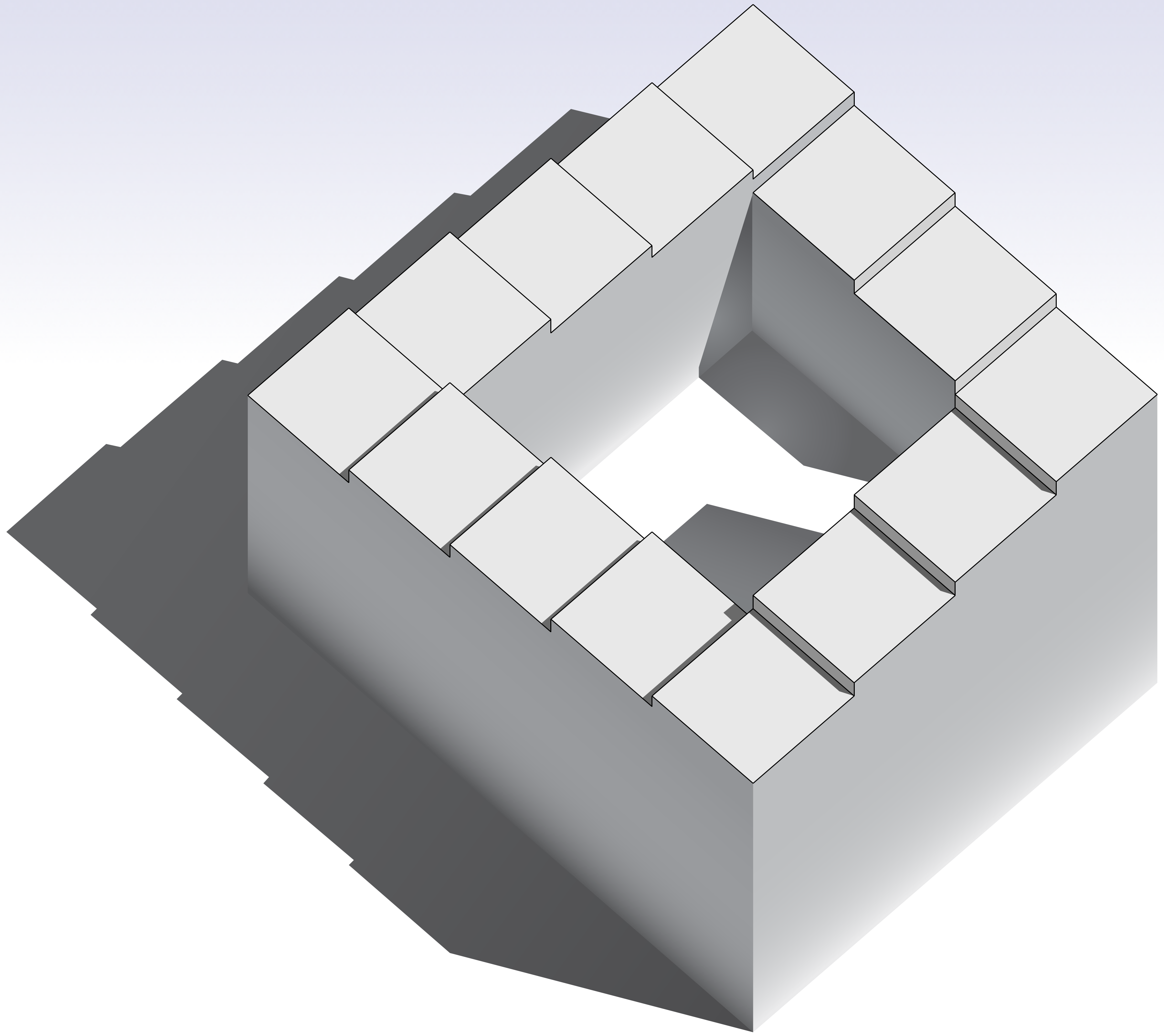
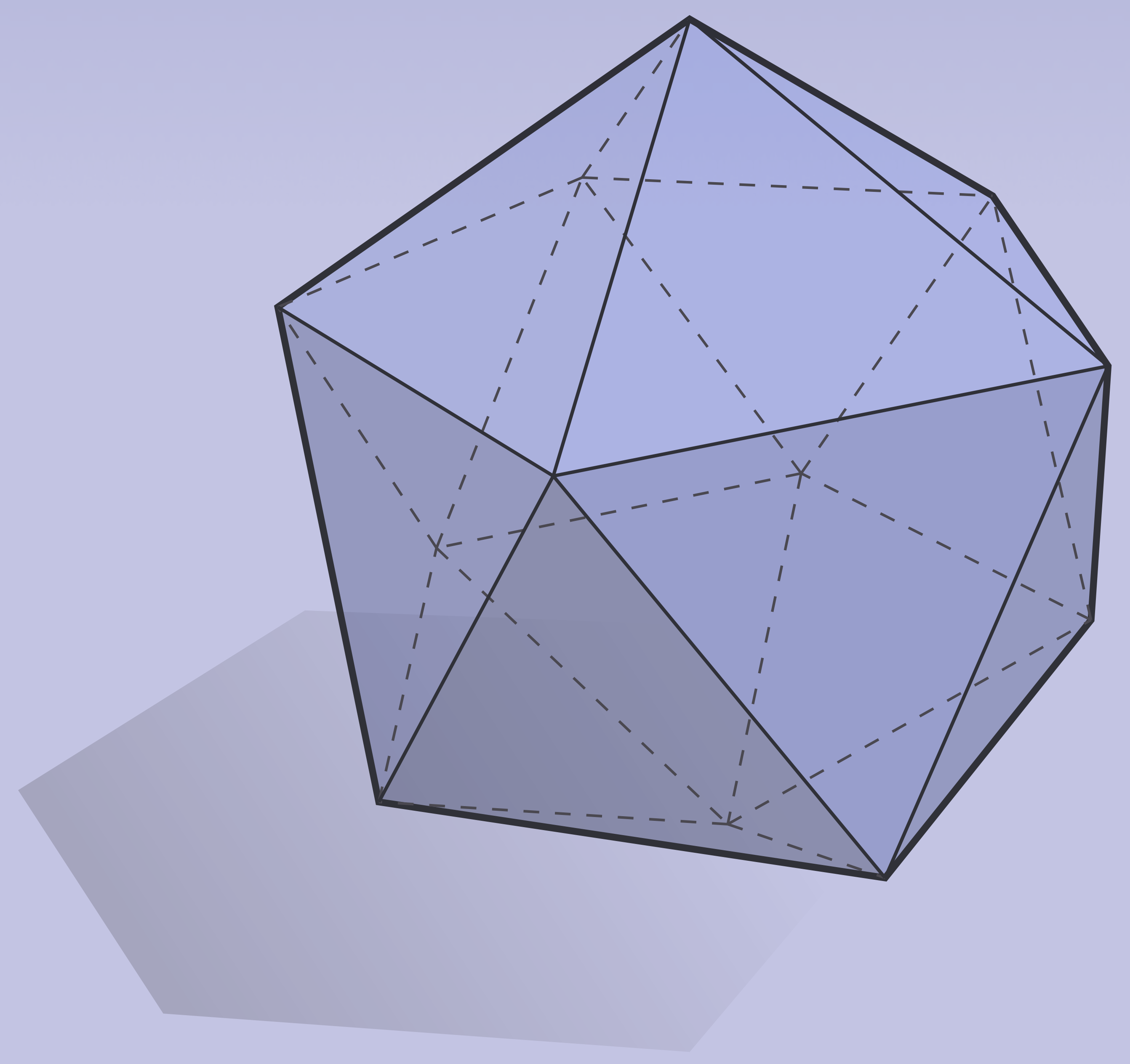


Discrete Differential Geometry: An Applied Introduction

CMU 15-458/858 • Keenan Crane

LECTURE 21:
GEODESICS

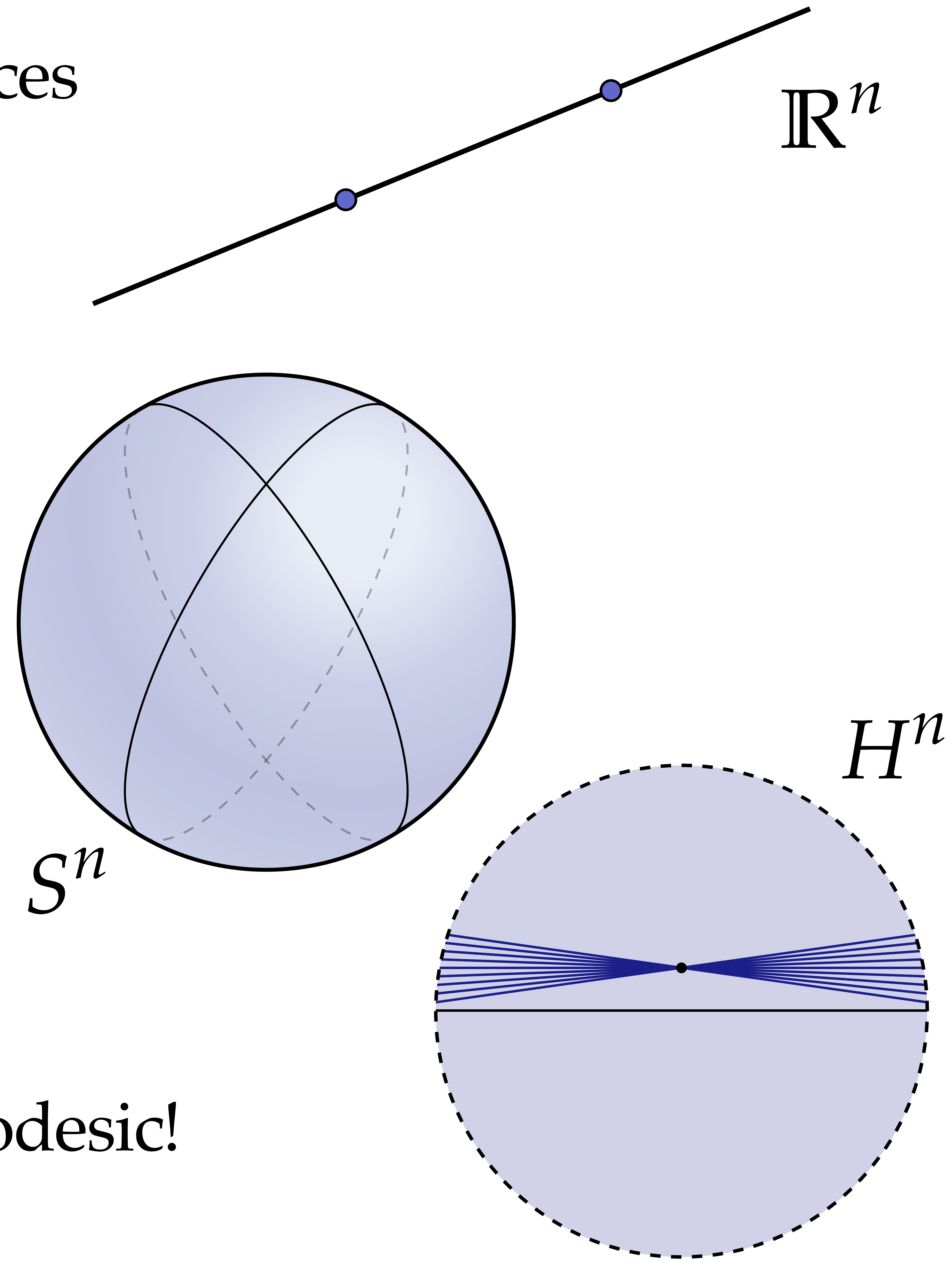


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

CMU 15-458/858 • Keenan Crane

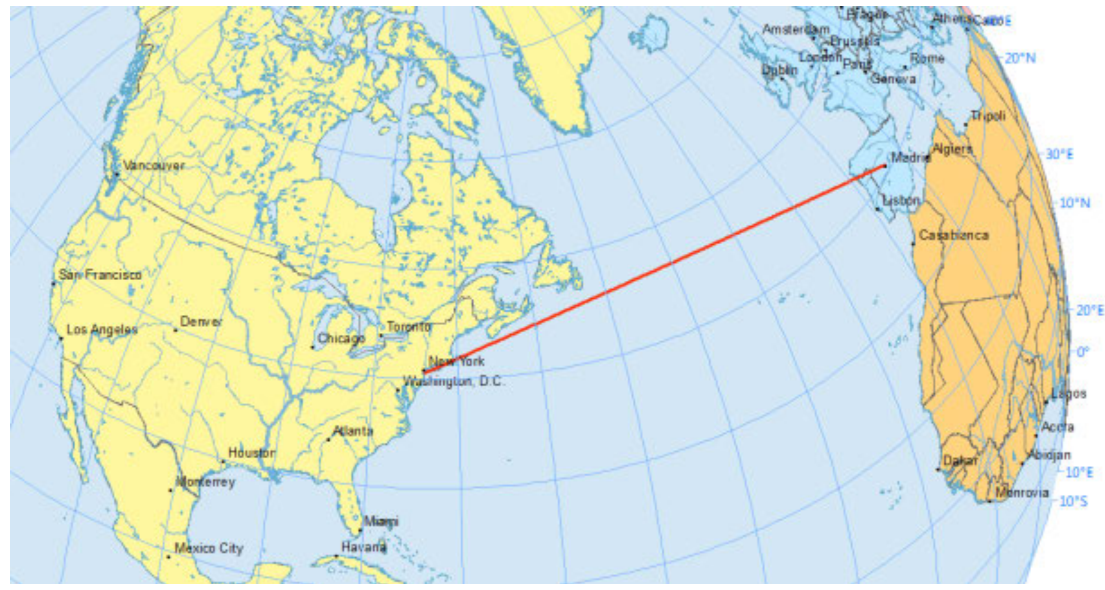
Geodesics — Overview

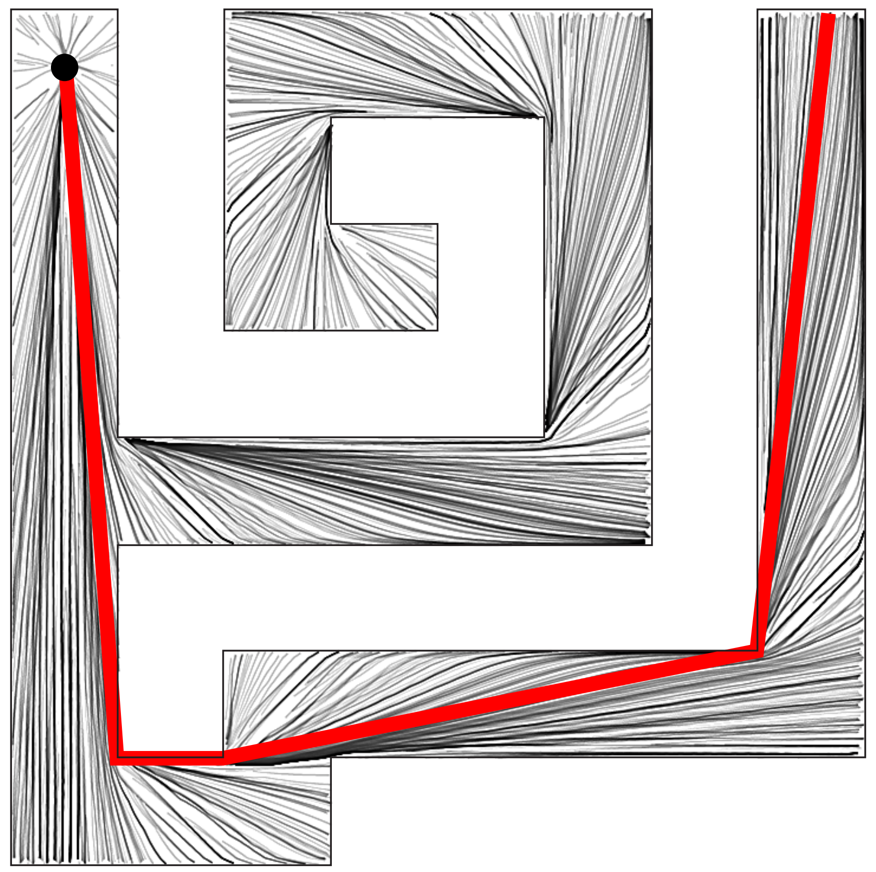
- Geodesics generalize the notion of a "line" to curved spaces
- Two basic features:
  1. **straightest** — no curvature / acceleration
  2. **shortest** — (locally) minimize length
- Can have very different behavior from Euclidean lines!
  - No parallel lines (spherical)
  - Multiple parallel lines through a point (hyperbolic)
- Part of the "origin story" of differential geometry…
- Also important in physics: all of life is motion along a geodesic!

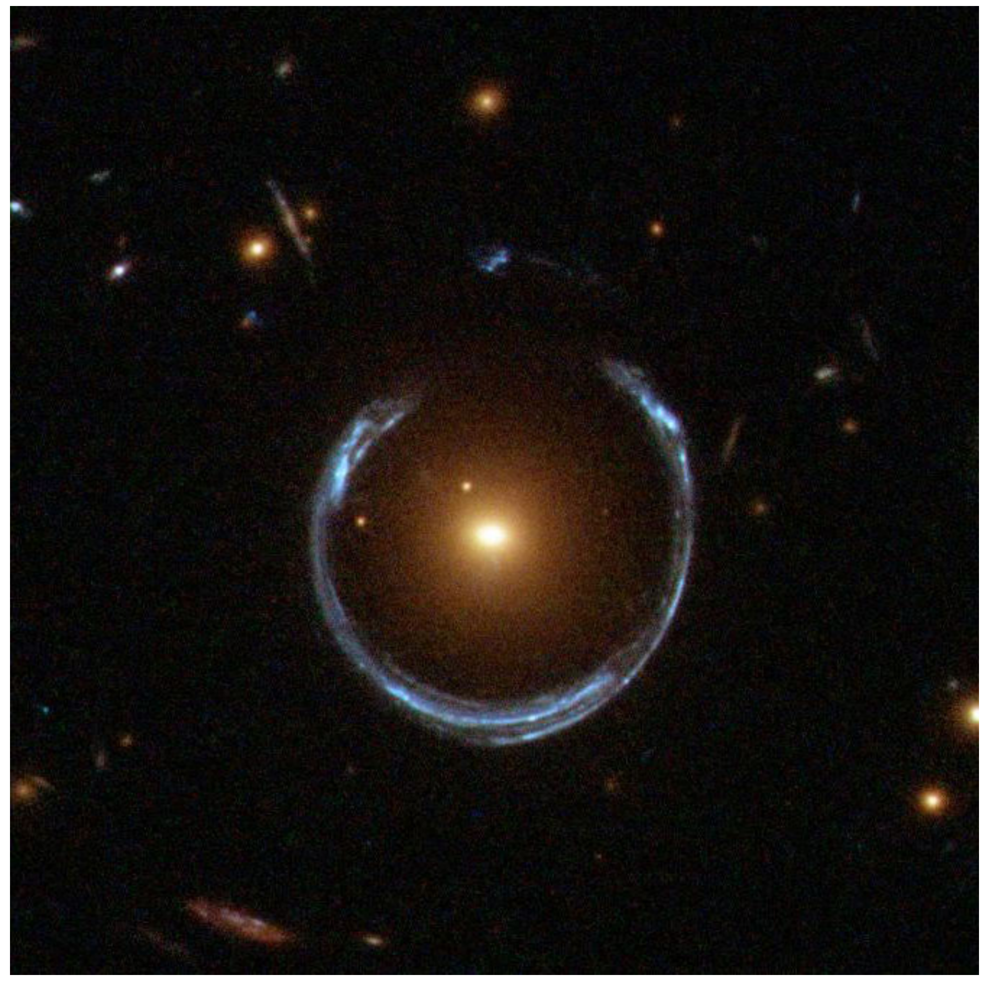


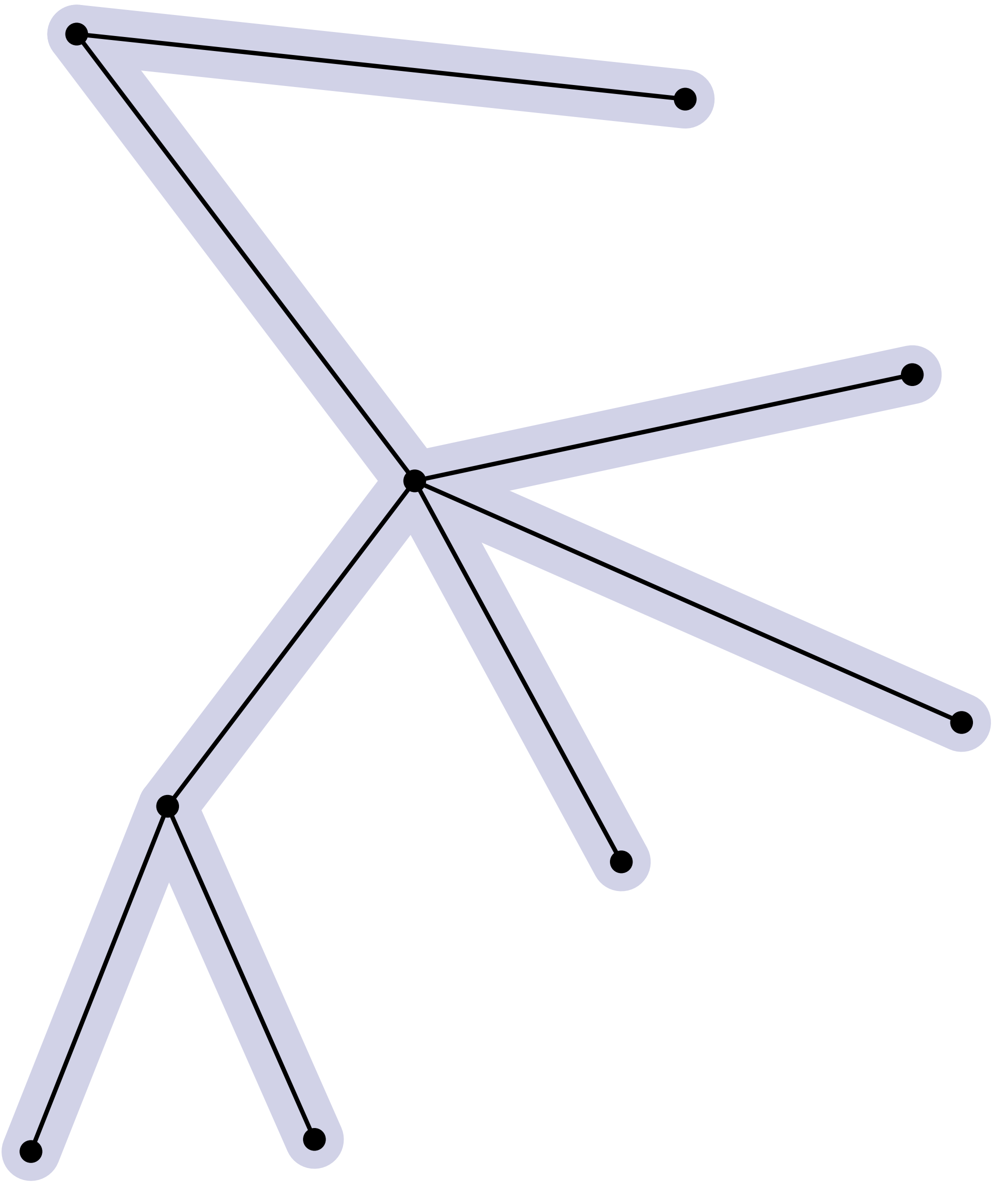
Examples of Geodesics

- Many familiar examples of geodesics:
  - straight line in the plane
  - great arc on circle (airplane trajectory)
  - shortest path in maze (path planning)
  - shortest path in *thickened* graph
  - light paths (gravitational lensing)



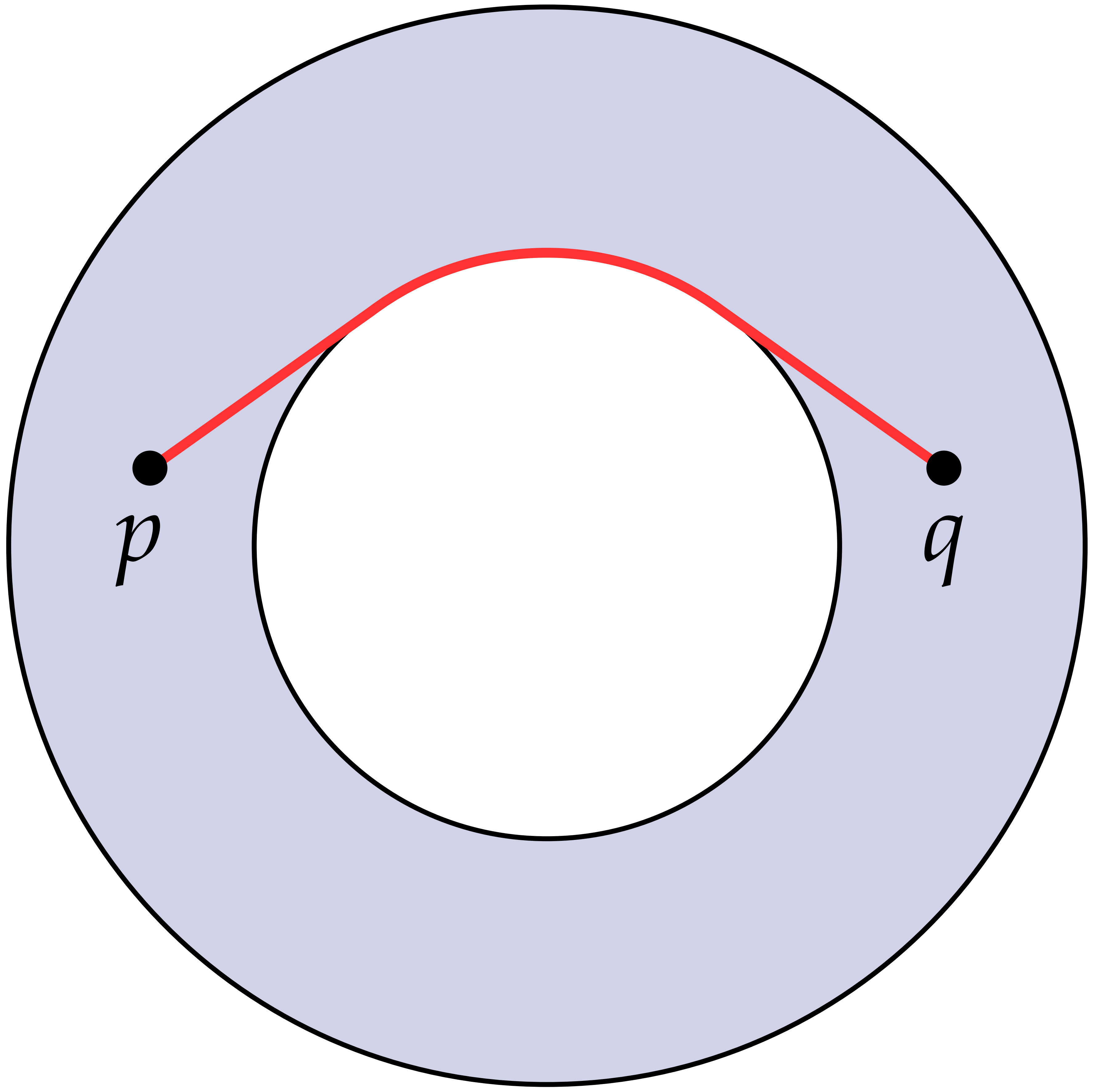






Aside: Geodesics on Domains with Boundary

- On domains with boundary, shortest path will not always be along a "straight" curve
- On the interior, path will still be both shortest & straightest
- May also "hug" pieces of the boundary (curvature will match boundary curvature, acceleration will match boundary normal)
- (For simplicity, we will mainly consider domains without boundary)

