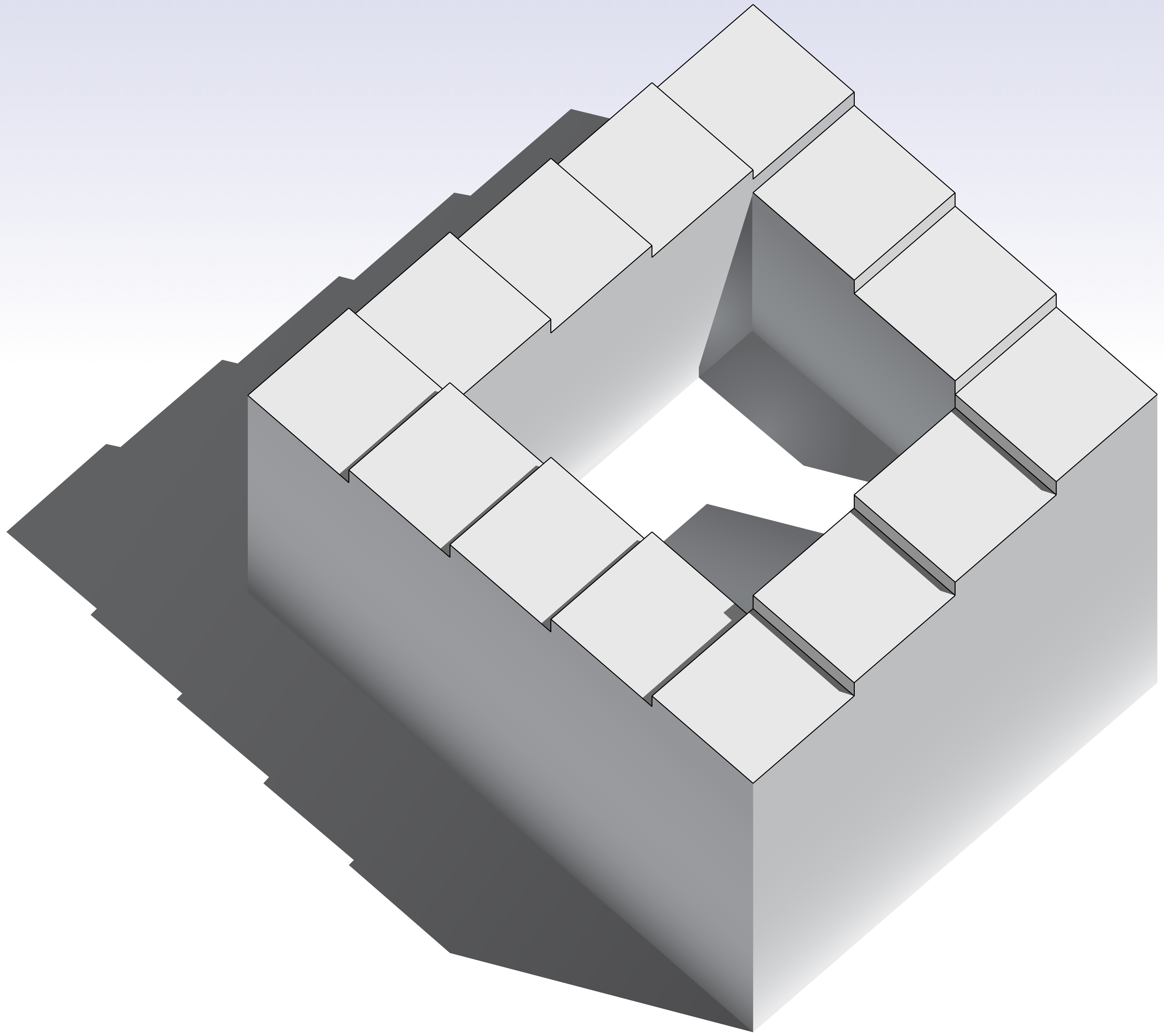
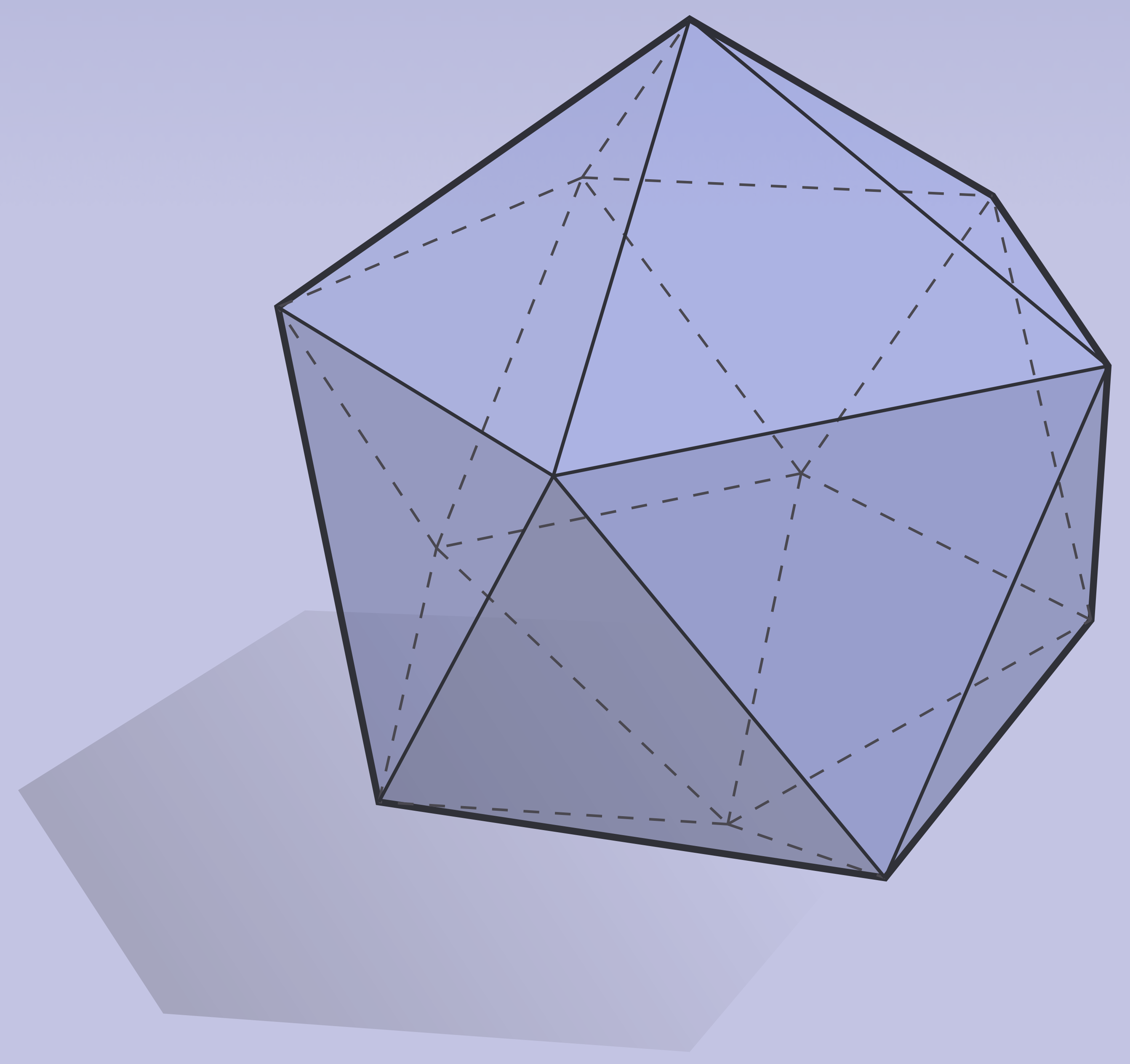


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 10:
Discrete Exterior Calculus



Discrete Differential Geometry: An Applied Introduction

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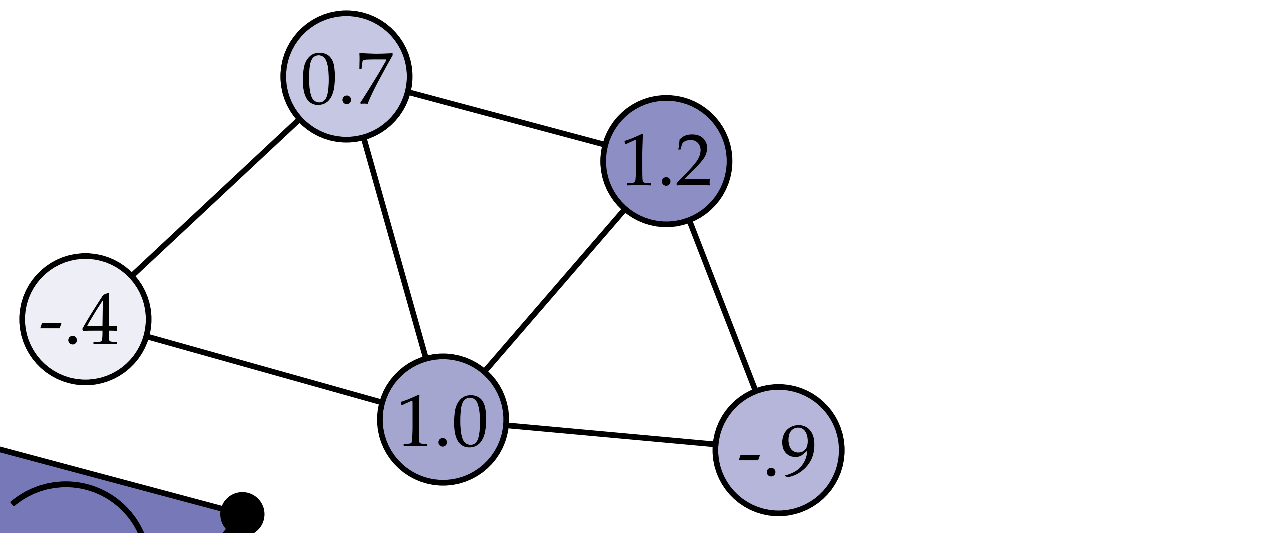
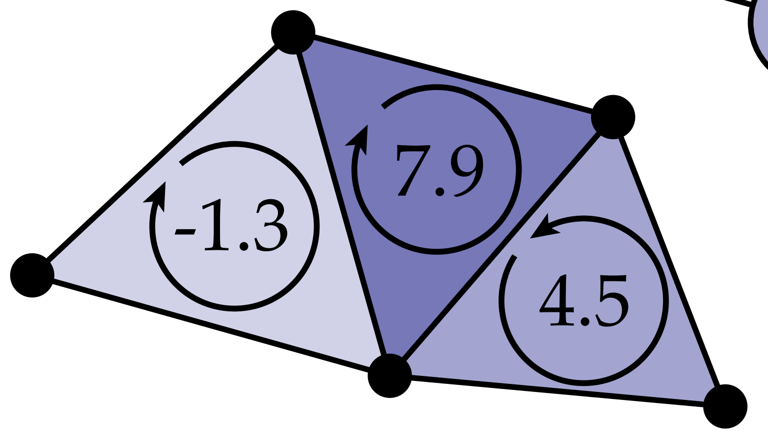
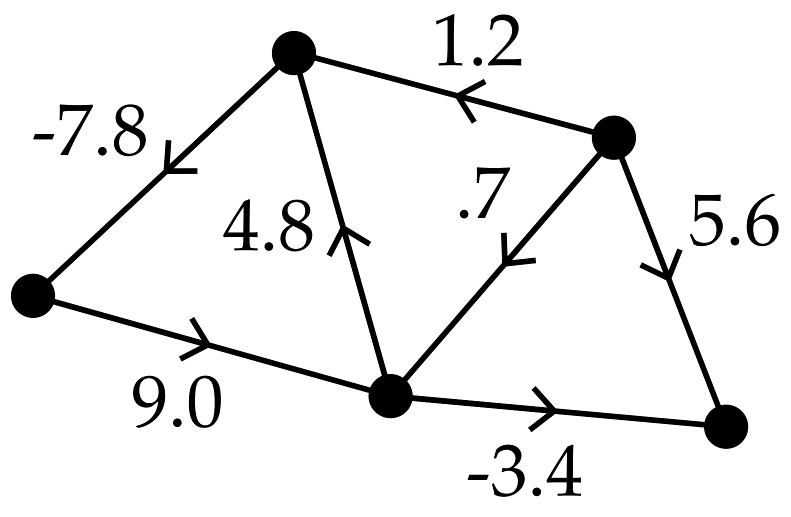
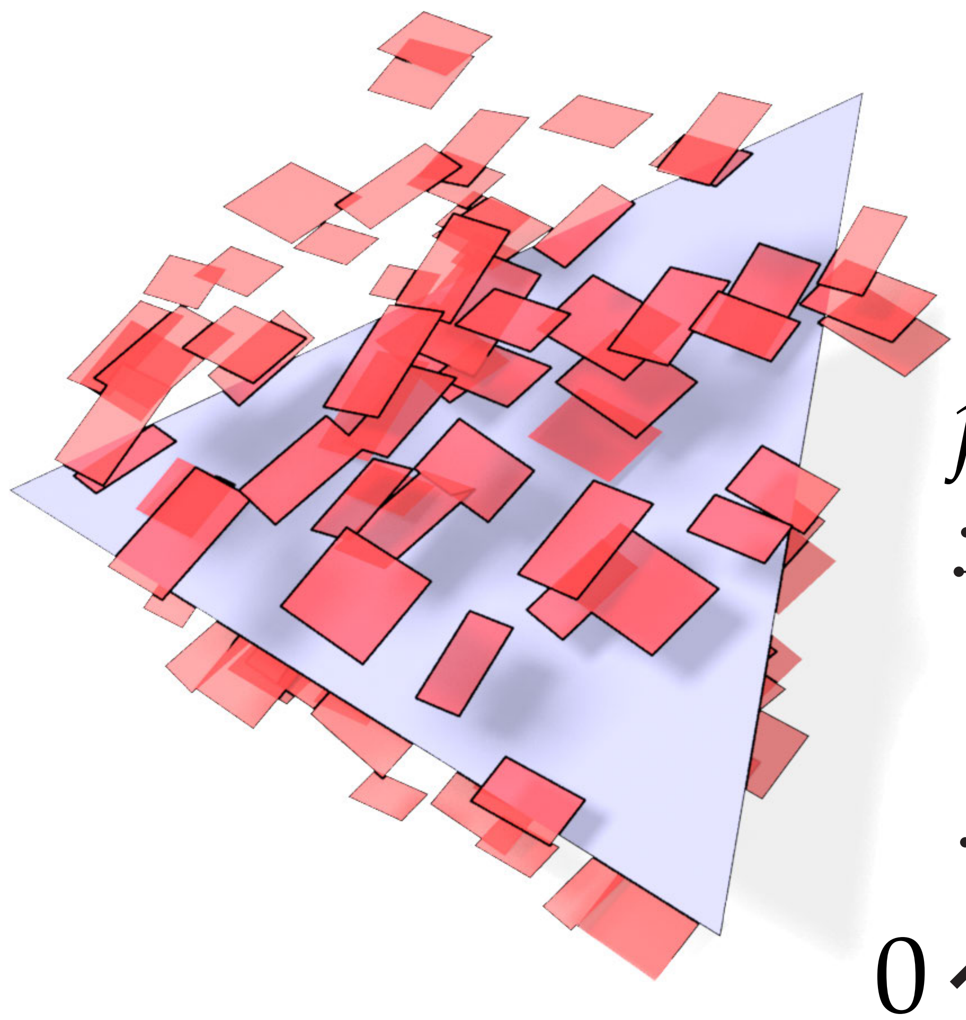
Review—Discrete Differential Forms

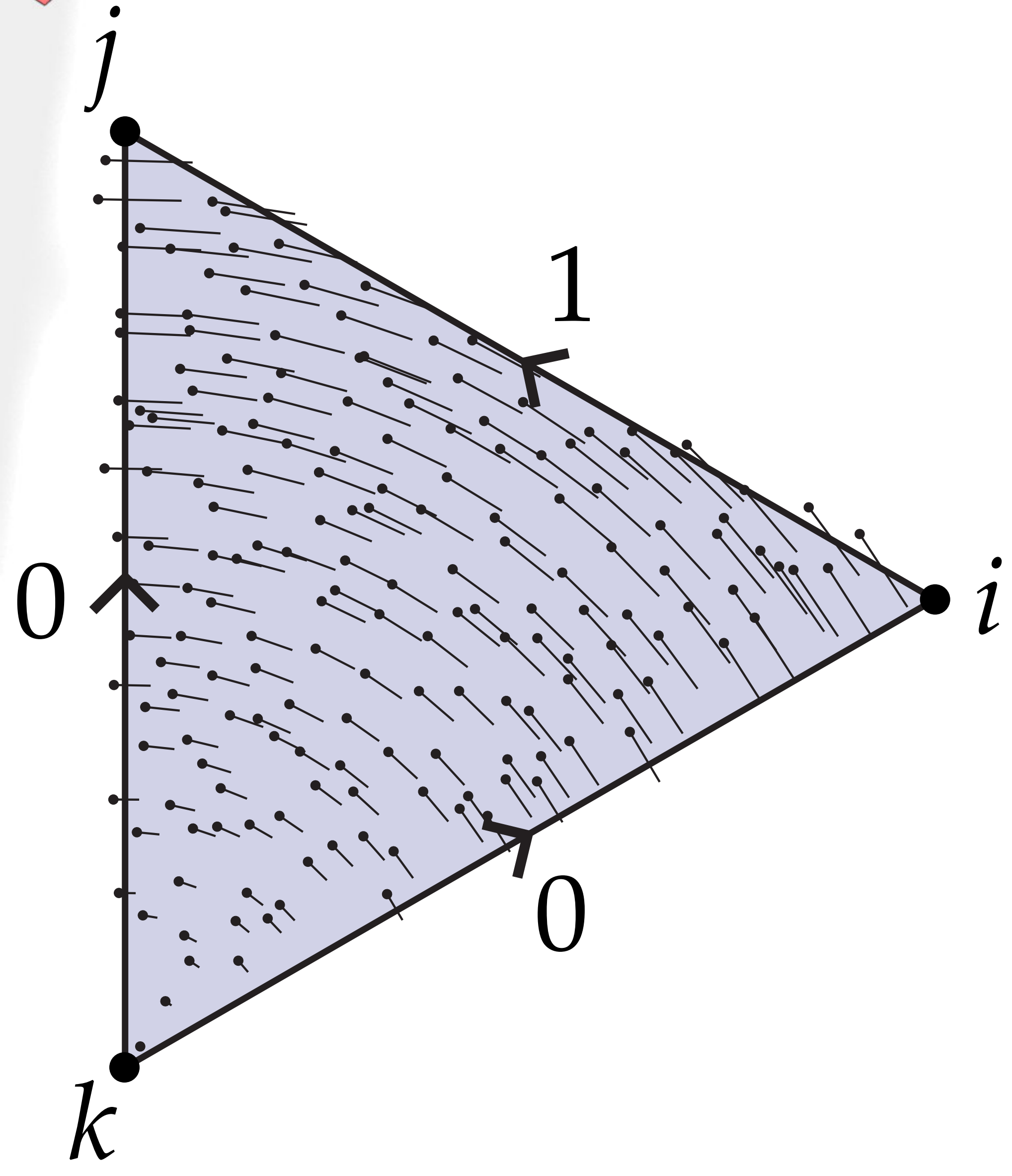
- A *discrete differential k-form* amounts to a value stored on each oriented *k*-simplex
- **Discretization:** given a smooth differential k-form, can approximate by a discrete differential k-form by integrating over each k-simplex
- *In practice,* almost never comes from direct integration. More typically, values start at vertices (samples of some function); 1-forms, 2-forms, *etc.*, arise from applying operators like the (discrete) exterior derivative.
- This lecture: **calculus** on discrete differential forms
    - differentiation—*discrete* exterior derivative
    - integration—just take sums!







