Written Assignment 5:
Geodesic Distance

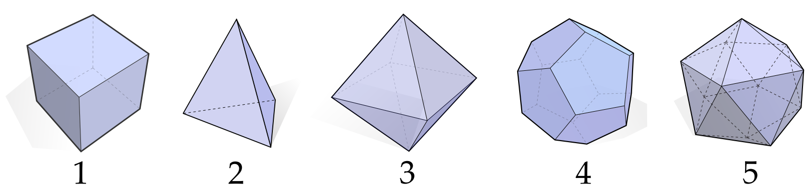
CMU 15-458/858

Submission Instructions. Please submit your solutions to the exercises (whether handwritten, LaTeX, etc.) as a **single PDF file** to Gradescope. Scanned images/photographs can be converted to a PDF using applications like *Preview* (on Mac) or a variety of free websites (*e.g.*, `http://imagetopdf.com`). Your graded submission will (hopefully!) be returned to you at least one day before the due date of the next written assignment.

Grading. Please clearly show your work. Partial credit **will** be awarded for ideas toward the solution, so please submit your thoughts on an exercise even if you cannot find a full solution. **Note that you are required to complete all the problems!**

If you don't know where to get started with some of these exercises, just ask! A great way to do this is to leave comments on the course webpage under this assignment; this way everyone can benefit from your questions. We are glad to provide further hints, suggestions, and guidance either here on the website, via email, or in person. Office hours are still TBD, but let us know if you'd like to arrange an individual meeting.

Late Days. Note that you have 5 no-penalty late days for the entire course, where a "day" runs from 6:00:00 PM Eastern to 5:59:59 PM Eastern the next day. No late submissions are allowed once all late days are exhausted. If you wish to claim one or more of your five late days on an assignment, please indicate which late day(s) you are using in your email submission. You must also draw **Platonic solids** corresponding to the late day(s) you are using (cube=1, tetrahedron=2, octahedron=3, dodecahedron=4, icosahedron=5). Use them wisely, as you cannot use the same polyhedron twice! If you are typesetting your homework on the computer, we have provided images that can be included for this purpose on the course webpage (in LaTeX these can be included with the `\includegraphics` command in the `graphicx` package).



Collaboration and External Resources. You are **strongly encouraged** to discuss all course material with your peers, including the written and coding assignments. You are especially encouraged to seek out new friends from other disciplines (CS, Math, Engineering, etc.) whose experience might complement your own. However, *your final work must be your own, i.e.*, direct collaboration on assignments is prohibited.

You are allowed to refer to any external resources *except* for homework solutions from previous editions of this course (at CMU and other institutions). If you use an external resource, cite such help on your submission. **If you are caught cheating, you will get a zero for the entire course.**

Warning! With probability 1, there are typos in this assignment. If *anything* in this handout does not make sense (or is blatantly wrong), let us know! We will be handing out extra credit for good catches. :-)

1 "Geodesics in Heat" Summary

Exercise 1. Read the paper "Geodesics in Heat: A New Approach to Computing Distance Based on Heat Flow" by Crane, Weischedel, and Wardetzky. Following the instructions for submitting a reading assignment on the course website.

See also this Youtube video for a more visual explanation.

2 Euler-Lagrange Equation Derivation

In the paper (p. 3), the authors have a vector field X on a surface M and seek to find $\phi : M \to \mathbb{R}$ which minimizes

$$E(\phi) := \int_M \|\nabla\phi - X\|^2 dA.$$

The paper then mentions that this minimum is achieved by precisely the solution to the *Euler-Lagrange equation* $\Delta\phi = \nabla \cdot X$. The following exercises informally derive this equation.

Exercise 2. Show that $E(\phi)$ is convex. That is, for all differentiable functions ϕ_1 and ϕ_2 and for all $\eta \in [0,1]$,

$$\eta E(\phi_1) + (1-\eta)E(\phi_2) \geq E(\eta\phi_1 + (1-\eta)\phi_2).$$

You are welcome to use the fact that a twice differentiable function on $\mathbb{R}^n$ is convex if and only if its Hessian is positive semidefinite. But you can't apply this directly to E since we haven't defined second derivatives of E.

Exercise 3. Assume that both $n \cdot X = 0$ and $n \cdot \nabla\phi = 0$, where n is the boundary normal. In other words, the normal component of both X and $\nabla\phi$ vanishes along the boundary of M (whether or not $\partial M = \emptyset$). Show that

$$E(\phi) = -\langle\langle \Delta\phi, \phi \rangle\rangle + 2\langle\langle \phi, \nabla \cdot X \rangle\rangle + \|X\|^2.$$

Hint: Use integration by parts like in the previous homeworks.