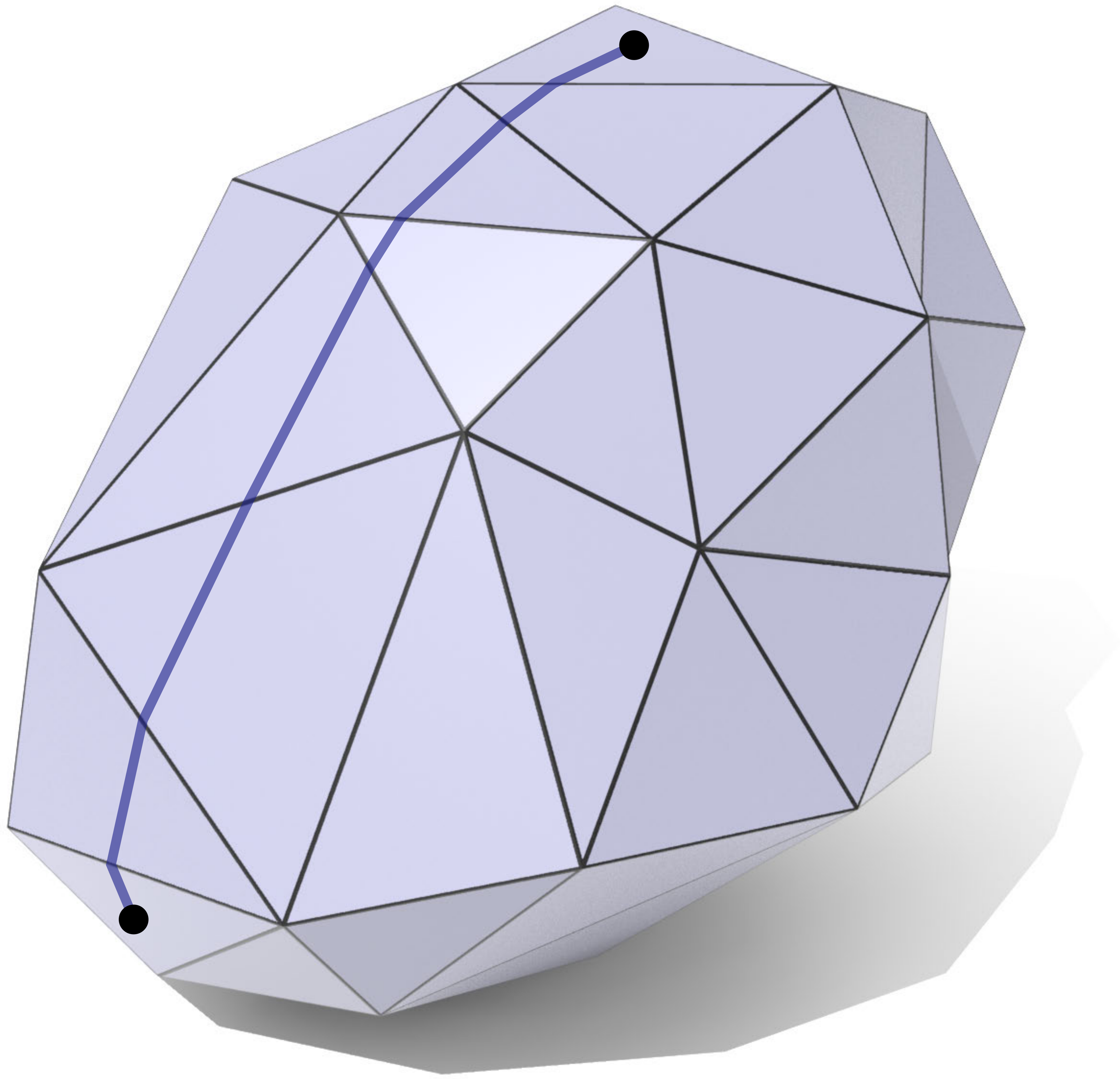


Isometry Invariance of Geodesics

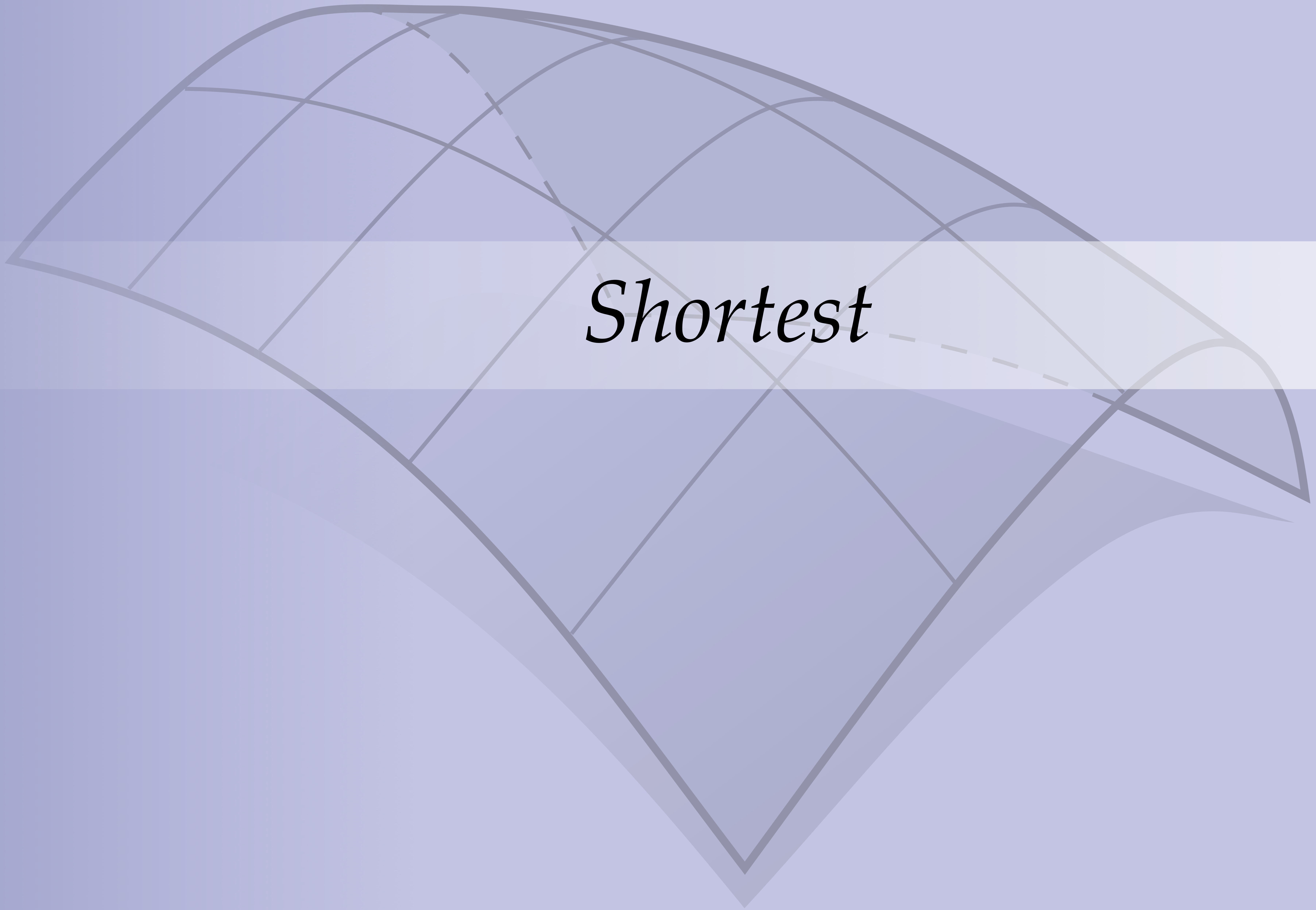
- *Isometries* are special deformations of curves, surfaces, etc., that don't change the "intrinsic" geometry, i.e., anything that can be measured using the Riemannian metric g
- For instance, rolling or folding up a map doesn't change the angle between tangent vectors pointing "north" and "south"
- Geodesics are also intrinsic: for instance, the shortest path between two cities will not change just because we roll up the map

Discrete Geodesics

- How can we approach a definition of *discrete* geodesics?
- Play "The Game" of DDG and consider different smooth starting points:
  - *zero acceleration*
  - *locally shortest*
  - *no geodesic curvature*
  - *harmonic map from interval to manifold*
  - *gradient of distance function*
  - …



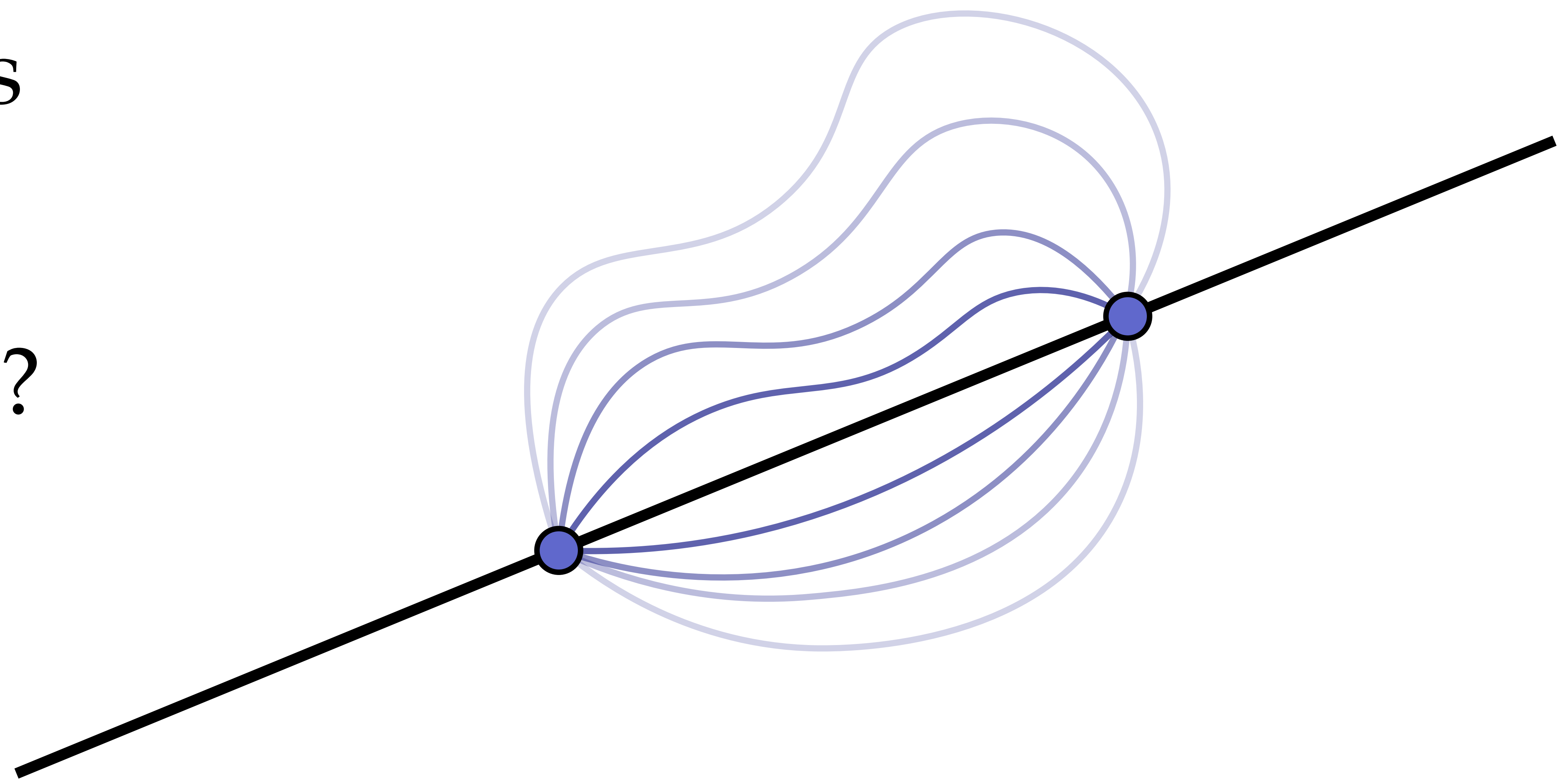
- Each starting point will have different consequences
- E.g., for simplicial surfaces will see that **shortest** and **straightest** disagree

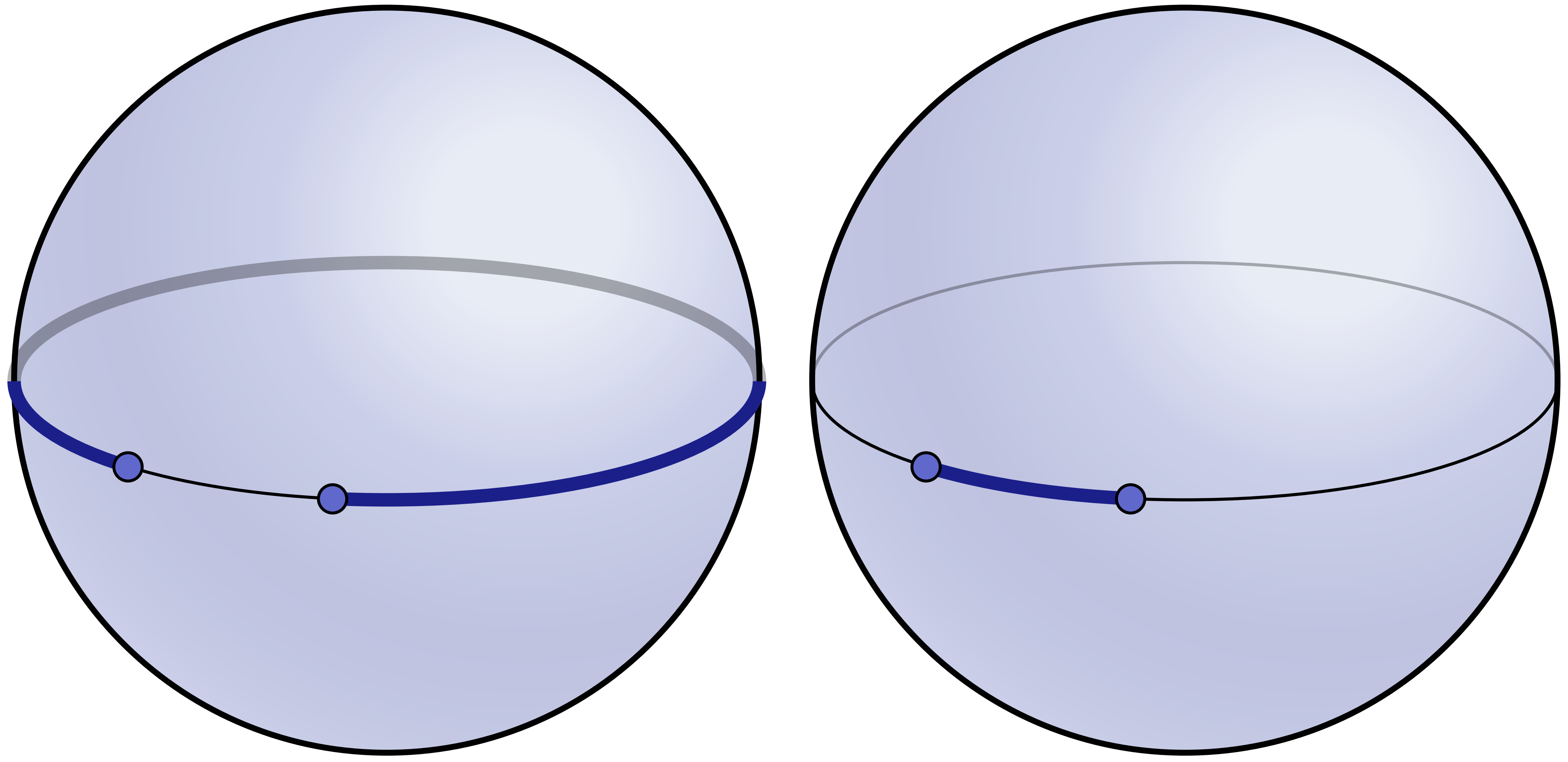


Shortest

Locally Shortest Paths

- A Euclidean line segment can be characterized as the shortest path between two distinct points
- How can we characterize a whole Euclidean line?
- Say that it's *locally shortest*: for any two "nearby" points on the path*, can't find a shorter route
- This description directly gives us one possible definition for (smooth) geodesics
- Note that *locally* shortest doesn't imply *globally* shortest! (But still critical points…)





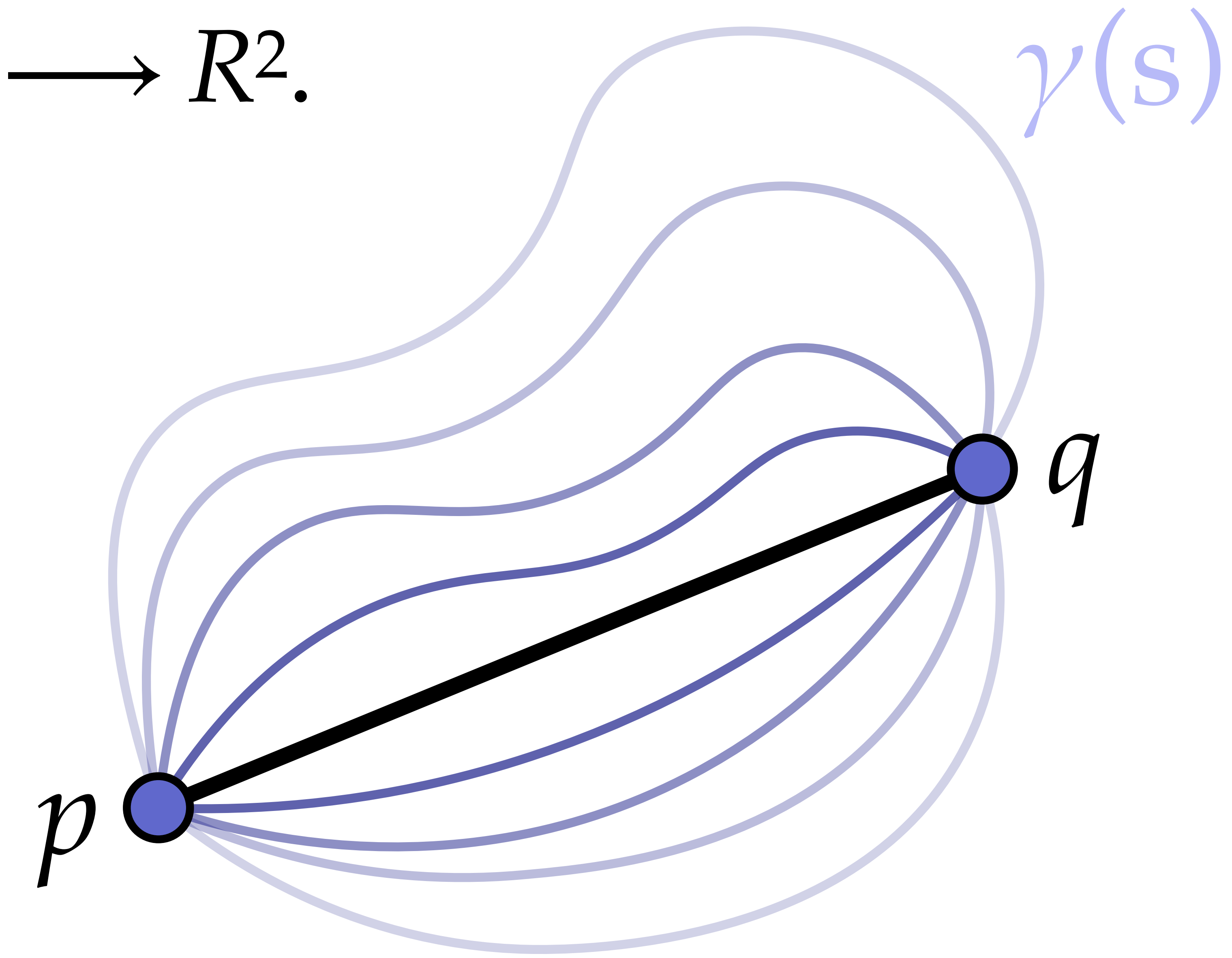
locally shortest **globally shortest**

*i.e., within the injectivity radius

Shortest Planar Curve—Variational Perspective

Consider an arc-length parameterized planar curve $\gamma(s)$: $[a,b] \longrightarrow R^2$.
Its squared length is given by the Dirichlet energy

$$L^2(\gamma) = \int_a^b |d\gamma|^2$$



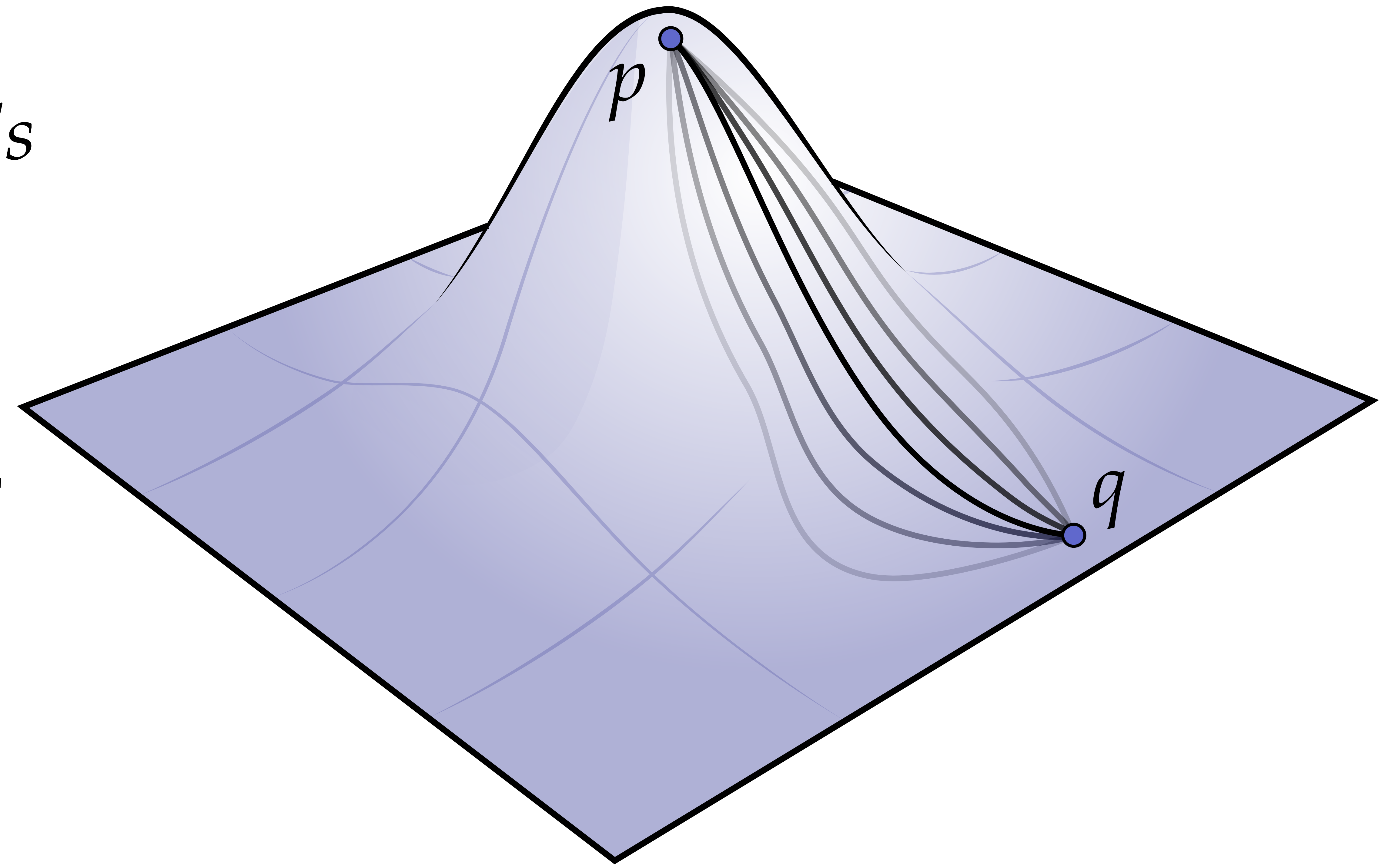
- We can get the shortest path between two points by minimizing this energy subject to fixed endpoints $\gamma(a) = p$ and $\gamma(b) = q$
- For planar curves, "setting the derivative to zero" yields a simple 1D Poisson equation.
- **Q:** What's the solution? Why does it make sense?

