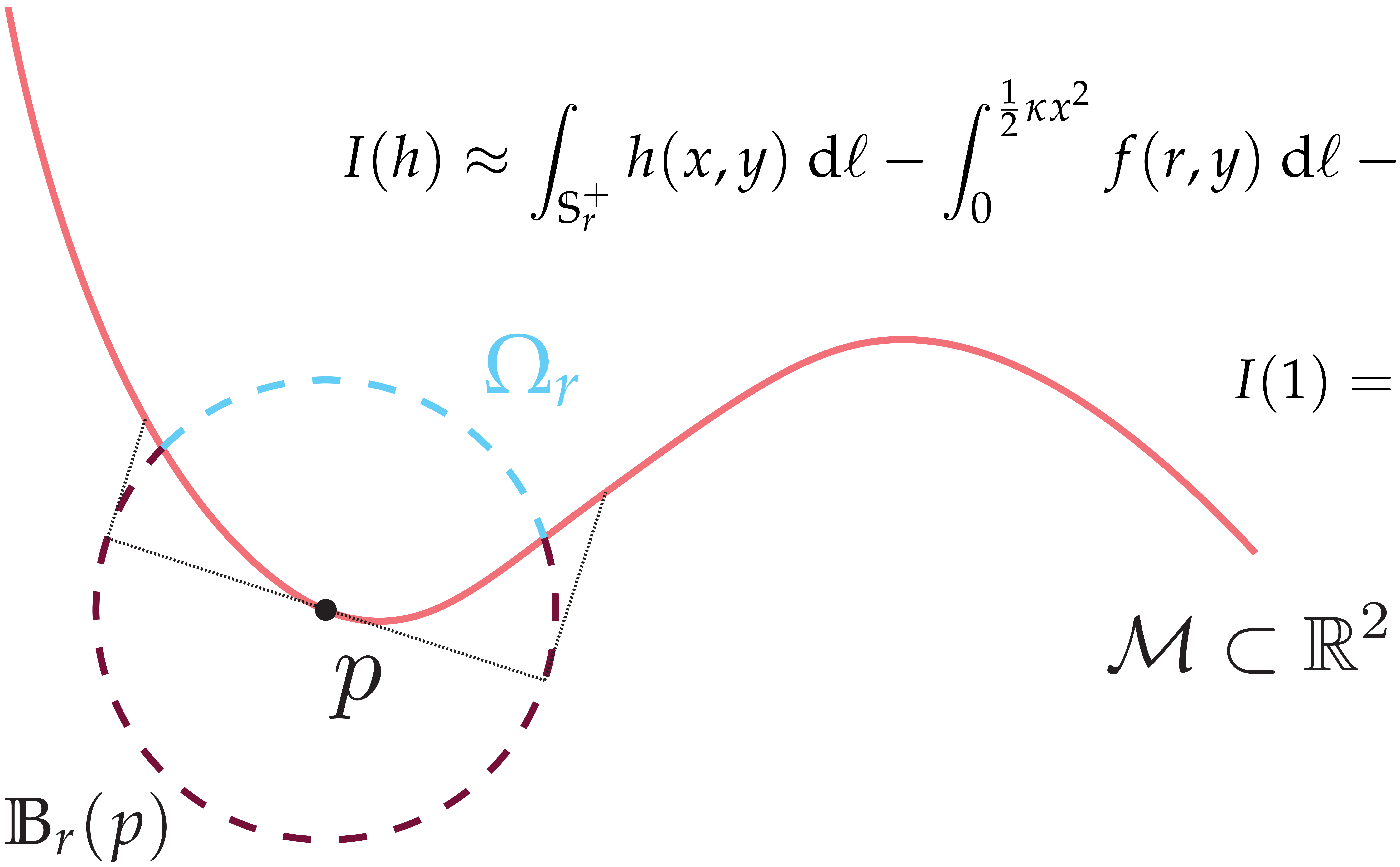
$$I(h) = \int_{\Omega_r} h(\mathbf{x})\, \mathrm{d}\mathbf{x}$$



