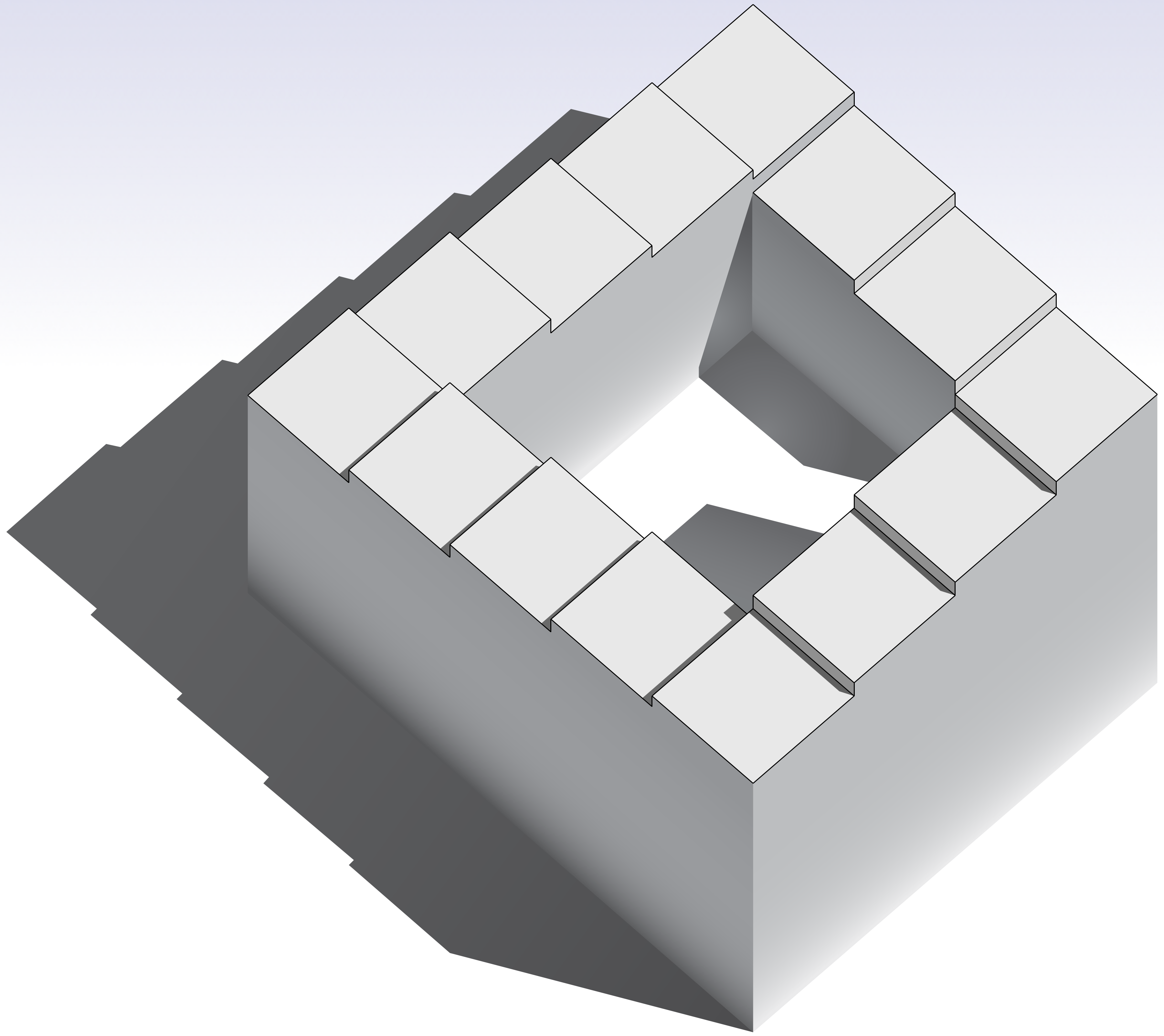
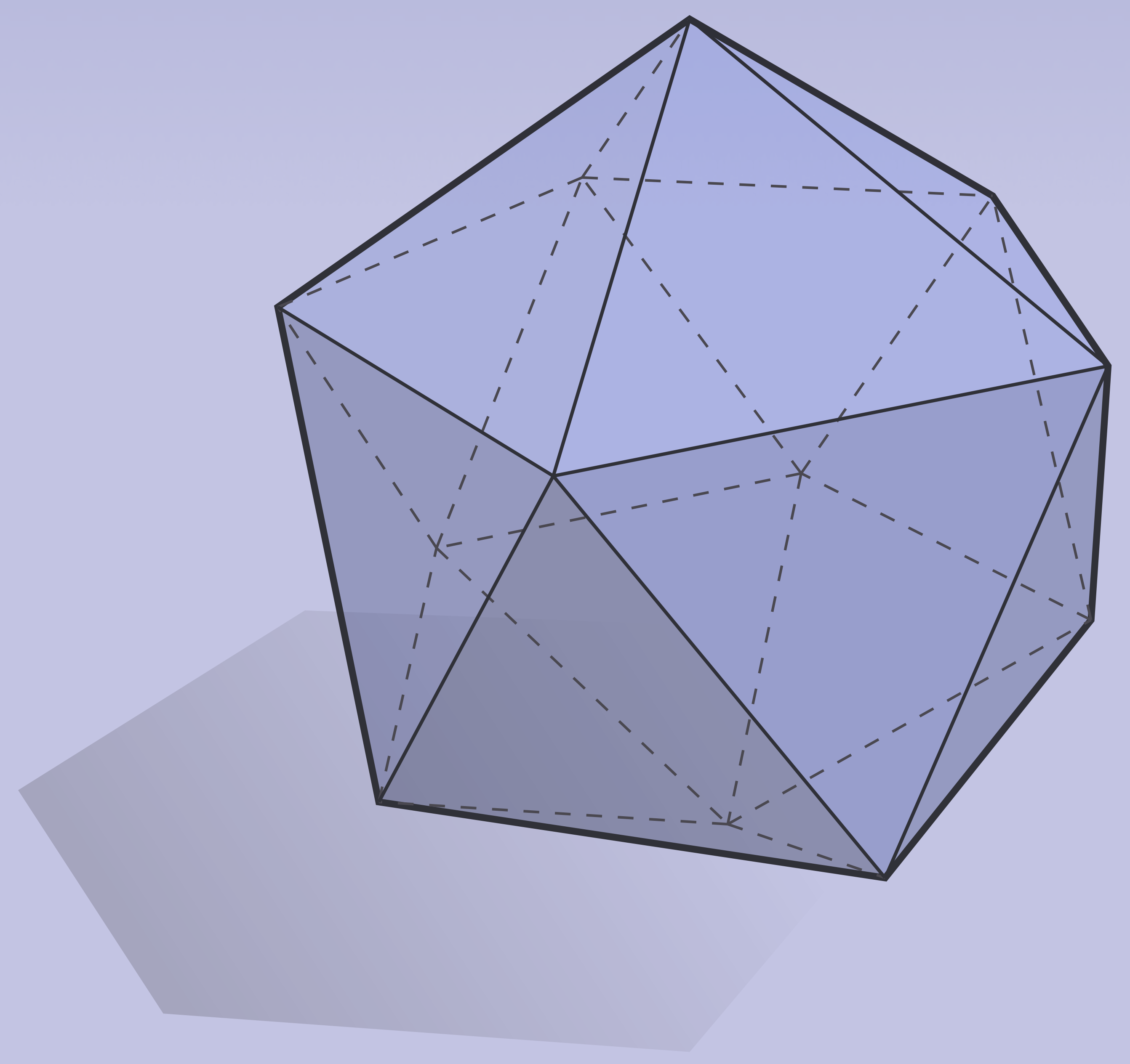


# Discrete Differential Geometry: An Applied Introduction

**Keenan Crane • CMU 15-458/858**

# Lecture 11:

# Discrete Curves

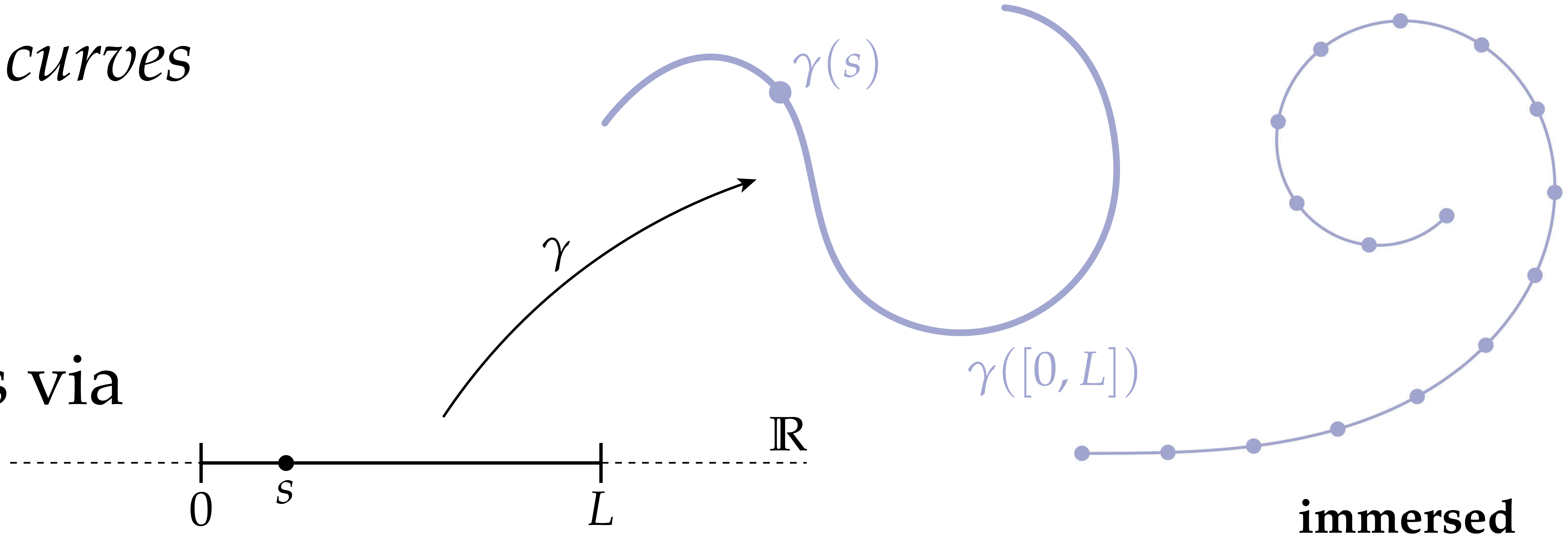


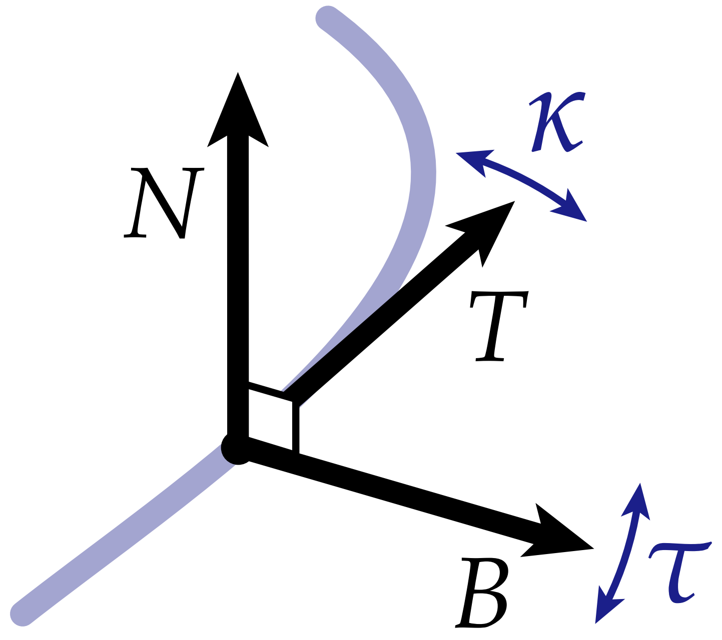
## Discrete Differential Geometry: An Applied Introduction

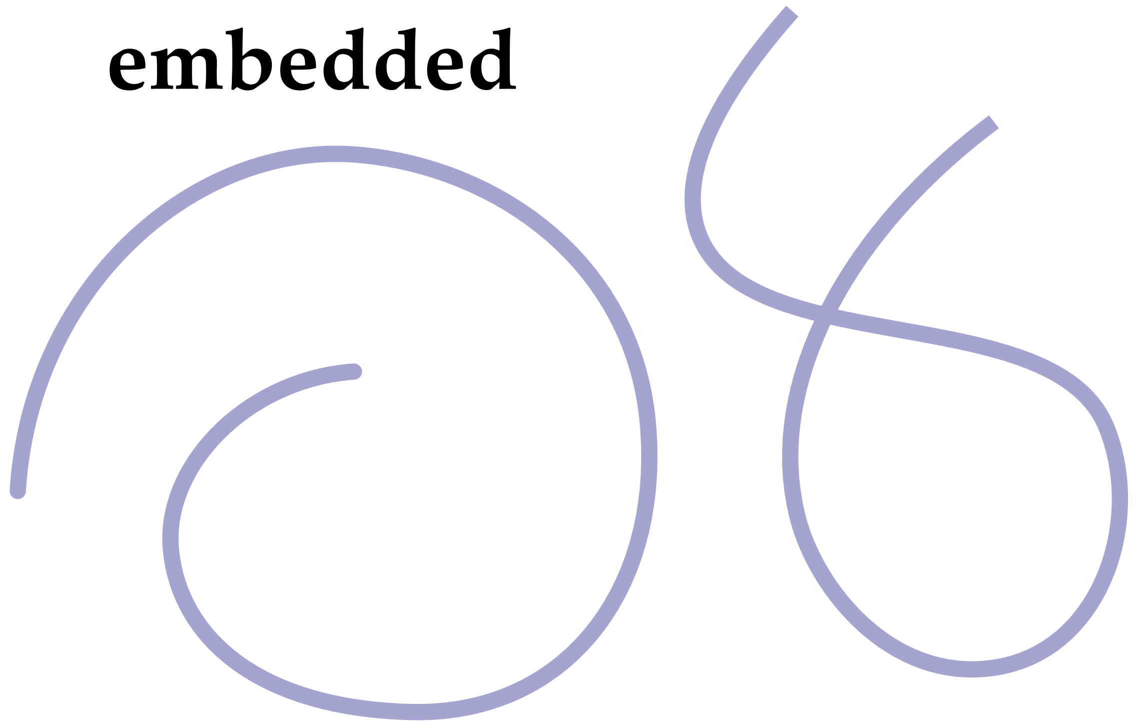
Keenan Crane • CMU 15-458/858

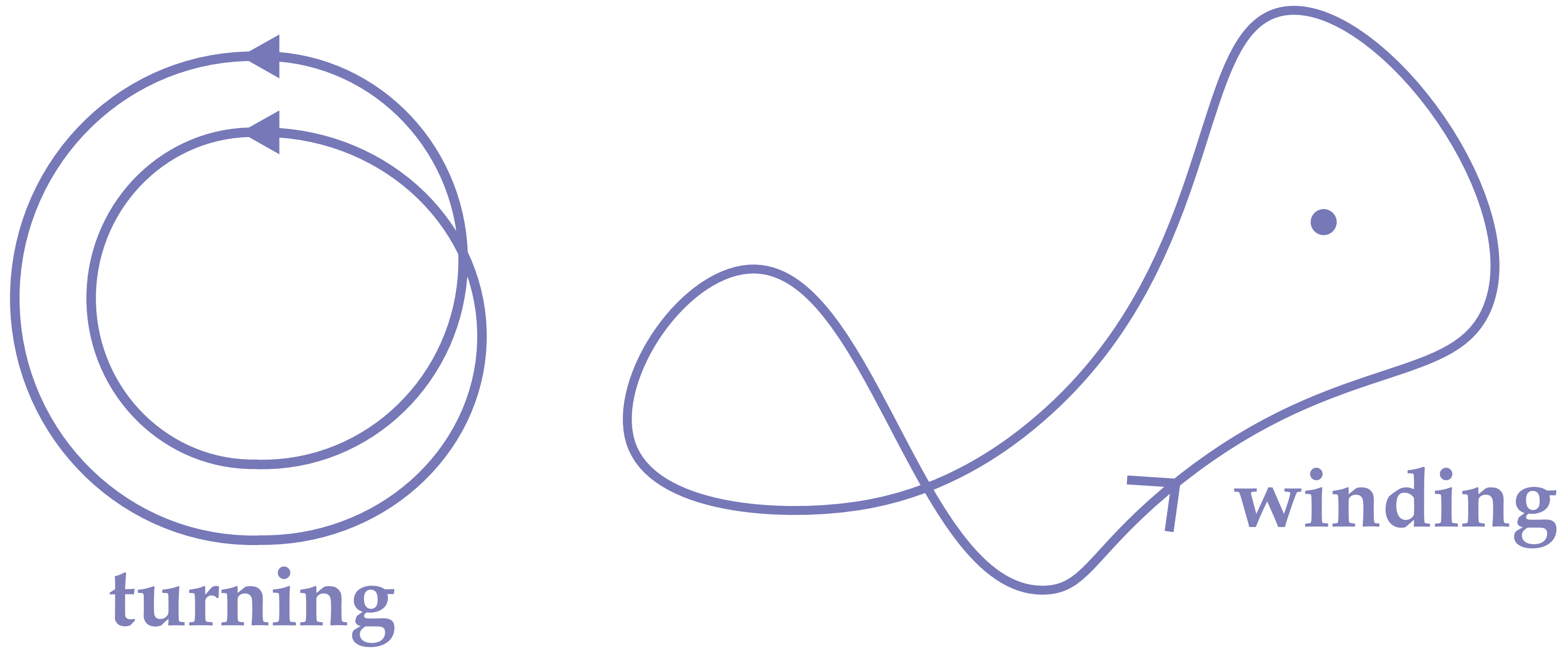
# *Recap — Smooth Curves*

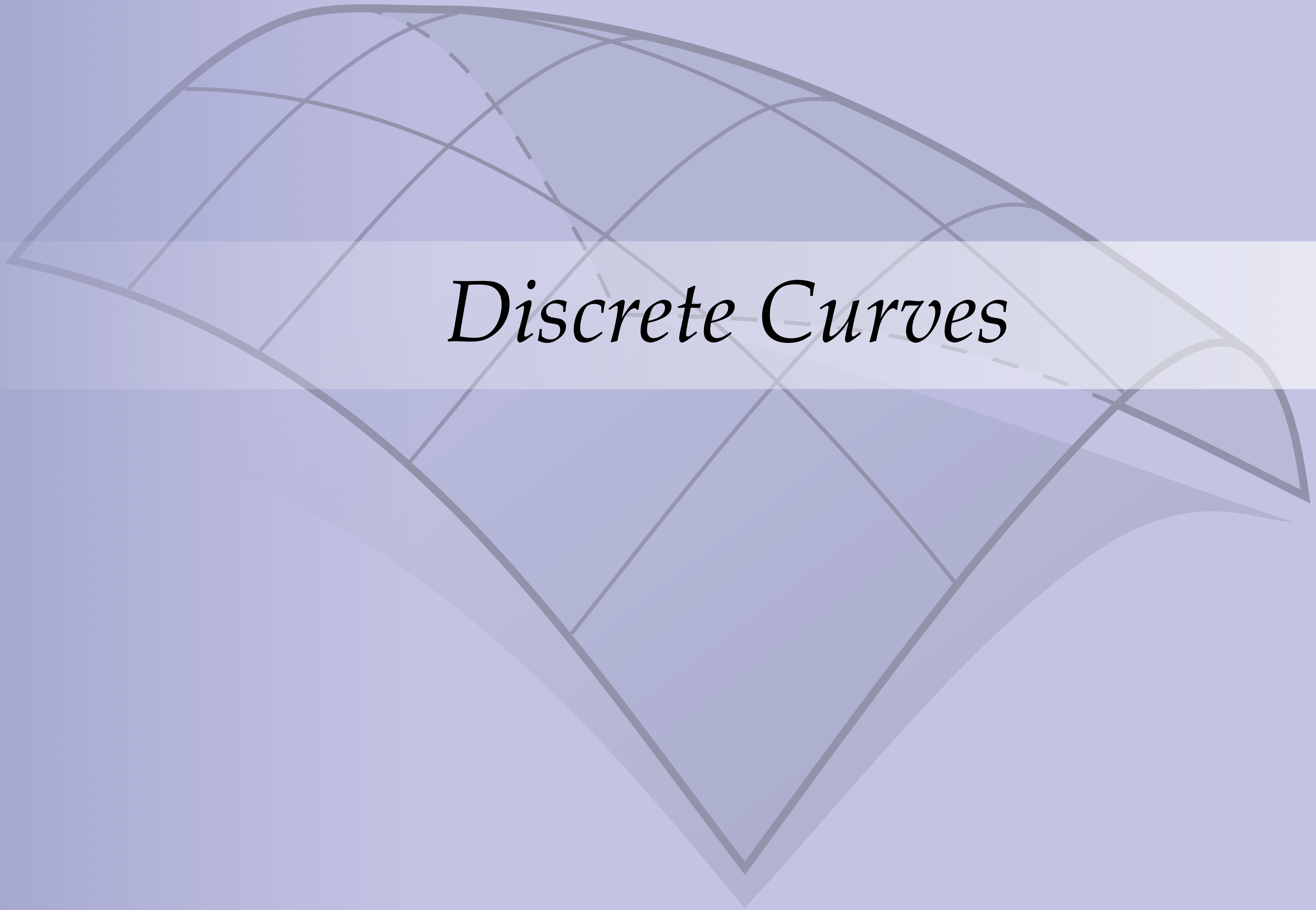
- Last time: introduced *parameterized curves*
  - every curve has *many* possible parameterizations
  - express important <u>local</u> quantities via derivatives of parameterization
  - tangent, normal, binormal (*Frenet frame*), curvature, torsion
- Embedded vs. immersed / regular
- *Turning number*—degree of tangent map
- *Winding number*—degree of map around point
- **Fundamental theorem:** recover from curvatures
- Today: discrete point of view!