$$\begin{aligned} & \frac{d^2}{ds^2}\gamma(s) = 0 \\ \text{s.t.} \quad & \gamma(a) = p \\ & \gamma(b) = q \end{aligned}$$

Shortest Geodesic—Variational Perspective

- In exactly the same way, we can characterize geodesics on curved manifolds as length-minimizing paths
- E.g., let M be a surface with Riemannian metric g, and let γ: [a,b] $\longrightarrow$ M be an arc-length parameterized curve. Its squared length is again given by the *Dirichlet energy*

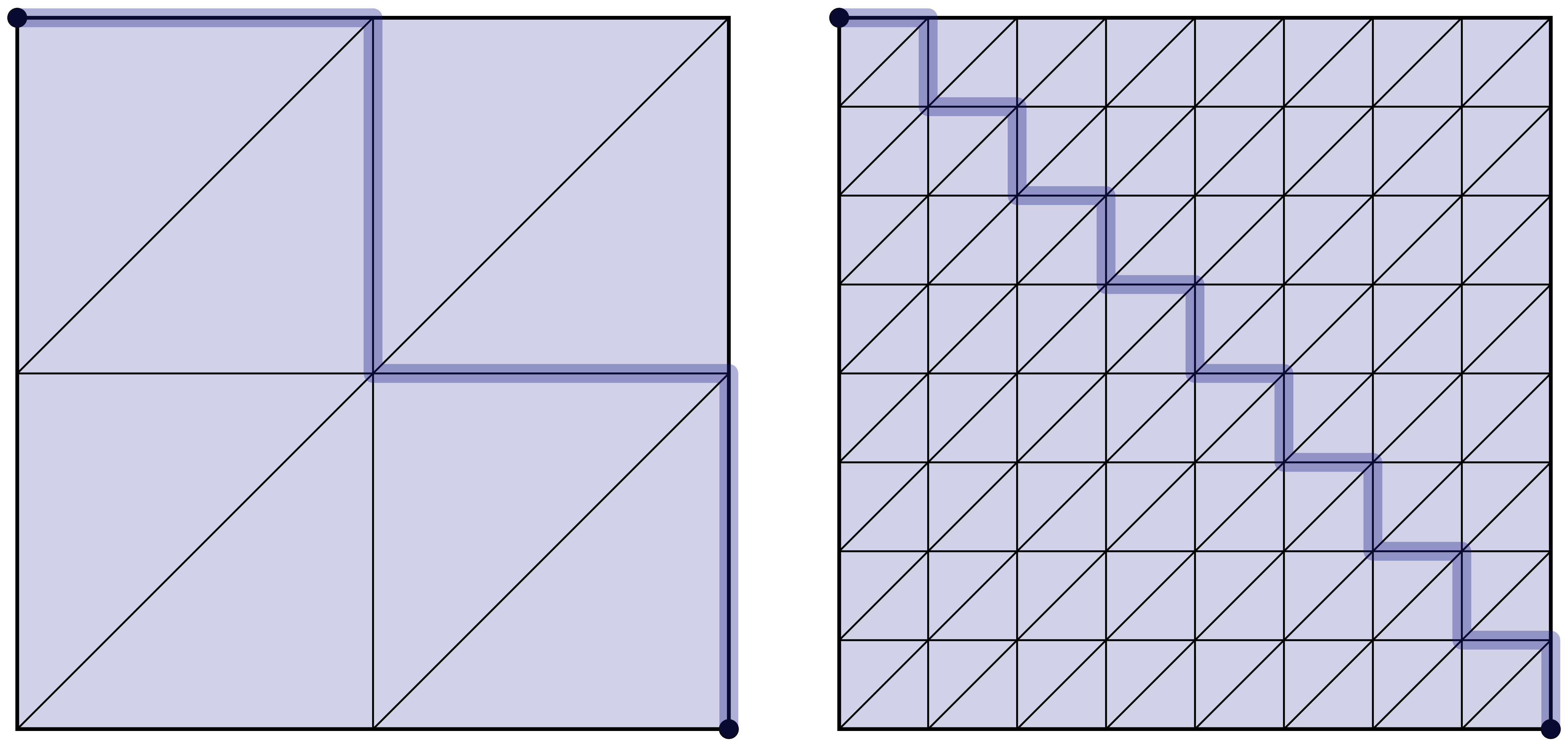
$$L(\gamma) := \int_a^b |d\gamma|^2 = \int_a^b g(d\gamma(\tfrac{d}{ds}), d\gamma(\tfrac{d}{ds}))\, ds$$

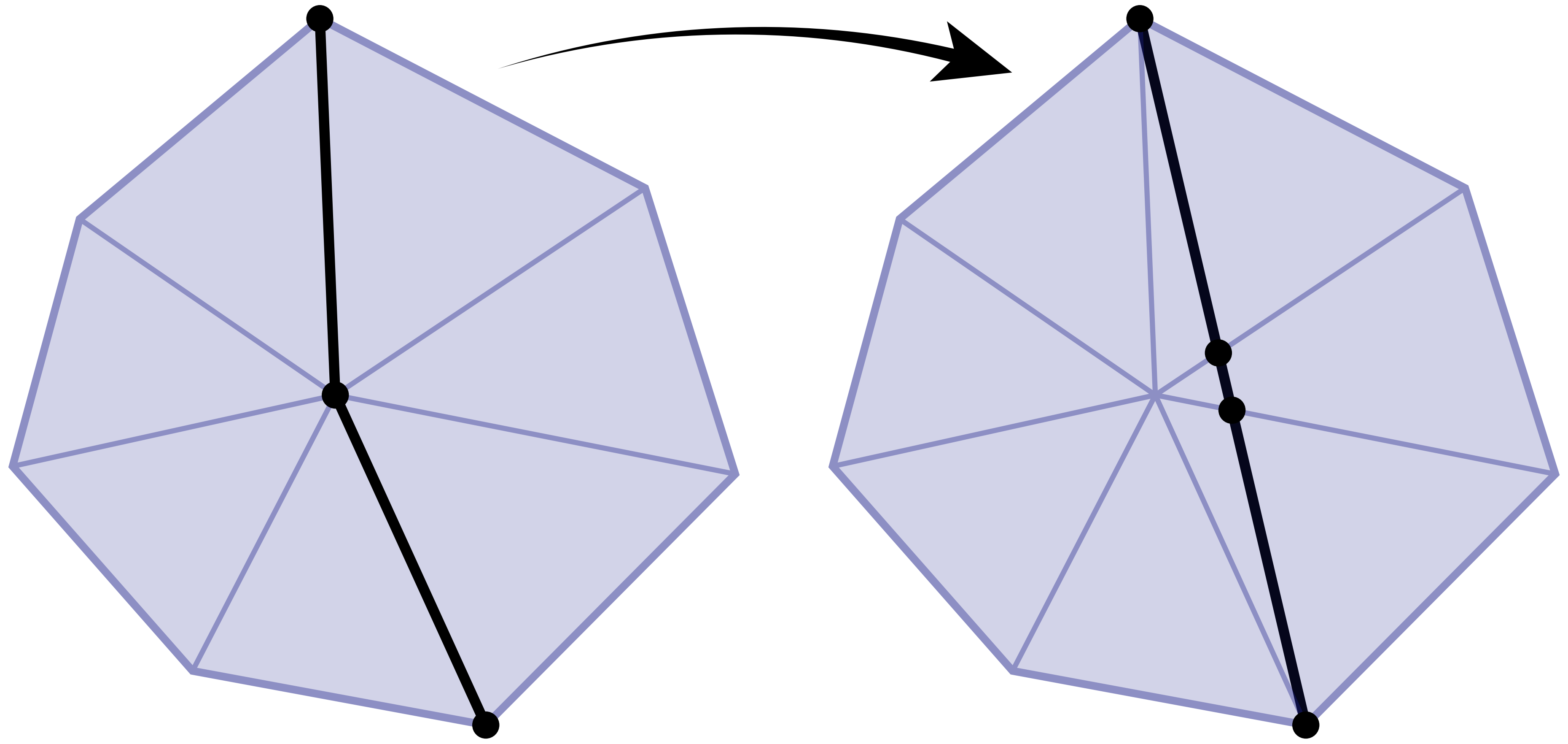


- Geodesics are still critical points (*harmonic*)
- But when *M* is curved, critical points no longer found by solving easy linear equations…
- In general, really need numerical algorithms!

Discrete Shortest Paths—Boundary Value Problem

- How can we find a shortest path in the discrete case?
- Dijkstra's algorithm obviously comes to mind, but a shortest path in the edge graph is almost never geodesic (even if you refine the mesh!)
- One can still start with a Dijkstra path and iteratively shorten local pieces until path is *locally* shortest
- However, no reason local shortening should always give a *globally* shortest path…



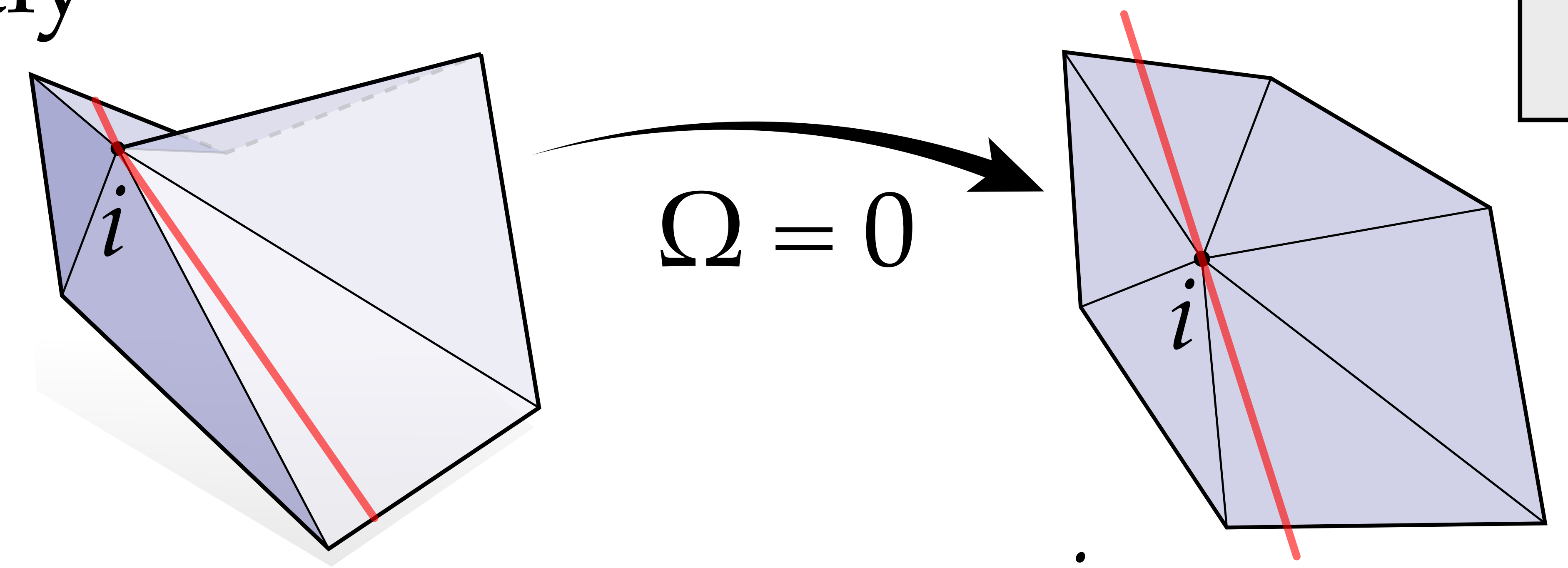


Martínez et al, "Computing Geodesics on Triangular Meshes" (2005)

Discrete Shortest Paths — Vertices

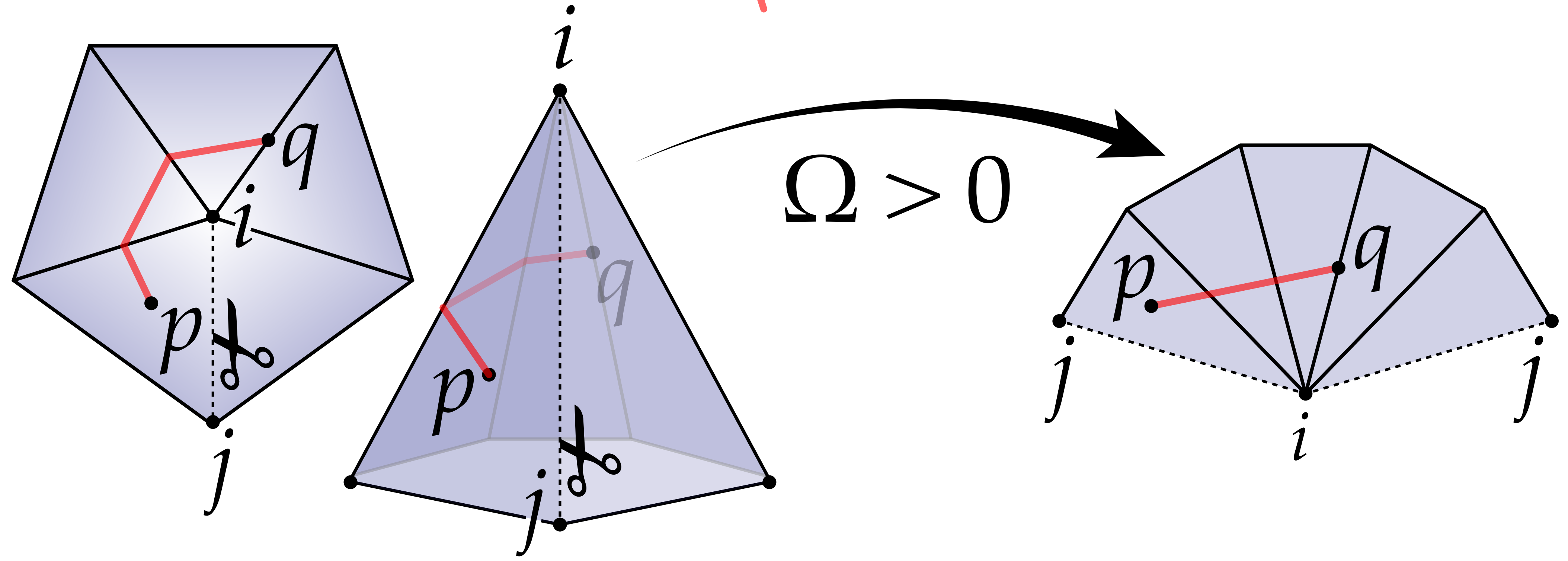
$$\Omega_i = 2\pi - \sum_{ijk} \theta_i^{jk}$$

- Even *locally* straightest paths near vertices require some care—behave differently depending on angle defect Ω



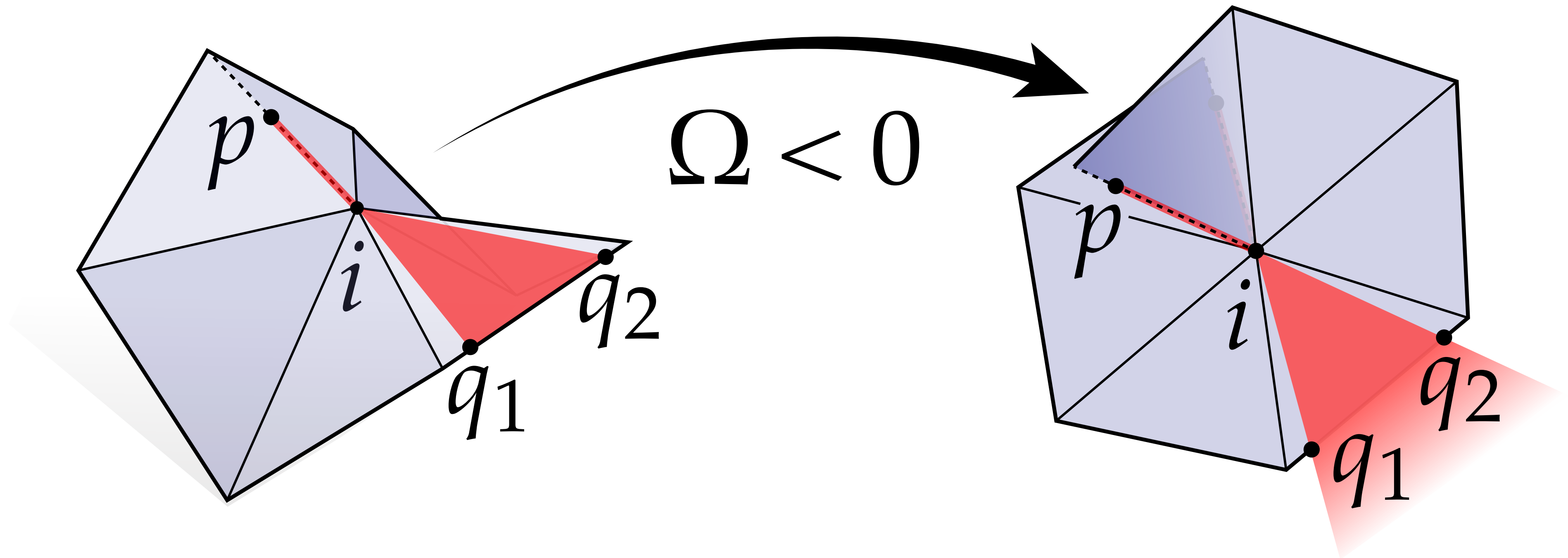
- **Flat ($\Omega = 0$)**

  Can lay out in plane; shortest path simply goes straight through vertex



- **Cone ($\Omega > 0$)**

  Always faster to go around one side or the other



- **Saddle ($\Omega < 0$)**

  Always faster to go through the vertex, but not unique!

Algorithms for Shortest Polyhedral Geodesics

- Algorithms for *shortest* polyhedral geodesics largely based on two closely related methods:

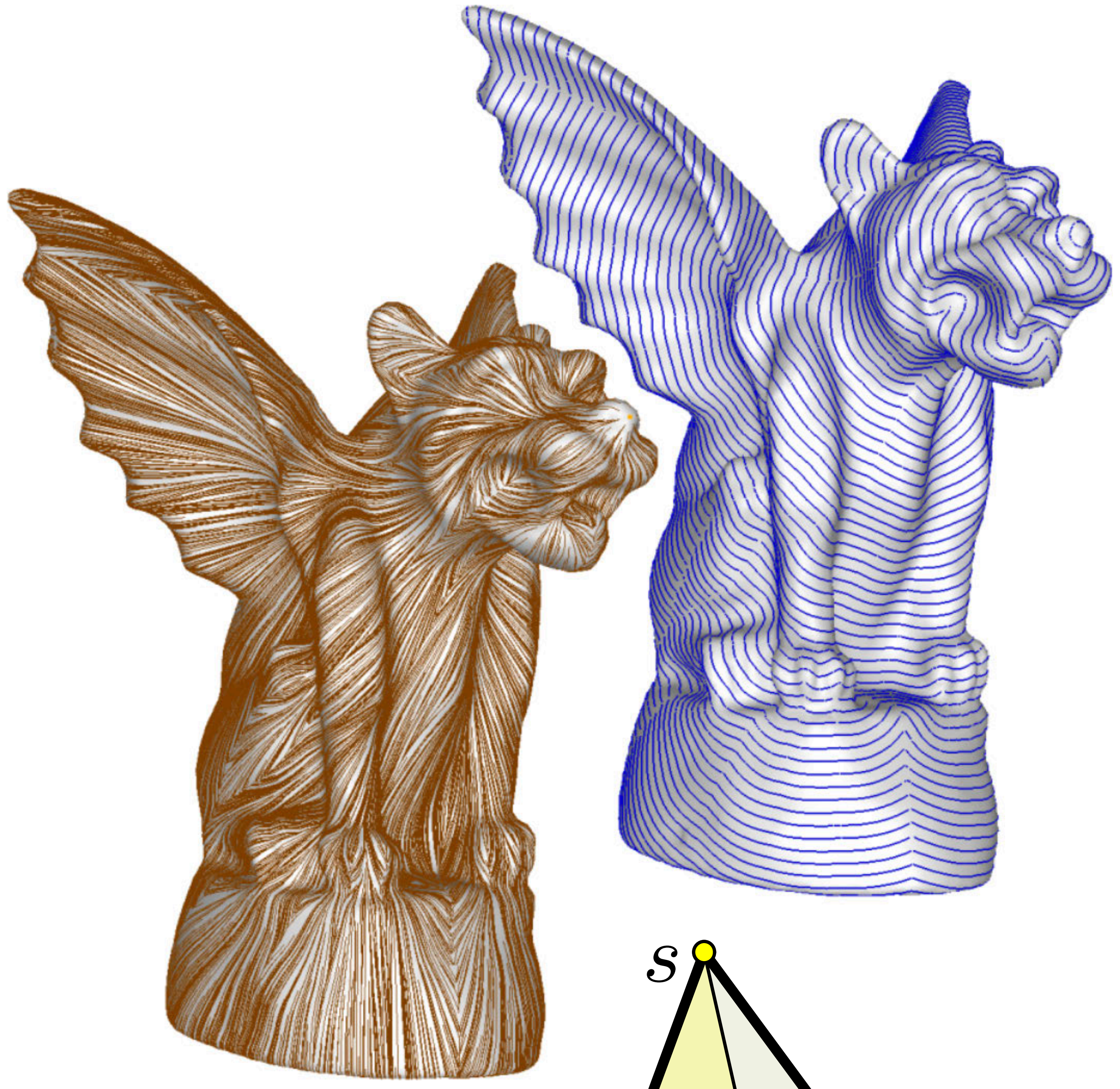
  1. Mitchell, Mount, Papadimitrou (MMP)
  "The Discrete Geodesic Problem" (1986) — $O(n^2 \log n)$

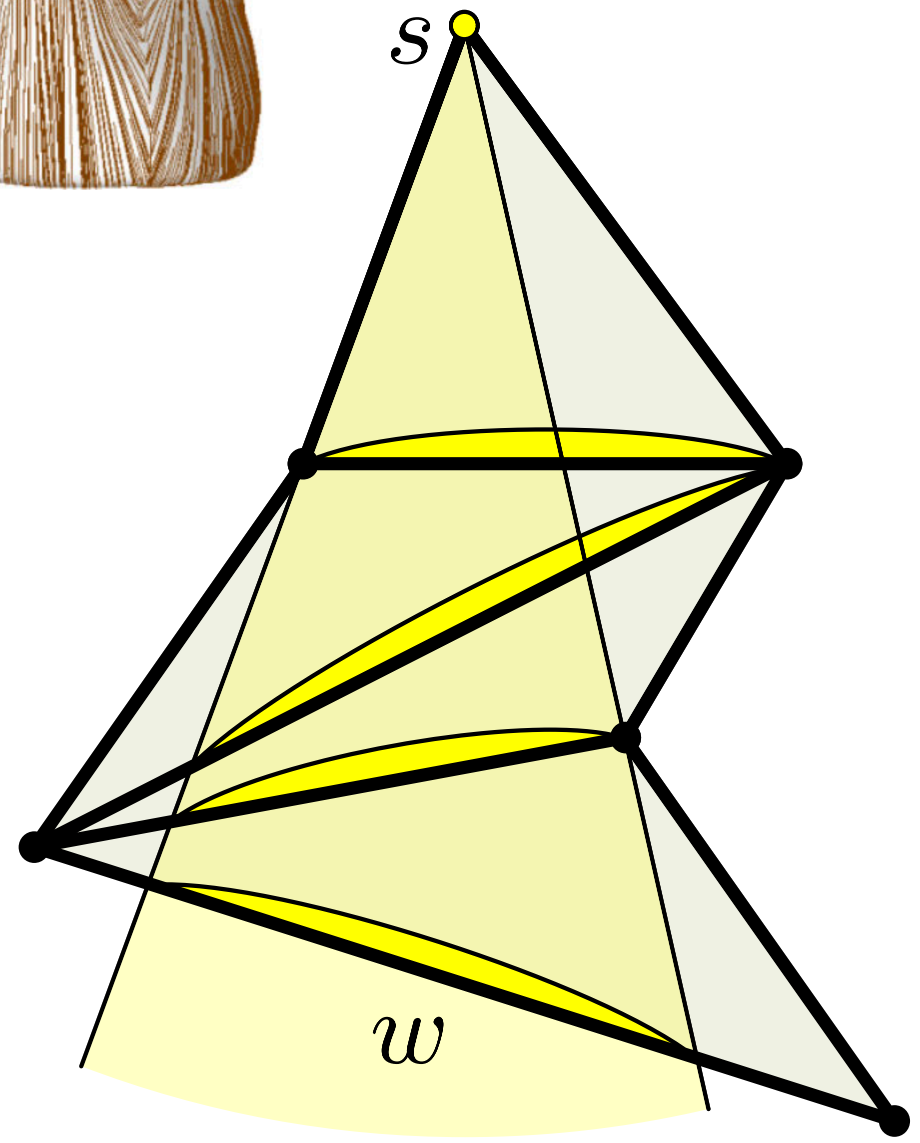
  2. Chen & Han (CH)
  "Shortest Paths on a Polyhedron" (1990) — $O(n^2)$

- Basic idea: track intervals or "windows" of common geodesic paths

- Great deal of work on improving efficiency by pruning windows, approximation, … though still fairly expensive.