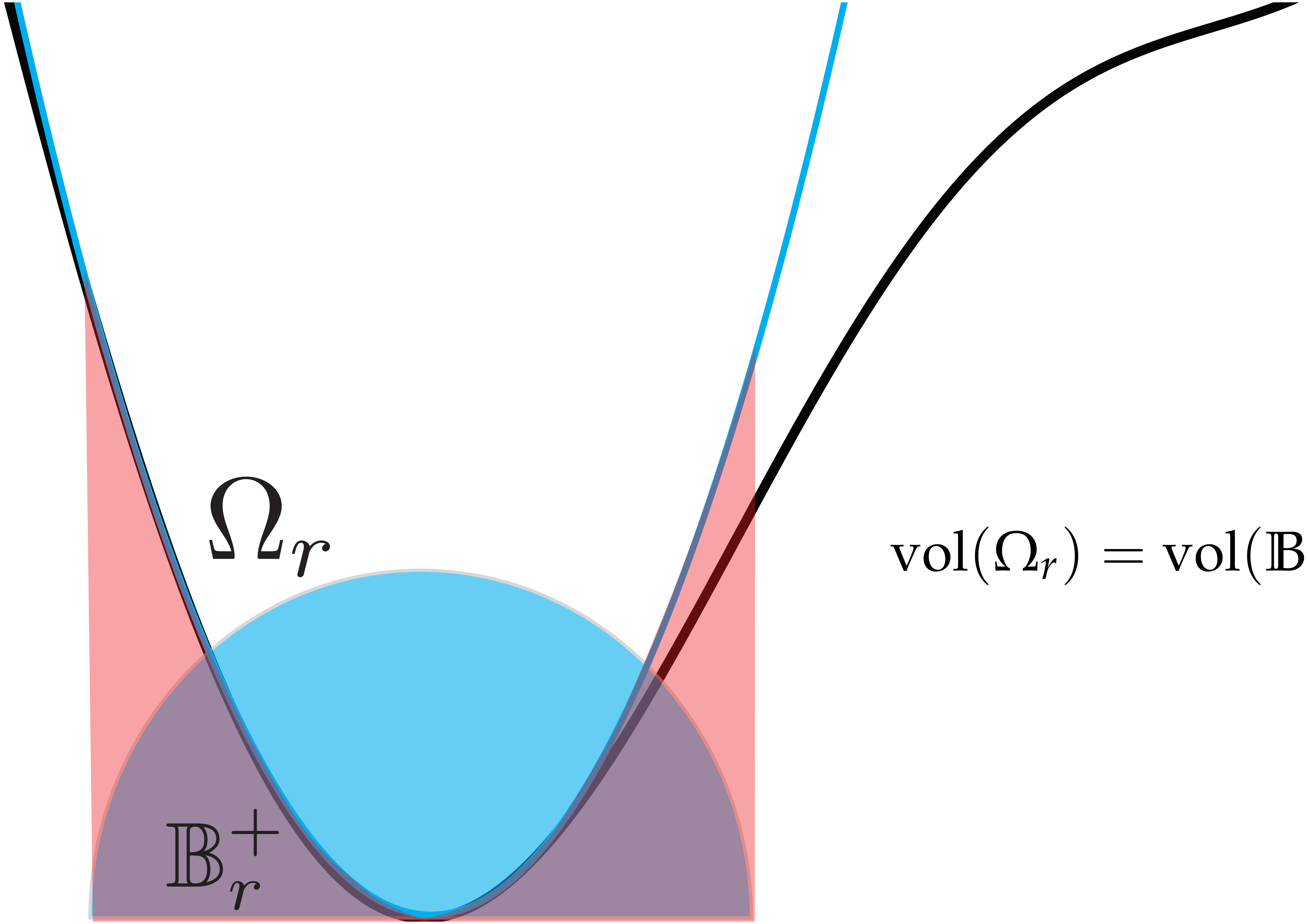
Curvature Estimation - Integral Invariants

$$I(h) \approx \int_{\mathbb{S}_r^+} h(x,y)\,\mathrm{d}\ell - \int_0^{\frac{1}{2}\kappa x^2} f(r,y)\,\mathrm{d}\ell - \int_0^{\frac{1}{2}\kappa x^2} f(-r,y)\,\mathrm{d}\ell$$

$$I(1) = \int_{\Omega_r} \mathrm{d}\ell \approx \pi r - \kappa r^2$$



Curvature Estimation - Integral Invariants



$$\mathrm{vol}(\Omega_r) = \mathrm{vol}(\mathbb{B}_r^+) - \frac{\mathcal{H}\pi r^4}{4} + \mathcal{O}(r^5)$$