Exercise 4. Even though E takes functions as input, we can still discuss the gradient of E, but it requires a little machinery. First, we define the *directional* derivative to be

$$D_\psi E(\phi) = \lim_{\epsilon \to 0} \frac{E(\phi + \epsilon\psi) - E(\phi)}{\epsilon}.$$

Using the formula derived in the previous exercise, compute $D_\psi E(\phi)$.

Exercise 5. We define the gradient $\nabla E(\phi)$ to be the function such that

$$\langle\langle \nabla E(\phi), \psi \rangle\rangle = D_\psi E(\phi)$$

Compute $\nabla E(\phi)$. Use this to show that $\nabla E(\phi) = 0$ only if $\Delta\phi = \nabla \cdot X$.

3 Discretization of Divergence

How does one discretize a vector field on a surface? Like in the paper (see p.4), we assume that the vector field is piecewise constant with respect to the faces.

Exercise 6. Derive

$$\nabla \cdot X = \frac{1}{2} \sum_j \cot\theta_1 (e_1 \cdot X_j) + \cot\theta_2 (e_2 \cdot X_j)$$

by calculating the integrated divergence of X out of the dual cell for vertex i. (Hint: use Stokes' theorem.)

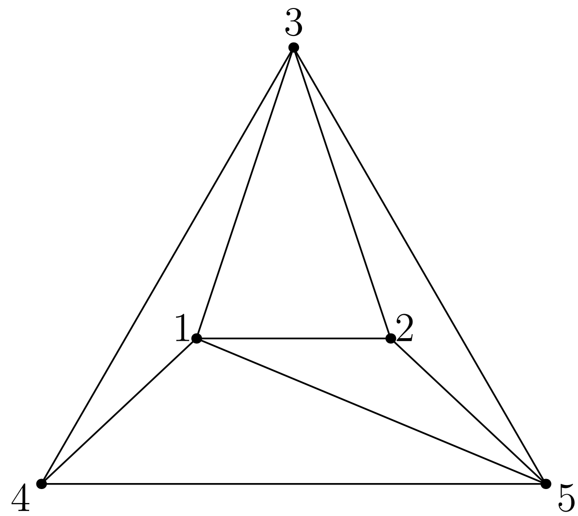
4 Discretization of Boundary Conditions

In the coding assignments, you have been working with the discrete Poisson problem

$$\Delta \phi = f,$$

where ϕ and f are discrete 0-forms (functions on the vertices of the mesh), and Δ is the cotan-Laplace operator. When the mesh M has no boundary (e.g., when it is a topological sphere), then this equation can be directly solved. In the case that M has boundary, some additional work in needed to discretize the boundary conditions, whether they are Dirichlet or Neumann. For further information, see the slides on the Laplace-Beltrami Operator.

In this section, we solve our discrete PDEs on the following mesh.



For simplicity of computation, assume that **all triangles are equilateral with unit length**. (There is an embedding in $\mathbb{R}^3$ for which this is the case!)

We further assume that

$$f_1 = 1, f_2 = 0, f_3 = -\tfrac{130}{33}, f_4 = \tfrac{108}{11}, f_5 = -\tfrac{130}{33}.$$

4.1 Dirichlet Boundary Conditions

Recall that Dirichlet boundary conditions specify the value of ϕ on ∂M.

Exercise 7. Consider the discrete Dirichlet boundary conditions

$$\phi_3 = 1, \phi_4 = -2, \phi_5 = 1.$$

Write down a *simplified* system of equations:

$$A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = b$$

and solve for ϕ_1 and ϕ_2. Remember the mass matrix!

4.2 Neumann Boundary Conditions

Recall that Neumann boundary conditions specify the normal derivative of ϕ on ∂M.

Exercise 8. Consider the Neumann boundary conditions

$$g_{34} = 1, g_{45} = 2, g_{53} = -1.$$

Write down the system of equations (in matrix form) corresponding to these constraints. As you will get a 5-variable system of equations, you do **not** need solve for ϕ.

4.3 Dirichlet and Neumann Boundary Conditions

Exercise 9. Can we specify both the Dirichlet and Neumann boundary conditions from the previous two exercises? If not, what goes wrong? (Your argument does not need to be incredibly formal. In particular, you do not need to solve the system(s) of equations.)