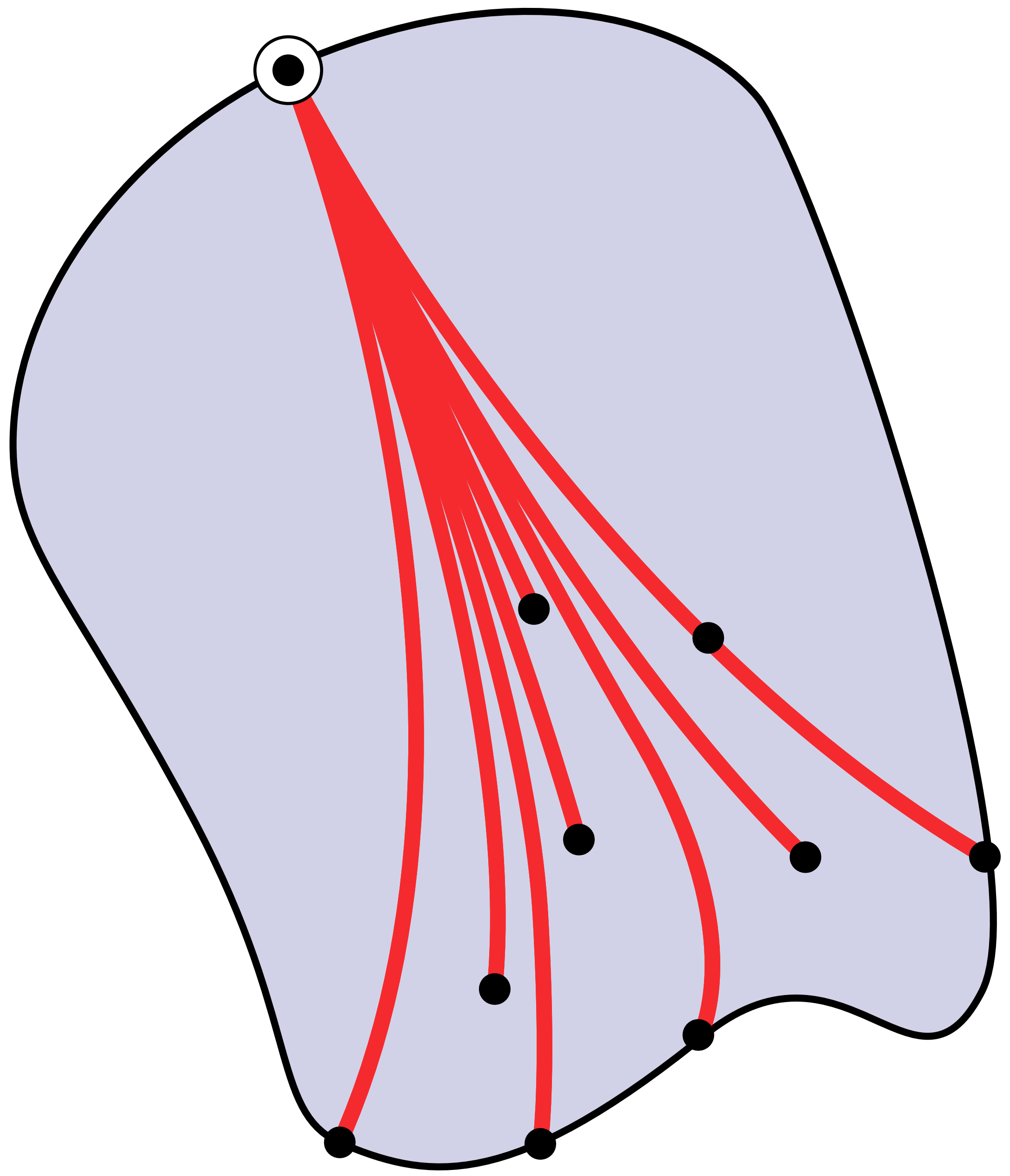
- Good intro in Surazhsky et al.
"Fast Exact and Approximate Geodesics on Meshes" (2005)

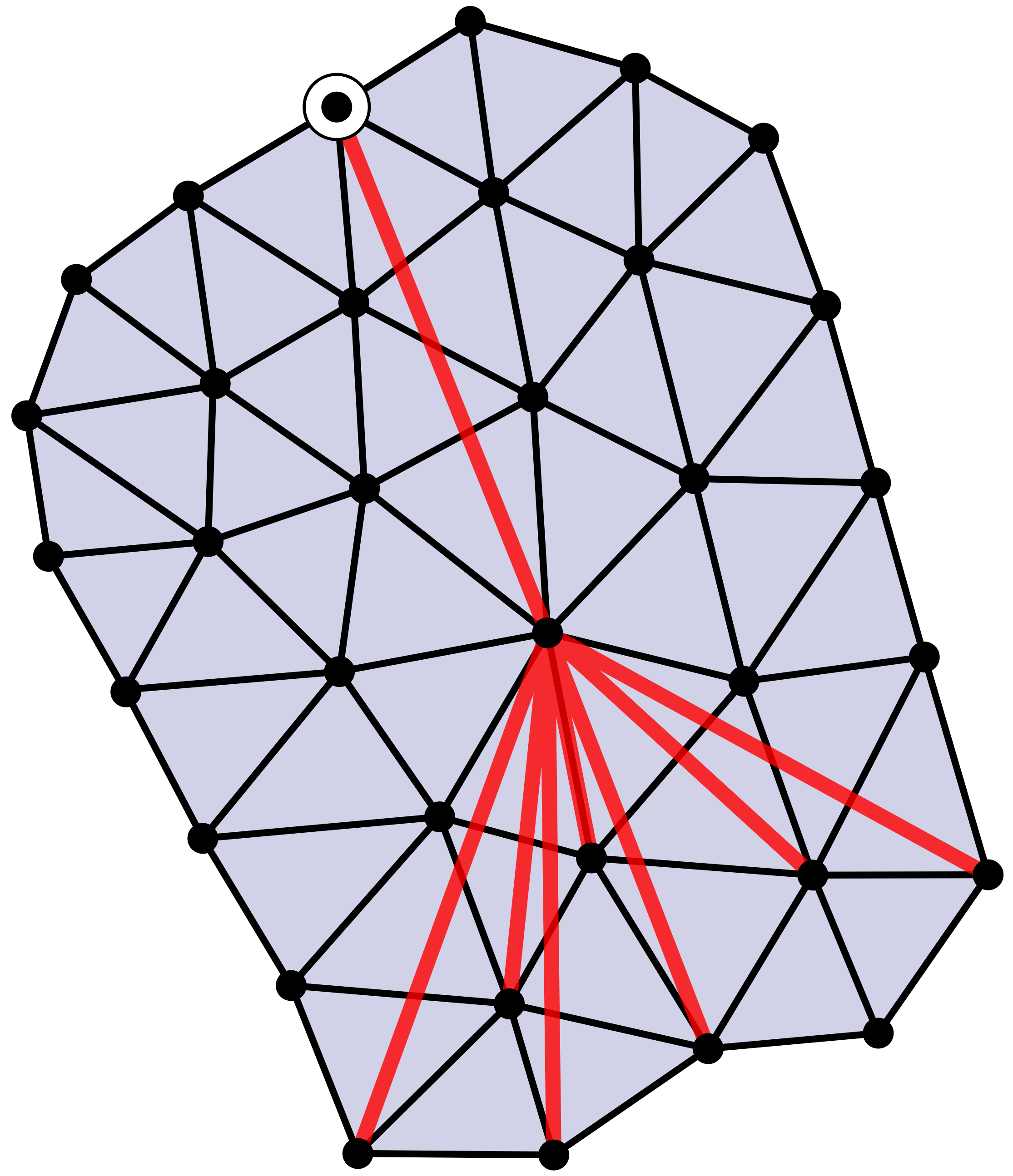




Shortest Geodesics—Smooth vs. Discrete

- **Smooth:** two minimal geodesics γ_1, γ_2 from a source p to distinct points p_1, p_2 (resp.) intersect only if $\gamma_1 \subseteq \gamma_2$ or $\gamma_2 \subseteq \gamma_1$
- **Discrete:** many geodesics can coincide at saddle vertex (*"pseudo-source"*)

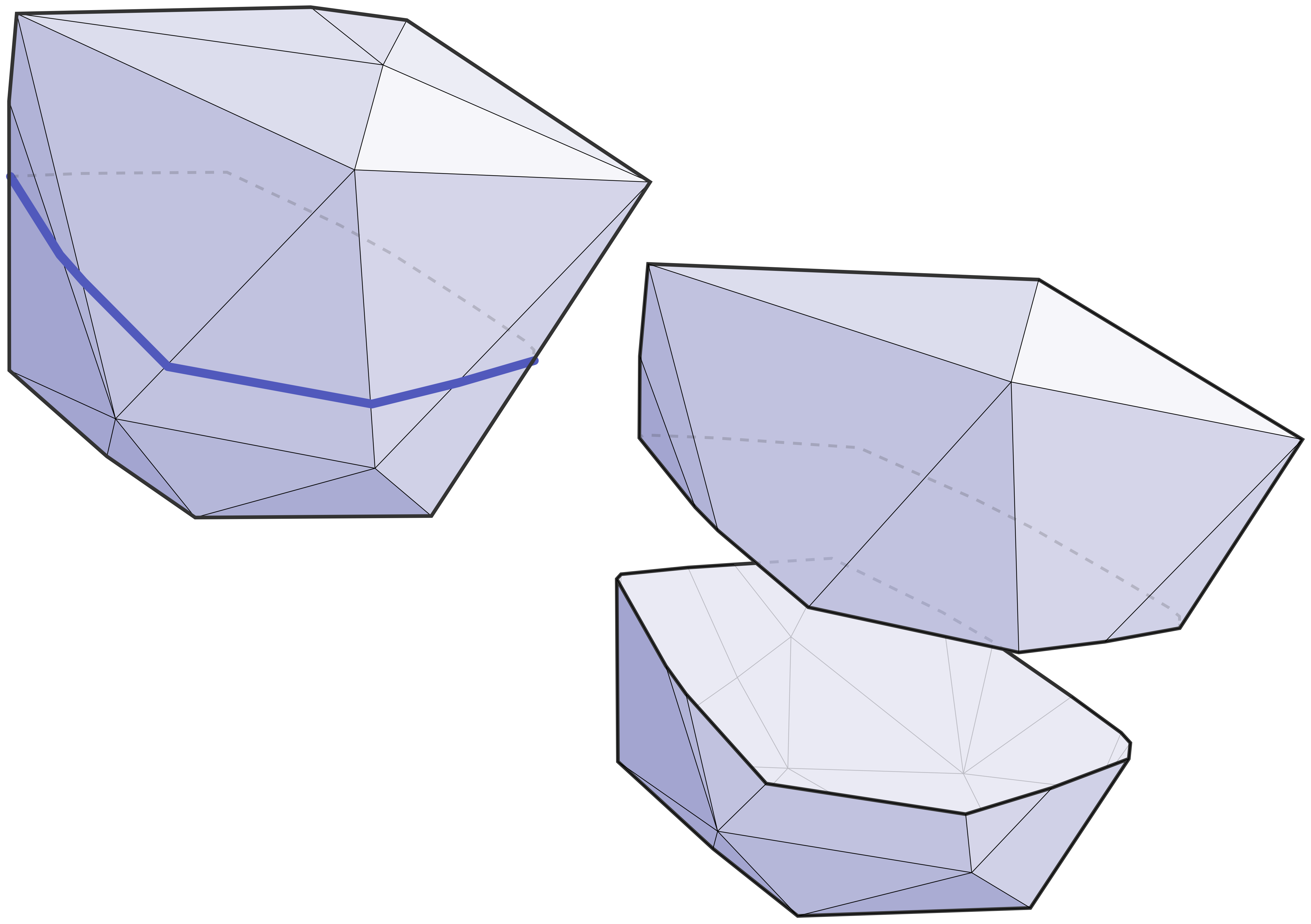




N.B. *Shortest polyhedral geodesics may not faithfully capture behavior of smooth ones!*

Closed Geodesics

- **Theorem.** (Birkhoff 1917) Every smooth convex surface contains a simple closed geodesic, *i.e.*, a geodesic loop that does not cross itself (*"Birkhoff equator"*)
- **Theorem.** (Luysternik & Shnirel'man 1929) Actually, there are at least three—and this result is sharp (*only* three on some smooth surfaces).
- **Theorem.** (Galperin 2002) *Most* convex polyhedra do not have simple closed geodesics (in the sense of discrete *shortest* geodesics).
- *Shortest* characterization of discrete geodesics again fails to capture properties from smooth setting…



A *shortest* discrete geodesic can't pass through convex vertices; by discrete Gauss-Bonnet, has to partition vertices into two sets that each have total angle defect of **<u>exactly</u>** 2π.

Cut Locus

- Given a source point p on a smooth surface M, the *cut locus* is the set of all points q such that there is not a unique (globally) shortest geodesic between p and q.
- *E.g.*, on a sphere the cut locus of any point $+p$ is just the antipodal point $-p$.
- In general can be *much* more complicated…

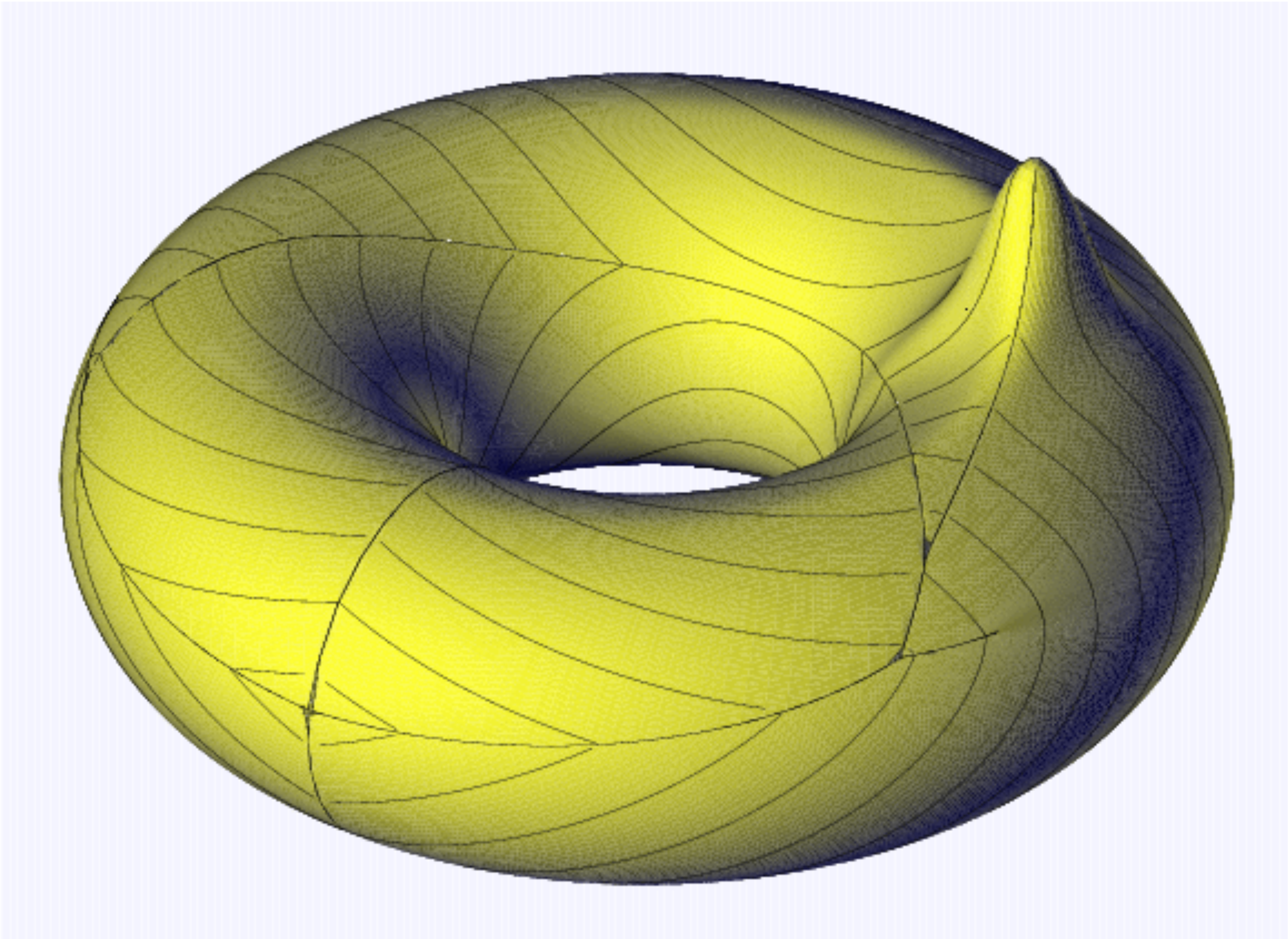
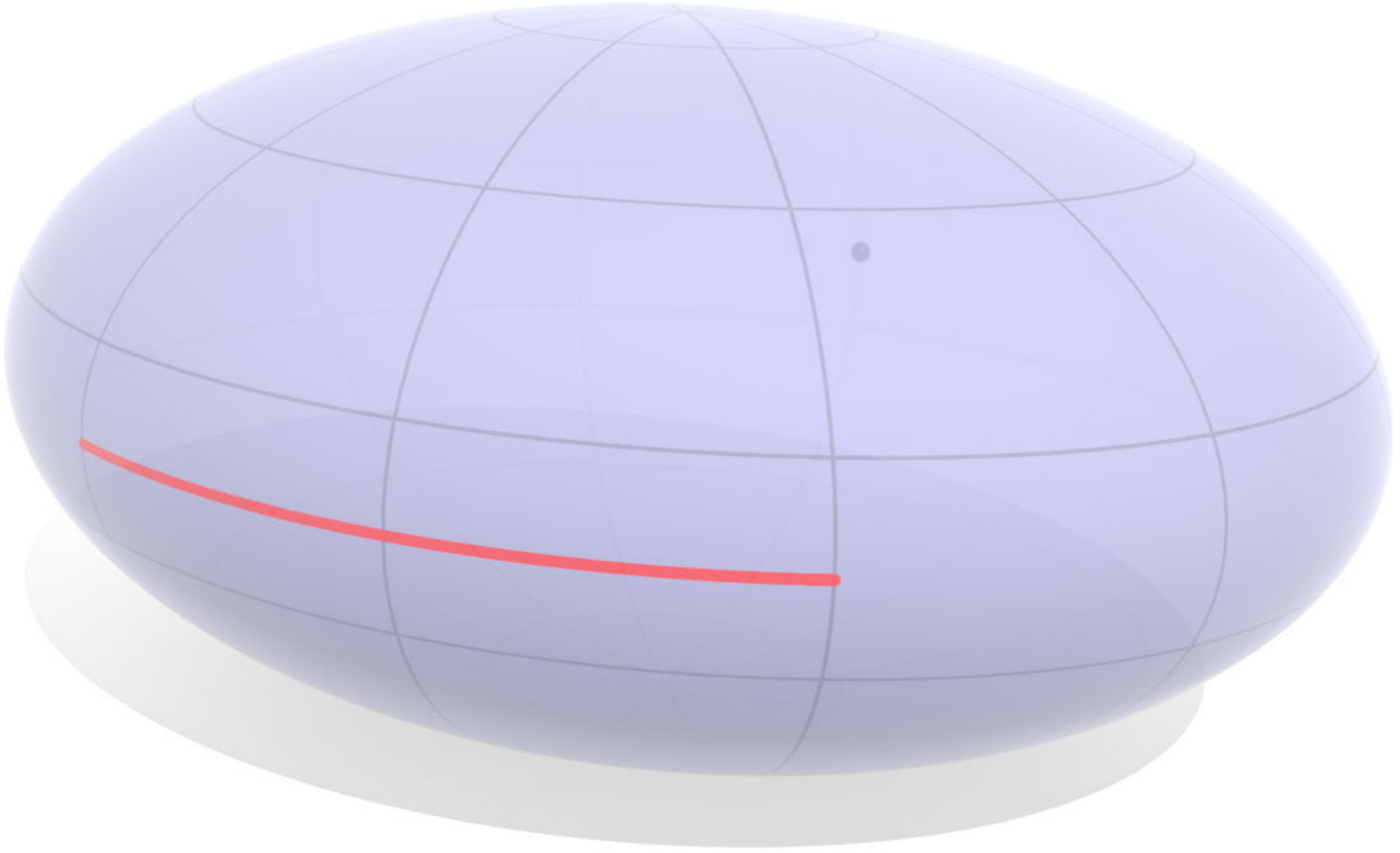
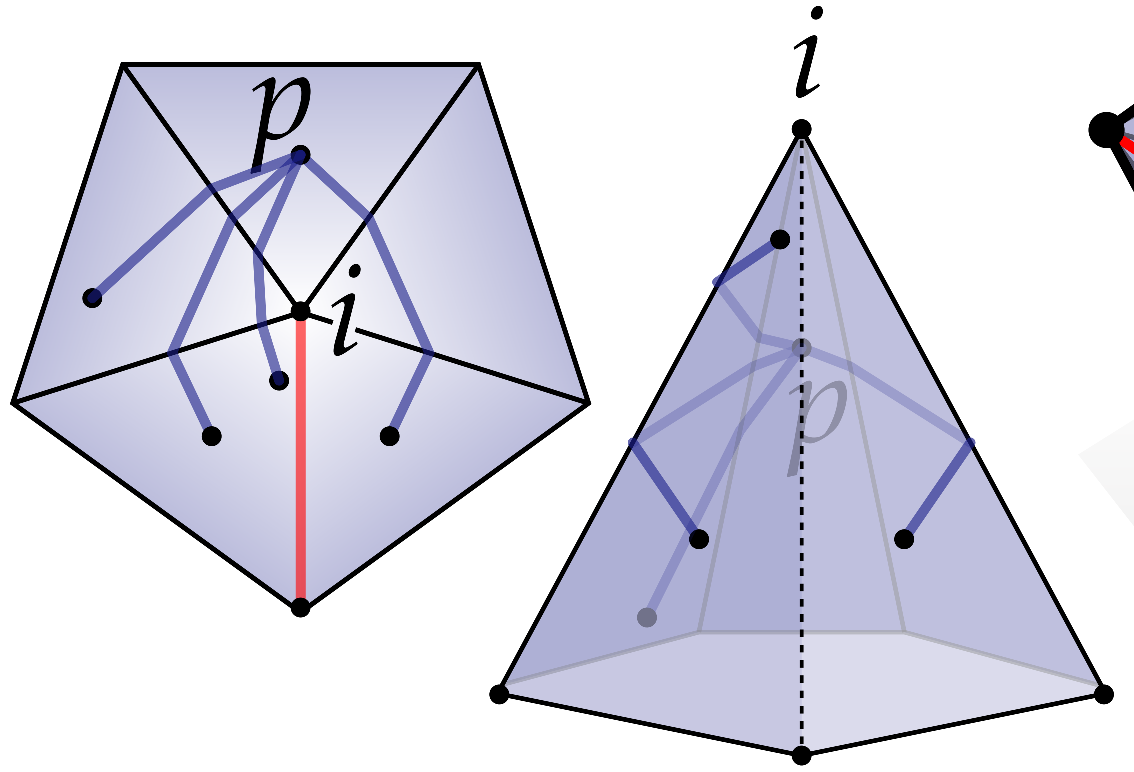


Image credit: S. Markvorsen and P.G. Hjorth (The Cut Locus Project)

Discrete Cut Locus

- What does cut locus look like for polyhedral surfaces?
- Recall that it's always shorter to go "around" a cone-like vertex (i.e., vertex with positive curvature $\Omega_i > 0$)
- Hence, polyhedral cut locus will contain <u>every</u> cone vertex in the entire surface
- Can look *very* different from smooth cut locus!
- *E.g.*, sphere vs. polyhedral sphere?