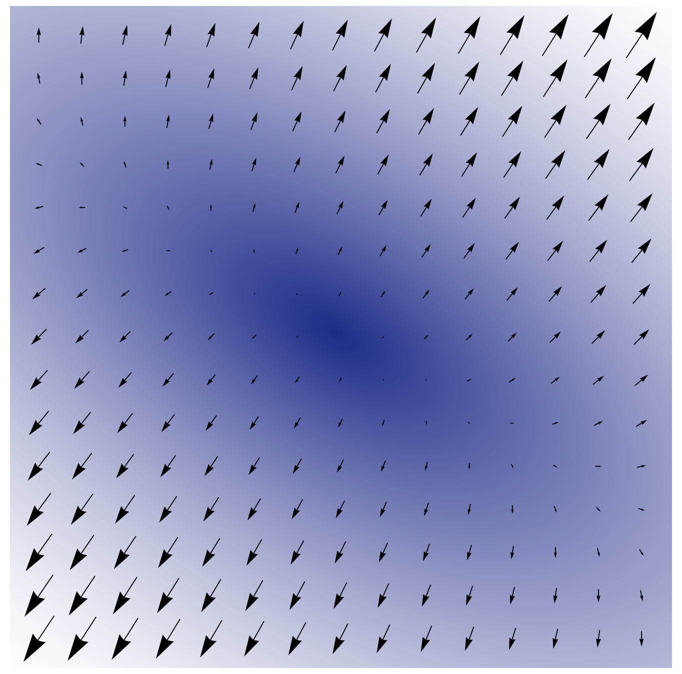




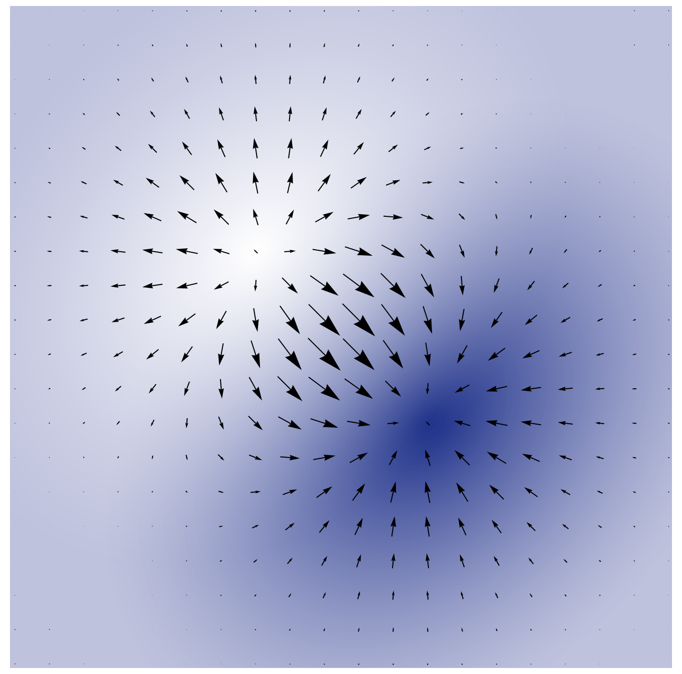
Discrete Exterior Derivative

Reminder: Exterior Derivative

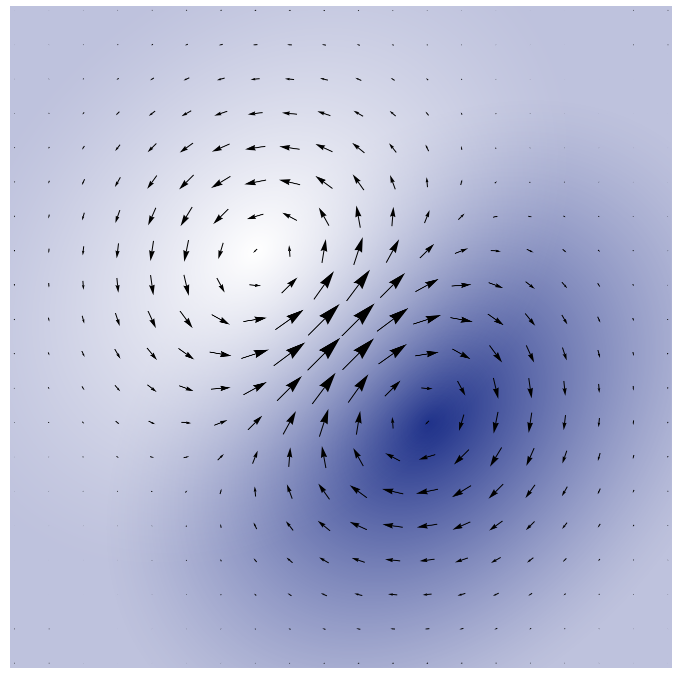
- Recall that in the smooth setting, the exterior derivative…
  - …maps differential k-forms to differential (k+1)-forms
  - …satisfies a product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
  - …yields zero when you apply it twice: $d \circ d = 0$
  - …is similar to the *gradient* for 0-forms
  - …is similar to *curl* for 1-forms
  - …is similar to *divergence* when composed w/ Hodge star
- To get **discrete** exterior derivative, we are simply going to evaluate the smooth exterior derivative and integrate the result over (oriented) simplices



$(d\phi)^\sharp$



$\star d(\star X^\flat)$

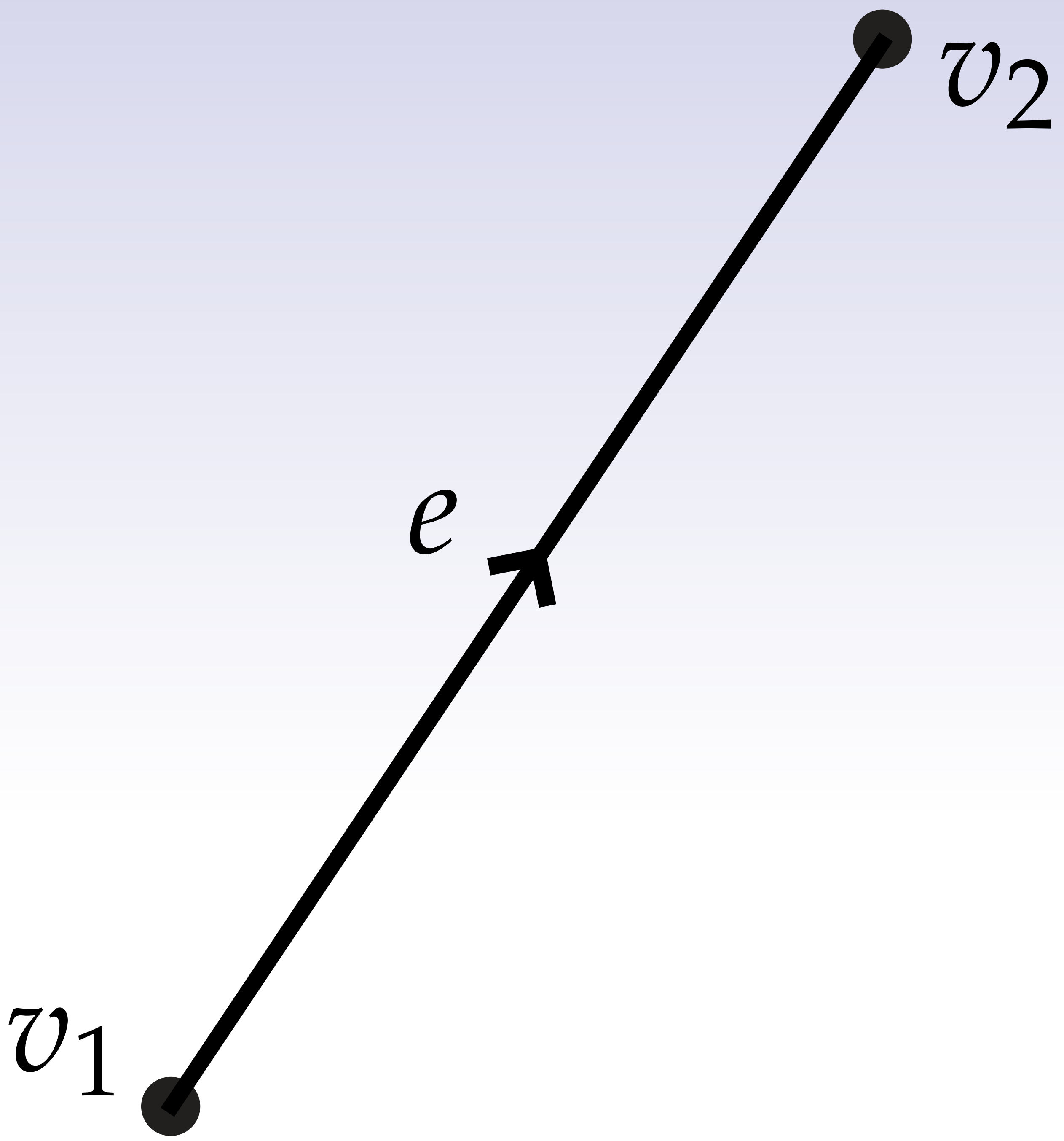


$(\star(dX^\flat))^\sharp$

Discrete Exterior Derivative (0-Forms)

ϕ - *primal 0-form* (vertices)

$d\phi$ - *primal 1-form* (edges)

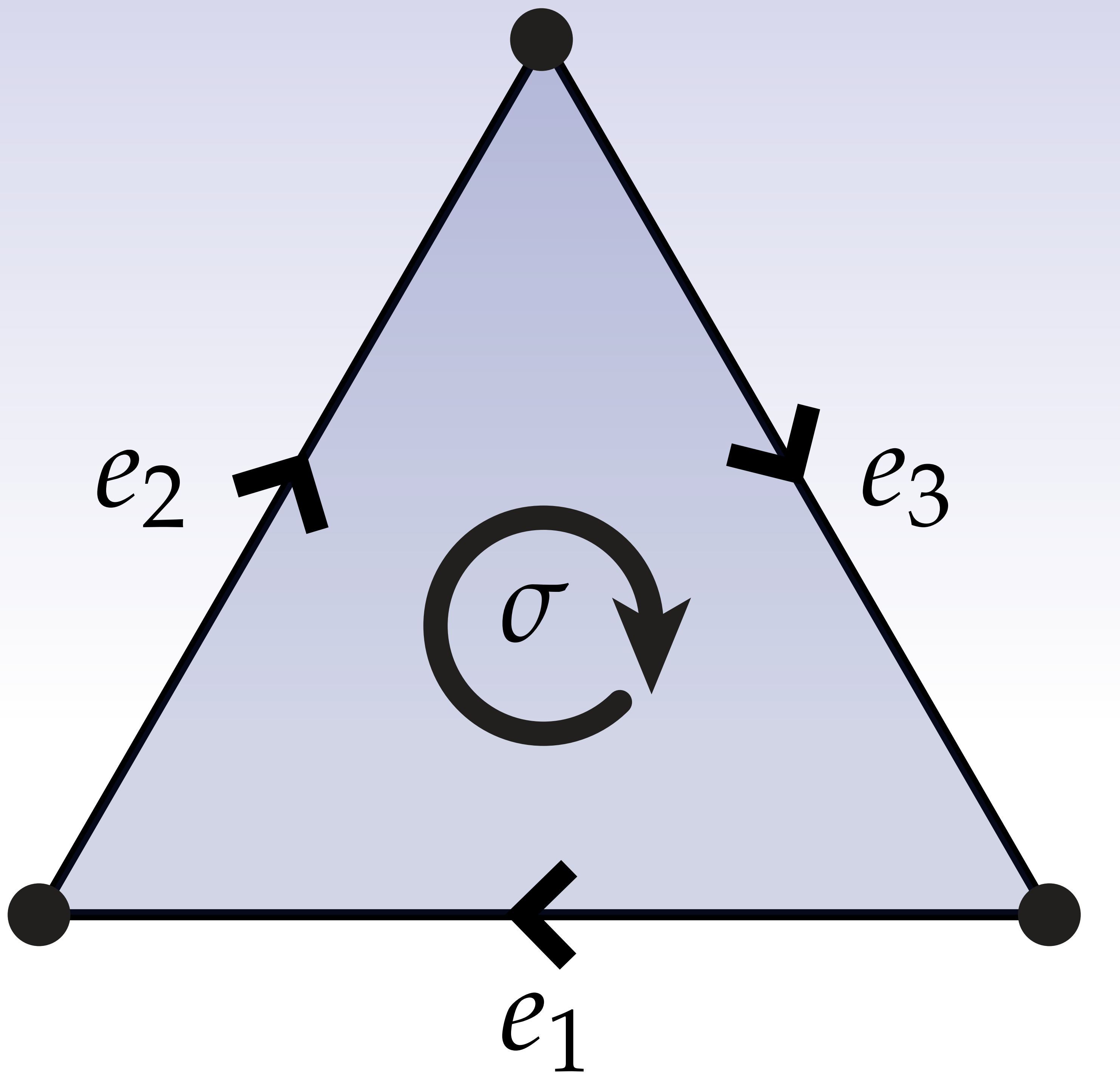


$$(\widehat{d\phi})_e = \int_e d\phi = \int_{\partial e} \phi = \hat{\phi}_2 - \hat{\phi}_1$$

Discrete Exterior Derivative (1-Forms)

α - *primal 1-form* (edges)

$d\alpha$ - *primal 2-form* (triangles)



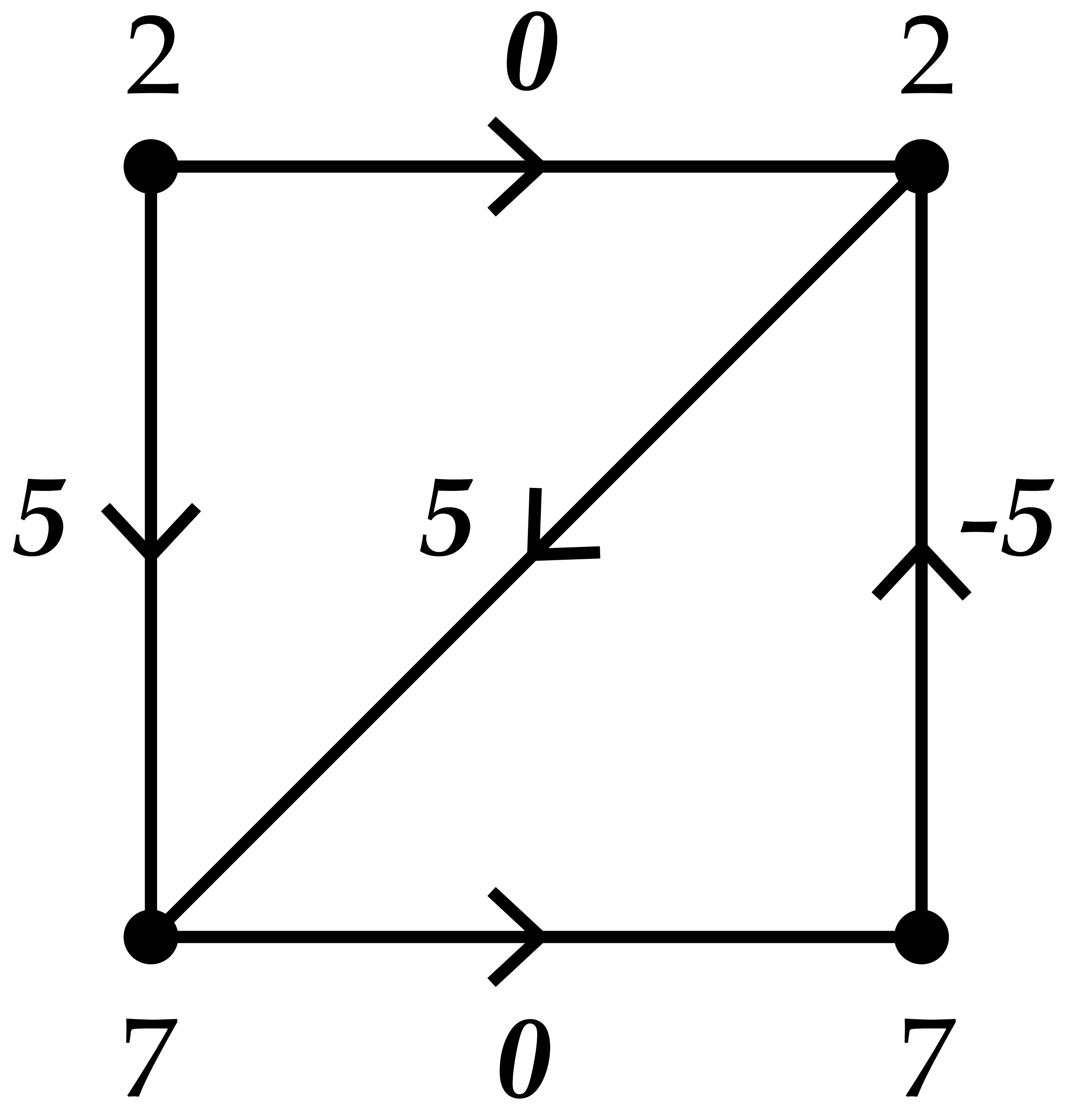
$$(\widehat{d\alpha})_\sigma = \int_\sigma d\alpha = \int_{\partial\sigma} \alpha = \sum_{i=1}^{3} \int_{e_i} \alpha = \sum_{i=1}^{3} \hat{\alpha}_i$$

In general: discrete exterior derivative is *coboundary* operator for *cochains*.

Discrete Exterior Derivative—Examples

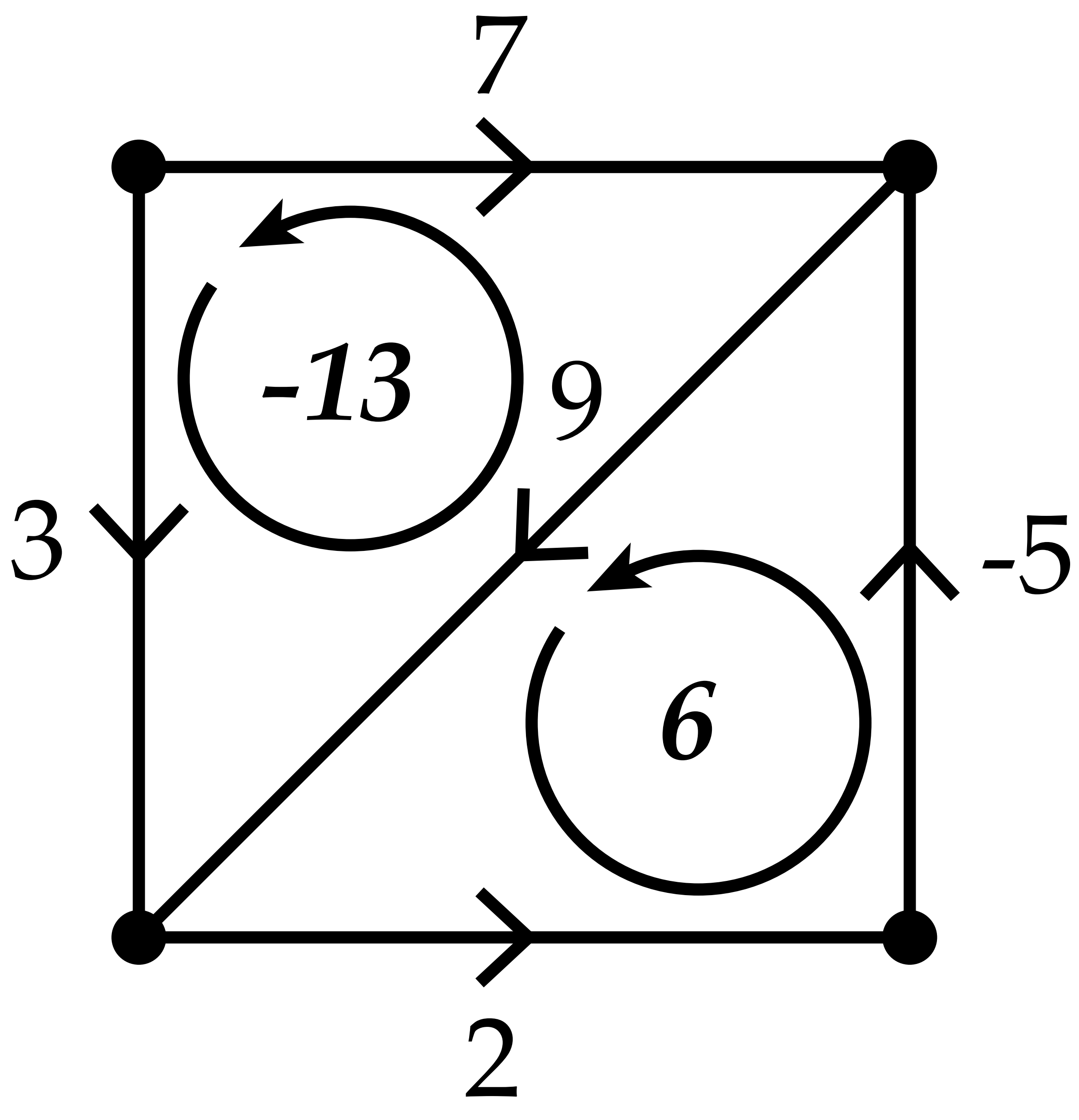
When applying the discrete exterior derivative, must be careful to take *orientation* into account.

