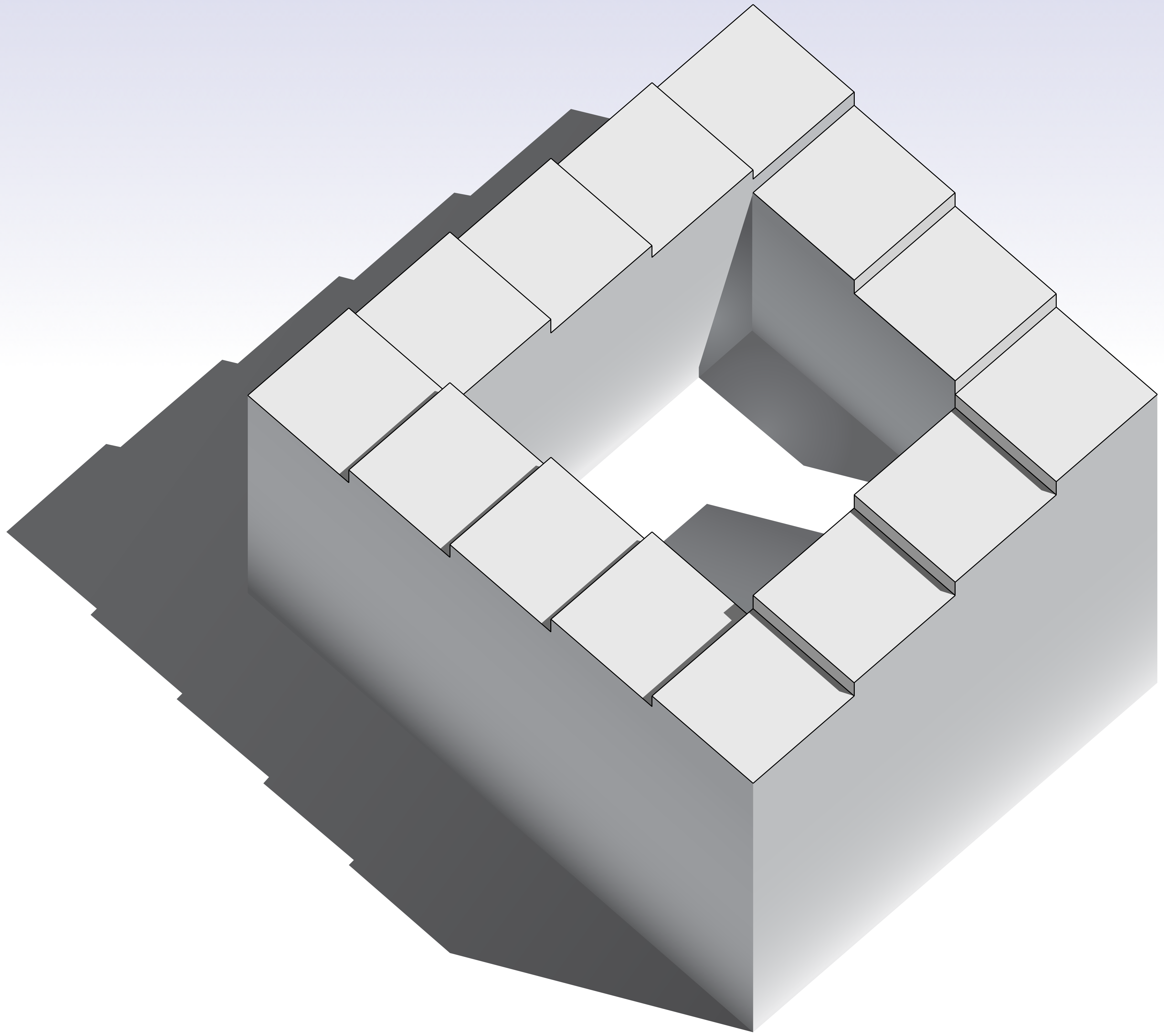
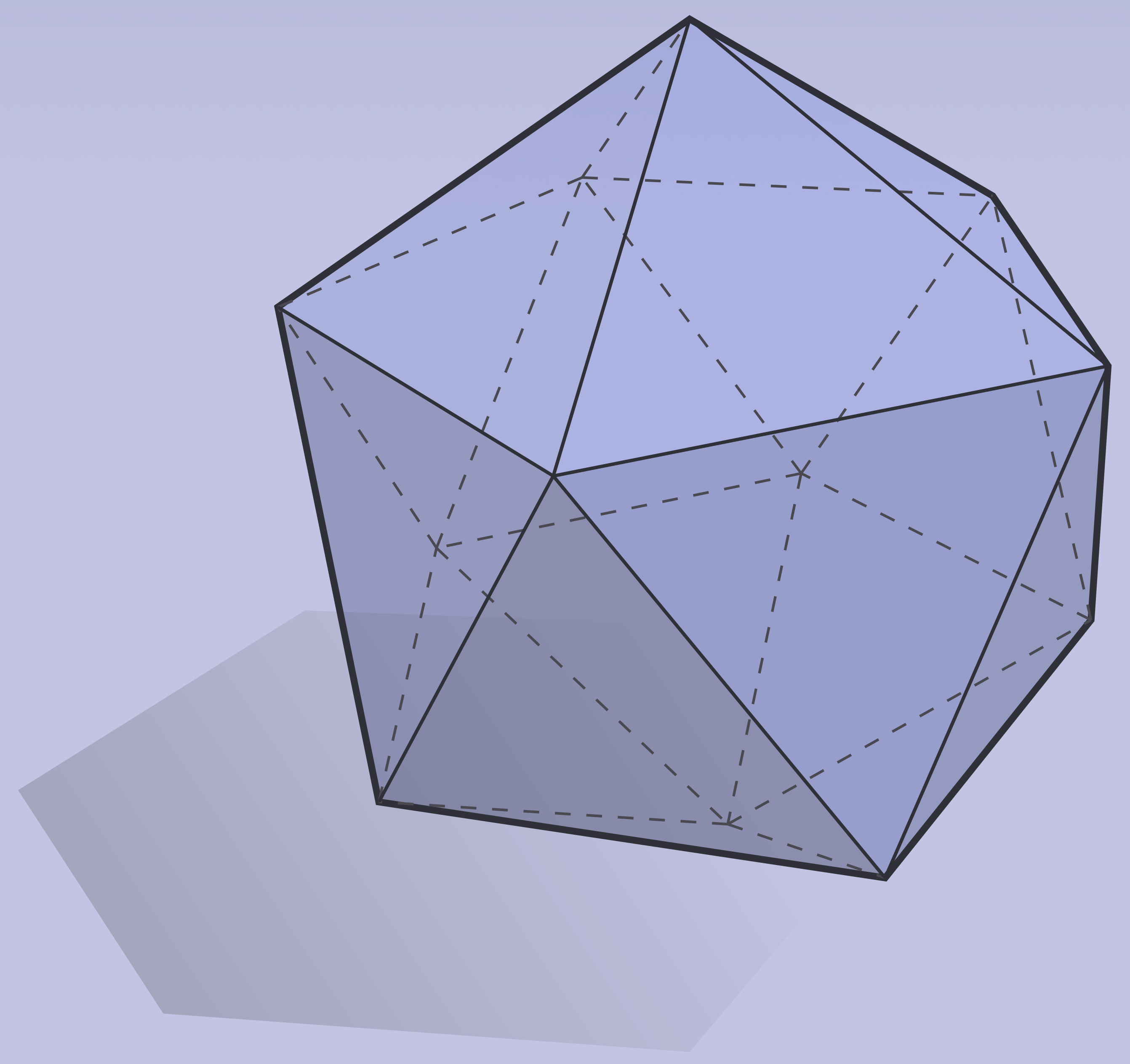


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 6:
EXTERIOR DERIVATIVE



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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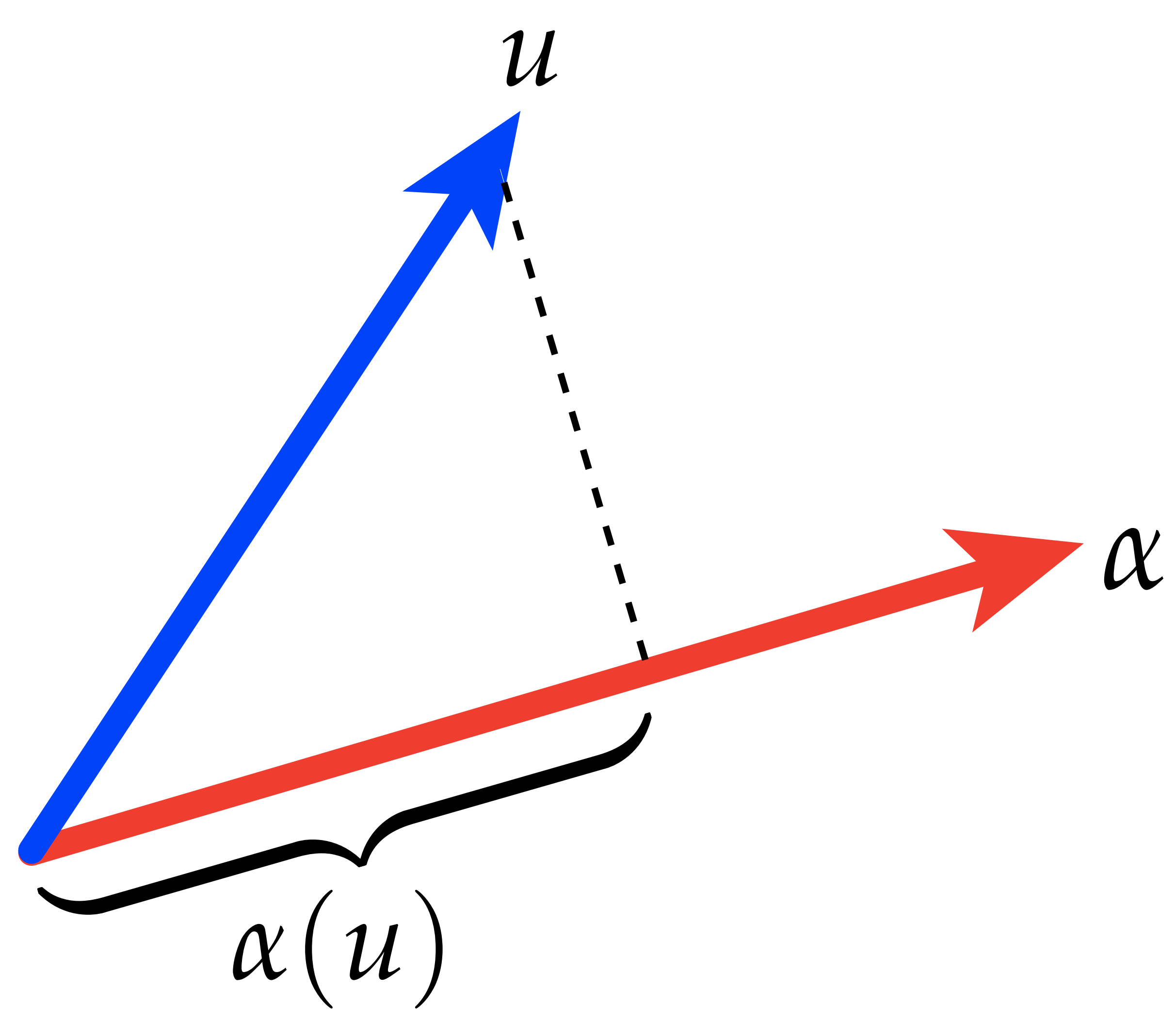
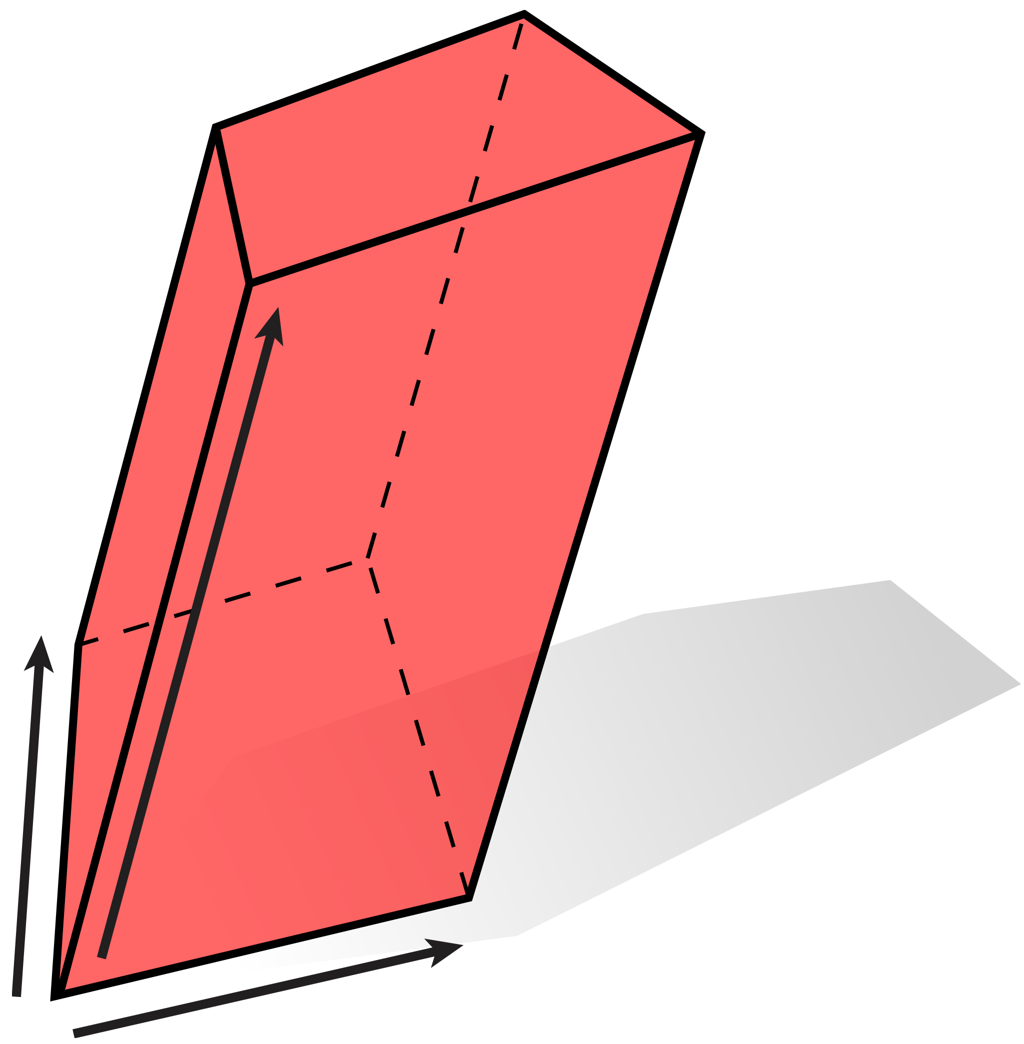
Exterior Calculus—Overview

- *Previously:*
  - **1-form**—linear measurement of a vector
  - ***k*-form**—multilinear measurement of volume
  - **differential *k*-form**—*k*-form at each point

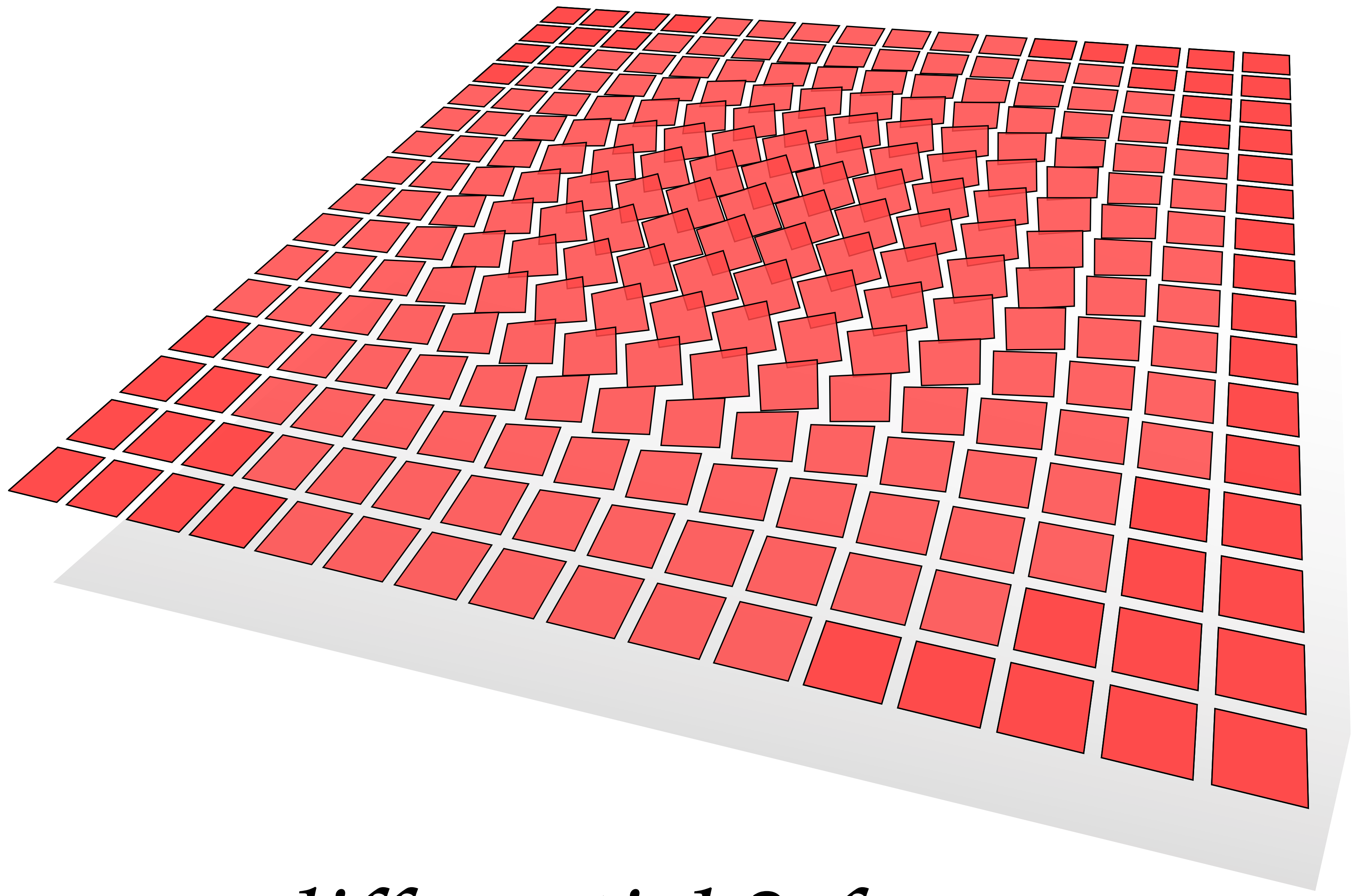
- *Today:* ***exterior calculus***
  - how do *k*-forms *change*?
  - how do we *integrate* *k*-forms?