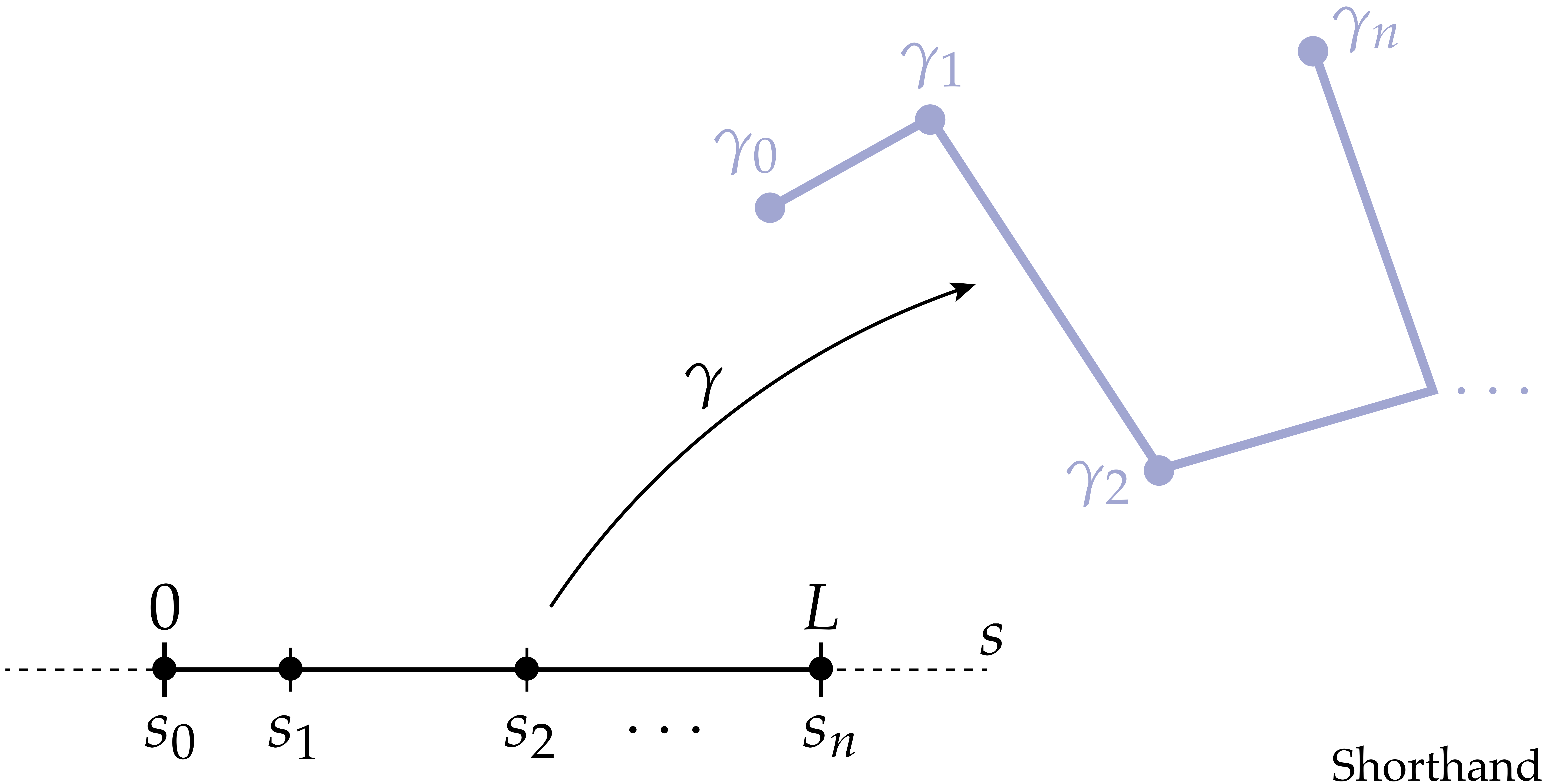





# *Discrete Curves*

# *Discrete Curves in the Plane*

We'll define a **discrete curve** as a *piecewise linear* parameterized curve, *i.e.*, a sequence of points connected by straight line segments:

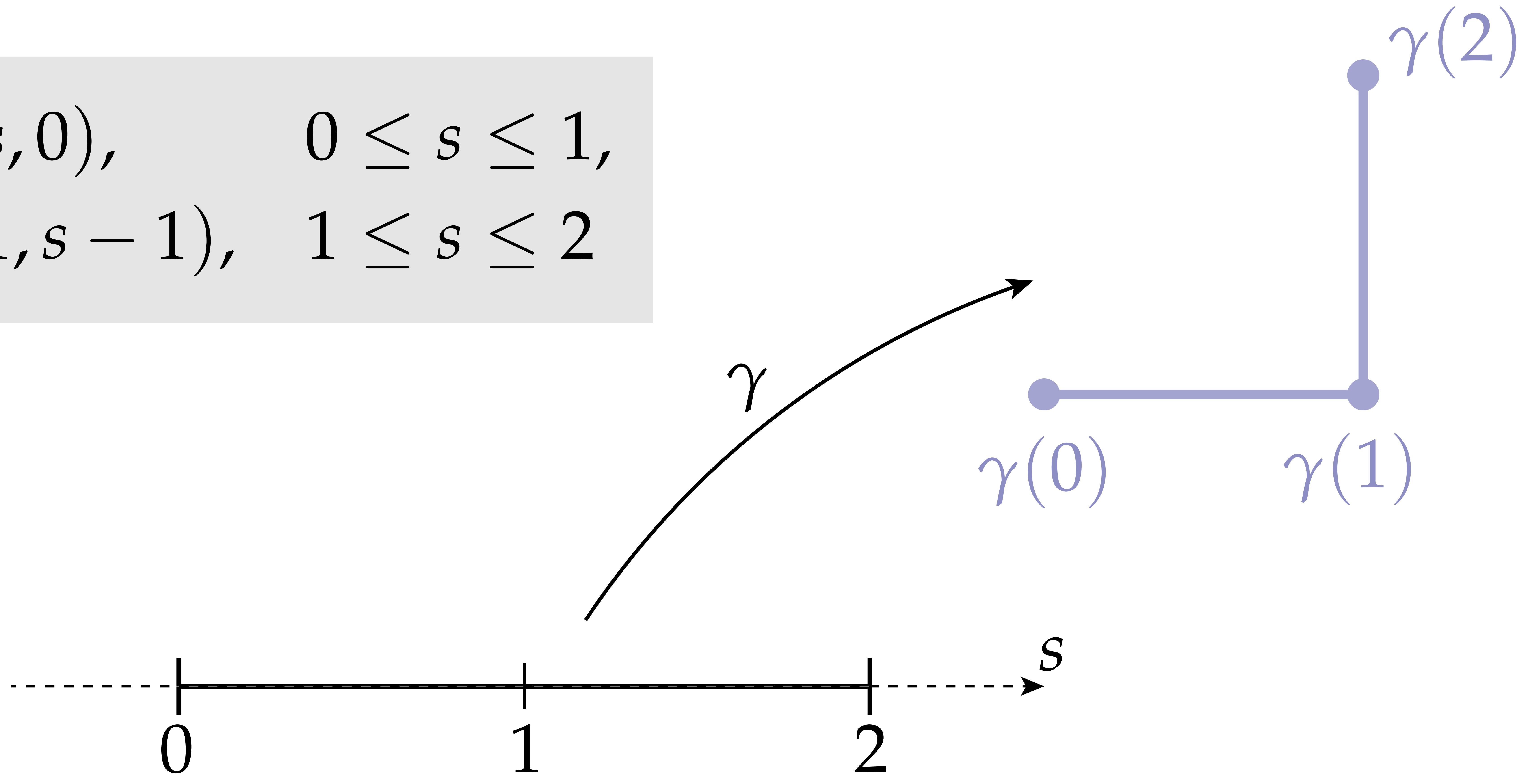



Shorthand: $\gamma_i := \gamma(s_i)$

# *Discrete Curves in the Plane—Example*

A simple example is a curve comprised of two segments:

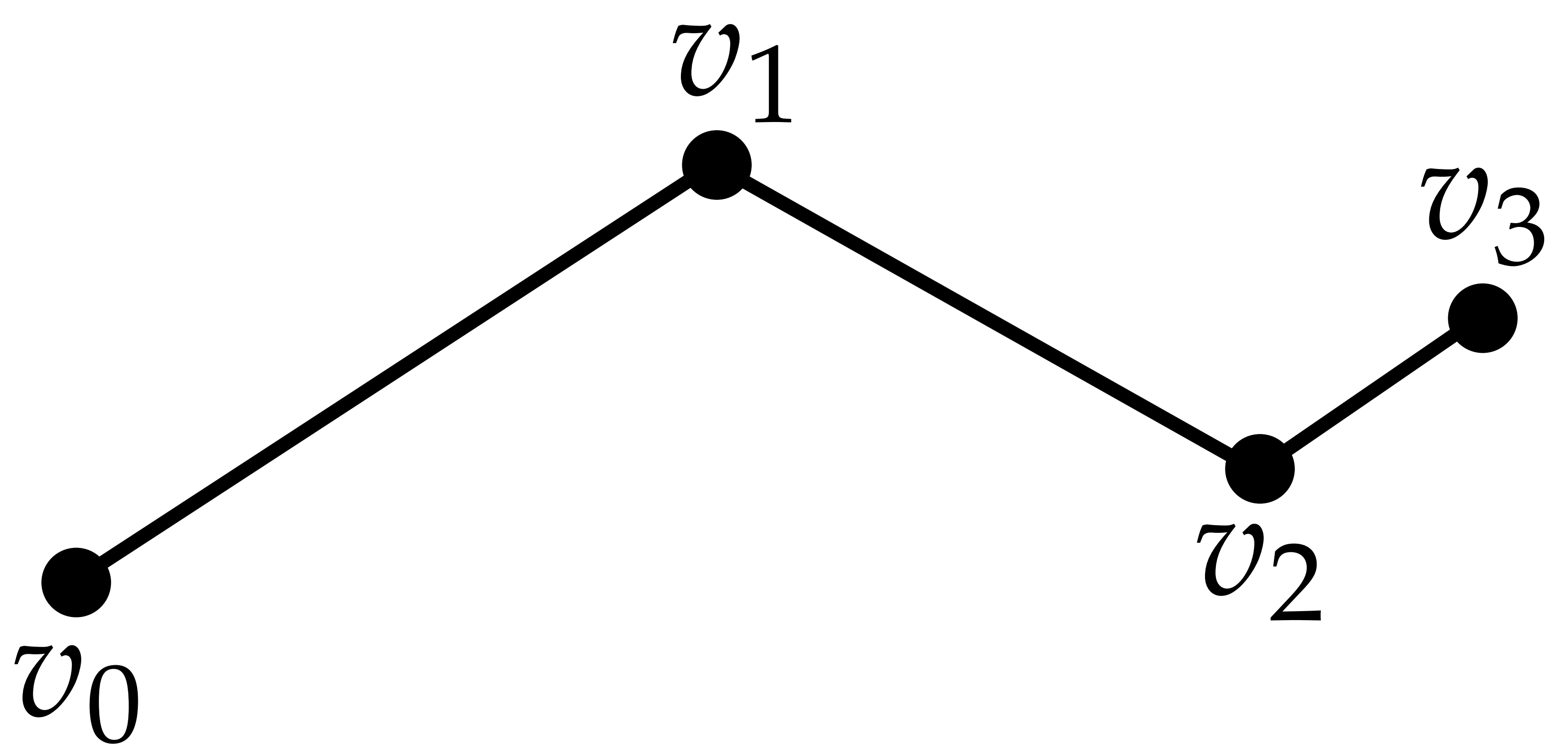
$$\gamma(s) := \begin{cases} (s,0), & 0 \leq s \leq 1, \\ (1, s-1), & 1 \leq s \leq 2 \end{cases}$$




**Key idea:** a "discrete curve" is also a continuous map… but fairly atypical to write it this way.

# *Discrete Curves and Discrete Differential Forms*

- Equivalently, a discrete curve is determined by a discrete, $\mathbb{R}^n$-valued 0-form $\gamma$ on a (manifold, oriented) abstract simplicial 1-complex
- The 0-form values give the location of the vertices; interpolation by Whitney bases (hat functions) gives the map from each edge to $\mathbb{R}^n$



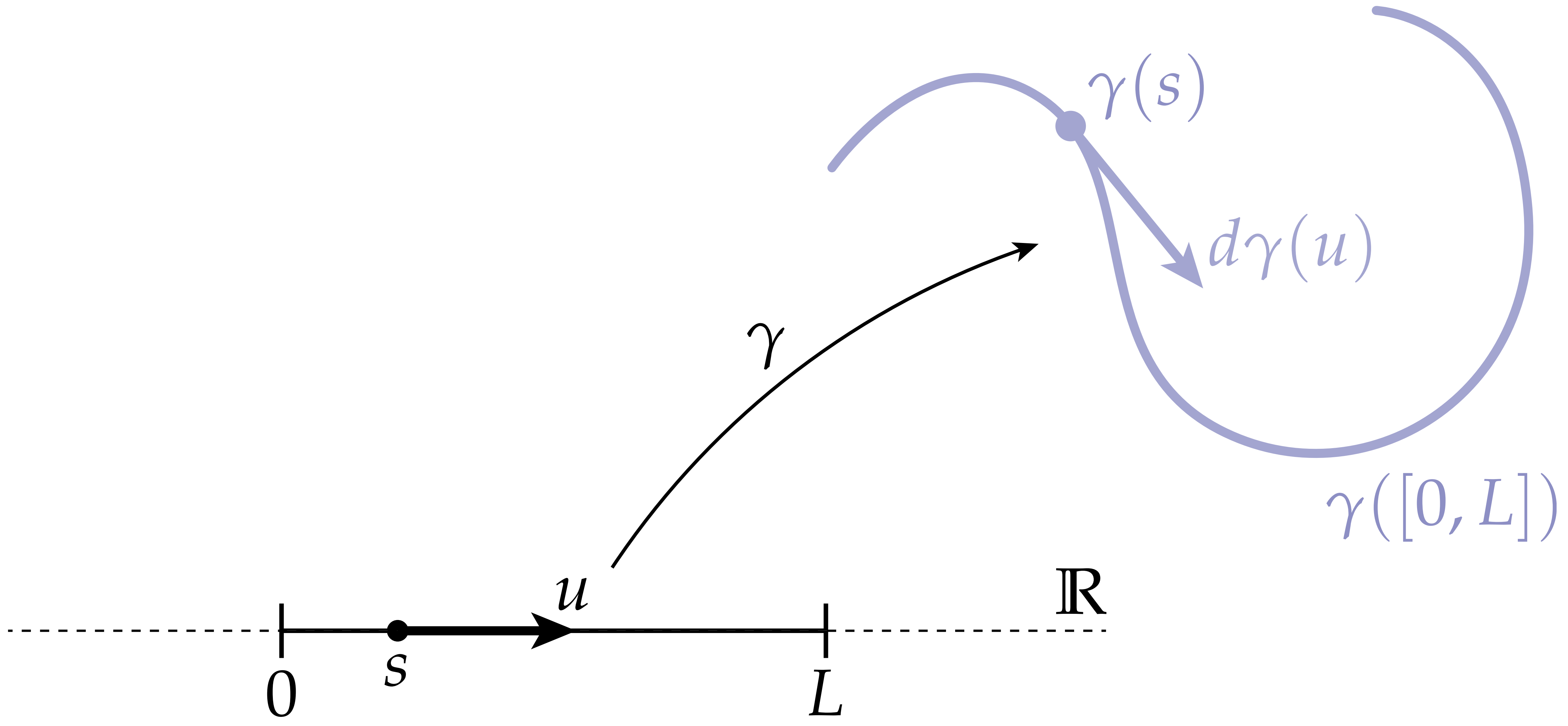

$$K = \{\, (v_0,v_1), (v_1,v_2), (v_2,v_3), (v_0), (v_1), (v_2), (v_3), \emptyset \,\}$$

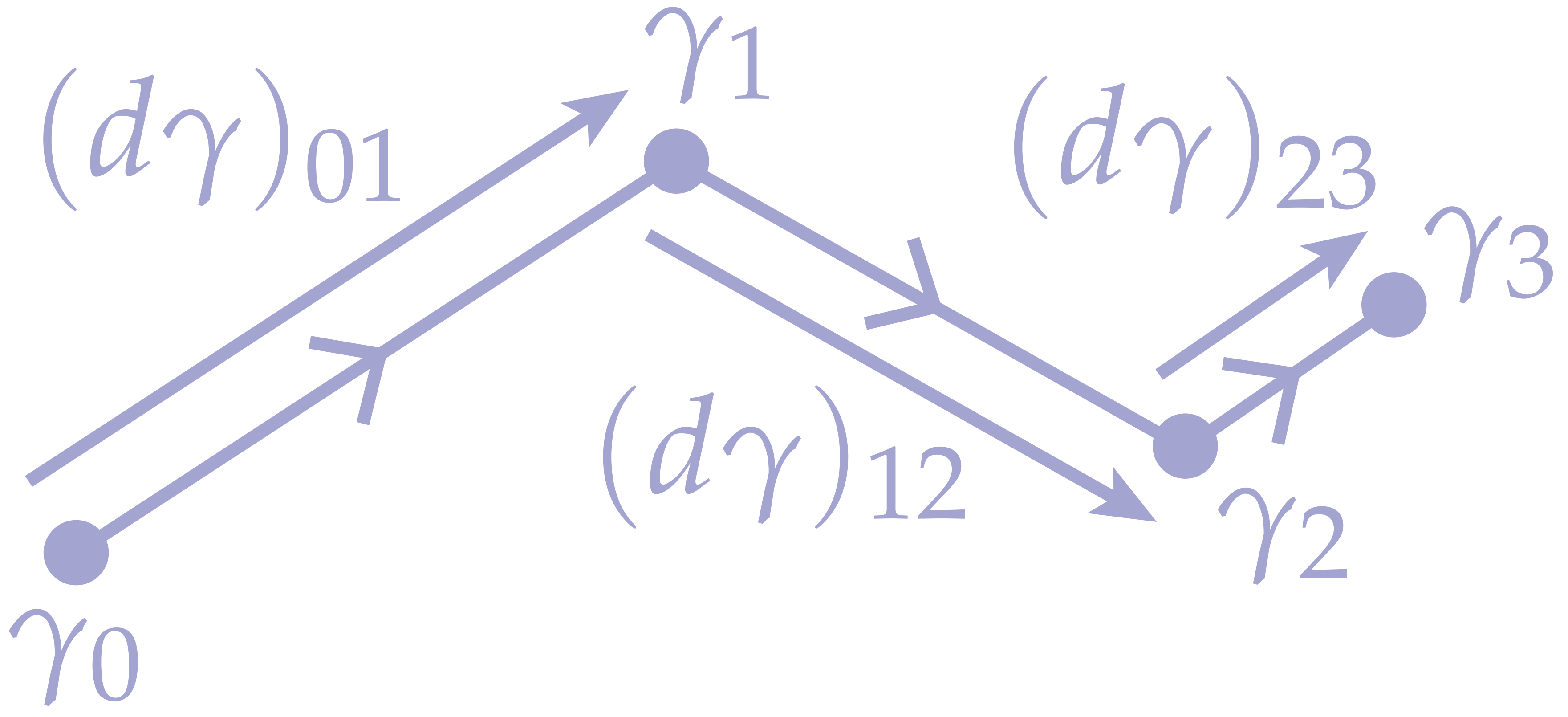
$$\begin{aligned} \gamma(v_0) &= (33, 66) \\ \gamma(v_1) &= (79, 36) \\ \gamma(v_2) &= (118, 58) \\ \gamma(v_3) &= (134, 47) \end{aligned}$$

# *Differential of a Discrete Curve*

- We can now directly translate statements about **smooth** curves expressed via **smooth** exterior calculus into statements about **discrete** curves expressed using **discrete** exterior calculus

- Simple example: the *differential* just becomes the edge vectors:

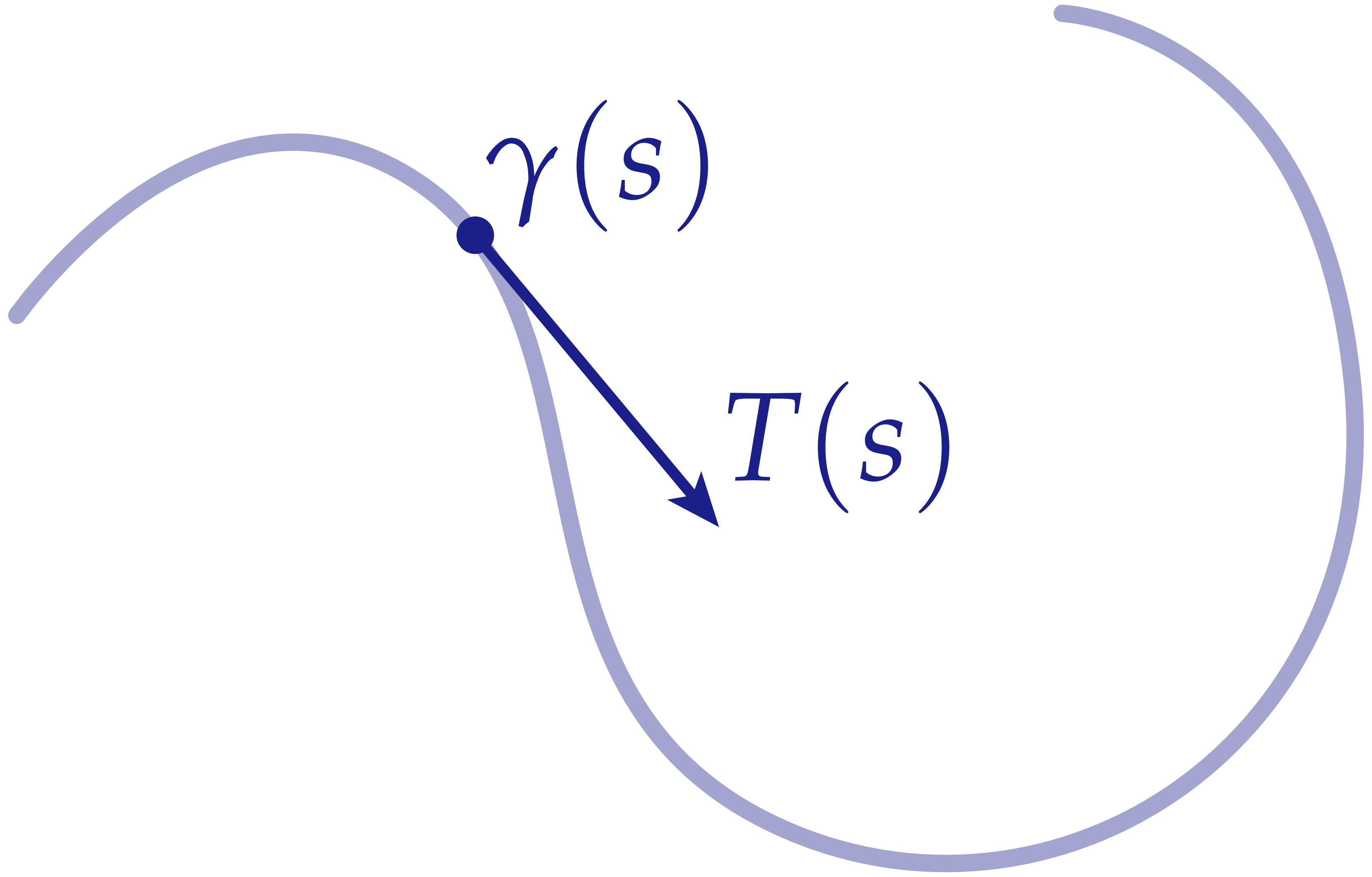


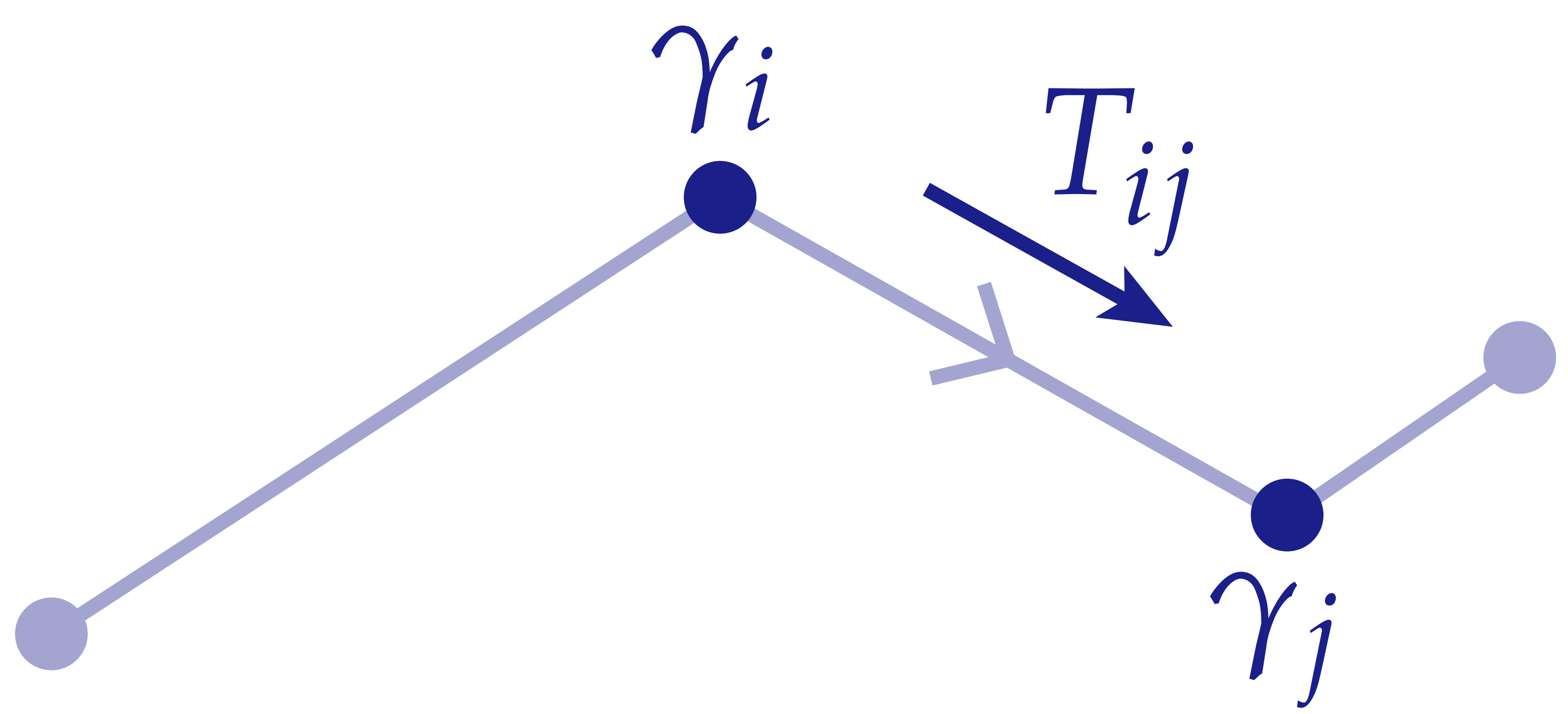




$$(d\gamma)_{ij} = \gamma_j - \gamma_i$$

# Discrete Tangent

As in smooth setting, can simply normalize differential to obtain tangents, yielding a vector per edge*