Example (0-form)



(Also notice that exterior derivative has *nothing* to do with length!)

Example (1-form)

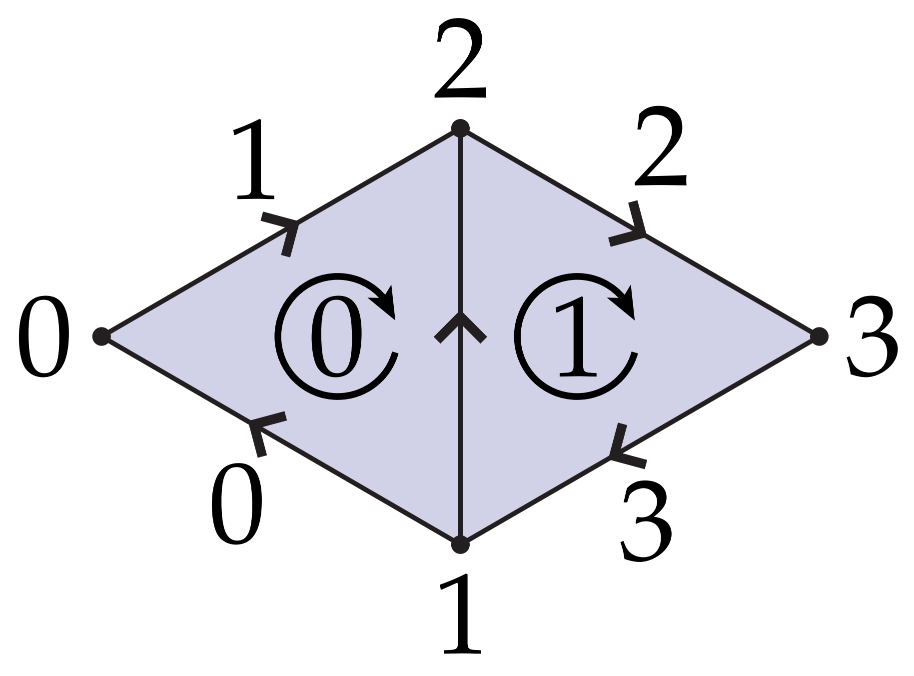


$3 - 9 - 7 = \boldsymbol{-13}$

$9 + 2 + (-5) = \boldsymbol{6}$

Discrete Exterior Derivative—Matrix Representation

- The discrete exterior derivative on *k*-forms, which we will denote by d_k, is a linear map from values on *k*-simplices to values on *(k+1)*-simplices:
  - d_0 maps values on vertices to values on edges
  - d_1 maps values on edges to values on triangles
  - d_2 maps values on triangles to values on tetrahedra
  - …
- We can encode each operator to a matrix, by assigning an indices to mesh elements (just as when we encoded discrete *k*-forms as column vectors)
- This matrix turns out to be just a *signed incidence matrix*, which we saw in our discussion of the oriented simplicial complex



$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{rrrr} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{array}\right] \end{array}$$

$$E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{rrrrr} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Discrete Exterior Derivative d_0—Example

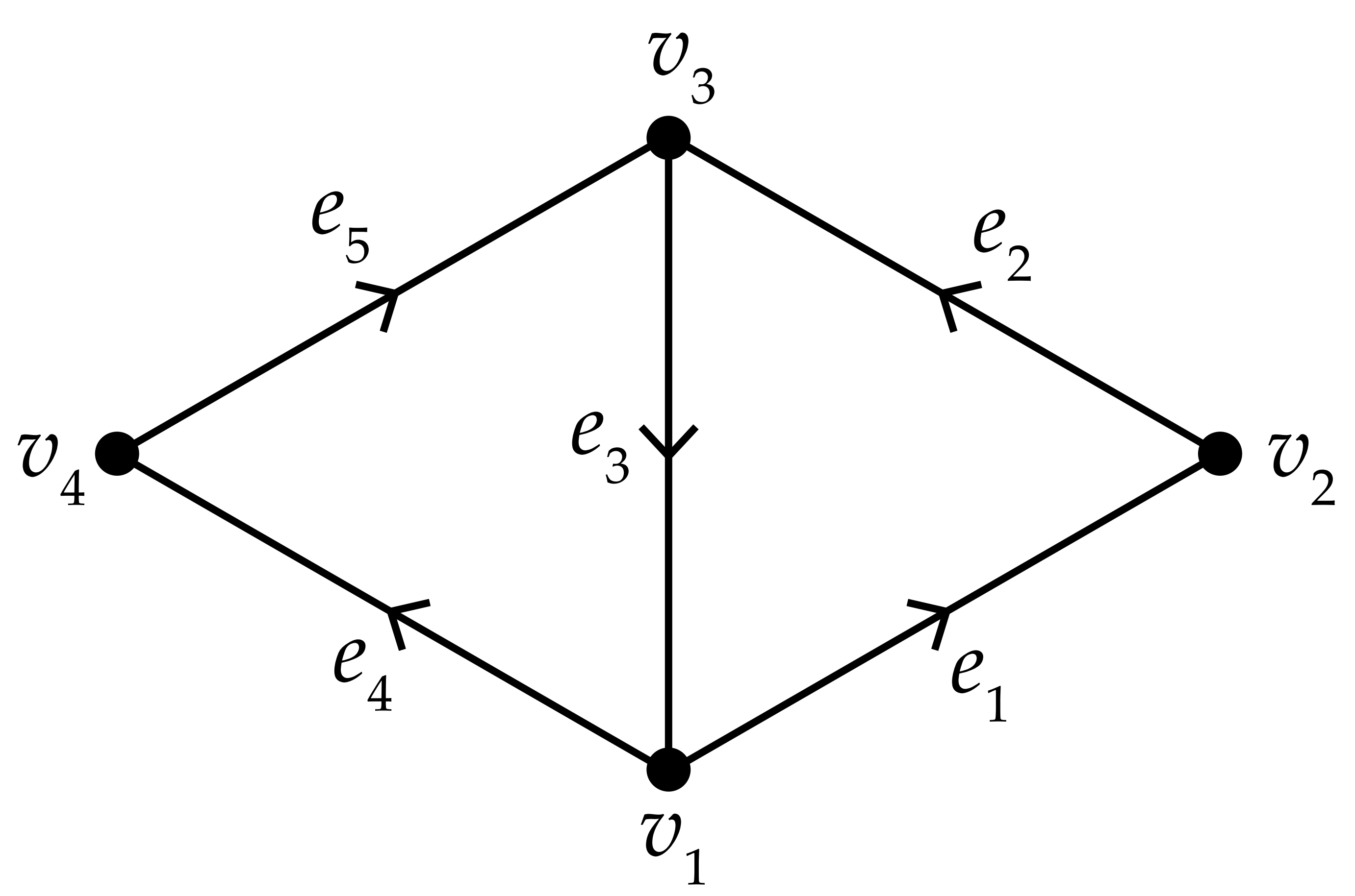
- To build the exterior derivative on 0-forms, we first need to assign an index to each *vertex* and each *edge*
  - A discrete 0-form is a list of $|V|$ values (one per vertex)
  - A discrete 1-form is a list of $|E|$ values (one per edge)
- The discrete exterior derivative d_0 is therefore a $|E| \times |V|$ matrix, taking values at vertices to values at edges

Example.

$$\phi \in \mathbb{R}^{|V|}$$

$$\alpha \in \mathbb{R}^{|E|}$$

$$d_0 \in \mathbb{R}^{|E|\times|V|}$$



$$\underbrace{\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}}_{d_0} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}}_{\alpha}$$

Discrete Exterior Derivative d_1—Example

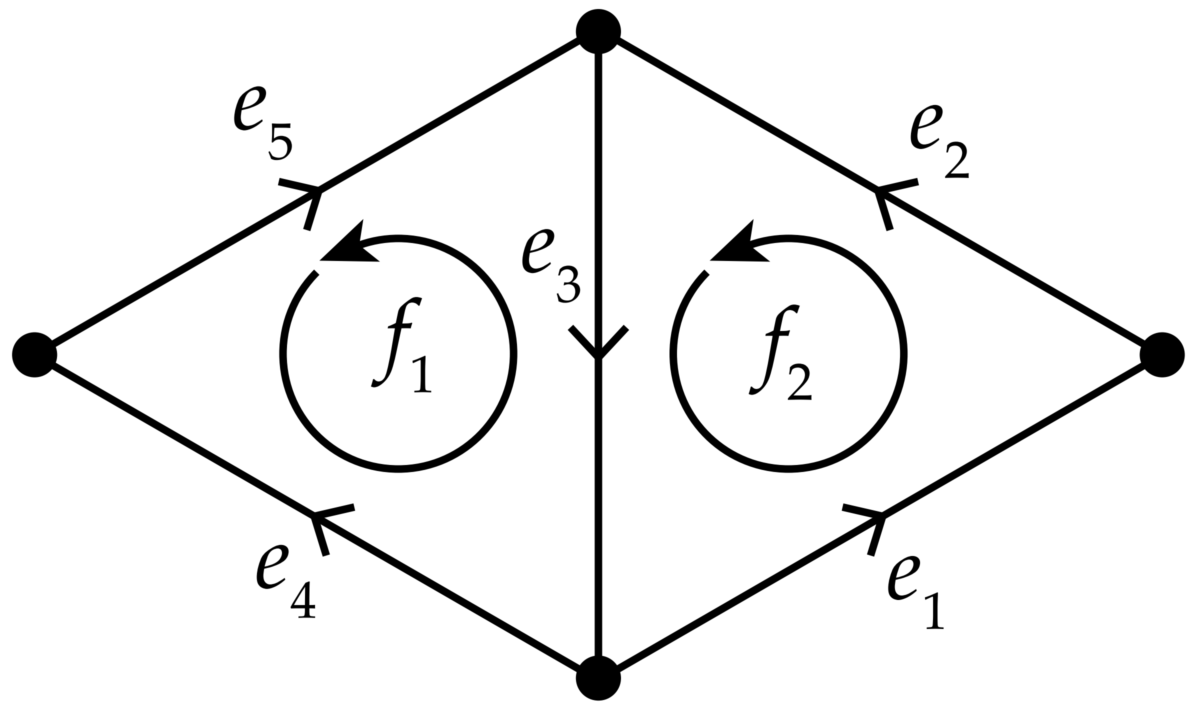
- To build the exterior derivative on 1-forms, we first need to assign an index to each *edge* and each *face*
  - A discrete 0-form is a list of $|E|$ values (one per edge)
  - A discrete 1-form is a list of $|F|$ values (one per face)
- The discrete exterior derivative d_1 is therefore a $|F|$x$|E|$ matrix, taking values at edges to values at faces
- This time, we need to be more careful about *relative orientation*

Example.

$$\alpha \in \mathbb{R}^{|E|}$$

$$\omega \in \mathbb{R}^{|F|}$$

$$d_1 \in \mathbb{R}^{|F| \times |E|}$$

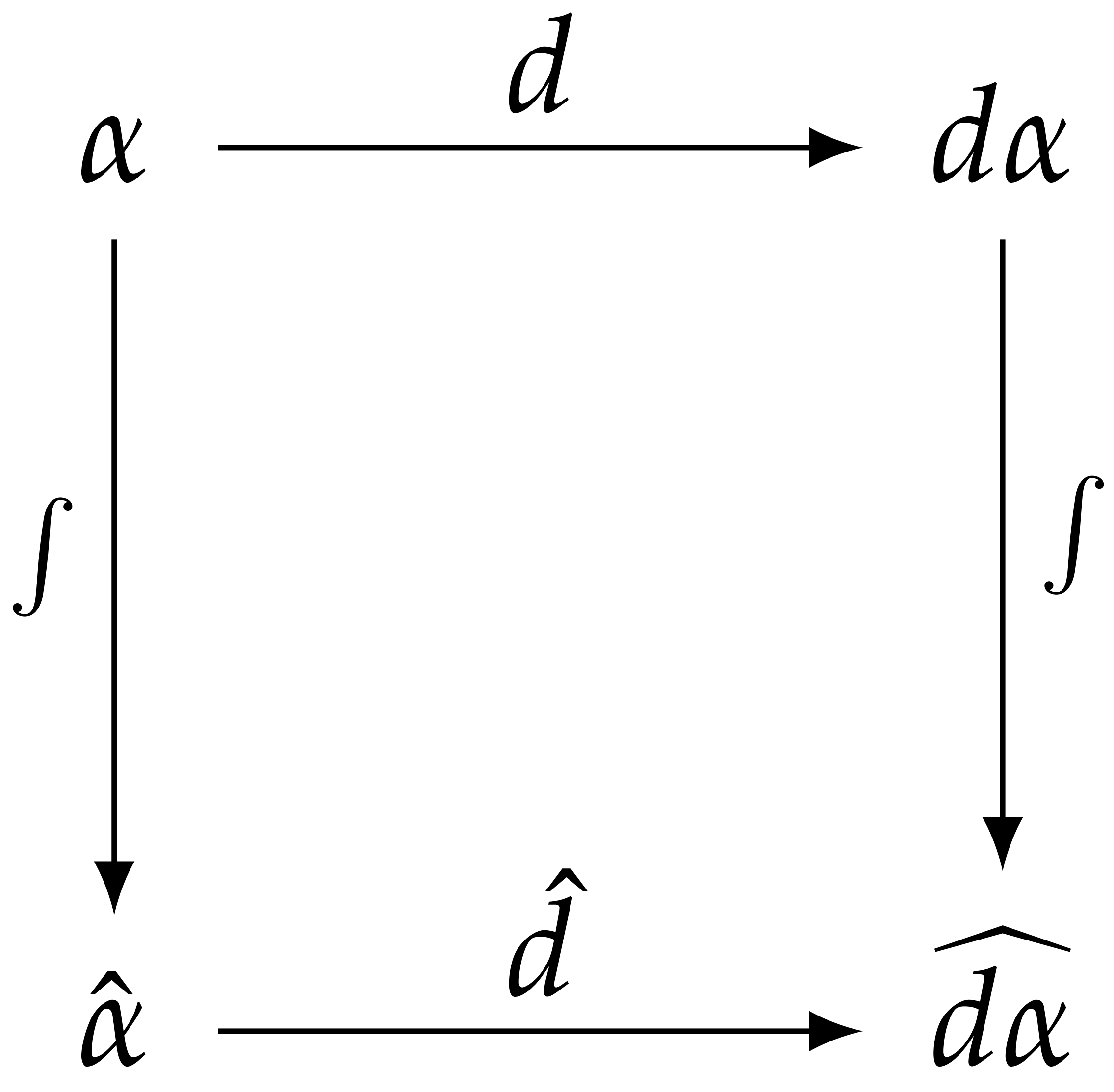


$$\underbrace{\begin{array}{c} \\ f_1 \\ f_2 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}}_{d_1} \underbrace{\left[\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{array}\right]}_{\alpha} = \underbrace{\left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right]}_{\omega}$$

Exterior Derivative Commutes w/ Discretization

- By definition, the discrete exterior derivative satisfies a very important property:

> Taking the **smooth** exterior derivative and then discretizing yields the same result as *discretizing* and then applying the **discrete** exterior derivative.



d	—	smooth exterior derivative
$\hat{d}$	—	discrete exterior derivative
$\int$	—	de Rham map (discretization)
α	—	smooth k-form
$\hat{\alpha}$	—	discrete k-form
$d\alpha$	—	smooth (k+1)-form
$\widehat{d\alpha}$	—	discrete (k+1)-form

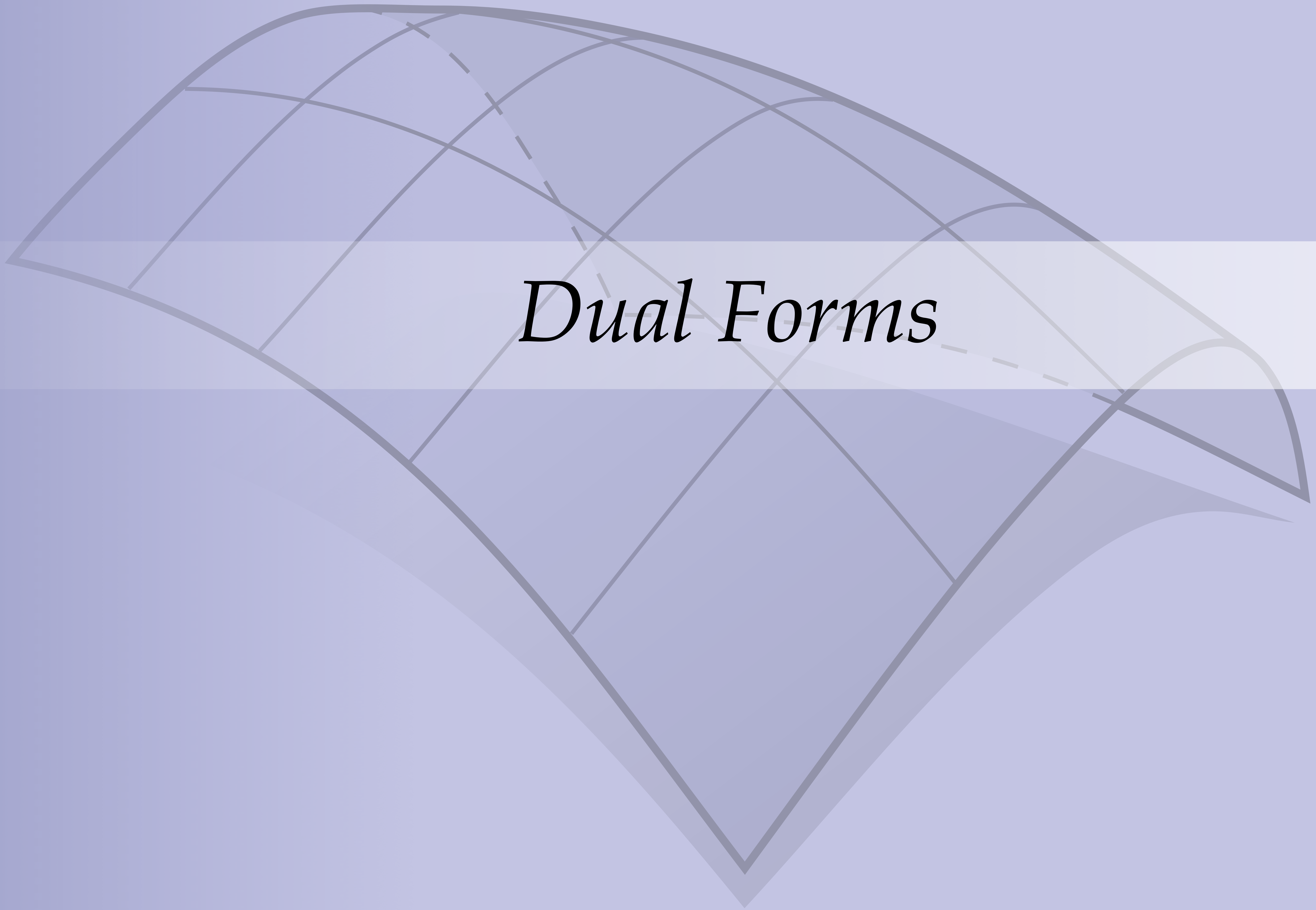
Corollary: applying discrete d twice yields zero (why?)