1-form



3-form



differential 2-form

Integration and Differentiation

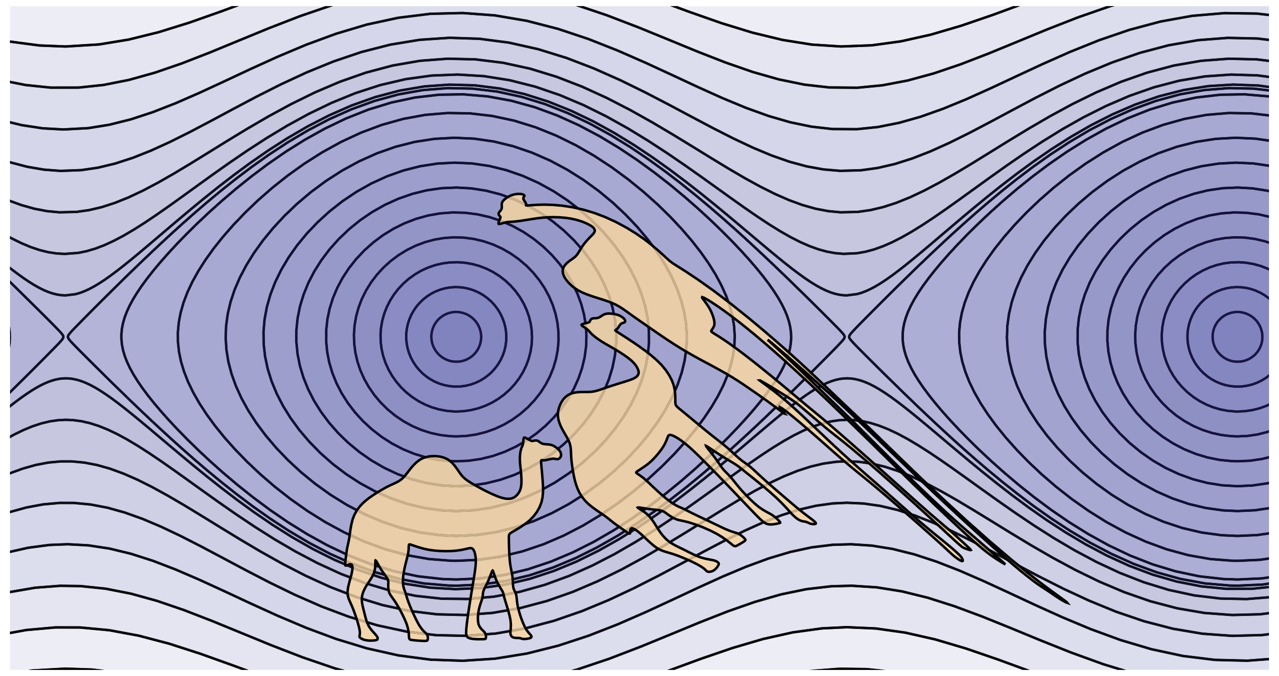
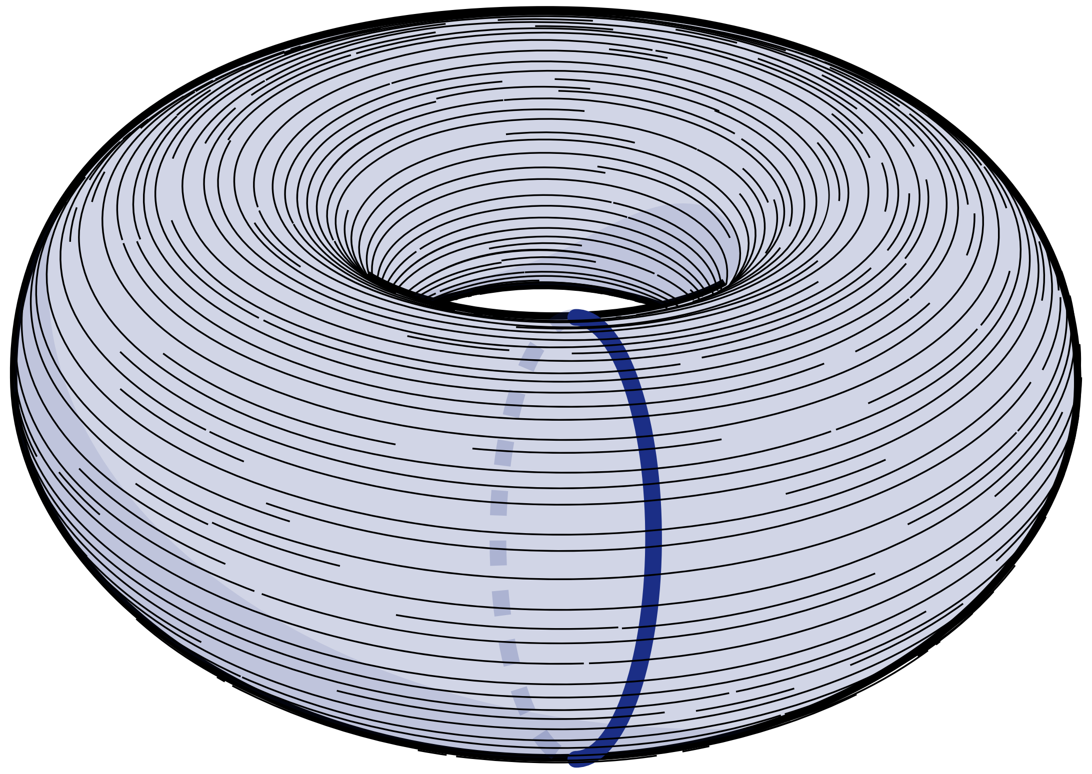
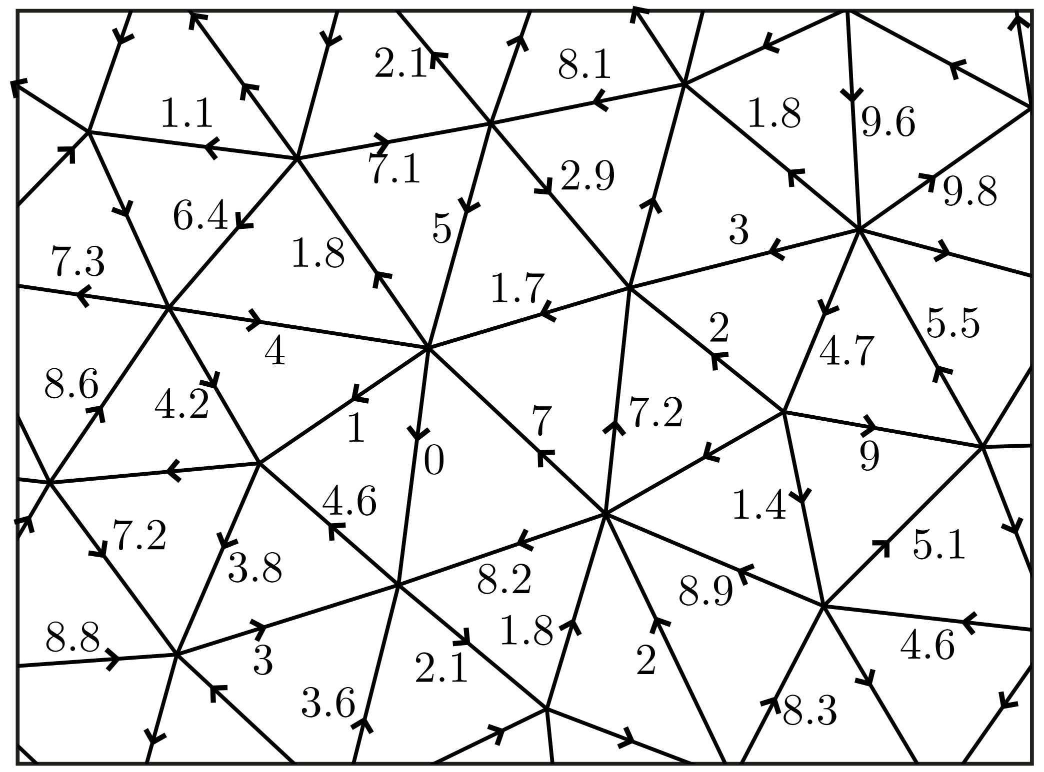
- Two big ideas in calculus:
  - **differentiation**
  - **integration**
  - linked by *fundamental theorem of calculus*
- Exterior calculus generalizes these ideas
  - **differentiation of k-forms** (exterior derivative)
  - **integration of k-forms** (measure volume)
  - linked by *Stokes' theorem*
- **Goal:** integrate differential forms over meshes to get *discrete exterior calculus (DEC)*

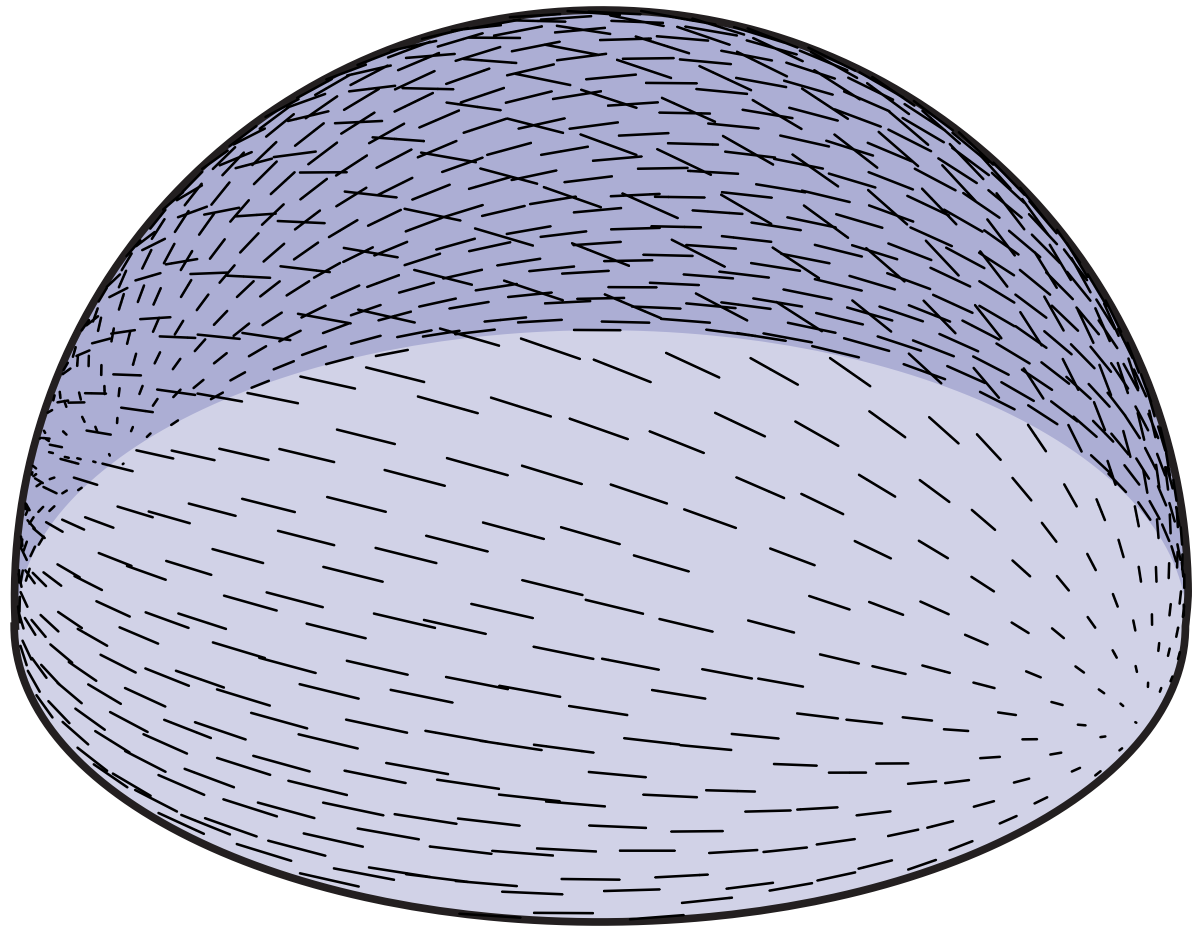
$$\int_a^b f'dx = f(b) - f(a)$$

$$\int_M d\alpha = \int_{\partial M} \alpha$$

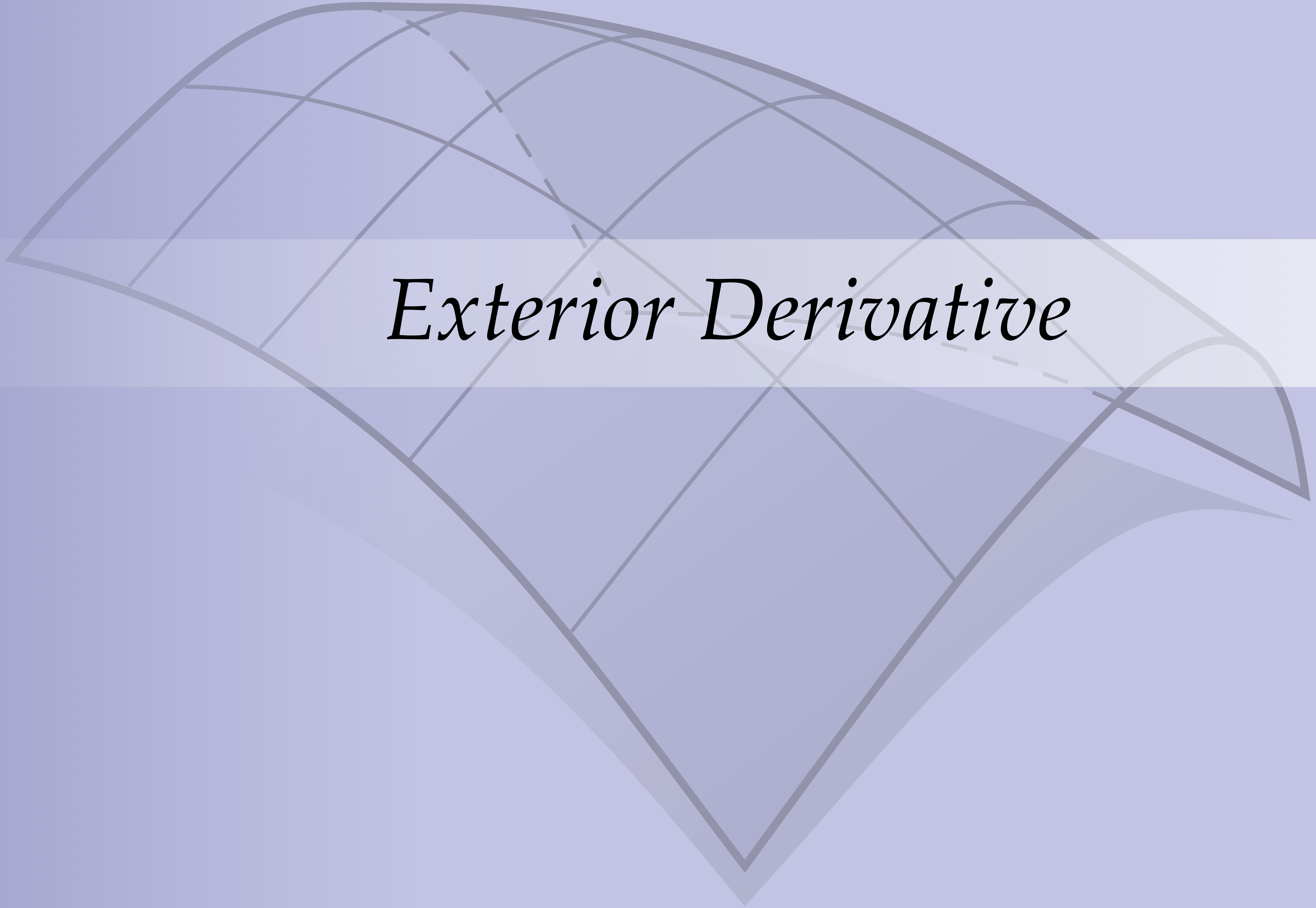
Motivation for Exterior Calculus

- Why generalize **vector** calculus to **exterior** calculus?
  - Hard to measure change in *volumes* using basic vector calculus
  - Duality clarifies the distinction between different concepts/quantities
  - **Topology**: notion of differentiation that does not require metric (e.g., *cohomology*)
  - **Geometry**: clear language for calculus on *curved* domains (Riemannian manifolds)
  - **Physics**: clear distinction between physical quantities (e.g., *velocity* vs. *momentum*)
  - **<u>Computer Science</u>**: *Leads directly to discretization/computation!*

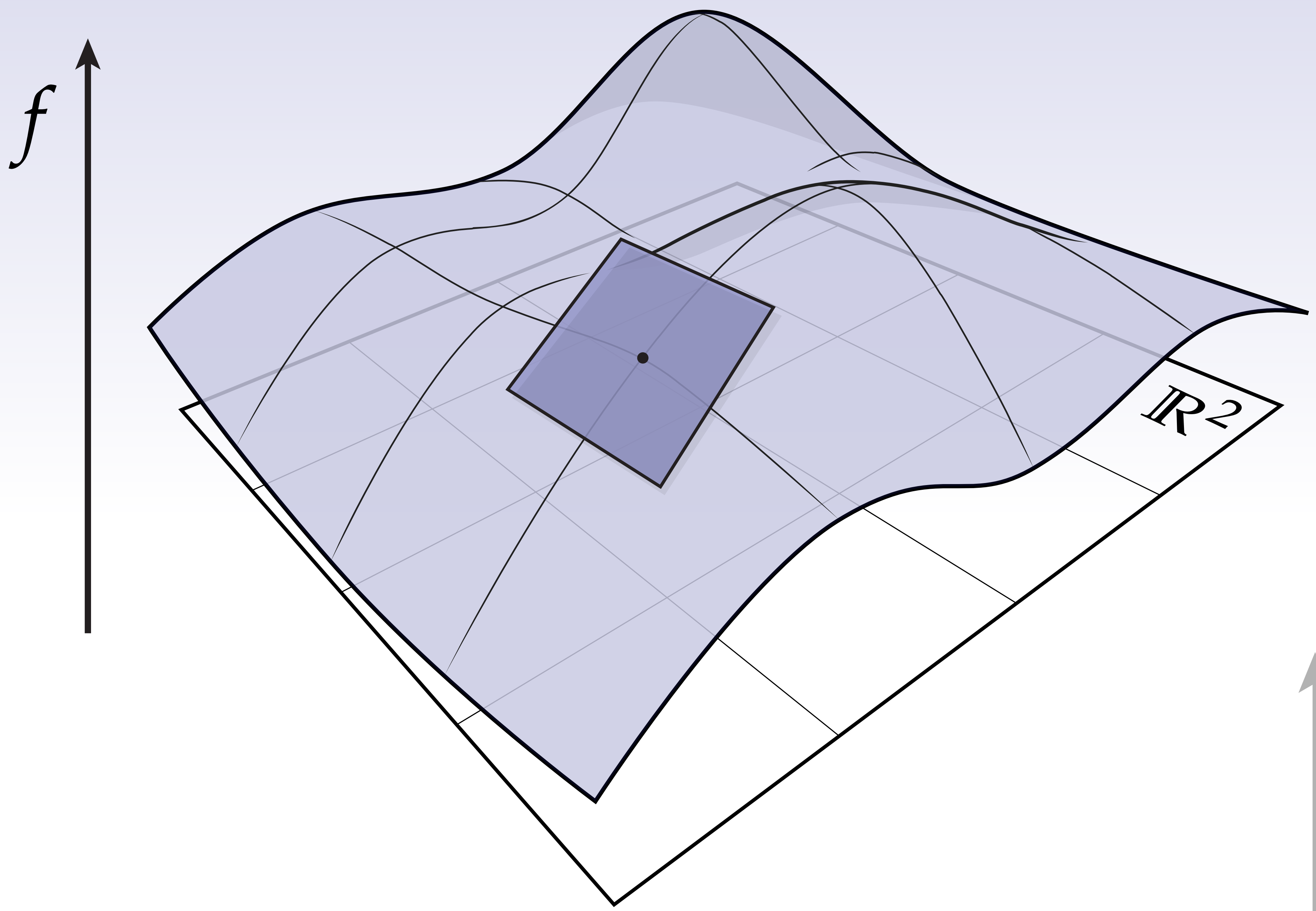


Exterior Derivative

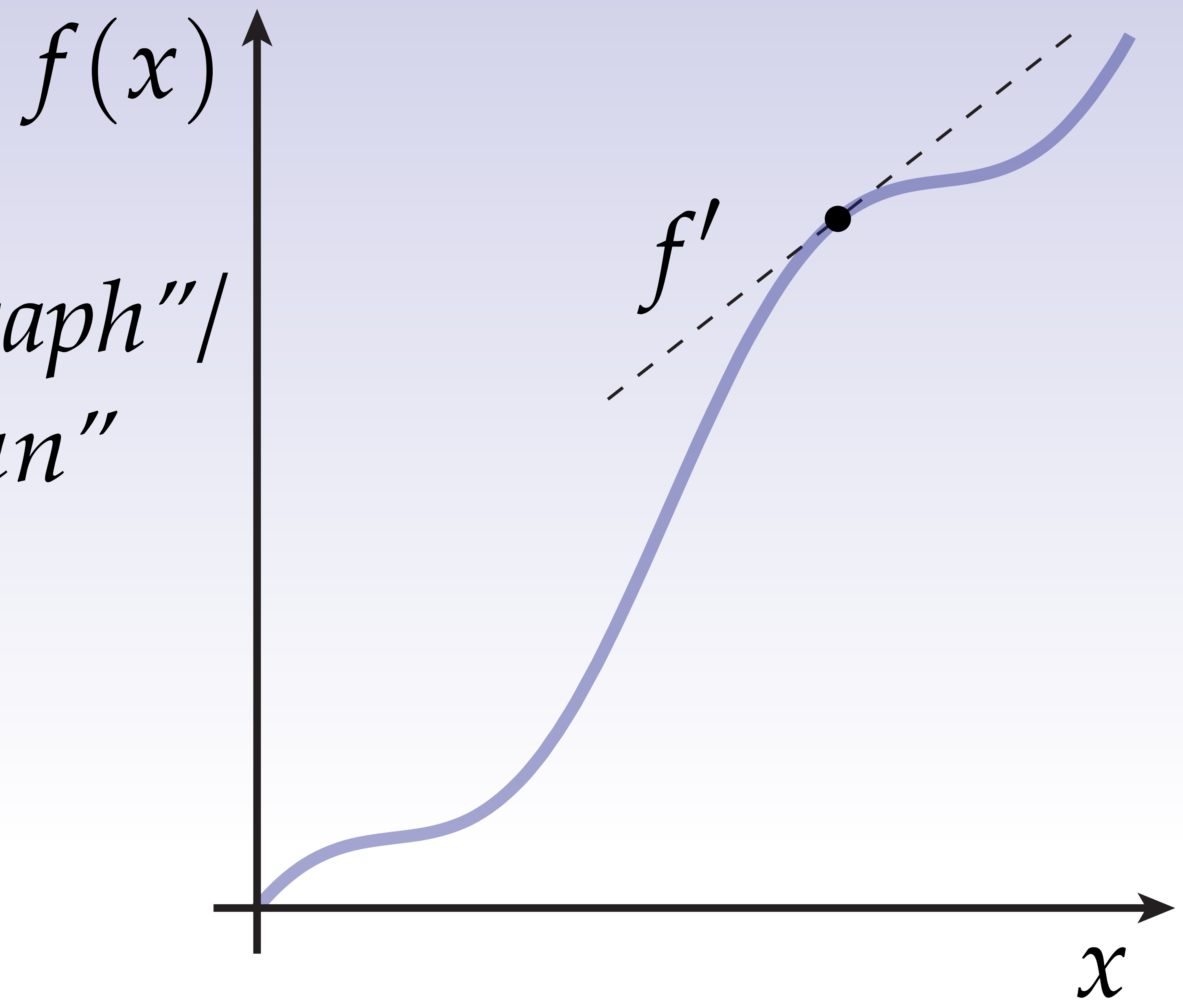


Derivative—Many Interpretations…

“best linear approximation”