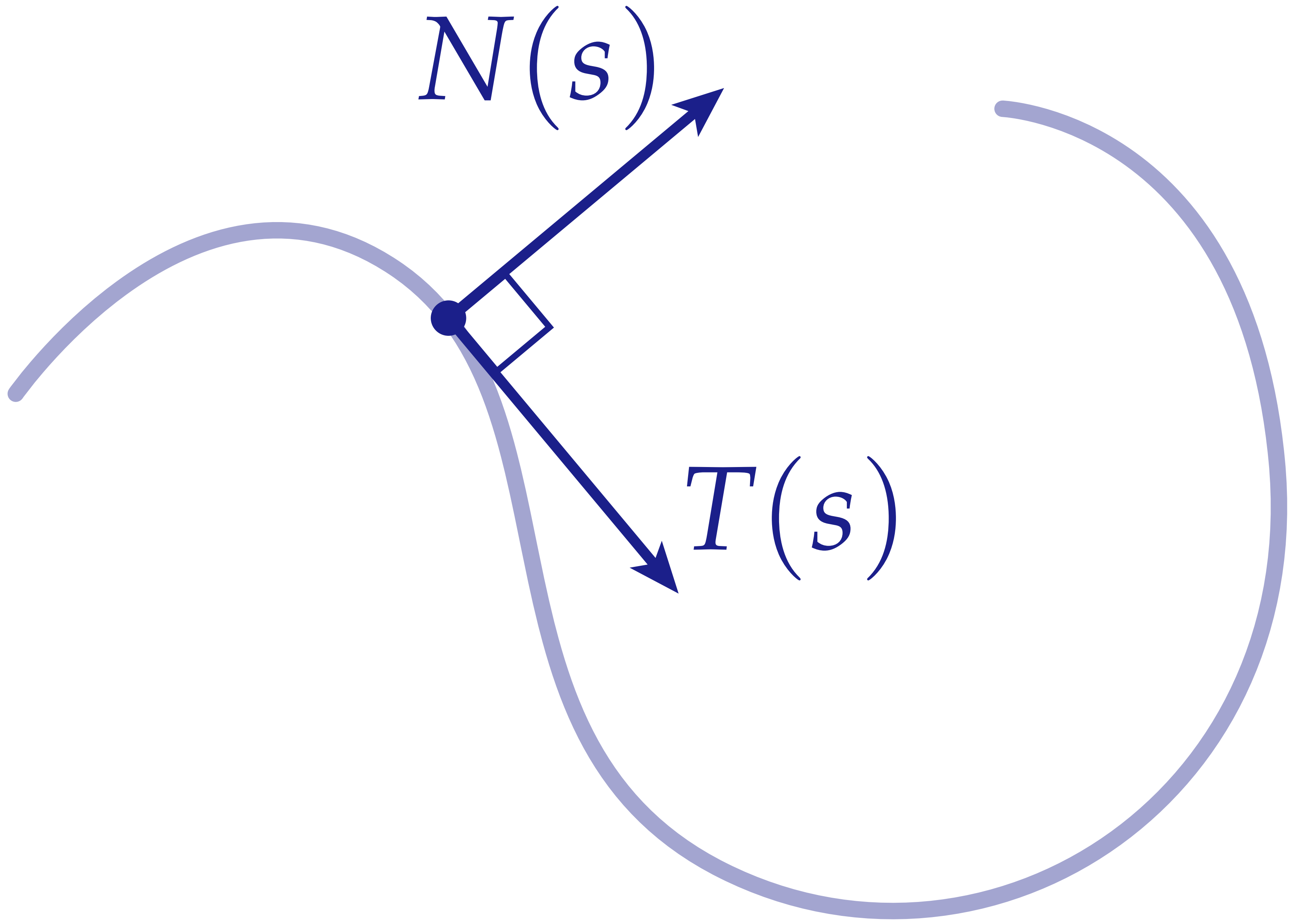

$$T(s) := d\gamma(\tfrac{d}{ds})/|d\gamma(\tfrac{d}{ds})|$$

$$T_{ij} := (d\gamma)_{ij}/|(d\gamma)_{ij}|$$

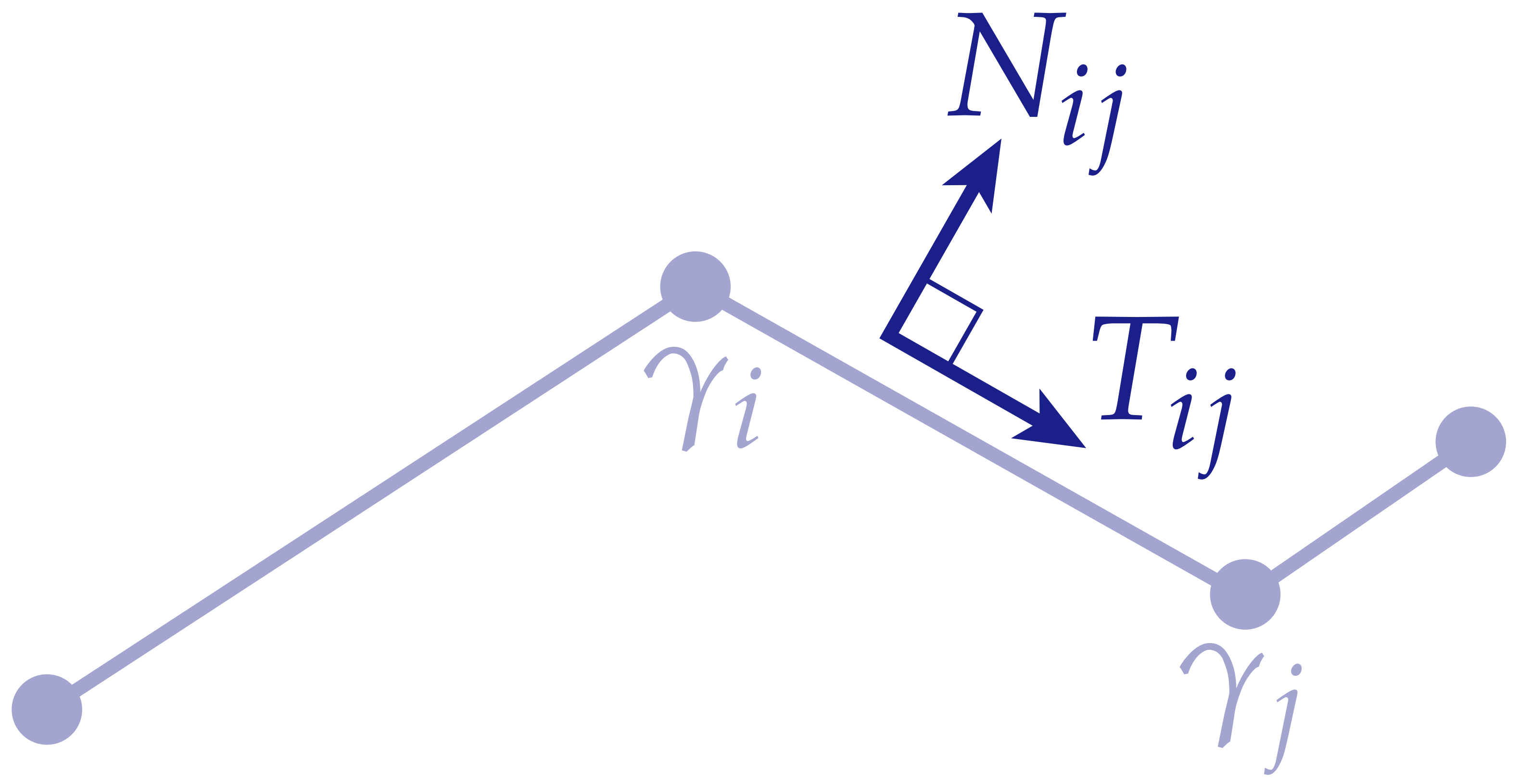
*And no definition of the tangent at vertices!

# *Discrete Normal*

As in the smooth setting, we can express the (discrete) normals of a planar curve as a 90-degree rotation of the (discrete) tangent:



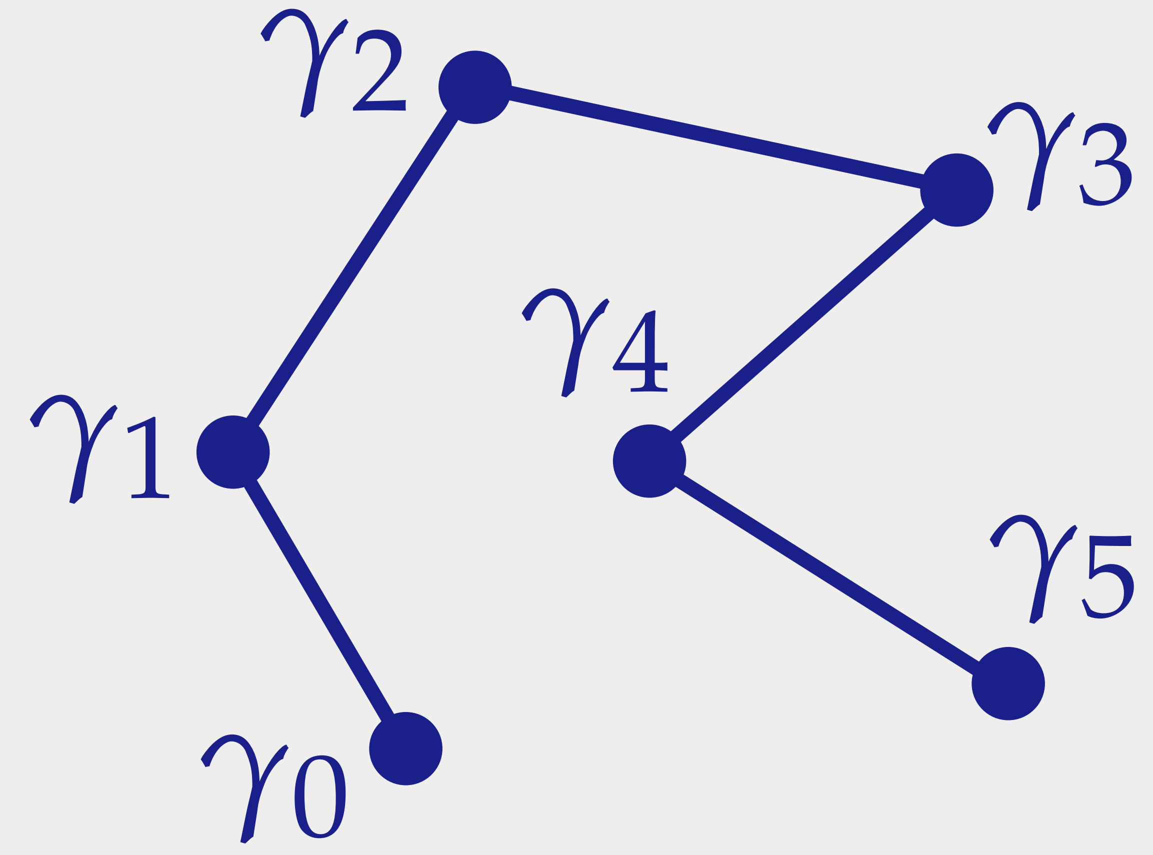

$$N(s) = \mathcal{J}T(s)$$



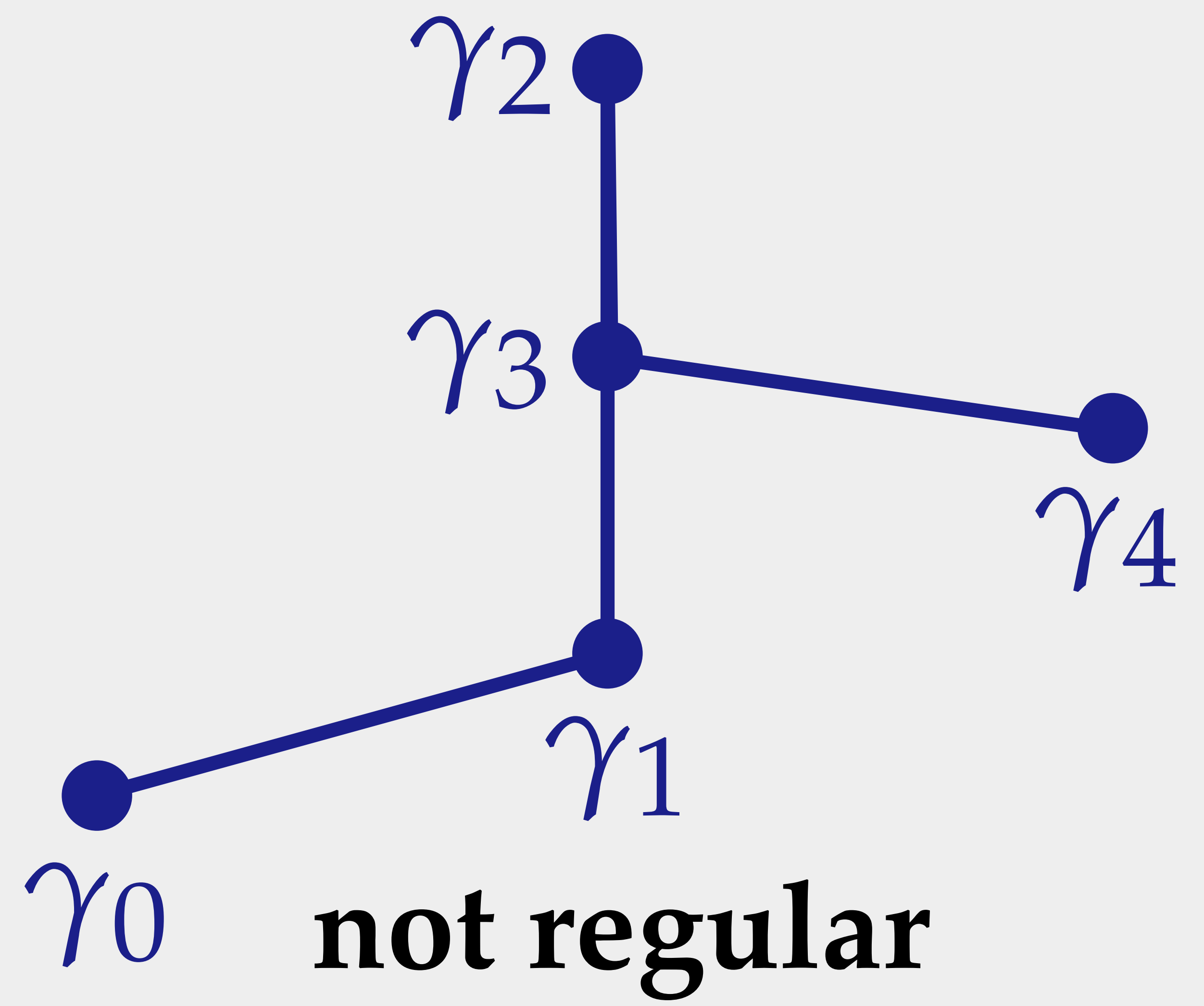

$$N_{ij} = \mathcal{J}T_{ij}$$

# *Regular Discrete Curve / Discrete Immersion*

- Recall that a smooth curve is *regular* if its differential is nonzero; this condition helps avoid "bad behavior" like sharp cusps
  - equivalently: parameterization is *locally injective*
- Discrete case: nonzero differential prevents zero edge lengths, but not zero angles
  - "regular motion" can change turning number!
  - need something stronger…
- In particular, will say a *regular discrete curve* or *discrete immersion* is a discrete curve that is a **locally injective map**
  - rules out zero edge lengths <u>*and*</u> zero angles