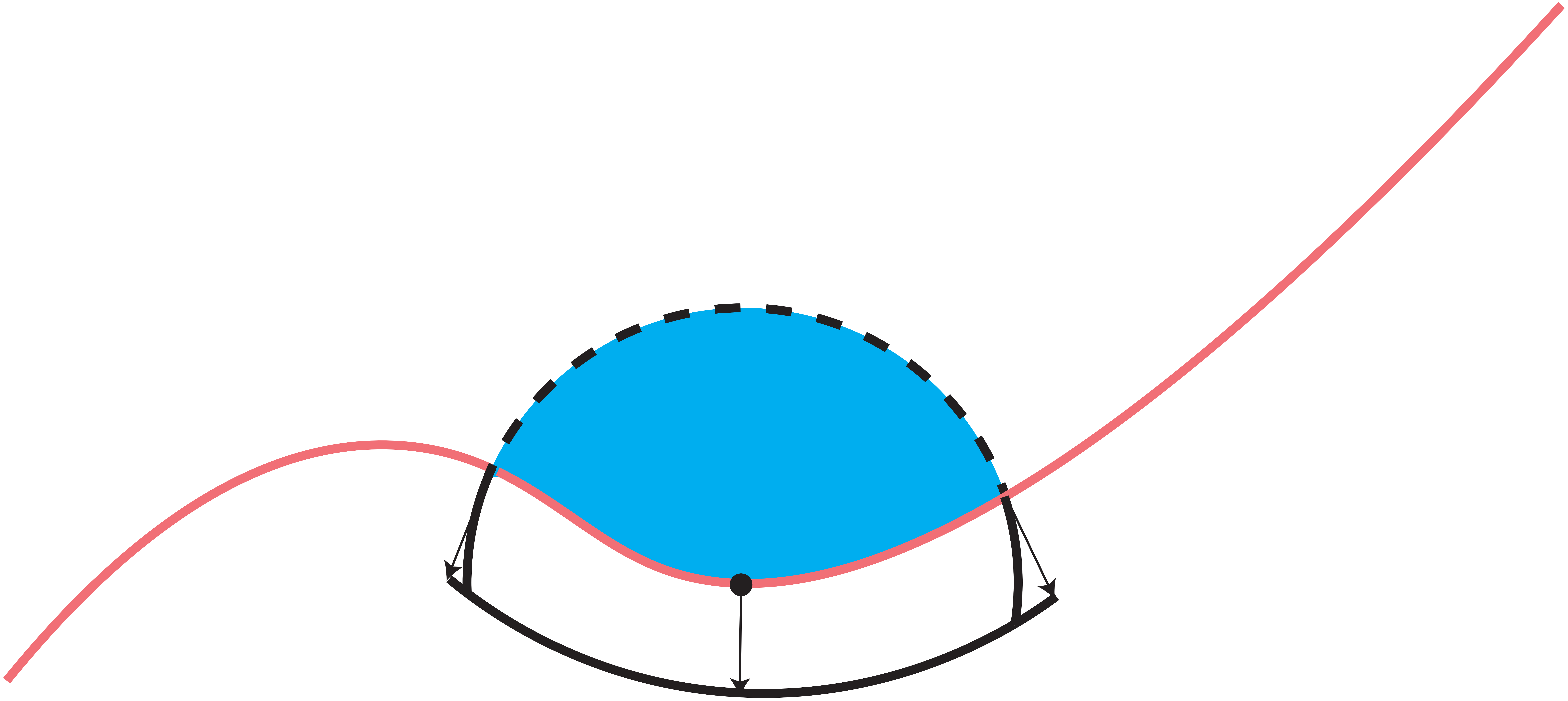
Curvature Estimation - Integral Invariants

$$I(h) = \int_{\Omega_r} h(x)\,\mathrm{d}x \approx \int_{\mathbb{B}_r^+} h(x)\,\mathrm{d}x - \int_{|x_1|\leq r} \int_{x_2=0}^{x_2=\frac{1}{2}\kappa x_1^2} h(x)\,\mathrm{d}x_2 \mathrm{d}x_1$$

For arbitrary hypersurfaces, I have shown that the mean curvature can be estimated as:

$$\widehat{\mathcal{H}}^{(r)} \approx \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot (m+2)}{\pi^{\frac{m+1}{2}} r^{m+2}} \left(\mathrm{vol}(\mathbb{B}_r^+) - \mathrm{vol}(\Omega_r) + \mathcal{O}(r^{m+3})\right)$$

Analyzing the Effects of Noise



What's next?

- Error bounds of different integral invariants
- Analyzing the effects of noise in higher codimension
- Implementation