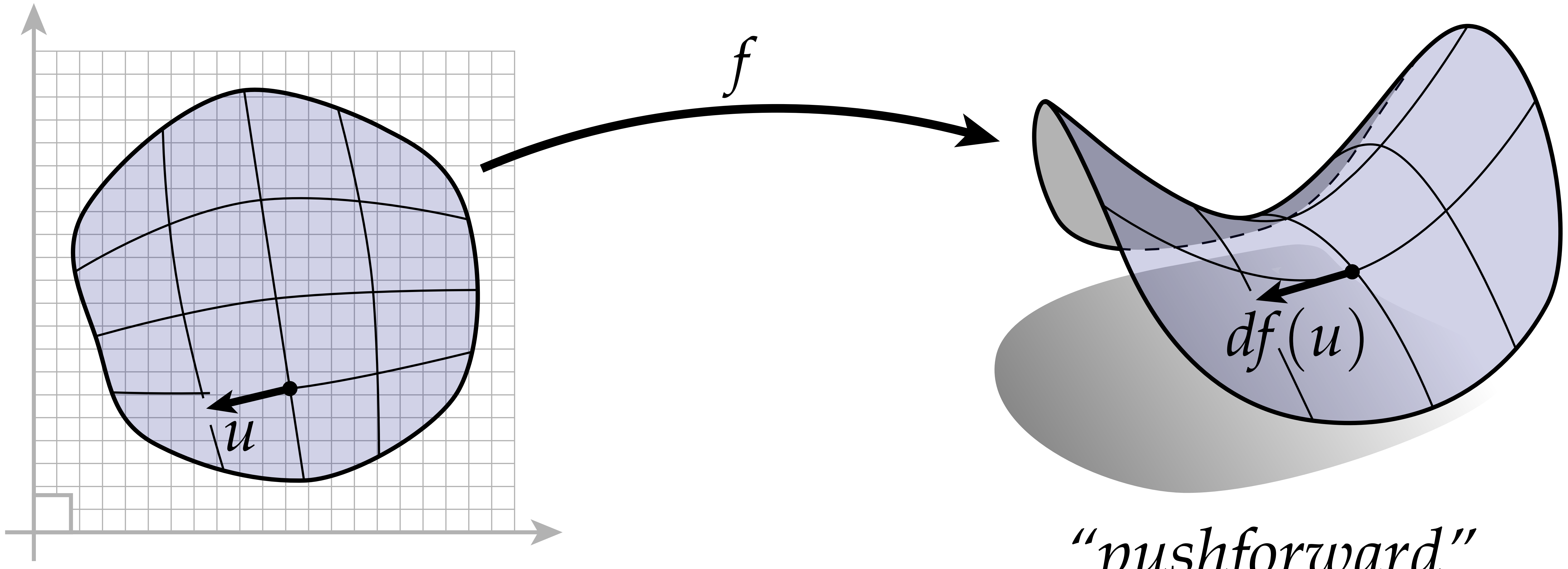


“slope of the graph”/
“rise over run”



$$f'(x) := \lim_{\varepsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\varepsilon}$$

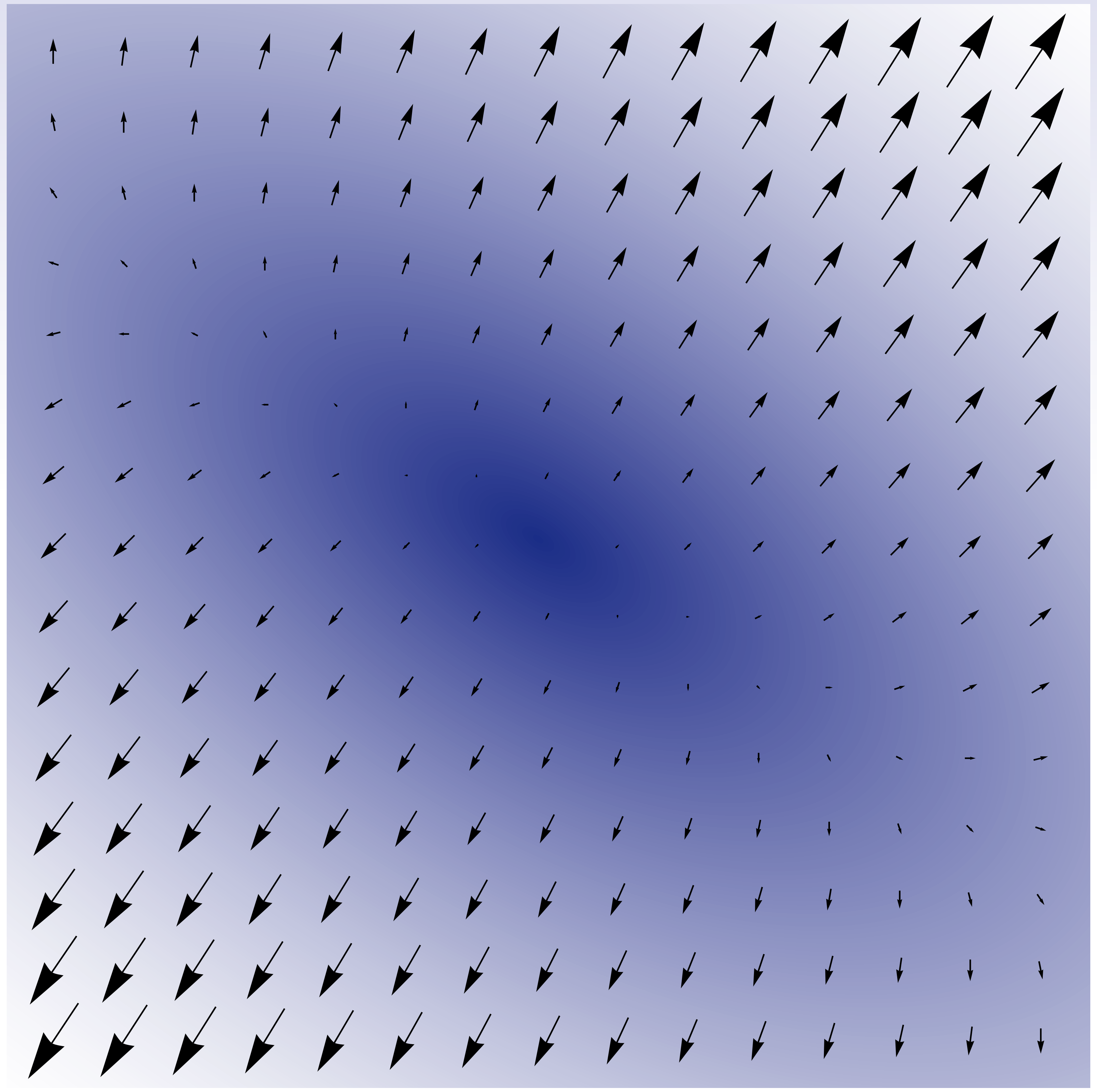
“rate of change”



“pushforward”

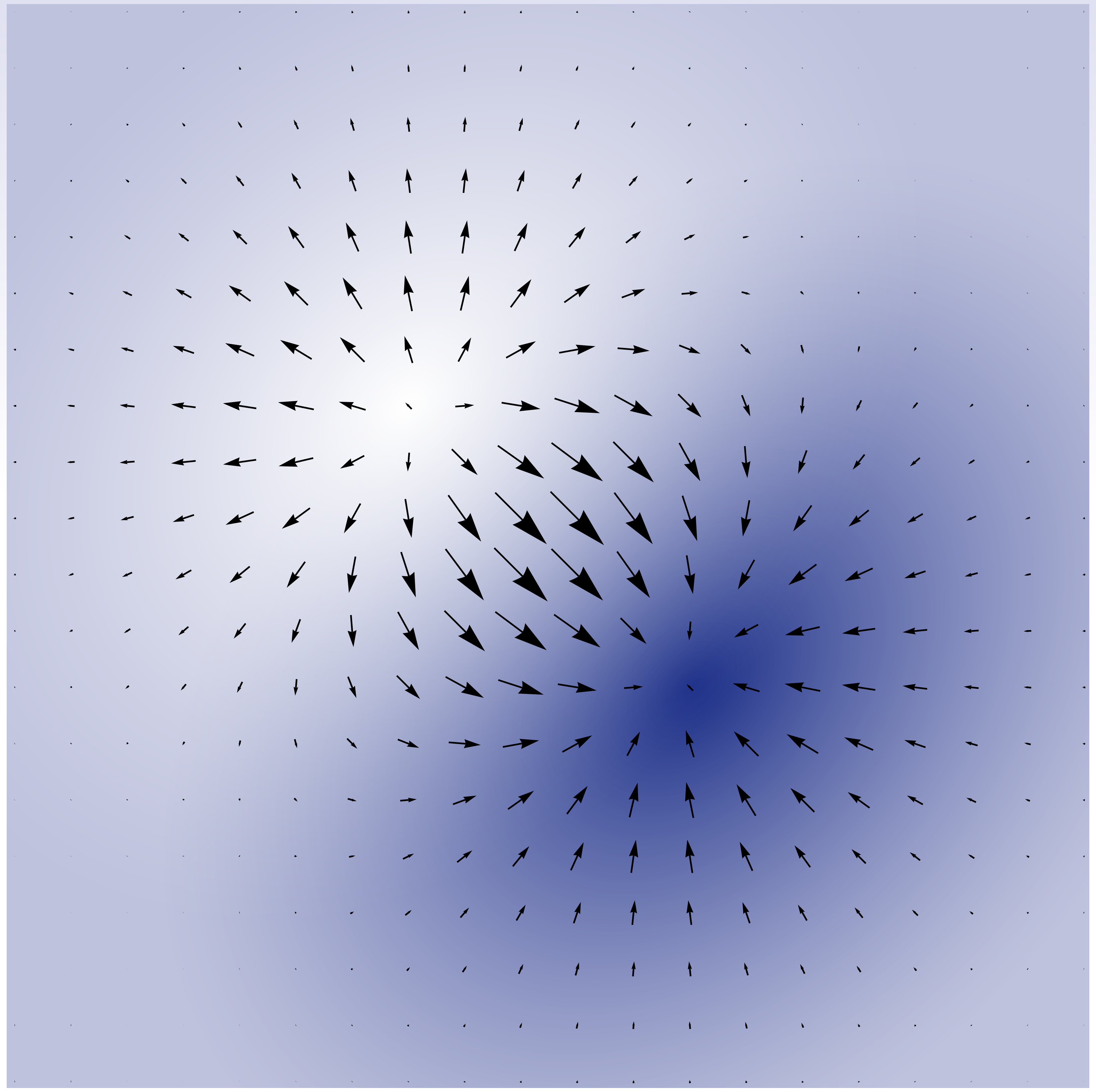
Review: Vector Derivatives

ϕ



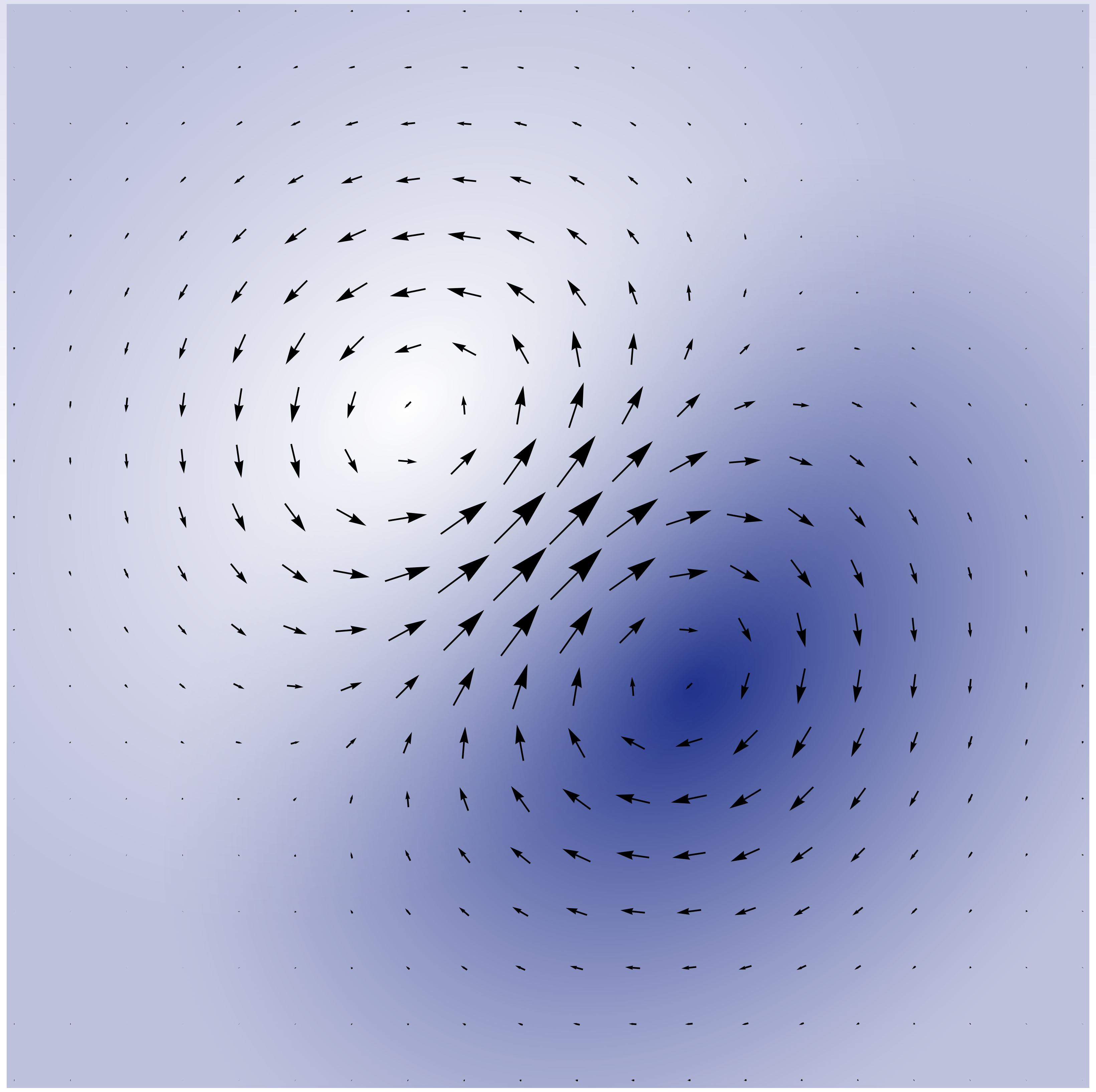
grad ϕ

X



div X

Y



curl Y

Review: Vector Derivatives in Coordinates

How do we express grad, div, and curl in coordinates?

Consider a scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$ and a vector field

$$X = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

where $u, v, w : \mathbb{R}^n \to \mathbb{R}$ are coordinate functions that vary over the domain, and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are the standard basis vector fields.

grad

$$\nabla\phi = \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}$$

div

$$\nabla\cdot X = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

curl

$$\nabla\times X = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial}{\partial z}$$

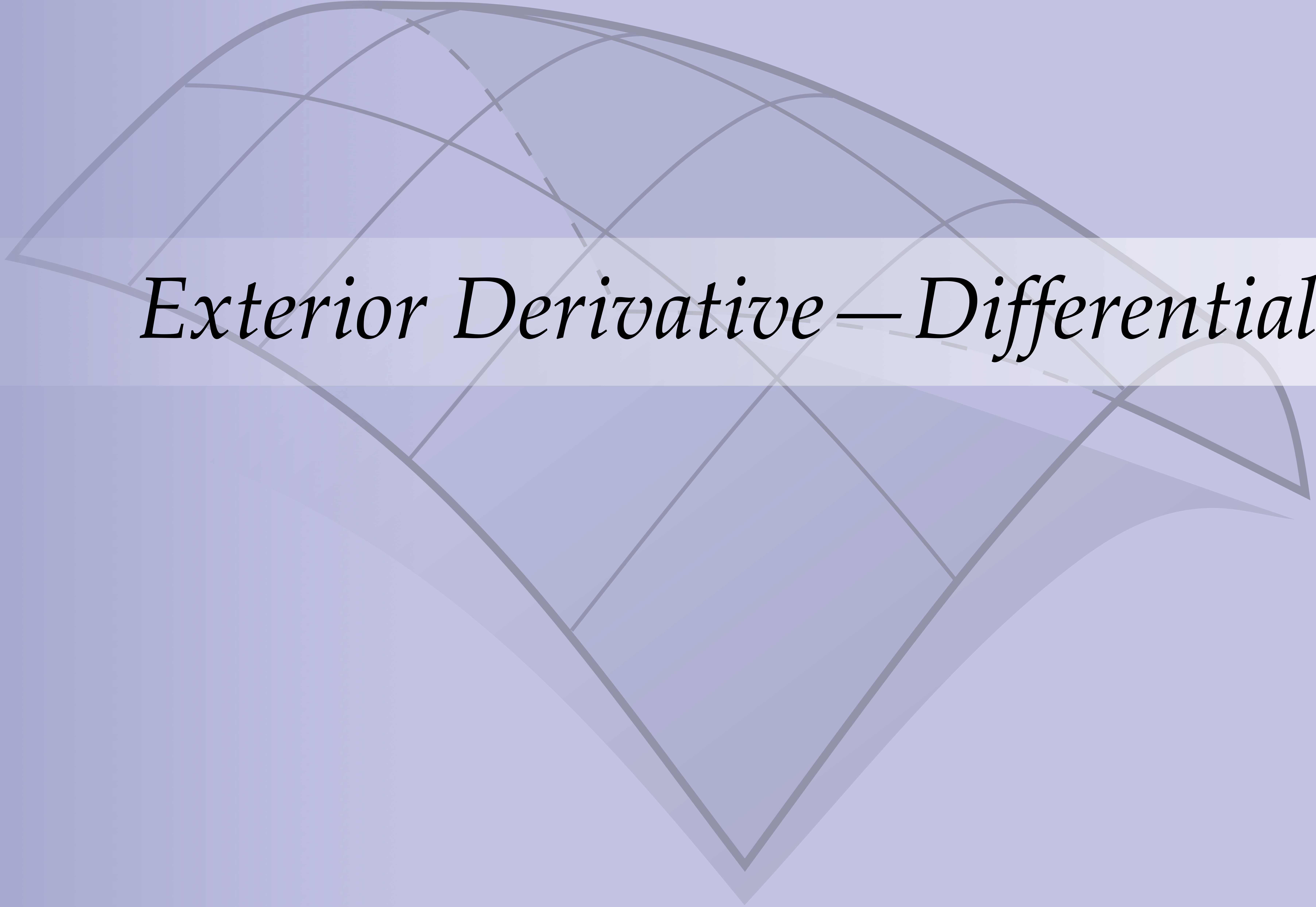
Exterior Derivative

(Ω^k — space of all differential k-forms)

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Where do these rules come from?
(What's the *geometric* motivation?)