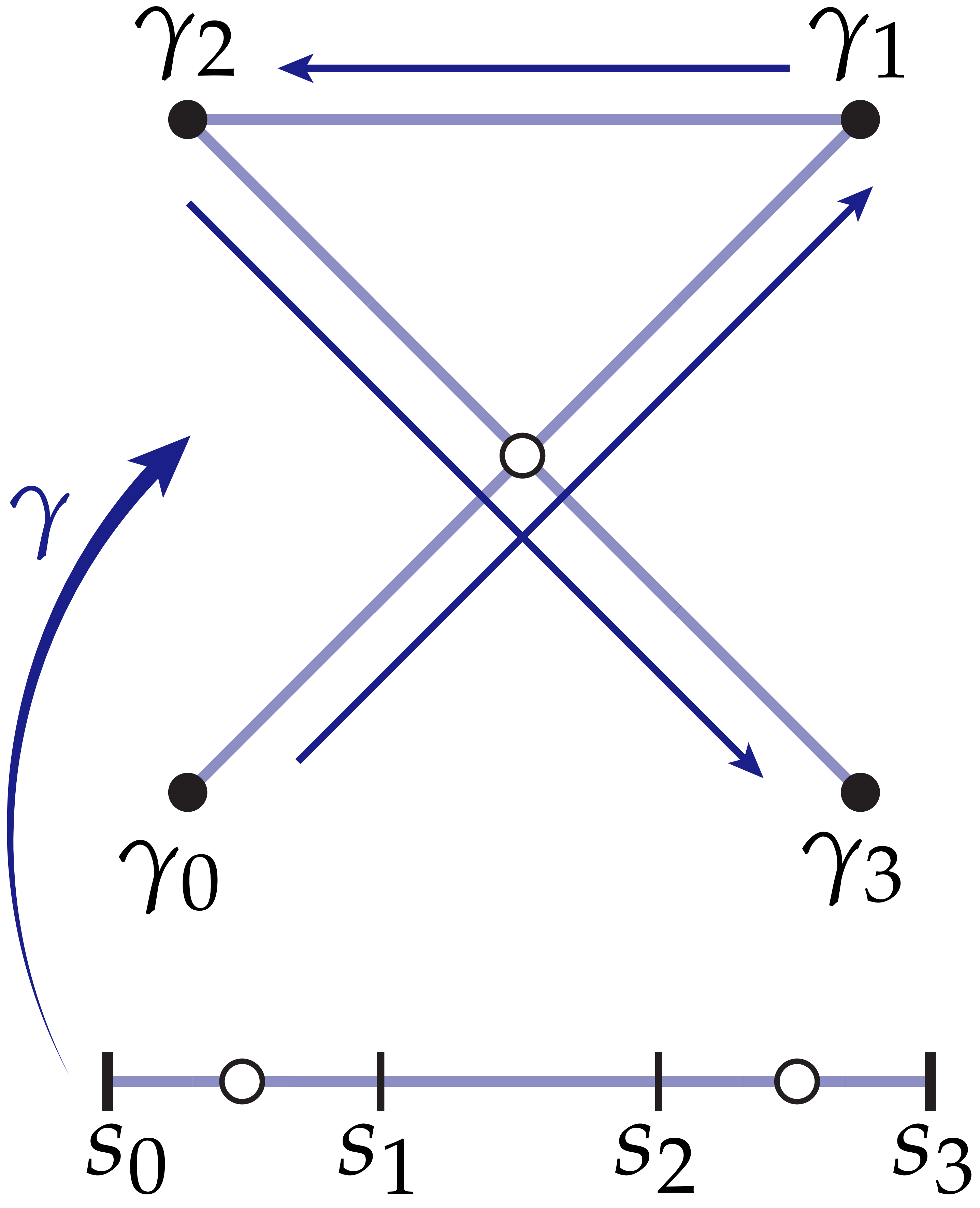



**regular**

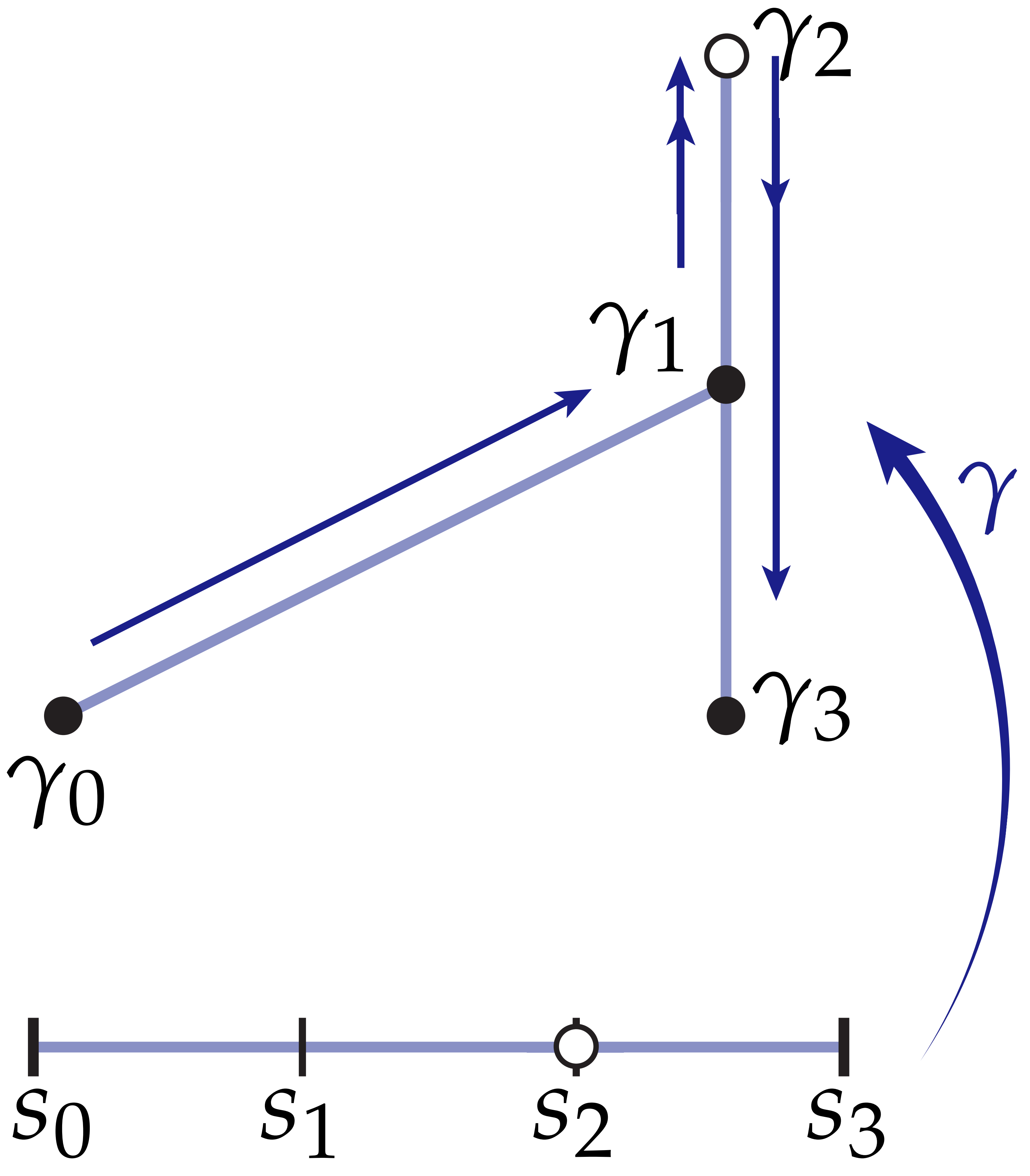



**not regular**

# *Discrete Regularity—Examples*




**locally injective**

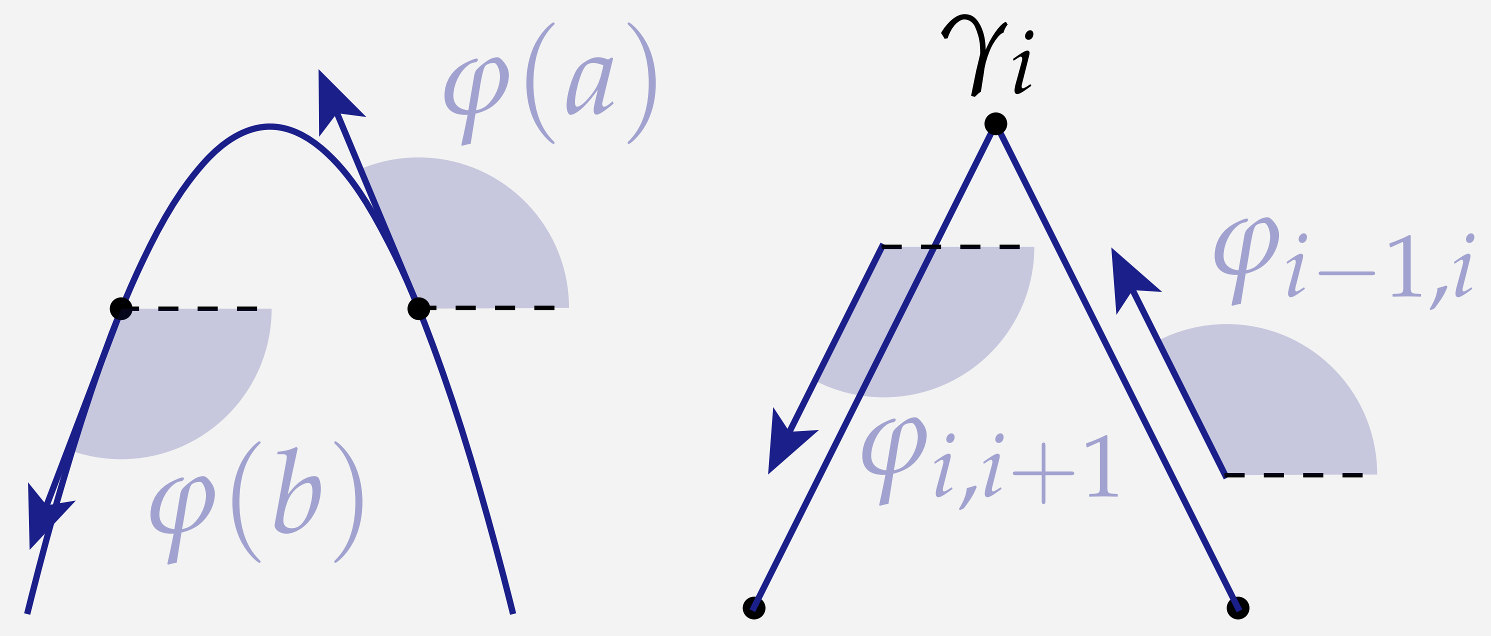



**not locally injective**

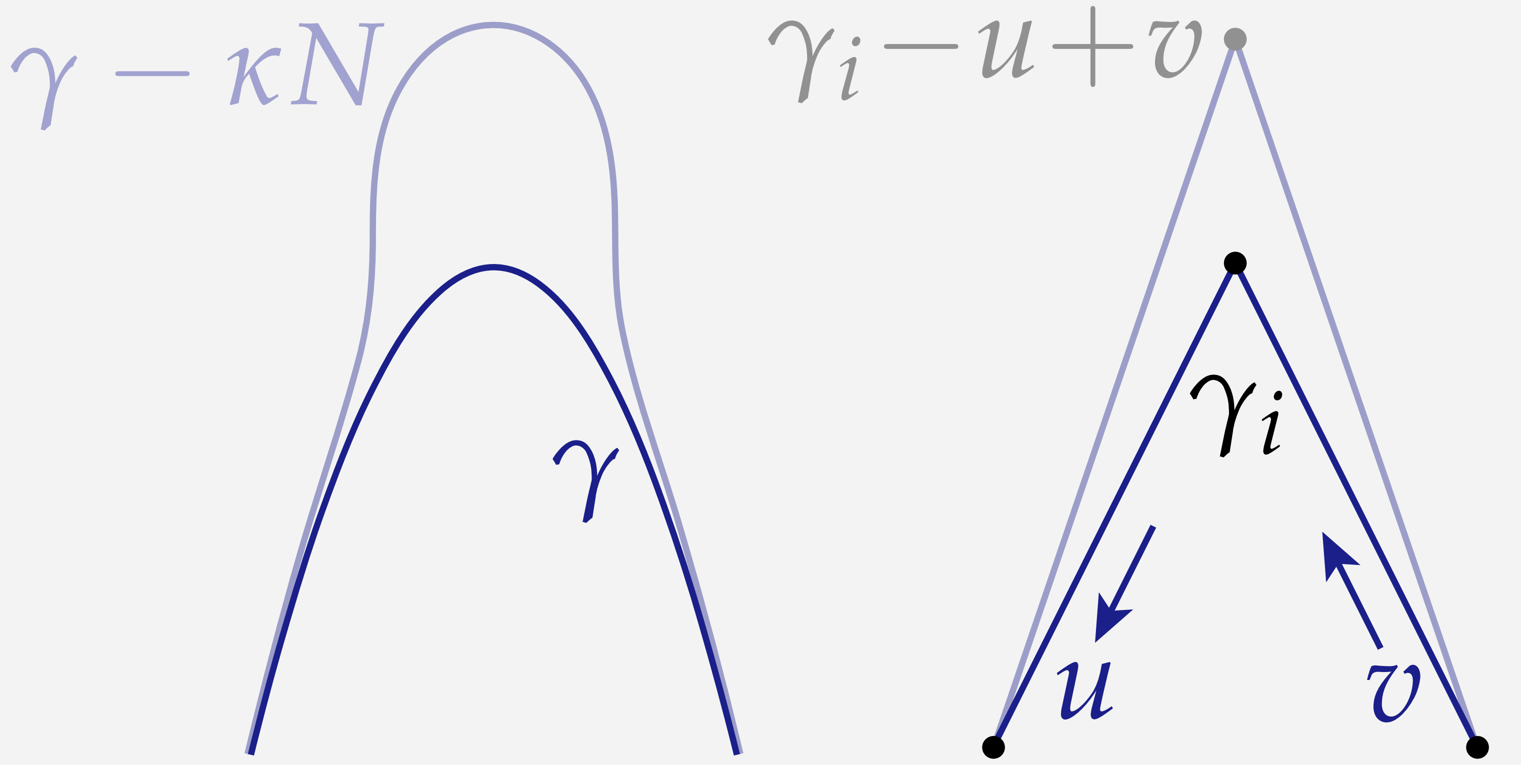
# *Discrete Curvature*

Recall that discrete curvature has several definitions:

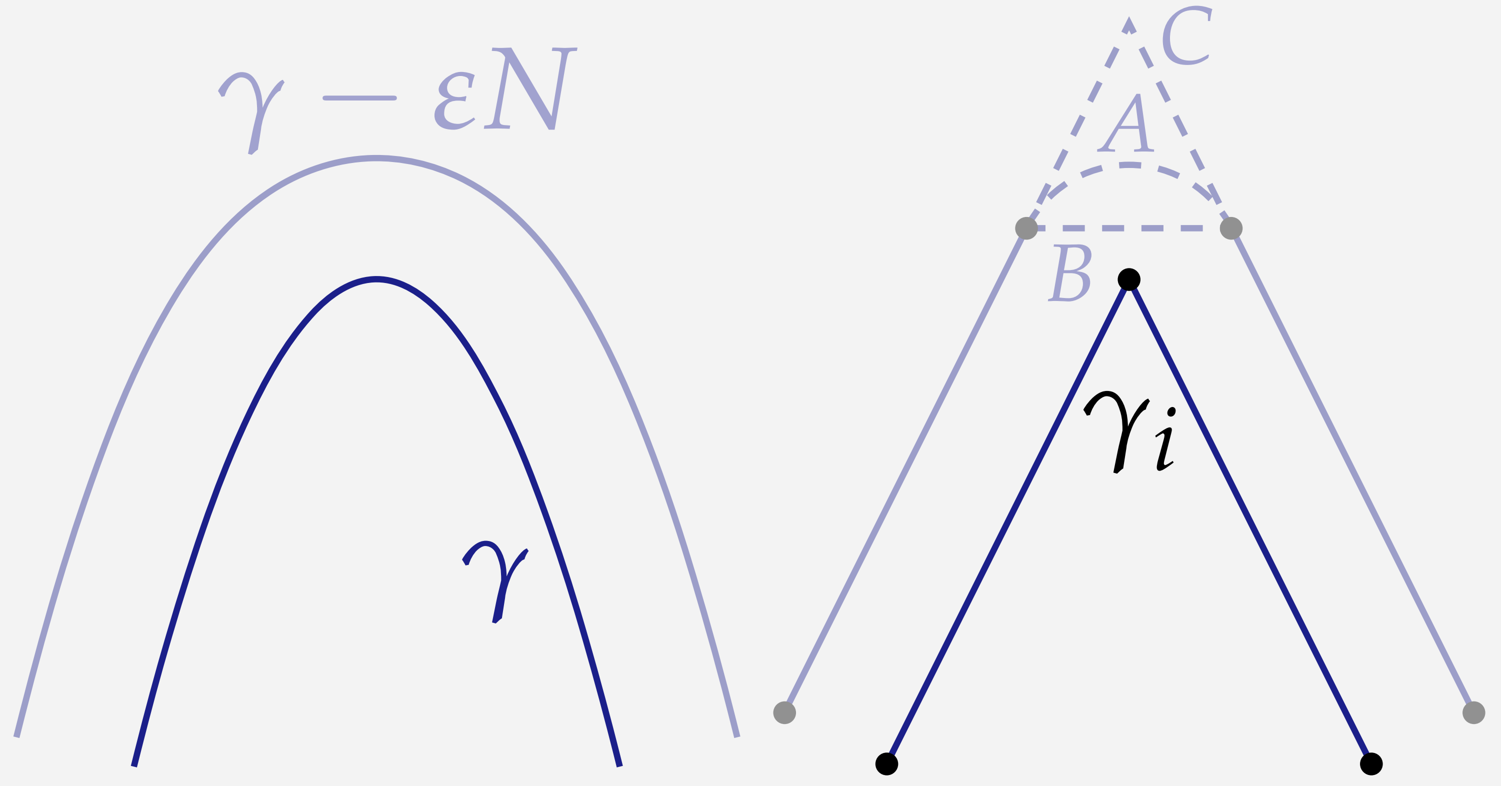
**Turning Angle**



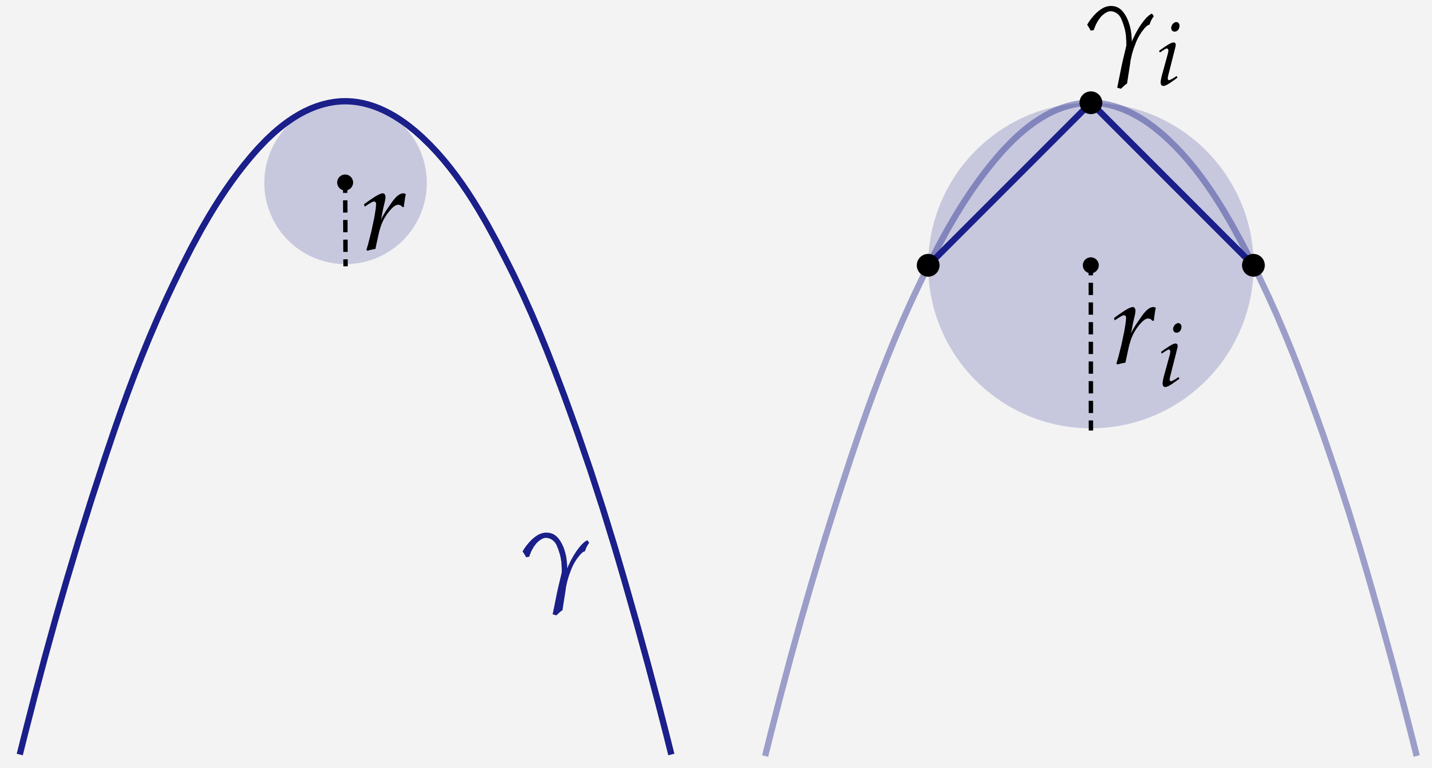

**Length Variation**




**Steiner Formula**




**Osculating Circle**

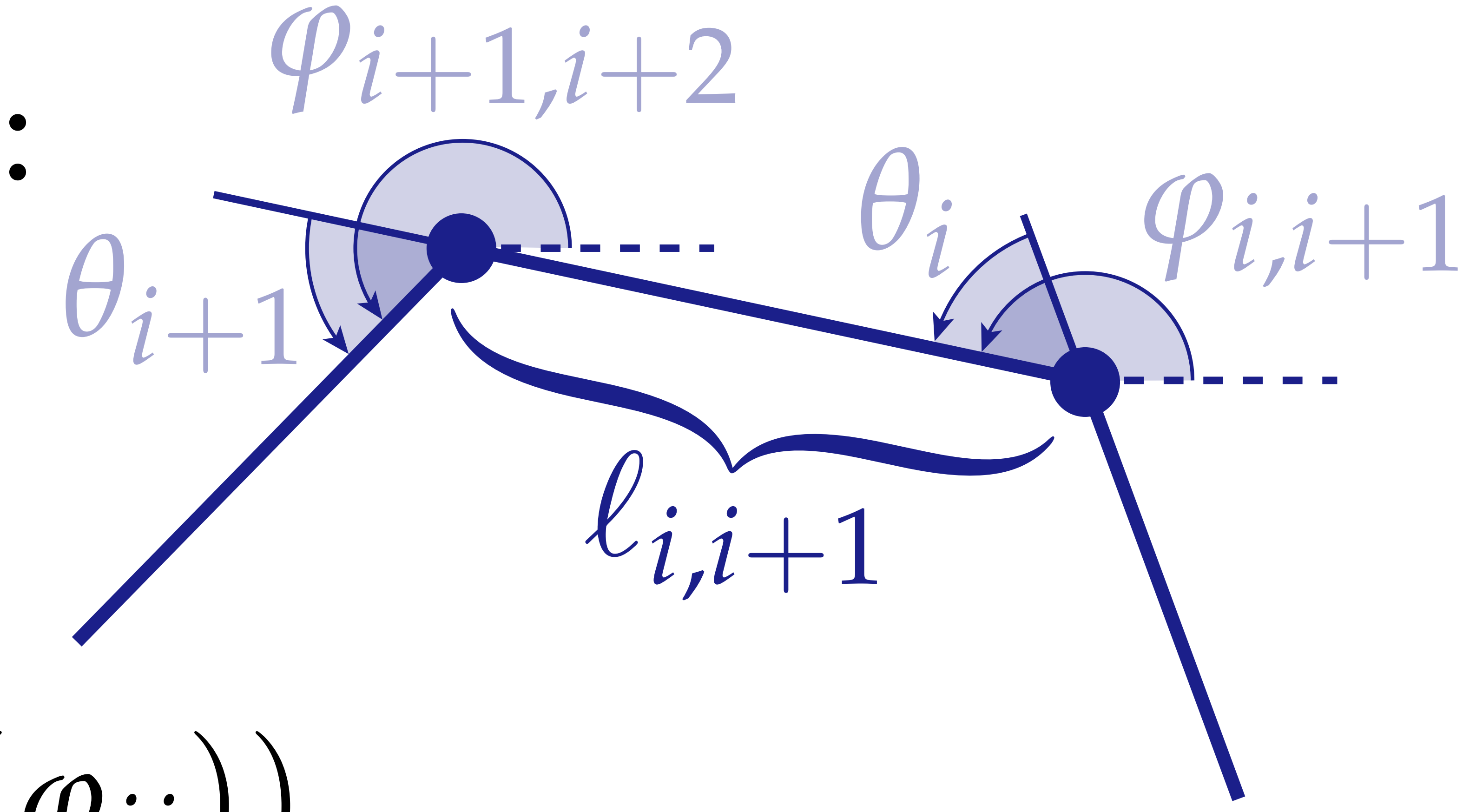


# *Fundamental Theorem of Discrete Plane Curves*

**Fact.** Up to rigid motions, a regular discrete plane curve is uniquely determined by its edge lengths and turning angles.

**Q:** Given only this data, how can we recover the curve?

**A:** Mimic the procedure from the smooth setting:




*Sum curvatures to get angles:* $\varphi_{i,i+1} := \sum_{k=1}^{i} \theta_k$

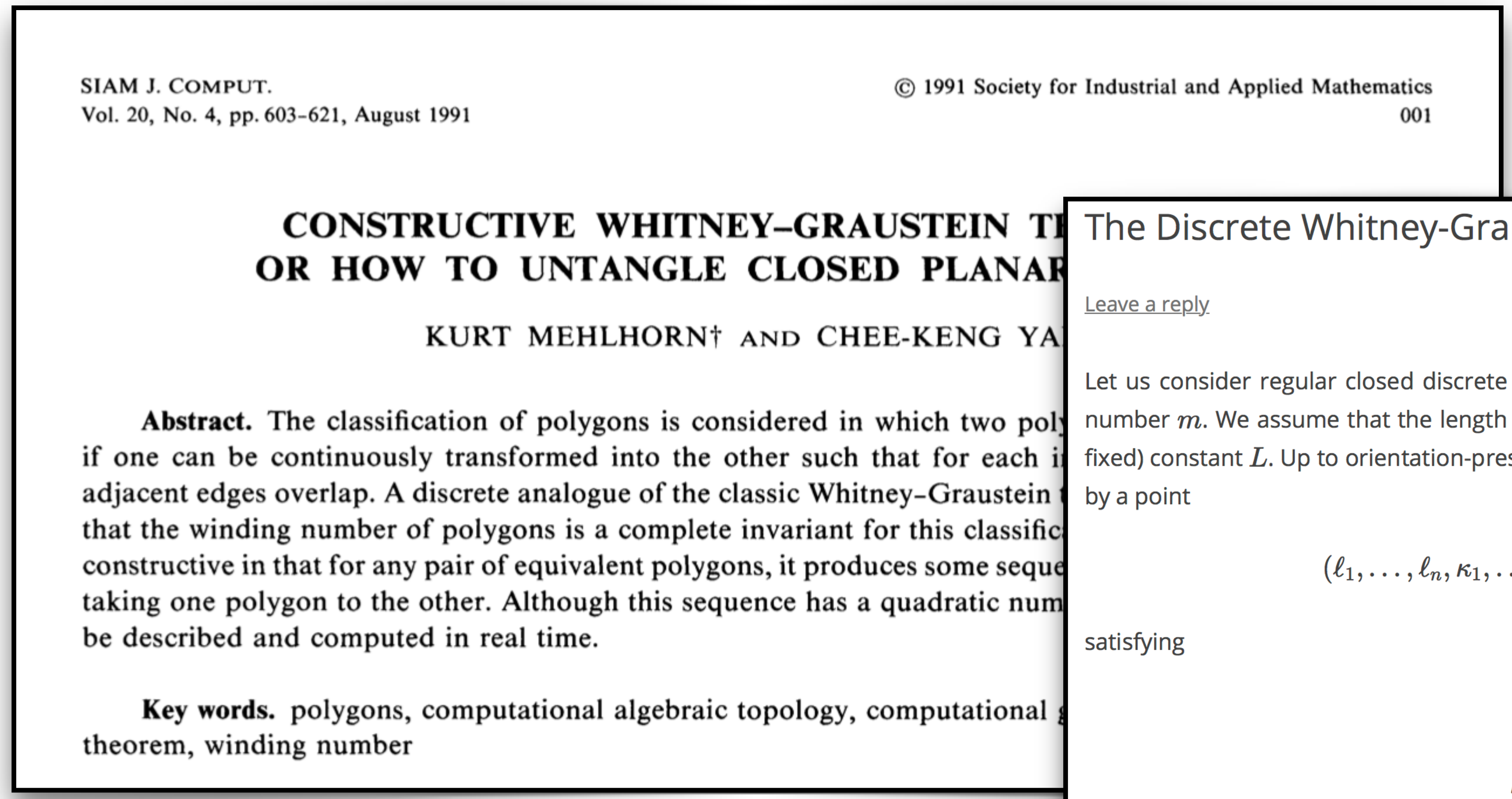
*Evaluate unit tangents:* $T_{ij} := (\cos(\varphi_{ij}), \sin(\varphi_{ij}))$

*Sum tangents to get curve:* $\gamma_i := \sum_{k=1}^{i} \ell_{k,k+1} T_{k,k+1}$

**Q:** Rigid motions?

# Discrete Whitney Graustein

- If we adopt the definition of a discrete regular curve as one that is *locally injective*, then there is a discrete version of Whitney-Graustein that exactly mirrors the smooth one
- Has been carefully studied from several perspectives:
  - Constructive algorithm (case analysis) by Mehlhorn & Yap (1991)
  - Simpler argument in Pinkall, *"The Discrete Whitney Graustein Theorem"* via convex polyhedra
- Both use central strategy from differential geometry: to find a "path" connecting two objects, find path from both objects to a canonical one, then compose… (uniformization, Delaunay, …)