Exactness of Discrete Exterior Derivative

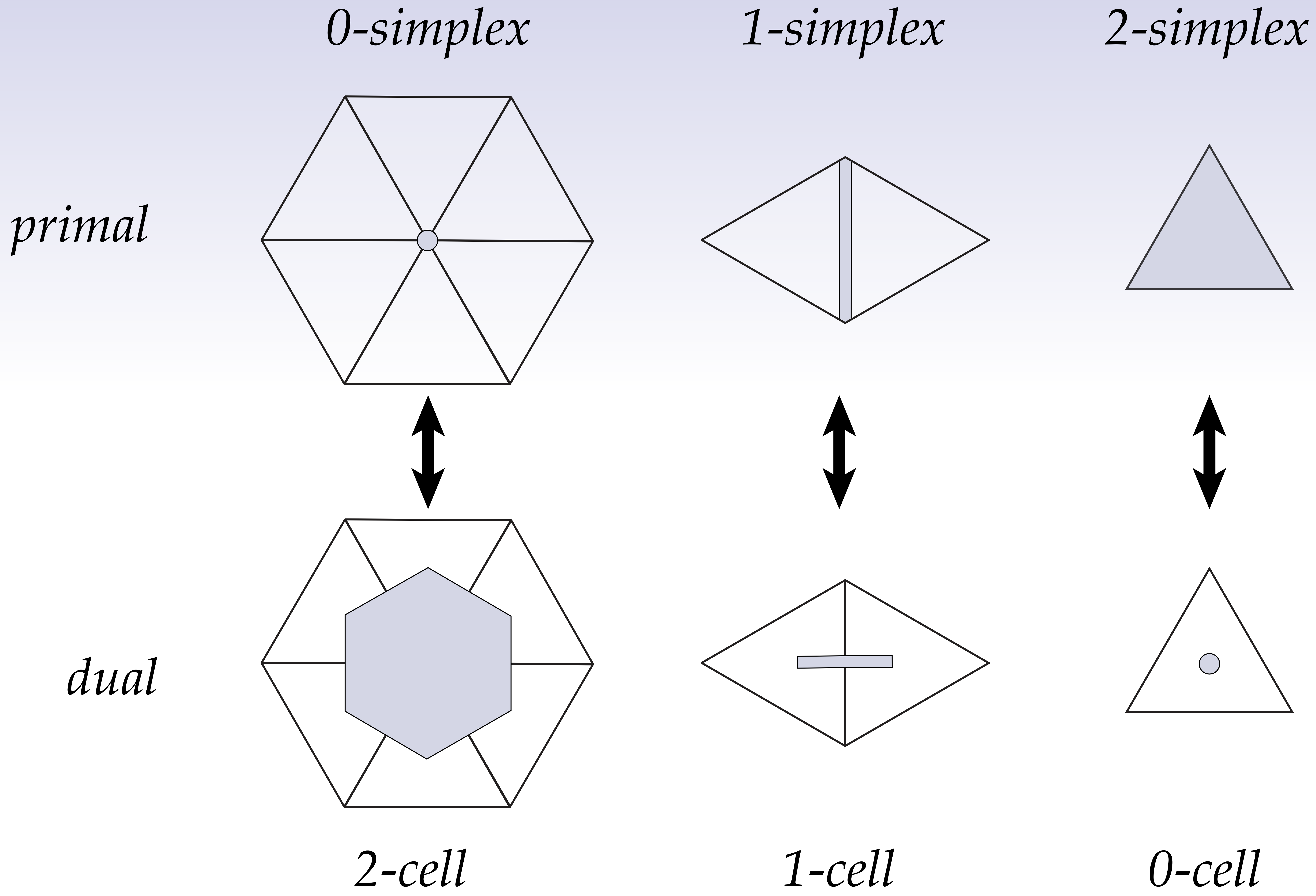
- To confirm that applying discrete exterior derivative twice yields zero, we can just multiply the exterior derivative matrices for 0- and 1-forms:

$$d_1 d_0 = \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Dual Forms



Reminder: Poincaré Duality

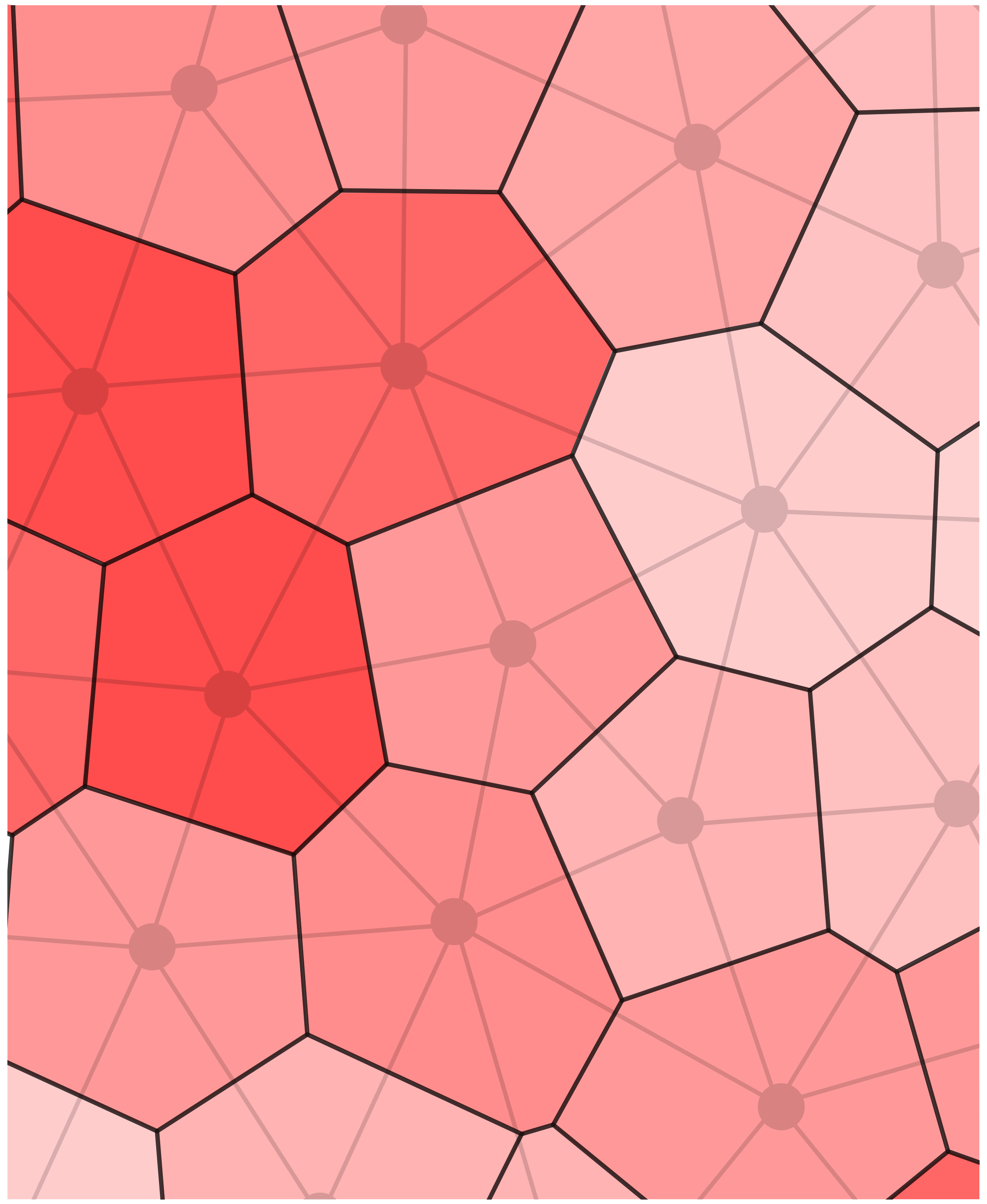


Dual Discrete Differential k-Form

Consider the (Poincaré) dual $K^\star$ of a manifold simplicial complex K.

Just as a discrete differential k-form was a value per k-simplex, a *dual discrete differential k-form* is a value per k-cell:

- a dual **0-form** is a value **dual vertex**
- a dual **1-form** is a value per **dual edge**
- a dual **2-form** is a value per **dual cell**



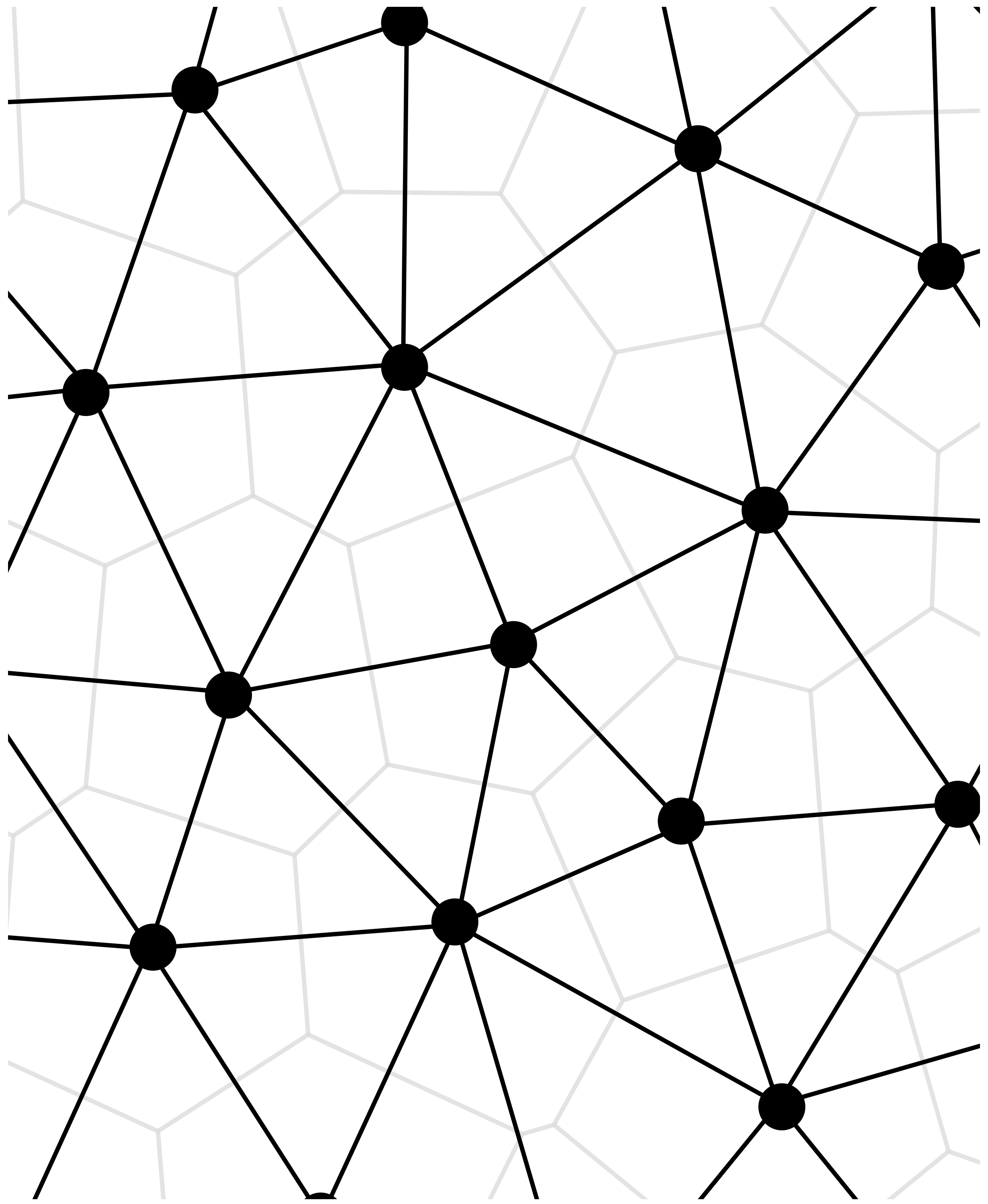
dual 2-form

(Can also formalize via dual chains, dual cochains…)

Primal vs. Dual Discrete Differential k-Forms

Let's compare primal and dual discrete forms on a triangle mesh:

	primal	**dual**
0-forms	vertices	dual vertices (*triangles*)
1-forms	edges	dual edges (*edges*)
2-forms	triangle	dual cells (*vertices*)

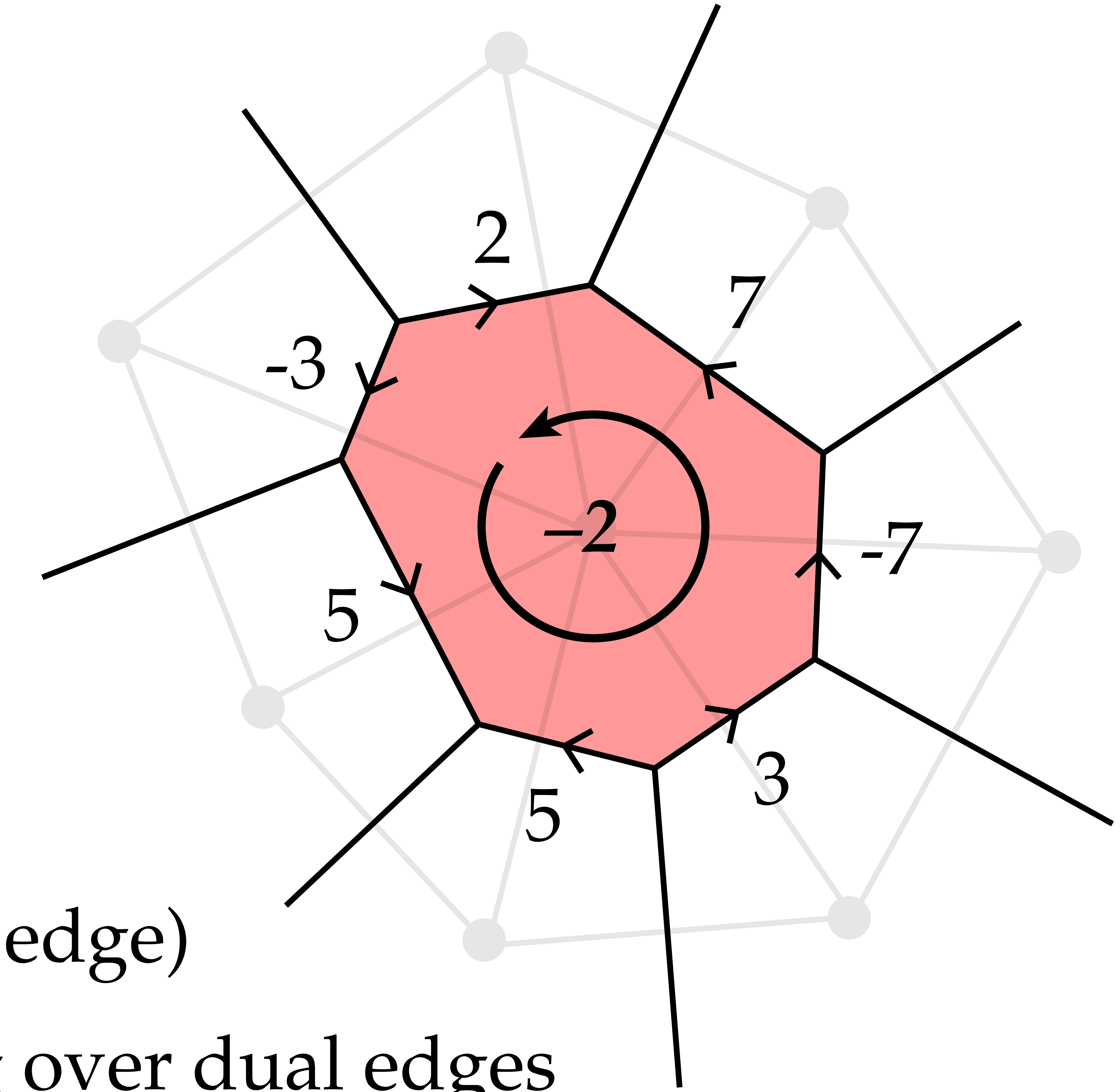


Note: no such thing as "primal" and "dual" forms in smooth setting!

Q: Is the dimension of primal and dual k-forms always the same?

Dual Exterior Derivative

- Discrete exterior derivative on *dual* k-forms works in essentially the same way as for primal forms:
  - To get the derivative on a $(k+1)$-cell, sum up values on each k-cell along its boundary
  - Sign of each term in the sum is determined by relative orientation of $(k+1)$-cell and k-cell



<u>Example.</u>

Let α be a dual discrete 1-form (one value per dual edge)

Then $d\alpha$ is a value per 2-cell, obtained by summing over dual edges

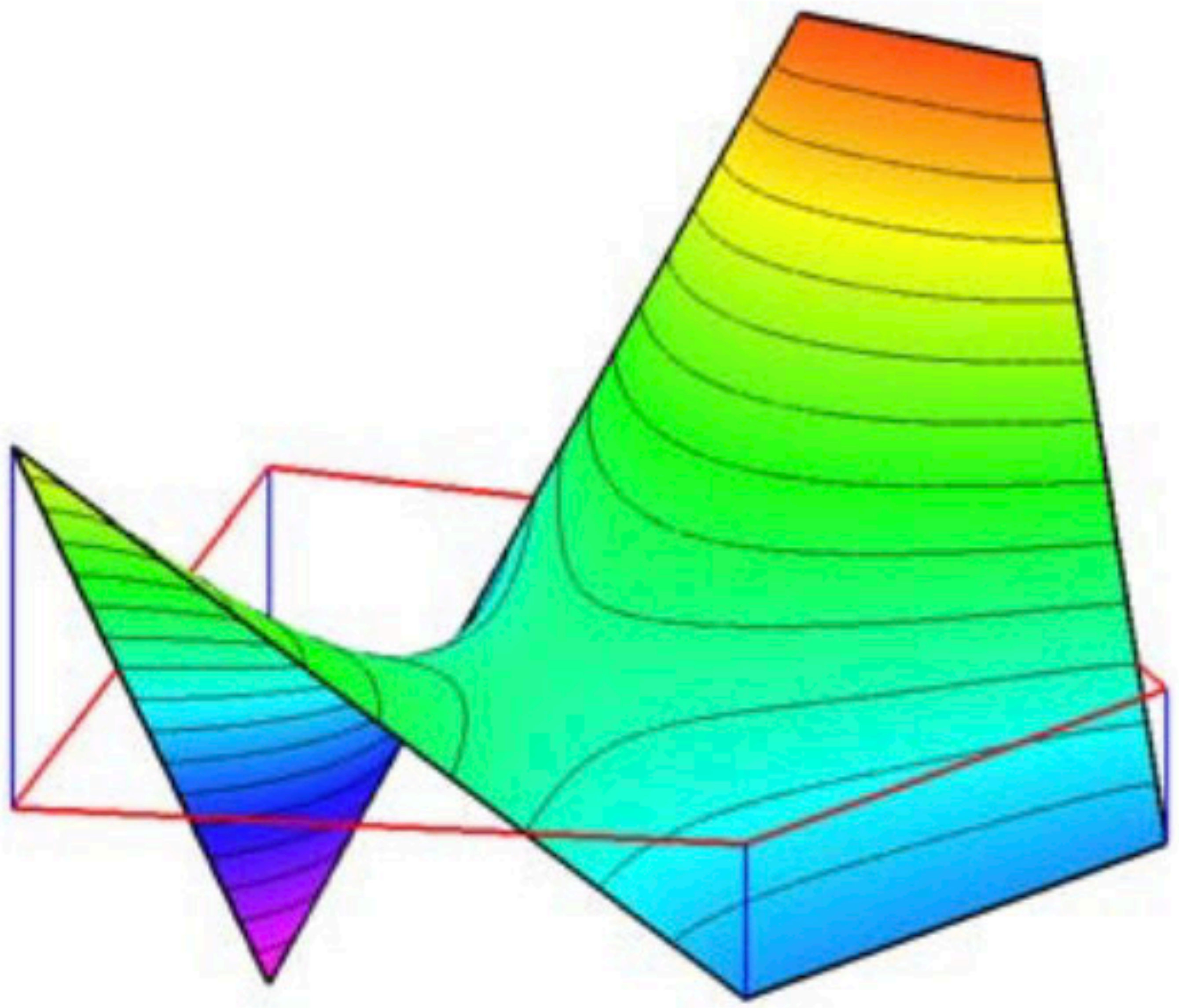
(As usual, relative orientation matters!)