Exterior Derivative — Differential

Review: Directional Derivative

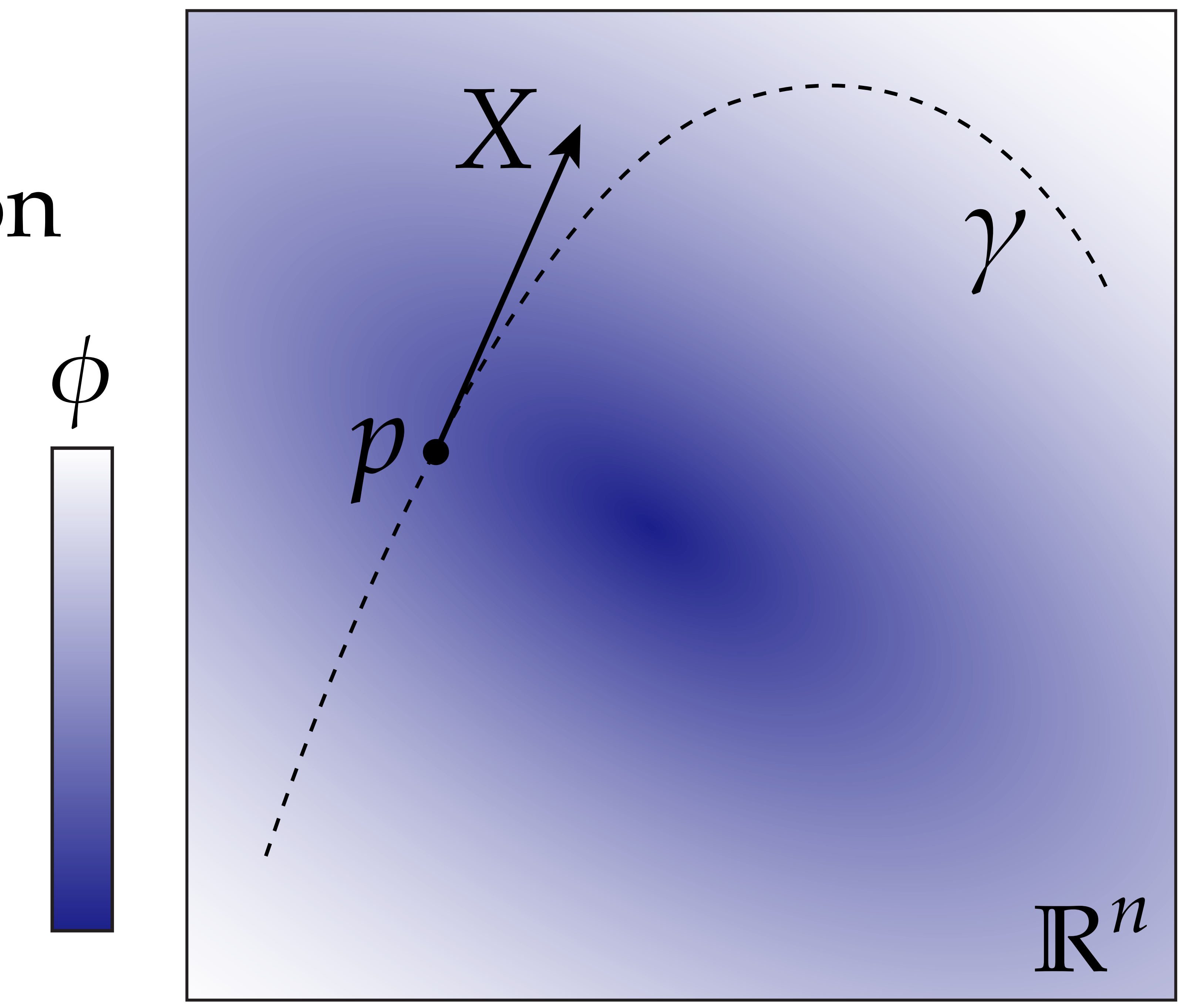
- The *directional derivative* of a scalar function ϕ at a point p with respect to a vector X is the rate at which that function increases as we walk away from p with velocity X.
- More precisely:

$$D_X\phi\Big|_p := \lim_{\varepsilon\to 0}\frac{\phi(p+\varepsilon X)-\phi(p)}{\varepsilon}$$

- Alternatively, suppose that X is a *vector field*, rather than just a vector at a single point. Then we can write just:

$$D_X\phi$$

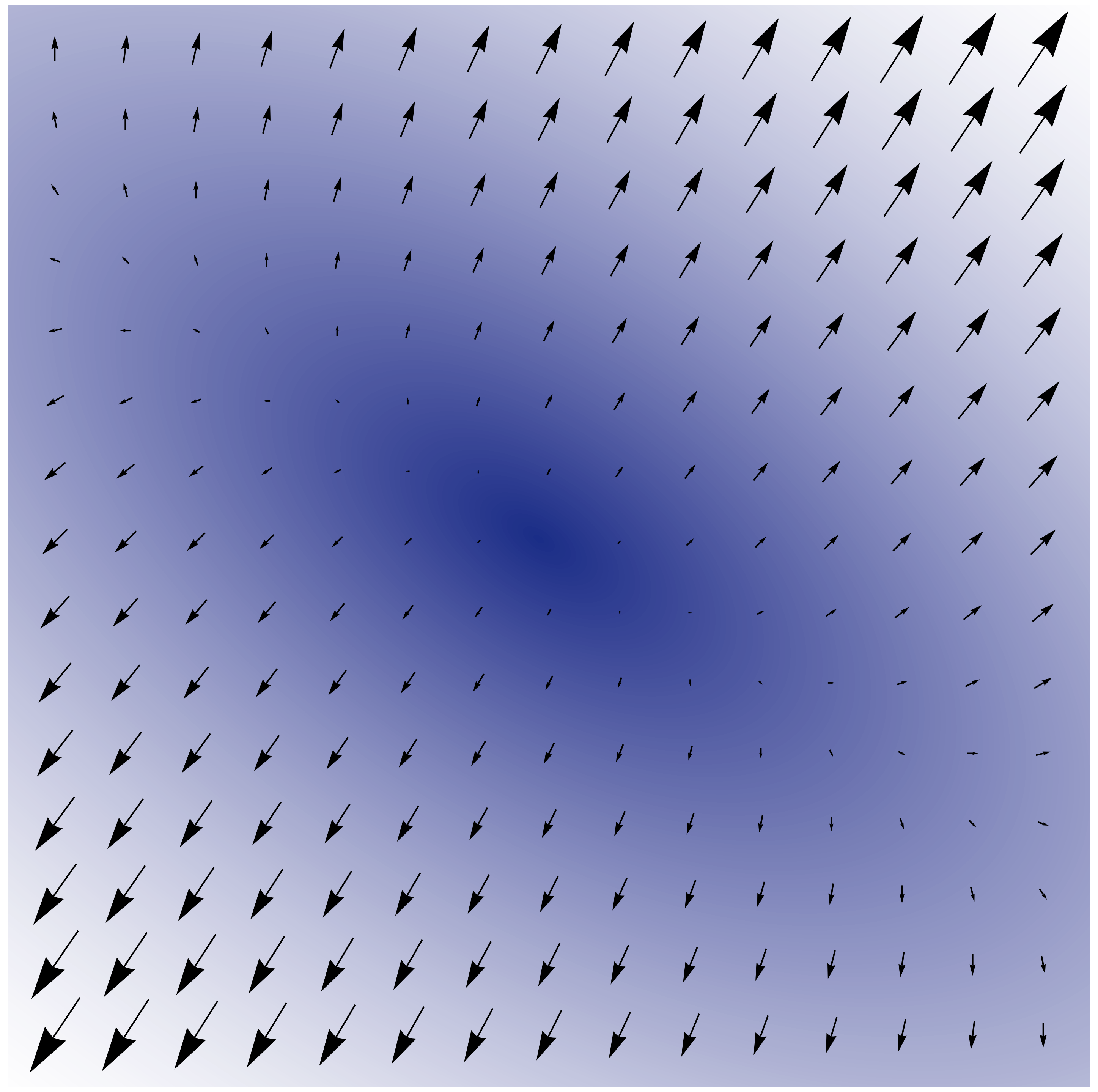
- The result is a *scalar function*, whose value at each point p is the directional derivative along the vector $X(p)$.



$\phi : \mathbb{R}^n \to \mathbb{R}$

Intuition: as we walk along a curve γ tangent to X, how fast will an observed quantity ϕ change as we pass through p?

Review: Gradient



Let $\phi : \mathbb{R}^n \to \mathbb{R}$. What is the *gradient* of ϕ?

Geometric intuition. "Uphill direction."

Coordinate approach. In Euclidean $\mathbb{R}^n$, list of partials:

$$\nabla\phi = \frac{\partial\phi}{\partial x^1}\frac{\partial}{\partial x^1} + \cdots + \frac{\partial\phi}{\partial x^n}\frac{\partial}{\partial x^n} = \begin{bmatrix} \frac{\partial\phi}{\partial x^1} & \cdots & \frac{\partial\phi}{\partial x^n} \end{bmatrix}^\mathsf{T}$$

Coordinate-free approach.

$$\langle \nabla\phi, X \rangle = D_X\phi \quad \text{for all } X$$

I.e., the gradient is the unique* vector field $\nabla\phi$ whose inner product with any vector field X yields the directional derivative $D_X\phi$ along X.

*If it exists! I.e., assuming the function is *differentiable*.

Differential of a Function

- Recall that differential 0-forms are just ordinary scalar functions
- Change in a scalar function can be measured via the *differential*
- Two ways to define differential:

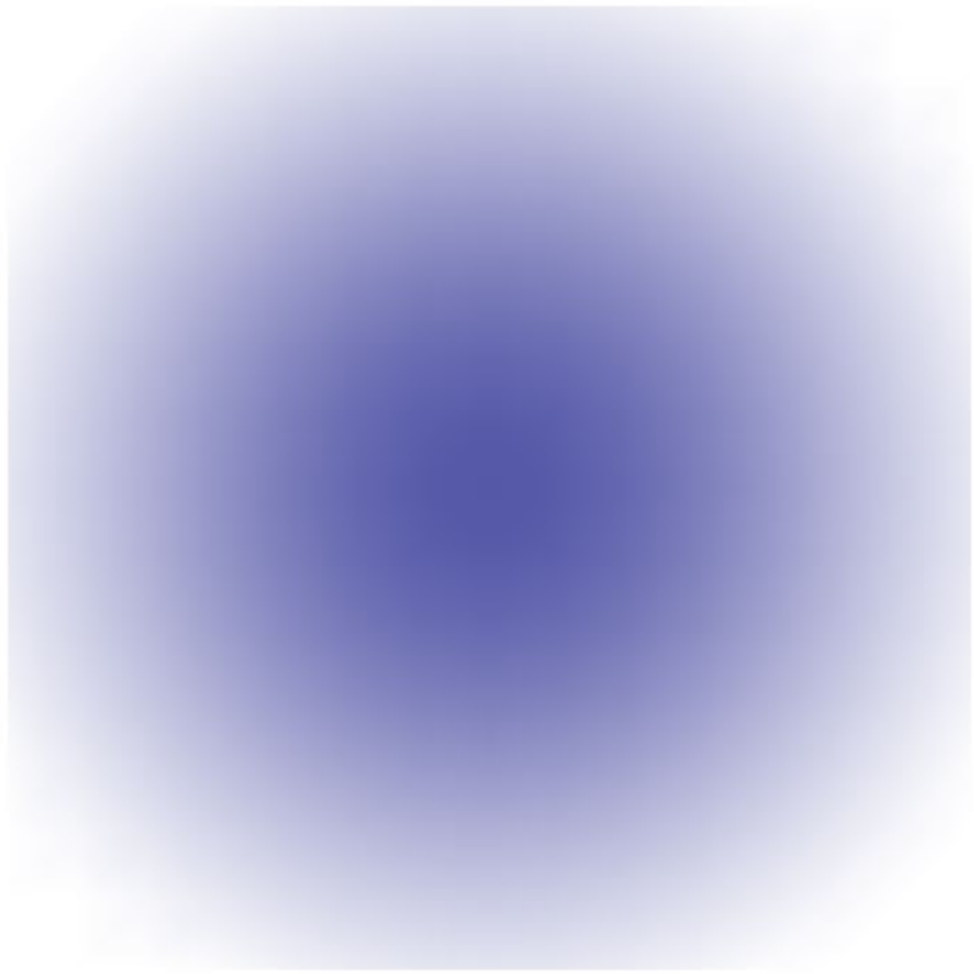
1. As unique 1-form such that applying to any vector field gives directional derivative along those directions:

$$d\phi(X) = D_X\phi$$

2. In coordinates:

$$d\phi := \frac{\partial \phi}{\partial x^1}dx^1 + \cdots + \frac{\partial \phi}{\partial x^n}dx^n$$

…but wait, isn't this just the same as the *gradient?*



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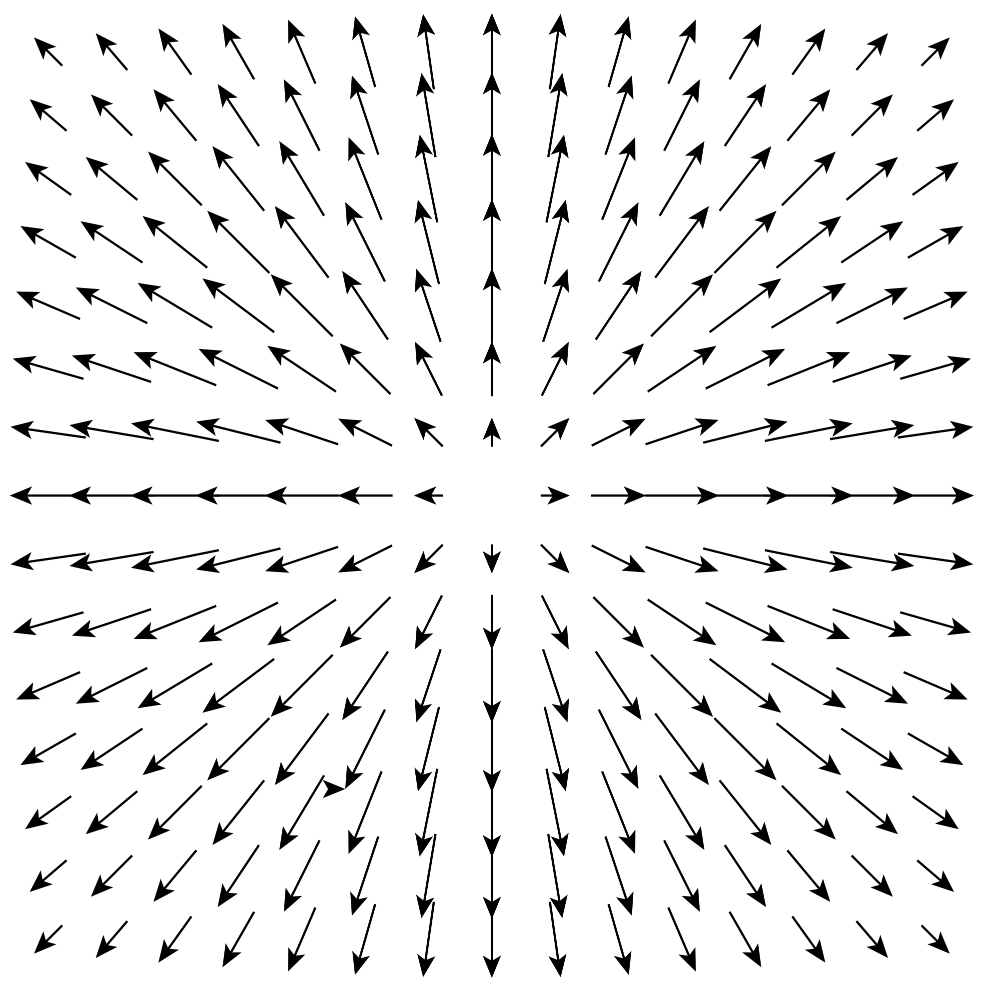


$d\phi$

Gradient vs. Differential

- Superficially, gradient and differential look quite similar (but not identical!):

$$\langle \nabla\phi, X\rangle = D_X\phi \qquad\qquad d\phi(X) = D_X\phi$$

- Especially in $\mathbb{R}^n$:

$$\nabla\phi = \frac{\partial\phi}{\partial x^1}\frac{\partial}{\partial x^1} + \cdots + \frac{\partial\phi}{\partial x^n}\frac{\partial}{\partial x^n} \qquad\qquad d\phi = \frac{\partial\phi}{\partial x^1}dx^1 + \cdots + \frac{\partial\phi}{\partial x^n}dx^n$$

- So what's the difference?
  - For one thing, one is a *vector field*; the other is a *differential 1-form*
  - More importantly, gradient depends on *inner product*; differential doesn't

$$(d\phi)^\sharp = \nabla\phi \iff \boxed{d\phi(\,\cdot\,) = \langle\nabla\phi,\,\cdot\,\rangle} \iff (\nabla\phi)^\flat = d\phi$$

Makes a *big* difference when it comes to curved geometry, numerical optimization, …

Exterior Derivative—Product Rule

Exterior Derivative

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

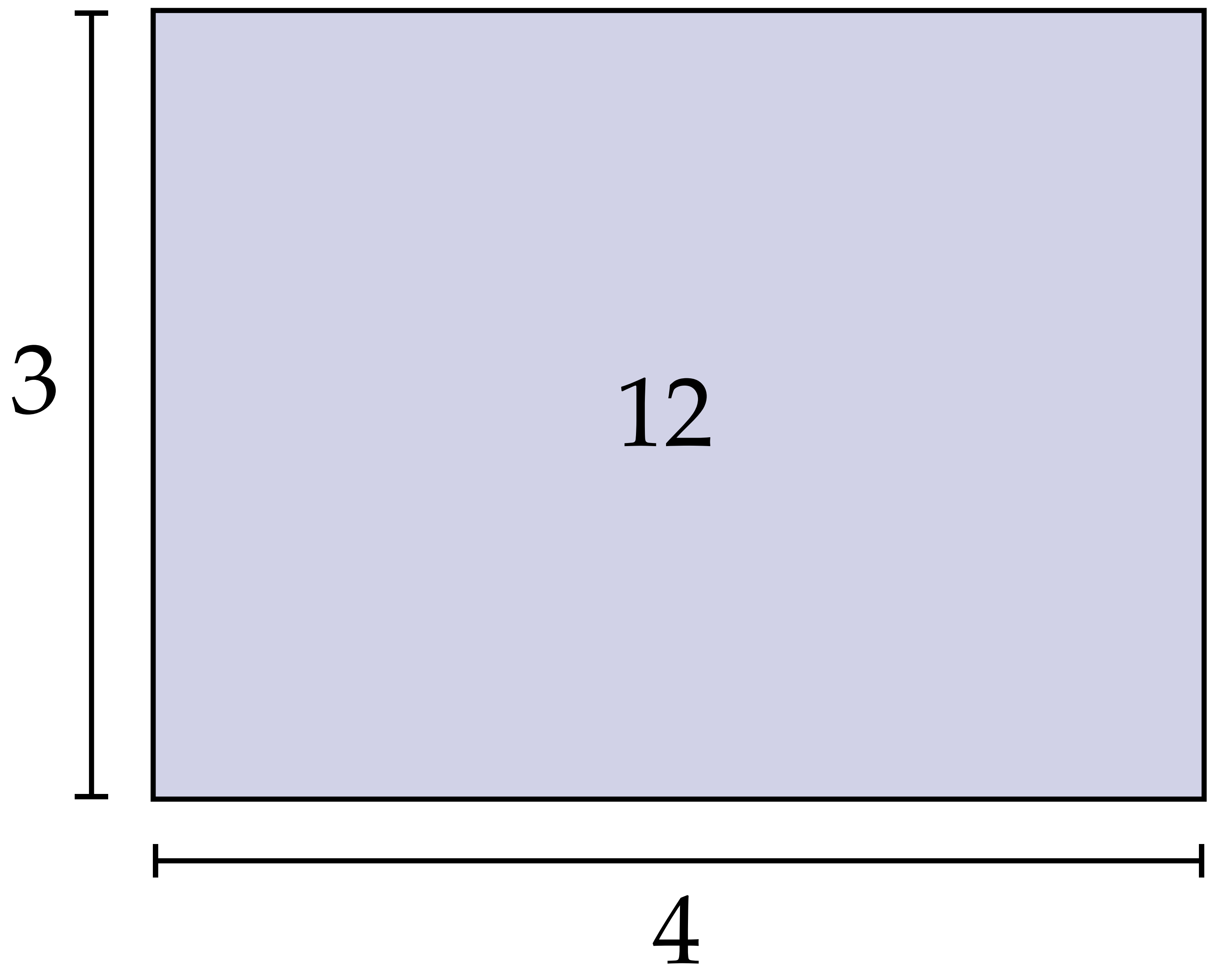
differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Review: Product of Numbers

Q: Why is it true that $ab = ba$ for any two real numbers a, b?