SIAM J. COMPUT.
Vol. 20, No. 4, pp. 603–621, August 1991

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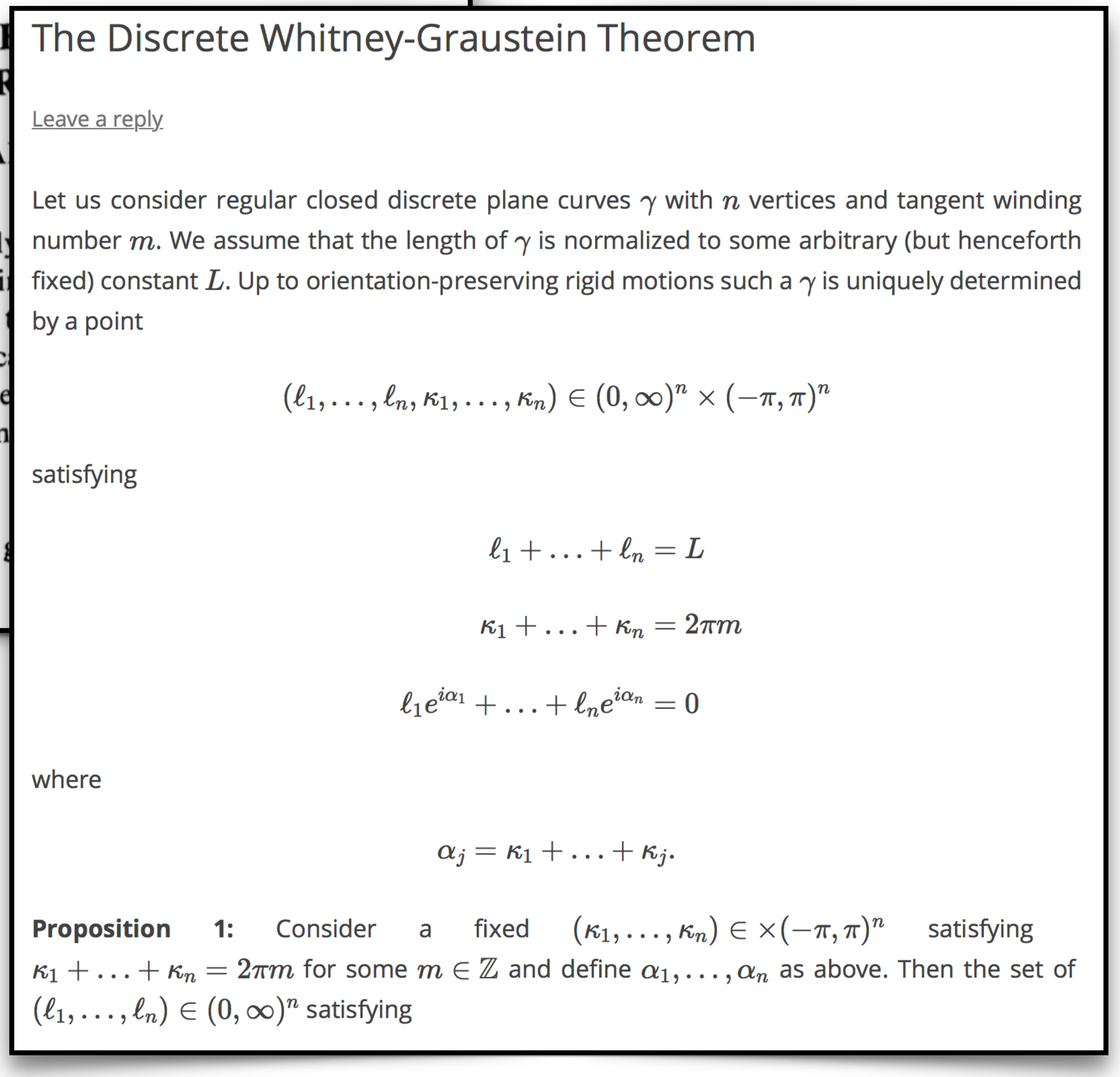
001

CONSTRUCTIVE WHITNEY–GRAUSTEIN TI
OR HOW TO UNTANGLE CLOSED PLANAR

KURT MEHLHORN† AND CHEE-KENG YA

**Abstract.** The classification of polygons is considered in which two poly if one can be continuously transformed into the other such that for each i adjacent edges overlap. A discrete analogue of the classic Whitney–Graustein t that the winding number of polygons is a complete invariant for this classific constructive in that for any pair of equivalent polygons, it produces some seque taking one polygon to the other. Although this sequence has a quadratic num be described and computed in real time.

**Key words.** polygons, computational algebraic topology, computational theorem, winding number


The Discrete Whitney-Graustein Theorem

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Let us consider regular closed discrete plane curves $\gamma$ with $n$ vertices and tangent winding number $m$. We assume that the length of $\gamma$ is normalized to some arbitrary (but henceforth fixed) constant $L$. Up to orientation-preserving rigid motions such a $\gamma$ is uniquely determined by a point

$$(\ell_1,\dots,\ell_n,\kappa_1,\dots,\kappa_n)\in(0,\infty)^n\times(-\pi,\pi)^n$$

satisfying

$$\ell_1+\ldots+\ell_n=L$$

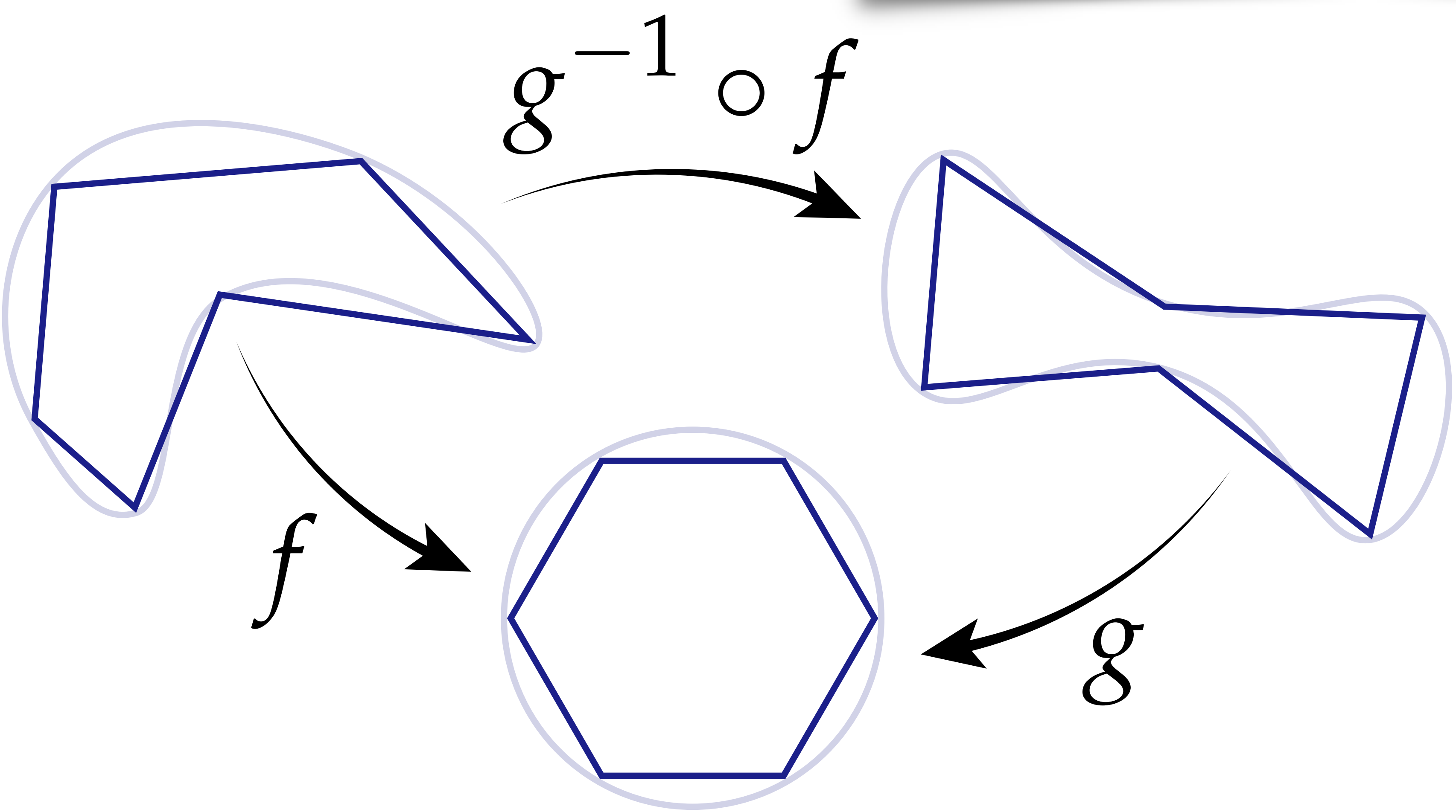
$$\kappa_1+\ldots+\kappa_n=2\pi m$$

$$\ell_1e^{i\alpha_1}+\ldots+\ell_ne^{i\alpha_n}=0$$

where

$$\alpha_j=\kappa_1+\ldots+\kappa_j.$$

**Proposition 1:** Consider a fixed $(\kappa_1,\dots,\kappa_n)\in\times(-\pi,\pi)^n$ satisfying $\kappa_1+\ldots+\kappa_n=2\pi m$ for some $m\in\mathbb{Z}$ and define $\alpha_1,\dots,\alpha_n$ as above. Then the set of $(\ell_1,\dots,\ell_n)\in(0,\infty)^n$ satisfying

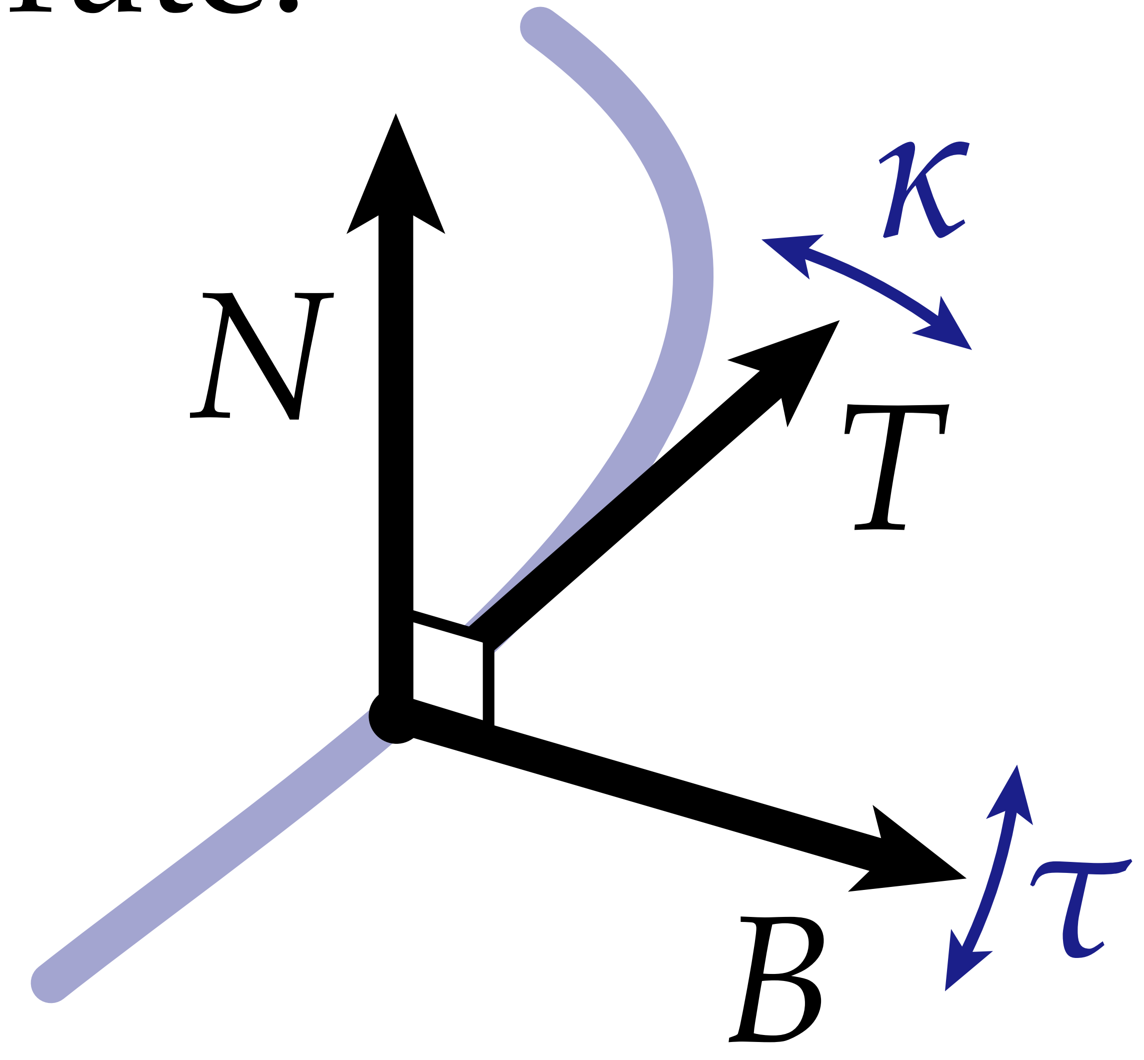


# *Discrete Space Curves*

# *Review: Fundamental Theorem of Space Curves*

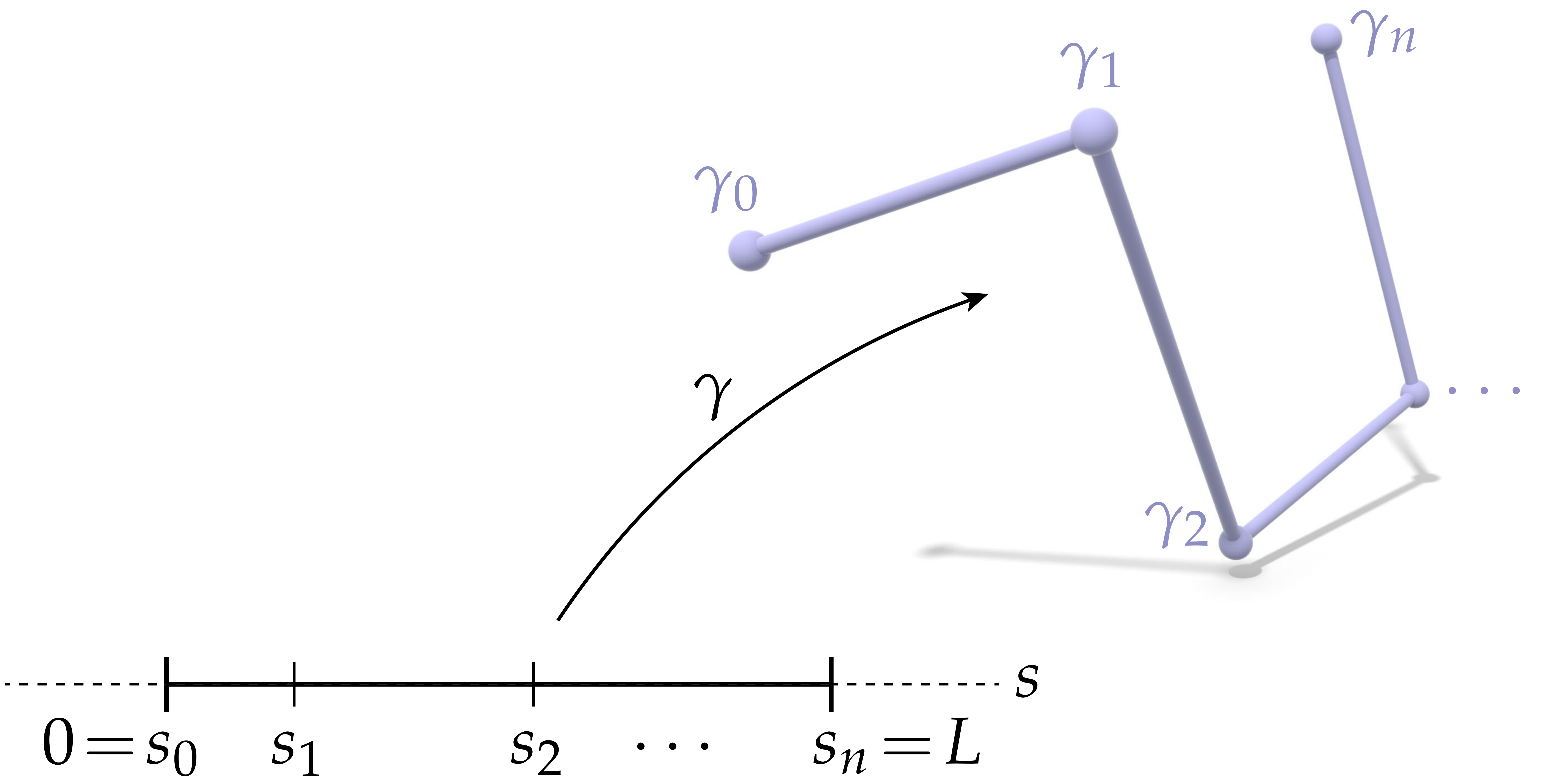
- The *fundamental theorem of space curves* tells that given the curvature $\kappa(s)$ and torsion $\tau(s)$ of an arc-length parameterized space curve, we can recover the curve itself
- Formally: integrate the *Frenet-Serret equations*; intuitively: start drawing a curve, bend & twist at prescribed rate.

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$



# *Discrete Space Curve*

A **discrete space curve** is simply a discrete curve in $\mathbb{R}^3$ rather than $\mathbb{R}^2$, *i.e.*, a piecewise linear parameterized curve $\gamma : [0, L] \to \mathbb{R}^3$




$\gamma_i := \gamma(s_i)$

# *Fundamental Theorem of Discrete Space Curves*

**Q:** How can we discretize the fundamental theorem for space curves?

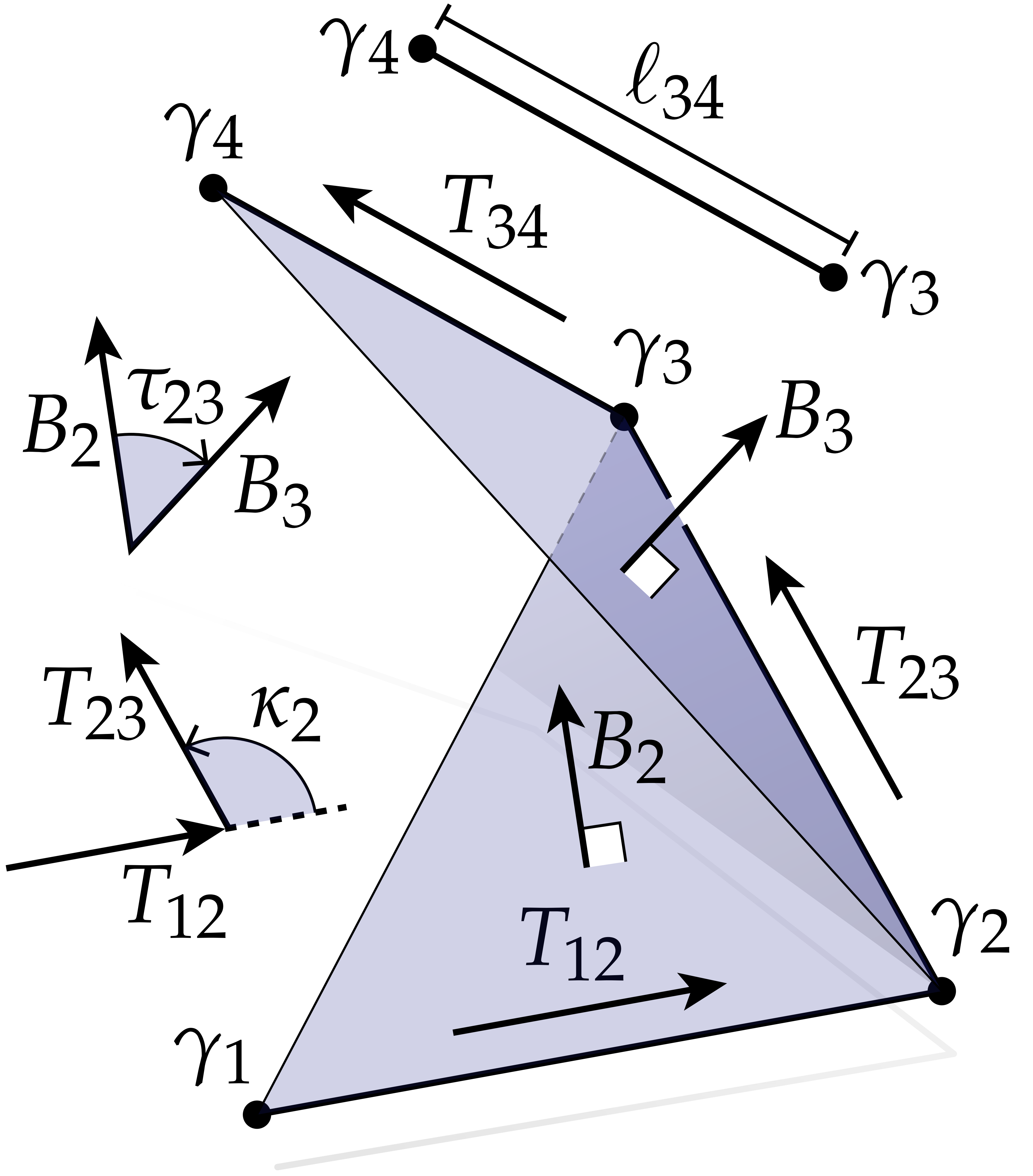
**A:** One possibility (*"reduced coordinates"*):

– arc length ⇒ lengths $\ell_{ij}$ at edges $ij$

– curvature ⇒ exterior angles $\kappa_i$ at vertices $i$

– torsion ⇒ angles $\tau_{ij}$ at edges $ij$

**Theorem.** Discrete space curve is determined by this data, up to rigid motion.




**Notice:** curve is determined by curvature, torsion, <u>and parameterization</u>.

# *Discrete Space Curve—Reconstruction*

**Given:**

- edge lengths $\ell_{ij}$, curvatures $\kappa_i$, torsions $\tau_{ij}$
- initial point, tangent, and normal $\gamma_0, T, N \in \mathbb{R}^3$

**Find:** vertex positions $\gamma_i$

**rotate by $\theta$ around axis $u$**

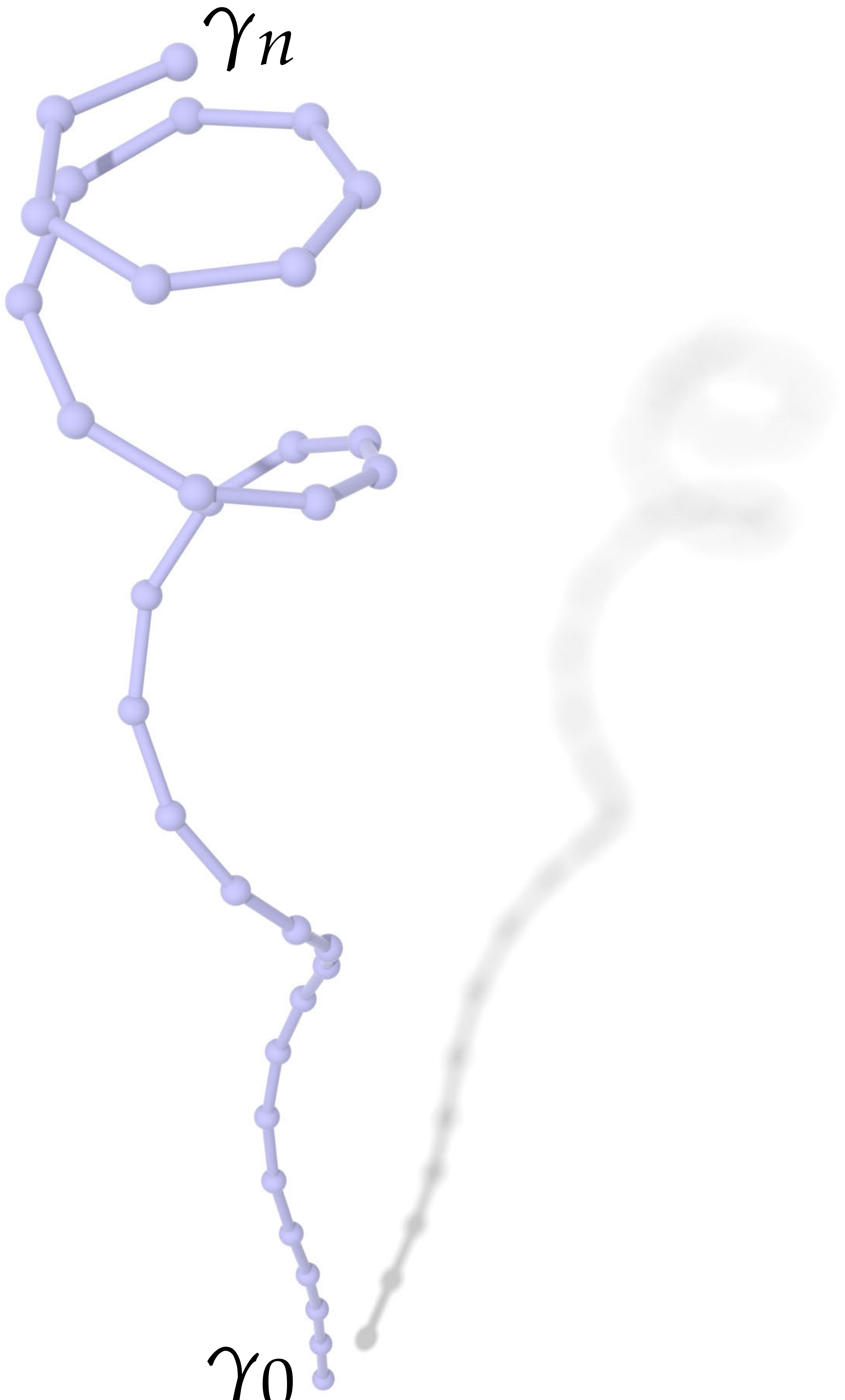
$$\hat{u} := \begin{bmatrix} 0 & u_z & -u_y \\ -u_z & 0 & u_x \\ u_y & -u_x & 0 \end{bmatrix}$$

$$R(u,\theta) := \exp(\theta\hat{u})$$

Algorithm:

**for** $i = 1, \ldots, n$:

- $\gamma_i \leftarrow \gamma_{i-1} + \ell_{i-1,i}T$ move to the next vertex
- $T \leftarrow R(N, \kappa_i)T$ rotate tangent in-plane
- $N \leftarrow R(T, \tau_{i,i+1})N$ rotate normal to new plane




$\kappa_i = i/n$

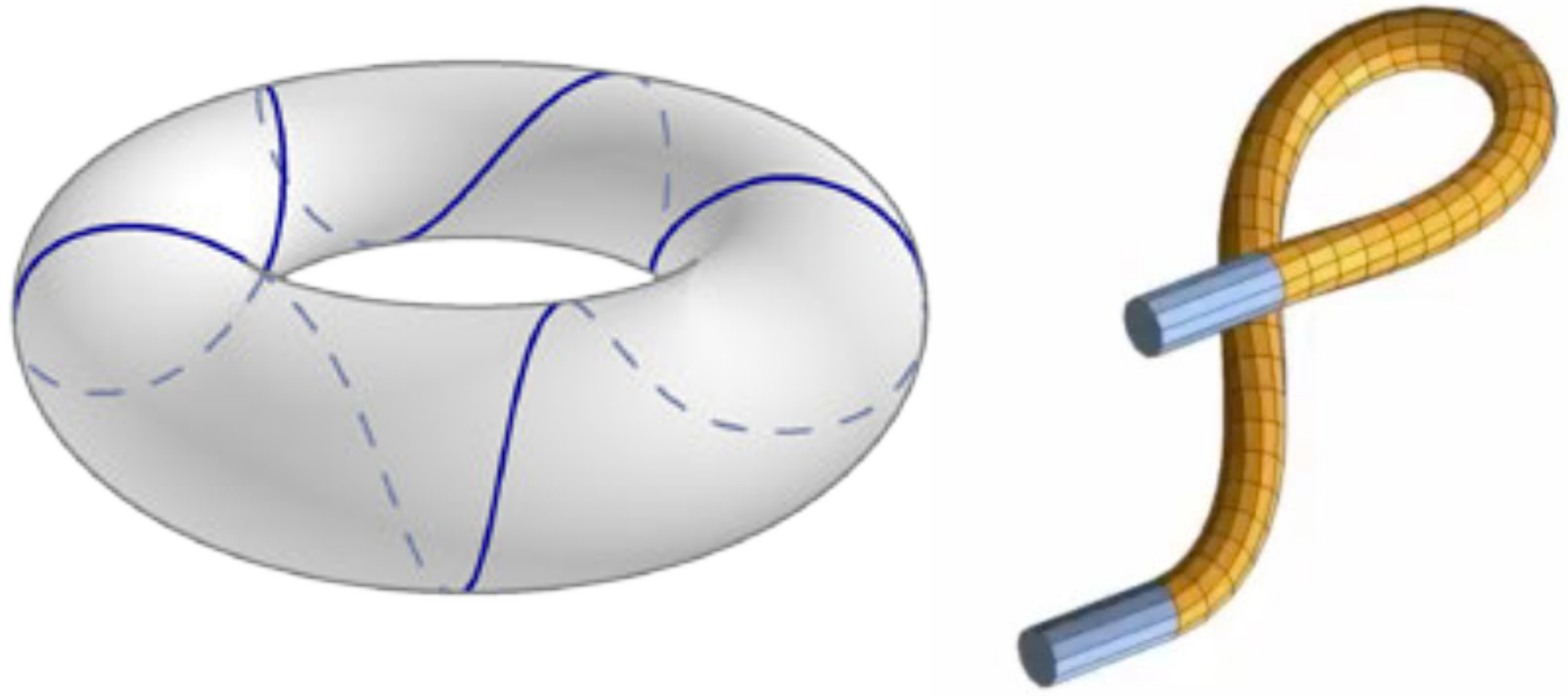
$\tau_i = (n-i)/n$

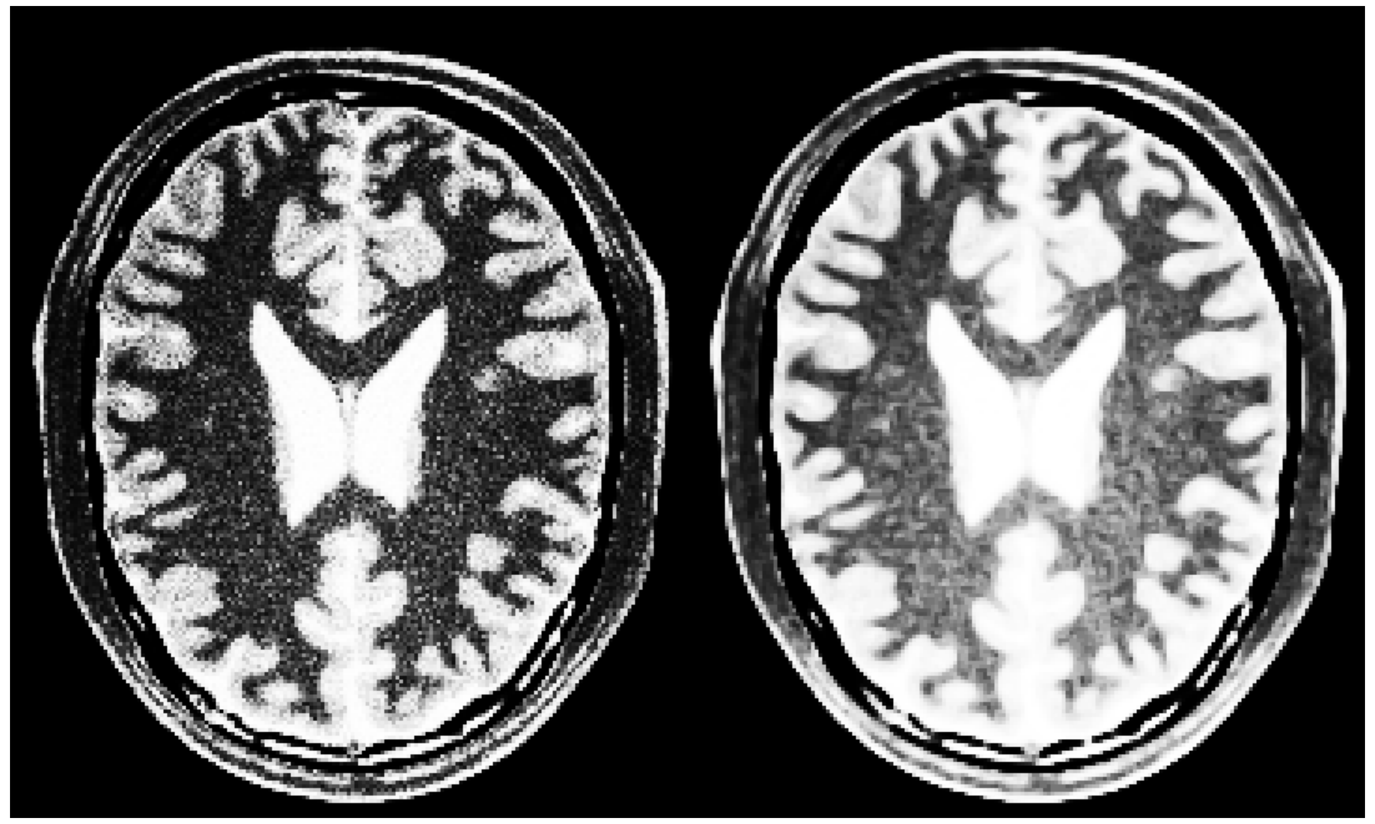
**Note:** much easier than solving Frenet-Serret equations!

# *Curvature Flow & Dynamics*

# *Curvature Flow on Curves*

- A *curvature flow* is a time evolution of a curve (or surface) driven by some function of its curvature
- Such flows model physical *elastic rods*, can be used to find shortest curves (*geodesics*) on surfaces, or might be used to smooth noisy data (*e.g.*, image contours)
- Basic idea: energy $E(\gamma)$ assigns a "cost" to each curve (*e.g.*, total length); follow the gradient so that the energy becomes smaller
- Two simple examples: *length-shortening flow* and *elastic flow*





**gradient flow**

$$\frac{d}{dt}\gamma = -\nabla_\gamma E(\gamma)$$

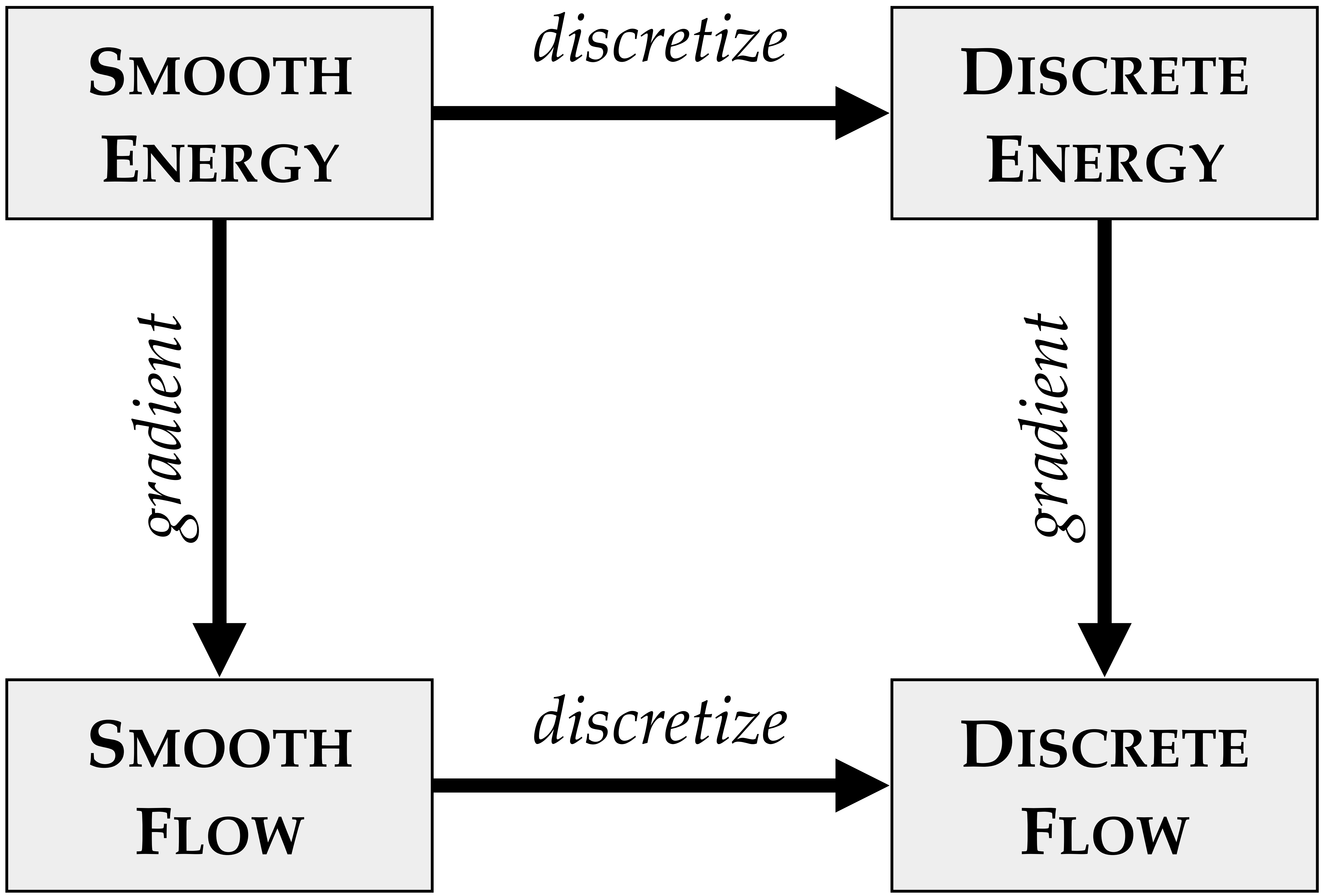
# *Discretizing a Gradient Flow*

- Two possible paths for discretizing any gradient flow:

  1. **First** derive the gradient of the energy in the smooth setting, **then** discretize the resulting evolution equation.

  2. **First** discretize the energy itself, **then** take the gradient of the resulting discrete objective.

- In general, ***will not*** *lead to the same numerical scheme/algorithm!*




(In general, does **NOT** commute.)

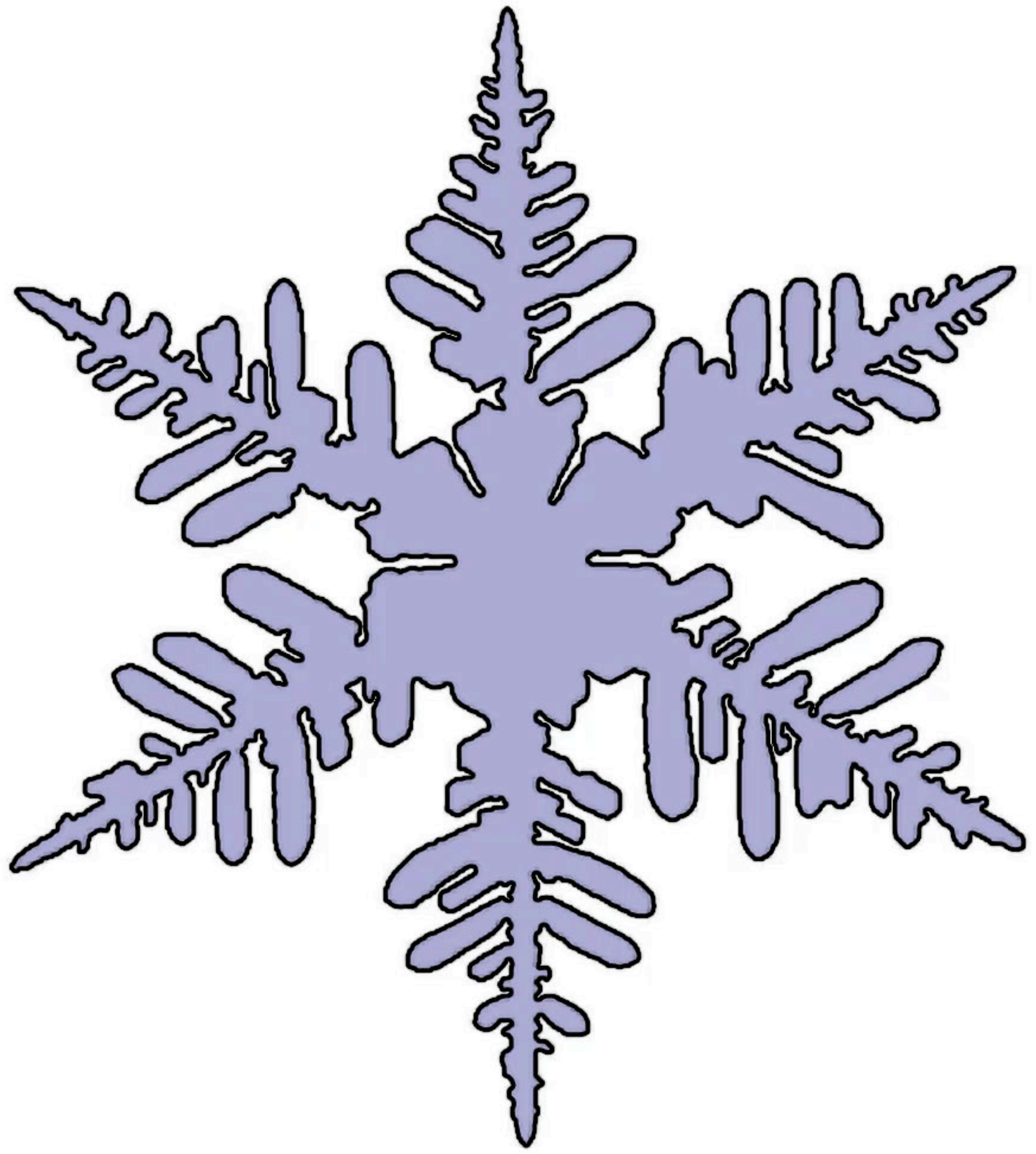
# *Length Shortening Flow*

- The energy for length shortening flow is simply the total length of the curve
- Recall that length gradient is curvature normal—hence, curve shortening moves faster where there are small bumps
- For closed curves, several interesting features (Gage-Grayson-Hamilton):
  - center of mass is preserved
  - curves flow to "round points"
  - embedded curves remain embedded

$$\text{length}(\gamma) := \int_0^L \left|\tfrac{d}{ds}\gamma\right| ds$$

$$\tfrac{d}{dt}\gamma = -\nabla_\gamma \text{length}(\gamma)$$

$$\tfrac{d}{dt}\gamma = -\kappa N$$



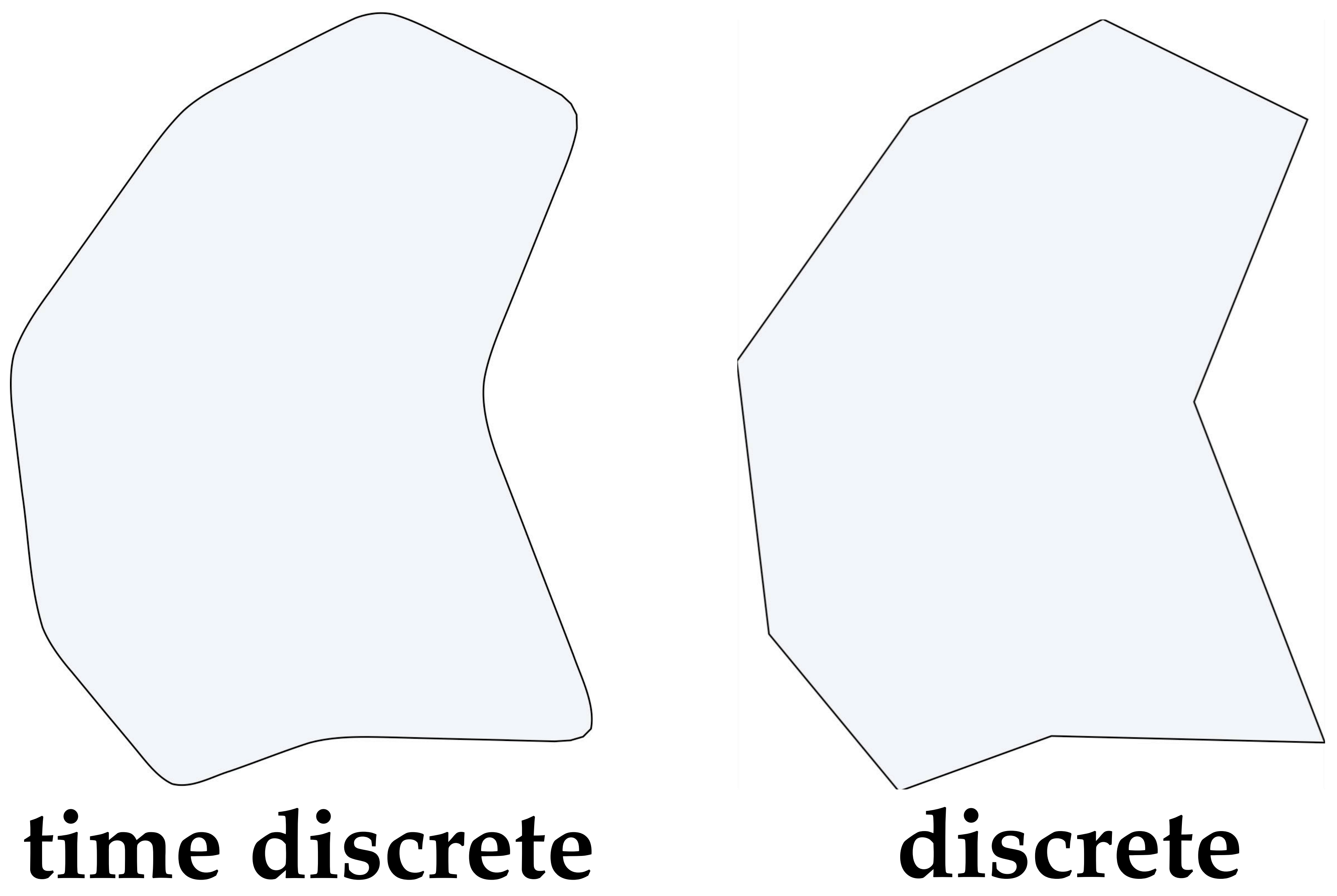
# *Length Shortening Flow—Discretized*

- At each moment in time, move curve in normal direction with speed proportional to curvature
- "Smooths out" curve (*e.g.*, noise), eventually becoming circular
- Discrete version:
  - replace time derivative with difference in time
  - replace smooth curvature with one (of many) curvatures
- "Repeatedly add a little bit of $\kappa N$"

$$\frac{d}{dt}\gamma(s,t) = -\kappa(s,t)N(s,t)$$

$$\frac{\gamma^{k+1}(s) - \gamma^{k}(s)}{\tau} = -\kappa^{k}(s)N^{k}(s)$$

$$\Longrightarrow \gamma_i^{k+1} = \gamma_i^{k} - \tau\kappa_i^{k}N_i^{k}$$

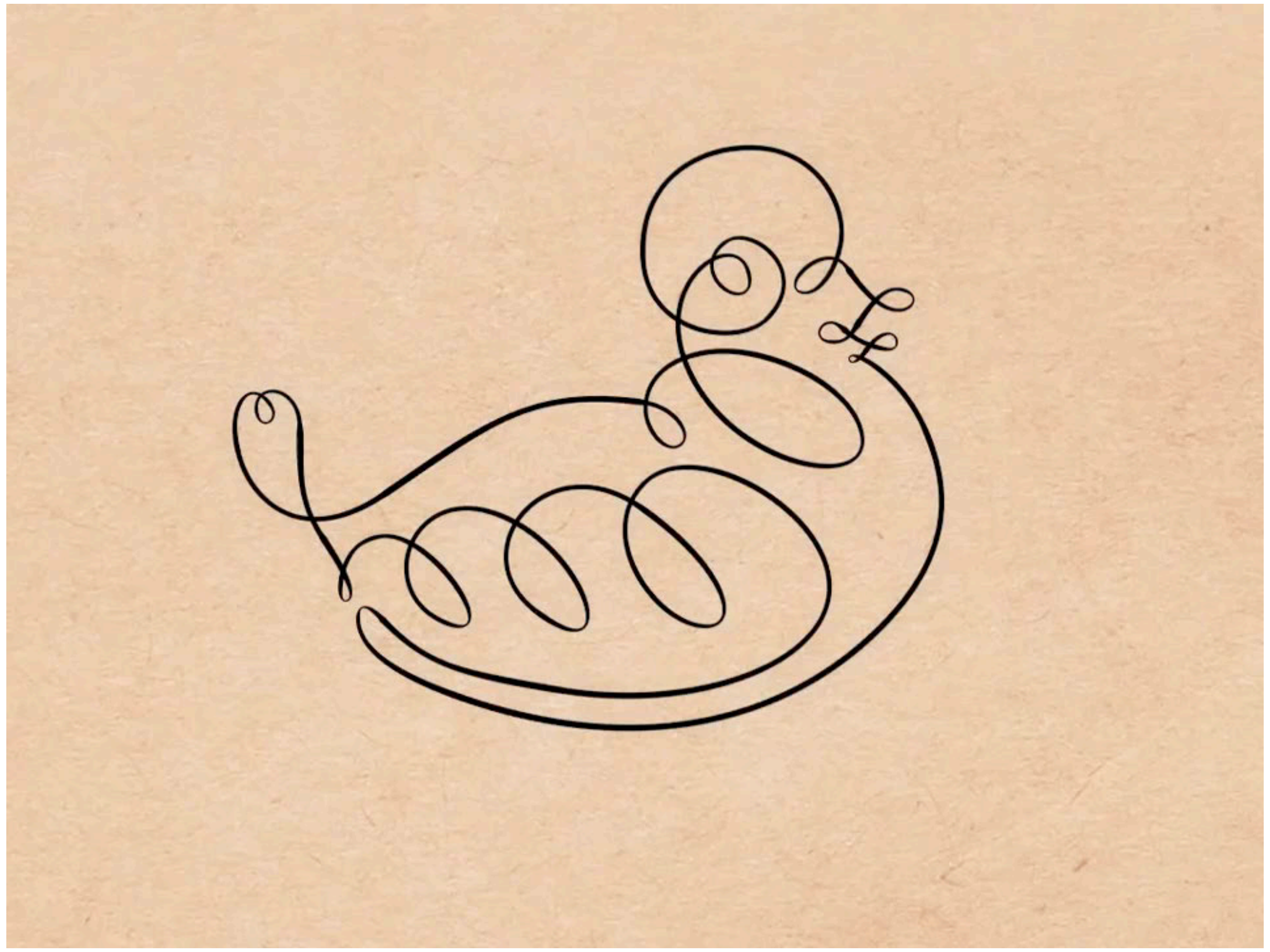


# *Elastic Flow*

- Basic idea: rather than shrinking *length*, try to reduce *bending* (*i.e.*, curvature)
- Energy is integral of squared curvature; elastic flow is then gradient flow on this objective
- Minimizers are called *elastic curves* or *Euler elastica*—model real elastic strips
- **Discrete:** express energy via *turning angles*
  - discrete minimizers converge to smooth ones under refinement