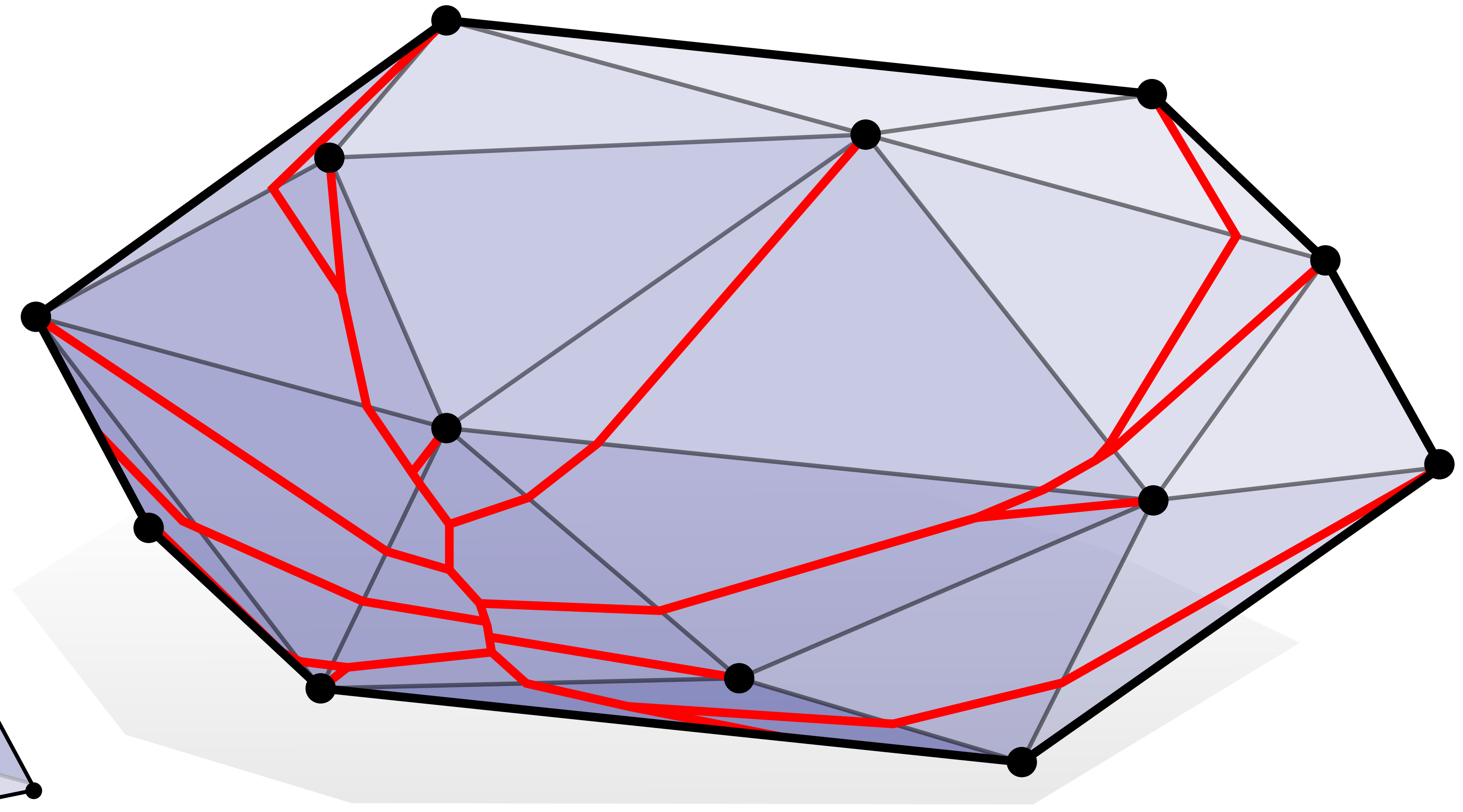
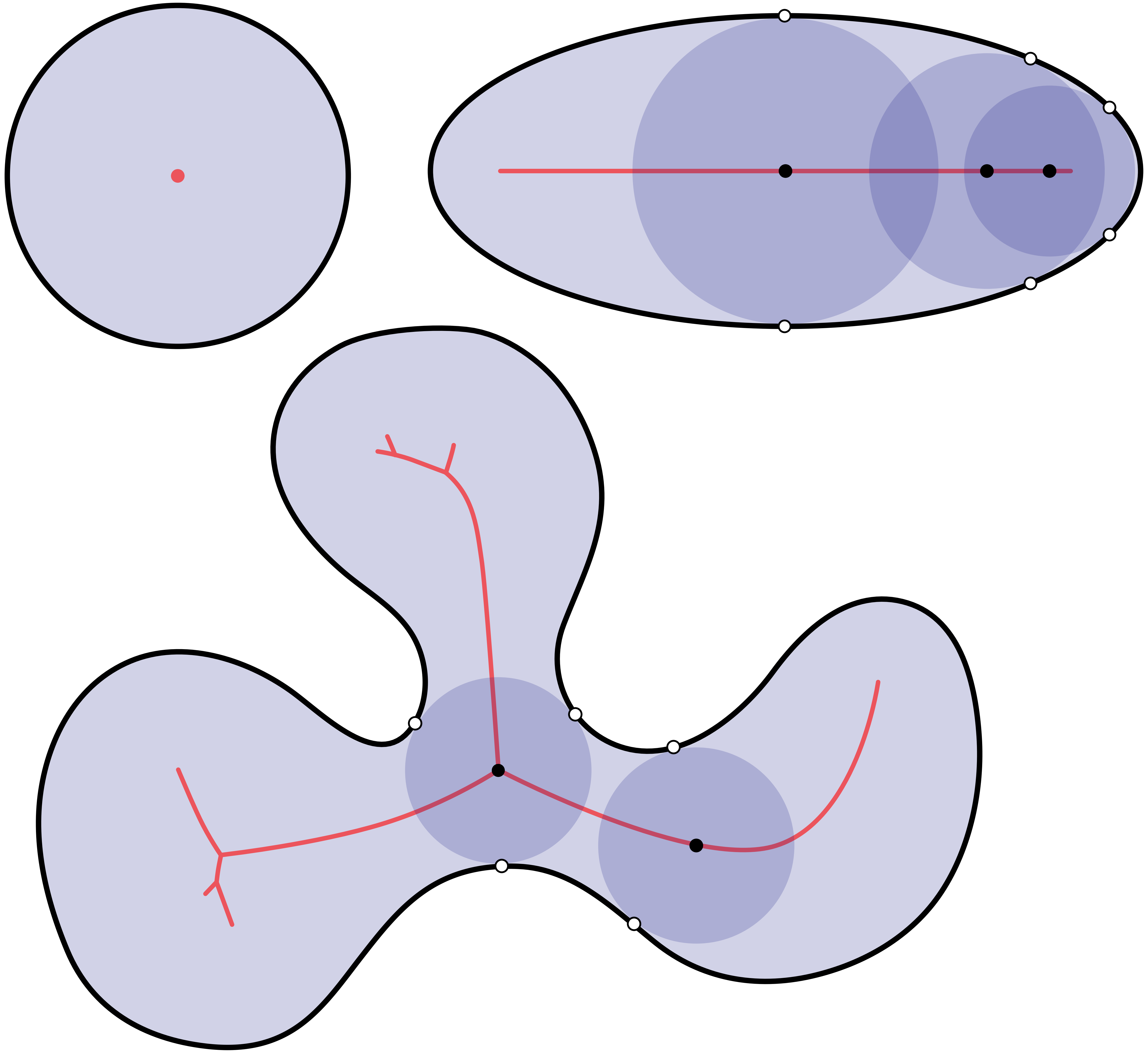


Image adapted from Itoh & Sinclair, "Thaw: A Tool for Approximating Cut Loci on a Triangulation of a Surface"

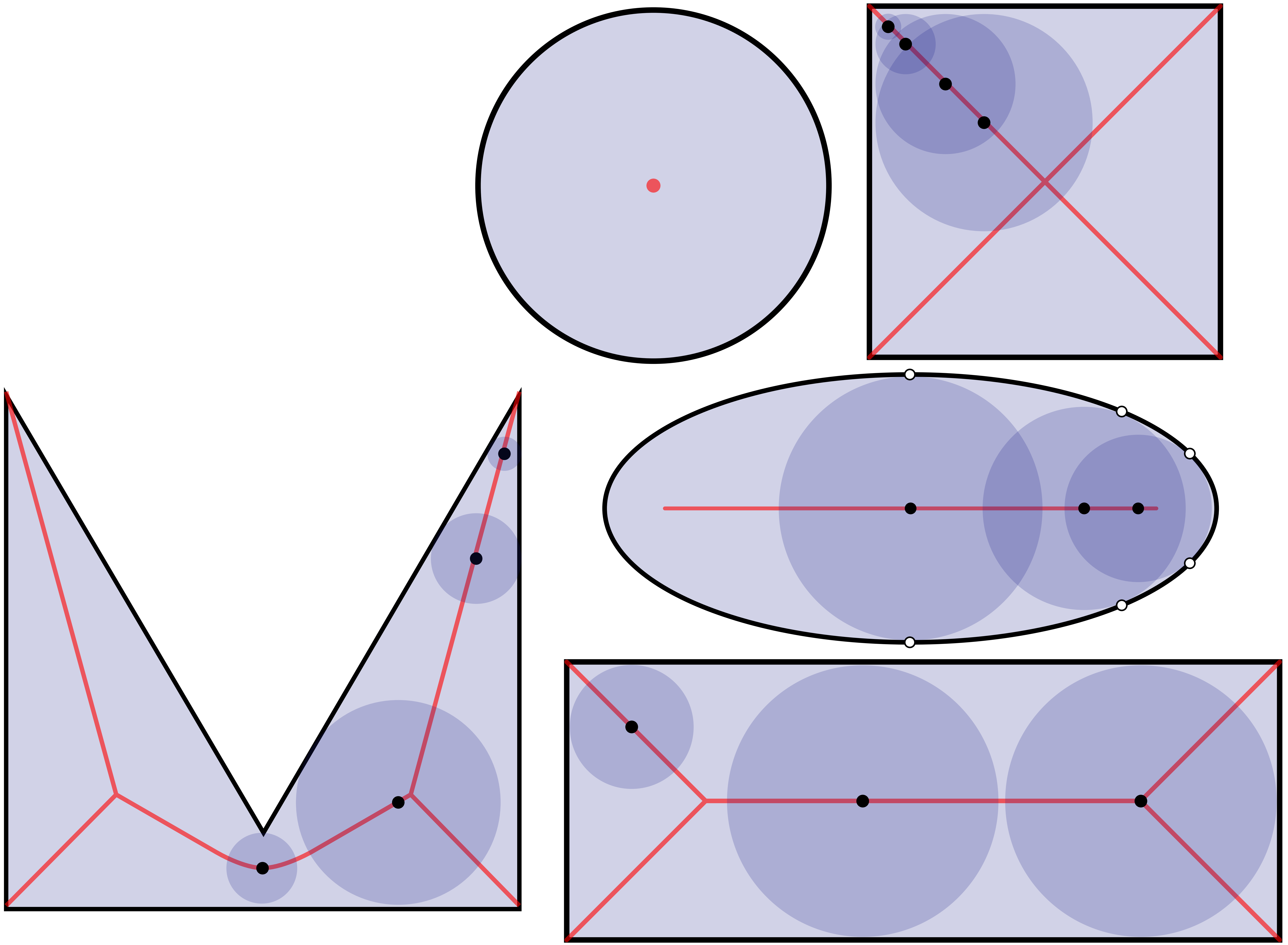
Medial Axis

- Similar to the cut locus, the *medial axis* of a curve or surface $M \subset R^n$ is the set of all points q that do not have a unique closest point on M
- A *medial ball* is a point on the medial axis, with radius given by the distance to the closest point
- Typically three branches (*why?*)
- Provides a "dual" representation: can recover original shape from
  - medial axis
  - radius function



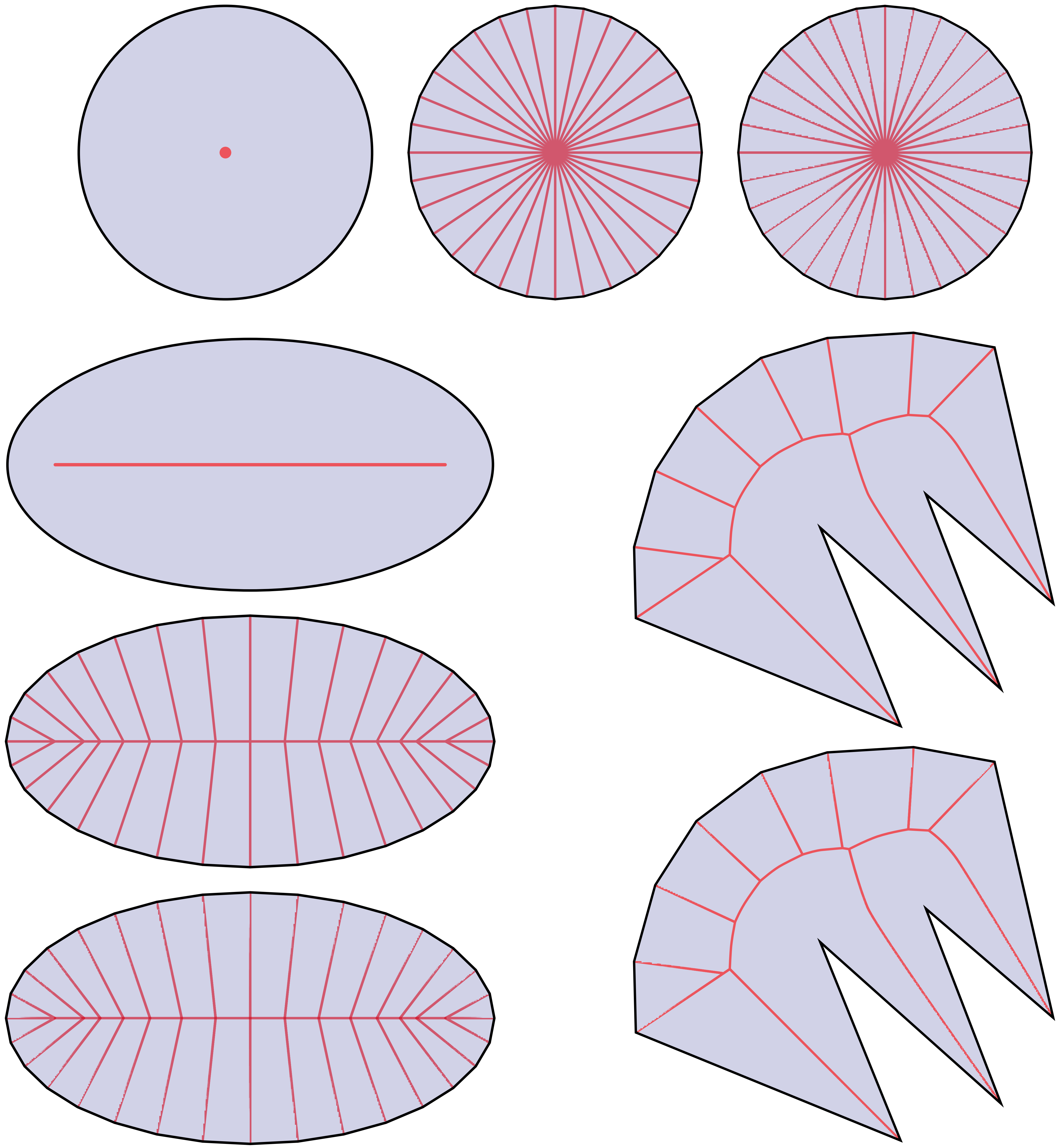
Discrete Medial Axis

- What does the medial axis of a discrete domain look like?
- Let's start with a square. (What did the medial axis for a circle look like?)
- What about a rectangle? (What did an ellipse look like?)
- How about a nonconvex polygon?

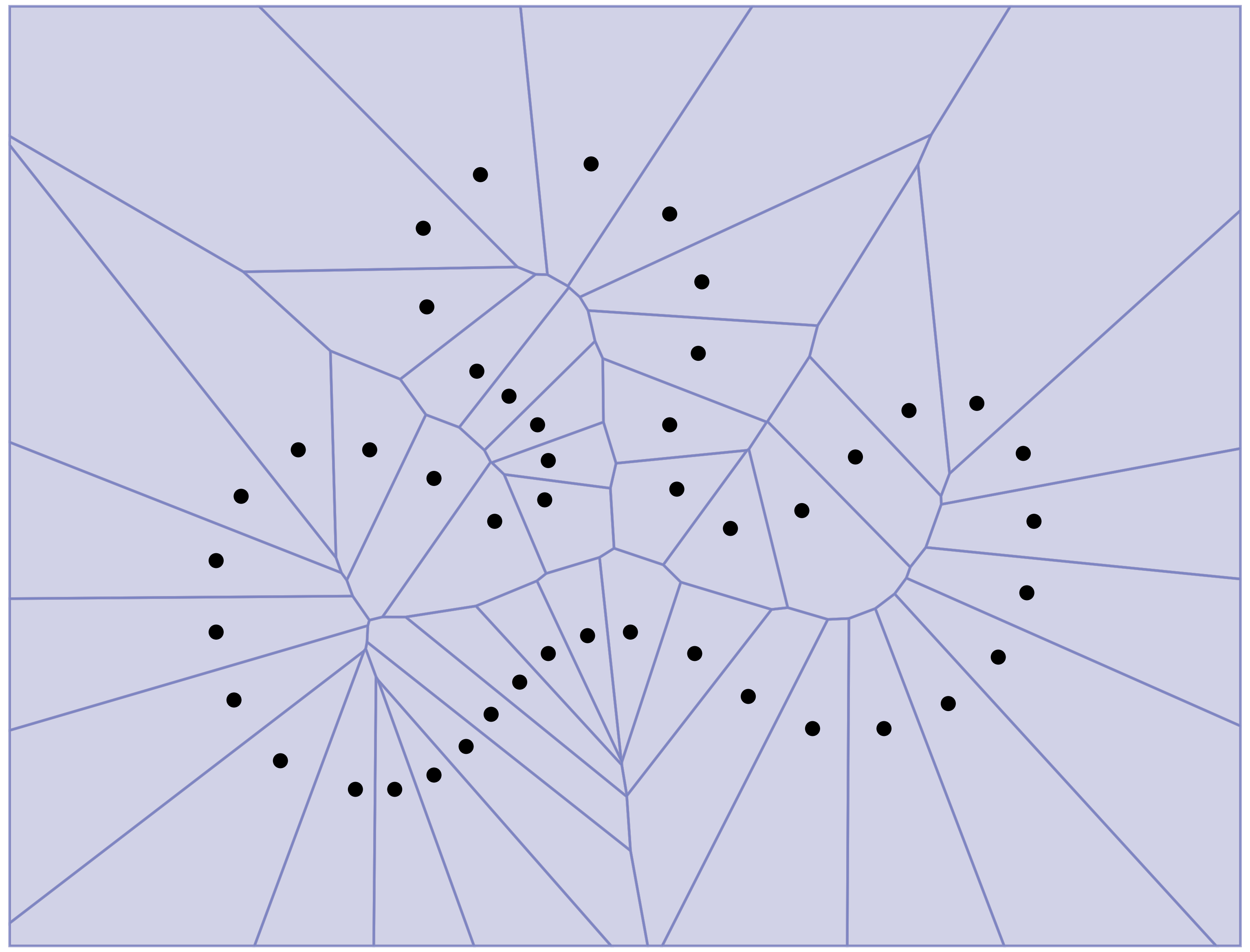


Discrete Medial Axis

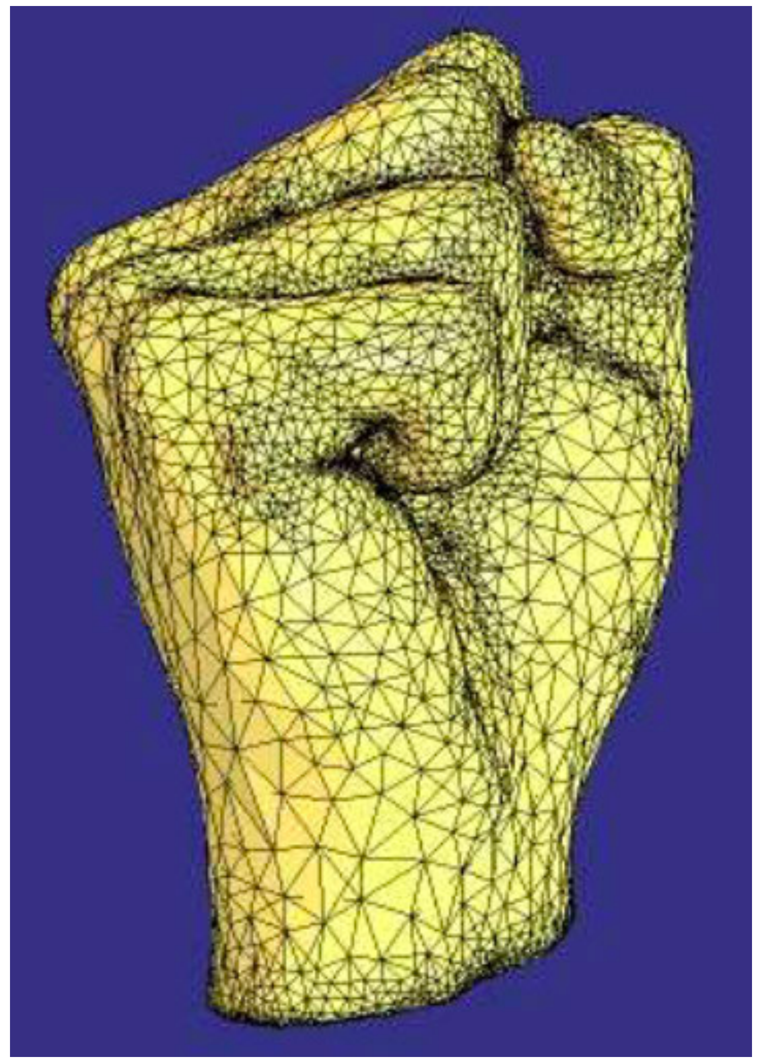
- In general, medial axis touches *every* convex vertex
- May not look much like true (smooth) medial axis!
- One idea: "filter" using radius function…
  - still hard to say exactly which pieces should remain
  - lots of work on alternative "shape skeletons" for discrete curves & surfaces

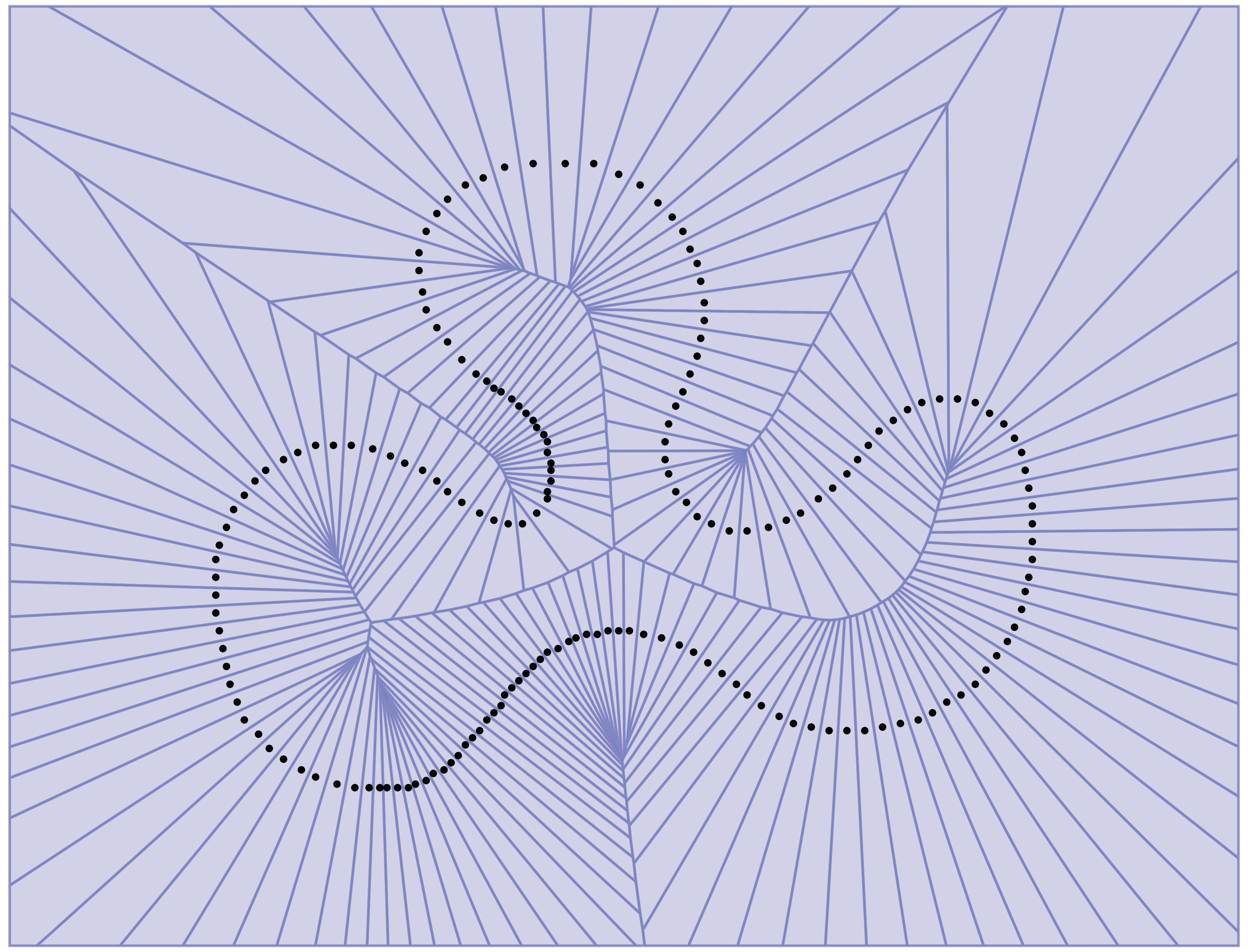


Computing the Medial Axis



- **Many** algorithms for computing/approximating medial axis & other "shape skeletons"
- One line of thought: use *Voronoi diagram* as starting point:
  - densely sample boundary points
  - compute Voronoi diagram
  - keep "short" facets of tall/skinny cells
- Works in 2D, 3D, …
- Very similar algorithm gives surface reconstruction from points



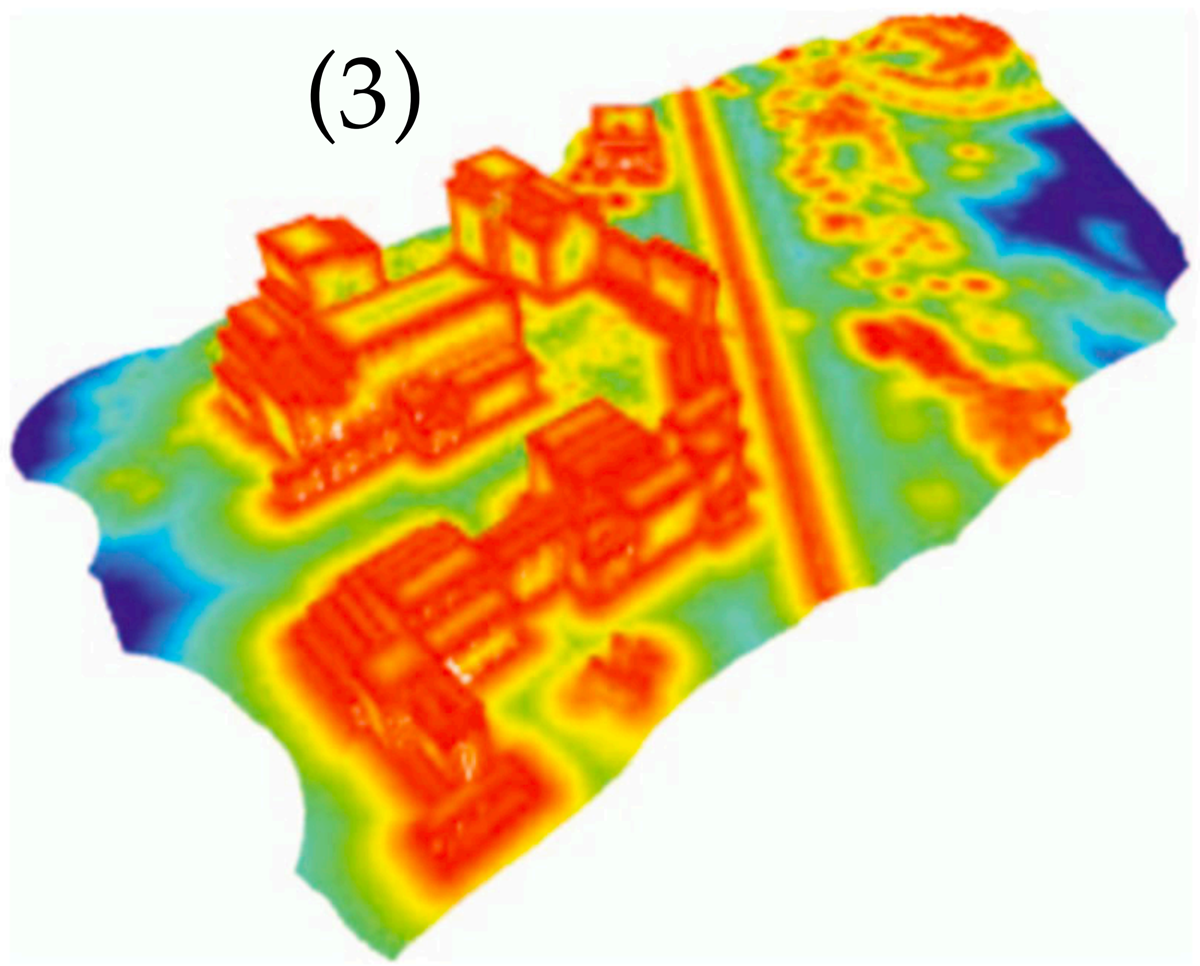


Amenta et al, "A New Voronoi-Based Surface Reconstruction Algorithm"

Medial Axis — Applications

- Many applications of medial axis
  - shape skeletons
  - local feature size
  - fast collision detection
  - fluid particle re-seeding
  - …

(3)



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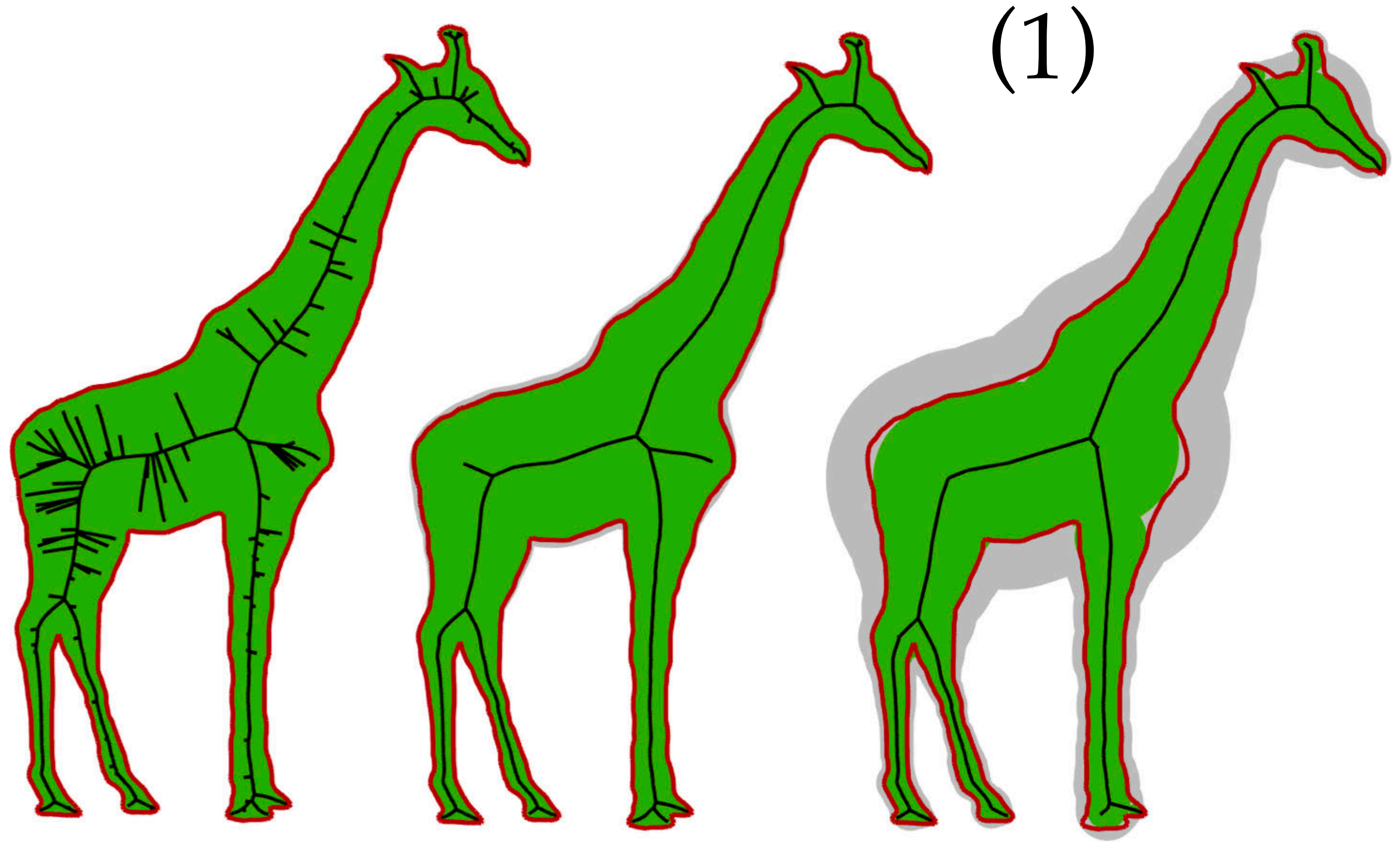

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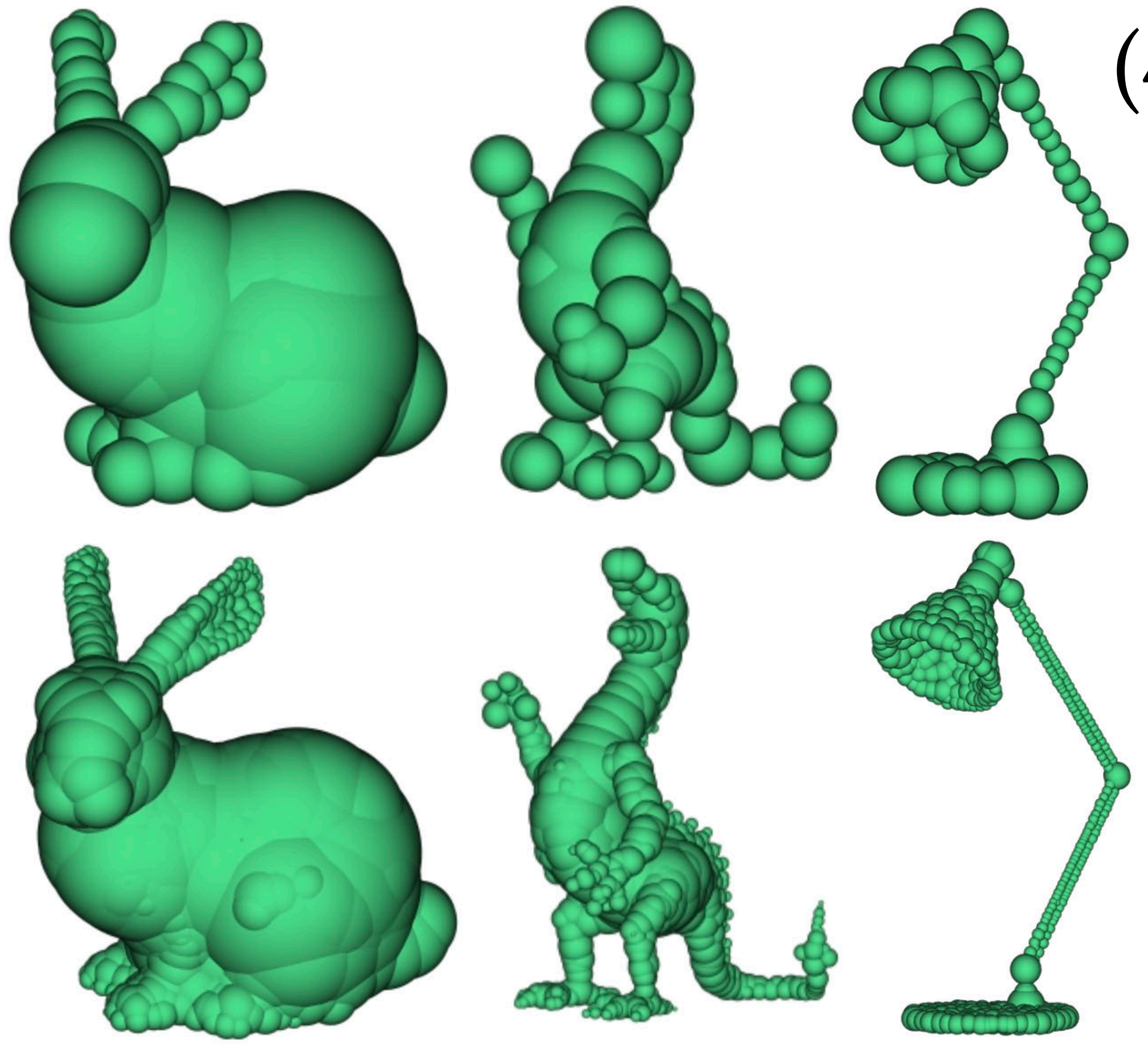

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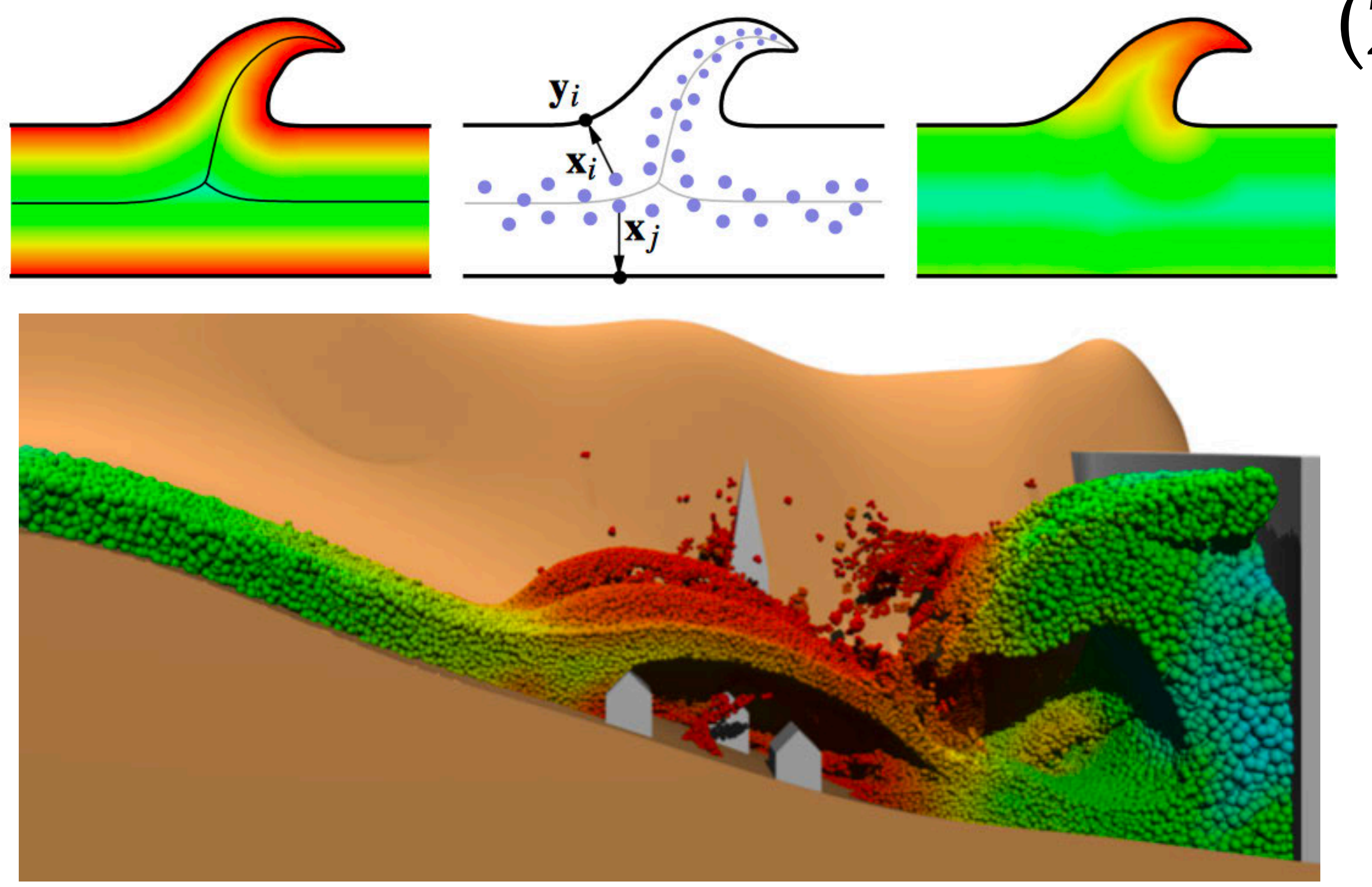
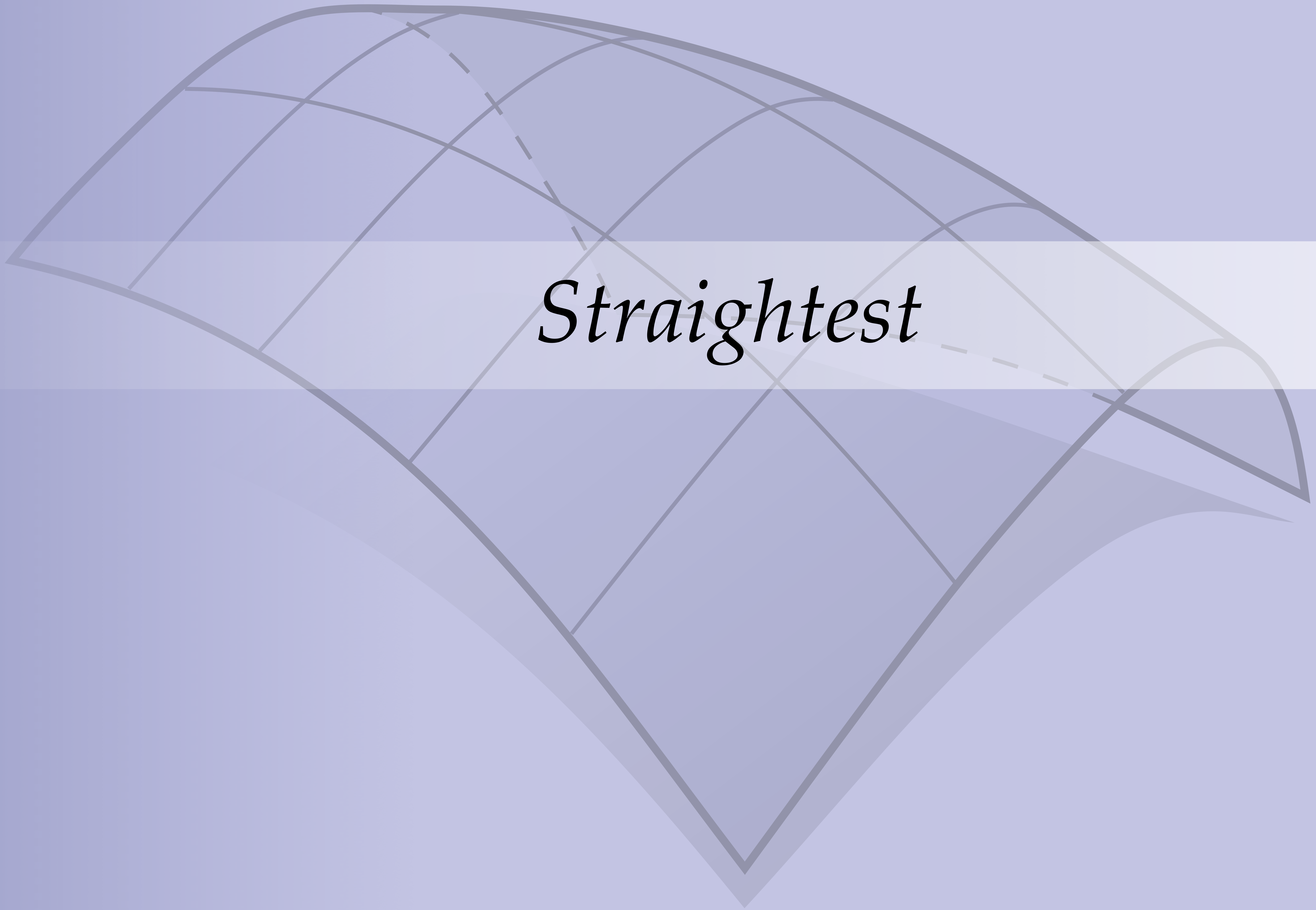


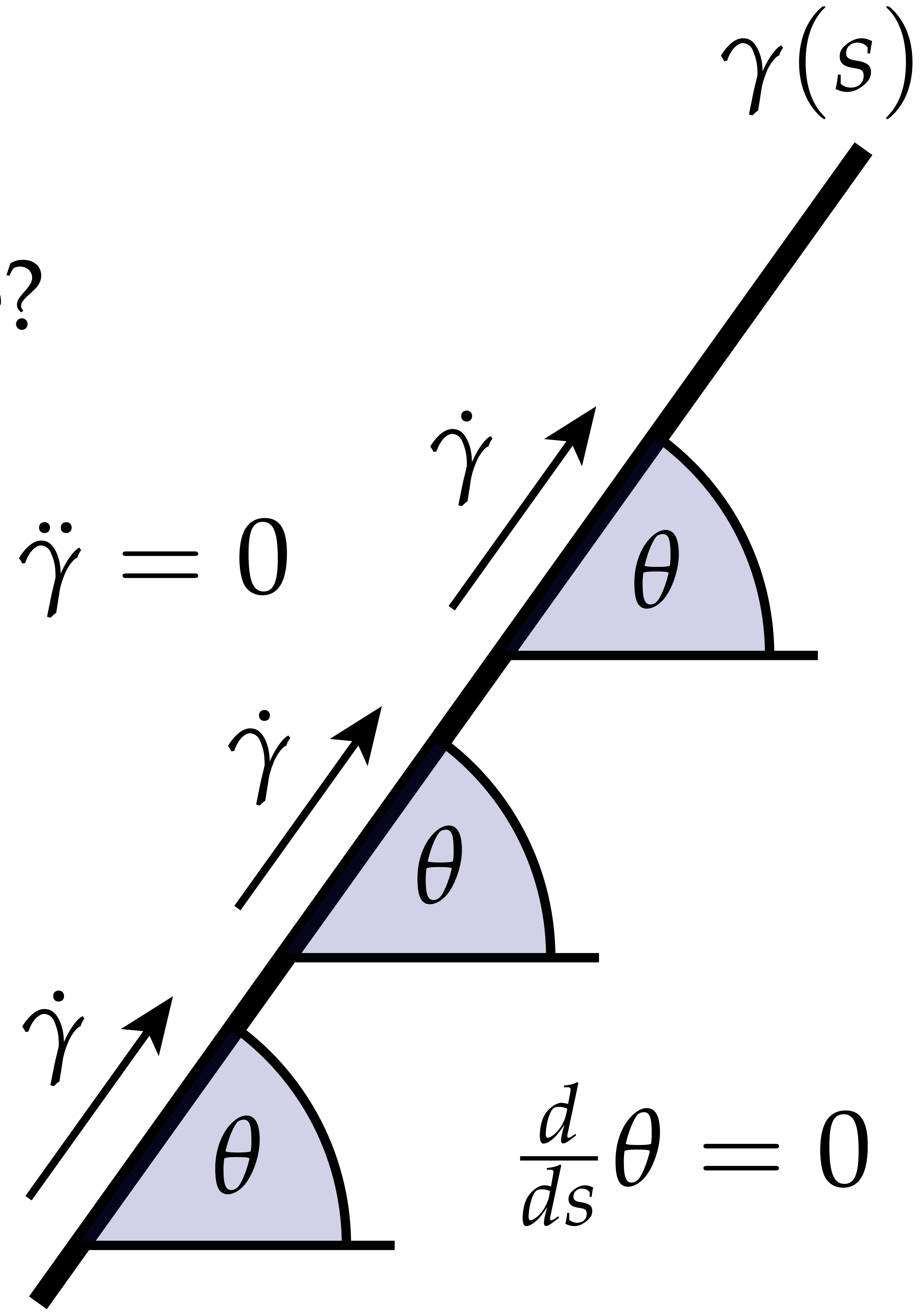
(1) Giesen et al, *"The Scale Axis Transform"*
(2) Adams et al, *"Adaptively Sampled Particle Fluids"*
(3) Peters & Ledoux, *"Robust approximation of the Medial Axis Transform of LiDAR point clouds"*
(4) Bradshaw & Sullivan, *"Adaptive Medial-Axis Approximation for Sphere-Tree Construction"*

Straightest



Straightest Paths

- A Euclidean line can be characterized as a curve that is "as straight as possible"
- **Q:** How can we make this statement more precise?
  - **geometrically**: no curvature
  - **dynamically**: no acceleration
- How can we generalize to curves in manifolds?
  - **geometrically**: no *geodesic curvature*
  - **dynamically**: zero *covariant derivative*

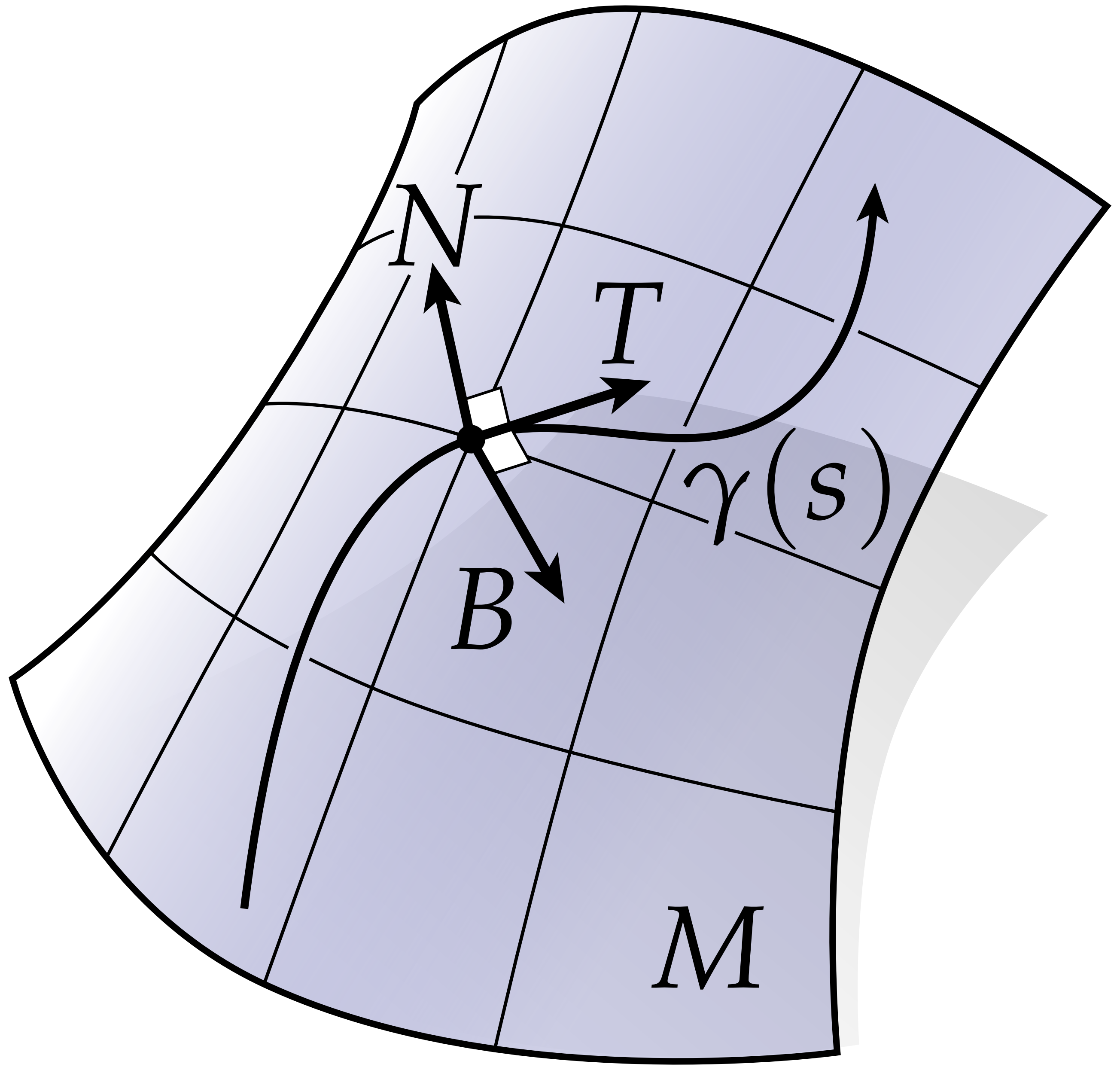


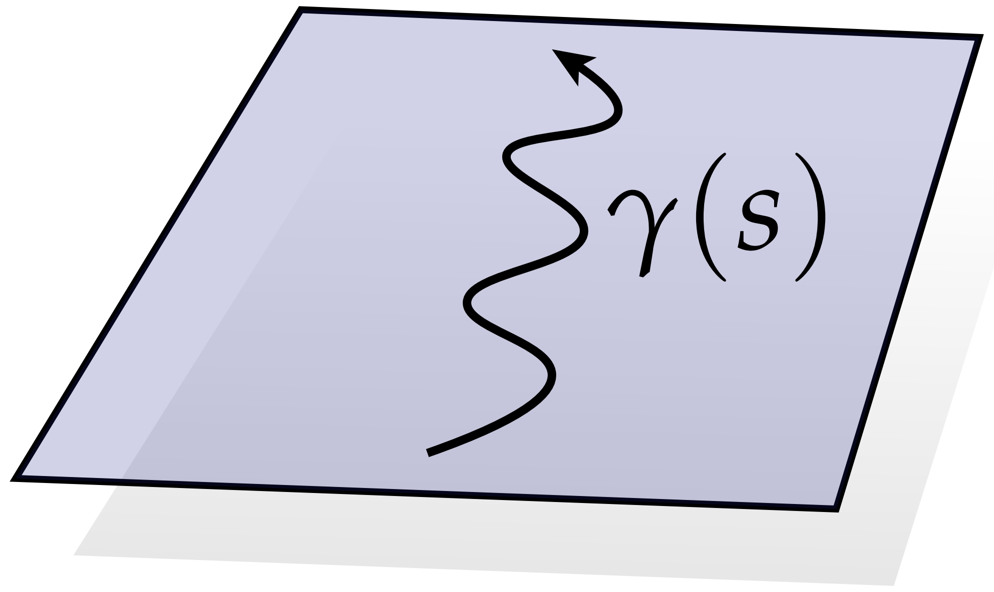
Straightness—Geometric Perspective

- Consider a curve $\gamma(s)$ with tangent T in a surface with normal N, and let $B := T \times N$.
- Can decompose "bending" into *normal curvature* κ_n and *geodesic curvature* κ_g:

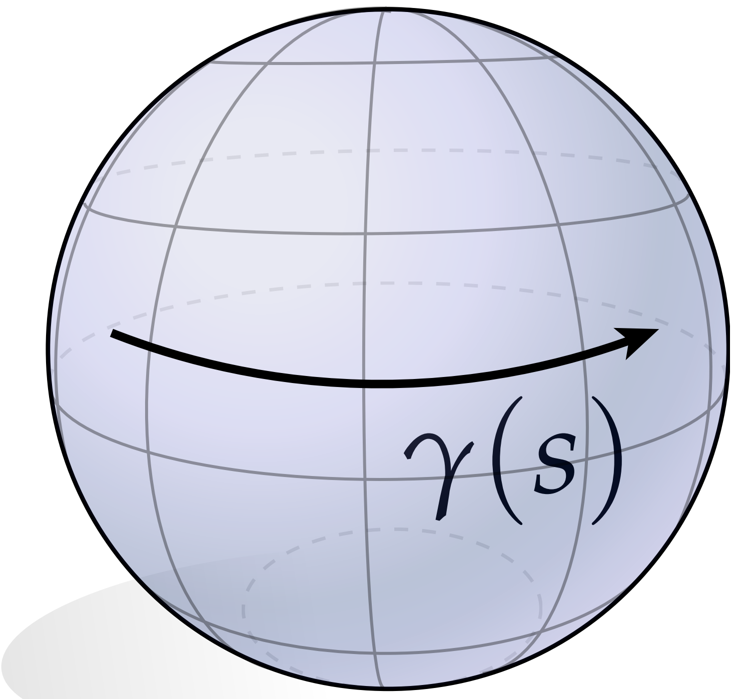
$$\begin{aligned} \kappa_n &:= \langle N, \tfrac{d}{ds} T \rangle \\ \kappa_g &:= \langle B, \tfrac{d}{ds} T \rangle \end{aligned}$$

- Curve is "forced" to have normal curvature due to curvature of M
- Any additional bending beyond this minimal amount is geodesic curvature
- *Geodesic* is curve such that $\kappa_g = 0$





large κ_g;
small κ_n



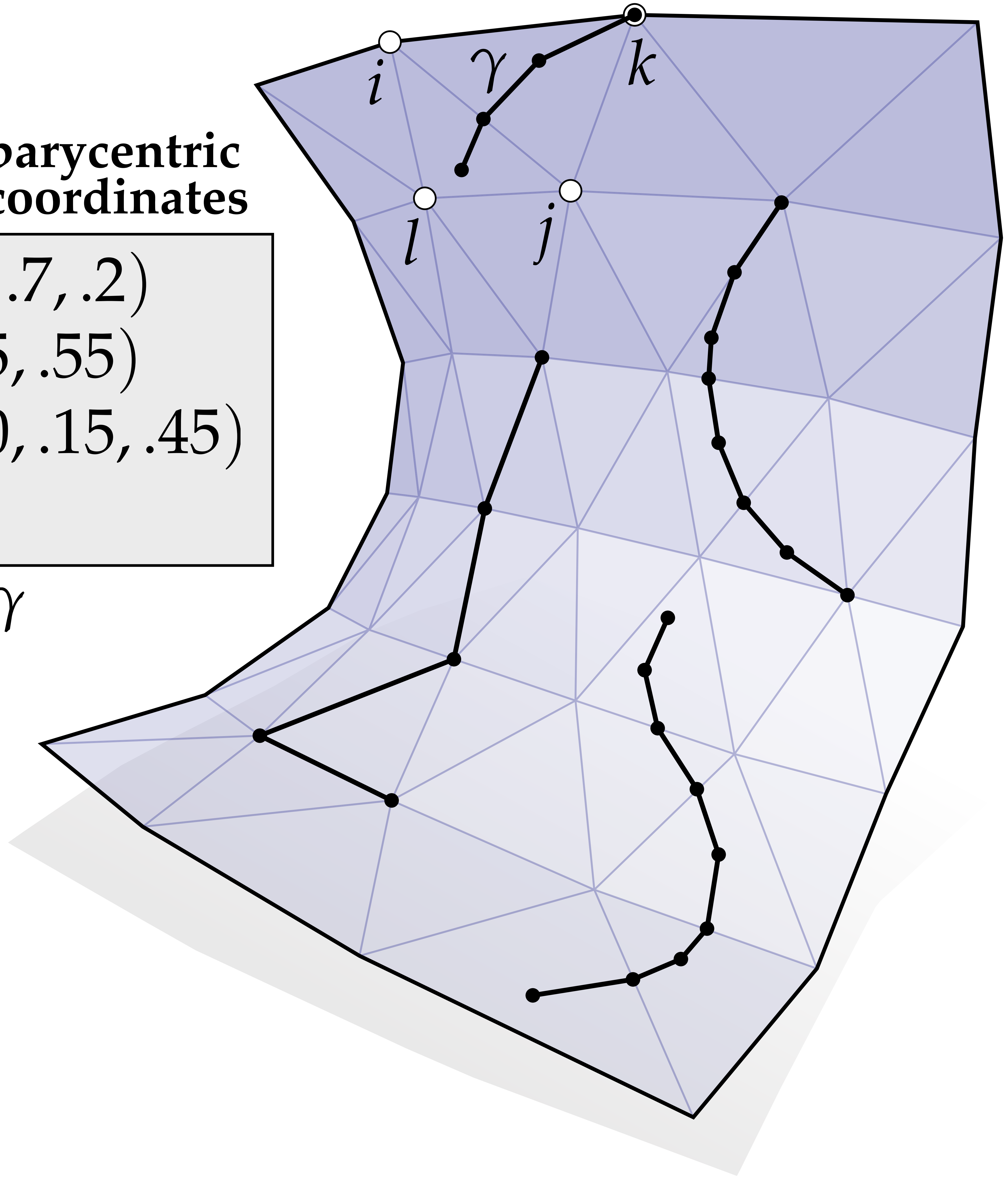
large κ_n;
small κ_g

Discrete Curves on Discrete Surfaces

- To understand straightest curves on discrete surfaces, first have to define what we mean by a *discrete curve*
- One definition: a discrete curve in a simplicial surface M is any continuous curve γ that is <u>piecewise</u> linear in each simplex
- Doesn't have to be a path of edges: could pass through faces, have multiple vertices in one face, …
- Practical encoding: sequence of k-simplices (not all same dimension), and barycentric coordinates for each simplex

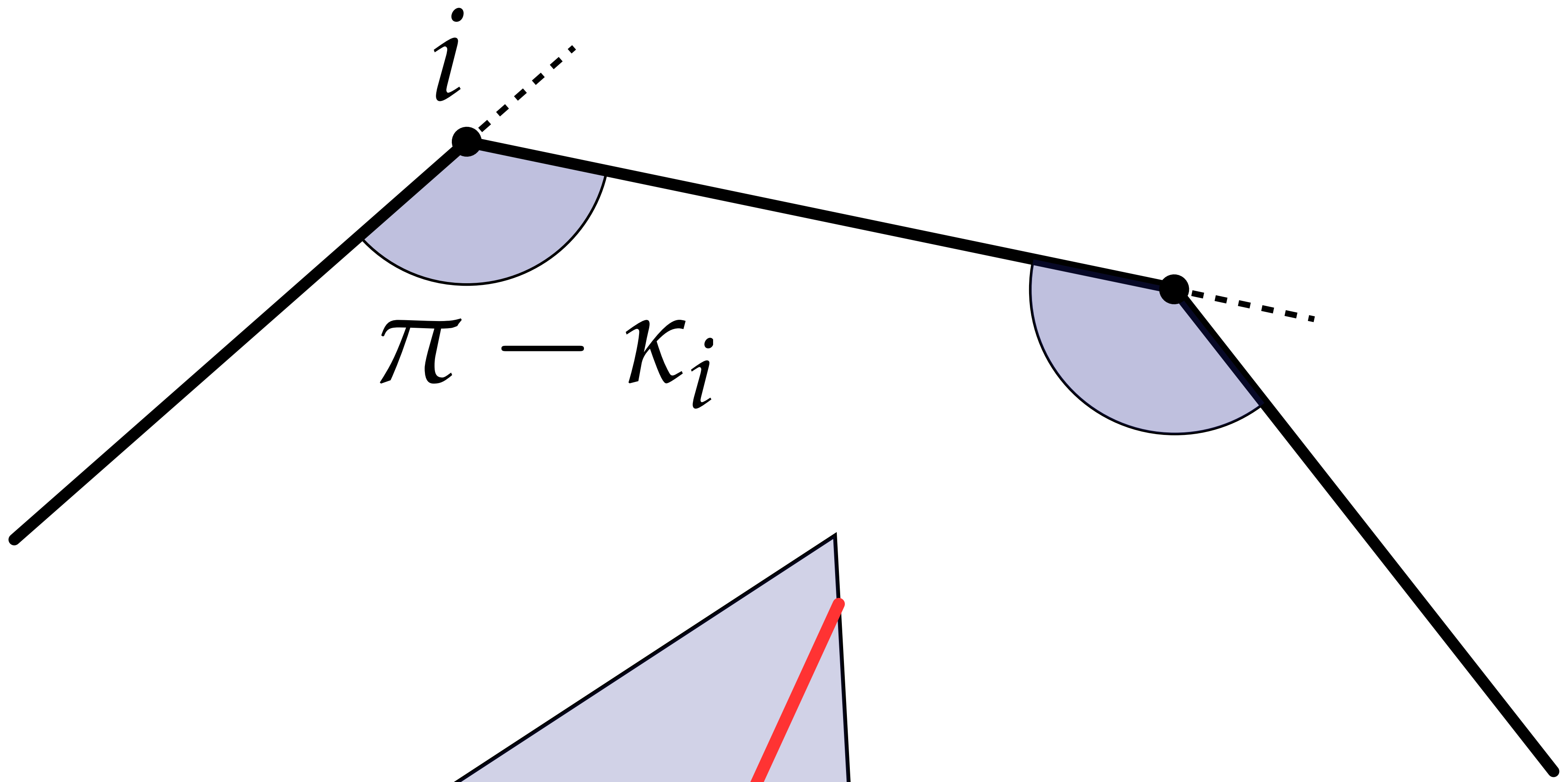
simplex	**barycentric coordinates**
ilj	$(.1, .7, .2)$
ij	$(.45, .55)$
ijk	$(.40, .15, .45)$
k	(1)

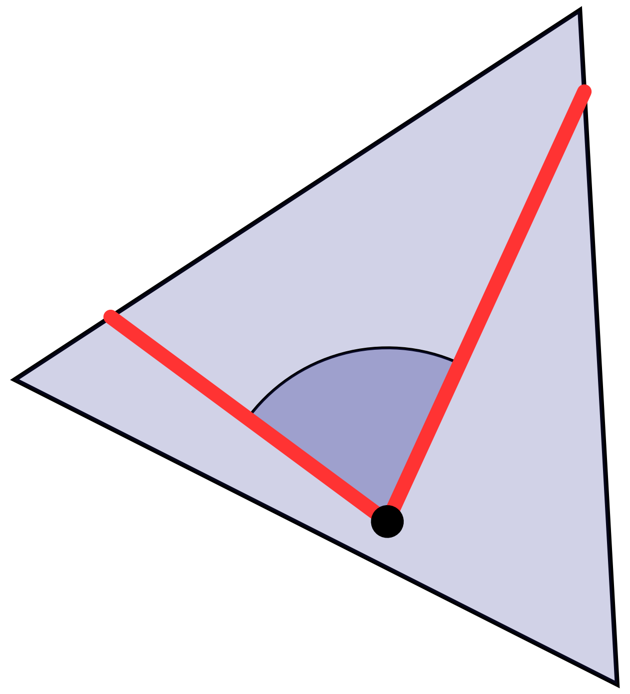
γ

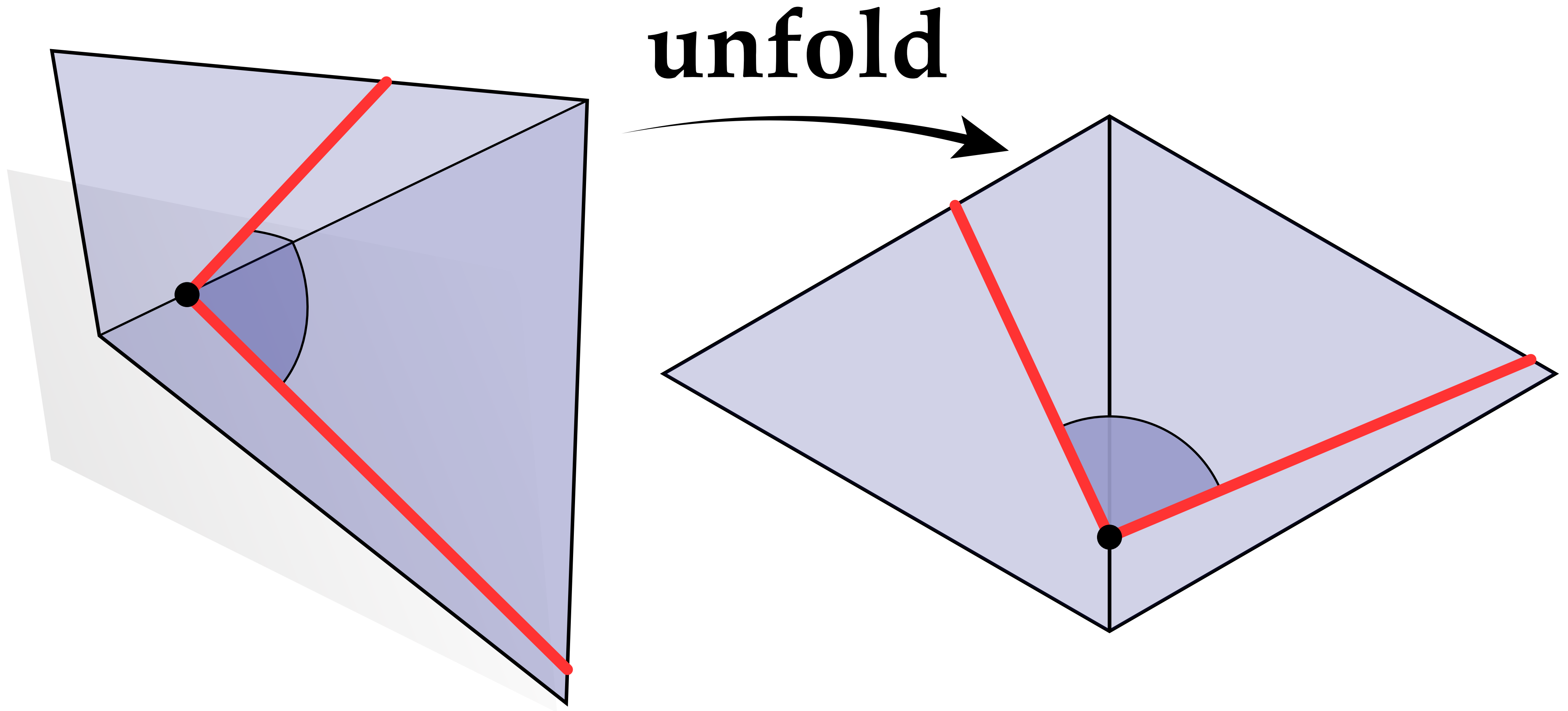


Discrete Geodesic Curvature

- For planar curve, one definition of discrete curvature was *exterior angle* (or π-interior)
- Since most points of a simplicial surface are *intrinsically* flat, can adopt this same definition for discrete geodesic curvature
- *Faces*: just measure angle between segments
- *Edges*: "unfold" and measure angle
- *Vertices*: not as simple—can't unfold!
  - Recall trouble w/ **shortest** geodesics…

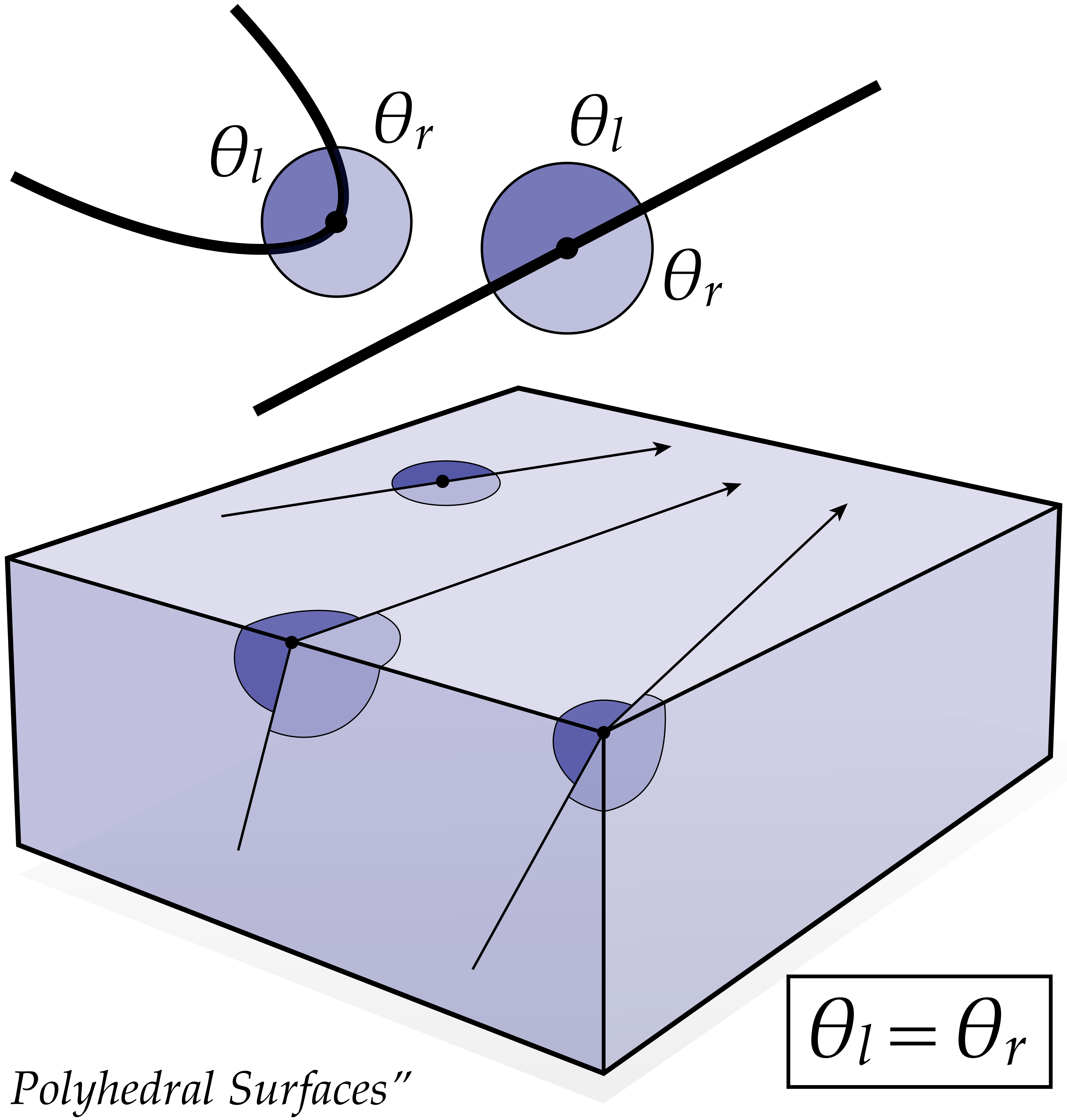






Discrete Straightest Geodesics

- In the smooth setting, characterized geodesics as curves with zero geodesic curvature
- In the discrete setting, have a hard time defining geodesic curvature at vertices
- Alternative smooth characterization: just have <u>same</u> angle on either side of the curve
- Translates naturally to the discrete setting: equal angle sum on either side of the curve
- Provides definition of discrete **straightest** geodesics (Polthier & Schmies 1998)

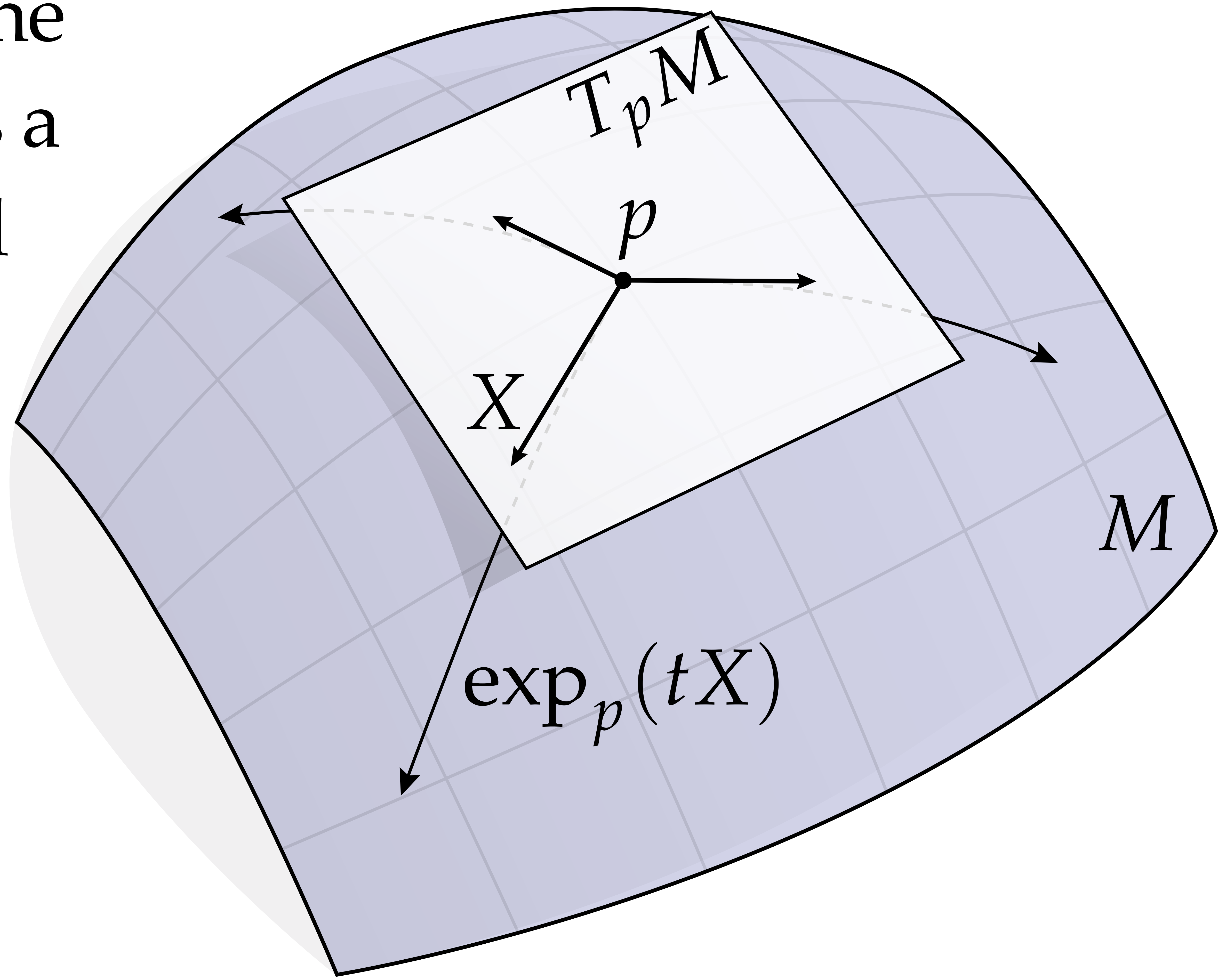


$$\boxed{\theta_l = \theta_r}$$

Image adapted from Polthier & Schmies, "Straightest Geodesics on Polyhedral Surfaces"

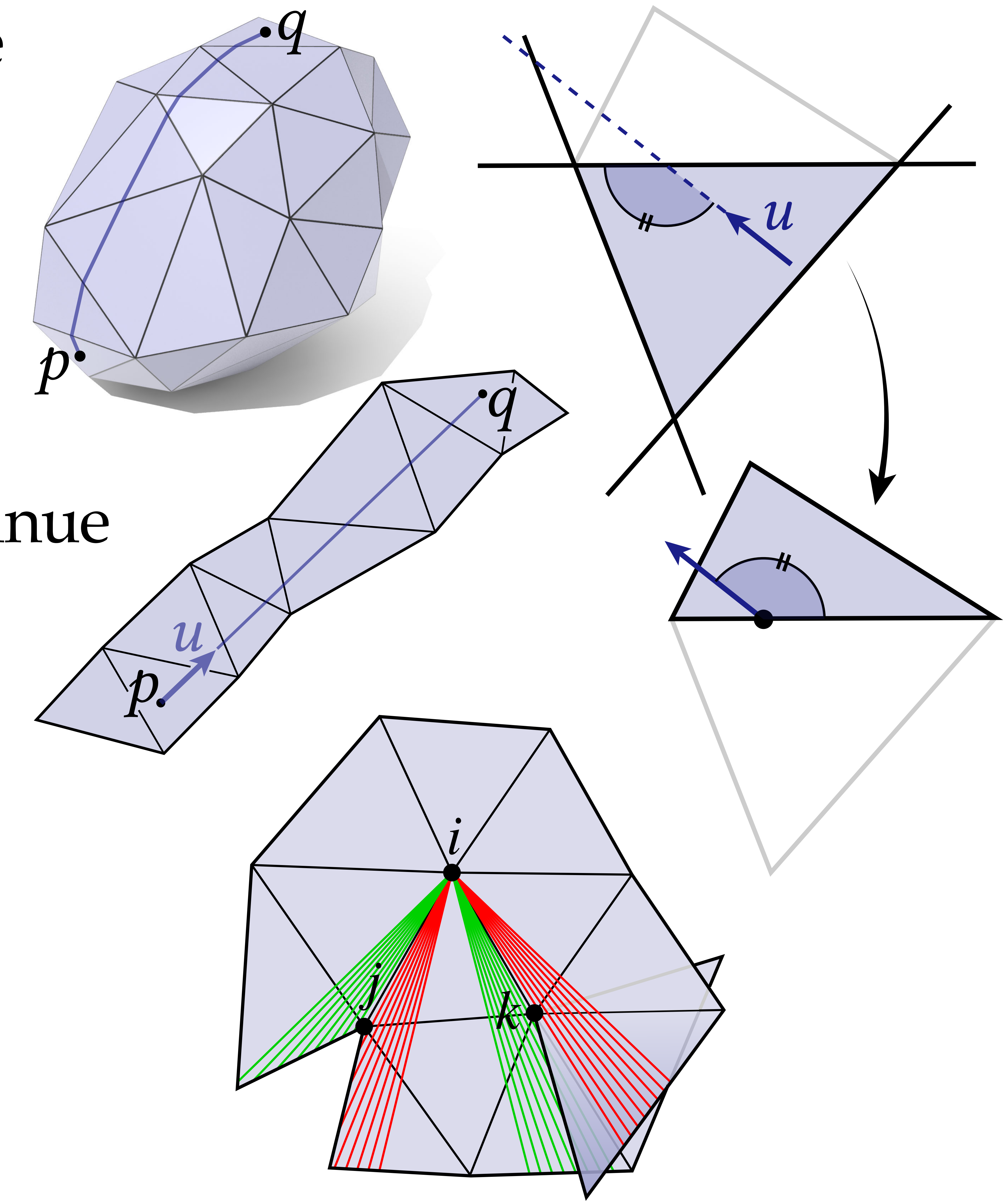
Exponential Map

- At a point p of a smooth surface M, the *exponential map* $\exp_p\colon T_pM \to M$ takes a tangent vector X to the point reached by walking along a geodesic in the direction $X/|X|$ for distance $|X|$
- Can also view as a map "wrapping" the tangent plane around the surface
- **Q:** Is this map surjective? Injective?
- *Injectivity radius* at p is radius of largest ball where $\exp_p$ is injective



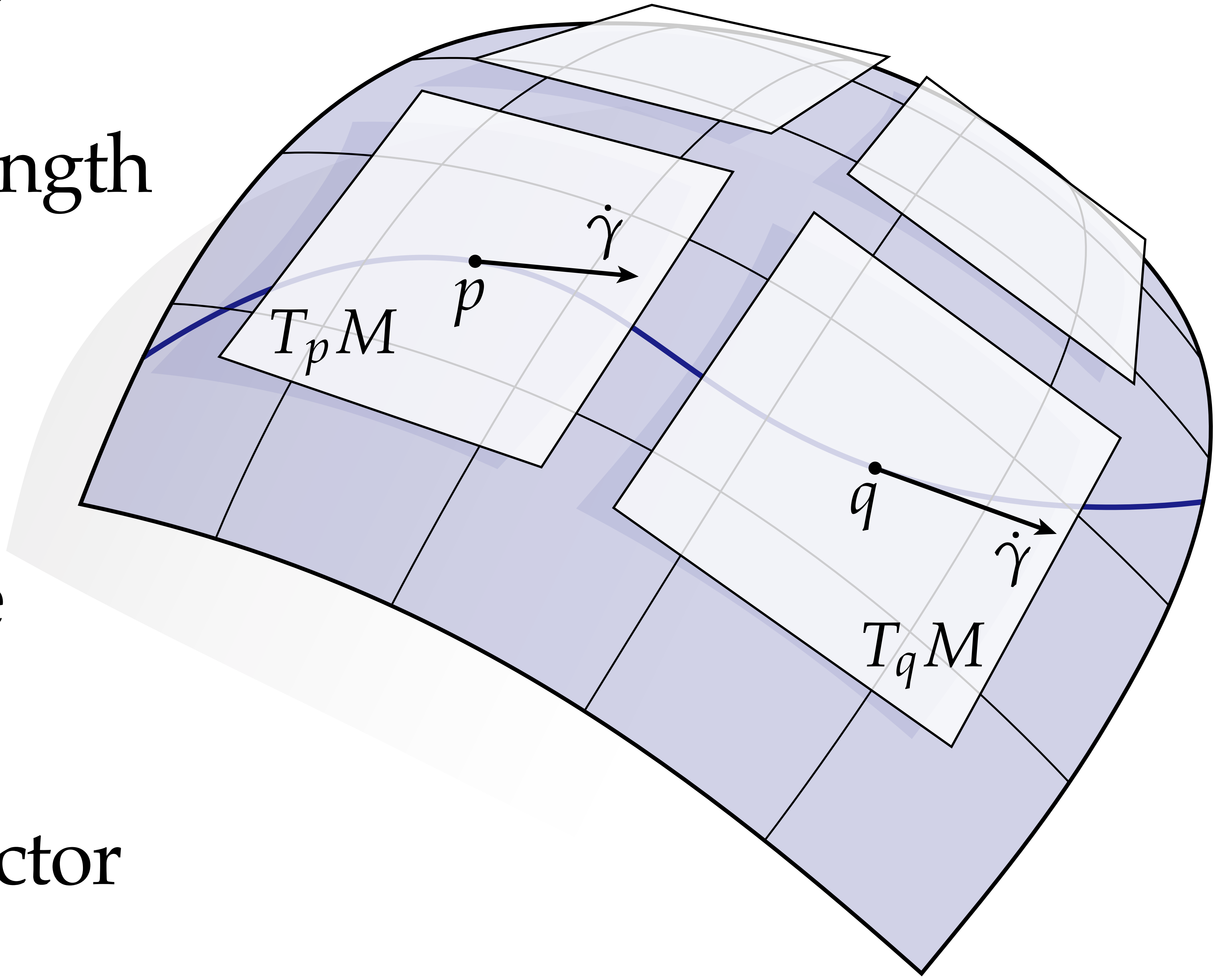
Discrete Exponential Map

- Not so hard to evaluate exponential map on discrete surface
- Given point and tangent vector, start walking along vector
  - “walking” amounts to 2D ray tracing
- At vertices, *straightest* definition tells us how to continue
- (Still have to think about what it means to *start* at a vertex—what are tangent vectors?)
- **Q:** How big is the injectivity radius?
- **A:** Just the distance to the closest vertex!
- **Q:** Is the discrete exponential map surjective?
- **A:** No! Consider a saddle vertex…



Straightness—Dynamic Perspective

- Dynamically, geodesic has *zero <u>tangential</u> acceleration*
- How exactly do we define "tangential acceleration"?
- Consider curve $\gamma(t)$: $[a,b] \longrightarrow M$ (*not* necessary arc-length parameterized)
- Tangential *velocity* is simply the tangent to the curve
- Tangential acceleration should be something like the "change in the tangent," but:
  - **extrinsically**, change in tangent is not a <u>tangent</u> vector
  - **intrinsically**, tangents belong to different vector spaces
- So, how do we measure acceleration?



Covariant Derivative

- Since geodesics are intrinsic, can define "straightness" using only the metric g
- *Covariant derivative* ∇ measures the change of one tangent vector field along another.
- For any function ϕ, tangent vector fields X, Y, Z, operator ∇ uniquely determined by

$$\begin{aligned} \nabla_Z(X+Y) &= \nabla_Z X + \nabla_Z X \\ \nabla_{X+Y} Z &= \nabla_X Z + \nabla_Y Z \\ \nabla_{fX} Y &= f \nabla_X Y \\ \nabla_X(fY) &= df(X)Y + f\nabla_X Y \end{aligned}$$

$$\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Can really "solve" these equations for ∇ in terms of g (*Christoffel symbols*). We won't!

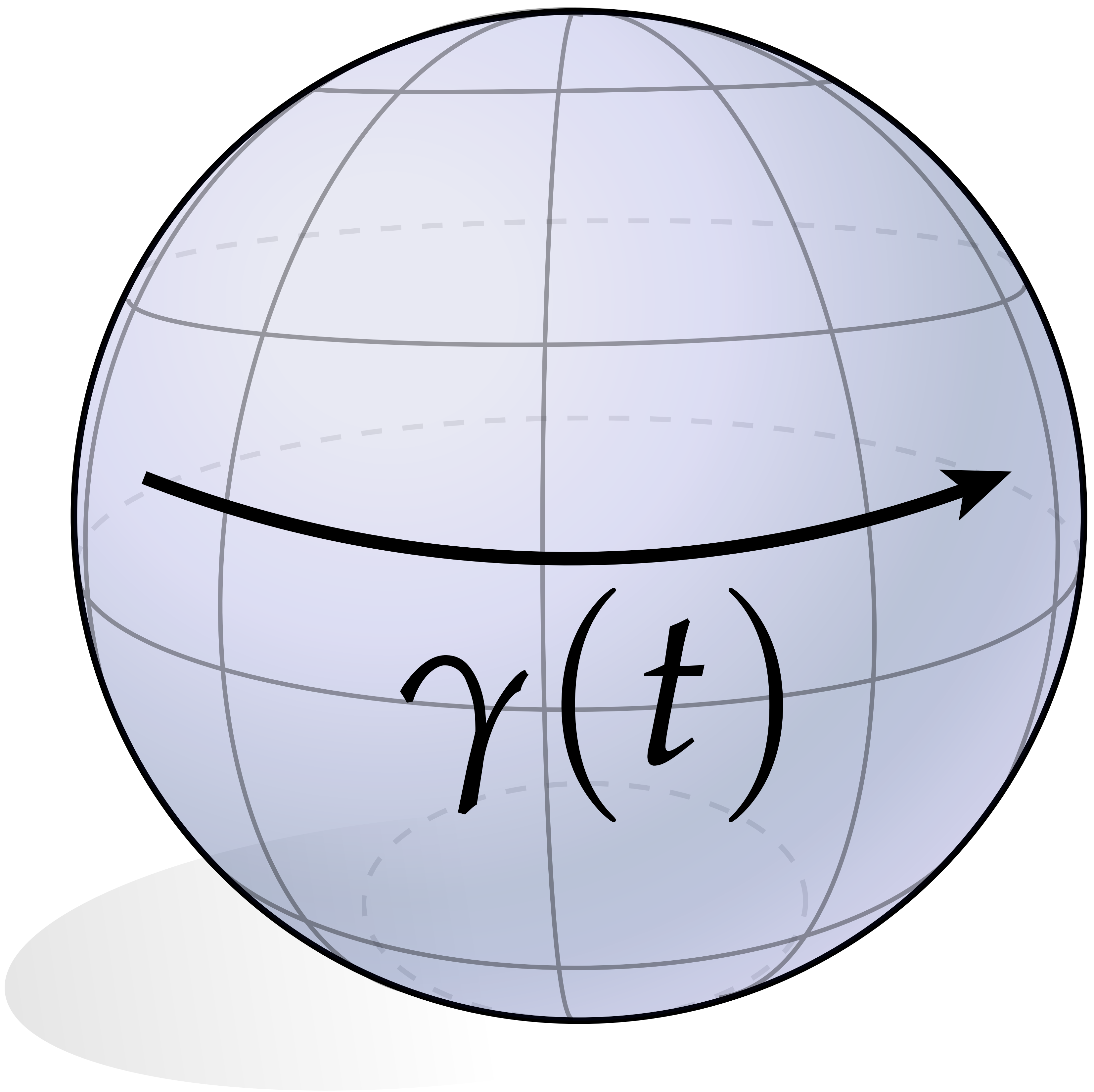
Geodesic Equation

Covariant derivative provides another, quite classic characterization of geodesics:

tangent to curve

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

"tangent doesn't turn"



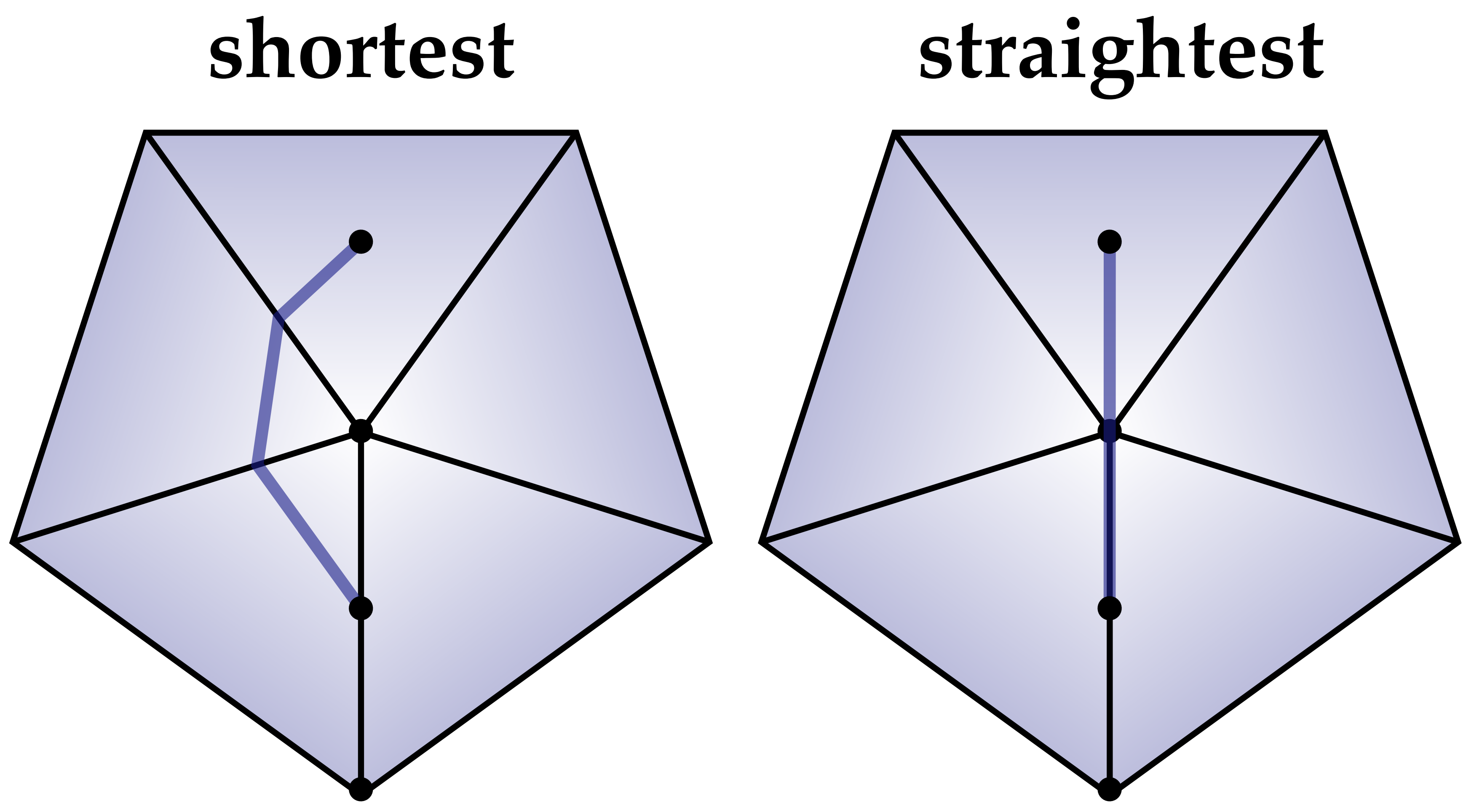
Q: Does this characterization suggest another approach to discrete geodesics?

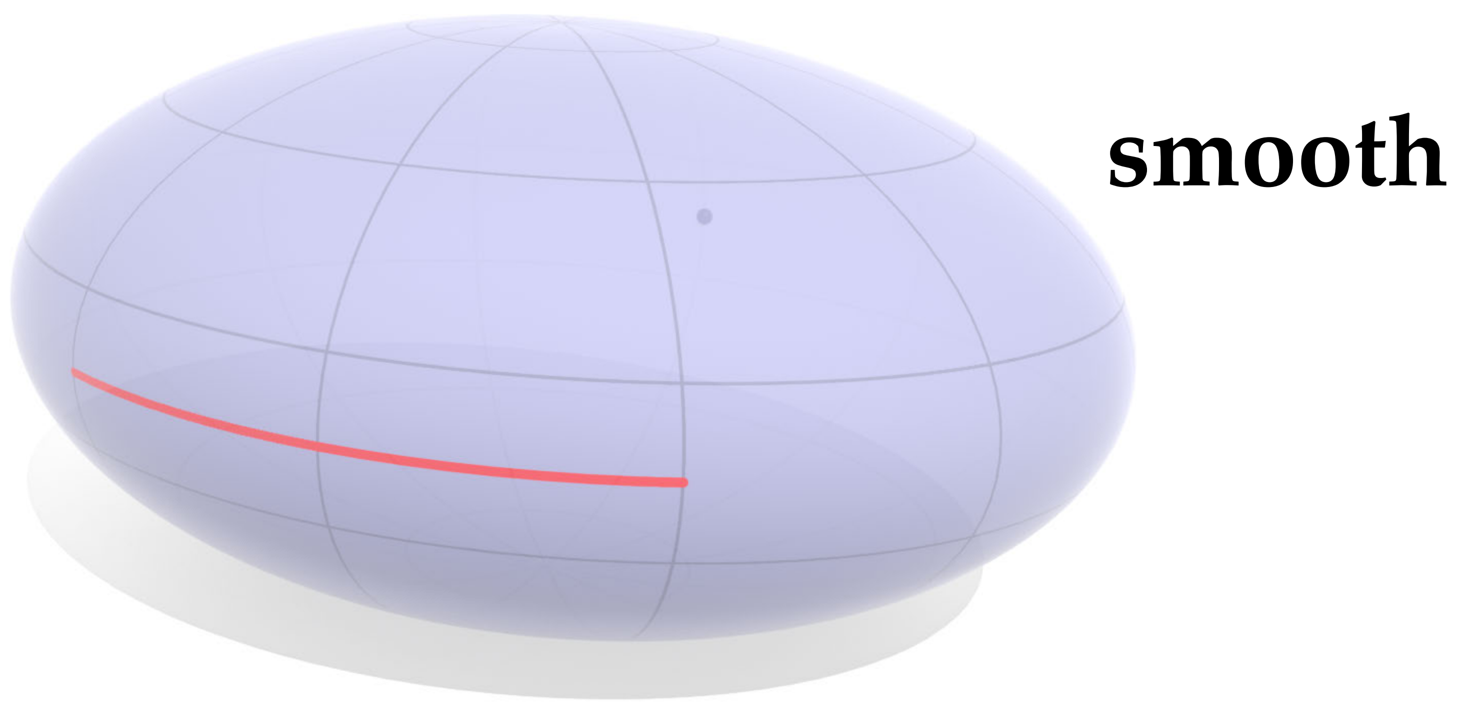
A: *Maybe*—though to go down that road we'll need *discrete connections* (later…)

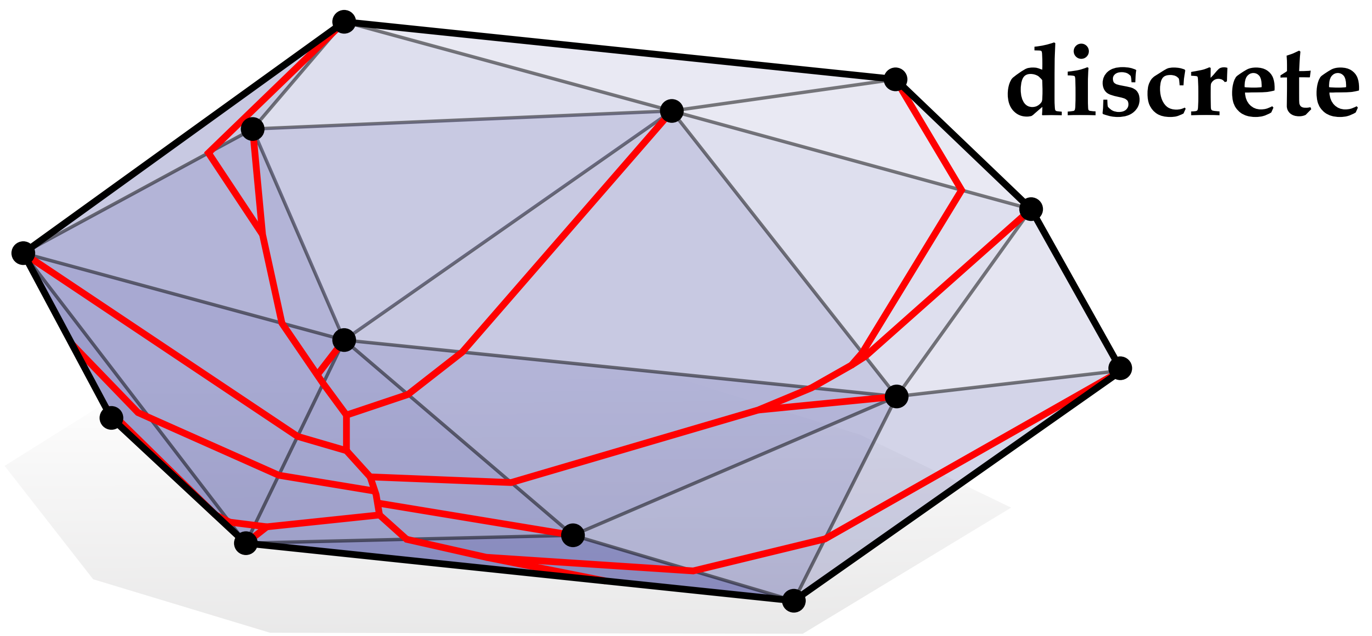
Summary

Geodesics—Shortest vs. Straightest, Smooth vs. Discrete

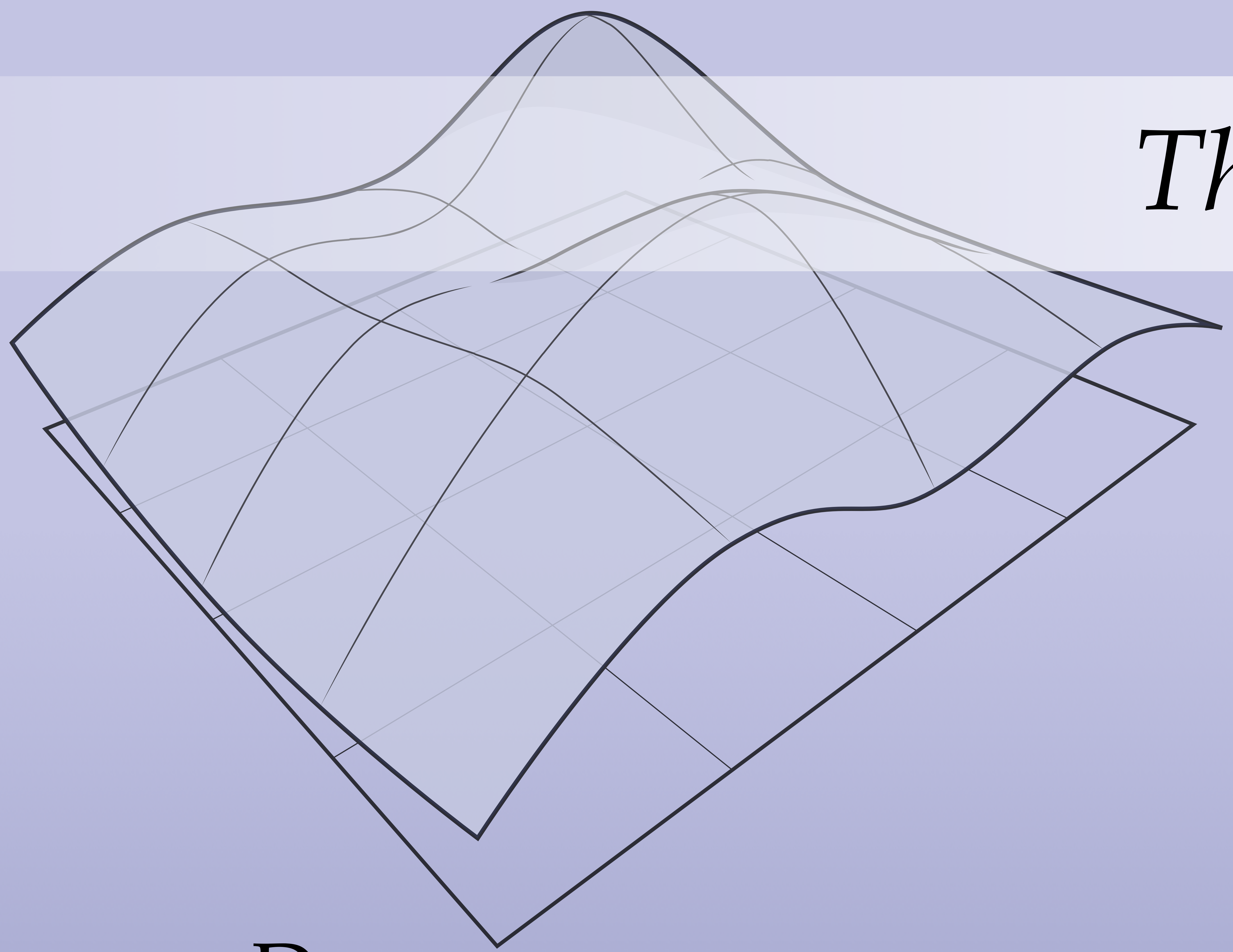
- In smooth setting, several equivalent characterizations:
  - shortest (harmonic)
  - straightest (zero curvature, zero acceleration)
- In discrete setting, characterizations no longer agree!
  - **shortest** natural for boundary value problem
  - **straightest** natural for initial value problem
  - *convex:* shortest paths are straightest (but not vice versa)
  - *nonconvex:* shortest may not even be straightest! (saddles)
- *Neither* definition faithfully captures all smooth behavior:
  - (shortest) cut locus/medial axis touches *every* convex vertex
  - (straightest) exponential map is not surjective
- Use the right tool for the job (*and look for other definitions!*)







Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

CMU 15-458/858 • Keenan Crane