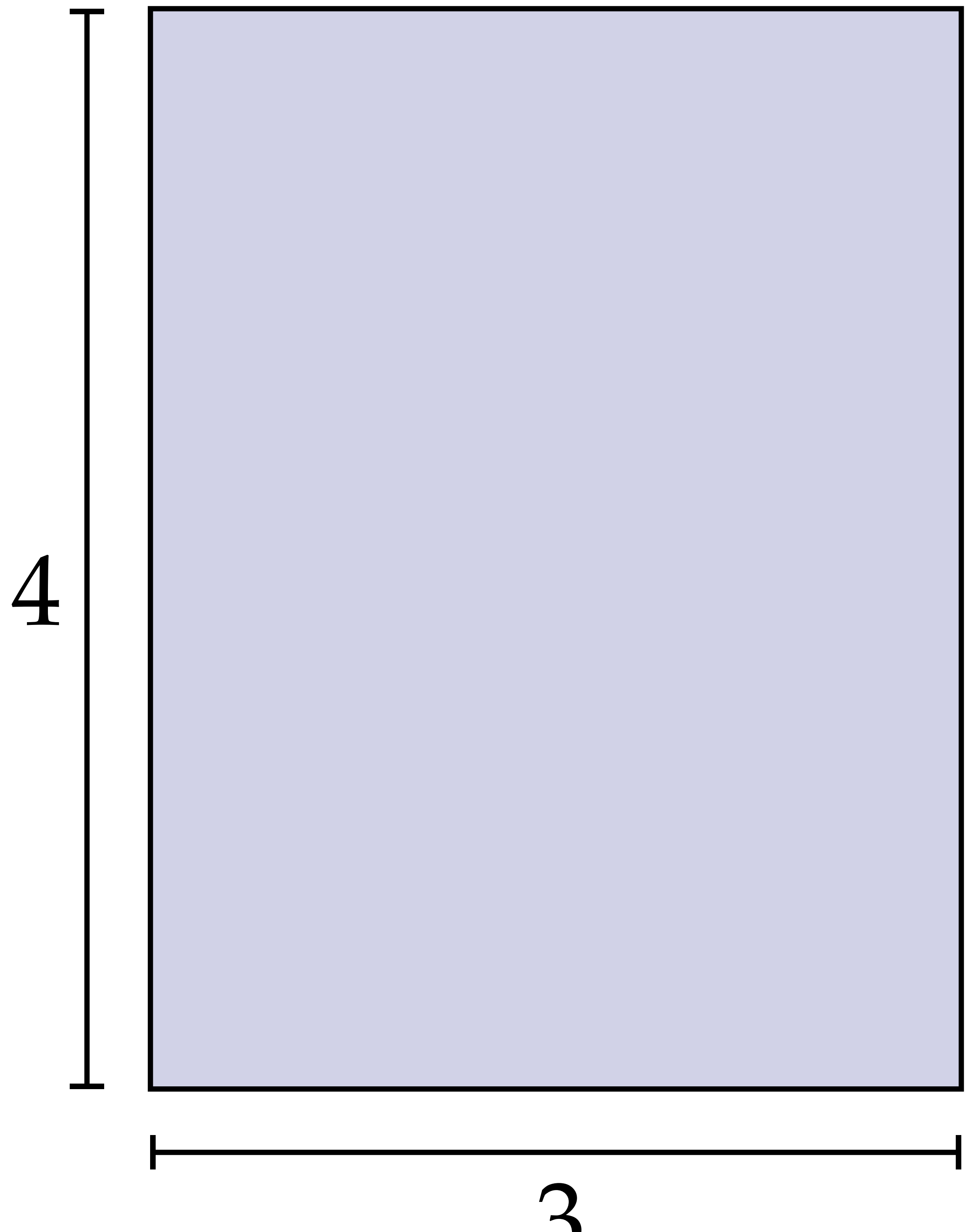
Q: What's the geometric interpretation of the statement "$4 \times 3 = 12$"?

A:



Q: How about "$3 \times 4 = 12$"?

A:

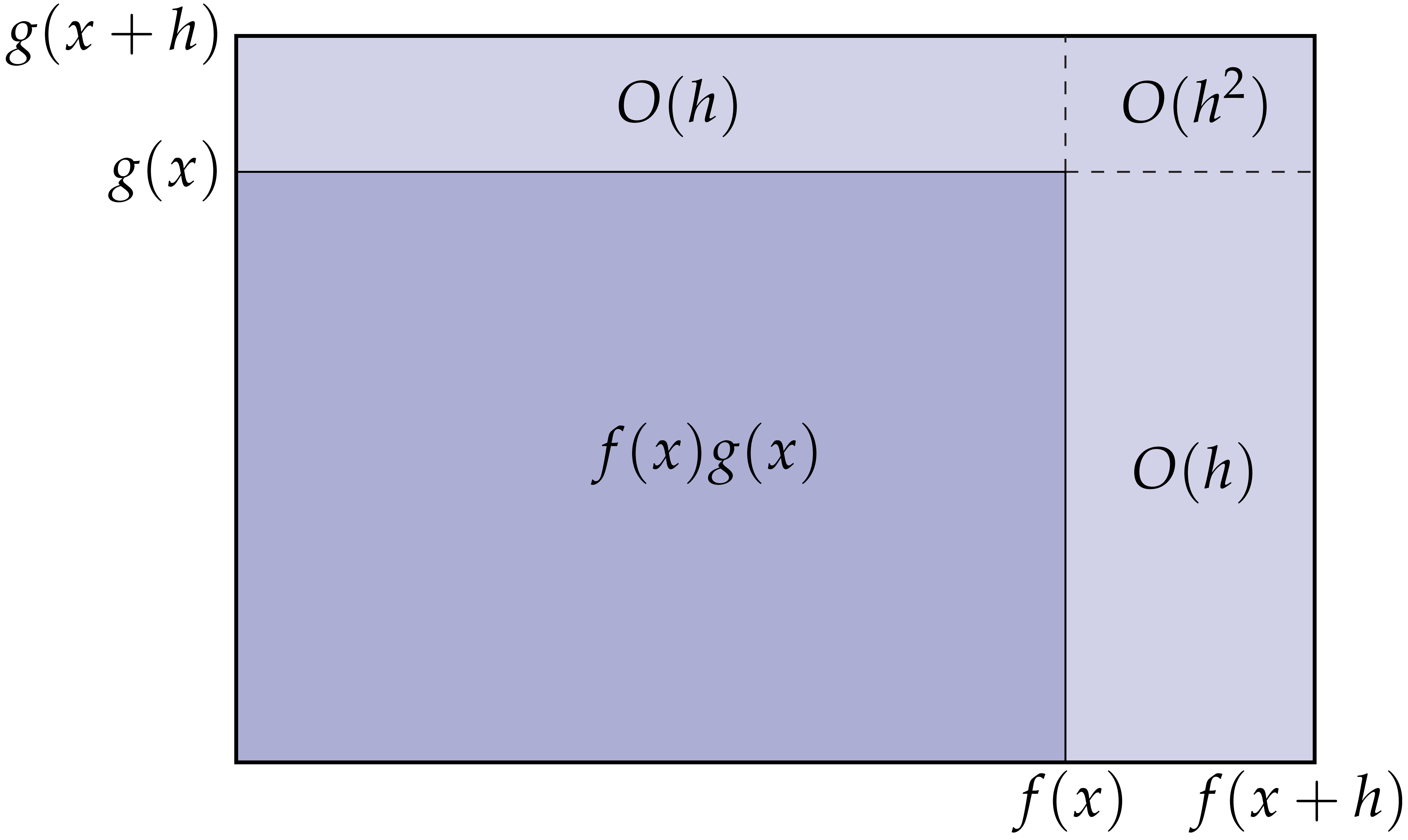


Product Rule—Derivative

For any differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$, $(fg)' = f'g + fg'$.

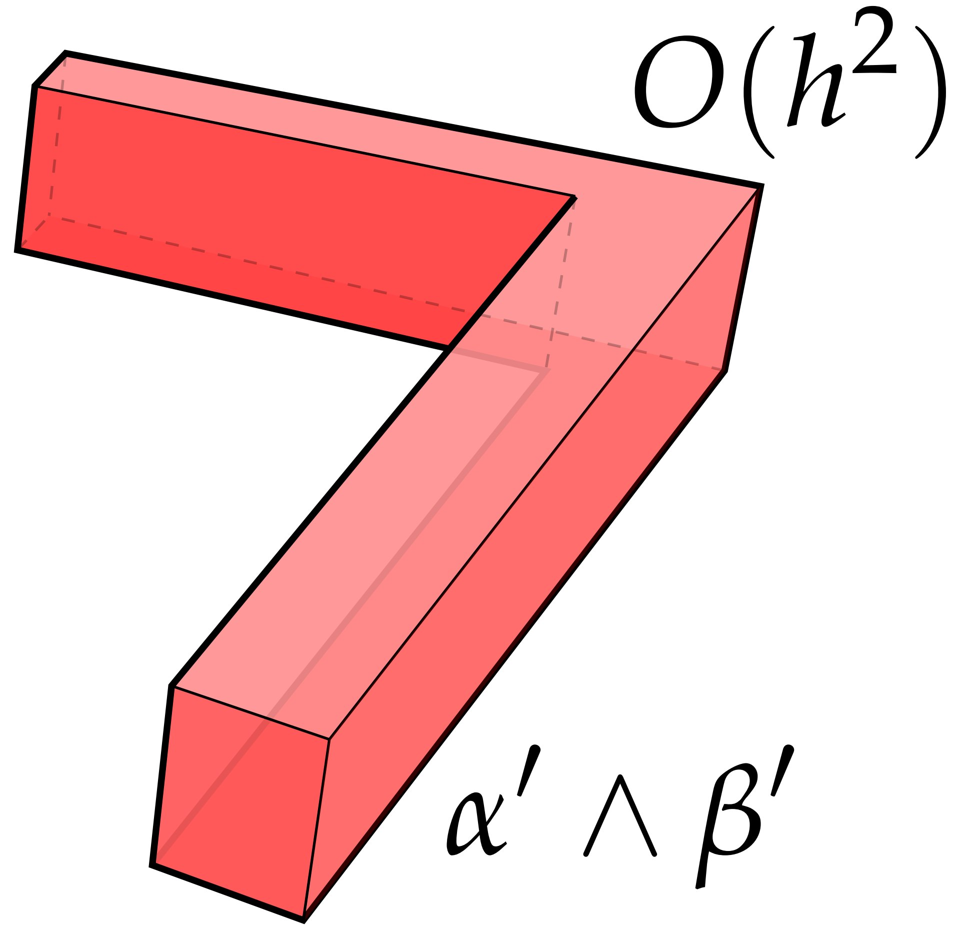
Q: Why? What's the *geometric* interpretation?

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$



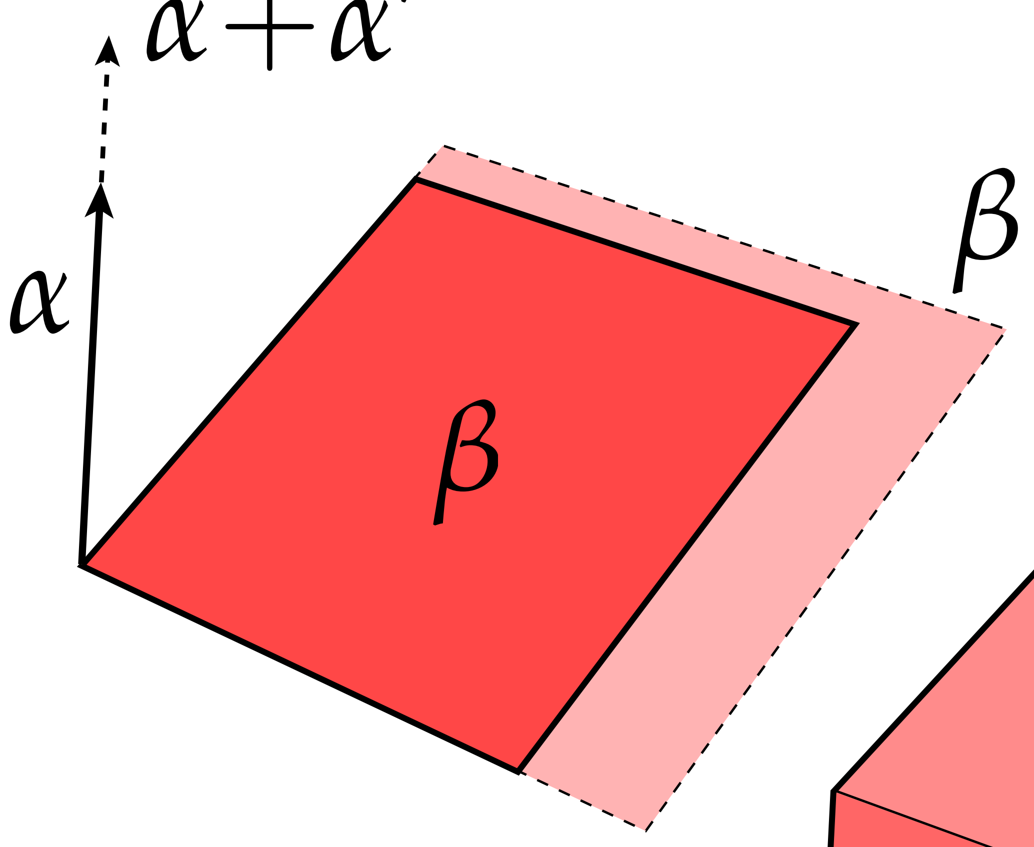
Product Rule—Exterior Derivative

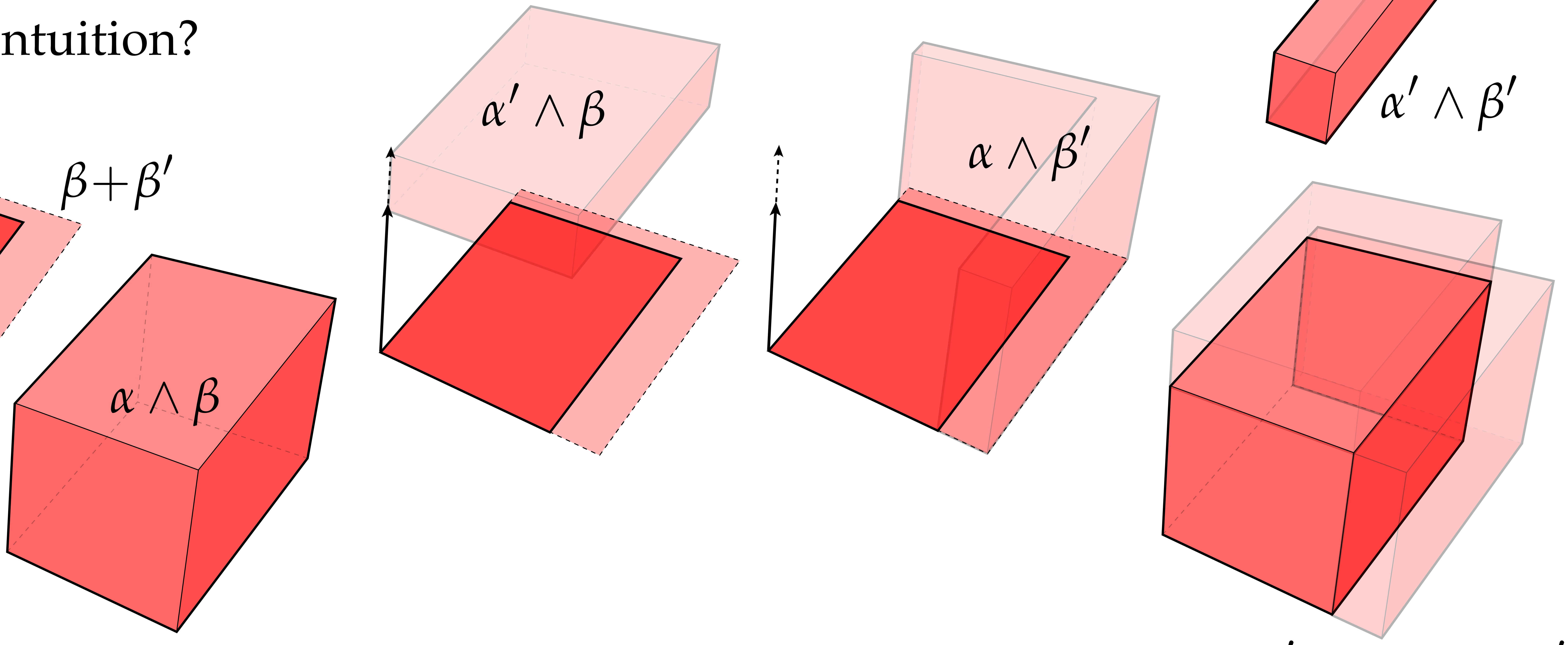
Let α be a k-form and let β be an ℓ-form. Then



$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

Q: Geometric intuition?





(Does this cartoon depict the *exterior derivative*? Or a *directional derivative*?)

Product Rule—"Recursive Evaluation"

Example. Let $\alpha := u\,dx$, $\beta := v\,dy$, and $\gamma := w\,dz$ be differential 1-forms on $\mathbb{R}^n$, where $u, v, w : \mathbb{R}^n \to \mathbb{R}$ are 0-forms, *i.e.*, scalar functions. Also, let $\omega := \alpha \wedge \beta$. Then

$$d(\omega \wedge \gamma) = (d\omega) \wedge \gamma + (-1)^2 \omega \wedge (d\gamma).$$

We can then "recursively" evaluate derivatives that appear on the right-hand side:

$$\begin{aligned}
d\omega &= (d\alpha) \wedge \beta + (-1)^1 \alpha \wedge (d\beta), \\
d\alpha &= (du) \wedge dx + (-1)^0 u \cancelto{0}{(ddx)}, \\
d\beta &= (dv) \wedge dy + (-1)^0 v \cancelto{0}{(ddy)}, \\
d\gamma &= (dw) \wedge dz + (-1)^0 w \cancelto{0}{(ddz)}.
\end{aligned}$$

Key idea: The "base case" is the 0-forms, *i.e.*, computing the final result boils down to taking the differential of ordinary scalar functions.

Exterior Derivative—Examples

Example. Let $\phi(x,y) := \frac{1}{2}e^{-(x^2+y^2)}$. Then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$

$$= -2\phi(xdx + ydy)$$

Example. Let $\alpha(x,y) = xdx + ydy$. Then $d\alpha =$

$$(\tfrac{\partial x}{\partial x}dx + \tfrac{\partial x}{\partial y}dy) \wedge dx + (\tfrac{\partial y}{\partial x}dx + \tfrac{\partial y}{\partial y}dy) \wedge dy$$

$$= dx \wedge dx + dy \wedge dy \;= 0 + 0 \;= 0.$$

Example. Again let $\alpha(x,y) = xdx + ydy$. Then $d \star \alpha = d(x \star dx + y \star dy)$