**Euler-Bernoulli energy**

$$E(\gamma) := \int_0^L \kappa(s)^2 \, ds$$



Scholtes, Schumacher, Wardetzky, *"Variational Convergence of Discrete Elasticae"*

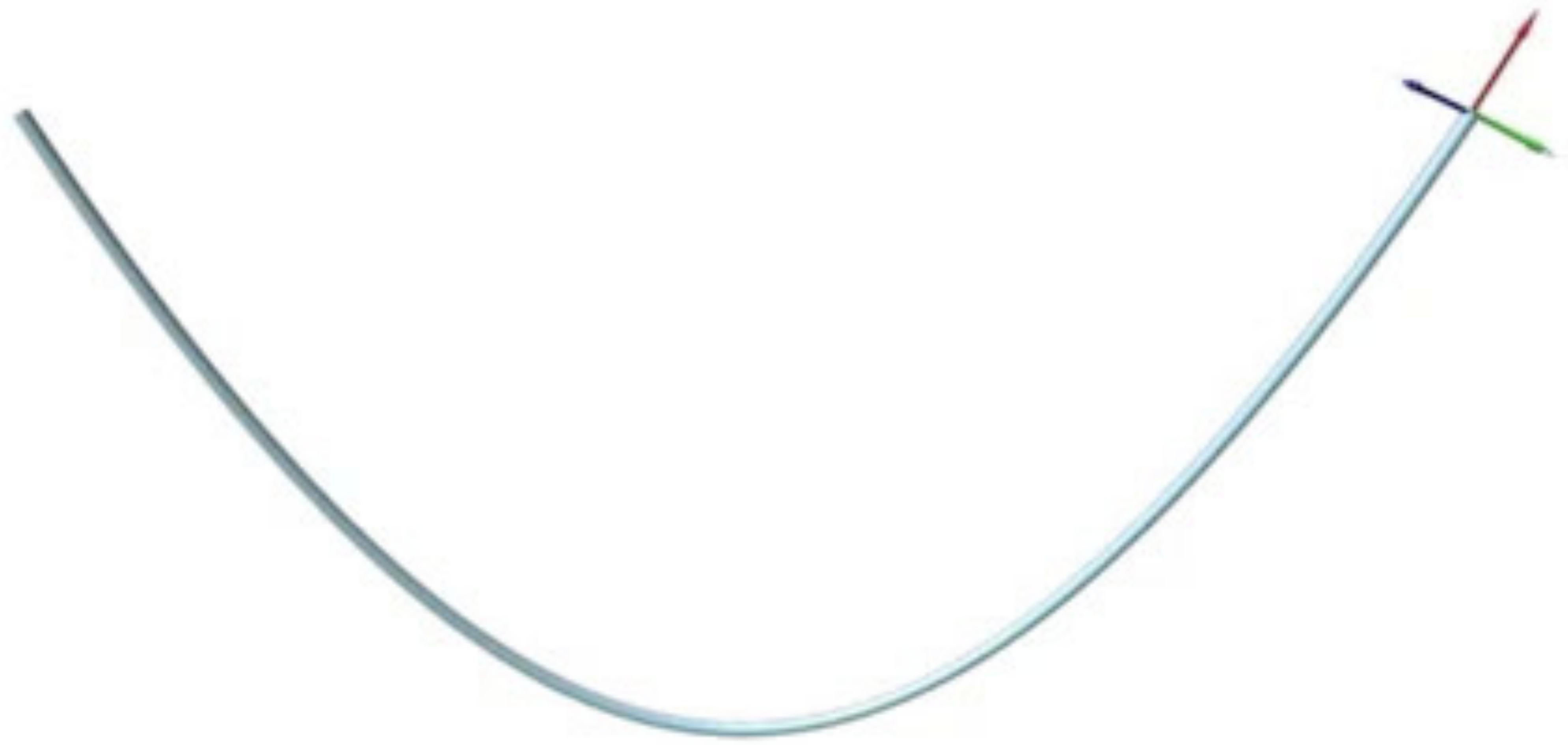
# *Isometric Elastic Flow*

- Different way to smooth out a curve is to directly "shrink" curvature
- Discrete case: scale down turning angles $\kappa_i$, then use the **fundamental theorem of discrete plane curves** to reconstruct
- Numerically stable; exactly preserves edge lengths
- Challenge: how do we make sure closed curves remain closed?

Crane, Pinkall, Schröder, *"Robust Fairing via Conformal Curvature Flow"*

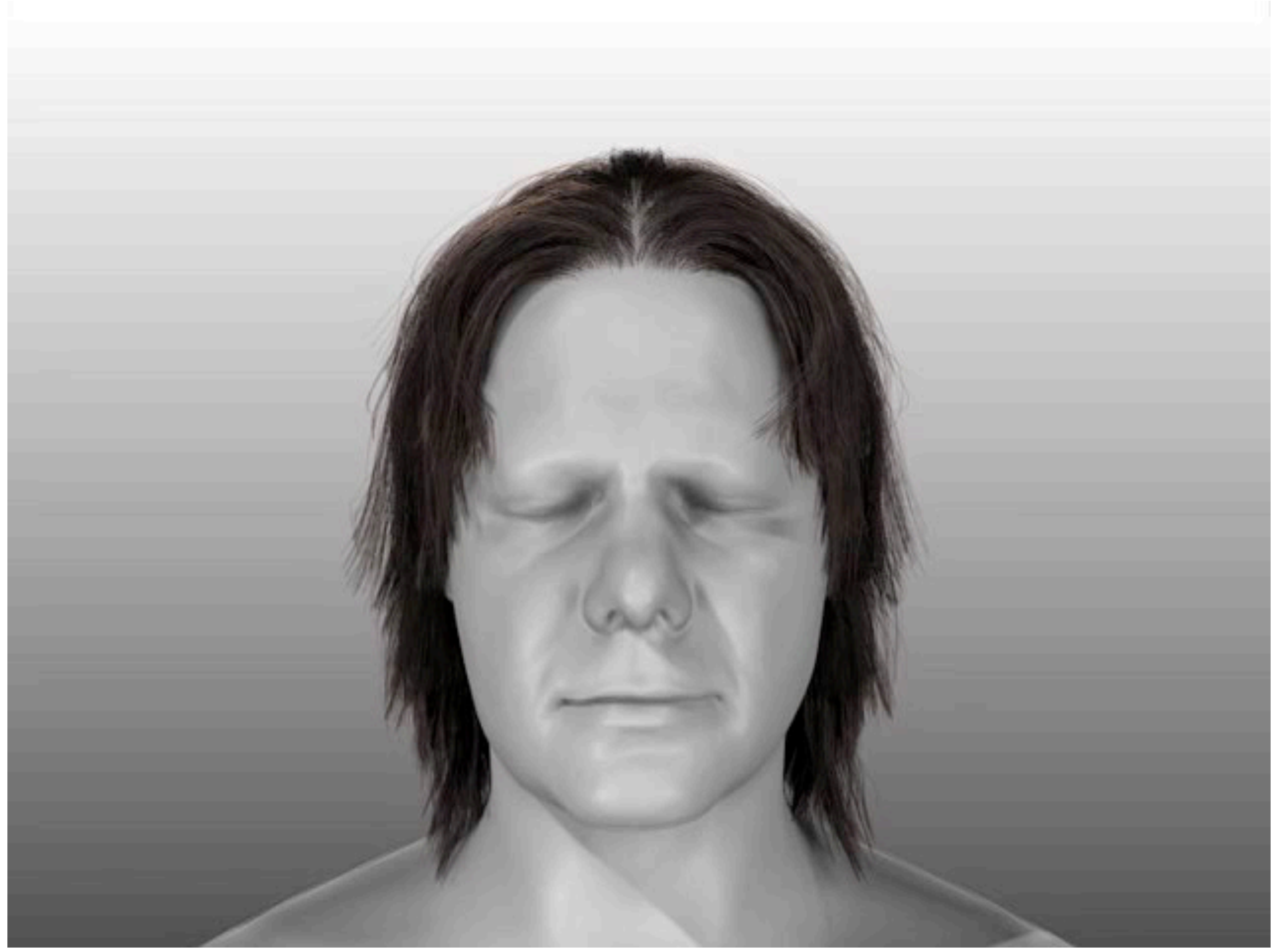
# *Elastic Rods*

- For space curves, can also try to minimize both curvature $\kappa$ **and** torsion $\tau$
- Both in some sense measure "non-straightness" of curve
- Provides rich model of *elastic rods*
- Lots of interesting applications (simulating hair, laying cable, …)

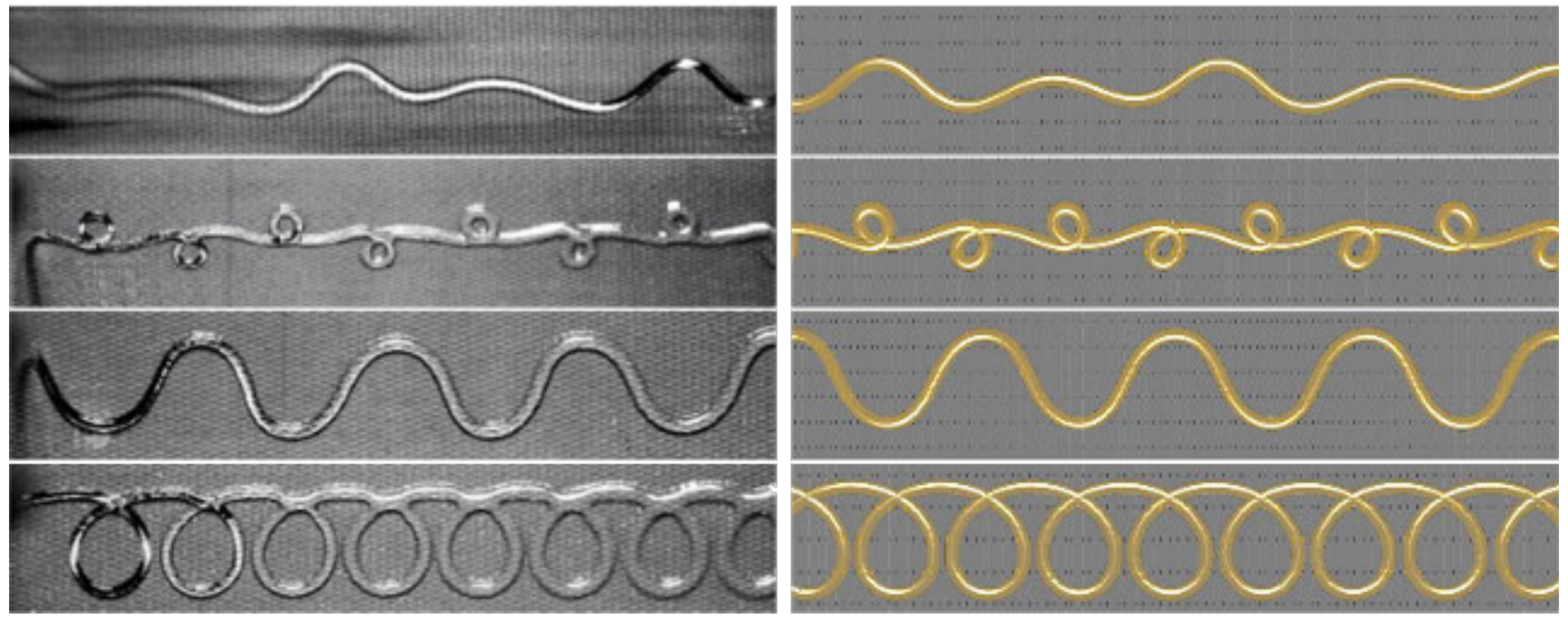


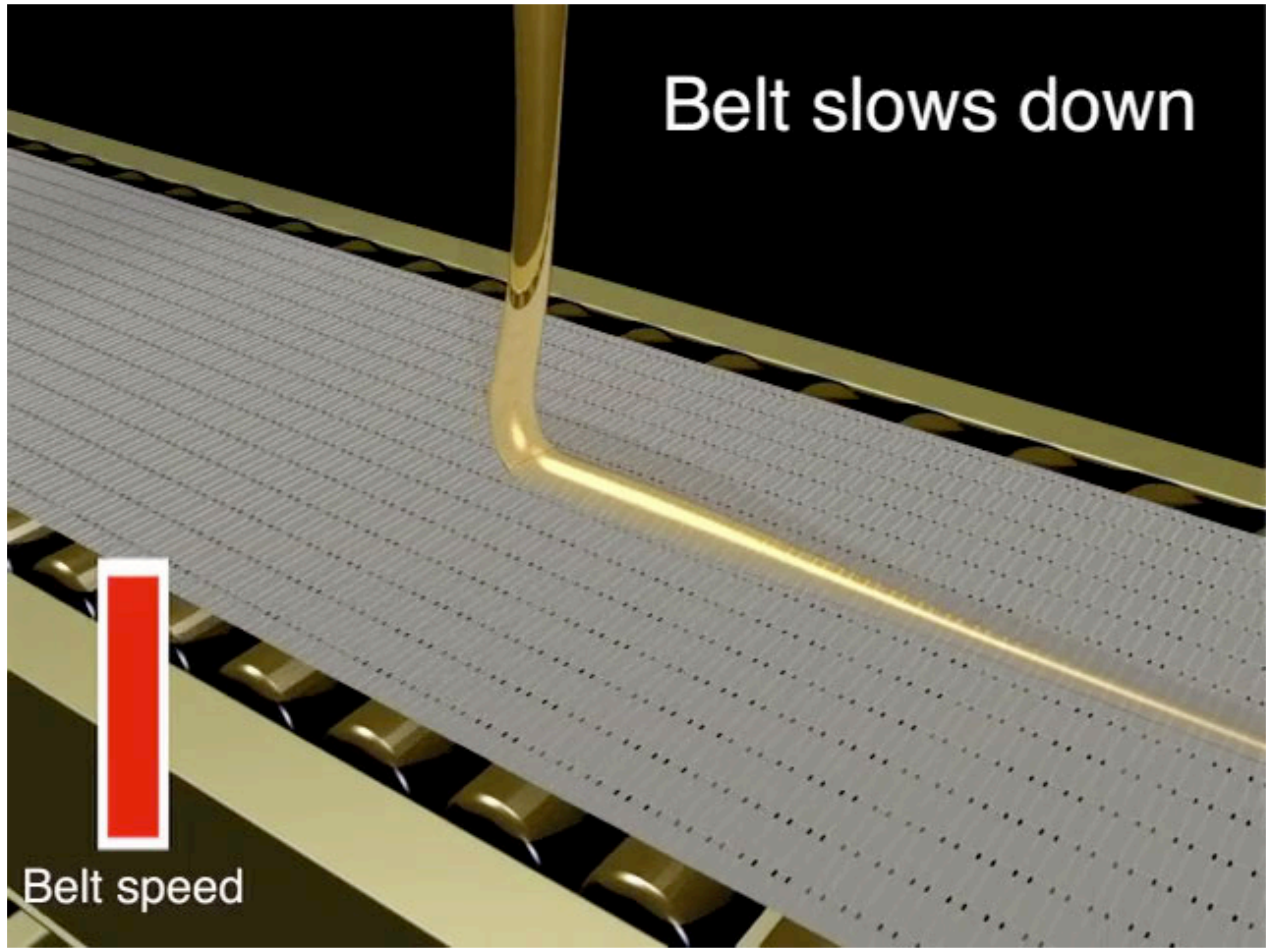
Bergou, Wardetzky, Robinson, Audoly, Grinspun, *"Discrete Elastic Rods"*

# *Viscous Threads*



**elastic rods**






**viscous threads**

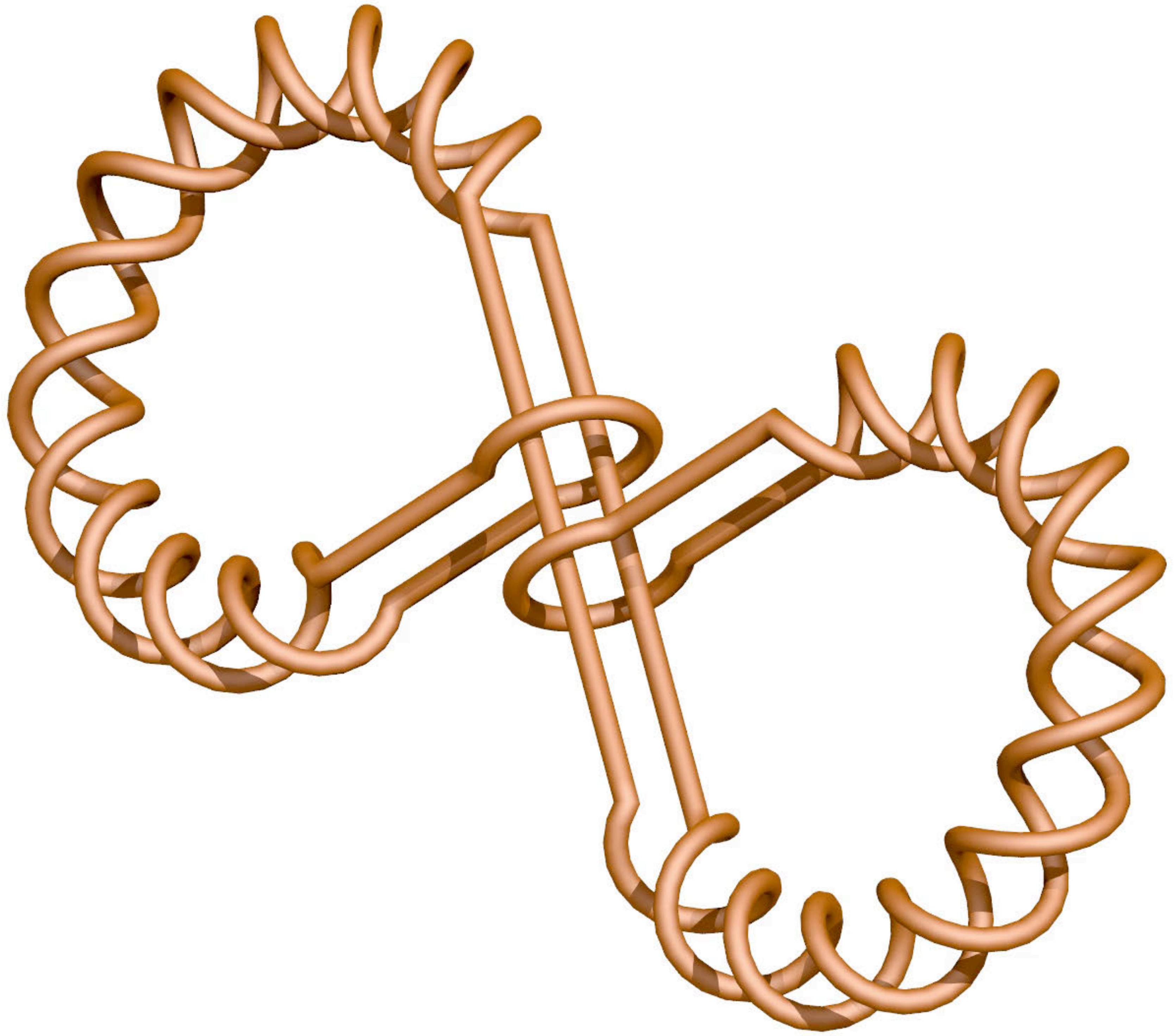
Bergou, Audoly, Vouga, Wardetzky, Grinspun, *“Discrete Viscous Threads”*

