DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
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Vector Valued $k$-Forms

- Originally defined $k$-form as linear map from $k$ vectors to real numbers
  - To encode geometry, need functions that describe points in space
  - Will therefore generalize to vector-valued $k$-forms

**Definition.** A *vector-valued $k$-form* is a fully antisymmetric multi-linear map from $k$ vectors in a vector space $V$ to another vector space $U$.

- Have already seen many $\mathbb{R}$-valued $k$-forms on $\mathbb{R}^n$ ($V = \mathbb{R}^n$, $U = \mathbb{R}$)
- A $\mathbb{R}^3$-valued 2-form on $\mathbb{R}^2$ would instead be a multilinear, fully-antisymmetric map from a pair of vectors $u,v$ in $\mathbb{R}^2$ to a single vector in $\mathbb{R}^3$:

  \[
  \alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \quad \alpha(u,v) = -\alpha(v,u)
  \]

  \[
  \alpha(au + bv, w) = a\alpha(u, w) + b\alpha(v, w), \quad \forall u, v, w \in \mathbb{R}^2, a, b \in \mathbb{R}
  \]

**Q:** What kind of object is a $\mathbb{R}^2$-valued 0-form on $\mathbb{R}^2$?
Vector-Valued $k$-forms — Example

Consider for instance the following $\mathbb{R}^3$-valued 1-form on $\mathbb{R}^2$:

$$\alpha := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^1 + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} e^2$$

Q: What do we get if we evaluate this 1-form on the vector

$$u := e_1 - e_2$$

A: Evaluation is not much different from real-valued forms:

$$\alpha(u) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^1(e_1 - e_2) + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} e^2(e_1 - e_2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Key idea: most operations just look like scalar case, applied to each component
• Most important change is how we evaluate wedge product for vector-valued forms.

• Consider for instance a pair of \( \mathbb{R}^3 \)-valued 1-forms:

\[
\alpha, \beta : V \to \mathbb{R}^3
\]

• To evaluate their wedge product on a pair of vectors \( u,v \) we would normally write:

\[
(\alpha \wedge \beta)(u,v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)
\]

• If \( \alpha \) and \( \beta \) were real-valued, then \( \alpha(u), \beta(v), \alpha(v), \beta(u) \), would just be real numbers, so we could just multiply the two pairs and take their difference.

• But what does it mean to take the “product” of two vectors from \( \mathbb{R}^3 \)?

• Many possibilities (e.g., dot product), but if we want result to be an \( \mathbb{R}^3 \)-valued 2-form, the product we choose must produce another vector in \( \mathbb{R}^3 \)!
Wedge Product of $\mathbb{R}^3$-Valued k-Forms

• When working with 3D geometry:
  – $k$-forms are $\mathbb{R}^3$-valued
  – use cross product to multiply vectors in $\mathbb{R}^3$

$$\alpha, \beta : V \to \mathbb{R}^3$$

$$\alpha \wedge \beta : V \times V \to \mathbb{R}^3$$

$$(\alpha \wedge \beta)(u, v) := \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u)$$
**R³-valued 1-forms: Antisymmetry & Symmetry**

With real-valued forms, we observed antisymmetry in both the wedge product of 1-forms as well as the application of the 2-form to a pair of vectors, i.e.,

\[(\alpha \wedge \beta)(u, v) = -(\alpha \wedge \beta)(v, u)\]
\[(\beta \wedge \alpha)(u, v) = -(\alpha \wedge \beta)(u, v)\]

What happens w/ \(\mathbb{R}³\)-valued 1-forms? Since cross product is antisymmetric, we get

\[
(\alpha \wedge \beta)(v, u) = \alpha(v) \times \beta(u) - \alpha(u) \times \beta(v)
\]
\[
= -(\alpha(u) \times \beta(v) - \alpha(v) \times \beta(u))
\]

\[
\Rightarrow (\alpha \wedge \beta)(u, v) = -(\alpha \wedge \beta)(v, u)
\]

(same as with real-valued forms)

\[
(\beta \wedge \alpha)(u, v) = \beta(u) \times \alpha(v) - \beta(v) \times \alpha(u)
\]
\[
= \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u)
\]
\[
= (\alpha \wedge \beta)(u, v)
\]

\[
\Rightarrow \alpha \wedge \beta = \beta \wedge \alpha
\]

(no sign change)

**Key idea:** “antisymmetries cancel”
\( \mathbb{R}^3 \)-valued 1-forms: Self-Wedge

Likewise, we saw that wedging a real-valued 1-form with itself yields zero:

\[ \alpha \wedge \alpha = 0 \]

Q: What was the geometric interpretation?

A: Parallelogram made from two copies of the same vector has zero area!

No longer true with \((\mathbb{R}^3, \times)\)-valued 1-forms:

\[ (\alpha \wedge \alpha)(u, v) = \alpha(u) \times \alpha(v) - \alpha(v) \times \alpha(u) = 2\alpha(u) \times \alpha(v) \neq 0 \]

Q: Geometric meaning?

A: Vector with (twice) area of “stretched out” parallelogram.
Vector-Valued Differential $k$-Forms

• Just as we distinguished between a $k$-form (value at a single point) and a differential $k$-form (value at each point), will say that a vector-valued differential $k$-form is a vector-valued $k$-form at each point.

• Just like any differential form, a vector-valued differential $k$-form gets evaluated on $k$ vector fields $X_1, \ldots, X_k$.

• Example: an $\mathbb{R}^3$-valued differential 1-form on $\mathbb{R}^2$:

$\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dx + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} dy$

Q: What does $\alpha$ do to a given vector field $U$ in the plane?

A: It turns it into a 3D vector field that “sticks out” of the plane.
Unlike the wedge product, not much changes with the exterior derivative. For instance, if we have an $\mathbb{R}^n$-valued $k$-form we can simply imagine we have $n$ real-valued $k$-forms and differentiate as usual.

**Example.**

Consider an $\mathbb{R}^2$-valued differential 0-form $\phi_{(x,y)} := \begin{bmatrix} x^2 \\ xy \end{bmatrix}$

Then $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \begin{bmatrix} 2x \\ y \end{bmatrix} dx + \begin{bmatrix} 0 \\ x \end{bmatrix} dy$

**Example.**

Consider an $\mathbb{R}^2$-valued differential 1-form $\alpha_{(x,y)} := \begin{bmatrix} x^2 \\ xy \end{bmatrix} dx + \begin{bmatrix} xy \\ y^2 \end{bmatrix} dy$

Then $d\alpha = \left( \begin{bmatrix} 2x \\ y \end{bmatrix} dx + \begin{bmatrix} 0 \\ x \end{bmatrix} dy \right) \wedge dx + \left( \begin{bmatrix} y \\ 0 \end{bmatrix} dx + \begin{bmatrix} x \\ 2y \end{bmatrix} dy \right) \wedge dy = \begin{bmatrix} y \\ -x \end{bmatrix} dx \wedge dy$