$$= d(xdy - ydx)$$

$$= dx \wedge dy - dy \wedge dx$$

$$= 2dx \wedge dy.$$

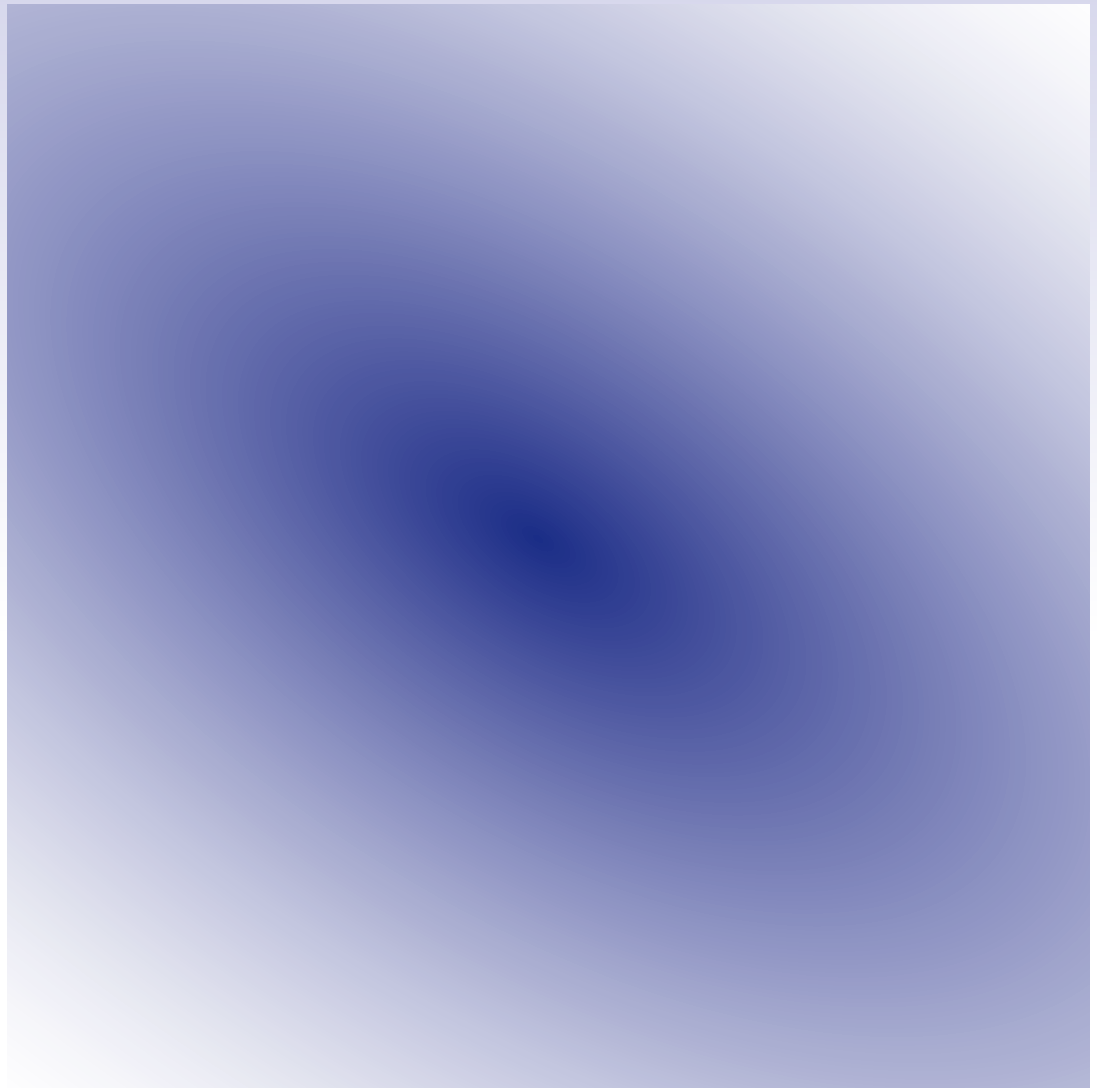
Exterior Derivative—Exactness

Exterior Derivative

Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Review: Curl of Gradient



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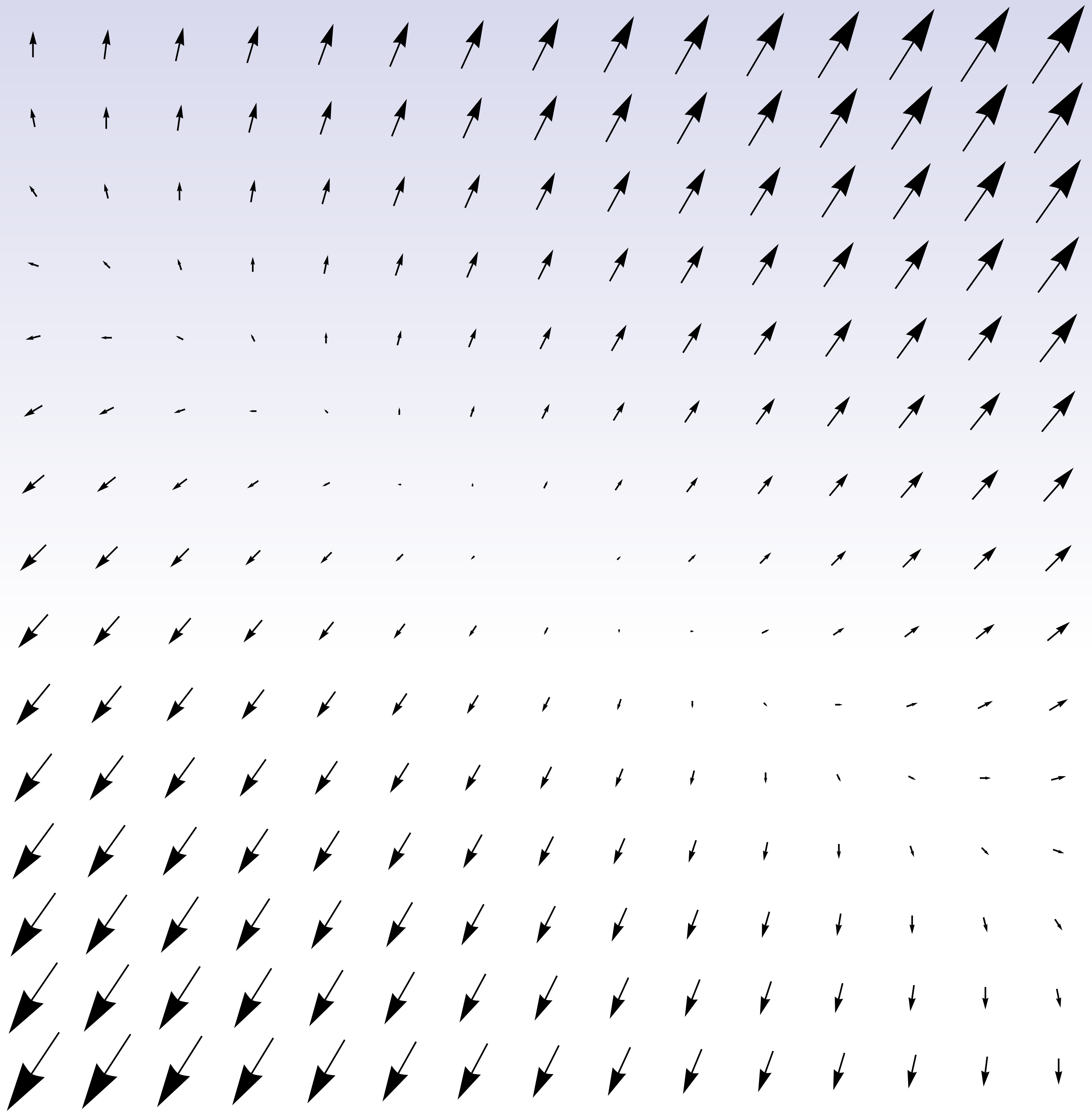
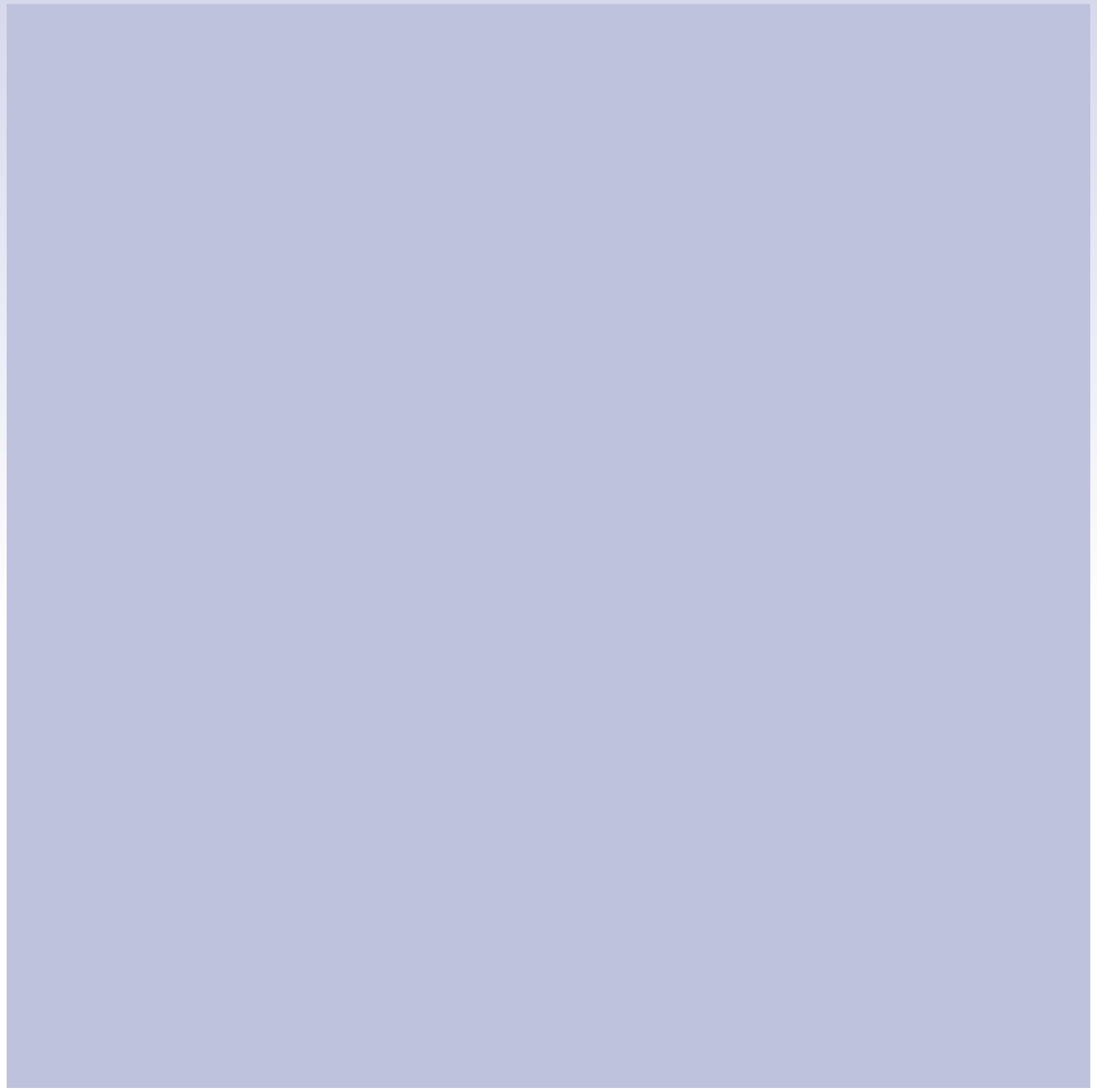


grad ϕ



curl $\circ$ grad ϕ

Key idea: exterior derivative should capture a similar idea.

What Happens if $d \circ d = 0$?

Q: Consider a 1-form $\alpha = udx + vdy + wdz$, where the coefficients u, v, w are each scalar functions $\mathbb{R}^3 \to \mathbb{R}$. What is the exterior derivative $d\alpha$ in coordinates x, y, z?

A: $d\alpha = d(udx + vdy + wdz) = du \wedge dx + u\cancelto{0}{ddx} + dv \wedge dy + v\cancelto{0}{ddy} + dw \wedge dz + w\cancelto{0}{ddz}$

$$= \begin{array}{l} (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz) \wedge dx + \\ (\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz) \wedge dy + \\ (\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz) \wedge dz \end{array} = \begin{array}{l} (\cancelto{0}{\frac{\partial u}{\partial x}dx \wedge dx} + \frac{\partial u}{\partial y}dy \wedge dx + \frac{\partial u}{\partial z}dz \wedge dx) + \\ (\frac{\partial v}{\partial x}dx \wedge dy + \cancelto{0}{\frac{\partial v}{\partial y}dy \wedge dy} + \frac{\partial v}{\partial z}dz \wedge dy) + \\ (\frac{\partial w}{\partial x}dx \wedge dz + \frac{\partial w}{\partial y}dy \wedge dz + \cancelto{0}{\frac{\partial w}{\partial z}dz \wedge dz}) \end{array}$$

$$= -\frac{\partial u}{\partial y}dx \wedge dy + \frac{\partial u}{\partial z}dz \wedge dx + \frac{\partial v}{\partial x}dx \wedge dy - \frac{\partial v}{\partial z}dy \wedge dz - \frac{\partial w}{\partial x}dz \wedge dx + \frac{\partial w}{\partial y}dy \wedge dz$$

$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})dy \wedge dz + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz \wedge dx + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx \wedge dy.$$

Q: Does this operation remind you of anything (*perhaps from vector calculus*)?

Exterior Derivative and Curl

Suppose we have a *vector field*

$$X := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

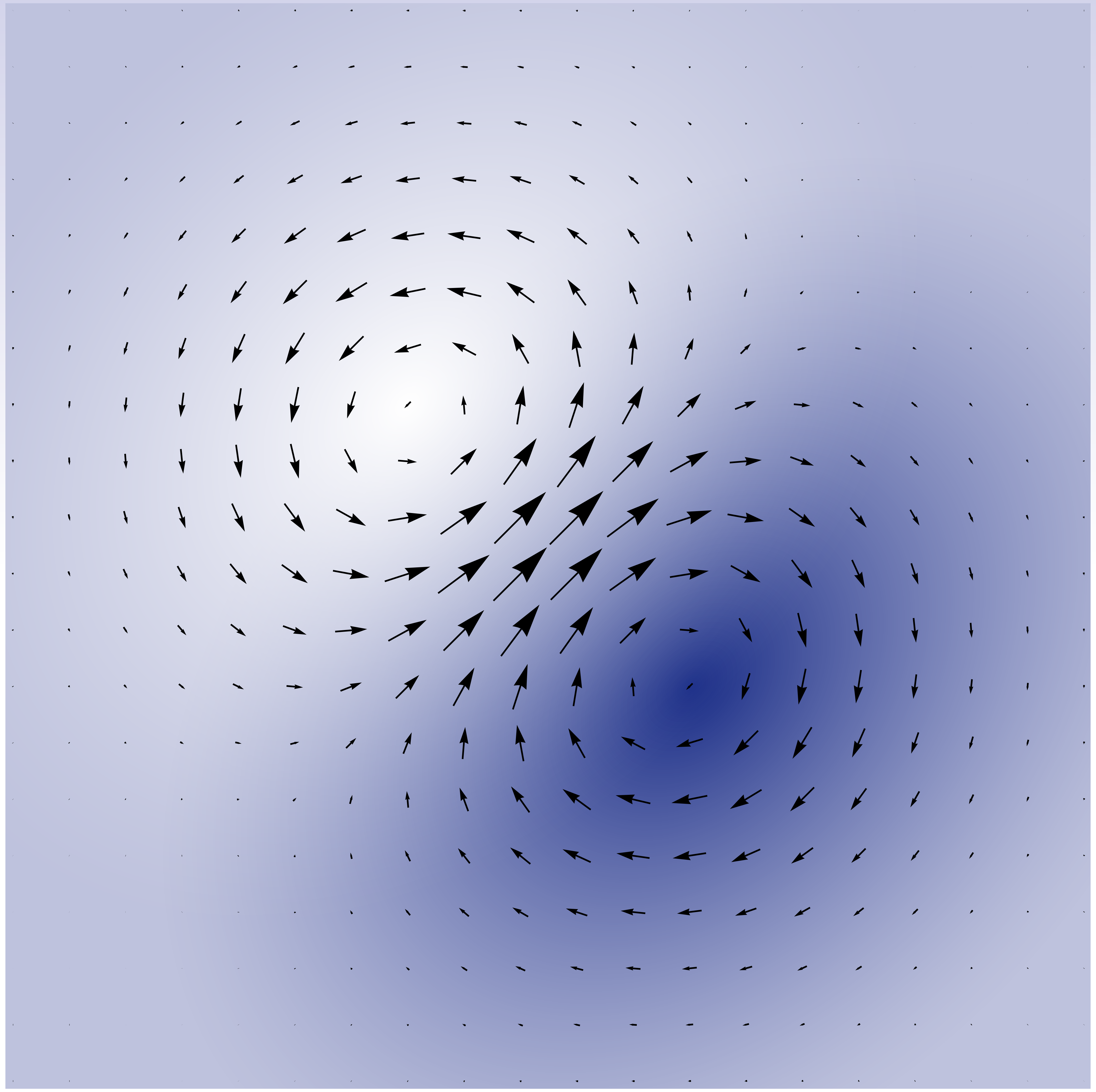
Its *curl* is then

$$\nabla \times X = \begin{array}{lcll} (\partial w/\partial y & - & \partial v/\partial z) & \frac{\partial}{\partial x} \quad + \\ (\partial u/\partial z & - & \partial w/\partial x) & \frac{\partial}{\partial y} \quad + \\ (\partial v/\partial x & - & \partial u/\partial y) & \frac{\partial}{\partial z} \end{array}$$

Looks an awful lot like…

$$d\alpha = \begin{array}{lcll} (\partial w/\partial y & - & \partial v/\partial z) & dy \wedge dz \quad + \\ (\partial u/\partial z & - & \partial w/\partial x) & dz \wedge dx \quad + \\ (\partial v/\partial x & - & \partial u/\partial y) & dx \wedge dy \end{array}$$

Especially if we then apply the *Hodge star*.



$$\nabla \times X \iff \star d\alpha$$

$$\nabla \times X = (\star d X^\flat)^\sharp$$

$d \circ d = 0$

ϕ $d\phi$ $dd\phi$

Intuition: in $\mathbb{R}^n$, first d behaves just like gradient; second d behaves just like curl.

Exterior Derivative in 3D (1-forms)

Q: How about $d \star \alpha$? (Still for $\alpha = udx + vdy + wdz$.)

A:
$$
\begin{aligned}
d \star \alpha &= d(\star(udx + vdy + wdz)) \\
&= d(udy \wedge dz + vdz \wedge dx + wdx \wedge dy) \\
&= du \wedge dy \wedge dz + dv \wedge dz \wedge dx + dw \wedge dx \wedge dy \\
&= \tfrac{\partial u}{\partial x} dx \wedge dy \wedge dz + \tfrac{\partial v}{\partial y} dy \wedge dz \wedge dx + \tfrac{\partial w}{\partial z} dz \wedge dx \wedge dy \\
&= (\tfrac{\partial u}{\partial x} + \tfrac{\partial v}{\partial y} + \tfrac{\partial w}{\partial z}) dx \wedge dy \wedge dz
\end{aligned}
$$

Q: Does this operation remind you of anything (*perhaps from vector calculus*)?

Exterior Derivative and Divergence

Suppose we have a *vector field*

$$X := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

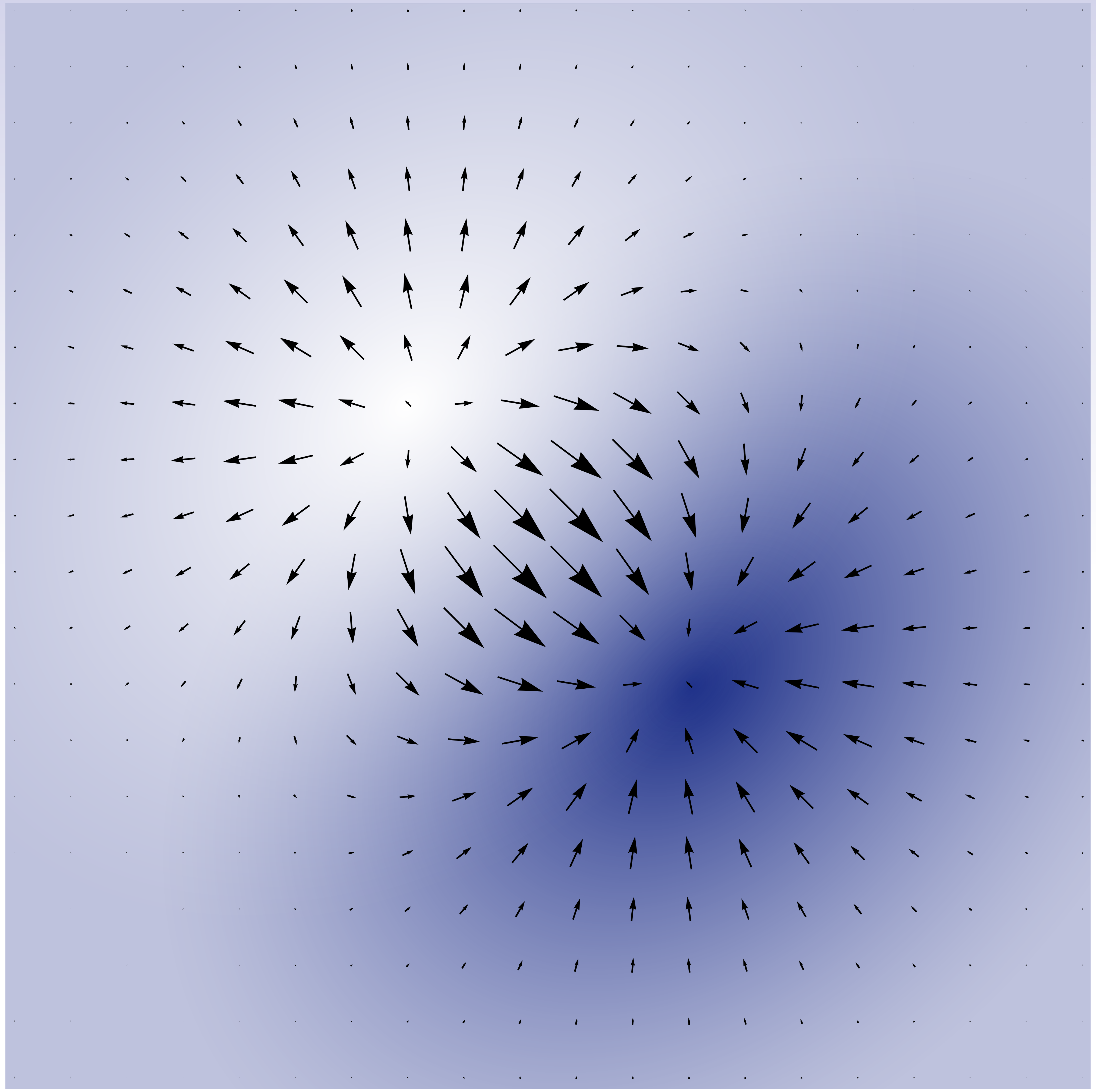
Its *divergence* is then

$$\nabla \cdot X = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial u} + \frac{\partial w}{\partial z}$$

Looks an awful lot like...

$$d \star \alpha = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx \wedge dy \wedge dz$$