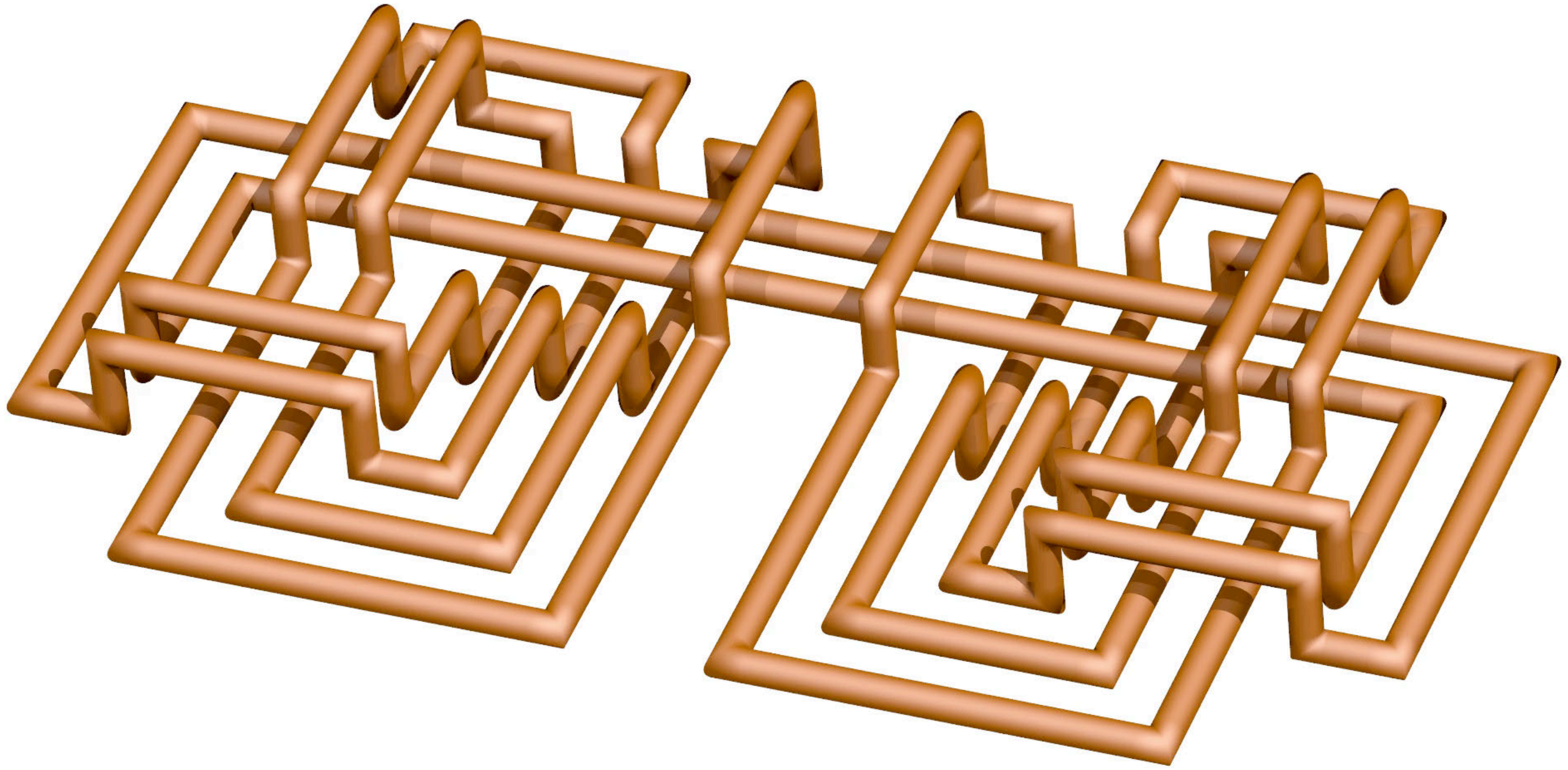
# *Untangling Knots*

- Is a given curve "knotted?"
- Minimize bending *and* penalize self-collision
- *Might* go to smoothest curve in same isotopy class

$$\int_0^L \int_0^L \frac{1}{|\gamma(s)-\gamma(t)|^2} - \frac{1}{d(s,t)^2}\, ds\, dt$$

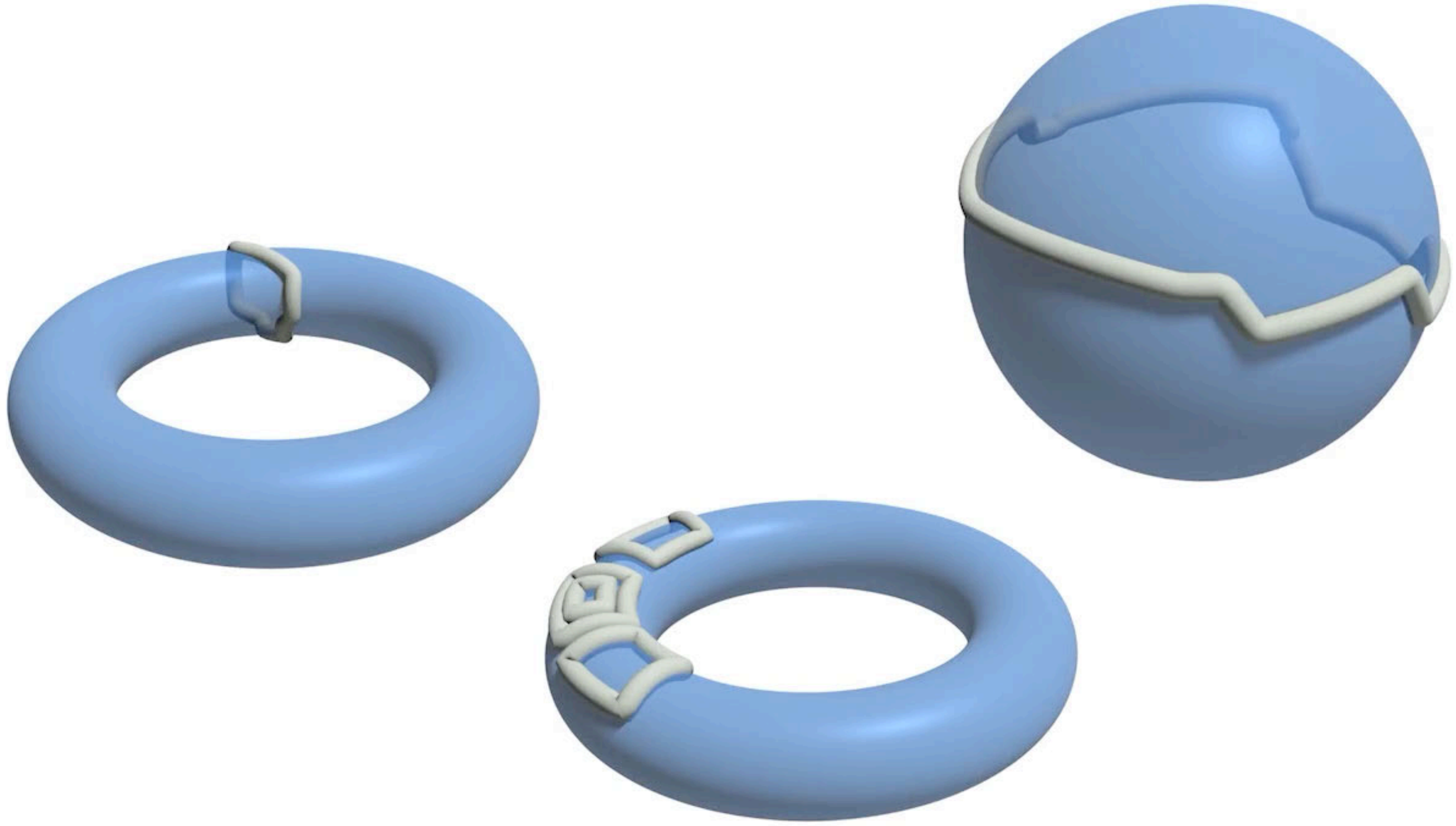
**Möbius energy**

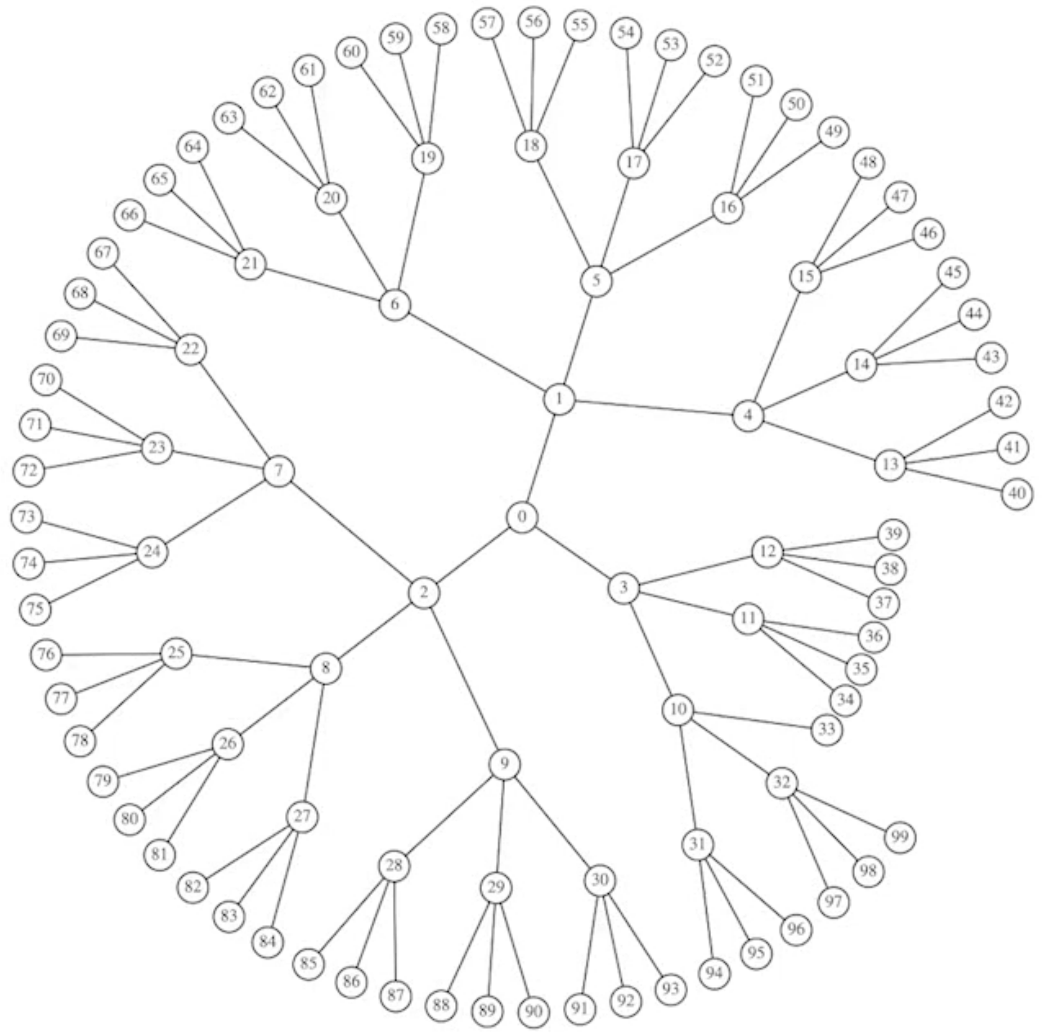




videos: Henrik Schumacher

# *Repulsive Curves*



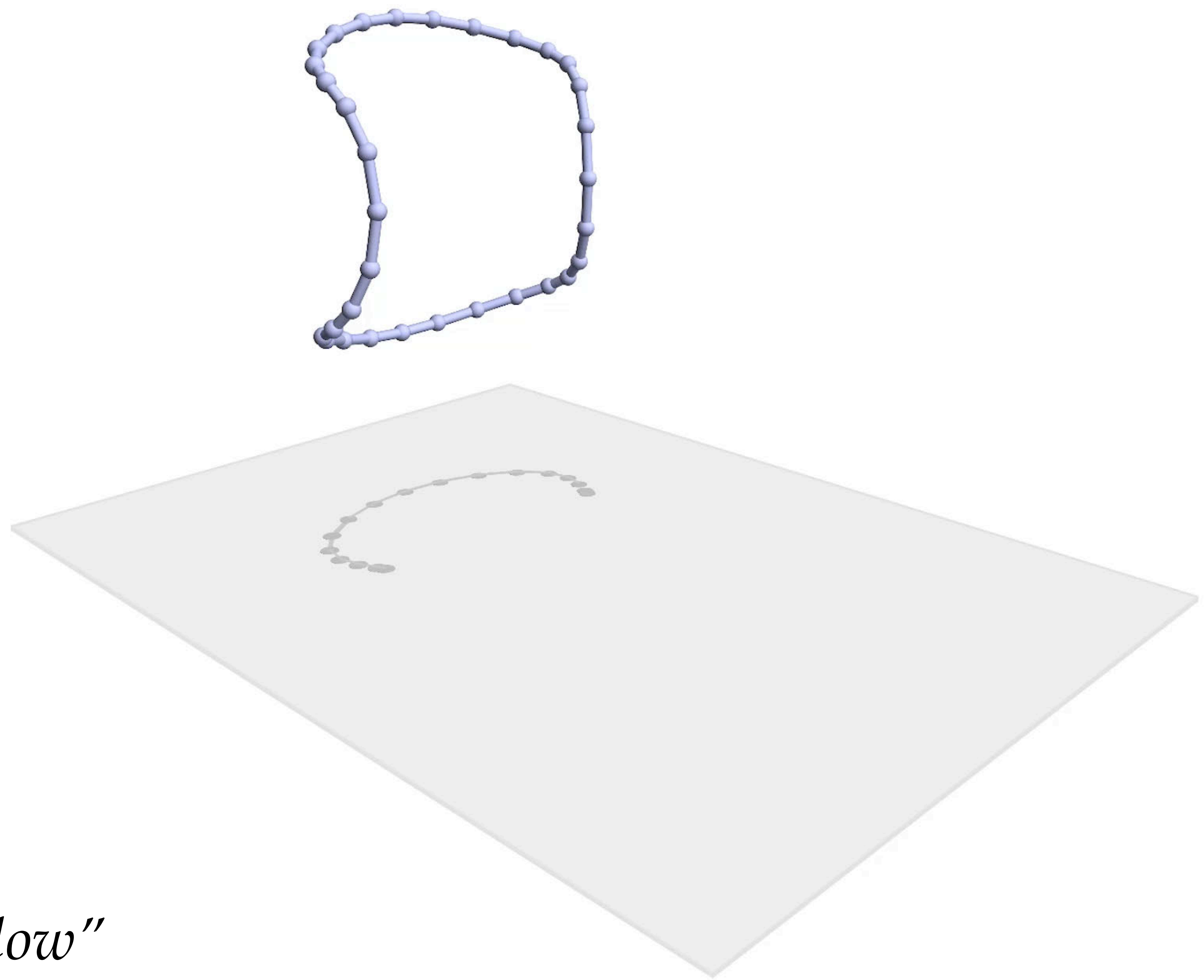


Yu, Schumacher, Crane, *"Repulsive Curves"*

# *Smoke Ring Flow*

- Roughly speaking, a *vortex filament* in a fluid isa curve along which the fluid is rapidly spinning (smoke rings, bubble rings, ...)
- Evolution captured by *Hashimoto flow*
  - easy to express for discrete curve via discrete binormal, curvature (as defined before)
  - take explicit time steps (as with curvature flow)
- More sophisticated discretization via special transformations (*Bäcklund*, *Darboux*) exactly preserves invariants of smooth flow

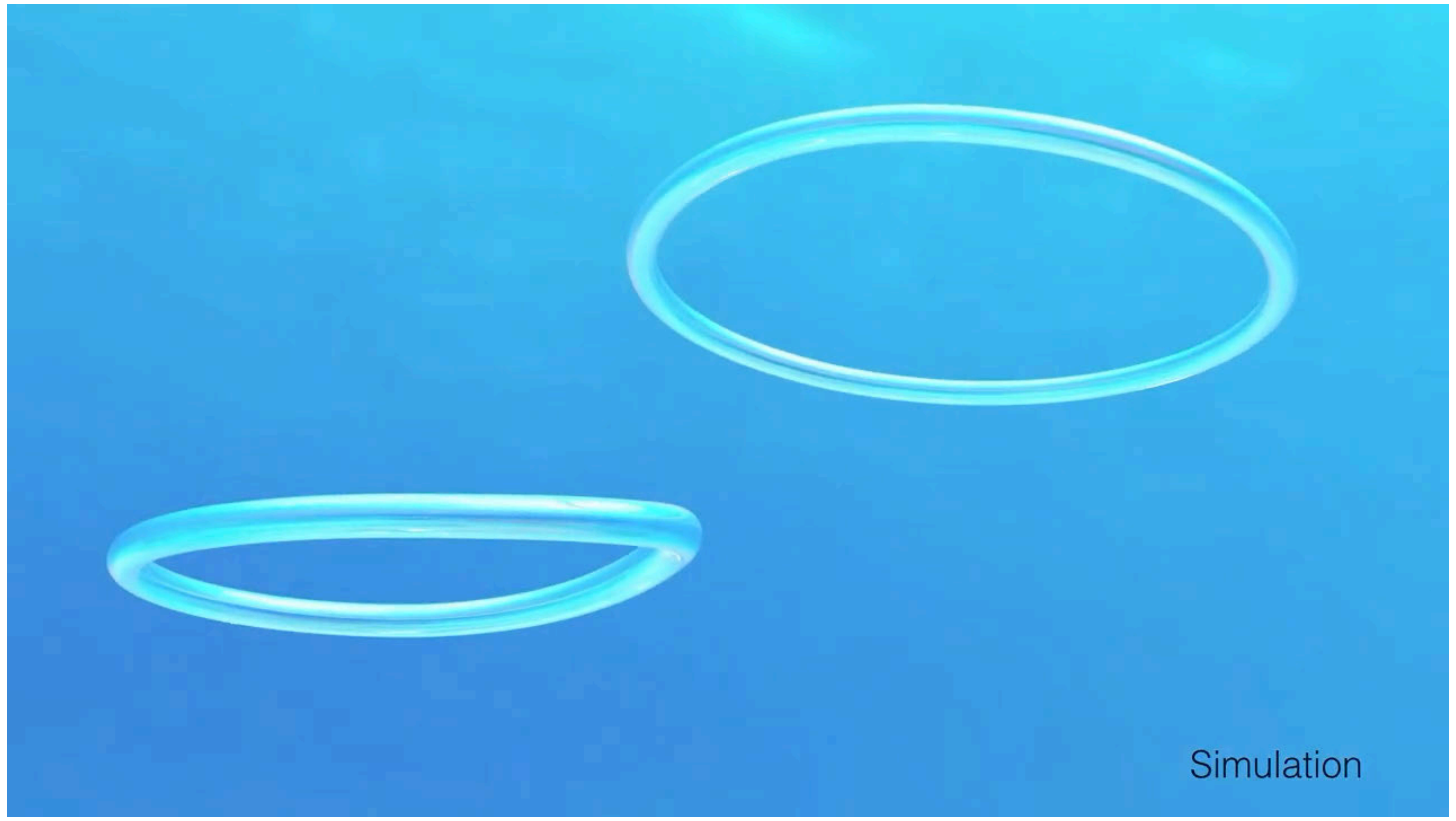
$$\begin{aligned} \tfrac{d}{dt}\gamma &= \gamma' \times \gamma'' \\ &= T \times \kappa N \\ &= \kappa B \end{aligned}$$

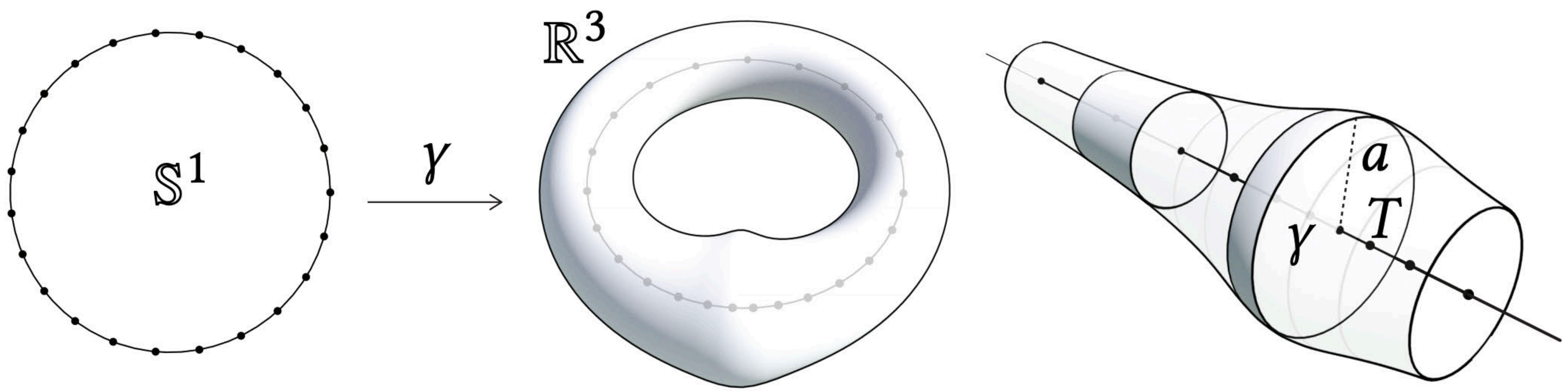


Hoffmann, *"Discrete Hashimoto Surfaces and a Doubly Discrete Smoke-Ring Flow"*

Pinkall, Springborn, Weißmann, *"A New Doubly Discrete Analogue of Smoke Ring Flow"*

# *Bubble Rings and Ink Chandeliers*

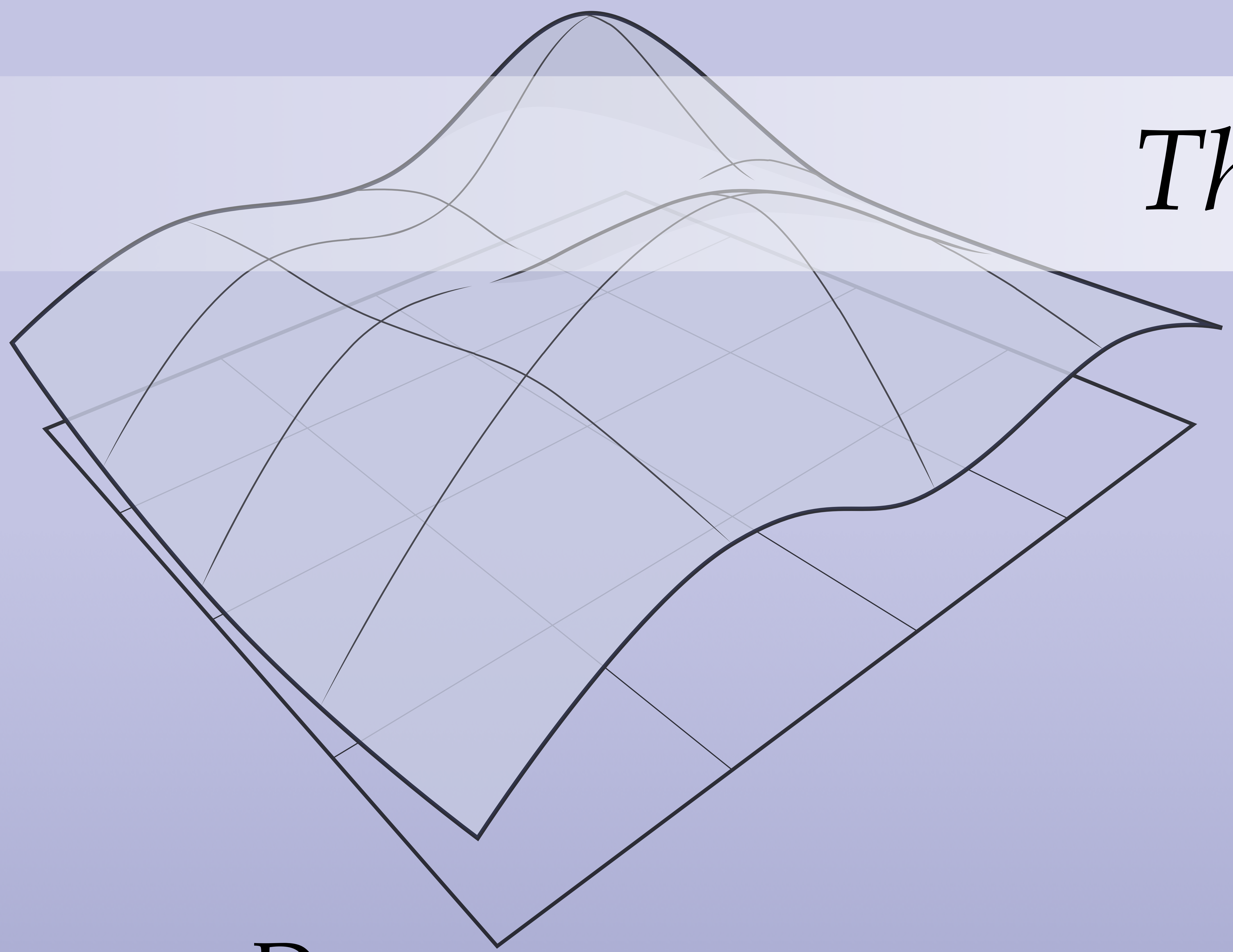






Real footage Simulation

Padilla, Chern, Knöppel, Pinkall, Schröder
*"On Bubble Rings and Ink Chandeliers"* (2019)

# *Thanks!*



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

**Keenan Crane • CMU 15-458/858**