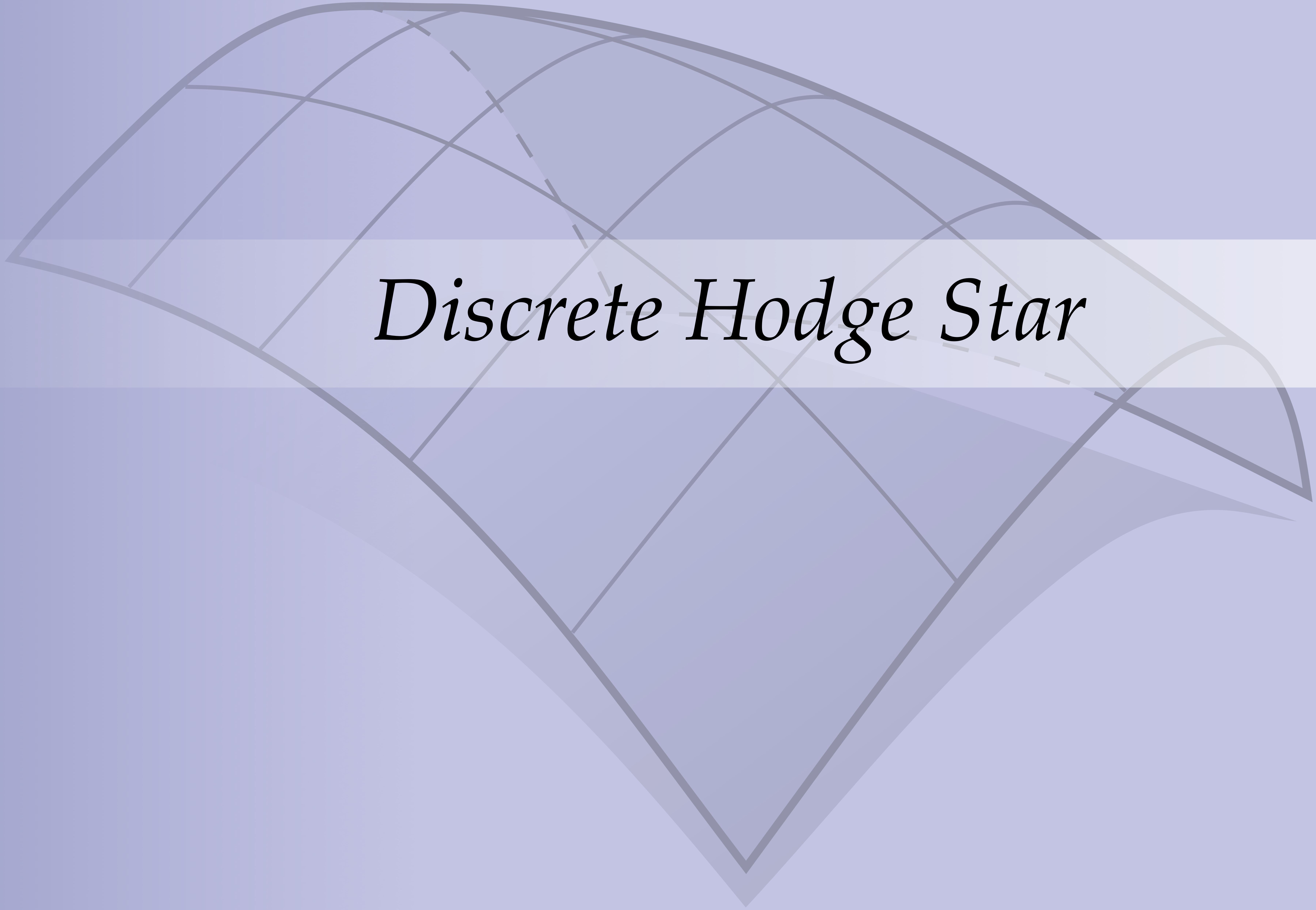
$$-7 + 7 - 2 + (-3) + 5 - 5 + 3 = -2$$

Notice: as with primal d, we don't need lengths, areas, …

Dual Forms: Interpolation & Discretization

- For primal forms, it was easy to make connection to smooth forms via *interpolation*
  - *k*-simplices have clear geometry: *convex hull of vertices*
  - *k*-forms have straightforward basis: *Whitney forms*
- Not so clear cut for dual forms!
  - e.g., can't interpolate dual 0-form with linear function
    - nonconvex cells even more challenging…
    - leads to question of *generalizing* barycentric coordinates
  - k-cells may not sit in a k-dimensional linear subspace
    - e.g., 2-cells in 3D can be non-planar
- Nonetheless, still easy to work with dual forms formally / abstractly (e.g., *d*)

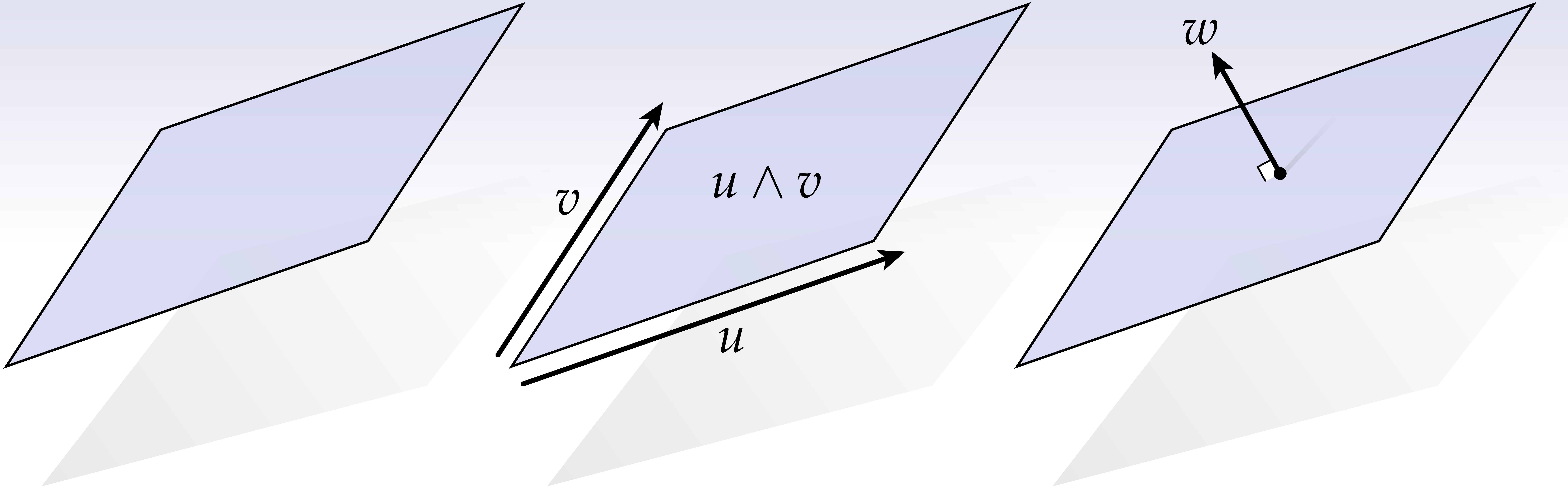




Discrete Hodge Star

Reminder: Hodge Star $(\star)$

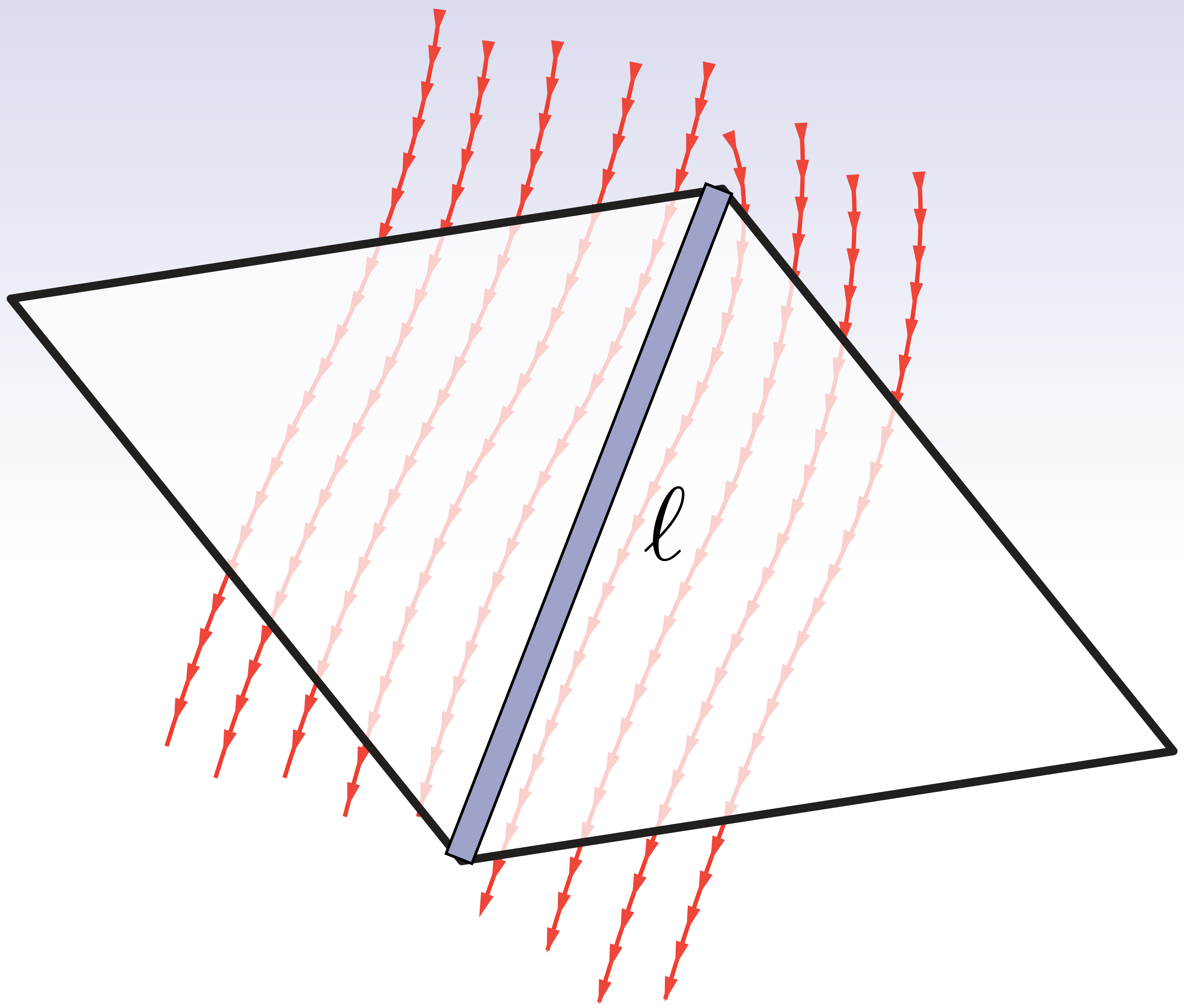
$$\star(u \wedge v) = w$$



Analogy: *orthogonal complement*

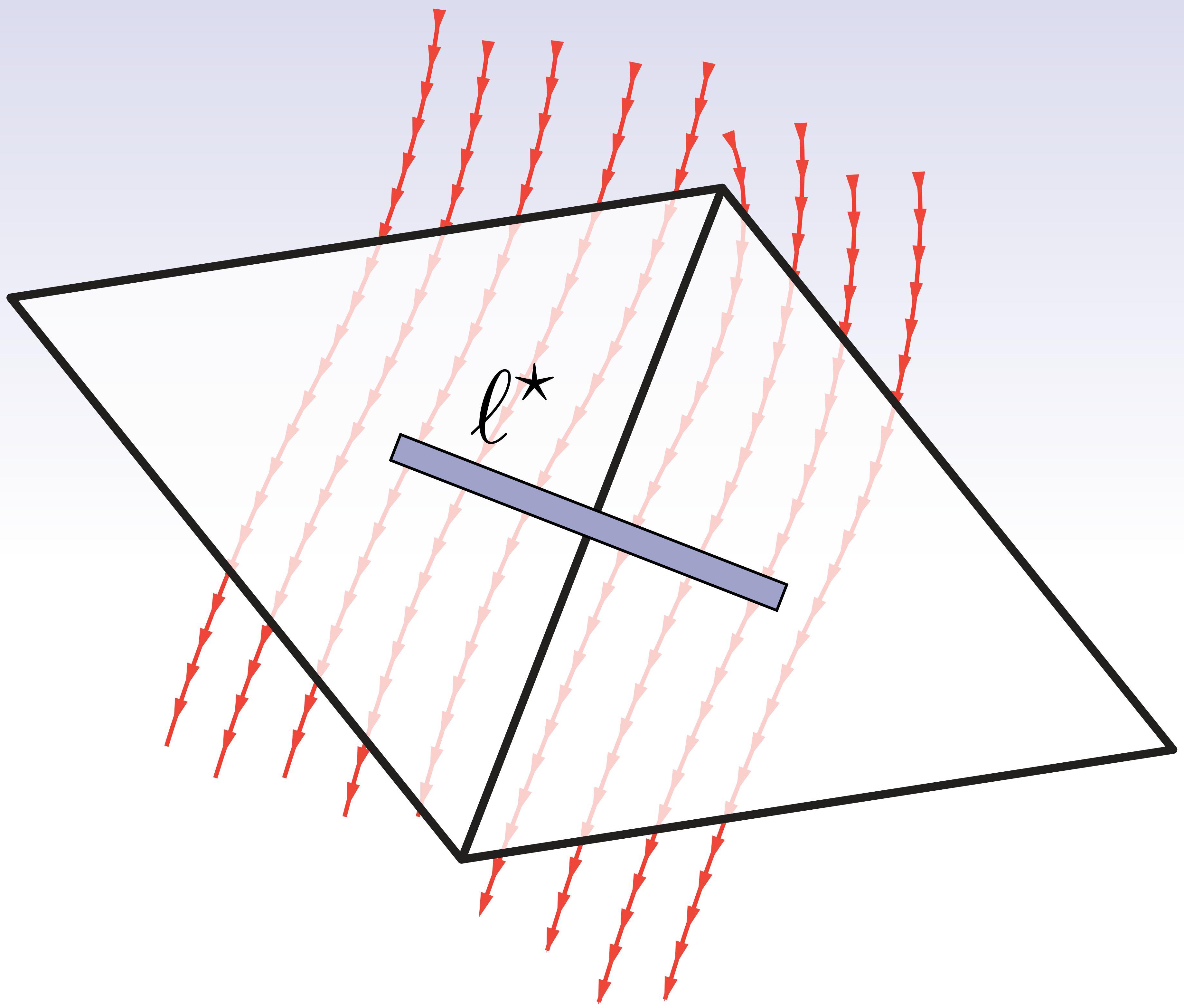
$$k \mapsto (n-k)$$

Discrete Hodge Star — 1-forms in 2D



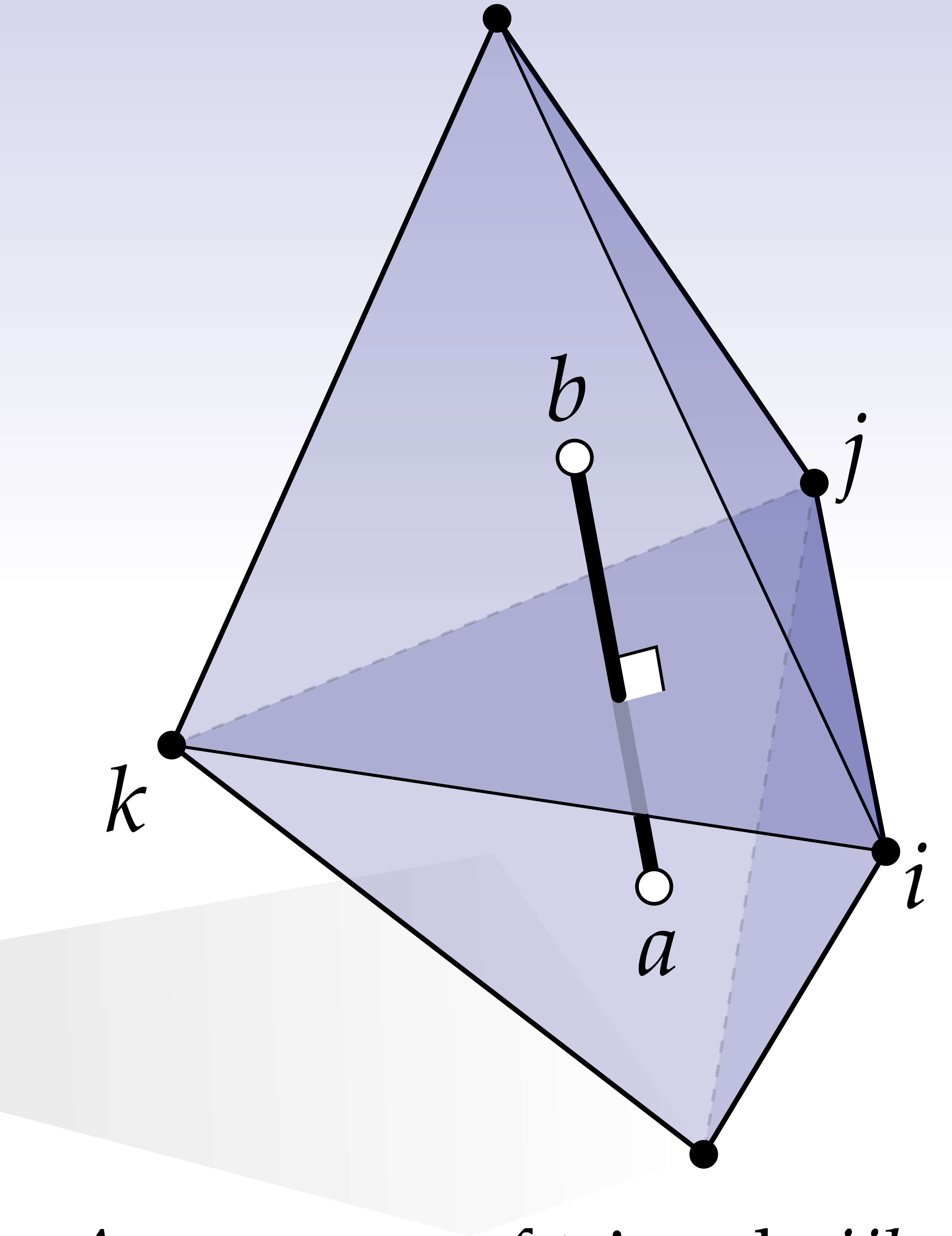
primal 1-form
(circulation)

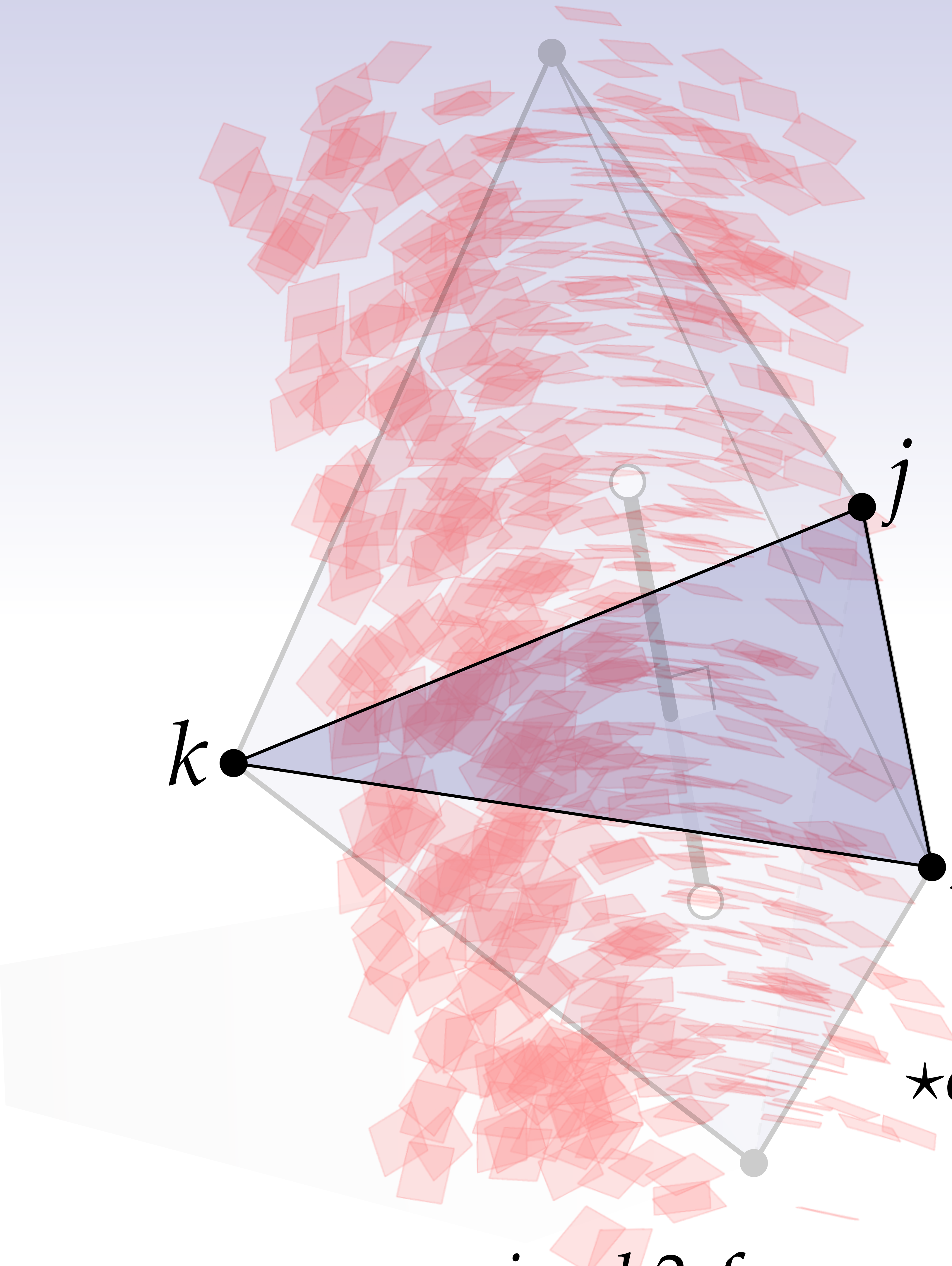
$$\star\hat{\alpha}_e = \frac{\ell^\star}{\ell}\hat{\alpha}$$



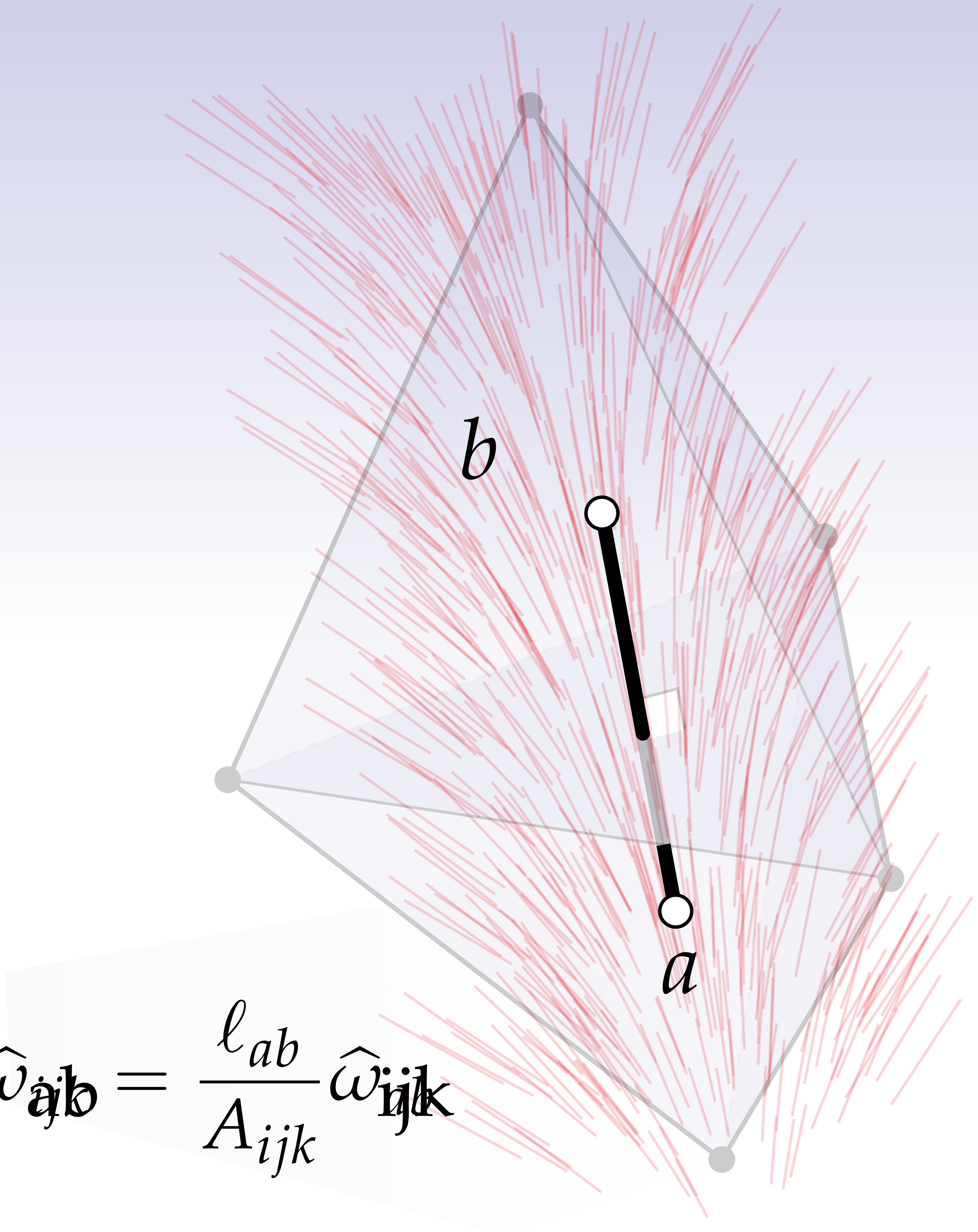
dual 1-form
(flux)

Discrete Hodge Star — 2-forms in 3D





$$\star\widehat{\omega}_{ab} = \frac{\ell_{ab}}{A_{ijk}}\widehat{\omega}_{ijk}$$



A_{ijk} — area of triangle *ijk*
ℓ_{ab} — length of dual edge *ab*

primal 2-form

dual 1-form

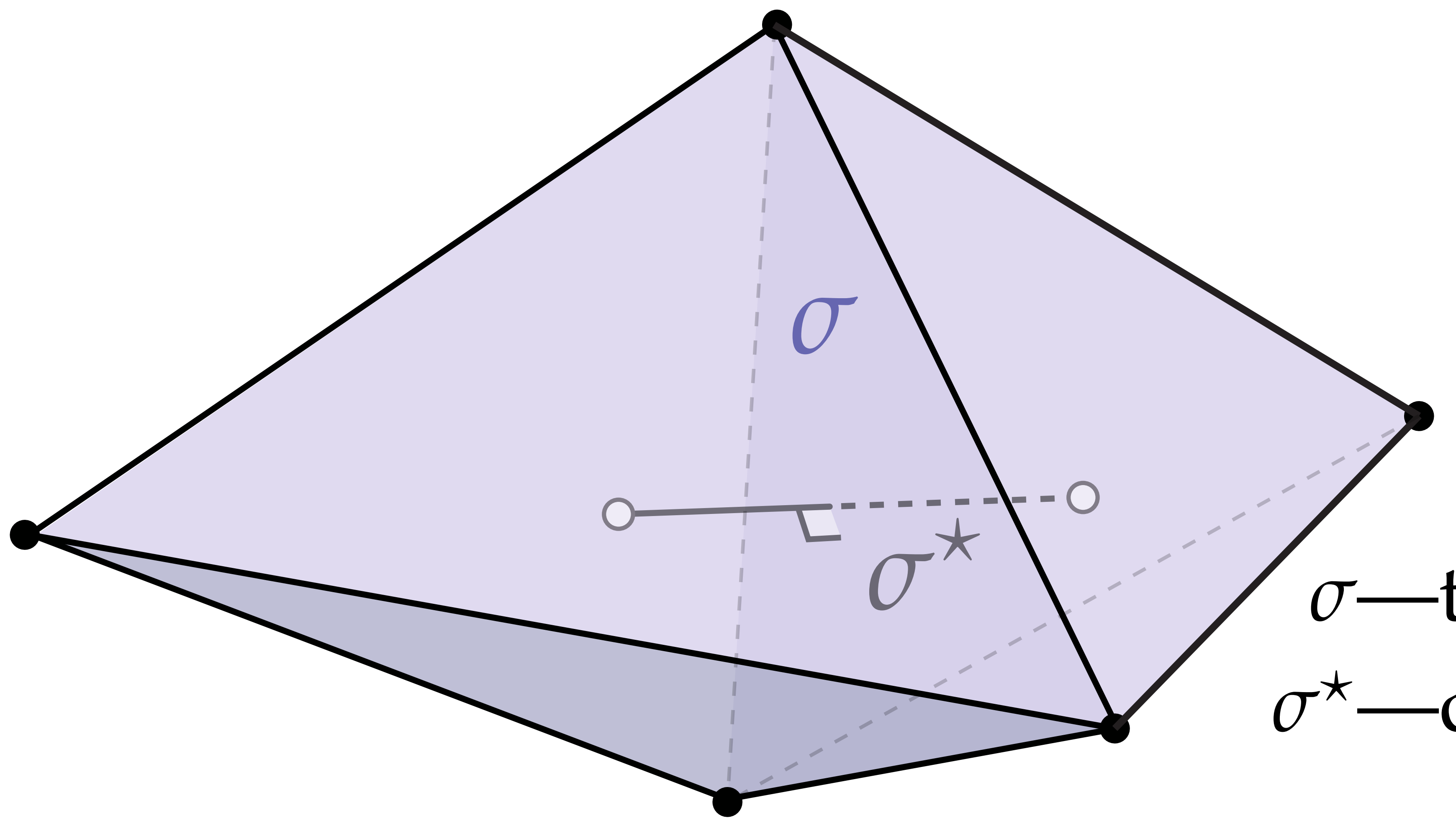
Diagonal Hodge Star

Definition. Let Ω_k and $\Omega^\star_{n-k}$ denote the primal k-forms and dual $(n-k)$ forms (respectively on an n-dimensional simplicial manifold M. The *diagonal Hodge star* is a map $\star : \Omega_k \to \Omega^\star_{n-k}$ determined by

$$\star\alpha(\sigma^\star) = \frac{|\sigma^\star|}{|\sigma|}\alpha(\sigma)$$

for each k-simplex σ in M, where $\sigma^\star$ is the corresponding dual cell, and $|\cdot|$ denotes the volume of a simplex or cell.

Key idea: divide by primal area, multiply by dual area. (Why?)



σ—triangle
$\sigma^\star$—dual edge

Matrix Representation of Diagonal Hodge Star

- Since the diagonal Hodge star on *k*-forms simply multiples each discrete k-form value by a constant (the volume ratio), it can be encoded via a *diagonal* matrix

$$\star_k := \begin{bmatrix} \frac{|\sigma_1^\star|}{|\sigma_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{|\sigma_N^\star|}{|\sigma_N|} \end{bmatrix} \in \mathbb{R}^{N \times N}$$

$\sigma_1, \ldots, \sigma_N$ — k-simplices in the primal mesh

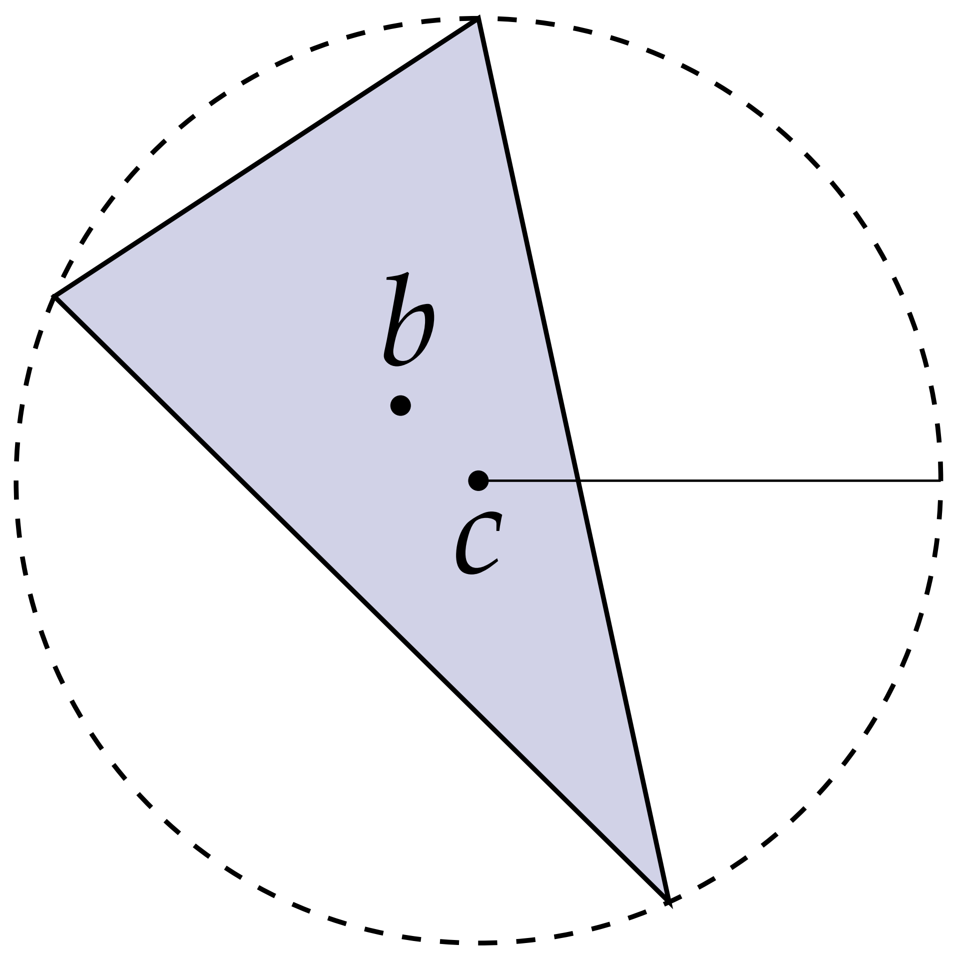
$\sigma_1^\star, \ldots, \sigma_N^\star$ — $(n-k)$-cells in the dual mesh

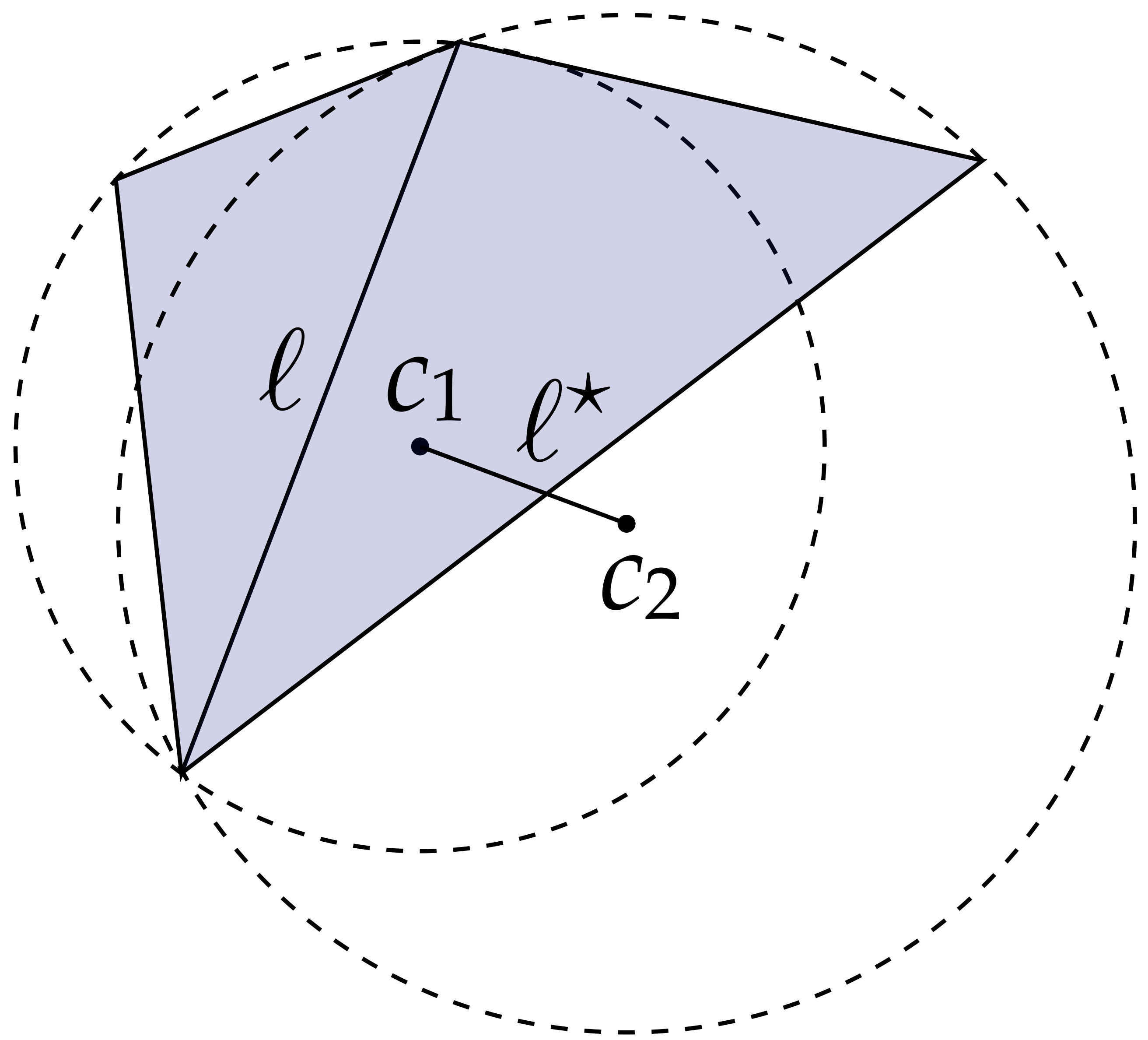
$|\cdot|$ — volume of a simplex or cell

$\star_k \in \mathbb{R}^{N \times N}$ — matrix for Hodge star on primal k-forms

Geometry of Dual Complex

- For exterior derivative, needed only *connectivity* of the dual cells
- For Hodge star, also need a specific *geometry*
- Many possibilities for location of dual vertices:
  - **circumcenter** (c) — center of sphere touching all vertices
    - most typical choice
    - pros: primal & dual are orthogonal (greater accuracy)
    - cons: can yield, e.g., negative lengths/areas/volumes…
  - **barycenter** (b) — average of all vertex coordinates
    - pros: dual volumes are always positive
    - cons: primal & dual not orthogonal (lower accuracy)





Possible Choices for Discrete Hodge Star

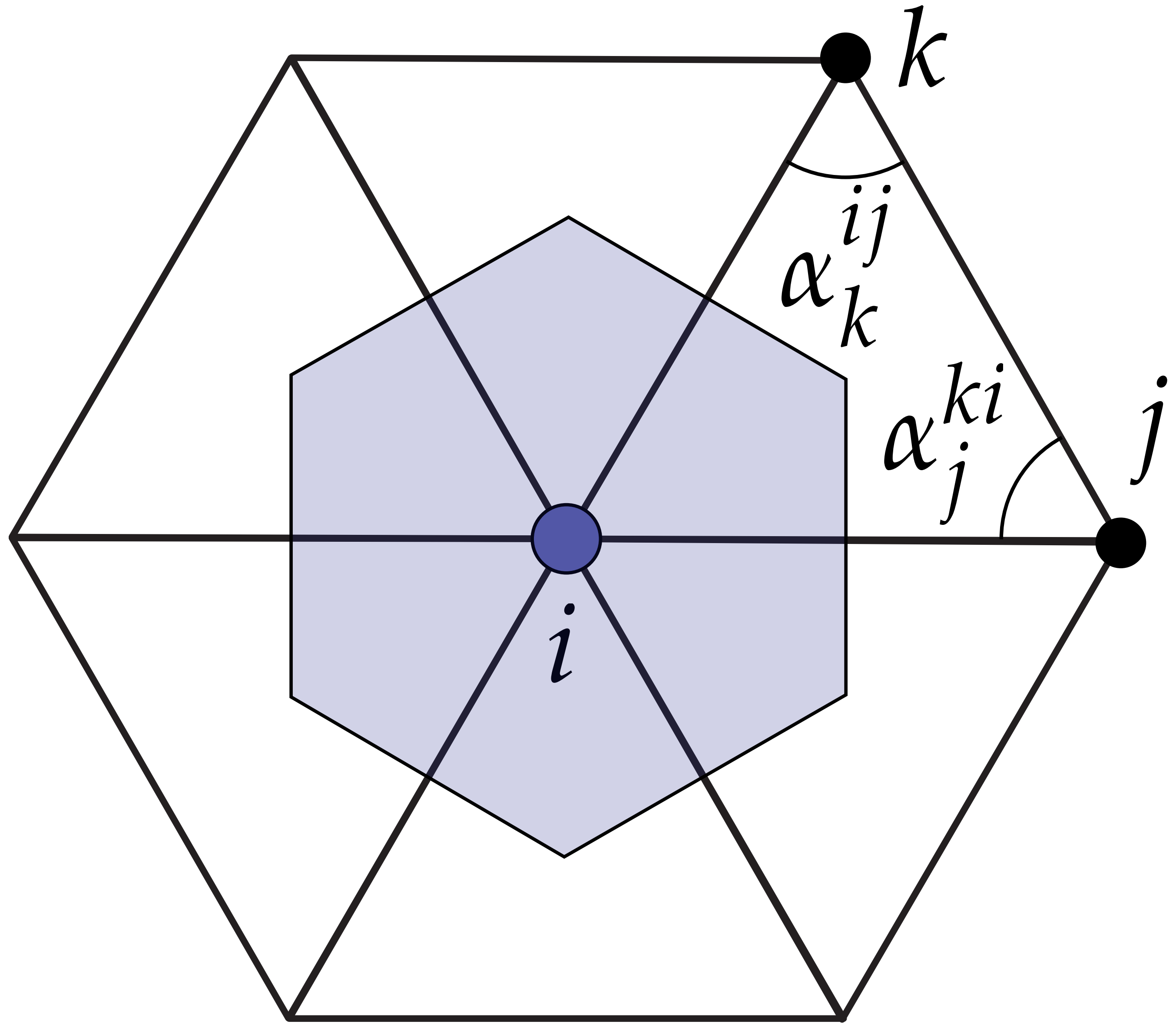
- Many choices—*none* give exact results!
- **Volume ratio**
  - diagonal matrix; most typical choice in DEC (Hirani, Desbrun et al)
    - typical choice: circumcentric dual (Delaunay/Voronoi)
    - more general orthogonal dual (weighted triangulation/power diagram)
    - can also use barycentric dual (e.g., Auchmann & Kurz, Alexa & Wardetzky)
- **Galerkin Hodge star**
  - L_2 norm on Whitney forms
    - non-diagonal, but still sparse; standard in, e.g., FEEC (Arnold et al).
    - appropriate "mass lumping" again yields circumcentric Hodge star

(Thanks: Fernando de Goes)

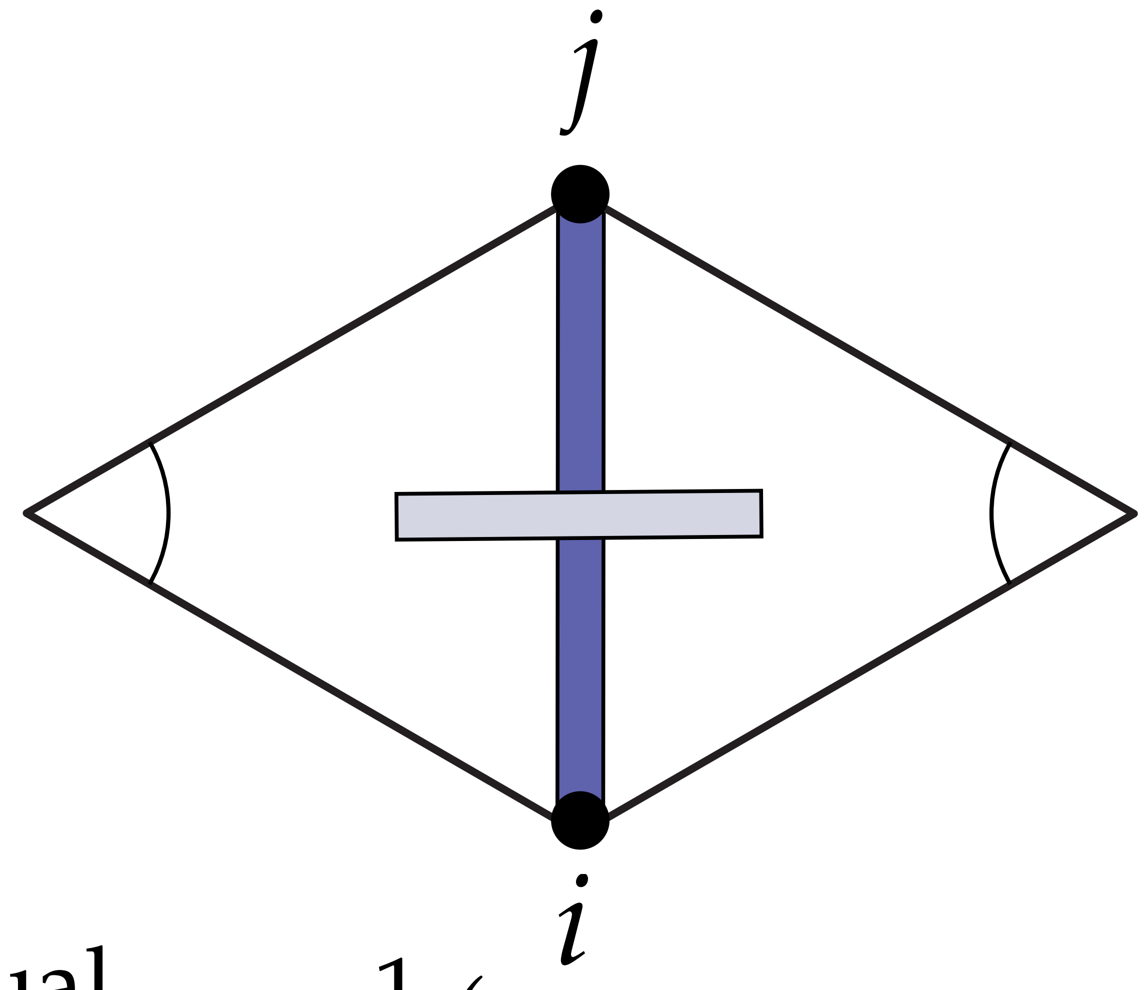
Computing Volumes

- Evaluating the Hodge star boils down to computing ratios of dual/primal volumes
- These ratios often have simple closed-form expressions (*don't compute circumcenters!*)

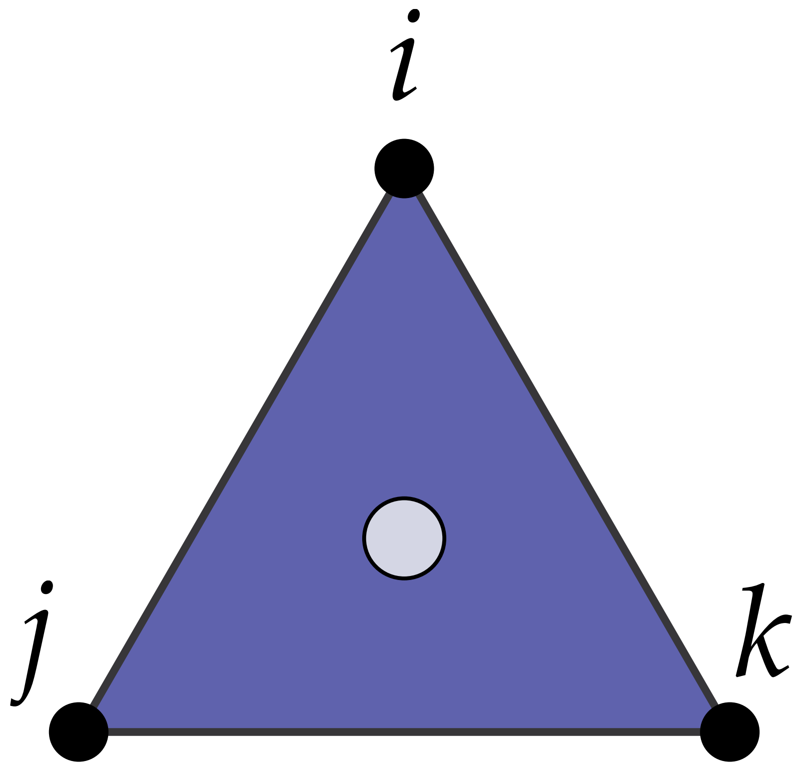
<u>Example: 2D circumcentric dual</u>



$$\frac{A_{\text{dual}}}{1} = \frac{1}{8}\sum_{ijk\in F}\left(\ell_{ij}^2 \cot\alpha_k^{jk} + \ell_{ik}^2 \cot\alpha_j^{ki}\right)$$



$$\frac{\ell_{\text{dual}}}{\ell_{\text{primal}}} = \frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij})$$



$$\frac{1}{A_{ijk}} = \frac{1}{\sqrt{s(s-\ell_{ij})(s-\ell_{jk})(s-\ell_{ki})}}$$

$$s = \frac{1}{2}(\ell_{ij} + \ell_{jk} + \ell_{ki})$$

Summary

Discrete Exterior Calculus—Basic Operators

- Basic operators can be summarized in a very useful diagram (here in 2D):

$$
\begin{array}{ccccc}
\Omega_0 & \xrightarrow{d_0} & \Omega_1 & \xrightarrow{d_1} & \Omega_2 \\
\star_0 \big\downarrow & & \star_1 \big\downarrow & & \star_2 \big\downarrow \\
\Omega_2^\star & \xleftarrow[d_0^\mathsf{T}]{} & \Omega_1^\star & \xleftarrow[d_1^\mathsf{T}]{} & \Omega_0^\star
\end{array}
$$

Ω_k — primal k-forms

$\Omega_k^\star$ — dual k-forms

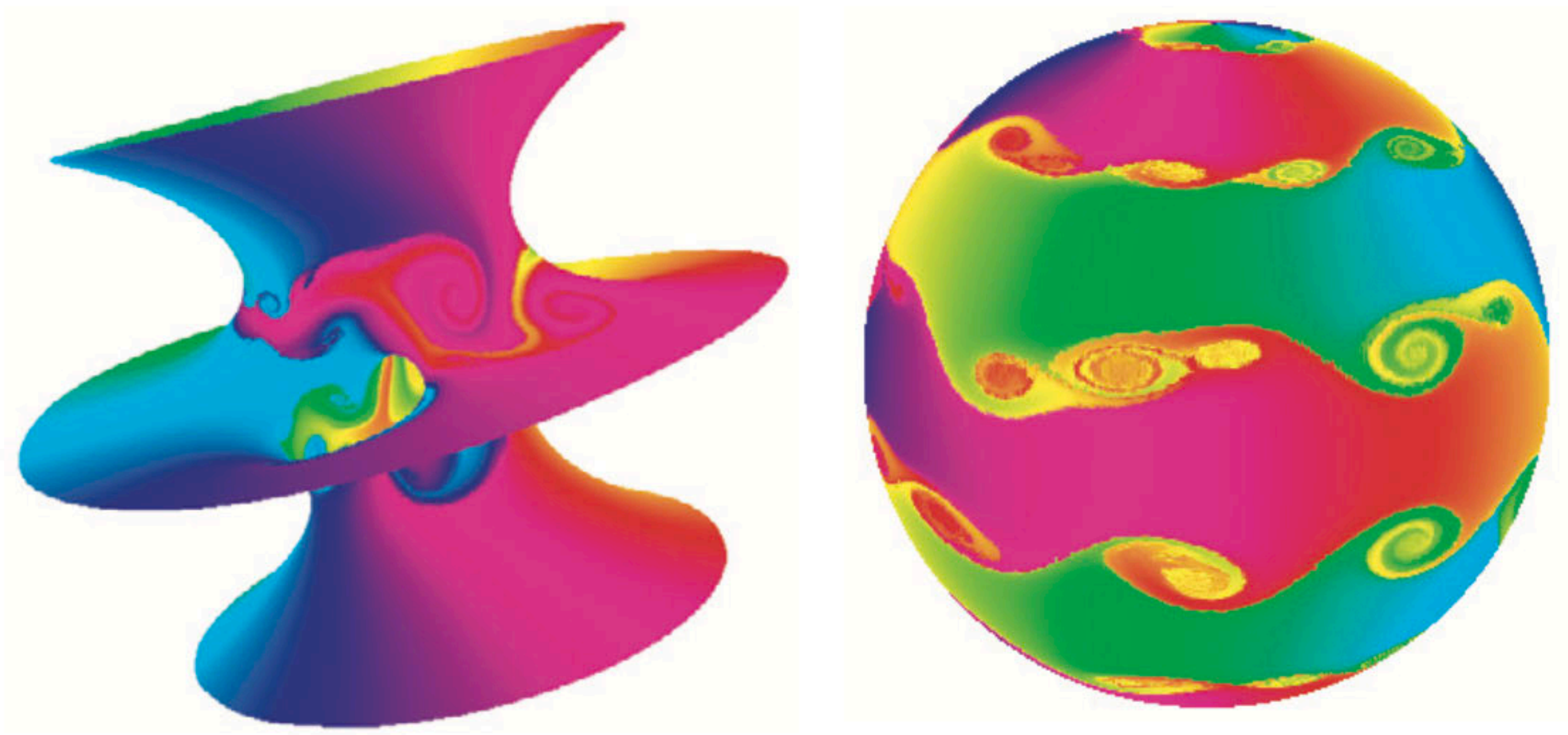
Composition of Operators

- By composing matrices, we can easily solve equations involving operators like those from vector calculus (grad, curl, div, Laplacian…) but in much greater generality (e.g., curved surfaces, k-forms…) and on complicated domains (meshes)

$$\text{grad} \longrightarrow d_0 \qquad \text{curl} \longrightarrow \star_2 d_1$$

$$\text{div} \longrightarrow \star_0^{-1} d_0^T \star_1$$

$$\Delta \longrightarrow \star_0^{-1} d_0^T \star_1 d_0$$

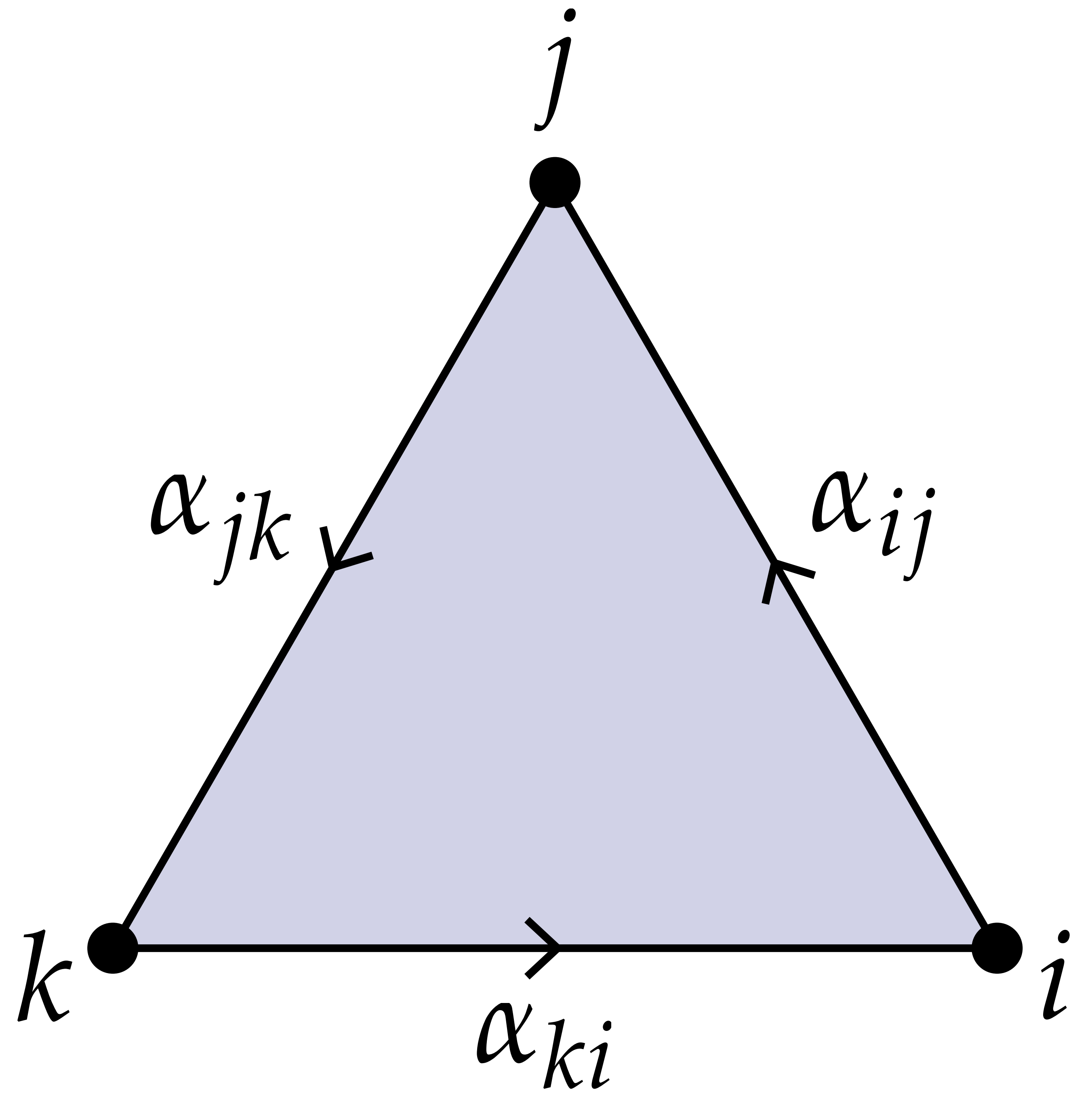


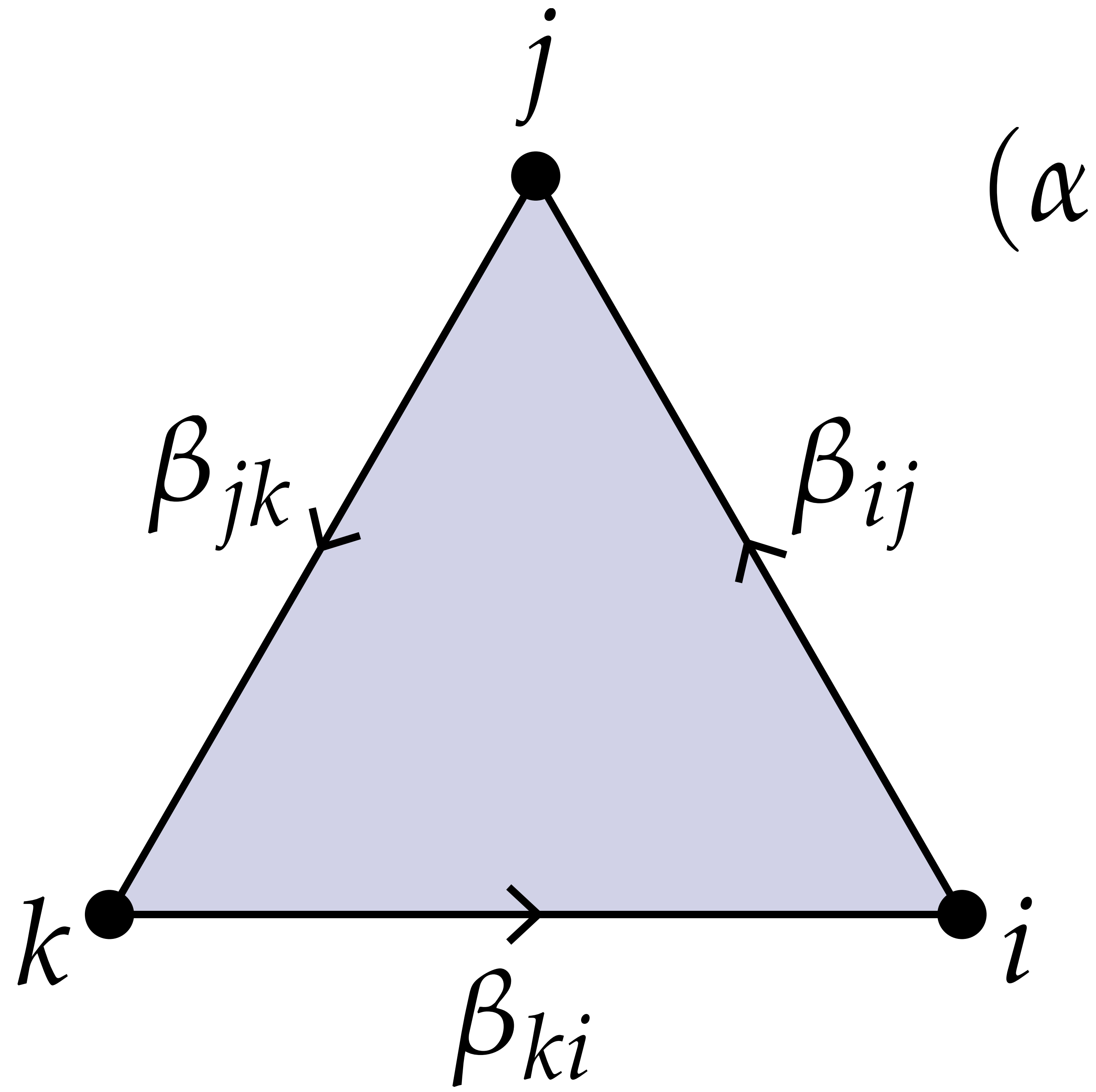
$$\Delta_k \longrightarrow d_{k-1} \star_{k-1}^{-1} d_{k-1}^T \star_k + \star_k^{-1} d_k^T \star_{k+1} d_k$$

Basic recipe: load a mesh, build a few basic matrices, solve a linear system.

Other Discrete Operators

- Many other operators in exterior calculus (wedge, sharp, flat, Lie derivative, …)
- E.g., wedge product on two discrete 1-forms:



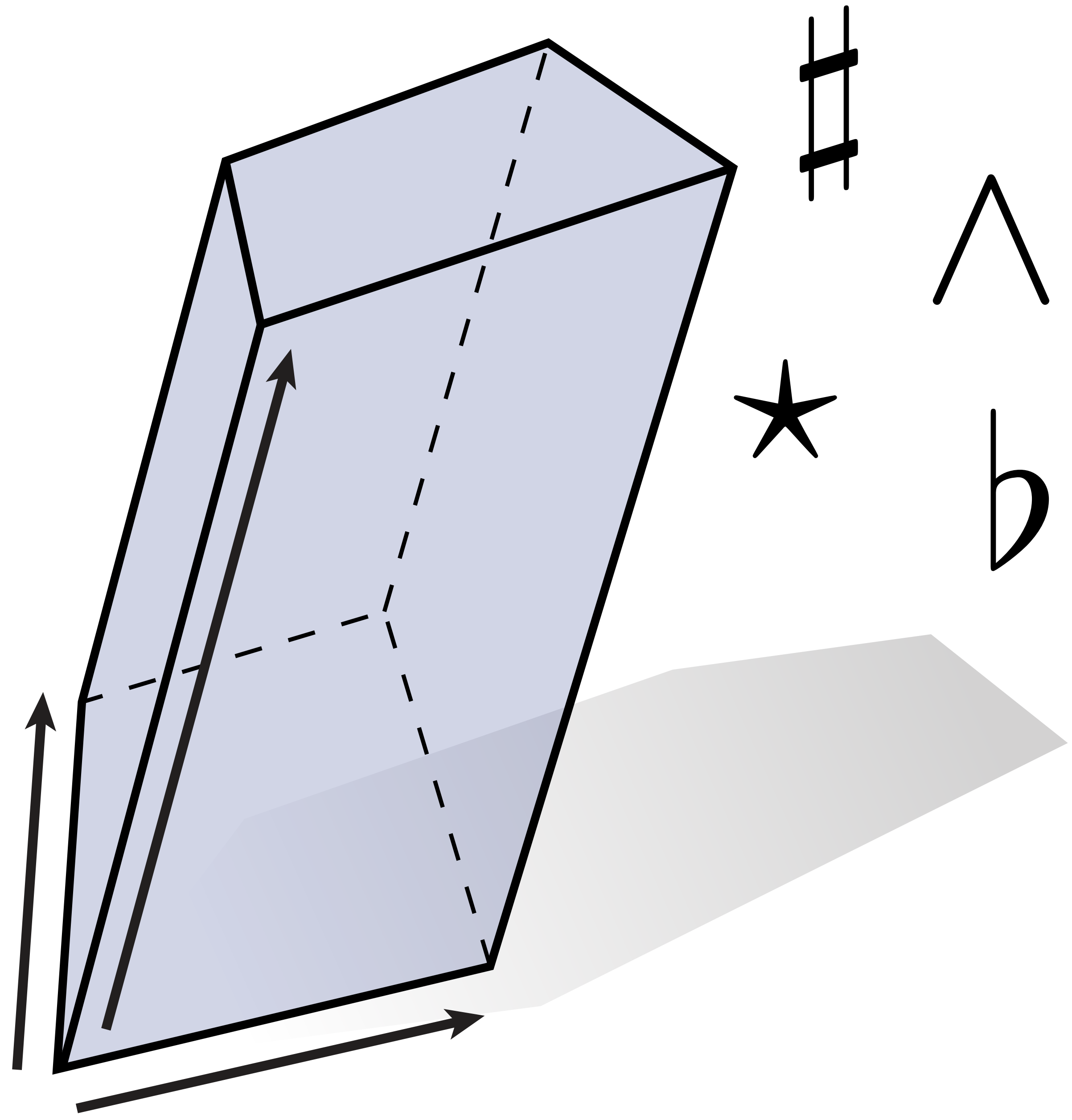


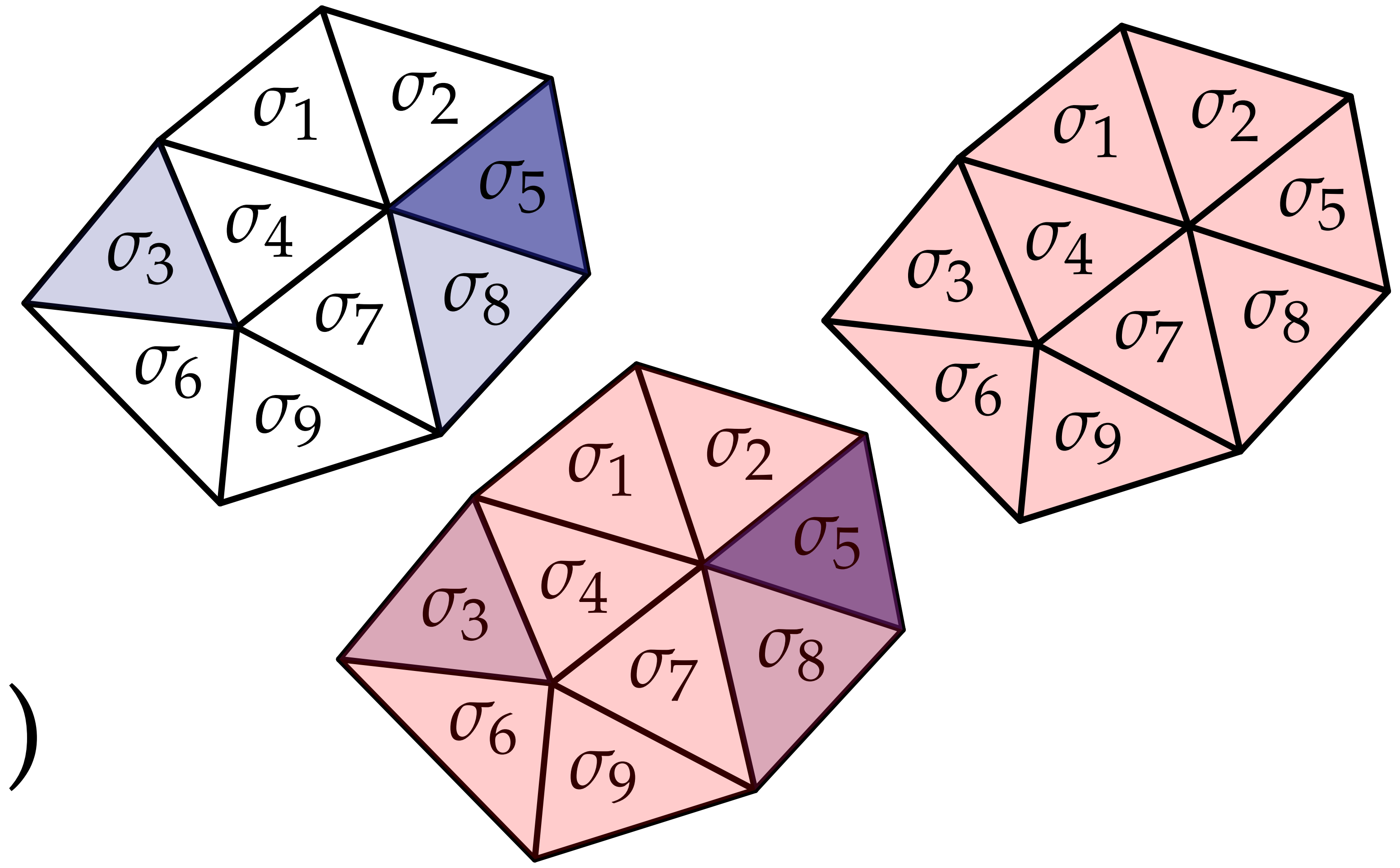
$$(\alpha \wedge \beta)_{ijk} := \frac{1}{6}\big(\alpha_{ij}\beta_{jk} - \alpha_{jk}\beta_{ij} + \alpha_{jk}\beta_{ki} - \alpha_{ki}\beta_{jk} + \alpha_{ki}\beta_{ij} - \alpha_{ij}\beta_{ki} \big)$$

(More broadly, many open questions about how to discretize exterior calculus…)

Discrete Exterior Calculus - Summary

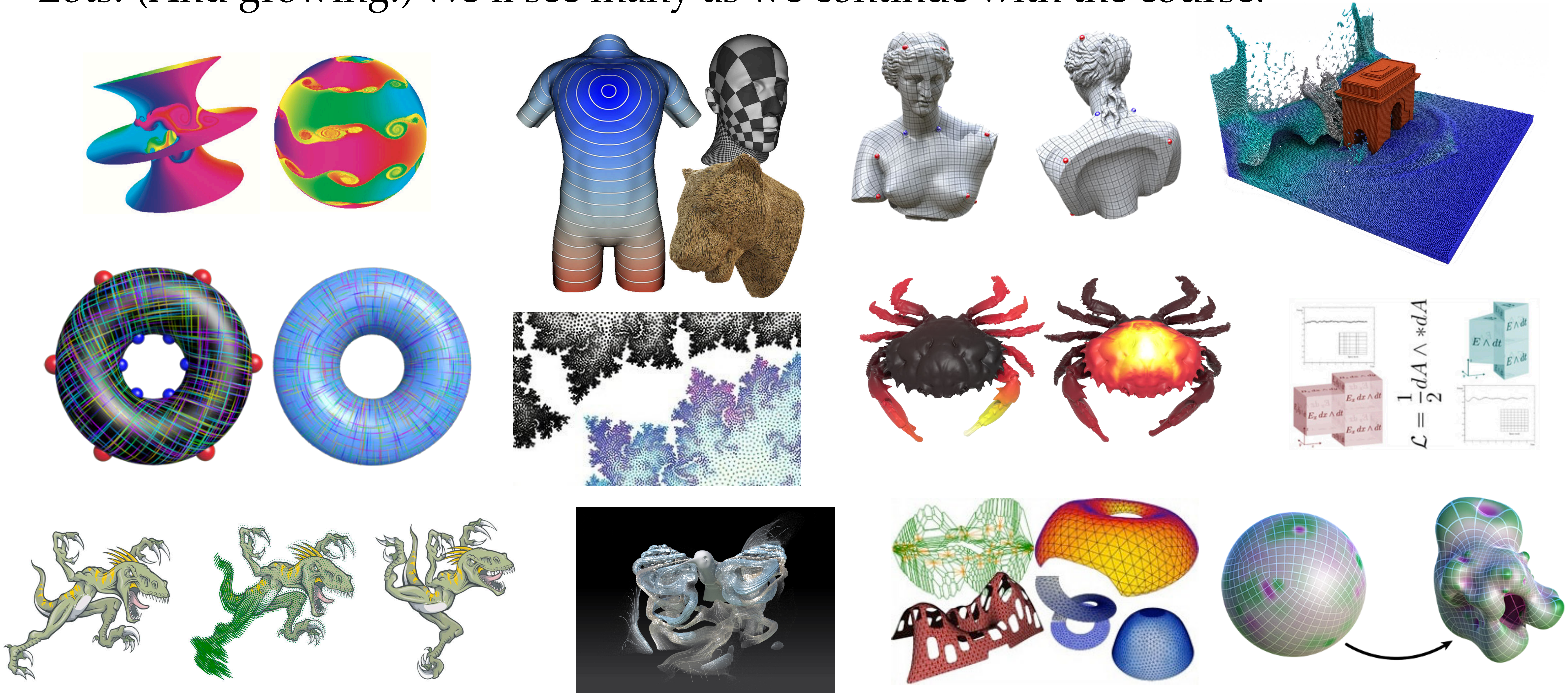
- integrate k-form over k-simplices
  - result is *discrete* k-form
  - sign changes according to orientation
- can also integrate over dual elements (*dual* forms)
- Hodge star converts between primal and dual (*approximately*!)
  - multiply by ratio of dual/primal volume
- discrete exterior derivative is just a sum
  - gives *exact* value (via Stokes' theorem)
- *Still plenty missing!* (Wedge, sharp, flat, Lie derivative, ...)



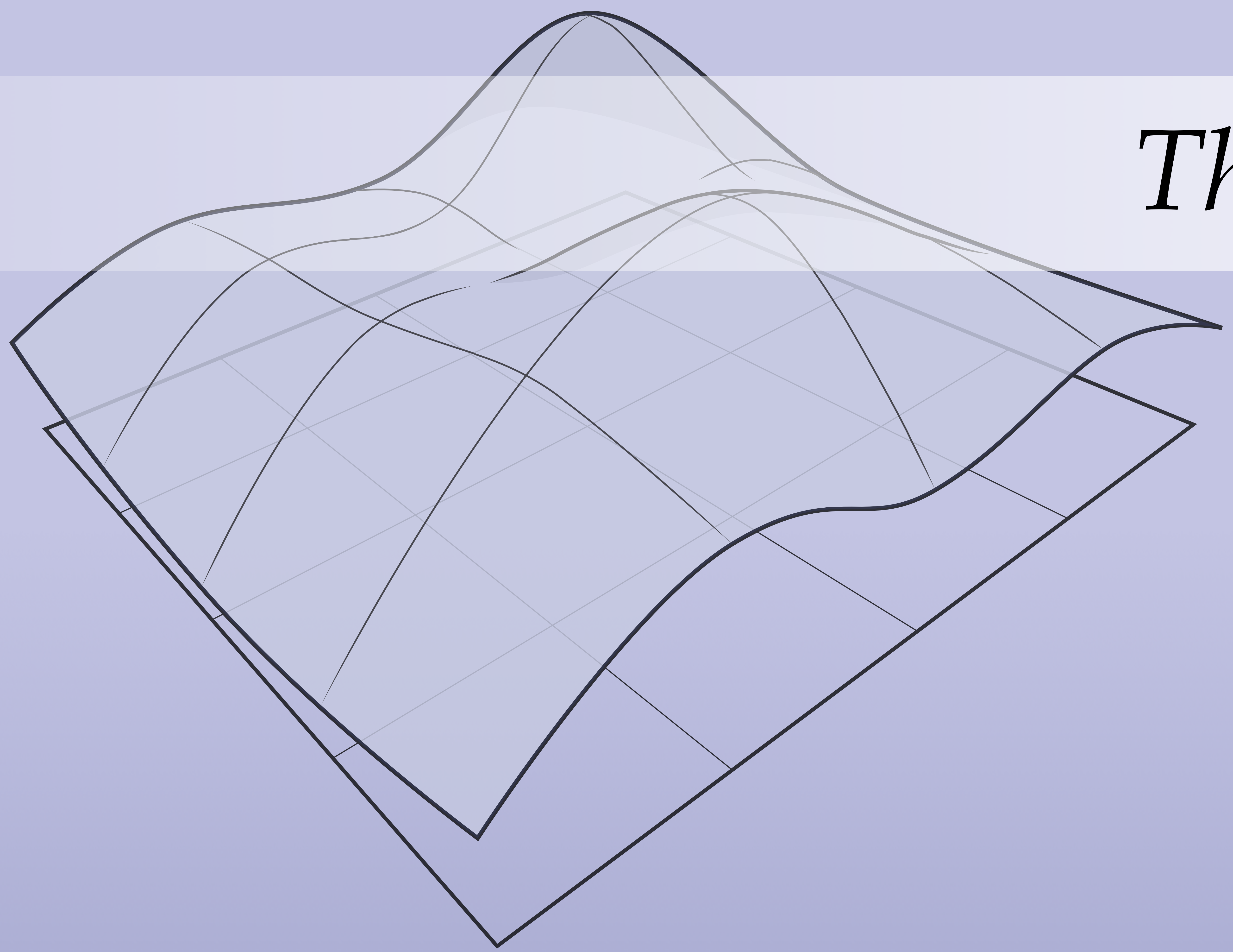


Applications

- Lots! (And growing.) We'll see many as we continue with the course.



Thanks!



Discrete Differential Geometry
An Applied Introduction