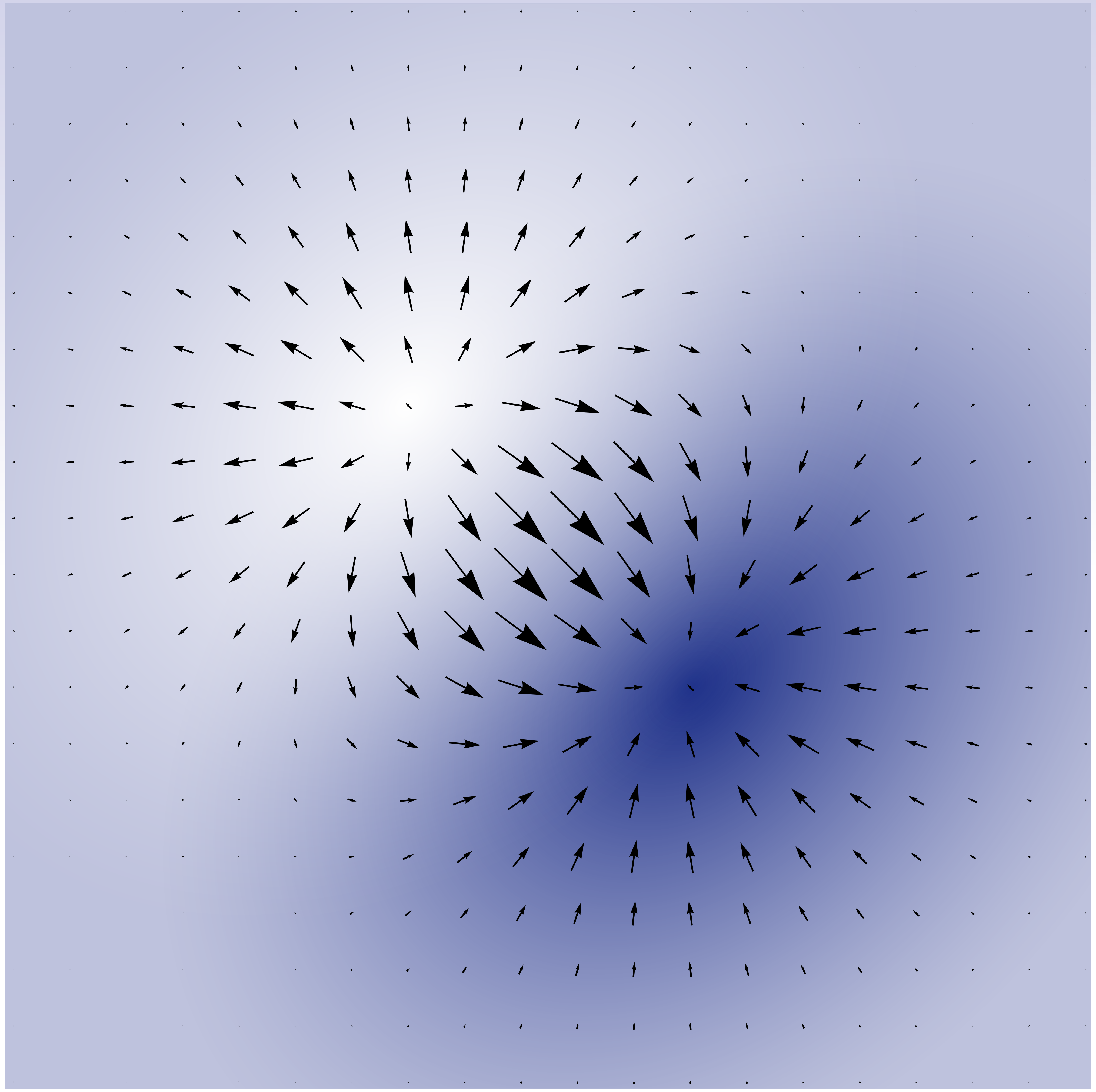
Especially if we then apply the *Hodge star.*



$$\nabla \cdot X \quad \Longleftrightarrow \quad \star d \star \alpha$$

$$\nabla \cdot X = \star d \star X^\flat$$

Exterior Derivative - Divergence

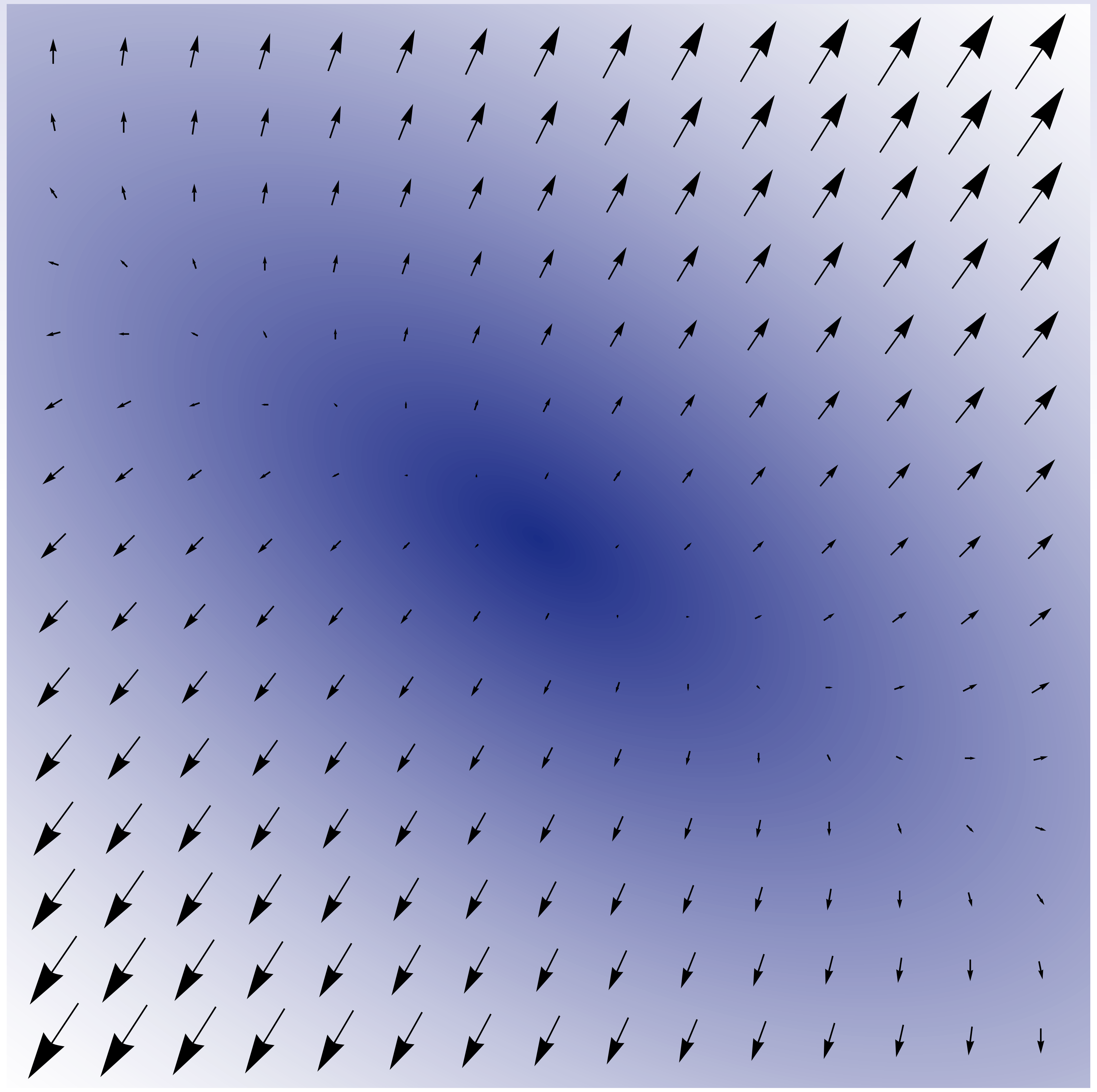


$$\nabla \cdot X = \star d(\star X^\flat)$$

(*codifferential:* $\delta := \star d \star$)

Exterior vs. Vector Derivatives — Summary

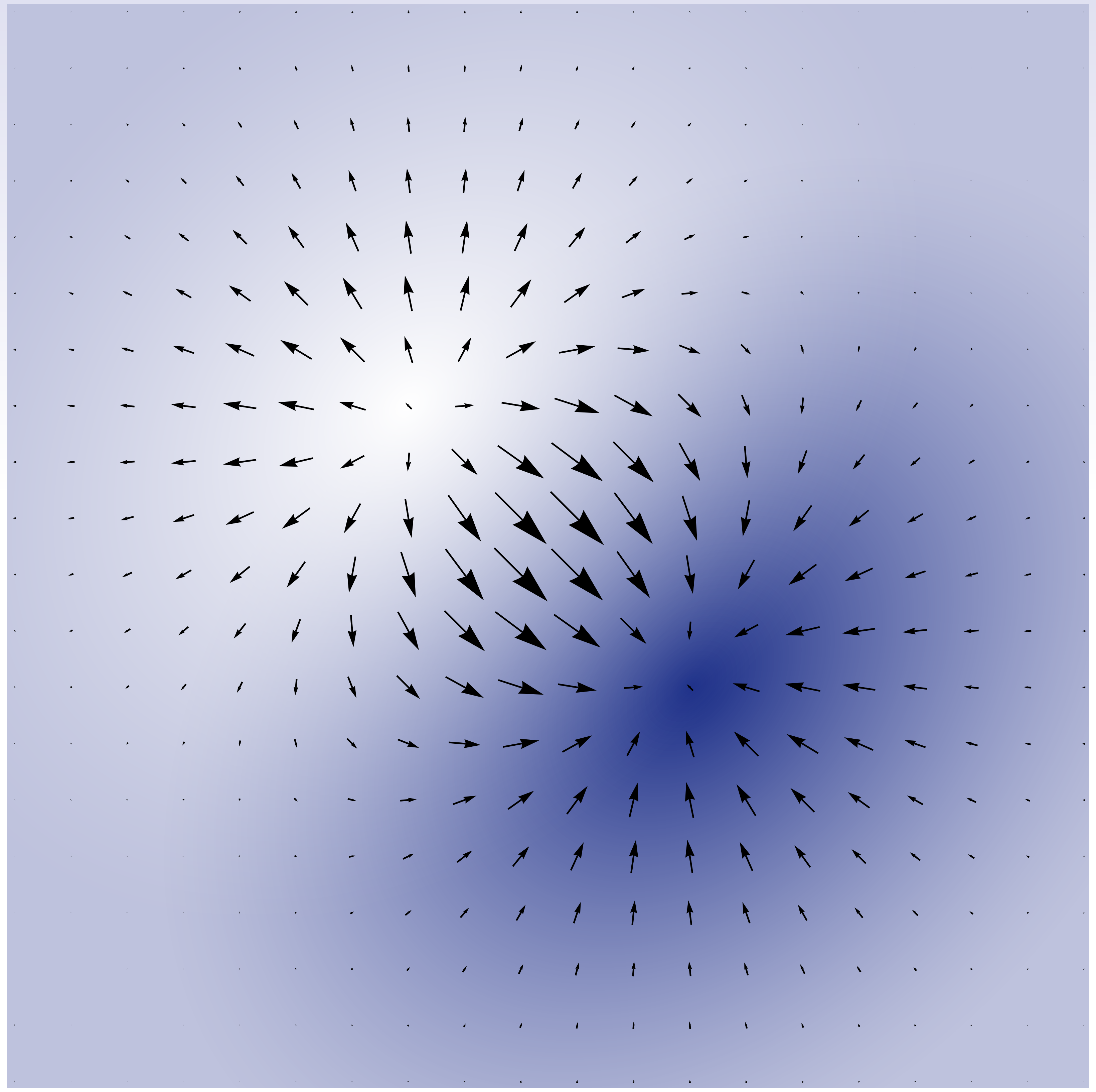
ϕ



grad ϕ

$(d\phi)^\sharp$

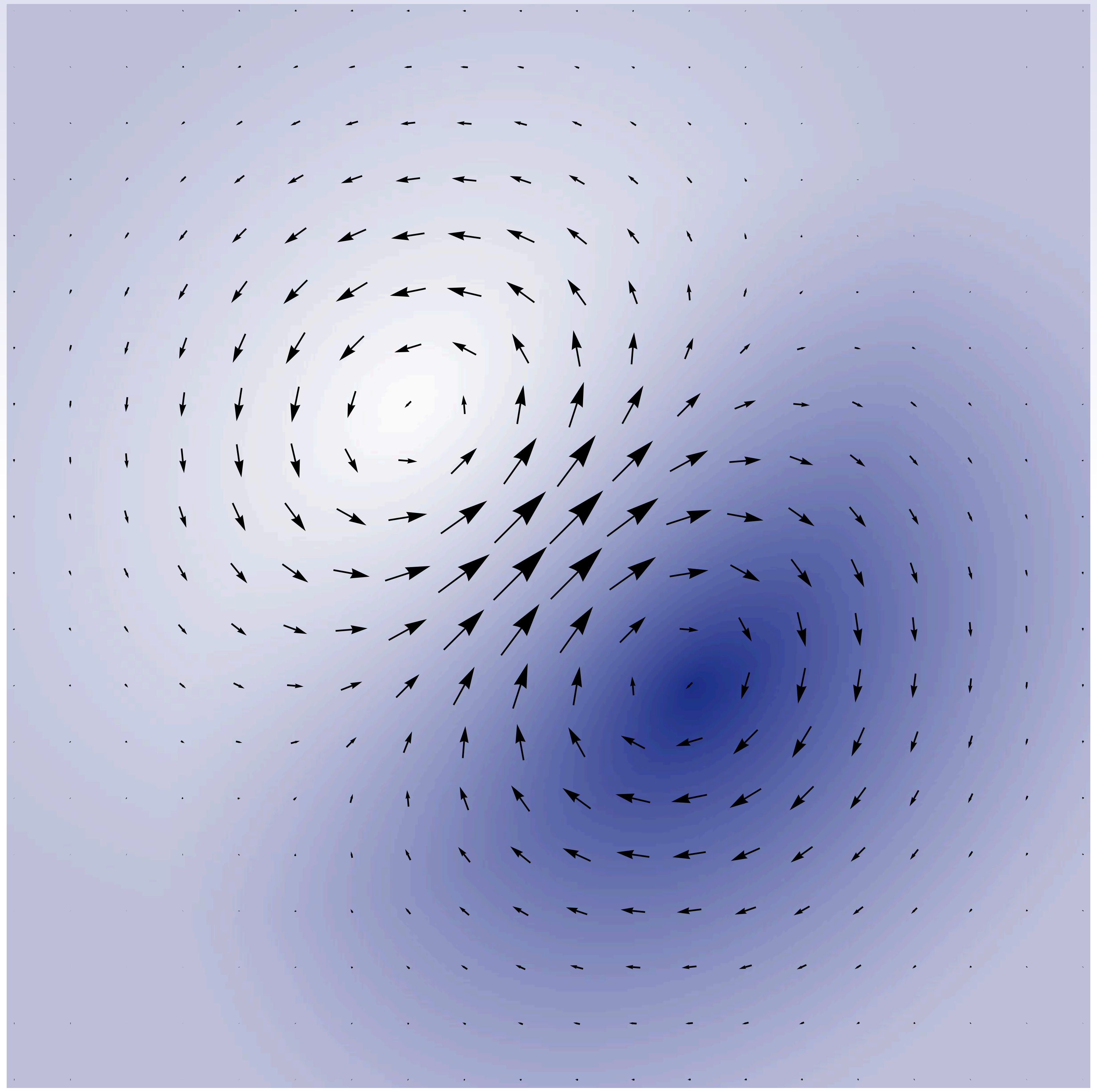
X



div X

$\star d(\star X^\flat)$

Y



curl Y

$(\star(dX^\flat))^\sharp$

Exterior Derivative—Summary

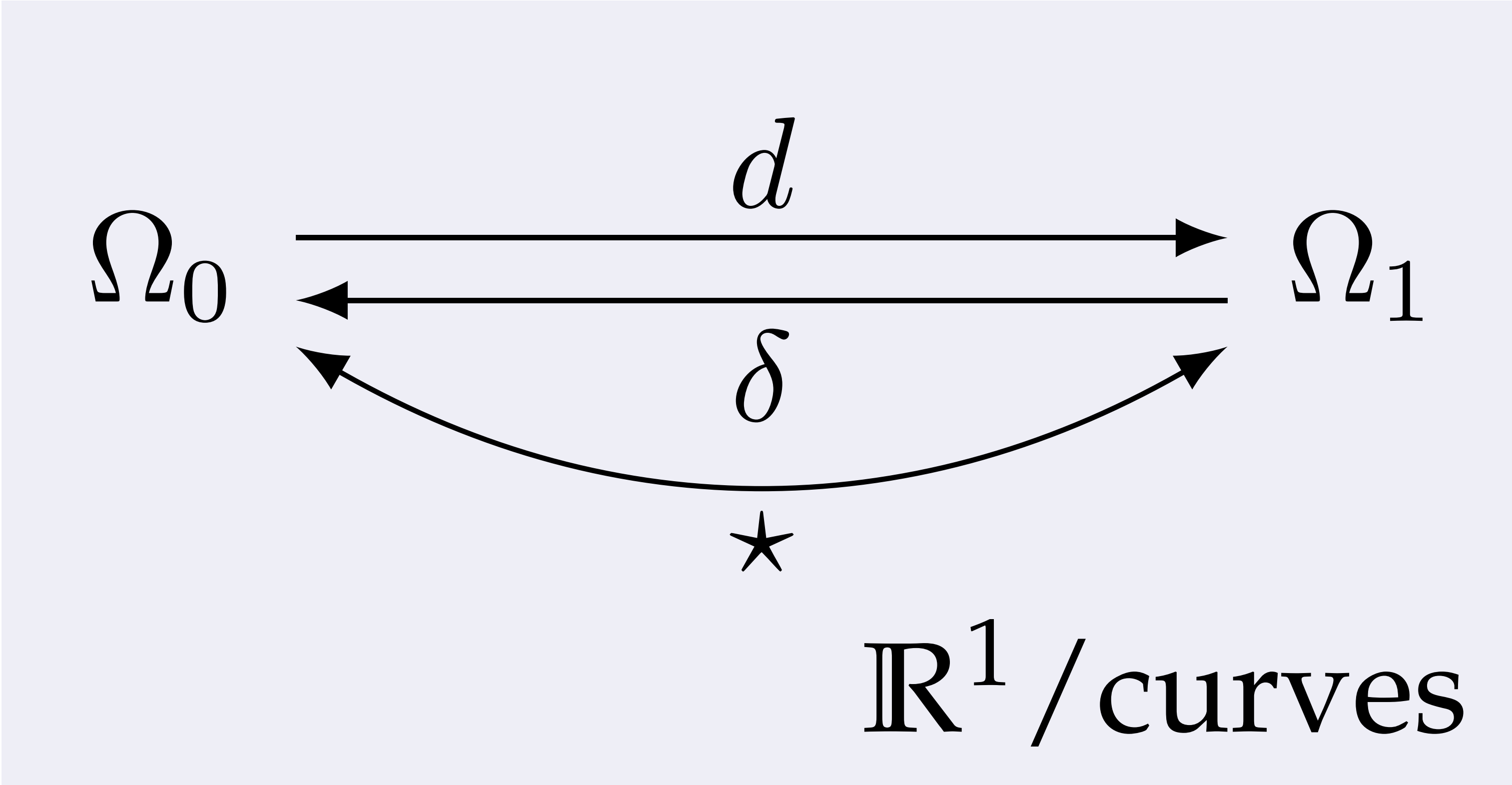
Exterior Derivative

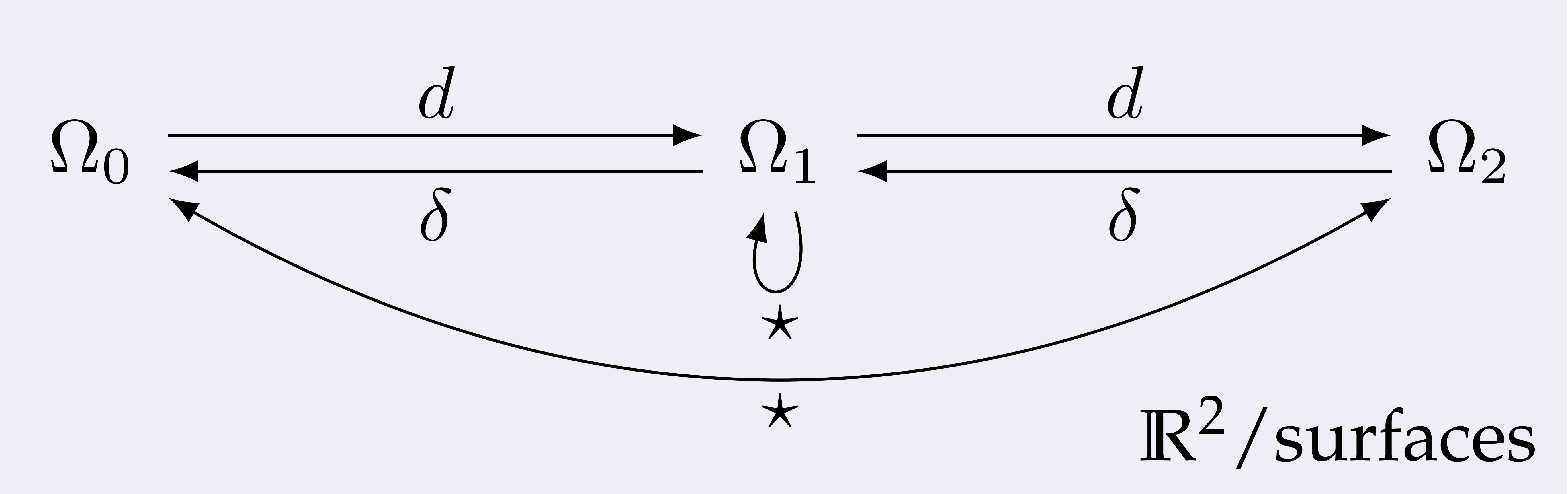
Unique *linear* map $d : \Omega^k \to \Omega^{k+1}$ such that

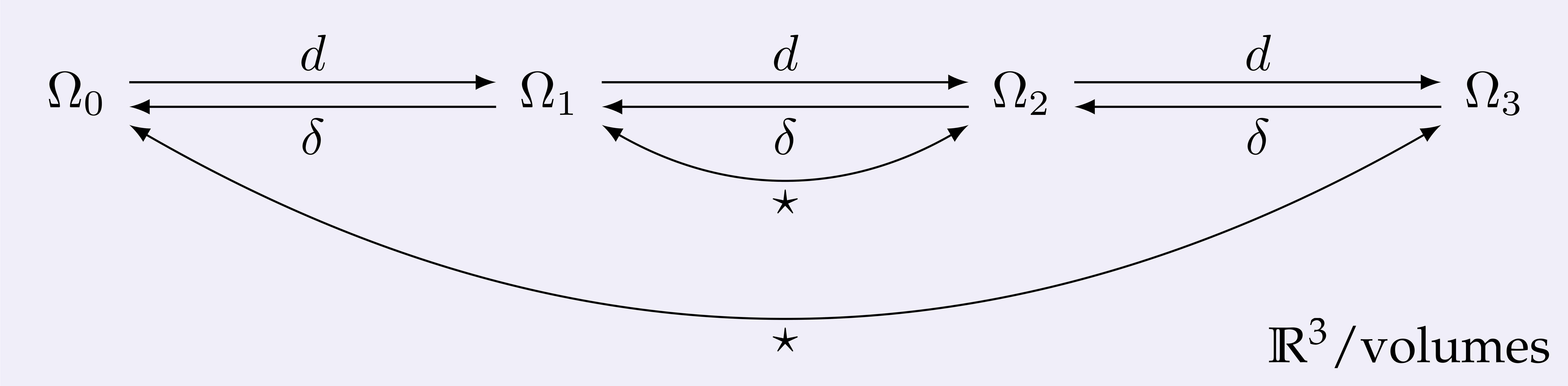
differential	for $k = 0$, $d\phi(X) = D_X\phi$
product rule	$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
exactness	$d \circ d = 0$

Exterior Calculus—Diagram View

- Taking a step back, we can draw many of the operators seen so far as diagrams:







Ω_k—differential k-forms

Laplacian

- Can now compose operators to get other operators
- E.g., *Laplacian* from vector calculus:

$$\Delta := \text{div} \circ \text{grad}$$

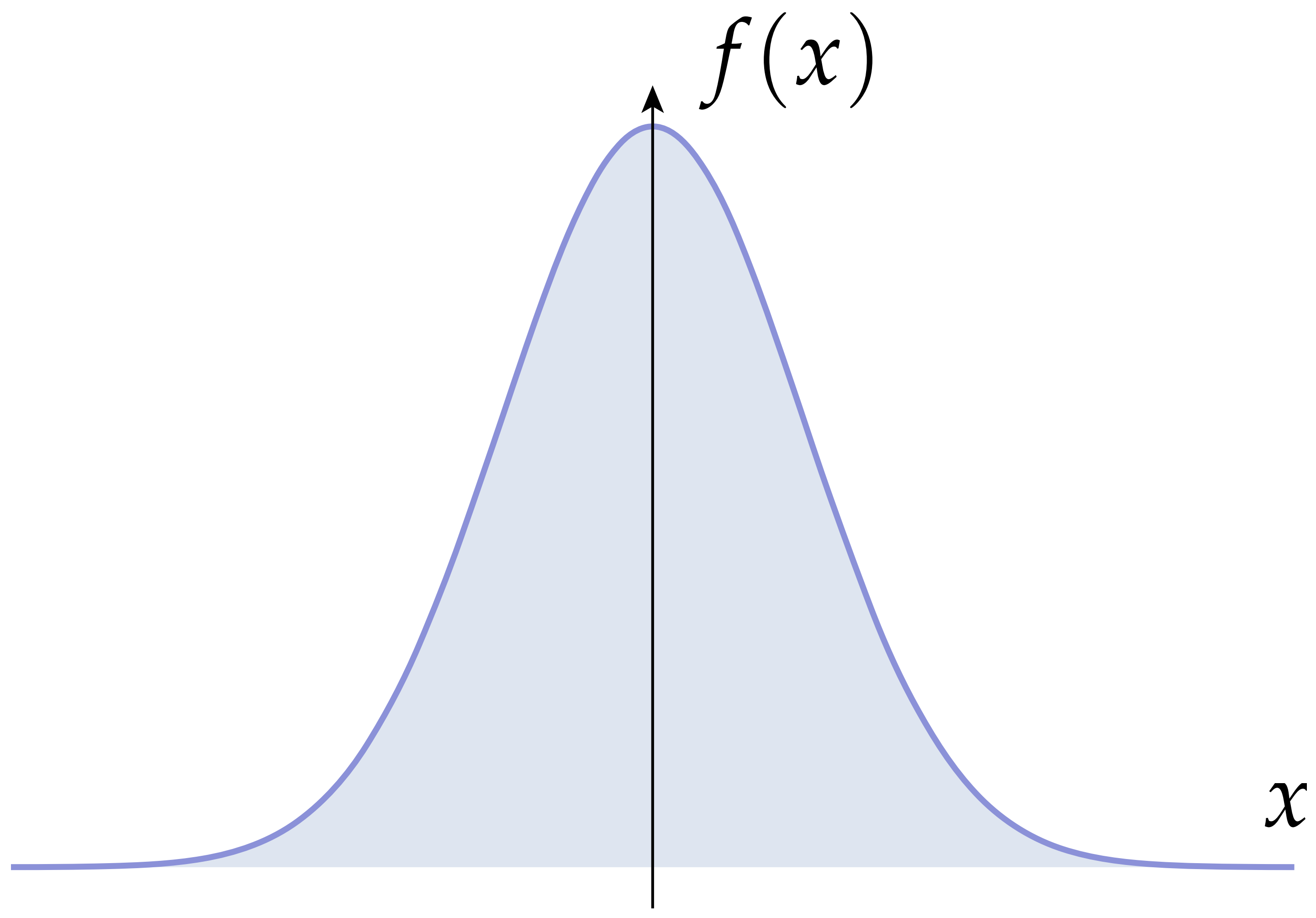
- Can express exact same operator via exterior calculus:

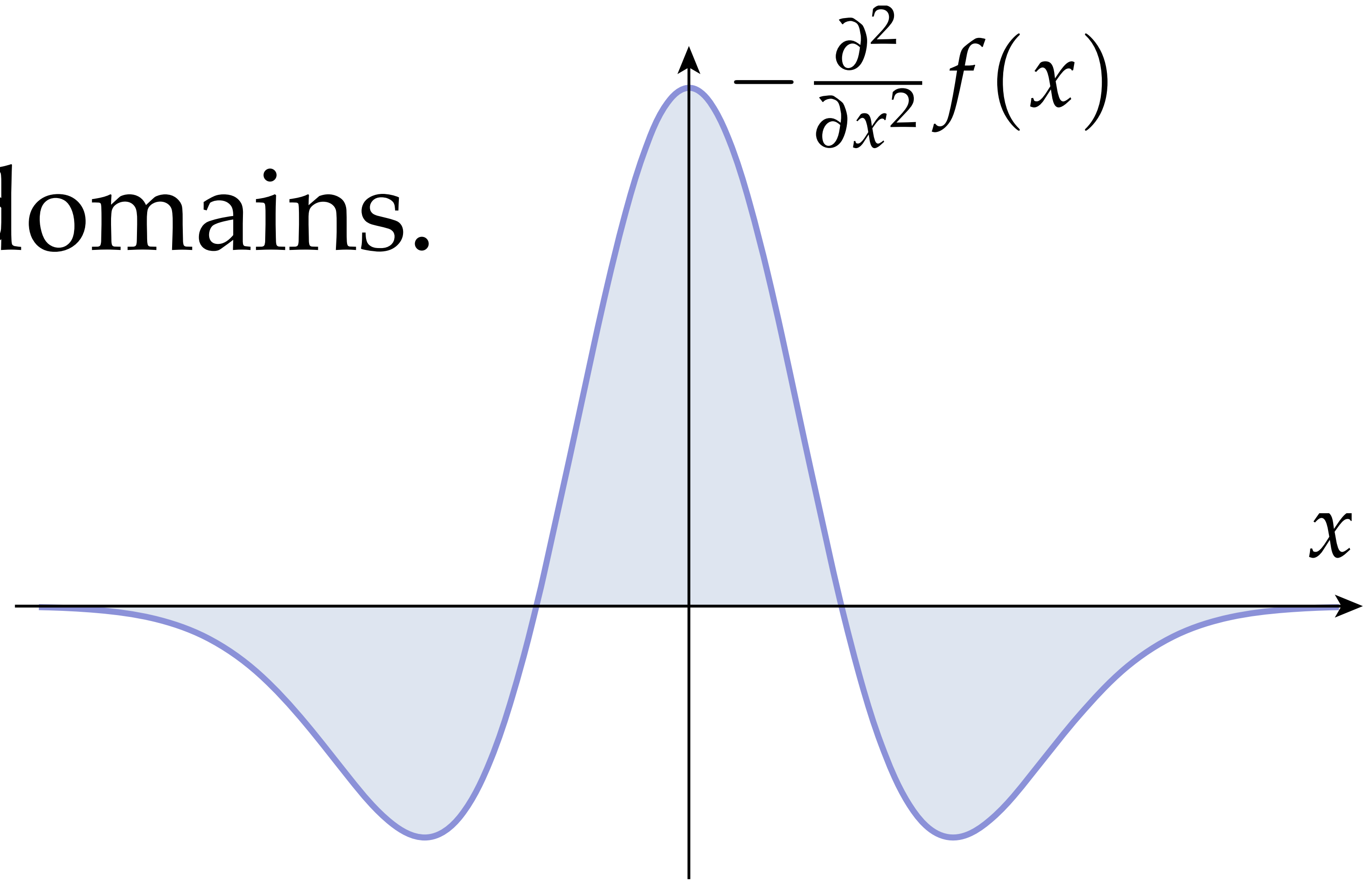
$$\Delta = \star d \star d$$

- …except that this expression easily generalizes to curved domains.
- Can also generalize to k-forms:

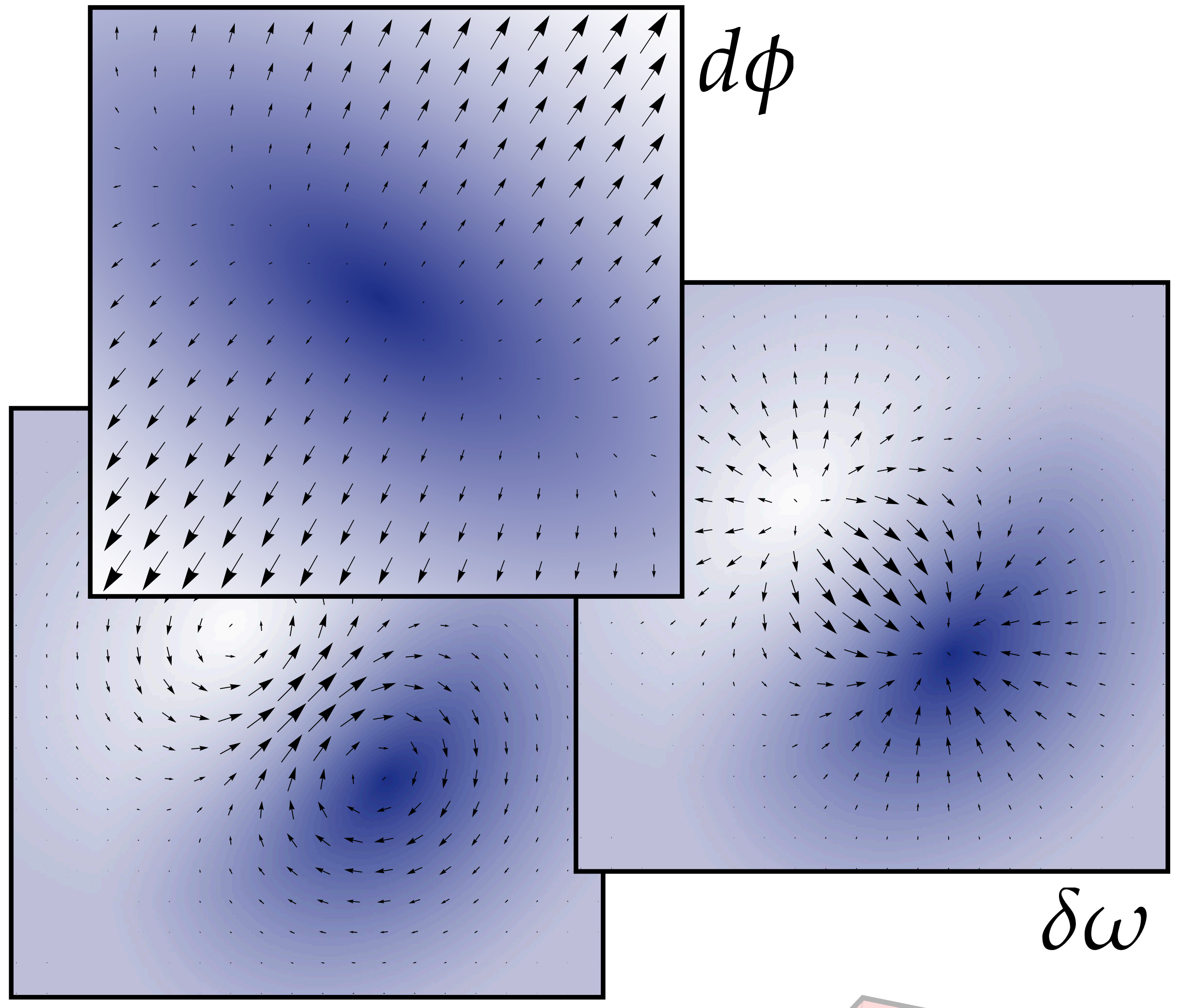
$$\Delta := \star d \star d + d \star d \star$$

- Will have **much** more to say about the Laplacian later on!



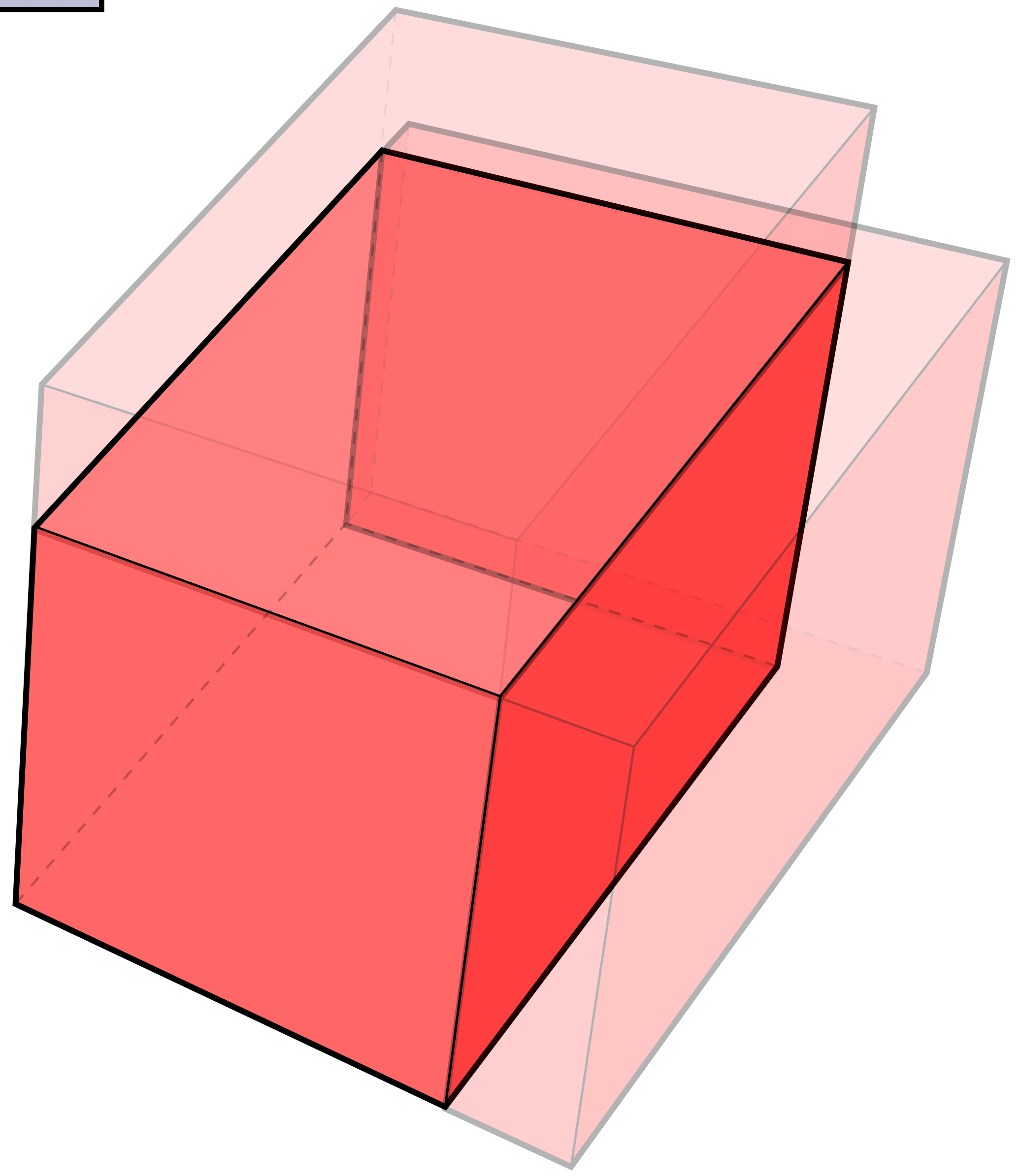


Exterior Derivative - Summary

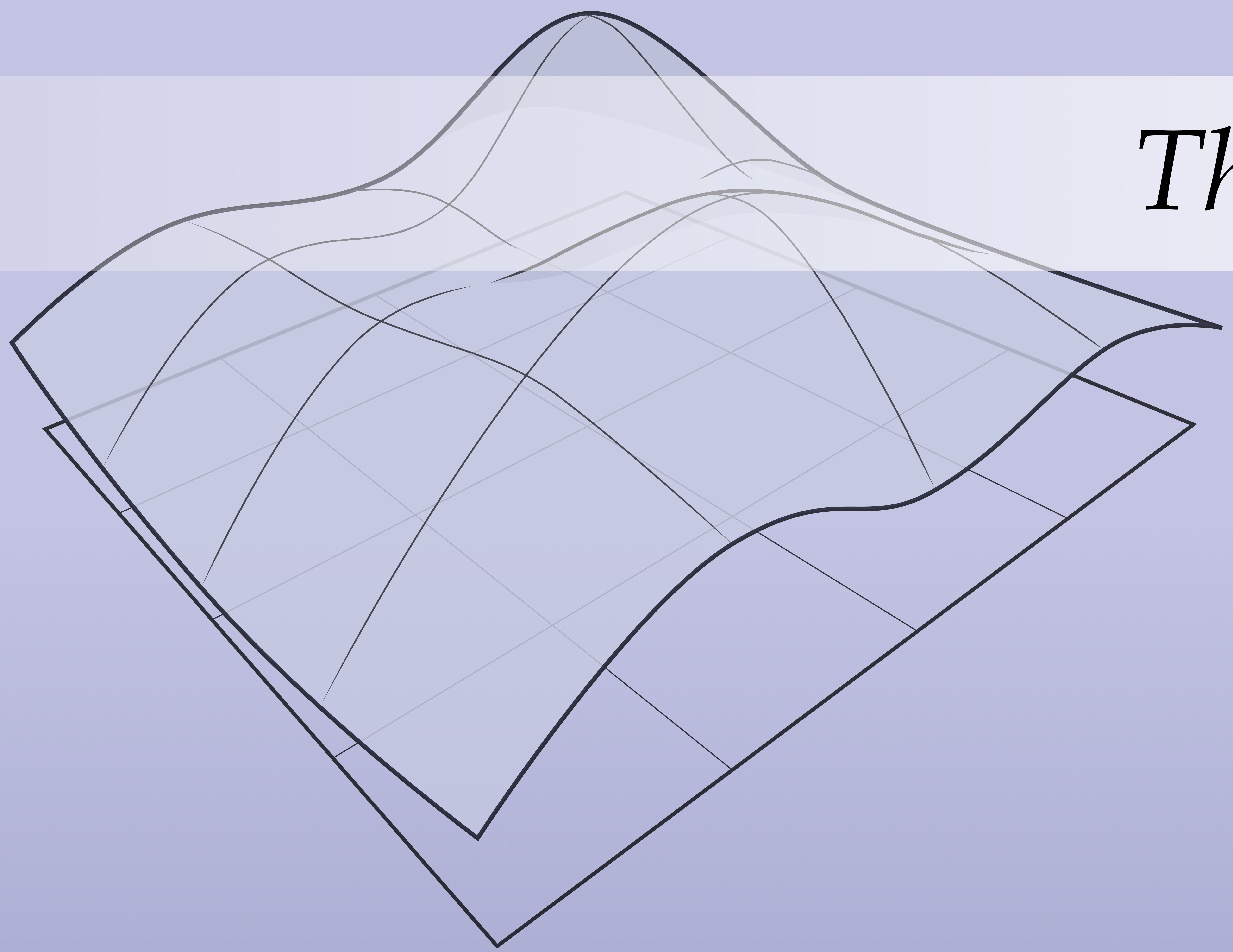


- Exterior derivative d used to differentiate k-forms
  - 0-form: "gradient"
  - 1-form: "curl"
  - 2-form: "divergence" (codifferential δ)
  - and more...
- Natural product rule
- d of d is zero
  - Analogy: curl of gradient
  - More general picture (soon!) via *Stokes' theorem*

$$\Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega_2$$



Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION